COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **7**, Number **6**, November **2008**

Website: http://AIMsciences.org

pp. **1393–1414**

ASYMPTOTIC BEHAVIOR OF A HYPERBOLIC SYSTEM ARISING IN FERROELECTRICITY

Sergio Frigeri

Dipartimento di Matematica "F. Casorati"
Università di Pavia
Via Ferrata, 1, I-27100 Pavia, Italy

(Communicated by Alain Miranville)

Abstract. We consider a coupled hyperbolic system which describes the evolution of the electromagnetic field inside a ferroelectric cylindrical material in the framework of the Greenberg-MacCamy-Coffman model. In this paper we analyze the asymptotic behavior of the solutions from the viewpoint of infinite-dimensional dissipative dynamical systems. We first prove the existence of an absorbing set and of a compact global attractor in the energy phase-space. A sufficient condition for the decay of the solutions is also obtained. The main difficulty arises in connection with the study of the regularity property of the attractor. Indeed, the physically reasonable boundary conditions prevent the use of a technique based on multiplication by fractional operators and bootstrap arguments. We obtain the desired regularity through a decomposition technique introduced by Pata and Zelik for the damped semilinear wave equation. Finally we provide the existence of an exponential attractor.

1. Introduction.

In this work we investigate the asymptotic behavior of a coupled hyperbolic system of the form

$$\begin{cases} u_{tt} + u_t + p_t - \Delta u = 0 \\ p_{tt} + p_t - u_t - \Delta p + p + \phi(p) = 0 \end{cases} \tag{1.1}$$

in $\Omega \times (0, \infty)$, where $\Omega \subset \mathbb{R}^2$ is a bounded and connected domain with smooth boundary $\partial\Omega$. This system of equations governs the evolution of the electromagnetic field inside a ferroelectric material occupying the cylindrical region $\Omega \times \mathbb{R}$, according with a physical model recently proposed by Greenberg et al. (see [10]). In this model u represents a field connected to the electric field ε (directed along the z axis) and to the components of the magnetic field h_1, h_2 (lying in the x, y plane) by the relations

$$\varepsilon = u_t \qquad h_1 = -cu_y \qquad h_2 = cu_x$$

c being the velocity of light in vacuum, while p is the polarization field inside the material (directed along the z axis). In [10] the nonlinearity ϕ is a smooth globally lipschitz function satisfying a certain coercivity property. For the sake of more generality, here we shall assume ϕ with polynomial growth of finite and arbitrary order and all the results will be deduced under this general assumption. In [10] a quite detailed analysis of the stationary states is performed. However, the study

2000 *Mathematics Subject Classification.* 35B40, 35B41, 35L05, 35Q60, 35R15, 37L05, 37L25.
Key words and phrases. Absorbing sets, global attractor, damped semilinear wave equation, exponential attractor.

of the asymptotic behavior is still at a preliminary stage. In the present paper we want to deepen the analysis of the long time behavior by studying global asymptotic properties such that the existence of a bounded absorbing set (thus showing the dissipative feature of the system), of the global attractor and its regularity, as well as the existence of an exponential attractor of optimal regularity. The results will be obtained under the physically significant Dirichlet-Neumann boundary conditions

$$u = \partial_{\mathbf{n}} p = 0, \qquad \text{on } \partial\Omega \times (0, \infty),$$

$\partial_{\mathbf{n}}$ being the outward normal derivative. The Dirichlet-Dirichlet boundary conditions

$$u = p = 0, \qquad \text{on } \partial\Omega \times (0, \infty)$$

can also be considered. Indeed, these latter conditions, although less significant from the physical point of view, allow to obtain more easily some results concerning the regularity of the attractor and the existence of the exponential attractor, through the use of standard techniques. Such results will be obtained in this paper under the Dirichlet-Neumann boundary conditions exploiting a recent approach due to Pata and Zelik (see [16]) based on suitable decomposition of the evolution semigroup. The formal problem we want to analyze is therefore the following

Problem P. *Find* (u, p) *solution to the system (1.1) in* $\Omega \times (0, \infty)$ *with boundary conditions* $u = \partial_{\mathbf{n}} p = 0$ *on* $\partial\Omega \times (0, \infty)$*, and initial conditions* $u(0) = u_0$*,* $u_t(0) = u_1$*,* $p(0) = p_0$*,* $p_t(0) = p_1$ *in* Ω*.*

This paper is organized in the following way. In Section 2 we introduce some functional notation and state Gronwall type lemmas useful in the following. Section 3 is devoted to the weak formulation of Problem **P** and to the well-posedness result. In particular, we associate with **P** a strongly continuous semigroup acting on the energy phase-space $\mathcal{H}_0$. In other words, **P** is interpreted, in the theory of infinite dimensional dynamical systems, as a generator of trajectories in the energy phase-space. This result, as well as the nonlinear feature of the system (causing the great sensibility to the initial conditions) suggests that the correct approach to study its asymptotic dynamics is geometric. In Section 4 the dissipativity of the system is deduced by proving the existence of a bounded absorbing set on the energy phase-space, as well as the existence of a bounded absorbing set in the second order phase-space $\mathcal{H}_1$. In this section we also deduce a sufficient condition on the nonlinearity ϕ which ensures the uniform decay of the trajectories departing from every bounded subset of $\mathcal{H}_0$ (see Proposition 4.1). The existence of the global attractor is shown in Section 5. The remarkable fact is that this result can be obtained by means of the same hypotheses used for the well-posedness result in $\mathcal{H}_0$ (see (H1)-(H3)). No further assumption is needed. In Section 6 we consider the regularity of the attractor, for the Dirichlet-Neumann case, obtained, adding further regularity assumptions on the nonlinearity, through the use of the Pata-Zelik technique. Such technique can obviously be applied for the Dirichlet-Dirichlet case as well. The last section is devoted to the existence of an exponential attractor. For this purpose we exploit the abstract result due to Efendiev, Miranville and Zelik (see [6, Proposition 1]) concerning the existence of exponential attractors for evolution equations.

2. **Functional setting and notation.**

We denote by $(\cdot, \cdot)$ and $\|\cdot\|$ the inner product and the norm on $L^2(\Omega)$, respectively. The symbol $\langle \cdot, \cdot \rangle$ will stand for the duality pairing between a Banach space and its dual. We introduce the operators

$A = -\Delta + I$ and $B = -\Delta$ (Δ is the laplacian in two dimensions) with domains $D(A) = \{v \in H^2(\Omega) : \partial_n v = 0 \text{ on } \partial\Omega\}$ and $D(B) = H^2(\Omega) \cap H^1_0(\Omega)$, respectively. As is well known, A and B are (unbounded) linear, strictly positive, self-adjoint operators with compact inverse. The spectral theory of this class of operators allows defining, for all $s \in \mathbb{R}$, the fractional operators A^s and B^s, with domains $D(A^s)$ and $D(B^s)$. Identifying $L^2(\Omega)$ with its dual $L^2(\Omega)'$, for any $s \in \mathbb{R}$ we consider the two families of Hilbert spaces $V^s := D(A^{s/2})$ and $V^s_0 := D(B^{s/2})$ with the natural inner products and norms. We recall that $V^s = V^s_0$ for $s \in (-1/2, 1/2)$. Moreover, $(V^s)' = V^{-s}$ and $(V^s_0)' = V^{-s}_0$. We observe that $V^0 = V^0_0 = L^2(\Omega)$, $V^1 = H^1(\Omega)$, $V^1_0 = H^1_0(\Omega)$ and $V^{-1}_0 = H^{-1}(\Omega)$. We have the continuous injections $V^s_0 \hookrightarrow V^s$, $V^{-s} \hookrightarrow V^{-s}_0$, for $s \geq 0$ and the compact and dense injections $V^s \hookrightarrow\hookrightarrow V^r$, $V^s_0 \hookrightarrow\hookrightarrow V^r_0$, for $s > r$. In particular, denoting by λ_A and λ_B the first eigenvalues of A and B respectively, we have the inequalities

$$\|A^{r/2}v\| \leq \lambda_A^{(r-s)/2}\|A^{s/2}v\|, \qquad \forall v \in V^s \tag{2.1}$$

$$\|B^{r/2}v\| \leq \lambda_B^{(r-s)/2}\|B^{s/2}v\|, \qquad \forall v \in V^s_0. \tag{2.2}$$

Formula (2.2), for $r = 0$ and $s = 1$, is the Poincaré inequality. Concerning the phase-space for our problem, we introduce, for $s \in \mathbb{R}$, the product Hilbert spaces

$$\mathcal{H}_s := V^{1+s}_0 \times V^s_0 \times V^{1+s} \times V^s,$$

with corresponding norms, induced by their inner products, given by

$$\|z\|^2_s := \|z\|^2_{\mathcal{H}_s} = \|B^{(1+s)/2}u\|^2 + \|B^{s/2}v\|^2 + \|A^{(1+s)/2}p\|^2 + \|A^{s/2}q\|^2$$

for all $z = (u, v, p, q) \in \mathcal{H}_s$. In particular we have

$$\mathcal{H}_0 = H^1_0(\Omega) \times L^2(\Omega) \times H^1(\Omega) \times L^2(\Omega).$$

From Section 5 on, we denote by $c \geq 0$ a generic constant, that may vary even from line to line within the same equation, depending on Ω and ϕ. Further dependences will be specified when necessary. Furthermore, we will use, sometimes without explicit reference, relations (2.1) and (2.2) as well as the Young and generalized Hölder inequalities and the usual Sobolev embeddings. We conclude this section with two technical lemmas that will be needed in the course of the investigation.

Lemma 2.1. *Let X be a Banach space, and $\mathcal{Z} \subset C([0, +\infty); X)$. Let be given a functional $E : X \to \mathbb{R}$ such that $\sup_{t\geq 0} E(z(t)) \geq -m$ and $E(z(0)) \leq M$, for some $m, M \geq 0$ and for every $z \in \mathcal{Z}$. In addition assume that $E(z(\cdot)) \in C^1([0, +\infty))$ for every $z \in \mathcal{Z}$ and that the differential inequality*

$$\frac{d}{dt}E(z(t)) + \delta_0\|z(t)\|^2_X \leq k$$

holds for all $t \geq 0$ and for some $\delta_0 > 0$, $k \geq 0$, both independent of $z \in \mathcal{Z}$. Then, for every $\epsilon > 0$ there is $t_0 = t_0(M, \epsilon) \geq 0$ such that, for every $z \in \mathcal{Z}$

$$E(z(t)) \leq \sup_{\zeta\in X}\{E(\zeta) : \delta_0\|\zeta\|^2_X \leq k + \epsilon\}, \qquad \forall t \geq t_0.$$

Furthermore, the time t_0 can be expressed by $t_0 = (M + m)/\epsilon$.

The proof can be found, for instance, in [2, Lemma 2.7]. The next result is a generalized version of the standard Gronwall lemma

Lemma 2.2. *Let $\Psi : [0, +\infty) \to [0, +\infty)$ be an absolutely continuous function satisfying*

$$\frac{d}{dt}\Psi(t) + 2\epsilon\Psi(t) \leq h(t)\Psi(t) + k$$

where $\epsilon > 0$, $k \geq 0$ and $\int_s^t h(\tau)d\tau \leq \epsilon(t-s) + m$, for all $t \geq s \geq 0$ and some $m \geq 0$. Then

$$\Psi(t) \leq \Psi(0)e^m e^{-\epsilon t} + \frac{ke^m}{\epsilon}, \qquad \forall t \geq 0.$$

For the proof, we refer the reader to [3, Lemma 2.1].

