# Persistence of gaps in the interacting anisotropic Hofstadter model

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We consider an interacting version of the Hofstadter model, which in absence of interactions has a spectrum given by a Cantor set with infinitely many gaps, provided that the ratio $\alpha$ between the magnetic length and the lattice constant is an irrational number. In the anisotropic situation where the hopping $t$ in one direction is smaller then the other one, we rigorously prove that the $n$-th gap persists in presence of interaction, even for interactions much stronger than the gap. We assume a Diophantine property for $\alpha$ and that $t, U$ are positive and smaller than some constant, weakly depending on $n$. The proof relies on a subtle interplay of Renormalization Group arguments combined with number-theoretic properties of the incommensurate frequencies.

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## I. INTRODUCTION

The energy spectrum of non-interacting electrons moving through magnetic fields in a lattice provides one of few example of fractals in quantum physics. A paradigmatic example is provided by the *Hofstadter model* [1], [2] describing non-interacting fermions hopping on a square lattice with a magnetic field in the orthogonal direction; one can consider also its anisotropic generalization where the hopping $t$ in one direction is smaller then the other one. The crucial parameter is the ratio $\alpha$ between the magnetic length (or cyclotron length) and the lattice constant. If $\alpha$ is rational the two lengths are commensurate and Bloch theory predicts a finite number of gaps. In correspondence of the gaps one has an integer Hall conductivity [2]. If one considers sequences of rationals $\alpha$ converging to an irrational, more and more gaps open and this gives indication that when $\alpha$ is irrational infinitely many gaps and a fractal spectrum appear. In the non-interacting case, the properties of the Hofstadter model can be deduced by the one dimensional single particle Harper or almost-Mathieu equation. A huge mathematical effort has been devoted to its analysis, starting from [3], [4] using Kolmogorov-Arnold-Moser (KAM) methods, and culminating in [5], where the proof that the spectrum is a Cantor set for any irrational $\alpha$ and $t$ was achieved (with all gaps open [6]).

The interest in the Hofstadter model has been renewed by recent experiments [7] (see also [8], [9], [10]) in which, using bilayer graphene, periodic structures with lattice periodicity comparable to magnetic length has been created and information on the gap is obtained by longitudinal and Hall conductivity measurements. Such experiments reveal also the presence of many body interaction. The natural question is therefore what is the influence of the interaction on the gaps; it is rather natural to expect that the gap persists when is larger than the interaction, but in the opposite situation, when the interaction is much stronger than the gap, the interaction can in principle radically alter or close it.

The mathematical techniques developed for the non-interacting Hofstader cannot be extended to the interacting model, as the problem has now infinite degrees of freedom. Most of previous studies on interactions in the Hofstadter model have been analyzed by approximate methods (e.g. of a mean-field type which effectively reduces the many-body effects) [11]-[15] or were restricted to very small system sizes. In [16]-[18] the anisotropic Hofstadter model was effectively described in terms of an array of wires, and the continuum limit, where the difference between the commensurate or incommensurate case is lost, makes possible a bosonization approach. Incommensurability effects are however known to be crucial in the Hofstadter model. The effect of interaction on one dimensional fermionic systems with quasi periodic potential has been studied in generalized Aubry-Andre' models in the extended regime in [19], and in the related case of interacting fermions with a Fibonacci potential in [20] [21]; the interacting Aubry-Andre' model in the localized regime has been considered in [22],[23] and in a dynamical context in [24]-[28]. In the one dimensional Fibonacci chain, a scenario was indeed suggested in [20], [21] according to which the gaps can be closed by interactions greater than the gaps (in the attractive case). The equivalence between Hofstadter and one dimensional models is lost in presence of interactions, but such results suggest that the effect of interaction can indeed qualitatively change the behavior.

In order to get information on the gaps in the interacting Hofstatder model we compute the large distance behavior of the thermodynamical correlations for values of the chemical potential corresponding to the gaps of the non-interacting case. The persistence of the gaps is signaled by the presence of a faster than any power large distance decay. We consider the anisotropic situation where the hopping $t$ in one direction is smaller then the other one, and we write the correlations as series in $t, U$. A very important point is that we get a convergent expansion. One has to face, even in the non-interacting case, with a *small divisor problem*, caused by processes involving large exchange of momentum such that, due to Umklapp, connect with arbitrary precision the Fermi points. Small divisors make the problem non-perturbative; physical properties cannot be understood by lowest order analysis but are encoded in the divergence

or convergence of the whole perturbative series. Typical examples of small divisor problems in classical mechanics are the Birkhoff series for prime integrals of perturbed integrable Hamiltonian system, which are typically diverging (Poincare' theorem), or the series for KAM tori which are instead convergent [29]. Our approach combines non-perturbative Renormalization Group (RG) methods with techniques coming from the analysis of Lindstedt series for KAM tori. The main difficulty relies in the fact that the incommensurability produces an infinite set of effective interactions almost connecting the Fermi points, and the persistence or not of gaps is connected by their relevance or irrelevance in the RG sense. We have a condition of smallness of the parameters, depending on $n$; we have however no condition on the relative size between $t$ and $U$ so that we get information not only when the gap is larger than the interaction, but in the opposite situation, when the interaction is much stronger than the gap.

The rest of this paper is organized as follows. In §II we introduce the model and we present the main result. In §III we recall the main features of the non interacting case. In §IV we analyze the Euclidean correlations of the interacting model by rigorous Renormalization Group methods. In §V we show the convergence of the RG iteration, using a Diophantine property for $\alpha$. Finally in §VI the main conclusions are presented.

## II. THE INTERACTING HOFSTADTER MODEL

We consider an interacting version of the Hofstadter model in which spinful fermions in a square lattice are subject to a vector potential $\vec{A} = (-Bx_2, 0, 0)$ and interact through a Hubbard interaction.

The Hamiltonian of the (anisotropic) Hofstadter-Hubbard model is $H = H_0 + V$ with $H_0 =$

$$\sum_{\vec{x},\sigma=\uparrow,\downarrow} \{-\frac{t_1}{2}(a^+_{\vec{x}+\vec{e}_1,\sigma} e^{-i2\pi\alpha x_2} a^-_{\vec{x},\sigma} + a^+_{\vec{x},\sigma} e^{i2\pi\alpha x_2} a^-_{\vec{x}+\vec{e}_1,\sigma}) - \frac{t_2}{2}(a^+_{\vec{x}+\vec{e}_2,\sigma} a^-_{\vec{x},\sigma} + a^+_{\vec{x},\sigma} a^-_{\vec{x}+\vec{e}_2,\sigma}) + \mu a^+_{\vec{x},\sigma} a^-_{\vec{x},\sigma})\} \quad (1)$$

where $a^{\pm}_{\vec{x},\sigma}$ are fermionic operators, $\sigma$ is the spin, $\vec{x} = (x_1, x_2)$ are points in a square lattice with step 1 ( pbc in the 1 direction and Direchelet in direction 2), $t_1, t_2$ are the hopping parameters, $\mu$ is the chemical potential, $B = 2\pi\alpha$ and the interaction is

$$V = U \sum_{\vec{x}} a^+_{\vec{x},\uparrow} a^-_{\vec{x},\uparrow} a^+_{\vec{x},\downarrow} a^-_{\vec{x},\downarrow} \quad (2)$$

with $U > 0$. In the $t_2 = 0$, the multi-wire limit, the system reduces to uncoupled one dimensional interacting chains parametrized by $x_2$. We choose a chemical potential such that $|\mu| < t_1$ and we define $\mu = t_1 \cos p_F$ where $p_F$ is the Fermi momentum.

