GLOBAL AND TRAJECTORY ATTRACTORS FOR A NONLOCAL CAHN-HILLIARD-NAVIER-STOKES SYSTEM

Sergio Frigeri

Dipartimento di Matematica *F. Enriques*

Università degli Studi di Milano

Milano I-20133, Italy

sergio.frigeri@unimi.it

Maurizio Grasselli

Dipartimento di Matematica *F. Brioschi*

Politecnico di Milano

Milano I-20133, Italy

maurizio.grasselli@polimi.it

Abstract

The Cahn-Hilliard-Navier-Stokes system is based on a well-known diffuse interface model and describes the evolution of an incompressible isothermal mixture of binary fluids. A nonlocal variant consists of the Navier-Stokes equations suitably coupled with a nonlocal Cahn-Hilliard equation. The authors, jointly with P. Colli, have already proven the existence of a global weak solution to a nonlocal Cahn--Hilliard-Navier-Stokes system subject to no-slip and no-flux boundary conditions. Uniqueness is still an open issue even in dimension two. However, in this case, the energy identity holds. This property is exploited here to define, following J.M. Ball's approach, a generalized semiflow which has a global attractor. Through a similar argument, we can also show the existence of a (connected) global attractor for the convective nonlocal Cahn-Hilliard equation with a given velocity field, even in dimension three. Finally, we demonstrate that any weak solution fulfilling the energy inequality also satisfies an energy inequality. This allows us to establish the existence of the trajectory attractor also in dimension three with a time dependent external force.

1   Introduction

Diffuse-interface methods in Fluid Mechanics are widely used by many researchers in order to describe the behavior of complex fluids (see, e.g., [5, 19] and references therein). A typical example is a mixture of two incompressible fluids like, e.g., oil and water. To describe the evolution of such a system a sufficiently simple model is the so-called H model (see [33], cf. also [30, 34, 43] and references therein). This consists in a suitable coupling of the Navier-Stokes equations for the (average) fluid velocity u, with a Cahn-Hilliard type equation for the order parameter φ (i.e., the relative concentration of one fluid or the difference of the two concentrations). Temperature variations are neglected and the density is supposed to be constant. This kind of system, called Cahn-Hilliard-Navier-Stokes system, has been analyzed by several authors both theoretically (see, for instance, [1, 3, 11, 24, 25, 26, 27, 49, 51]) and numerically (cf., e.g., [6, 13, 20, 36, 37, 39, 48]). Generalizations to unmatched densities and compressible case have also been investigated (see [2, 4, 12]). On the other hand, it is well know that the usual Cahn-Hilliard equation can be viewed as a local approximation of a nonlocal Cahn-Hilliard equation (see, for instance, [9, 10, 22, 23, 28, 29, 31, 40]). However, the corresponding nonlocal version of the Cahn-Hilliard-Navier-Stokes system has been analyzed only recently in [17]. Nonetheless it is worth mentioning that there exist some related works devoted to liquid-vapor phase transitions (i.e., the so-called Navier-Stokes-Korteweg systems) in which nonlocal energy functionals are considered (see [44, 45], cf. also [32]).

More precisely, we want consider the following system (see [17] for details)

$$\varphi_t + u \cdot \nabla\varphi = \Delta\mu, \tag{1.1}$$

$$\mu = a\varphi - J * \varphi + F'(\varphi), \tag{1.2}$$

$$u_t - \text{div}(\nu(\varphi)2Du) + (u \cdot \nabla)u + \nabla\pi = \mu\nabla\varphi + h(t), \tag{1.3}$$

$$\text{div}(u) = 0, \tag{1.4}$$

in $\Omega \times (0, \infty)$, where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain with a sufficiently smooth boundary and the density has been taken equal to one. Here $J : \mathbb{R}^d \to \mathbb{R}$ is the interaction kernel and

$$(J * \varphi)(x) := \int_\Omega J(x - y)\varphi(y)dy, \qquad a(x) := \int_\Omega J(x - y)dy, \qquad x \in \Omega. \tag{1.5}$$

2

We recall that F is the potential accounting for the presence of two phases, while $\nu > 0$ denotes the viscosity, π the pressure, $2Du := \nabla u + (\nabla u)^{tr}$ and h represents an external force acting on the mixture.