3. **Well-Posedness.** The assumptions we make on the nonlinearity are the following

(H1) $\phi \in C^1(\mathbb{R})$

(H2) $|\phi'(s)| \leq c_0(1 + |s|^{r-2}), \quad \forall s \in \mathbb{R}, \qquad 2 \leq r < +\infty, \qquad c_0 \geq 0$

(H3) $\liminf_{|s| \to +\infty} \frac{\phi(s)}{s} > -\lambda_A$.

Remark 3.1. For the well-posedness we can substitute condition (H3) with the weaker assumption

(H3*) $F(s) \geq -\alpha s^2 - \beta, \quad \forall s \in \mathbb{R}, \qquad \alpha, \beta \geq 0, \qquad F(s) := \int_0^s \phi(\sigma)d\sigma.$

We can now introduce the notion of weak solution to Problem **P**.

Definition 3.1. Let (H1) to (H3) hold. Suppose $z_0 := (u_0, u_1, p_0, p_1) \in \mathcal{H}_0$. For $T > 0$, setting $I := [0, T]$, we say that $z := (u, u_t, p, p_t)$ which fulfills

$$u \in C^0(I; V_0^1) \cap C^1(I; V^0) \cap C^2(I; V_0^{-1}), \;\; p \in C^0(I; V^1) \cap C^1(I; V^0) \cap C^2(I; V^{-1})$$

is a weak solution to **P** in I with initial data z_0 provided that

$$\langle u_{tt}, v_0 \rangle + (u_t, v_0) + (p_t, v_0) + (B^{1/2}u, B^{1/2}v_0) = 0 \tag{3.1}$$

$$\langle p_{tt}, v \rangle + (p_t, v) - (u_t, v) + (A^{1/2}p, A^{1/2}v) + (\phi(p), v) = 0 \tag{3.2}$$

for every $v_0 \in V_0^1$ and $v \in V^1$, almost everywhere in I, and $u(0) = u_0$, $u_t(0) = u_1$, $p(0) = p_0$, $p_t(0) = p_1$, almost everywhere in Ω.

Existence, uniqueness and continuous dependence, according to Definition 3.1, can be deduced by standard arguments based on a Faedo-Galerkin approximation scheme and on the use of the energy identity. We shall obtain energy type estimates in Section 4 (see (4.1), (4.14)). More precisely, we have the following well-posedness result

Theorem 3.2. *In the hypotheses (H1)-(H3), for any $T > 0$, Problem **P** has a unique (weak) solution z on the time interval $I = [0, T]$ with initial data $z_0 \in \mathcal{H}_0$. Moreover, if z_{01} and z_{02} are two sets of data in $\mathcal{H}_0$, and z_1 and z_2 are the two corresponding solutions on $[0, \infty)$, there exists $\theta_0 > 0$, depending (continuously and increasingly) only on the $\mathcal{H}_0$-norms of the data $\|z_{0i}\|_0$ for $i = 1, 2$ (besides on Ω and ϕ) such that*

$$\|z_2(t) - z_1(t)\|_0 \leq e^{\theta_0 t}\|z_{02} - z_{01}\|_0, \qquad \forall t \geq 0. \tag{3.3}$$

Since the system is autonomous, the well-posedness result immediately leads to the following

Corollary 3.1. *The one-parameter family of continuous (nonlinear) operators $S(t)$: $\mathcal{H}_0 \to \mathcal{H}_0$ defined by $S(t)z_0 := z(t)$, for every $t \geq 0$ and every $z_0 \in \mathcal{H}_0$, where $z(t)$ is the solution to **P** at time t with $z(0) = z_0$, is a strongly continuous semigroup on the phase-space $\mathcal{H}_0$.*

Remark 3.2. The particular time dependence in the estimate (3.3) holds if either (H3) or (H3*) with the further assumption $\alpha < \lambda_A/2$ are fulfilled. Indeed, in this case from the energy identity (see (4.14)) we easily get the control $\|z(t)\|_0 \leq \Lambda(\|z_0\|_0)$ for every $t \geq 0$ (see also Corollary 4.1). The assumption (H3*) with $\alpha \geq \lambda_A/2$ still yields well-posedness, but the time dependence in the estimate (3.3) is more involved.

Furthermore, due to standard regularity results, it is possible to prove that, under the assumptions (H3) and

(H4) $\phi \in C^2(\mathbb{R})$

(H5) $|\phi''(s)| \leq c_0'(1 + |s|^{r-3}), \qquad \forall s \in \mathbb{R}, \qquad 3 \leq r < +\infty, \quad c_0' \geq 0$

$S(t)$ is also a strongly continuous semigroup on the phase-space $\mathcal{H}_1$. In particular, the continuous dependence estimate holds in $\mathcal{H}_1$ in the following form: for every $R_1, T > 0$ there exists $C_1 = C_1(R_1, T) \geq 0$ such that

$$\|S(t)z_2 - S(t)z_1\|_1 \leq C_1\|z_2 - z_1\|_1 \tag{3.4}$$

for every $t \in [0, T]$ and every $z_1, z_2 \in \mathcal{H}_1$ with $\|z_1\|_1, \|z_2\|_1 \leq R_1$.

4. **Dissipativity.** Here we would like to see whether the trajectories originating from any given bounded subset of the phase-space $\mathcal{H}_0$ eventually fall, uniformly in time, into a fixed bounded subset, which we call *absorbing set*. For this purpose we shall need some uniform (with respect to time) energy type estimates.

Theorem 4.1. *Let (H1)-(H3) hold. Then, there exists a constant $R_0 > 0$ with the following property: given any $R > 0$, there exists $t_0 = t_0(R)$ such that, whenever $\|z_0\|_0 \leq R$, the inequality $\|S(t)z_0\|_0 \leq R_0$ holds for every $t \geq t_0(R)$. Consequently, the set*

$$\mathcal{B}_0 = \{z_0 \in \mathcal{H}_0 : \|z_0\|_0 \leq R_0\}$$

*is a bounded absorbing set for the semigroup $S(t)$ generated by Problem **P** on $\mathcal{H}_0$, that is, for every given bounded subset $\mathcal{B} \subset \mathcal{H}_0$, there exists a time $t_0 = t_0(\mathcal{B}) \geq 0$ such that $S(t)\mathcal{B} \subset \mathcal{B}_0$ for every $t \geq t_0$.*

Proof. We suppose to work within a proper approximation scheme, with regular data and solutions, to justify the formal estimates we derive below. We multiply $(1.1)_1$ and $(1.1)_2$ in $L^2(\Omega)$ by the auxiliary variables $\xi := u_t + \epsilon u$ and $\zeta := p_t + \epsilon p$, respectively, with $0 < \epsilon \leq \epsilon_0$ and ϵ_0 to be chosen later. Adding together the resulting equations, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{\|B^{1/2}u\|^2 + \|\xi\|^2 + \|A^{1/2}p\|^2 + \|\zeta\|^2\} + \epsilon\|B^{1/2}u\|^2 + (1-\epsilon)\|\xi\|^2\\&+\epsilon\|A^{1/2}p\|^2 + (1-\epsilon)\|\zeta\|^2 + \frac{d}{dt}\int_\Omega F(p) = \epsilon(1-\epsilon)(\xi, u) + \epsilon(1-\epsilon)(\zeta, p)\\&+\epsilon(\xi, p) - \epsilon(\zeta, u) - \epsilon(p, \phi(p)).\end{aligned} \tag{4.1}$$

By (H3) it is easy to show that there exists $\nu \in (0, 1]$ such that

$$(p, \phi(p)) \geq -(1-\nu)\|A^{1/2}p\|^2 - c_1, \qquad \forall p \in V^1 \tag{4.2}$$

and

$$2\int_\Omega F(p) \geq -(1-\nu)\|A^{1/2}p\|^2 - c_2, \qquad \forall p \in V^1. \tag{4.3}$$

We now introduce the following functional $E : \mathcal{H}_0 \to [0,+\infty)$, defined by

$$E(z) := \|B^{1/2}u\|^2 + \|v+\epsilon u\|^2 + \|A^{1/2}p\|^2 + \|q+\epsilon p\|^2 + 2\int_\Omega F(p) \tag{4.4}$$

for every $z = (u,v,p,q) \in \mathcal{H}_0$. Using (4.3) and the Young inequality, it is not difficult to show that

$$E(z) \geq \frac{\nu}{2}\|z\|_0^2 - c_2, \qquad \forall z \in \mathcal{H}_0 \tag{4.5}$$

provided that ϵ_0 is small enough, say $\epsilon_0 \in (0,\epsilon_0']$ (see Remark 4.1 below). From (H2), by means of relations (2.1), (2.2) and by the Sobolev embedding $H^1(\Omega) \hookrightarrow L^r(\Omega)$ for every $r \in [1,+\infty)$ $(N=2)$, we also get

$$E(z) \leq c_3\|z\|_0(1+\|z\|_0^{r-1}), \qquad \forall z \in \mathcal{H}_0. \tag{4.6}$$

By virtue of (4.2) and taking into account the definition of the functional E, from (4.1) we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}E(z) + \epsilon\|B^{1/2}u\|^2 + (1-\epsilon)\|\xi\|^2 + \epsilon\|A^{1/2}p\|^2 + (1-\epsilon)\|\zeta\|^2\\ &\leq \epsilon\|\xi\|\|u\| + \epsilon\|\zeta\|\|p\| + \epsilon\|\xi\|\|p\| + \epsilon\|\zeta\|\|u\| + \epsilon(1-\nu)\|A^{1/2}p\|^2 + \epsilon c_1.\end{aligned} \tag{4.7}$$

Using again the Young inequality and choosing ϵ_0 small enough, namely $\epsilon_0 \in (0,\epsilon_0'']$ (cf. Remark 4.1 again), we obtain from (4.7)

$$\frac{1}{2}\frac{d}{dt}E(z) + \frac{\epsilon\nu}{3}\{\|B^{1/2}u\|^2 + \|\xi\|^2 + \|A^{1/2}p\|^2 + \|\zeta\|^2\} \leq \epsilon c_1. \tag{4.8}$$

We now have

$$\|B^{1/2}u\|^2 + \|\xi\|^2 + \|A^{1/2}p\|^2 + \|\zeta\|^2 \geq \frac{1}{2}\|z\|_0^2, \quad z = (u,u_t,p,p_t) \tag{4.9}$$

provided we choose $\epsilon \in (0,\epsilon_0']$ small enough. Therefore, by means of (4.9), inequality (4.8) entails

$$\frac{d}{dt}E(z) + \delta_0\|z\|_0^2 \leq 2\epsilon_0 c_1 \tag{4.10}$$

where $\delta_0 = \epsilon_0\nu/3$ and $\epsilon_0 = \min\{\epsilon_0', \epsilon_0''\}$. The existence of a bounded absorbing set is now a direct consequence of (4.10), in light of Lemma 2.1. Actually, let us fix $R > 0$ and a set of initial data $z_0 \in \mathcal{H}_0$, with $\|z_0\|_0 \leq R$. By (4.6) we have the bound

$$E(z(0)) \leq c_3\|z_0\|_0(1+\|z_0\|_0^{r-1}) \leq c_3R(1+R^{r-1}). \tag{4.11}$$