For definiteness we set $t_1 = 1$ and $t_2 = t$. The Hamiltonian can be written as $H_0 = \sum_{k_1} H_0(k_1)$; the eigenfunctions of $H_0(k_1)$ are Slater determinants of the eigenfunctions of the (single-particle) one dimensional almost-Mathieu or Aubry-Andre' equation, parametrized by $k_1$

$$-t(u_{x_2-1} + u_{x_2+1}) - 2\cos(k_1 - 2\pi\alpha x_2) u_{x_2} = E u_{x_2} \quad (3)$$

The existence of quasi-Floquet states was proven by KAM methods in [3],[4] assuming that $\alpha$ is *Diophantine*, that is there exists $C_0, \tau$ such that

$$||2k\pi\alpha|| \geq C_0 |k|^{-\tau}, \qquad k \neq 0 \quad (4)$$

$||.||$ being the norm on the one dimensional $2\pi$ torus, $k$ integer. Any irrational except a zero measure set verifies such a property for some $C_0, \tau$. In particular in [4] the existence of gaps was proved in correspondence of rotation number $n\pi\alpha$ for small $t$. After several developments, it was proved that the spectrum is a Cantor set [5] for any irrational $\alpha$ and any $t$. The above properties says that the gaps of $H_0$ are located in correspondence of Fermi momenta of the form

$$p_F = n\pi\alpha \mod 2\pi \quad (5)$$

with $n$ integer; equivalently (5) can be written as $N/N_0 = n_F\alpha + s$, with $N$ the number if fermions, $N_0$ the maximal number of fermions and $s$ integer.

When the interaction is present the system is not reducible to a one dimensional one. Information on the spectrum can be obtained by the large distance decay of imaginary time correlations. If $a^{\pm}_{\underline{\vec{x}},\sigma} = e^{Hx_0} a^{\pm}_{\vec{x},\sigma} e^{-Hx_0}$ with $\underline{x} = (\mathbf{x}, x_2)$, $\mathbf{x} = x_0, x_1$, the zero temperature 2-point is $S(\underline{x}, \underline{y}) =< a^-_{\underline{\vec{x}},\sigma} a^+_{\underline{y},\sigma} >$ with $< O >= \lim_{\beta\to\infty, L\to\infty} Tre^{-\beta H} TO / Tre^{-\beta H}$, $T$ is time ordering. Let us fix the Fermi level in a gap and switch on the interaction.

As the interaction changes the Fermi momentum, we choose $\mu = \cos p_F + \nu_{x_2}$ and we choose $\nu_{x_2}$ so that the Fermi momentum is equal to (5). Our main result is the following

**Theorem** *Assume $p_F$ verifying (5) and $\alpha$ verifying (4); there exists $\varepsilon_0$ such that for a suitable $\nu_{x_2}$ and assuming $0 < t, U < \varepsilon_0$ then , for any $N$*

$$|S(\underline{x}, \underline{y})| \leq \frac{1}{|\mathbf{x} - \mathbf{y}|^{1+\eta}} \frac{C_N}{1 + (\bar{\Delta}|x_2 - y_2| + \sigma_n |\mathbf{x} - \mathbf{y}|)^N} \quad (6)$$

*with $\bar{\Delta} = |\log t|$ and, if $\sigma^0_n$ is $\sigma_n$ at $U = 0$*

$$\sigma_n = \sigma^0_n + R_U \quad |R_U| \leq t^n C_n U^2 \quad (7)$$

*and $3^{-n} t^n \leq \sigma^0_n \leq C_n t^n$.*

The faster than any power decay in the imaginary time signals the presence of a gap in the spectrum of the interacting Hofstadter model; the decay rate $\sigma_n$ provides an estimate of the gap. The interacting gap is equal to the non interacting one $\sigma^0_n$ plus a correction $R_U$ expressed by a convergent expansion in $U, t$. From (7) we

see that interacting and bare gap have the same size not only when $U << \sigma_n^0$ but also in the opposite situation $U >> \sigma_n^0$ (it is sufficient to require $3C_nU^2 < 1$). Therefore gaps persist even in situation when the interaction $U$ is much larger than the gap, so excluding a scenario like the one in [20], [21]. The result of persistence of gaps is proved in the anisotropic case and assuming $t, U$ small; our estimate on $\varepsilon_0$ can be obtained by collecting all the constants in §5, like (43), and it depends on $n, C_0, \tau$. It is likely that such dependence is spurious and due to the use of KAM methods for getting convergence; it is indeed known to be absent in the $U = 0$ case [5] but the methods used there cannot be extended to infinitely many particles (even extensions to 2 particles are hard [30]). The dependence we got in $\varepsilon_0$ on $n$ is weak and the result proves that there is a region of parameters for which, for fixed $t, U$, the gap with $n$ not too large persists even if $U$ is much larger than the gap; convergence allows us to exclude non perturbative effects, quite possible in presence of small divisors.

The analysis is based on a multiscale expansion expressing quantities in terms of running coupling constants. There are two phenomena apparently spoiling convergence producing factorials in the bounds; one is the small divisor problem and the other are the presence of fermionic loops in presence of interaction again producing factorials. Convergence is achieved using determinant bounds for fermionic expectations (what eliminates the $k!$ present in Feynman graph expansions at order $k$ which spoil convergence) and using techniques coming from the analysis of Lindstedt KAM series to deal with small divisors. One has also to control the flow of the effective coupling, and the partial asymptotic freedom is what makes necessary the condition $U > 0$. The condition of smallness of $t, U$ is due to the fact that we expand in $t, U$. One cannot however rely on results on the non-interacting case (expanding only in $U$ and not on $t, U$ as we do) as the theory is not analytically close to the non-interacting one, due to the presence of anomalous exponents, see (6) and Luttinger liquid behavior in the $t = 0$ case; an expansion in $U$ would be convergent only for $U$ smalller than $O(\log \sigma_n^0)$.