In [17], jointly with P. Colli, we have proven the existence of a global weak solution for system (1.1)-(1.4) endowed with the following boundary and initial conditions

$$\frac{\partial \mu}{\partial n} = 0, \quad u = 0, \quad \text{on } \partial\Omega \times (0, T), \tag{1.6}$$

$$u(0) = u_0, \quad \varphi(0) = \varphi_0, \quad \text{in } \Omega, \tag{1.7}$$

where n is the unit outward normal to $\partial\Omega$. This result has been obtained by assuming that F is sufficiently smooth and of arbitrary polynomial growth. In addition, we have shown some regularity properties of the solution provided that F satisfies a reasonable coercivity condition. In particular, such properties entail the validity of an energy identity in dimension two. However, even in this case, uniqueness is still an open issue. This is due to the lack of regularity of the order parameter φ which is a consequence of the presence of the nonlocal term in place of the usual Laplace operator acting on φ (see [17] for details). On the other hand, finding stronger solutions does not seem straightforward as well. Thus, even in two dimensions, the analysis of the (global) longtime behavior appears to be rather challenging. Fortunately, at least in this case, we have an energy equality so we have already observed that, in the autonomous case, the existence of a global attractor might be established by using the notion of generalized semiflow introduced by J.M. Ball (see [17, Rem. 7]). This is exactly the first (and main) result of this contribution. Namely, if $d = 2$ and h does not depend on time, we prove that (1.1)-(1.4) with (1.6)-(1.7) defines a generalized semiflow which is point dissipative and possesses a compact attractor. An interesting consequence is that we can also prove the existence of a global attractor for the nonlocal Cahn-Hilliard equation with convection assuming $u \in L^\infty(\Omega)^d$ is given and independent of time. This can be achieved even in the case $d = 3$ with a restriction on the growth of F (still including the classical smooth double-well potential). The reason is that, for the Cahn-Hilliard equation alone, the energy equality also holds in three dimensions. In addition, in this case, we can prove uniqueness so that we can define a semiflow and the related global attractor is connected. The last result of this paper is of interest, in particular, for the three dimensional nonautonomous case. Indeed, we first demonstrate a suitable generalization of an integral form of Gronwall's lemma. This inequality allows us to show that any weak solution satisfies a dissipative estimate also in dimension three. Moreover, we can show that there is a weak solution satisfying the energy estimate for any initial time on, with some growth restrictions on F if $d = 3$. Using this fact we can establish the existence of the trajectory attractor following the theory presented in [14] (cf. [25] for the local Cahn-Hilliard-Navier-Stokes system).

The plan of the paper goes as follows. In the next Section 2 we introduce the assumptions and we briefly restate the results obtained in [17]. Then, in Section 3, we proceed to proving the main result by recalling first some basic notions on generalized semiflows. The convective nonlocal Cahn-Hilliard equation case is discussed in Section 4, while the generalized Gronwall lemma and the dissipative estimate are proven in Section 5. The final Section 6 is devoted to the existence of the trajectory attractor.

2 Functional setup and known results

For $d = 2, 3$ we introduce the classical Hilbert spaces for the Navier-Stokes equations (see, e.g., [50])

$$G_{div} := \overline{\{u \in C_0^\infty(\Omega)^d : \ \operatorname{div}(u) = 0\}}^{L^2(\Omega)^d},$$

and

$$V_{div} := \{u \in H_0^1(\Omega)^d : \ \operatorname{div}(u) = 0\}.$$

We also set $H = L^2(\Omega)$, $V = H^1(\Omega)$ and denote by $\| \cdot \|$ and $(\cdot, \cdot)$ the norm and the scalar product, respectively, on both H and G_{div}. H will also be used for L^2 spaces of vector or matrix valued functions. The notation $\langle \cdot, \cdot \rangle$ will stand for the duality pairing between a Banach space and its dual. V_{div} is endowed with the scalar product

$$(u, v)_{V_{div}} = (\nabla u, \nabla v), \qquad \forall u, v \in V_{div}.$$

Let us also recall the definition of the Stokes operator $A : D(A) \cap G_{div} \to G_{div}$ in the case of no-slip boundary condition (1.6), i.e. $A = -P\Delta$ with domain $D(A) = H^2(\Omega)^d \cap V_{div}$, where $P : L^2(\Omega)^d \to G_{div}$ is the Leray projector. Notice that we have

$$(Au, v) = (u, v)_{V_{div}} = (\nabla u, \nabla v), \qquad \forall u \in D(A), \quad \forall v \in V_{div}.$$

We also recall that $A^{-1} : G_{div} \to G_{div}$ is a self-adjoint compact operator in G_{div} and by the classical spectral theorems there exists a sequence λ_j with $0 < \lambda_1 \leq \lambda_2 \leq \cdots$ and $\lambda_j \to \infty$, and a family of $w_j \in D(A)$ which is orthonormal in G_{div} and such that $Aw_j = \lambda_j w_j$. We also define the map $\mathcal{A} : V_{div} \times H \to V_{div}'$ in the following way. For every $u \in V_{div}$ and every $\varphi \in H$ we set

$$\langle \mathcal{A}(u, \varphi), v \rangle := (\nu(\varphi) 2Du, Dv), \qquad \forall v \in V_{div},$$

where ν is a continuous function satisfying $\nu_1 \leq \nu(s) \leq \nu_2$, for all $s \in \mathbb{R}$, with $\nu_1, \nu_2 > 0$. Notice that if $\nu = 1$ we have

$$\langle \mathcal{A}(u, \varphi), v \rangle = (2Du, Dv) = (\nabla u, \nabla v), \qquad \forall u, v \in V_{div},$$