We now take $X = \mathcal{H}_0$ and $\mathcal{Z} \subset C^0([0,+\infty);\mathcal{H}_0)$ given by the family of the trajectories departing from the initial data $z_0 \in \mathcal{H}_0$ with $\|z_0\|_0 \leq R$. From Lemma 2.1 we therefore conclude that there exists a time $t_0 = t_0(R) > 0$ such that

$$E(z(t;z_0)) \leq \sup\{E(\zeta) : \zeta \in \mathcal{H}_0, \;\; \delta_0\|\zeta\|_0^2 \leq 2\epsilon_0 c_1 + 1\} \tag{4.12}$$

for every $t \geq t_0$ and for every $z_0 \in \mathcal{H}_0$ with $\|z_0\|_0 \leq R$. Here we have set $z(t;z_0) := S(t)z_0$. Estimating the right hand side of (4.12) by means of (4.6) and taking account of (4.5), we conclude that there exists $R_0 > 0$ such that

$$\|z(t;z_0)\|_0 \leq R_0, \quad \forall t \geq t_0(R), \quad \forall z_0 \in \mathcal{H}_0, \;\; \|z_0\|_0 \leq R. \tag{4.13}$$

It follows that $\mathcal{B}_0 = \{z \in \mathcal{H}_0 : \|z\|_0 \leq R_0\}$ is a bounded absorbing set for the semigroup. This completes the proof. □

A straightforward consequence of Theorem 4.1 is

Corollary 4.1. *Let (H1)-(H3) hold. Then, for every $R > 0$, there exists a positive constant $\Lambda = \Lambda(R)$ such that, whenever $\|z_0\|_0 \le R$, the corresponding solution fulfills $\|z(t)\|_0 \le \Lambda$ for all $t \ge 0$.*

Another important corollary, that will be useful in the following, provides the uniform control of the dissipation integral, namely,

Corollary 4.2. *Let (H1)-(H3) hold. Then, for every $R > 0$, there exists a positive constant $\Lambda = \Lambda(R)$ such that, whenever $\|z_0\|_0 \le R$, there holds*

$$\int_0^{+\infty} (\|u_t(\tau)\|^2 + \|p_t(\tau)\|^2) d\tau \le \Lambda.$$

Proof. We write (4.1) for $\epsilon = 0$ getting the energy identity

$$\frac{1}{2}\frac{d}{dt}\|z\|_0^2 + \|u_t\|^2 + \|p_t\|^2 + \frac{d}{dt}\int_\Omega F(p) = 0. \tag{4.14}$$

Integrating with respect to time and using (4.3) we are led to

$$\frac{\nu}{2}\|z(t)\|_0^2 + \int_0^t (\|u_t(\tau)\|^2 + \|p_t(\tau)\|^2) d\tau \le \frac{1}{2}\|z_0\|_0^2 + \int_\Omega F(p_0) + \frac{c_2}{2}.$$

From this inequality, letting $t \to \infty$, we deduce the thesis. ☐

Remark 4.1. We can furnish an estimate for the radius R_0 of the absorbing set $\mathcal{B}_0$ in terms of the parameters of the problem. First let us introduce the following notation. Given $a_1, a_2, a_3 \ge 0$ and $r \ge 2$, we denote by $\gamma = \gamma(a_1, a_2, a_3; r)$ a nonnegative constant such that $a_1 t^r + a_2 t^2 + a_3 t \le \gamma(t + t^r)$ for every $t \ge 0$. We indicate by $\widetilde{c}_0$ a nonnegative constant such that $|F(s)| \le \widetilde{c}_0(|s|+|s|^r)$ for every $s \in \mathbb{R}$. With the previous notation we can also take $\widetilde{c}_0 = \gamma(c_0/2, c_0/2, |\phi(0)|; r)$, where c_0 is the constant appearing in (H2). For the constants ν, c_1 and c_2 in (4.2), (4.3) we can assume $\nu = \min\{(\lambda_A - \widetilde{\lambda})/2\lambda_A, 1\}$, $c_1 = |\Omega| \max\{C, 0\}$ and $c_2 = |\Omega| D$, where $\widetilde{\lambda} = -\liminf_{|s|\to+\infty} \frac{\phi(s)}{s} \in \mathbb{R} \cup \{-\infty\}$, $C = -\min_{|r|\le\rho} r\phi(r)$ and $D = -2\min_{|r|\le\rho} F(r)$, with $\rho \ge 0$ such that $r\phi(r) \ge -\max\{(\lambda_A + \widetilde{\lambda})/2, 0\} r^2$ for every $|r| \ge \rho$. The constant c_3 in (4.6) can be calculated to give $c_3 = \gamma(2\widetilde{c}_0 c_i^r, 2, 2\widetilde{c}_0 |\Omega|^{1/2}/\sqrt{\lambda_A}; r)$, where $c_i \ge 0$ is the constant of the continuous embedding $H^1(\Omega) \hookrightarrow L^r(\Omega)$ (i.e. $\|w\|_{L^r(\Omega)} \le c_i \|w\|_{H^1(\Omega)}$ for every $w \in H^1(\Omega)$. We recall that we can assume $c_i = \sqrt{2}[r/2]\|P\|$, where $[x]$ is the smallest integer greater or equal to x, for every $x > 0$, and $P : H^1(\Omega) \to H^1(\mathbb{R}^2)$ is a bounded and linear extension operator). The constant c_i depends on Ω and r. Finally, the values of $\epsilon_0', \epsilon_0''$ can be given by $\epsilon_0'^2 = \min\{\nu\lambda_A, \lambda_B\}/4$ and $\epsilon_0'' = (1 + \lambda_B^{-1} + (\nu\lambda_A)^{-1} + \nu/3)^{-1}$. We observe that in (4.6) the constant c_3 is given as above, provided $\epsilon \le \epsilon_0'$. From (4.5), (4.6), (4.12) we thus can deduce the required estimate

$$R_0^2 = \frac{2}{\nu}\Big[c_3\Big(\frac{2\epsilon_0 c_1 + 1}{\delta_0}\Big)^{1/2}\Big(1 + \Big(\frac{2\epsilon_0 c_1 + 1}{\delta_0}\Big)^{(r-1)/2}\Big) + c_2\Big], \qquad \delta_0 = \frac{\epsilon_0 \nu}{3}.$$

Remark 4.2 (Uniform decay of the trajectories)**.** Exploiting the considerations of the previous remark and using Lemma 2.1 once more, we can immediately deduce a sufficient condition which ensures the decay in $\mathcal{H}_0$ of the trajectories uniformly from every bounded subset $B \subset \mathcal{H}_0$. Indeed, let $\eta > 0$ be fixed arbitrary. Then, the quantity $R_{0,\eta} > 0$ given by

$$R_{0,\eta}^2 = \frac{2}{\nu}\Big[c_3\Big(\frac{2\epsilon_0 c_1 + \eta}{\delta_0}\Big)^{1/2}\Big(1 + \Big(\frac{2\epsilon_0 c_1 + \eta}{\delta_0}\Big)^{(r-1)/2}\Big) + c_2\Big]$$

is the radius of an absorbing ball $\mathcal{B}_0 = B_{\mathcal{H}_0}(0, R_{0,\eta})$. Therefore, for every $R > 0$, there exists a time $t_0 = t_0(R,\eta) > 0$, such that, for every $t \geq t_0(R,\eta)$ we have $\|z(t;z_0)\|_0 \leq R_{0,\eta}$ for any $z_0 \in \mathcal{H}_0$ with $\|z_0\|_0 \leq R$. From Lemma 2.1, (4.5) and (4.6) we can also infer $t_0 = [c_3 R(1 + R^{r-1}) + c_2]/\eta$. We hence recognize at once the desired sufficient condition, that is $c_1 = c_2 = 0$, and we can state the following proposition

Proposition 4.1. *Let (H1) to (H3) hold. In addition, suppose that the following conditions*

$$\min_{|r|\leq\rho} r\phi(r) \geq 0, \qquad \min_{|r|\leq\rho} F(r) = 0 \tag{4.15}$$

be satisfied, with $\rho \geq 0$ as in Remark 4.1. Then, for every $R > 0$, we have

$$\|z(t;z_0)\|_0 \to 0 \qquad \text{as } t \to +\infty$$

uniformly for $\|z_0\|_0 \leq R$. In particular, assumption (4.15) holds if $s\phi(s) \geq 0$ for every $s \in \mathbb{R}$.

Remark 4.3. The assumptions of Proposition 4.1 are satisfied if we take, for example, $\phi(s) = s|s|^{r-2}/(r-1)$, $r \geq 2$.

We conclude this section with the result concerning the existence of a bounded absorbing set in $\mathcal{H}_1$ for the semigroup $S(t) : \mathcal{H}_1 \to \mathcal{H}_1$. This will be guaranteed by the next theorem whose proof, based on the generalized Gronwall lemma recalled in Section 2 (see Lemma 2.2), requires, besides (H3)-(H5), the following further (but reasonable) assumption

(H6) $\phi'(s) \geq -l, \qquad \forall s \in \mathbb{R}, \qquad l \geq 0.$

Theorem 4.2. *Let (H3)-(H6) hold. Given $R_0 \geq 0$ and $R_1 \geq 0$ such that $\|z_0\|_0 \leq R_0$ and $\|z_0\|_1 \leq R_1$, there exist constants $C = C(R_0)$, $K = K(R_1)$, depending increasingly and continuously on R_0 and R_1 respectively, and $\epsilon_1 > 0$ such that, for every $t \geq 0$,*

$$\|z(t)\|_1 \leq K e^{-\epsilon_1 t} + C.$$

Proof. We rewrite system (1.1) in the form

$$\begin{cases} u_{tt} + u_t + p_t + Bu = 0 \\ p_{tt} + p_t - u_t + Ap + \psi(p) = lp \end{cases} \tag{4.16}$$

where $\psi(s) := \phi(s) + ls$. For $z_0 \in \mathcal{H}_1$, we consider the linear nonhomogeneous problem

$$\begin{cases} v_{tt} + v_t + q_t + Bv = 0 \\ q_{tt} + q_t - v_t + Aq + \psi'(p)q = lp_t \\ v(0) = u_1 =: v_0, \qquad v_t(0) = -u_1 - p_1 - Bu_0 =: v_1 \\ q(0) = p_1 =: q_0, \qquad q_t(0) = -p_1 + u_1 - Ap_0 - \phi(p_0) =: q_1 \end{cases} \tag{4.17}$$

obtained by differentiation of the above system with respect to time. Since we have $(v_0, v_1, q_0, q_1) \in \mathcal{H}_0$, by virtue of standard well-posedness results for linear equations (see, e.g., [17]), problem (4.17) admits a unique weak solution $w := (v, v_t, q, q_t)$ such that $w \in C^0([0,\infty);\mathcal{H}_0)$. By comparision with the solution to Problem **P** we obtain $v(t) = u_t(t)$, $q(t) = p_t(t)$ and hence $w(t) = z_t(t)$. Now, for $\epsilon > 0$ to be determined later, we multiply in $L^2(\Omega)$ $(4.17)_1$ by $\xi := v_t + \epsilon v$, $(4.17)_2$ by $\zeta := q_t + \epsilon q$ and we add together the resulting equations. After some calculations we obtain

$$\frac{1}{2}\frac{d}{dt}\Big(\|B^{1/2}v\|^2 + \|\xi\|^2 + \|A^{1/2}q\|^2 + \|\zeta\|^2 + (\psi'(p)q,q)\Big) + \epsilon\|B^{1/2}v\|^2 + (1-\epsilon)\|\xi\|^2$$
$$-\epsilon(1-\epsilon)(v,\xi) + \epsilon\|A^{1/2}q\|^2 + (1-\epsilon)\|\zeta\|^2 - \epsilon(1-\epsilon)(q,\zeta) + \epsilon(\psi'(p)q,q)$$
$$= \frac{1}{2}(\psi''(p)p_t,q^2) + l(p_t,\zeta) + \epsilon(p_t,\xi) - \epsilon(u_t,\zeta). \tag{4.18}$$