## III. SMALL DIVISORS AND FEYNMAN GRAPHS

The persistence of gaps is studied expanding the imaginary-time correlations around the point $U = t = 0$, where the system reduces to a collection of independent fermionic wires labeled by $x_2$ with dispersion relation $\cos(k_1 - 2\pi\alpha x_2)$; the Fermi points are given by, if $p_F = \pi n_F \alpha$

$$p_{\pm}^{x_2} = \pm p_F + 2\pi\alpha x_2 \tag{8}$$

if $\mu = \cos p_F$. The 2-point function $S(\underline{x}, \underline{y})|_{t=U=0} \equiv \bar{g}(\underline{x}, \underline{y})$ is

$$\bar{g}(\underline{x}, \underline{y}) = \delta_{x_2, y_2} \int d\mathbf{k} e^{i\mathbf{k}(\mathbf{x}-\mathbf{y})} \widehat{g}_{x_2}(\mathbf{k}) \tag{9}$$

where

$$\widehat{g}_{x_2}(\mathbf{k}) = \frac{1}{-ik_0 + \cos(k_1 - 2\pi\alpha x_2) - \cos p_F} \tag{10}$$

We call $g_{x_2}(\mathbf{x})$ the Fourier transform of $\widehat{g}_{x_2}(\mathbf{k})$. It is convenient to write the imaginary-time correlations in terms of the following Grassmann integral

$$e^{W(\phi)} = \int P(d\psi) e^{-T-V-N-(\psi,\phi)} \tag{11}$$

with

$$\begin{aligned} T &= \sum_{x_2,\sigma} \int d\mathbf{x} (\psi^+_{\mathbf{x},x_2+1,\sigma} \psi^-_{\mathbf{x},x_2,\sigma} + \psi^+_{\mathbf{x},x_2-1,\sigma} \psi^-_{\mathbf{x},x_2,\sigma}) \\ V &= U \sum_{x_2,\sigma} \int d\mathbf{x} \psi^+_{\mathbf{x},x_2,\uparrow} \psi^-_{\mathbf{x},x_2,\uparrow} \psi^+_{\mathbf{x},x_2,\downarrow} \psi^-_{\mathbf{x},x_2,\downarrow} \\ N &= \sum_{x_2,\sigma} \nu_{x_2} \int d\mathbf{x} \psi^+_{\mathbf{x},x_2,\sigma} \psi^-_{\mathbf{x},x_2,\sigma} \end{aligned} \tag{12}$$

and $(\psi, \phi) = \sum_{x_2,\sigma} \int d\mathbf{x} (\psi^+_{\mathbf{x},x_2,\sigma} \phi^-_{\mathbf{x},x_2,\sigma} + \psi^-_{\mathbf{x},x_2,\sigma} \phi^+_{\mathbf{x},x_2,\sigma})$. The term $N$ has been introduced writing the chemical potential as $\mu = \cos p_F + \nu_{x_2}$, in order to take into account its possible renormalization due to the interaction. The 2-point function is given by $S(\underline{x}, \underline{y}) = \frac{\partial^2 W}{\partial \phi^+_{\underline{x}} \partial \phi^-_{\underline{x}}}|_0$. One can write the correlation in terms of Feynman diagrams with propagators (9) ; examples are in Fig. 1. The small divisors problem is clearly exhibited already in the non-interacting case $U = 0$. Consider a chain graph contributing to the effective potential $\int d\mathbf{k} \phi^+_{x_2,\mathbf{k}} W_2(\mathbf{k}) \phi^-_{x'_2,\mathbf{k}}$

FIG. 1: A graph with four external lines of order $t^3U^3$ and another with two external lines of order $t^4$.

with $x_2' = x_2 + \sum_{k=1}^{n} \varepsilon_k$, $\varepsilon_k = \pm 1$ and

$$W_2(\mathbf{k}) = t^n \prod_{k=1}^{n-1} \frac{1}{-ik_0 + \cos(k - 2\pi\alpha(x_2 + \varepsilon_k)) - \cos p_F} \tag{13}$$

The infrared divergences in many body perturbation theory are associated with the repetitions of propagators with the same momentum $k'$ measured from the Fermi points, that is $k_1 = k' + p_\omega^{x_2}$, if $\omega = \pm$; if $x_2$ and $x_2'$ are the coordinates associated to two propagators, this happens if $x_2 = x_2'$, $\omega = \omega'$, $\omega = \pm$ or, if $p_F = n_F \pi\alpha$, if $x_2 - x_2' = -\omega n_F$ and $\omega = -\omega'$: in such cases the subgraph are resummed in the self energy or the mass terms. If $\alpha$ is rational, if $x_2 - x_2' \neq 0, \omega n_F$ the denominators differ by a finite quantity $O(1/q)$ if $\alpha = p/q$ with $p, q$ coprime. If $\alpha$ is irrational, however, $2\pi\alpha(x_2 - x_2')$ can be *arbitrarily close* mod. $2\pi$ to 0 or $2n_F\pi\alpha$; in other words, due to Umklapp terms involving the exchange of $2\pi$, there are propagators with almost the same size which cannot be resummed in self energy or mass terms. This produces an accumulation of small divisors which could cause a failure of the expansion.

Consider for instance the case $\varepsilon_k = 1$ in (13); then the momenta flowing in the propagators would be all different. By the diophantine condition, if $k = n_F\pi\alpha$ we can bound each propagator by

$$|\widehat{g}_{x_2}(\mathbf{k})| \leq \frac{C}{||2\pi\alpha x_2 + 2\pi\alpha n_F||} \leq C|x_2 + n_F|^\tau \tag{14}$$

so that

$$|W_2(\mathbf{k})| \leq C^n t^n \prod_{k=1}^{n} k^\tau \leq C^n t^n n!^\tau \tag{15}$$

The appearance of such factorials, possibly breaking the convergence of the series, is what is known in classical mechanics as small divisors. Physical information cannot be decided on the basis of lowest order analysis, but it depends on the convergence or divergence of the whole series. Formal series for prime integrals in perturbed integrable Hamiltonian systems are order by order finite but typically non convergent, that is no prime integrals except the energy exists (Poincare' theorem). In other cases, instead, the bounds can be improved and the factorials cancel out; this is what happens in Lindstedt series for KAM tori. This is also what happens in the Hofstadter model, where convergence of perturbation theory is implied by results on the almost Mathieu equation using KAM methods. The persistence of the gap in the interacting Hofstadter model depends on the convergence or divergence of its series expansions, which contains also graphs with loops in addition to chain graphs, and cannot be decided on the basis of lowest order perturbative considerations.

## IV. RENORMALIZATION GROUP ANALYSIS

We study the 2-point function of the interacting Hofstatder model by exact RG methods. The starting point is the multiscale decomposition of the propagator

$$g_{x_2}(\mathbf{x}, \mathbf{x}') = g_{x_2}^{(1)}(\mathbf{x}, \mathbf{x}') + \sum_{\omega=\pm} g_{\omega;x_2}^{(\leq 0)}(\mathbf{x}, \mathbf{x}') \tag{16}$$

where $\widehat{g}_{\omega;x_2}^{(\leq 0)}(\mathbf{k})$ has support in a region around $(0, p_\omega^{x_2})$, $\omega = \pm$, and $\widehat{g}_{x_2}^{(1)}(\mathbf{k})$ in the complement of such regions.