If we define

$$\Phi := \|B^{1/2}v\|^2 + \|\xi\|^2 + \|A^{1/2}q\|^2 + \|\zeta\|^2 + (\psi'(p)q,q) \tag{4.19}$$

it is easy to see that there holds

$$k_1\|w(t)\|_0^2 \le \Phi(t) \le k_2\|w(t)\|_0^2, \qquad \forall t \ge 0 \tag{4.20}$$

provided ϵ is small enough, where k_1 and k_2 are positive constants, with only k_2 depending on the $\mathcal{H}_0$-norm of the initial data z_0. To prove (4.20) one uses (H2), the control $\|z(t)\|_0 \le \Lambda(\|z_0\|_0)$ (provided by Corollary 4.1), Young inequality and the fact that $\psi'(s) \ge 0$ for every $s \in \mathbb{R}$. From now on, in the course of this proof, we denote by k some positive constant depending, increasingly and continuously, on $\|z_0\|_0$. After exploiting Corollary 4.1 to estimate the last three terms on the right hand side of (4.18) as

$$l(p_t,\zeta) + \epsilon(p_t,\xi) - \epsilon(u_t,\zeta) \le \epsilon\|\xi\|^2 + \epsilon\|\zeta\|^2 + \frac{k}{2},$$

we can write for ϵ small enough

$$\epsilon\|B^{1/2}v\| + (1-2\epsilon)\|\xi\|^2 - \epsilon(1-\epsilon)(v,\xi) + \epsilon\|A^{1/2}q\|^2 + (1-2\epsilon)\|\zeta\|^2$$
$$-\epsilon(1-\epsilon)(q,\zeta) + \epsilon(\psi'(p)q,q) \ge \frac{\epsilon}{2}\Phi. \tag{4.21}$$

Finally, using (H5) and Corollary 4.1 once more, the first term on the right hand side of (4.18) can be estimated as follows

$$(\psi''(p)p_t,q^2) \le c(1+\|A^{1/2}p\|^{r-3})\|p_t\|\|A^{1/2}q\|^2 \le k\|p_t\|\|A^{1/2}q\|^2. \tag{4.22}$$

Now, by means of (4.21), (4.22) and (4.19), we get from (4.18) the following differential inequality

$$\frac{d}{dt}\Phi + \epsilon\Phi \le k\|p_t\|\Phi + k \tag{4.23}$$

and we observe that, on account of the integral bound provided by Corollary 4.2, for every $0 \le s \le t$, there holds

$$\int_s^t \|p_t(\tau)\|d\tau \le \Lambda(\|z_0\|_0)(t-s)^{1/2} \le \frac{\epsilon}{2}(t-s) + k. \tag{4.24}$$

Therefore, we can apply Lemma 2.2 and, due to (4.20), we deduce

$$\|w(t)\|_0 \le k\|w_0\|_0 e^{-\frac{\epsilon}{4}t} + k \tag{4.25}$$

with $w_0 := w(0)$. To conclude the proof we observe that, from $(4.17)_3$ and $(4.17)_4$ we can write

$$\|w_0\|_0 \le c(\|z_0\|_1) \tag{4.26}$$

with $c(\cdot)$ a nondecreasing continuous function, whereas using $(4.16)_1$ and $(4.16)_2$ and the fact that $\|z_0\|_0 \le R_0$, for every $t \ge 0$, we have

$$\|z(t)\|_1 \le c\|w(t)\|_0 + k. \tag{4.27}$$

The thesis follows from (4.25), (4.26) and (4.27). □

Corollary 4.3. *Let the assumptions of Theorem 4.2 hold. Then, the (closed) ball in $\mathcal{H}_1$ given by*

$$\mathcal{B}_0^{(1)} := \{z_0 \in \mathcal{H}_1 : \|z_0\|_1 \le 2C(R_0)\}$$

where R_0 is the radius of a bounded absorbing set in $\mathcal{H}_0$ (given by Theorem 4.1), is a bounded absorbing set in $\mathcal{H}_1$ for the semigroup $S(t) : \mathcal{H}_1 \to \mathcal{H}_1$.

5. The global attractor.

The aim of this section is to prove the existence of the global attractor for the semigroup $S(t)$ on $\mathcal{H}_0$. We consider Problem **P**, that is system (1.1) endowed with Dirichlet-Neumann boundary conditions. Nevertheless the same procedure, with few modifications, can be repeated in the case of Dirichlet-Dirichlet boundary conditions as well. We recall that the global attractor is the (unique) compact set $\mathcal{A} \subset \mathcal{H}_0$ which is fully invariant, i.e., $S(t)\mathcal{A} = \mathcal{A}$ for every $t \ge 0$, and attracting in the sense of the Hausdorff semidistance. See, for instance, [1, 11, 12, 17] for reference on the general theory of dissipative infinite-dimensional dynamical systems. Let us state the main result of this section.

Theorem 5.1. *In the hypotheses (H1)-(H3), the semigroup $S(t)$ on $\mathcal{H}_0$ associated to Problem **P** possesses the global attractor.*

Proof. We decompose the solution $z = (u, u_t, p, p_t)$ to **P** with initial data $z_0 = (u_0, u_1, p_0, p_1) \in \mathcal{H}_0$ as $z = z_d + z_c$, where $z_d = (u_d, \partial_t u_d, p_d, \partial_t p_d)$ and $z_c = (u_c, \partial_t u_c, p_c, \partial_t p_c)$ are the solutions to the problems

$$\begin{cases} \partial_{tt}u_d + \partial_t u_d + \partial_t p_d + Bu_d = 0 \\ \partial_{tt}p_d + \partial_t p_d - \partial_t u_d + Ap_d = 0 \\ z_d(0) = z_0 \end{cases} \tag{5.1}$$

and

$$\begin{cases} \partial_{tt}u_c + \partial_t u_c + \partial_t p_c + Bu_c = 0 \\ \partial_{tt}p_c + \partial_t p_c - \partial_t u_c + Ap_c + \phi(p) = 0 \\ z_c(0) = 0. \end{cases} \tag{5.2}$$

It is easy to check that problems (5.1) and (5.2) are well posed. The thesis of the theorem follows from the general theory of dynamical systems once we show that, uniformly for $z_0 \in \mathcal{B}_0$ $z_d(t) \to 0$ in $\mathcal{H}_0$ as $t \to +\infty$, whereas (for $z_0 \in \mathcal{B}_0$) $z_c(t)$ lies in a compact subset of $\mathcal{H}_0$ (possibly depending on t) for all $t \ge 0$. This is precisely the content of Lemma 5.2 and Lemma 5.3. □

Lemma 5.2. *The solution z_d to (5.1) fulfills*

$$\lim_{t\to+\infty} [\sup_{z_0 \in \mathcal{B}_0} \|z_d(t; z_0)\|_0] = 0.$$

Proof. The thesis can be deduced immediately from Proposition 4.1 for the case $\phi = 0$. □

Lemma 5.3. *The solution z_c to (5.2) fulfills*

$$z_c(t; z_0) \in \mathcal{K}(t), \qquad \forall t \ge 0, \qquad \forall z_0 \in \mathcal{B}_0$$

where $\mathcal{K}(t)$ is a compact subset of $\mathcal{H}_0$.

Proof. We multiply $(5.2)_1$ by $B^s\partial_t u_c$ and $(5.2)_2$ by $A^s\partial_t p_c$, where we fix $s \in (0,1/2)$. Adding the resulting identities we are led to the differential equality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\{\|B^{(1+s)/2}u_c\|^2 + \|B^{s/2}\partial_t u_c\|^2 + \|A^{(1+s)/2}p_c\|^2 + \|A^{s/2}\partial_t p_c\|^2\} \\ &+\|B^{s/2}\partial_t u_c\|^2 + \|A^{s/2}\partial_t p_c\|^2 \\ = \quad & -(B^s\partial_t u_c, \partial_t p_c) + (A^s\partial_t p_c, \partial_t u_c) - (A^s\partial_t p_c, \phi(p)). \end{aligned} \tag{5.3}$$

Observe now that

$$\begin{aligned} (A^s\partial_t p_c, \partial_t u_c) \;&\le\; \|B^{-s/2}A^s\partial_t p_c\|\|B^{s/2}\partial_t u_c\| \\ &\le\; c\|A^{-s/2}A^s\partial_t p_c\|\|B^{s/2}\partial_t u_c\| = c\|A^{s/2}\partial_t p_c\|\|B^{s/2}\partial_t u_c\| \end{aligned} \tag{5.4}$$

where we have exploited the embedding $D(B^{s/2}) \hookrightarrow D(A^{s/2})$, for every $s > 0$, from which follows $D(A^{-s/2}) \hookrightarrow D(B^{-s/2})$, for every $s > 0$. Moreover

$$-(B^s\partial_t u_c, \partial_t p_c) \le \|A^{-s/2}B^s\partial_t u_c\|\|A^{s/2}\partial_t p_c\| \le c\|B^{s/2}\partial_t u_c\|\|A^{s/2}\partial_t p_c\| \tag{5.5}$$

where we have used the fact that, for $s \in (0,1/2)$, $D(A^{s/2}) = D(B^{s/2})$ and the operator $A^{-s/2}B^{s/2}$ is bounded (i.e., $\|A^{-s/2}B^{s/2}w\| \le c\|w\|$ for every $w \in D(B^{s/2})$). Integrating (5.3) with respect to time from 0 to t and taking into account $(5.2)_3$, (5.4), (5.5) we get

$$\begin{aligned} &\frac{1}{2}\{\|B^{(1+s)/2}u_c\|^2 + \|B^{s/2}\partial_t u_c\|^2 + \|A^{(1+s)/2}p_c\|^2 + \|A^{s/2}\partial_t p_c\|^2\} \\ &+\int_0^t\{\|B^{s/2}\partial_t u_c\|^2 + \|A^{s/2}\partial_t p_c\|^2\}d\tau \\ \le \quad & c\int_0^t \|B^{s/2}\partial_t u_c\|\|A^{s/2}\partial_t p_c\|d\tau - \int_0^t (A^s\partial_t p_c, \phi(p))d\tau. \end{aligned} \tag{5.6}$$

Now, we integrate by parts the last term on the right side of (5.6) and get

$$-\int_0^t (A^s\partial_t p_c, \phi(p))d\tau = -(A^s p_c, \phi(p)) + \int_0^t (A^s p_c, \phi'(p)p_t)d\tau. \tag{5.7}$$

It is easy to see that the first term on the right side of (5.7) is bounded. Indeed, for $z_0 \in \mathcal{B}_0$, denoting henceforth by c a nonnegative constant depending on R_0 (besides on Ω and ϕ), we have $\|A^s p_c(t)\| \le c\|A^{1/2}p_c(t)\| \le c$, for all $t \ge 0$, as a consequence of the decay to zero of p_d (Lemma 5.2), and of Corollary 4.1 (which implies $\|A^{1/2}p(t)\| \le c$ for every $t \ge 0$). Also, by (H2), $\|\phi(p(t))\| \le c(1+\|A^{1/2}p(t)\|^{r-1}) \le c$ for all $t \ge 0$. Furthermore, we have

$$\begin{aligned} &\int_0^t (A^s p_c, \phi'(p)p_t)d\tau \le \int_0^t\int_\Omega |A^s p_c||\phi'(p)||p_t|dxd\tau \\ &\le \int_0^t \|A^s p_c\|_{L^{2/s}}\|\phi'(p)\|_{L^{2/(1-s)}}\|p_t\|d\tau \le c\int_0^t \|A^{(1+s)/2}p_c\|d\tau \end{aligned} \tag{5.8}$$

where we have used the embedding $D(A^{(1-s)/2}) \hookrightarrow L^{2/s}(\Omega)$ and the fact that, for $z_0 \in \mathcal{B}_0$, $\|\phi'(p)\|_{L^{2/(1-s)}}$, $\|p_t\| \le c$, by virtue of Corollary 4.1. Substituting (5.8) into (5.6), we obtain the differential inequality

$$\Phi(t) \le ct + c\int_0^t \Phi(\tau)d\tau$$

where $\Phi := \|B^{(1+s)/2}u_c\|^2 + \|B^{s/2}\partial_t u_c\|^2 + \|A^{(1+s)/2}p_c\|^2 + \|A^{s/2}\partial_t p_c\|^2$. The standard Gronwall lemma yields $\Phi(t) \le c(t)$ for all $t \ge 0$, where $c(t)$ generally depends on t, but it is independent of z_0 (provided $z_0 \in \mathcal{B}_0$). Hence $z_c(t) \in B_{\mathcal{H}_s}(0, c(t))$, for every $t \ge 0$ and every $z_0 \in \mathcal{B}_0$. Hence, the thesis follows from the compact embedding $\mathcal{H}_s \hookrightarrow\hookrightarrow \mathcal{H}_0$. □

6. **Smooth attracting sets.** We now establish the existence of a bounded subset in $\mathcal{H}_1$, denoted by $\mathcal{B}_1$, which attracts the bounded subsets in $\mathcal{H}_0$ exponentially fast. This circumstance, on one hand, will provide the regularity of the attractor, and, on the other hand, will turn to be very useful in the construction of the exponential attractor. We point out that, through the use of a different proof of Lemma 5.3 obtained by multiplying $(5.2)_1$ and $(5.2)_2$ by $B^s\partial_t u_c + \epsilon B^s u_c$ and $A^s\partial_t p_c + \epsilon A^s p_c$, respectively, and taking ϵ small enough, it can be shown that the solution z_c of (5.2) actually fulfills

$$\|z_c(t; z_0)\|_s \le c_s, \qquad \forall t \ge 0, \qquad \forall z_0 \in \mathcal{B}_0, \qquad s \in (0, 1/2).$$

This implies that the global attractor $\mathcal{A}$ is a bounded subset of the Hilbert space $\mathcal{H}_s$, for $s \in (0, 1/2)$, which is compactly embedded into the phase-space $\mathcal{H}_0$. In this section we show that the bounded inclusion $\mathcal{A} \subset \mathcal{H}_s$ can be pushed up to $s = 1$. For this purpose, the application of the technique based on the multiplication by fractional operators and on bootstrap arguments, which works perfectly for the the system (1.1) with Dirichlet-Dirichlet boundary conditions, is problematic in the case of the Dirichlet-Neumann boundary conditions, due essentially to the different domains of $A^{s/2}$ and $B^{s/2}$ for $s \ge 1/2$, and to the presence of the coupling terms. In order to achieve the existence of $\mathcal{B}_1$ and hence the $\mathcal{H}_1$-regularity of the attractor for Problem **P** we therefore employ a different approach, which consists in the application to system (1.1) of a new technique due to Pata and Zelik (see [16]), whose key step is a suitable decomposition of the solution semigroup. This decomposition has been recently employed successfully in other recent works (see, e.g., [8, 9, 18]). Notice that this procedure can also be intended as a proof of existence of the global attractor $\mathcal{A}$. Nevertheless, the only existence of $\mathcal{A}$ can be deduced under weaker assumptions, like those of Section 5 and Section 4. Following [16], we shall need, besides (H3), the assumptions (H4), (H5) and (H6). Here is the result we want to prove

Theorem 6.1. *Let (H3)-(H6) hold. Then, there exists a subset $\mathcal{B}_1 \subset \mathcal{H}_1$ closed and bounded in $\mathcal{H}_1$ and $\nu > 0$ such that*

$$dist_{\mathcal{H}_0}(S(t)\mathcal{B}_0, \mathcal{B}_1) \le Me^{-\nu t}, \qquad \forall t \ge 0 \tag{6.1}$$

for some $M > 0$ depending on R_0.

As a straightforward consequence, by virtue of the minimality property of the global attractor, we have the following

Corollary 6.1. *In the hypotheses (H3)-(H6), the global attractor $\mathcal{A}$ of the semigroup on $\mathcal{H}_0$ associated with Problem **P** is contained and bounded in $\mathcal{H}_1$.*

In order to prove Theorem 6.1 we consider the initial data $z_0 \in \mathcal{B}_0$ and we decompose the solution z to **P** into the sum $z = z_1 + z_2$, where $z_1 = (u_1, u_{1t}, p_1, p_{1t})$

and $z_2 = (u_2, u_{2t}, p_2, p_{2t})$ are the solutions to the problems, respectively,

$$\begin{cases} u_{1tt} + u_{1t} + p_{1t} + Bu_1 = 0 \\ p_{1tt} + p_{1t} - u_{1t} + Ap_1 + \psi(p) - \psi(p_2) = 0 \\ z_1(0) = z_0 \end{cases} \tag{6.2}$$

and

$$\begin{cases} u_{2tt} + u_{2t} + p_{2t} + Bu_2 = 0 \\ p_{2tt} + p_{2t} - u_{2t} + Ap_2 + \psi(p_2) = \theta p \\ z_2(0) = 0. \end{cases} \tag{6.3}$$

Here we have set

$$\psi(s) := \phi(s) + \theta s$$

with $\theta \geq l$, in order to have, by (H6), $\psi'(s) \geq 0$. The following lemmas will be needed for the proof of Theorem 6.1. We stress that $c \geq 0$ stands for a generic constant depending possibly only on R_0 (the radius of the absorbing set) and on Ω and ϕ, but neither on $z_0 \in \mathcal{B}_0$ nor on the time t.

Lemma 6.2. *We have $\|z_2(t)\|_0 \leq c$, for every $t \geq 0$.*

Proof. We can use the same argument of the proof of Theorem 4.1. Indeed, identity (4.1) can be obviously rewritten for system (6.3), replacing $F(p)$ with $\Psi(p_2) := \int_0^{p_2} \psi(\sigma)d\sigma$ on the left hand side of (4.1) and adding the additional term $\theta(p, \zeta_2)$ on the right hand side. It is immediate to verify that (H3) is still fulfilled with ϕ replaced by ψ and hence (4.3) still holds for $\Psi(p_2)$ in place of $F(p)$. Writing $\theta(p, \zeta_2) \leq \frac{1}{2}\|\zeta_2\|^2 + c\|p\|^2$, we are led to

$$\frac{d}{dt}E(z_2(t)) + \delta_0\|z_2(t)\|_0^2 \leq 2\epsilon_0 c_1 + c\|p\|^2 \leq c$$

for $z_0 \in \mathcal{B}_0$. By means of Lemma 2.1 (observe that here $z_2(0) = 0$) we immediately conclude the proof. □

Lemma 6.3. *For every $0 \leq s \leq t$ and every $\omega > 0$ we have*

$$\int_s^t (\|u_{2t}(\tau)\|^2 + \|p_{2t}(\tau)\|^2)d\tau \leq \omega(t-s) + \frac{c}{\omega} + c.$$

Proof. We multiply $(6.3)_1$ by u_{2t} and $(6.3)_2$ by p_{2t}. Adding the resulting equations we get

$$\begin{aligned} &\frac{d}{dt}\Big\{\|B^{1/2}u_2\|^2 + \|u_{2t}\|^2 + \|A^{1/2}p_2\|^2 + \|p_{2t}\|^2 + 2\int_\Omega \Psi(p_2) - 2\theta(p, p_2)\Big\} \\ &+2\|u_{2t}\|^2 + 2\|p_{2t}\|^2 = -2\theta(p_t, p_2). \end{aligned} \tag{6.4}$$

Setting

$$\Lambda := \|B^{1/2}u_2\|^2 + \|u_{2t}\|^2 + \|A^{1/2}p_2\|^2 + \|p_{2t}\|^2 + 2\int_\Omega \Psi(p_2) - 2\theta(p, p_2)$$

it is easy to see that $\Lambda(t) \leq c$, for all $t \geq 0$, as a consequence of Corollary 4.1, Lemma 6.2 and (H5). We now can write

$$\frac{d\Lambda}{dt} + 2\|u_{2t}\|^2 + 2\|p_{2t}\|^2 = -2\theta(p_t, p_2) \leq \frac{c}{\omega}\|p_t\|^2 + 2\omega \tag{6.5}$$

where we have used Lemma 6.2 once more. Integrating (6.5) with respect to time between s and t and taking account of Corollary 4.2 and of the bound $\Lambda(t) \leq c$, for every $t \geq 0$, we get the thesis. □

Collecting the above results, for all $z_0 \in \mathcal{B}_0$ and all $t \geq s \geq 0$ we have the bounds

$$\|z(t)\|_0 + \|z_2(t)\|_0 \leq c \tag{6.6}$$

$$\int_s^t (\|u_t(\tau)\|^2 + \|p_t(\tau)\|^2 + \|u_{2t}(\tau)\|^2 + \|p_{2t}(\tau)\|^2) d\tau \leq \omega(t-s) + \frac{c}{\omega} + c \tag{6.7}$$

for every $\omega > 0$.