It is convenient to measure the momenta from the Fermi points writing $k_1 = k' + \omega p_F + 2\pi\alpha x_2$; therefore $\psi = \psi^{(1)} + \sum_{\omega=\pm} e^{ip_\omega^{x_2} x_1} \bar{\psi}_\omega^{(\leq 0)}$ and the propagator of $\bar{\psi}_\omega^{(\leq 0)}$ is

$$g_\omega^{(\leq 0)}(\underline{x}, \underline{x}') = \delta_{x_2, x_2'} \int d\mathbf{k}' \frac{\chi_0(\mathbf{k}') e^{i\mathbf{k}'(\mathbf{x}-\mathbf{x}')}}{-ik_0 + \omega v_F k' + r(k')} \tag{17}$$

with $r(k') = O(k'^2)$ and $\chi_0(\mathbf{k}')$ has support around $\mathbf{k}' = 0$.

Integrating the scales $\leq 0$ we get a sequence of effective potentials sum of terms of the form $\int W_n \prod_{i=1}^{n} \psi_{\omega_i, \mathbf{k}_i', x_{2,i}}^{\varepsilon_i}$, $\varepsilon, \omega = \pm$, with momenta $\mathbf{k}_i'$ verifying the relations

$$\sum_i \varepsilon_i k_i' = \sum_i \varepsilon_i \omega_i p_F + \sum_i \varepsilon_i 2\pi\alpha x_{2,i} \quad \text{mod.} 2\pi \tag{18}$$

Note that the momenta measured from the Fermi points are not conserved unless the r.h.s. of the above expression is vanishing.

After the integration of $\psi^{(1)}$ a mass term, which was absent in the original interaction, is generated, of the form

$$\sum_{x_2} \int d\mathbf{k}' W_2^0(\mathbf{k}') (\psi_{+,x_2-n_F,\mathbf{k}'}^{(\leq 0),+} \psi_{-,x_2,\mathbf{k}'}^{(\leq 0),-} + \psi_{-,x_2,\mathbf{k}'}^{(\leq 0),+} \psi_{+,x_2-n_F,\mathbf{k}'}^{(\leq 0),-}) \tag{19}$$

which connect fields in chains $x_2, x_2 - n_F$, with momenta near $p_{x_2-n_F}^+ = p_F + 2\pi\alpha(x_2 - n_F)$ to $p_{x_2}^- = -p_F + 2\pi\alpha(x_2) = p_{x_2-n_F}^+$. The lowest order contribution to $W_2^0(\mathbf{k})$ is the chain graph , see Fig.2, with

$$G_{x_2-n_F,x_2}(\mathbf{k}) = t^{n_F} g_{x_2-n_F+1}^{(1)}(\mathbf{k}) g_{x_2-n_F+2}^{(1)}(\mathbf{k}) ... g_{x_2-1}^{(1)}(\mathbf{k}) \tag{20}$$

and the contribution to $\alpha$ is obtained by computing it at $\mathbf{k} = \mathbf{p}_{x_2}^+ \equiv (0, \mathbf{p}_{x_2}^+)$. This chain graph is independent from $U$; regarding the lowest order contribution in $U$, there are no linear terms in $U$ as the interaction connect only fields with the same $x_2$. The second order contribution in $U$ is given by, see Fig. 2, if $G_{x_2,y_2}(\mathbf{k})$ is defined in (20)

$$A(\mathbf{p}_{x_2}^+) = \int d\mathbf{k}_1 d\mathbf{k}_2 G_{x_2-n_F,x_2}(\mathbf{k}_1) \times$$
$$G_{x_2,x_2-n_F}(\mathbf{k}_2) G_{x_2,x_2-n_F}(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{p}_{x_2}^+) \tag{21}$$

$x_2$ $x_2 - n_F$

FIG. 2: The upper graph is a contribution to the mass of order $U^2 t^{3n_F}$; the lower graph is a contribution $t^{n_F}$.

with $\mathbf{p}^+_{x_2} = (0, 0, p^+_{x_2})$. Similar contributions appears integrating out the lower scales. It is convenient to add and subtract a factor

$$M = \sum_{x_2} \alpha_{x_2} \int d\mathbf{x} (\psi^+_{+,\mathbf{x},x_2-n_F} \psi^-_{-,\mathbf{x},x_2} + \psi^+_{-,\mathbf{x},x_2} \psi^-_{+,\mathbf{x},x_2-n_F}) \tag{22}$$

which is included in the free integration. We include such term in the free integration, and we set $P(d\psi^{\le 0}) e^M \equiv \widetilde{P}(d\psi^{\le 0})$, with $\widetilde{P}(d\psi^{\le 0})$ with propagator, if $\omega_1 = -; \omega_2 = +$ and $\delta_1 = 0, \delta_2 = -1$

$$< \psi^-_{\omega_i,\mathbf{k}',x_2+\delta_i n_F} \psi^+_{\omega_j,\mathbf{k}',y_2+\delta_j n_F} >= \delta_{x_2,y_2} \chi_0(\mathbf{k}') \times \tag{23}$$

$$\begin{pmatrix} -ik_0 - v_F \sin k' + c(k') & \sigma_{x_2} \\ \sigma_{x_2} & -ik_0 + v_F \sin k' + c(k') \end{pmatrix}^{-1}_{i,j}$$

We consider $\sigma_{x_2}$ and $\alpha_{x_2}$ as independent, and we will choose $\alpha_{x_2}$ as function of $U$ and $\sigma$ so that the flow of the corresponding coupling is bounded; at the end we impose the condition

$$\sigma_{x_2} = \alpha_{x_2} \tag{24}$$

We describe our RG analysis inductively. We write $\psi^{(\le 0)}_\omega = \sum_{h=-\infty}^0 \psi^{(h)}_\omega$ and the corresponding propagator has cut-off $f_h$ with support in $\gamma^{h-1} \le |\mathbf{k}'| \le \gamma^{h+1}$ with $\gamma > 1$ a momentum scale.

After the integration of $\psi^{(0)}, ... \psi^{(h-1)}$ one gets that the generating function has the form

$$\int P(d\psi^{(\le h)}) e^{\mathcal{V}^{(h)}(\psi^{\le h}, \phi)} \tag{25}$$

where the propagator is

$$< \psi^-_{\omega_i,\mathbf{k}',x_2+\delta_i n_F} \psi^+_{\omega_j,\mathbf{k}',y_2+\delta_j n_F} >= \frac{\delta_{x_2,y_2}}{Z^{(h)}_1} \chi_h(\mathbf{k}') \tag{26}$$

$$\begin{pmatrix} -ik_0 - v_h \sin k' + c(k') & \sigma_{x_2} \\ \sigma_{x_2} & -ik_0 + v_h \sin k' + c(k') \end{pmatrix}^{-1}_{i,\mathrm{J}}$$

and $\mathcal{V}^{(h)}(\psi, 0) =$

$$\sum_{m,\underline{\omega}} \sum_{x_{2,1},..,x_{2,m}} \int d\mathbf{k}'_1 ... d\mathbf{k}'_m W^{(h)}_m(\underline{\mathbf{k}}') \prod_i \psi^{\varepsilon_i(\le h)}_{\omega_i,\mathbf{k}'_i,\mathbf{x}_{2,i}} \delta_m \tag{27}$$

where $\delta_m$ vanishing in correspondence of (18); $Z_h$ is a wave function renormalization, $v_h$ is an effective Fermi velocity and $\chi_h = \sum_{k \le h} f_k$ with support in $|\mathbf{k}'| \le \gamma^{h+1}$; $\mathcal{V}^{(h)}(\psi, \phi)$ as a similar expression as (27) with some of the fields $\psi$ replaced by external fields $\phi$.