We are now in a position to prove

Lemma 6.4. *There exists $\nu > 0$ such that*

$$\|z_1(t)\|_0 \leq ce^{-\nu t}, \qquad \forall t \geq 0, \qquad \forall z_0 \in \mathcal{B}_0.$$

Proof. For $\epsilon \in (0,1)$ to be determined later, we multiply $(6.2)_1$ by $u_{1t} + \epsilon u_1$, $(6.2)_2$ by $p_{1t} + \epsilon p_1$, respectively, and add the resulting equations. After some calculations we obtain

$$\begin{aligned}
&\frac{d}{dt}\{\|B^{1/2}u_1\|^2 + \|u_{1t}\|^2 + \epsilon\|u_1\|^2 + 2\epsilon(u_{1t}, u_1) + \|A^{1/2}p_1\|^2 + \|p_{1t}\|^2 \\
&+\epsilon\|p_1\|^2 + 2\epsilon(p_{1t}, p_1) + 2(\psi(p) - \psi(p_2), p_1) - (\psi'(p)p_1, p_1)\} \\
&+2\epsilon\|B^{1/2}u_1\|^2 + 2(1-\epsilon)\|u_{1t}\|^2 + 2\epsilon\|A^{1/2}p_1\|^2 + 2(1-\epsilon)\|p_{1t}\|^2 \\
&+2\epsilon(p_{1t}, u_1) - 2\epsilon(u_{1t}, p_1) + 2\epsilon(p_1, \psi(p) - \psi(p_2)) \\
&= 2((\psi'(p) - \psi'(p_2))p_{2t}, p_1) - (\psi''(p)p_t, p_1^2).
\end{aligned} \tag{6.8}$$

Introducing the functionals

$$\begin{aligned}
\Lambda := &\|B^{1/2}u_1\|^2 + \|u_{1t}\|^2 + \epsilon\|u_1\|^2 + 2\epsilon(u_{1t}, u_1) + \|A^{1/2}p_1\|^2 + \|p_{1t}\|^2 \\
&+\epsilon\|p_1\|^2 + 2\epsilon(p_{1t}, p_1) + 2(\psi(p) - \psi(p_2), p_1) - (\psi'(p)p_1, p_1),
\end{aligned} \tag{6.9}$$

and

$$\begin{aligned}
\Gamma := &\epsilon\|B^{1/2}u_1\|^2 + (2-3\epsilon)\|u_{1t}\|^2 - \epsilon^2\|u_1\|^2 - 2\epsilon^2(u_{1t}, u_1) + \frac{\epsilon}{2}\|A^{1/2}p_1\|^2 \\
&+(2-3\epsilon)\|p_{1t}\|^2 - \epsilon^2\|p_1\|^2 - 2\epsilon^2(p_{1t}, p_1) + \epsilon(\psi'(p)p_1, p_1) \\
&+2\epsilon(p_{1t}, u_1) - 2\epsilon(u_{1t}, p_1),
\end{aligned} \tag{6.10}$$

then, identity (6.8) can be rewritten in the form

$$\frac{d\Lambda}{dt} + \epsilon\Lambda + \frac{\epsilon}{2}\|A^{1/2}p_1\|^2 + \Gamma = 2((\psi'(p) - \psi'(p_2))p_{2t}, p_1) - (\psi''(p)p_t, p_1^2). \tag{6.11}$$

On the other hand, we obtain, using (H6)

$$2(\psi(p) - \psi(p_2), p_1) = 2(\phi(p) - \phi(p_2), p_1) + 2\theta\|p_1\|^2 \geq 2(\theta - l)\|p_1\|^2 \tag{6.12}$$

and also

$$\begin{aligned}
|(\phi'(p)p_1, p_1)| &\leq c(1 + \|A^{1/2}p\|^{r-2})\|p_1\|\|A^{1/2}p_1\| \leq c\|p_1\|\|A^{1/2}p_1\| \\
&\leq \frac{1}{2}\|A^{1/2}p_1\|^2 + c\|p_1\|^2
\end{aligned} \tag{6.13}$$

where we have used the fact that, for $z_0 \in \mathcal{B}_0$, we have $\|A^{1/2}p(t)\| \leq c$ for all $t \geq 0$ (see (6.6) or Corollary 4.1). From (6.12) and (6.13) we deduce

$$2(\psi(p) - \psi(p_2), p_1) - (\psi'(p)p_1, p_1) \geq -\frac{1}{2}\|A^{1/2}p_1\|^2 \tag{6.14}$$

where we suppose to fix θ large enough ($\theta \geq 2l + c$). By means of the Young inequality, from (6.9) and (6.14) it is now straightforward to prove that, choosing

$0 < \epsilon \le \epsilon_0$ with ϵ_0 small enough, there holds (always for $z_0 \in \mathcal{B}_0$ and θ fixed as above)

$$\Lambda(t) \ge c_1 \|z_1(t)\|_0^2, \qquad \forall t \ge 0. \tag{6.15}$$

Furthermore, we have

$$\begin{aligned} 2|(\psi(p) - \psi(p_2), p_1)| &\le c\|p_1\|^2 + c(\|A^{1/2}p\|^{r-2} + \|A^{1/2}p_2\|^{r-2})\|A^{1/2}p_1\|^2 \\ &\le c\|A^{1/2}p_1\|^2 \end{aligned} \tag{6.16}$$

and

$$|(\psi'(p)p_1, p_1)| \le c\|p_1\|^2 + c\|A^{1/2}p\|^{r-2}\|A^{1/2}p_1\|^2 \le c\|A^{1/2}p_1\|^2 \tag{6.17}$$

where we have exploited (6.6) once again (or Corollary 4.1) and Lemma 6.2. Therefore, by means of (6.16) and (6.17), we obtain at once (always for $z_0 \in \mathcal{B}_0$)

$$\Lambda(t) \le c_2 \|z_1(t)\|_0^2, \qquad \forall t \ge 0. \tag{6.18}$$

As far as the functional Γ is concerned, we first observe that $(\psi'(p)p_1, p_1) \ge 0$ (for $\theta \ge l$) and

$$\frac{\epsilon}{2}\|A^{1/2}p_1\|^2 - \epsilon^2\|p_1\|^2 \ge \frac{\epsilon}{4}\|A^{1/2}p_1\|^2, \qquad (\text{for } 0 < \epsilon < 1/4).$$

Thus, using also the Poincaré inequality, we can write

$$\begin{aligned} \Gamma \ge{}& \epsilon(1 - \frac{2\epsilon}{\lambda_B})\|B^{1/2}u_1\|^2 + (2 - 3\epsilon - \epsilon^2)\|u_{1t}\|^2 + \frac{\epsilon}{4}\|A^{1/2}p_1\|^2 \\ &+ (2 - 3\epsilon - \epsilon^2)\|p_{1t}\|^2 - \epsilon^2\|p_1\|^2 - 2\epsilon\|p_{1t}\|\|u_1\| - 2\epsilon\|u_{1t}\|\|p_1\|. \end{aligned}$$

From this last inequality it is easy to infer that

$$\Gamma \ge c\|z_1\|_0^2 \tag{6.19}$$

for ϵ small enough. Finally, the first and the second term on the right hand side of (6.11) can be controlled by

$$c(1 + \|A^{1/2}p\|^{r-3} + \|A^{1/2}p_2\|^{r-3})\|p_{2t}\|\|A^{1/2}p_1\|^2 \le c\|p_{2t}\|\|A^{1/2}p_1\|^2 \tag{6.20}$$

and by

$$c(1 + \|A^{1/2}p\|^{r-3})\|p_t\|\|A^{1/2}p_1\|^2 \le c\|p_t\|\|A^{1/2}p_1\|^2 \tag{6.21}$$

respectively. Therefore, collecting (6.11), (6.20) and (6.21), we can write

$$\begin{aligned} \frac{d\Lambda}{dt} + \epsilon\Lambda + \frac{\epsilon}{2}\|A^{1/2}p_1\|^2 + \Gamma &\le c(\|p_{2t}\| + \|p_t\|)\|A^{1/2}p_1\|^2 \\ &\le \left[\frac{\epsilon}{2} + c(\|p_{2t}\|^2 + \|p_t\|^2)\right]\|A^{1/2}p_1\|^2 \end{aligned} \tag{6.22}$$

and, by means of (6.15) and (6.19), (6.22) leads us to the differential inequality

$$\frac{d\Lambda}{dt} + \epsilon\Lambda \le c(\|p_{2t}\|^2 + \|p_t\|^2)\Lambda. \tag{6.23}$$

Now we observe that, by virtue of (6.7), the hypotheses for the application of Lemma 2.2 to inequality (6.23) are satisfied (with the choice $k = 0$, $h = \|p_{2t}\|^2 + \|p_t\|^2$ and $\omega = \epsilon/2c$). Thus, we deduce

$$\Lambda(t) \le c\Lambda(0)e^{-\epsilon t/2}, \qquad \forall t \ge 0, \qquad z_0 \in \mathcal{B}_0 \tag{6.24}$$

and, by means of (6.15) and (6.18), we get the thesis. ◻

Lemma 6.5. *We have*

$$\|z_2(t)\|_1 \le c, \qquad \forall t \ge 0, \qquad \forall z_0 \in \mathcal{B}_0.$$

Proof. We differentiate $(6.3)_1$ and $(6.3)_2$ with respect to time and we set $v := u_{2t}$ and $q := p_{2t}$. We obtain

$$\begin{cases} v_{tt} + v_t + q_t - \Delta v = 0 \\ q_{tt} + q_t - v_t - \Delta q + q + \psi'(p_2)q = \theta p_t. \end{cases} \tag{6.25}$$

We multiply $(6.25)_1$ by $v_t + \epsilon v$, $(6.25)_2$ by $q_t + \epsilon q$, with $\epsilon > 0$ to be chosen later, and we add the resulting equations. Performing the same kind of calculations done in the proof of Lemma 6.4, we get

$$\begin{aligned} &\frac{d\Lambda}{dt} + \epsilon\Lambda + \epsilon\|B^{1/2}v\|^2 + (2-3\epsilon)\|v_t\|^2 - \epsilon^2\|v\|^2 - 2\epsilon^2(v_t, v) + \epsilon\|A^{1/2}q\|^2 \\ &+(2-3\epsilon)\|q_t\|^2 - \epsilon^2\|q\|^2 - 2\epsilon^2(q_t, q) + \epsilon(\psi'(p_2)q, q) + 2\epsilon(q_t, v) - 2\epsilon(v_t, q) \\ &= (\psi''(p_2)p_{2t}, q^2) + 2\theta(p_t, q_t) + 2\epsilon\theta(p_t, q) \end{aligned}$$

where

$$\begin{aligned} \Lambda := &\|B^{1/2}v\|^2 + \|v_t\|^2 + \epsilon\|v\|^2 + 2\epsilon(v_t, v) + \|A^{1/2}q\|^2 + \|q_t\|^2 \\ &+\epsilon\|q\|^2 + 2\epsilon(q_t, q) + (\psi'(p_2)q, q). \end{aligned} \tag{6.26}$$

If we set

$$\begin{aligned} \Gamma := &\epsilon\|B^{1/2}v\|^2 + (2-3\epsilon)\|v_t\|^2 - \epsilon^2\|v\|^2 - 2\epsilon^2(v_t, v) + \frac{\epsilon}{2}\|A^{1/2}q\|^2 \\ &+(1-3\epsilon)\|q_t\|^2 - \epsilon^2\|q\|^2 - 2\epsilon^2(q_t, q) + \epsilon(\psi'(p_2)q, q) \\ &+2\epsilon(q_t, v) - 2\epsilon(v_t, q) \end{aligned} \tag{6.27}$$

we are thus led to

$$\frac{d\Lambda}{dt} + \epsilon\Lambda + \Gamma + \frac{\epsilon}{2}\|A^{1/2}q\|^2 + \|q_t\|^2 = (\psi''(p_2)p_{2t}, q^2) + 2\theta(p_t, q_t) + 2\epsilon\theta(p_t, q). \tag{6.28}$$

We now have

$$\begin{aligned} (\psi''(p_2)p_{2t}, q^2) &\le c(1 + \|A^{1/2}p_2\|^{r-3})\|p_{2t}\|\|A^{1/2}q\|^2 \\ &\le c\|p_{2t}\|\|A^{1/2}q\|^2 \le (\frac{\epsilon}{2} + c\|p_{2t}\|^2)\|A^{1/2}q\|^2 \end{aligned} \tag{6.29}$$

by (H5) and Lemma 6.2, being $z_0 \in \mathcal{B}_0$. Therefore, from (6.28), (6.29) we deduce

$$\begin{aligned} \frac{d\Lambda}{dt} + \epsilon\Lambda + \Gamma &\le c\|p_{2t}\|^2\|A^{1/2}q\|^2 + c\|p_t\|^2 + c\|p_{2t}\|^2 \\ &\le c\|p_{2t}\|^2\|A^{1/2}q\|^2 + c. \end{aligned} \tag{6.30}$$

using Corollary 4.1 and Lemma 6.2 again in the last estimate. Now, it is not difficult to see, with the help of the Young inequality, that, for $0 < \epsilon \le \epsilon_0$ with ϵ_0 small enough, and $\theta \ge l$, we have

$$c_1\|(v, v_t, q, q_t)\|_0 \le \Lambda \le c_2\|(v, v_t, q, q_t)\|_0$$

and $\Gamma \ge 0$. From (6.30) we thus obtain

$$\frac{d\Lambda}{dt} + \epsilon\Lambda \le c\|p_{2t}\|^2\Lambda + c \tag{6.31}$$

and the application of Lemma 2.2 to the differential inequality (6.31) (cf. (6.7)) yields the bound $\Lambda(t) \le c$ for all $t \ge 0$, and for all $z_0 \in \mathcal{B}_0$, which entails

$$\|B^{1/2}u_{2t}(t)\| + \|u_{2tt}(t)\| + \|A^{1/2}p_{2t}(t)\| + \|p_{2tt}(t)\| \le c.$$

From this last bound we recover, with the help of $(6.3)_1$ and $(6.3)_2$, the further controls $\|Bu_2(t)\| \le c$ and $\|Ap_2(t)\| \le c$, from which we get the thesis. □

We can now conclude the proof of the main result of this section.

Proof of Theorem 6.1. Let us put $\mathcal{B}_1 := \{z \in \mathcal{H}_1 : \|z\|_1 \le c\}$ with c as in Lemma 6.5. Then, by using Lemma 6.4 and Lemma 6.5 we immediately get (6.1). □

7. **The exponential attractor.** The global attractor is not always a nice object to describe the longterm dynamics of a system. Actually, the rate of convergence of the trajectories to the attractor is not controlled in general and, in some concrete cases, may be arbitrarily small (see, e.g., [13]). This means that, when only the existence of the global attractor is proved, it is in general very difficult, if not impossible, to know the time needed to stabilize the system. A more precise information on the convergence rate can be achieved when, for instance, the stationary solutions are hyperbolic. In that case, the global attractor (the so-called regular attractor in the terminology of A.V. Babin and M.I. Vishik [1]) is regular and exponential (in a sense that we shall precise below). For the regular attractors the rate of convergence can also be estimated in terms of the hyperbolicity contants of the equilibria, but, even in this situation, it is usually very difficult to estimate these constants for concrete equations. We also point out that the global attractor presents other defaults. Indeed, it is very difficult to express the convergence rate in terms of the physical parameters of the problem and, in addition, the global attractor may be sensitive to perturbations. In order to overcome all these difficulties Eden, Foias, Nicolaenko and Temam (see [4], [5]) introduced the notion of exponential attractor. This is a compact, positively invariant subset of the phase-space of finite fractal dimension that attracts bounded subsets of initial data exponentially fast (with a rate independent from the chosen subset of initial data) and it is more robust under perturbations. Unfortunately, contrary to the global attractor, the exponential attractor is not unique. However, if there exists an exponential attractor $\mathcal{E}$, then the semigroup possesses a compact attracting set, and thus it has a global attractor $\mathcal{A} \subset \mathcal{E}$ of finite fractal dimension, being $\dim_F \mathcal{A} \le \dim_F \mathcal{E}$. In spite of its lack of uniqueness, the exponential attractor can be considered a good compromise between the necessity of confining the longterm dynamics in a small set, and the necessity of having a satisfactory time control of the convergence of the trajectories. For a more extensive discussion and a review on recent results on exponential attractors for evolution equations we refer the reader to, e.g., [14].
In this section we prove that system (1.1) subject to Dirichlet-Neumann omogeneous boundary conditions possesses an exponential attractor which attracts all bounded subsets of $\mathcal{H}_0$. We first recall, for the reader's convenience, the definition of exponential attractor, which is a generalization of the definition in [4], [5] justified by the fact that we can prove that the exponential attractor has a basin of attraction coinciding with the whole phase-space (see, e.g., [15]).

Definition 7.1. A compact set $\mathcal{E} \subset \mathcal{H}_0$ is called an *exponential attractor* for the semigroup $S(t)$ if the following conditions hold

(i) $\mathcal{E}$ is positively invariant, that is, $S(t)\mathcal{E} \subset \mathcal{E}$ for every $t \ge 0$;

(ii) $\mathcal{E}$ has finite fractal dimension, that is, $\dim_F \mathcal{E} < \infty$;

(iii) there exists an increasing function $J : [0, +\infty) \to [0, +\infty)$ and $\kappa > 0$ such that, for every $R > 0$ and for every set $\mathcal{B} \subset \mathcal{H}_0$ with $\sup_{z_0 \in \mathcal{B}} \|z_0\|_0 \leq R$ there holds

$$\mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{B}, \mathcal{E}) \leq J(R)e^{-\kappa t}. \tag{7.1}$$

We are now ready to state the main result of this section.

Theorem 7.2. *In the hypotheses (H3)-(H6) the semigroup $S(t)$ on $\mathcal{H}_0$ associated with problem $\boldsymbol{P}$ possesses an exponential attractor $\mathcal{E}$.*

In order to prove Theorem 7.2 we exploit the approach introduced in [6] by Efendiev, Miranville and Zelik, which allows the construction of the exponential attractor without the use of orthogonal projectors. For our purpose the abstract result we need is the following (see [15])

Lemma 7.3. *Let $\mathcal{X} \subset \mathcal{H}_0$ be a relatively compact (positively) invariant subset for the semigroup $S(t)$. Assume that there exists a time $t^* > 0$ such that*

(i) the map

$$(t, z) \mapsto S(t)z : [0, t^*] \times \mathcal{X} \to \mathcal{X}$$

is Lipschitz continuous (with the topology inherited from $\mathcal{H}_0$);

(ii) the map $S(t^*) : \mathcal{X} \to \mathcal{X}$ admits a decomposition of the form

$$S(t^*) = S_d + S_c, \qquad S_d : \mathcal{X} \to \mathcal{H}_0, \qquad S_c : \mathcal{X} \to \mathcal{H}_1$$

where S_d and S_c satisfy the conditions

$$\|S_d(z_2) - S_d(z_1)\|_0 \leq \frac{1}{8}\|z_2 - z_1\|_0, \qquad \forall z_1, z_2 \in \mathcal{X}$$

and

$$\|S_c(z_2) - S_c(z_1)\|_1 \leq C_* \|z_2 - z_1\|_0, \qquad \forall z_1, z_2 \in \mathcal{X}$$

for some $C_* > 0$.

Then there exists an invariant compact set $\mathcal{E} \subset \mathcal{X}$ such that

$$dist_{\mathcal{H}_0}(S(t)\mathcal{X}, \mathcal{E}) \leq J_0 e^{-\frac{log2}{t_*}t} \tag{7.2}$$

where

$$J_0 = 2L_* \sup_{z_0 \in \mathcal{X}} \|z_0\|_0 e^{\frac{log2}{t_*}}$$

and L_* is the Lipschitz constant of the map $S(t_*) : \mathcal{X} \to \mathcal{X}$. Moreover

$$dim_F \mathcal{E} \leq 1 + \frac{\log N_*}{\log 2} \tag{7.3}$$

where N_* is the minimum number of balls of radius $\frac{1}{8C_*}$ of $\mathcal{H}_0$ necessary to cover the unit ball of $\mathcal{H}_1$.

Besides Lemma 7.3 we recall another important abstract result which, in particular, shall be applied to show that the basin of attraction of the exponential attractor coincides with the whole phase-space. This result concerns the transitivity property of exponential attraction (see [7, Theorem 5.1]).

Lemma 7.4. *Let K_1, K_2, K_3 be subsets of $\mathcal{H}_0$ such that*

$$dist_{\mathcal{H}_0}(S(t)K_1, K_2) \leq L_1 e^{-\theta_1 t}, \qquad dist_{\mathcal{H}_0}(S(t)K_2, K_3) \leq L_2 e^{-\theta_2 t}$$

for some $\theta_1, \theta_2 > 0$ and $L_1, L_2 \geq 0$. Assume also that for all $z_1, z_2 \in \cup_{t \geq 0} S(t)K_j$ (with $j = 1, 2, 3$) there holds

$$\|S(t)z_2 - S(t)z_1\|_0 \leq L_0 e^{\theta_0 t} \tag{7.4}$$

for some $\theta_0, L_0 \geq 0$. Then we have

$$dist_{\mathcal{H}_0}(S(t)K_1, K_3) \leq L e^{-\theta t}$$

where $\theta = \frac{\theta_1 \theta_2}{\theta_0 + \theta_1 + \theta_2}$ and $L = L_0 L_1 + L_2$.

We now state and prove some lemmas that will be useful to verify the assumptions of Lemma 7.3.

Lemma 7.5. *There exists $C > 0$ such that*

$$\sup_{z_0 \in \mathcal{B}_1} \|z_t(t)\|_0 \leq C, \qquad \forall t \geq 0.$$

Proof. The proof follows immediately from the proof of Theorem 4.2. Indeed, by means of (4.25), (4.26) and recalling that $w(t) = z_t(t)$, we get the thesis. □

Now, if we set

$$\mathcal{X} := \bigcup_{t \geq 0} S(t)\mathcal{B}_1$$

then, $\mathcal{X}$ is positively invariant and furthermore we have

- $\mathcal{X}$ is a bounded subset of $\mathcal{H}_1$ and thus relatively compact in $\mathcal{H}_0$. This is an easy consequence of the existence of a bounded absorbing set $\mathcal{B}_0^{(1)}$ in $\mathcal{H}_1$ and of the continuous dependence estimate that holds in $\mathcal{H}_1$. Indeed, let $t_1 > 0$ be such that $\cup_{t \geq t_1} S(t)\mathcal{B}_1 \subset \mathcal{B}_0^{(1)}$. We only have to show that $\cup_{0 \leq t \leq t_1} S(t)\mathcal{B}_1$ is bounded in $\mathcal{H}_1$ as well. But this fact is immediately implied by (3.4).
- $\mathcal{X}$ satisfies
$$\text{dist}_{\mathcal{H}_0}(S(t)\mathcal{B}_0, \mathcal{X}) \leq M e^{-\nu t} \tag{7.5}$$
for some $M \geq 0$ and $\nu > 0$. Indeed, we have $\text{dist}_{\mathcal{H}_0}(S(t)\mathcal{B}_1, \mathcal{X}) = 0$ and, on account of Theorem 6.1 as well, we can apply Lemma 7.4 with the choice $K_1 = \mathcal{B}_0$, $K_2 = \mathcal{B}_1$ and $K_3 = \mathcal{X}$. Assumption (7.4) of Lemma 7.4 holds in light of the continuous dependence estimate in $\mathcal{H}_0$ ensured by Theorem 3.2 (see (3.3)).
- There exist $C > 0$ such that
$$\sup_{z_0 \in \mathcal{X}} \|z_t(t)\|_0 \leq C, \qquad \forall t \geq 0.$$
This is a direct consequence of Lemma 7.5.