We have to extract from the effective potential the relevant and marginal terms, which contribute to the corresponding running coupling constants. The scaling dimension of the theory is $D = 2 - n/2$, so all the terms with $n \ge 6$ are irrelevant. If we renormalize all the quartic terms $\psi^+_{\omega_1,x_{2,1}} \psi^-_{\omega_2,x_{2,2}} \psi^+_{\omega_2,x_{2,3}} \psi^-_{\omega_3,x_{2,4}}$ we would get a huge number of running coupling constants, one for any choice of $\omega_1, .., \omega_4$ and $x_{2,1}, .., x_{2,4}$. There is however a dramatic improving with respect to power counting, and a huge class of quadratic or quartic terms are indeed irrelevant, namely:

1. The terms such that the r.h.s. of (18) is non vanishing;

2. The quartic terms with different $x_{2,i}$, and the marginal quadratic terms with different $x_{2,i}$.

Condition (1) is quite natural in the commensurate case $\alpha = p/q$; indeed if it is violated than the corresponding process disappear at scales smaller that some energy scale $\bar{h} = O(\log 1/q)$ by conservation of momenta measured from the Fermi points. In the incommensurate case things are however more subtle. The l.h.s. of (18) can be arbitrarily small and there is no a finite scale below which such terms disappear. In other terms, there are quadratic processes which connect with arbitrary precision Fermi points $p^{x_2}_\omega$ can be arbitrarily close to $p^{x'_2}_{\omega'}$ for large $x_2 - x'_2$; deciding if they are relevant or irrelevant is a rather subtle issue which will be discussed below, and it can depend on the specific form of the considered quasi periodic system. Condition 2), on the other hand, depends on the presence of a gap.

We introduce a renormalization operation which acts on the quadratic or quartic terms. Regarding the quadratic terms, condition (1) says that the non irrelevant terms verify

$$(\omega_1 - \omega_2) p_F + 2\pi\alpha(x_{2,1} - x_{2,2}) = 0 \tag{28}$$

If $\omega_1 = \omega_2$ we define a renormalization operation $\mathcal{R}$ consisting in extracting from the kernel $W^h(\mathbf{k})$ the term $W^h(\mathbf{p}^{x_2}_\omega) + (k - p^{x_2}_\omega) \partial W^h(\mathbf{p}^{x_2}_\omega) + k_0 \partial W^h(0)$. The first term contributes to the renormalization of the chemical potential

$$F^{(h)}_\nu = \sum_{\omega,\sigma} \sum_{x_2} \int d\mathbf{x} \gamma^h \nu_{x_2} \psi^+_{\underline{x},\omega,\sigma} \psi^-_{\underline{x},\omega,\sigma} \tag{29}$$

while the other terms contribute to the wave function, that is $Z_{h-1} = Z_h (1 + \partial_0 W^h)$, and Fermi velocity renormalization.

On the other hand if $\omega_1 = -\omega_2 = \pm$ the r.h.s. of (18) is vanishing if $n_F = (x_{2,2} - x_{2,1})$ and $p^{x_2}_- = p^{x_2 - n_F}_+$; we

define the renormalization operation $\mathcal{R}$ in this case as the subtraction from the kernel $W^h(\mathbf{k})$ of the term $W^h(\mathbf{p}^{x_2}_\omega)$ and this produces an effective interaction

$$F^{(h)}_\alpha = \int d\mathbf{x} 2^h \alpha_{x_2} (\psi^+_{+,x_2-n_F}\psi^-_{-,x_2} + \psi^+_{-,x_2}\psi^-_{+,x_2-n_F}) \tag{30}$$

Regarding the quartic terms, the $\mathcal{R}$ operation is non trivial only on the quartic terms with the same $x_2$, and in such a case we extract from $W^h_4(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_2)$ the term $W^h_4(\mathbf{p}^{x_2}_{\omega_1},\mathbf{p}^{x_2}_{\omega_2},\mathbf{p}^{x_2}_{\omega_3},\mathbf{p}^{x_2}_{\omega 4})$. The effective potential can be therefore written as $\mathcal{V}^{(h)} = \mathcal{L}\mathcal{V}^{(h)} + \mathcal{R}\mathcal{V}^{(h)}$ where $\mathcal{L}\mathcal{V}^{(h)}$ is the relevant or marginal part

$$\mathcal{V}^{(h)}(\psi,0) = F^{(h)}_\nu + F^{(h)}_\alpha + F^{(h)}_1 + F^{(h)}_2 + F^{(h)}_4 \tag{31}$$

with

$$F^{(h)}_1 = \sum_{x_2,\sigma,\sigma',\omega} \int d\mathbf{x} g_{1,h,x_2} \psi^+_{\underline{x},\omega,\sigma}\psi^-_{\underline{x},-\omega,\sigma}\psi^+_{\underline{x},-\omega,\sigma'}\psi^-_{\underline{x},\omega,\sigma'}$$

$$F^{(h)}_2 = \sum_{x_2,\sigma,\sigma',\omega} \int d\mathbf{x} g_{2,h,x_2} \psi^+_{\underline{x},\omega,\sigma}\psi^-_{\underline{x},\omega,\sigma}\psi^+_{\underline{x},-\omega,\sigma'}\psi^-_{\underline{x},-\omega,\sigma'}$$

$$F^{(h)}_4 = \sum_{x_2,\sigma,\sigma',\omega} \int d\mathbf{x} g_{4,h,x_2} \psi^+_{\underline{x},\omega,\sigma}\psi^-_{\underline{x},\omega,\sigma}\psi^+_{\underline{x},\omega,\sigma'}\psi^-_{\underline{x},\omega,\sigma'}$$

Note that the quartic marginal terms in $\mathcal{L}V^h$ only connect fermions with the same $x_2$, that is in the same wire; all the processes connecting different wires are irrelevant. The only terms connecting different wires are the hopping terms. Integrating the field $\psi^h$ one gets an expression similar to (25) with $h$ replaced by $h-1$ and the procedure can be iterated.

We have to discuss the flow of the running coupling constants. Note that the RG flow stops at a scale $h^* = -\log\sigma$. One has first to fix the counterterms $\alpha,\nu$ so that the flow of the relevant running coupling constants is bounded. We write

$$\alpha_{h-1} = \gamma\alpha_h + \beta^h_\alpha \tag{32}$$

where in $\beta^h_\alpha$ one can separate two kinds of terms: a) the ones independent from $U$, which are $O(t^{n_F}\gamma^{\theta k})$ (the factor $\gamma^{\theta k}$, $0<\theta<1$ follows from the irrelevance of the $t$ vertices, see the following section); b) the ones with at least one $U$ or $g_{i,k}$ quartic coupling, which are at least quadratic in $U$ (both the initial interaction $V$ and the quartic effective interactions in $\mathcal{L}V^k$ involve fields with the same $x_2$) and $O(U^2\sigma^3\gamma^{-3h})$. Therefore we can choose $\alpha_0 \equiv \alpha_{x_2}$ so that the flow is bounded, that is $\alpha_0 = -\sum_{k=h^*}^0 \gamma^k\beta^k_\alpha$ and the r.h.s. is bounded by $\sum_{k=h^*}^0(\gamma^k t^{n_F}\gamma^{\theta k} + U^2\sigma^3\gamma^{-2k})$ and finally, extracting the dominant term

$$\alpha_{x_2} = t^{n_F}(a_{n_F} + R) \qquad |R| \le C(t+U^2) \tag{33}$$

and $t^{n_F}a_{n_F}$ is the contribution from the chain graph, see Fig. 2

$$a_{n_F} = \prod_{k=1}^{n_F-1} \frac{1}{\cos(-n_F\pi\alpha + 2\pi\alpha k) - \cos(n_F\pi\alpha)} \tag{34}$$

which is independent from $x_2$; moreover $\alpha_h$ behave as $t^{n_F}\gamma^{\theta h} + U^2\sigma^3\gamma^{-2h}$.

Similarly we have to control the flow of $\nu_h$; we write $\nu_h = \gamma\nu_{h+1} + \beta^\nu_h$ with $\beta^\nu_h$ is sum of terms $O(U\gamma^{\theta h})$ (the contributions independent on $t$, where the $\gamma^{\theta h}$ comes from a parity cancellation) and $O(t\gamma^{\theta h})$ (the terms containing $t$ vertices) or $O(U\sigma^2\gamma^{-2h})$; in order to have $\nu_h$ small we choose a $\nu_0$ so that $\nu_0 = -\sum_{k=h^*}^0 \gamma^k\beta_k$ and $|\nu_0| \le C(U+t)$ and $\nu_h$ behave as $(t+U)\gamma^{\theta h} + U\sigma^2\gamma^{-h}$.

In order to discuss the flow of the quartic running coupling constants $g_{1,h}, g_{2,h}, g_{4,h}$, we notice that we can write $g_{i,h-1} = g_{i,h} + \beta^h_{i,1} + \beta^h_{i,2}$ with $\beta^h_{i,1}$ sum of graphs containing only quartic vertices $g_{1,k}$ and $\beta^h_{i,2}$ with at least a vertex $t,\nu_k,\alpha_k,\sigma$. By iteration, if $i=2$ $g_{i,h-1} = g_{i,0} + \sum_{k=0}^h(\beta^k_{2,1}+\beta^k_{2,2})$ and the second addend is bounded by $\sum_{k=0}^h U^2(\alpha_h+\nu_h)$ hence is $O(U^2)$ while $\beta^h_{2,1}$ again is summable as is proportional to $g^2_{1,h}$; therefore $g_{2,h}, g_{4,h-1}$ tends to values which are $U+O(U^2)$. On the other hand $g_{1,h} \sim \frac{U}{1-aUh}$, that is tends to vanish for repulsive interactions while $v_h \to v_{-\infty} = v_F(1+O(U))$; finally the wave function renormalization behaves as $Z_h \sim \gamma^{\eta h}$ with $\eta = bU^2 + O(U^2), b>0$. By imposing the condition $\alpha = \sigma$ one gets the size of the gap in the interacting case.

It is finally convenient to compare the above flow with the in one dimensional models. In the interacting Aubry-Andre' model the flow of the gap term is linear in the effective coupling, as the quasi-periodic potential involve fermions on the same chain; therefore one has a contribution to the analogous of $\beta_\alpha$ of the form $\sigma\gamma^{-k}$ which corresponds to the generation of anomalous critical exponents in the gaps [19]. In interacting fermionic Fibonacci chains one considers infinitely many quadratic couplings and this produce a complex flow suggest the closure of all gaps except a finite number in the attractive case [20],[21].

## V. CONVERGENCE

As we discussed before the presence of small divisors in the expansions has the effect that information on persistence of gaps are encoded in the convergence or divergence of the whole renormalized series; in particular, one has to discuss the relevance or irrelevance of the Umklapp terms almost connecting Fermi points.

The kernels of the effective potential $V^h$ can be written as sum of graphs such that to each line connecting two points $\underline{x}$ with $\underline{y}$ is associated a scale $h$ and it corresponds to a propagator $\delta_{x_2,y_2}\bar{g}^{(h)}(\mathbf{x},\mathbf{y})$ defined by (27); to the vertices are associated the effective couplings $g_h, \nu_h, \alpha_h$ and the couplings $\lambda, t, \nu$. The scales induce a structure of *clusters* in the graph; each cluster $v$ with scale $h_v$ contains a connected subset of the graph, such that the internal propagators have scale $\le h_v$ and at least one of them scale $h_v$, and the external lines scales $> h_v$; the clusters can be represented as a tree $\tau$, see Fig.3. We call $S_v$ the number of subclusters $w$ in the cluster $v$, with $w' = v$,

connected by $S_v - 1$ propagators $g^{(h_v)}$. We associate a scale $h_v$ also to the end-points and $v'$ is the first cluster enclosing it; regarding the end-point $g_h, \delta_h, \nu_h$ one has $h_{v'} = h_v - 1$. We call $\bar{m}_v^i$, $i = t, g_h, \nu_h, \alpha_h$ the number of $i$ end-points in $v$ and not not contained in other smaller clusters, and $m_v^i$, $\alpha = t, g_h, \nu_h, \alpha_h$ the total number of $i$ end-points in $v$. To each cluster $v$ is associated a set of $p_v$ external lines with scale $< h_v$ and coordinate $\mathbf{x}_i$.


FIG. 4: A representation of a cluster $v$ and the $S_v$ subclusters. The lines internal to the blob have scale $h_v$, the lines external $h_{v'}$; the gray blobs have a similar structure and so on.

FIG. 3: A graph with its clusters and the corresponding tree

We can define two kind of clusters:

1. The *non resonant* clusters $v \in NR$ are such that $\sum_i \varepsilon_i p_{x_{2,i}}^{\omega_i} \neq 0$
2. The *resonant* clusters $v \in R$ are such that $\sum \varepsilon_i p_{x_{2,i}}^{\omega_i} = 0$; $v \in R1$ are such that all the $x_{2,i}$ of the external lines are equal; $v \in R2$ are such that all the $x_{2,i}$ of the external lines are not all equal.