Our purpose now is to verify assumptions (i) and (ii) of Lemma 7.3 taking for $\mathcal{X}$ the subset we have just constructed.

Lemma 7.6. *For every $T > 0$ the mapping $(t, z_0) \to S(t)z_0$ is Lipschitz continuous from $[0, T] \times \mathcal{X}$ with values in $\mathcal{H}_0$. Therefore, assumption (i) of Lemma 7.3 holds true.*

Proof. Let $z_1, z_2 \in \mathcal{X}$ and $t_1, t_2 \in [0,T]$. We write

$$\|S(t_2)z_2 - S(t_1)z_1\|_0 \leq \|S(t_2)z_2 - S(t_2)z_1\|_0 + \|S(t_2)z_1 - S(t_1)z_1\|_0.$$

Due to (3.3), we have $\|S(t_2)z_2 - S(t_2)z_1\|_0 \leq K\|z_2 - z_1\|_0$, where $K = e^{\theta_0 T}$ depends only on R_0 and T. Moreover, by Lemma 7.5,

$$\|S(t_2)z_1 - S(t_1)z_1\|_0 = \Big\| \int_{t_1}^{t_2} z_t(\tau; z_1)d\tau \Big\|_0 \leq \Big| \int_{t_1}^{t_2} \|z_t(\tau; z_1)\|_0 d\tau \Big| \leq C|t_2 - t_1|.$$

Hence

$$\|S(t_2)z_2 - S(t_1)z_1\|_0 \leq L(\|z_2 - z_1\|_0 + |t_2 - t_1|)$$

with $L = \max\{K, C\} = L(T) > 0$. ☐

Lemma 7.7. *Assumption (ii) of Lemma 7.3 holds true.*

Proof. We consider two trajectories departing from $\mathcal{X}$, namely, $z^1 = (u^1, u^1_t, p^1, p^1_t)$ and $z^2 = (u^2, u^2_t, p^2, p^2_t)$ corresponding to z_{10}, $z_{20} \in \mathcal{X}$. Then we set

$$\overline{z} := z^2 - z^1 =: (\overline{u}, \overline{u}_t, \overline{p}, \overline{p}_t), \qquad \overline{z}_0 := z_{20} - z_{10}$$

and we decompose $\overline{z}$ as $\overline{z} = \overline{z}_d + \overline{z}_c =: (\overline{u}_d, \partial_t\overline{u}_d, \overline{p}_d, \partial_t\overline{p}_d) + (\overline{u}_c, \partial_t\overline{u}_c, \overline{p}_c, \partial_t\overline{p}_c)$, where the components of $\overline{z}_d$ and $\overline{z}_c$ solve, respectively,

$$\begin{cases} \partial_{tt}\overline{u}_d + \partial_t\overline{u}_d + \partial_t\overline{p}_d + B\overline{u}_d = 0 \\ \partial_{tt}\overline{p}_d + \partial_t\overline{p}_d - \partial_t\overline{u}_d + A\overline{p}_d = 0 \\ \overline{z}_d(0) = \overline{z}_0 \end{cases} \tag{7.6}$$

and

$$\begin{cases} \partial_{tt}\overline{u}_c + \partial_t\overline{u}_c + \partial_t\overline{p}_c + B\overline{u}_c = 0 \\ \partial_{tt}\overline{p}_c + \partial_t\overline{p}_c - \partial_t\overline{u}_c + A\overline{p}_c = \phi(p^1) - \phi(p^2) \\ \overline{z}_c(0) = 0. \end{cases} \tag{7.7}$$

Arguing as in the proof of Lemma 5.2, for $\overline{z}_d$ we obtain

$$\|\overline{z}_d(t)\|_0 \leq c\|\overline{z}_0\|_0 e^{-\nu_0 t}, \qquad \forall t \geq 0$$

for some c and $\nu_0 > 0$. Hence, choosing $t = t^* = \frac{1}{\nu_0}\log(8c)$, the first part of assumption (ii) of Lemma 7.3 is fulfilled. In order to verify the second part of assumption (ii), from system (7.7), by multiplying the first equation by $B\partial_t\overline{u}_c$, the second by $A\partial_t\overline{p}_c$ and summing the resulting identities, we easily obtain

$$\frac{1}{2}\frac{d}{dt}\|\overline{z}_c(t)\|_1^2 + \|B^{1/2}\partial_t\overline{u}_c\|^2 + \|A^{1/2}\partial_t\overline{p}_c\|^2 = (\phi(p^1) - \phi(p^2), A\partial_t\overline{p}_c). \tag{7.8}$$

We now have to control the nonlinear term on the right hand side of (7.8) in terms of the $\mathcal{H}_0$-norm of the difference of the initial data $\overline{z}_0$. First we observe that

$$\begin{aligned} (\phi(p^1) - \phi(p^2), A\partial_t\overline{p}_c) &= \int_\Omega (\phi'(p^1)\nabla p^1 - \phi'(p^2)\nabla p^2)\cdot\nabla\partial_t\overline{p}_c \\ &+ \int_\Omega (\phi(p^1) - \phi(p^2))\partial_t\overline{p}_c \leq \|(\phi'(p^1) - \phi'(p^2))\nabla p^1\|\|\nabla\partial_t\overline{p}_c\| \\ &+ \|\phi'(p^2)(\nabla p^1 - \nabla p^2)\|\|\nabla\partial_t\overline{p}_c\| + \|\phi(p^1) - \phi(p^2)\|\|\partial_t\overline{p}_c\|. \end{aligned} \tag{7.9}$$

Concerning the first two terms on the right hand side of (7.9), with the aid of (H5) and of the equivalence of the H^2-norm and of the graph norm on $D(A)$, we get the estimates

$$\begin{aligned} \|(\phi'(p^1) - \phi'(p^2))\nabla p^1\| &\leq c\Big(\int_\Omega (1 + |p^1|^{2(r-3)} + |p^2|^{2(r-3)})|p^1 - p^2|^2|\nabla p^1|^2\Big)^{\frac{1}{2}} \\ &\leq c(1 + \|A^{1/2}p^1\|^{r-3} + \|A^{1/2}p^2\|^{r-3})\|Ap^1\|\|A^{1/2}\overline{p}\| \end{aligned} \tag{7.10}$$

and

$$\begin{aligned}\|\phi'(p^2)(\nabla p^1-\nabla p^2)\| &\le c\Big(\int_\Omega (1+|p^2|^{2(r-2)})|\nabla p^1-\nabla p^2|^2\Big)^{\frac12}\\ &\le c(1+\|Ap^2\|^{r-2})\|A^{1/2}\overline{p}\|. \end{aligned}\tag{7.11}$$

Obviously, we also have

$$\|\phi(p^1)-\phi(p^2)\|\le c(1+\|A^{1/2}p^1\|^{r-2}+\|A^{1/2}p^2\|^{r-2})\|A^{1/2}\overline{p}\|. \tag{7.12}$$

Exploiting the continuous dependence estimate on $\mathcal{H}_0$ and taking into account the fact that, for $z_{01}, z_{02}\in\mathcal{X}\subset\mathcal{H}_1$ we have $\|z^1(t)\|_1$, $\|z^2(t)\|_1\le c$ for every $t\ge 0$, by combination of (7.9)-(7.12) we deduce

$$\begin{aligned}(\phi(p^1)-\phi(p^2), A\partial_t\overline{p}_c) &\le c\|A^{1/2}\overline{p}\|(\|\nabla\partial_t\overline{p}_c\|+\|\partial_t\overline{p}_c\|)\\ &\le \frac12\|A^{1/2}\partial_t\overline{p}_c\|^2+c_T\|\overline{z}_0\|_0^2, \qquad \forall t\in[0,T] \end{aligned}\tag{7.13}$$

for every $T>0$. Therefore, from (7.8) and (7.13) we eventually get

$$\frac{d}{dt}\|\overline{z}_c(t)\|_1^2\le 2c_T\|\overline{z}_0\|_0^2, \qquad \forall t\in[0,T]$$

so that

$$\|\overline{z}_c(t^*)\|_1\le C_*\|\overline{z}_0\|_0$$

with $C_*=\sqrt{2t^*c_{t^*}}$. This implies the second part of assumption (ii). □

Proof of Theorem 7.2. By virtue of Lemma 7.6 and of Lemma 7.7 all the assumptions of Lemma 7.3 are fulfilled thus yielding the existence of a compact invariant subset $\mathcal{E}\subset\mathcal{X}\subset\mathcal{H}_1$ such that (7.2) and (7.3) hold. We now show that $\mathcal{E}$ is an exponential attractor for the semigroup $S(t)$ on $\mathcal{H}_0$. Indeed, being $\mathcal{E}$ of finite fractal dimension, there only remains to show that $\mathcal{E}$ attracts (exponentially fast) all bounded subset of the whole phase-space $\mathcal{H}_0$ (see (7.1)). Let $\mathcal{B}\subset\mathcal{H}_0$ be a bounded subset of $\mathcal{H}_0$ and $R>0$ be such that $\sup_{z_0\in\mathcal{B}}\|z_0\|_0\le R$. Recalling (7.5) and (7.2), we can apply the transitivity property of exponential attraction (Lemma 7.4) with $K_1=\mathcal{B}_0$, $K_2=\mathcal{X}$ and $K_3=\mathcal{E}$. Assumption (7.4) of Lemma 7.4 is easily checked due to (3.3) and to the fact that $\cup_{t\ge0}S(t)K_j$ is bounded in $\mathcal{H}_0$ for $j=1,2,3$. Hence, we obtain $\mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{B}_0,\mathcal{E})\le Le^{-\theta t}$, for all $t\ge0$, for some $L,\theta\ge0$ with only L depending (increasingly and continuously) on the radius R_0 of the absorbing set $\mathcal{B}_0$. Now, let $t_0=t_0(R)\ge0$ be such that $S(t)\mathcal{B}\subset\mathcal{B}_0$, for all $t\ge t_0$. Then

$$\mathrm{dist}_{\mathcal{H}_0}(S(2t)\mathcal{B},\mathcal{E})\le \mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{B}_0,\mathcal{E})\le Le^{-\theta t}, \qquad \forall t\ge t_0(R).$$

On the other hand, thanks to Corollary 4.1, we have $\mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{B},\mathcal{E})\le\widetilde{L}$, for all $t\ge0$, for some $\widetilde{L}=\widetilde{L}(R)$. Collecting the last two inequalities we get

$$\mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{B},\mathcal{E})\le Je^{-\frac{\theta}{2}t}, \qquad \forall t\ge0,$$

with $J=J(R):=L+\widetilde{L}(R)e^{\theta t_0(R)}$. This completes the proof. □

Acknowledgements. I would like to thank Professor Maurizio Grasselli for having proposed this subject to me and for many inspiring and stimulating discussions. I also thank the referee for careful reading and valuable comments.

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Received June 2007; revised January 2008.

E-mail address: sergio.frigeri@unipv.it