According to the previous definitions, the $\mathcal{R}$ operation acts non trivially only on the clusters $v \in R1$ with 2 or 4 external lines or $v \in R2$ with two external lines. In the quartic terms the action of $\mathcal{R}$ consists in replacing an external field $\psi_{\mathbf{x}}$ with $\psi_{\mathbf{x}} - \psi_{\mathbf{y}} = (\mathbf{x}-\mathbf{y}) \int_0^1 dt \partial\psi$; the same action is for the terms with two external lines $v \in R2$, while there is a replacement with the second difference when $v \in R1$ and two external lines. With respect to the $\mathcal{R} = 0$ case, this corresponds to an extra derivative on the external lines, giving a factor $\gamma^{h_{v'}}$ and an extra $(\mathbf{x} - \mathbf{y})$ which can be associated to the propagators $g^{h_v}$ and produces dimensionally a factor $\gamma^{-h_v}$. The same factor is obtained in quadratic terns $v \in R2$ while in the quadratic term $v \in R1$ the second difference produces a term $\gamma^{2(h_{v'}-h_v)}$. In conclusion the $\mathcal{R}$ operation produces a factor $\gamma^{z_v(h_{v'}-h_v)}$ with a) $z_v = 1$ if $p_v = 4$ $v \in R1$; b) $z_v = 2$ if $p_v = 2$ and $v \in R1$; b) $z_v = 1$ if $v \in R2$ and $p_v = 2$; $z_v = 0$ in all the other cases.

The size of a generic Feynman graph is easily obtained using that $|g^h(\mathbf{x})| \leq C\gamma^h$ and $\int d\mathbf{x}|g^h(\mathbf{x})| \leq C\gamma^{-h}$; by choosing in the graph a tree of propagators connecting the $S_v$ clusters or end-points, see Fig. 4, we get by integrating a factor $\gamma^{-2h_v(S_v-1)}$ while the remaining propagators are bounded by $\gamma^{h_v(n_v-S_v+1)}$, where $n_v$ is the number of propagators $g^{h_v}$: note that the sum over $x_2$ is done using the kronecker deltas in the propagator of the tree, causing that only one sum remain. The bound for the Feynman graph is proportional to, up to a constant $C^m$, $m$ is the number of vertices and not taking into account the $\mathcal{R}$ operation

$$\prod_v \gamma^{-2h_v(S_v-1)} \prod_v \gamma^{n_v h_v} \prod_v (\nu_{h_v}\gamma^{h_v})^{\bar{m}_v^\nu}$$
$$\prod_v t^{\bar{m}_v^t} \prod_v (\alpha_{h_v}\gamma^{h_v})^{\bar{m}_v^\alpha} = \gamma^{(2-n/2)h} \prod_v \gamma^{-(h_v-h_{v'})D_v}$$
$$\prod_v (t\gamma^{-h_v})^{\bar{m}_v^t} \prod_v (\nu_{h_v})^{\bar{m}_v^\nu} \prod_v (\alpha_{h_v})^{\bar{m}_v^\alpha} \tag{35}$$

with $D_v = 2 - n_v^e/2$ and $n_v^e$ is the number of external lines of $v$. In principle a bound on Feynman graphs is not enough for getting non-perturbative information; even if a finite bound is obtained at order $m$, one has to worry about extra combinatorial $m!$ due to the large number of graphs which could ruin convergence. It is however a well known fact that cancellations due to Pauli principle in fermionic expansions has the effect that such extra $m!$ are absent, see e.g. [23]. We get therefore the following estimate, if $\varepsilon = max(|U|, t^{\frac{1}{2}})$ and using that the $g_{i,h}$ are bounded by bare coupling $U$ times a constant, if $U > 0$, as discussed in the previous section

$$\frac{1}{L\beta} \int d\mathbf{x}|W^h(\mathbf{x})| \leq \sum_m \varepsilon^m \sum_{\tau, h_v, n_v} \gamma^{(2-n/2)h}$$
$$[\prod_v (\sigma\gamma^{-h_v})^{\bar{m}_v^\sigma}][\prod_v \gamma^{-(h_v-h_{v'})(D_v+z_v)} \prod_v (t^{\frac{1}{2}}\gamma^{-h_v})^{\bar{m}_v^t}$$

where we take into account the effect of the $\mathcal{R}$ operation and of the presence of non-diagonal propagators, giving extra factors $\prod_v (\sigma\gamma^{-h_v})^{\bar{m}_v^\sigma}$. One needs to sum over all the possible attributions of scales $h_v$; the sum would be finite of $D_v + z_v$ can be vanishing or negative,

what however is not the case. This lack of convergence is a manifestation of the small divisor problem, as it is due also to the fact that we have not renormalized the quadratic and quartic non resonant terms. In order to show that they give a finite contribution one has to improve the estimate by the Diophantine property of $\alpha$ (4). Let us consider a non resonant cluster $v \in NR$ with 2 external lines; we get, $\delta = 0, 1$

$$2\gamma^{h_{v'}} \geq ||k_1'|| + ||k_2'|| \geq ||k_1' - k_2'|| \\ \geq ||2\delta n_F \pi\alpha + 2\pi\alpha(x_2 - x_2')|| \geq C_0 |x_2 - x_2'|^{-\tau}$$

so that

$$|x_2 - x_2'| \geq C\gamma^{\frac{-h_{v'}}{\tau}} \tag{36}$$

This says that in order to have a cluster a low scales the difference of coordinates must be large. In addition, if we apply this to the $t$ vertices when $x_2 - x_2' = \pm 1$ it says that $h_v'$ is bounded by a constant so that

$$\prod_v (t^{\frac{1}{2}} \gamma^{-h_v})^{\bar{m}_v^t} \leq \prod_v (t^{\frac{1}{2}} C)^{\bar{m}_v^t} \tag{37}$$

Regarding the terms with 4 lines we can write

$$4\gamma^{h_{v'}} \geq ||\sum_i \varepsilon_i \mathbf{k}_i'|| \geq ||2\pi\alpha \sum_{i=1}^4 \varepsilon_i x_{2,i} + \sum_i \varepsilon_i \omega_i \pi n_F \alpha|| \\ \geq C_0 |\sum_{i=1}^4 \varepsilon_i x_{2,i} + \sum_i \varepsilon_i \omega_i n_F|^{-\tau} \geq C|\bar{x}_2 - \bar{x}_2'|^{-\tau}$$

where $|\bar{x}_2 - \bar{x}_2'|$ is the maximal difference of the $x_2$ of the incoming and outcoming lines; therefore

$$|\bar{x}_2 - \bar{x}_2'| \geq C\gamma^{\frac{-h_{v'}}{\tau}} \tag{38}$$

Note that there is a path of propagators connecting the external lines with coordinates $\bar{x}_2$ and $\bar{x}_2'$ and

$$|\bar{x}_2 - \bar{x}_2'| \leq n_F N_v + m_v^t \leq 2 n_F N_v \tag{39}$$

where $N_v$ is the number of vertices in the cluster $v$; the reason is that one modify the coordinate by non diagonal propagators or vertices $t$. In conclusion

$$N_v \geq C_0 \gamma^{\frac{-h_{v'}}{\tau}} / n_F^{\frac{1}{\tau}} \tag{40}$$

where $C_0, \tau$ are the parameters appear in in the Diophantine condition (5).

We can now associate to each vertex in the graph a constant $\bar{c} < 1$ (at the expense of a factor $\bar{c}^{-m}$ in the final bound). Moreover we can write $\bar{c} = \prod_{h=-\infty}^1 \bar{c}^{2^h/2}$ so that we can associate a factor $c^{2^h_v/2}$ to each of the $N_v$ vertices contained in a cluster $v$; therefore

$$\bar{c}^m \leq \prod_v \bar{c}^{N_v 2^{h_v}} \leq \prod_v \bar{c}^{N_v 2^{h_{v'}}} \tag{41}$$

and using (40) one gets

$$\bar{c}^m \leq \prod_{v \in NR} \bar{c}^{C_0 \gamma^{\frac{-h_{v'}}{\tau}} 2^{h_{v'}} / n_F^{\frac{1}{\tau}}} \leq \bar{C}^n \prod_{v \in NR} \gamma^{2(h_{v'} - h_v)} \tag{42}$$

provided that $\gamma^{\frac{1}{\tau}}/2 = \gamma^{\bar{\xi}}$. with $\bar{\xi} > 0$ ($\gamma > 1, \tau > 1$), and we have used $e^{-\alpha x} x^N \leq (Ne/\alpha)^N$ with $x = \gamma^{-\bar{\xi} h}$. We can choose for instance $\gamma^{\frac{1}{\tau}} = 4$, $\gamma^{\bar{\xi}} = 2$. It is sufficient to take $N = 2$ and

$$\bar{C} = (\frac{4\tau e n_F^{\frac{1}{\tau}}}{C_0 \log \bar{c}})^{4\tau} \tag{43}$$

We have finally to consider the quartic terms or the marginal quadratic terms $v \in R2$. We note first that due to the presence of a gap there is a scale $h^* = -\log \sigma$, with $\sigma = O(t^{n_F})$, such that the fields $\leq h^*$ can be integrated in a single step; that is, the iterative integration stops at $h^*$. As the external lines of the clusters $v \in R2$ have different coordinate $x_2$, necessarily contain a non diagonal propagator or a $t$ or $\alpha$ end-point; in the first case one of the factors (35) $(\sigma\gamma^{-h_v}) \leq \gamma^{(h^* - h_v)}$ provides the dimensional gain of all the clusters containing such non diagonal propagator. If there is a $t$ vertex we use $t^{\frac{1}{2}} \leq \gamma^{\frac{(h^* - h_v)}{2n_F}}$. Similarly is there is an $\alpha$ vertex we use that $\alpha_h$ is $O(\sigma^2 U \gamma^{-h})$ or $O(tU\gamma^{\theta h})$ one gets an extra $\gamma^{\frac{(h^* - h_v)}{2n_F}}$.

In conclusion

$$\frac{1}{L\beta} \int d\mathbf{x} |W^h(\mathbf{x})| \leq \\ \sum_m \sum_{\tau, h_v, n_v} \gamma^{(2-n/2)h} \varepsilon^m [\prod_v \gamma^{-(h_v - h_{v'})(D_v + \bar{z}_v)} \tag{44}$$

where

- $\bar{z}_v = 2$ if $v \in NR$ and $n_e^v = 4, 2$
- $\bar{z}_v = 1$ if $v \in R1$ and $n_e^v = 4$, $z_v = 2$ if $v \in R1$ and $n_e^v = 2$
- $z_v = 1 + 1/n_F$ if $v \in R2$ and $n_e^v = 2$; $z_v = 1/n_F$ if $v \in R2$ and $n_e^v = 4$ .

Therefore we can sum over the scales and one gets a convergent estimate for the effective potential; moreover the contributions with an irrelevant $t$ coupling have an extra $\gamma^{\theta h}$ due to the fact that the dimensions are all negative. Therefore for each contribution of order $n$ to the renormalized expansion we get a bound $C_1^n \varepsilon^n$, $C_1$ depending on $n_F, C_0, \tau$ (some dependence follows from from (42) and other from the sum over scales); this condition ensure convergence if the r.c.c. are small enough (and they are small for $U, t$ small, by the analysis of the flow of the previous section); the inverse of $C_1$ is just $\varepsilon_0$ in the main Theorem. Note that by (34)

$$2^{-n_F} \leq a_{n_F} \leq C_1^{n_F} \tag{45}$$

where the lower bound follows simply from the fact the denominators are larger than 2 and the upper by the multiscale analysis; hence by (24) $\sigma = t^{n_F} a_{n_F}(1 + a_{n_F}^{-1} R)$ and $|R| \leq 2C(U^2 + t)$.

It is immediate to get the large distance asymptotic decay of the 2-point function. The decay in $\mathbf{x}$ is an immediate consequence of the fact that there is a last scale $h^*$; the decay rate $\sigma$ provide an estimate in the gap of the interacting case, which is always non vanishing for $U$ small. The decay in the direction $x_2$ is faster than any power with rate $\log t$ because the contribution in $t$ starts from order $x_2 - x_2'$.

## VI. CONCLUSIONS

We have proven that there is a region of parameters for which, for fixed $t, U$, the gap with $n$ not too large persists even if $U$ is much stronger than the gap. The main difficulty relies in the presence of infinitely many processes which, due to Umklapp scattering and the incommensurability of the two periods, connect arbitrarily close the Fermi points. We can however rigorously establish the irrelevance of such terms by combining non perturbative RG methods with a strategy inspired by KAM problems and relying on number theoretical properties of irrationals. In principle an interaction much stronger than the non-interacting gap can destroy it, as in interacting Fibonacci chains [20], [21], but our result excludes this possibility. This seems of possible application to experiments, where gaps with $n$ too large are outside resolution but the many body interaction can be greater than the gap on which the Fermi level is set.

As we said the dependence on $n$ of $\varepsilon_0$ is quite weak and is due to the use of KAM methods; indeed our result can be seen as the analogue of of [3], [4] in an interacting situation. Getting results for $t$ and $U$ uniformly in $n$ is surely a challenging mathematical problem. Other interesting open issues include what happens to gaps in the case of attractive potential $U < 0$.

One could consider also the case of chemical potentials in the spectrum of the non interacting case, and investigate the question of the generation of gaps due to the interaction. The same argument explained above shows that the non resonant terms terms are irrelevant, but resonant terms connecting different wires are instead marginal and have a complicate flow which could exhibit non trivial fixed points. This opens the way to the a quantitative understanding starting from a microscopic lattice model of the opening of new gaps caused by the interaction, as it appears in experiments [7]-[10].


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