4

and hence in this case we have $\mathcal{A}(u, \varphi) = Au$ for every $u \in D(A)$. Moreover we have

$$\|\mathcal{A}(u, \varphi)\|_{V'_{div}} \leq \nu_2 \|u\|_{V_{div}}, \qquad \forall u \in V_{div}, \quad \forall \varphi \in H.$$

The trilinear form b which appears in the weak formulation of the Navier-Stokes equations is defined as usual

$$b(u, v, w) = \int_{\Omega} (u \cdot \nabla) v \cdot w, \qquad \forall u, v, w \in V_{div},$$

and the associated bilinear map $\mathcal{B}$ from $V_{div} \times V_{div}$ into V'_{div} as

$$\langle \mathcal{B}(u, v), w \rangle = b(u, v, w), \qquad \forall u, v, w \in V_{div}.$$

We shall need the following standard estimates which hold for all $u \in V_{div}$

$$\|\mathcal{B}(u, u)\|_{V'_{div}} \leq c \|\nabla u\|^{3/2} \|u\|^{1/2}, \qquad d = 3, \tag{2.1}$$

$$\|\mathcal{B}(u, u)\|_{V'_{div}} \leq c \|u\| \|\nabla u\|, \qquad d = 2. \tag{2.2}$$

The assumptions listed below are the same as in [17]. We report them for the reader's convenience.

(A1) $J \in W^{1,1}(\mathbb{R}^d)$, $\quad J(x) = J(-x)$, $\quad a \geq 0$ a.e. in Ω.

(A2) The function ν is locally Lipschitz on $\mathbb{R}$ and there exist $\nu_1, \nu_2 > 0$ such that

$$\nu_1 \leq \nu(s) \leq \nu_2, \qquad \forall s \in \mathbb{R}.$$

(A3) $F \in C^{2,1}_{loc}(\mathbb{R})$ and there exists $c_0 > 0$ such that

$$F''(s) + a(x) \geq c_0, \qquad \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega.$$

(A4) There exist $c_1 > \frac{1}{2} \|J\|_{L^1(\mathbb{R}^d)}$ and $c_2 \in \mathbb{R}$ such that

$$F(s) \geq c_1 s^2 - c_2, \qquad \forall s \in \mathbb{R}.$$

(A5) There exist $c_3 > 0$, $c_4 \geq 0$ and $p \in (1, 2]$ such that

$$|F'(s)|^p \leq c_3 |F(s)| + c_4, \qquad \forall s \in \mathbb{R}.$$

Remark 1. Since F is bounded from below, it is easy to see that (A5) implies that F has polynomial growth of order p', where $p' \in [2, \infty)$ is the conjugate index to p. Namely, there exist $c_5 > 0$ and $c_6 \geq 0$ such that

$$|F(s)| \leq c_5 |s|^{p'} + c_6, \qquad \forall s \in \mathbb{R}. \tag{2.3}$$

Observe that assumption (A5) is fulfilled by a potential of arbitrary polynomial growth. For example, (A3)–(A5) are satisfied for the case of the well-known double-well potential $F(s) = (s^2 - 1)^2$.

We also recall the notion of weak solution to system (1.1)-(1.4) with (1.6)-(1.7).

Definition 1. *Let $T > 0$, $h \in L^2(0, T; V'_{div})$, $u_0 \in G_{div}$, $\varphi_0 \in H$ with $F(\varphi_0) \in L^1(\Omega)$ be given. Then $[u, \varphi]$ is a weak solution to (1.1)-(1.4) on $(0, T)$ satisfying (1.6)-(1.7) if*

- *u, φ and μ satisfy*

$$u \in L^\infty(0, T; G_{div}) \cap L^2(0, T; V_{div}), \tag{2.4}$$
$$u_t \in L^{4/3}(0, T; V'_{div}), \qquad if \quad d = 3,$$
$$u_t \in L^{2-\gamma}(0, T; V'_{div}), \qquad \forall \gamma \in (0, 1), \quad if \quad d = 2,$$
$$\varphi \in L^\infty(0, T; H) \cap L^2(0, T; V),$$
$$\varphi_t \in L^{4/3}(0, T; V'), \quad if \quad d = 3, \tag{2.5}$$
$$\varphi_t \in L^{2-\delta}(0, T; V'), \quad \forall \delta \in (0, 1), \quad if \quad d = 2, \tag{2.6}$$
$$\mu \in L^2(0, T; V);$$

- *we have*
$$\mu = a\varphi - J * \varphi + F'(\varphi), \tag{2.7}$$

and for every $\psi \in V$, every $v \in V_{div}$ and for almost any $t \in (0, T)$

$$\langle \varphi_t, \psi \rangle + (\nabla \mu, \nabla \psi) = (u, \varphi \nabla \psi), \tag{2.8}$$
$$\langle u_t, v \rangle + (\nu(\varphi) 2Du, Dv) + b(u, u, v) = -(v, \varphi \nabla \mu) + \langle h, v \rangle; \tag{2.9}$$

- *the following initial conditions hold*

$$u(0) = u_0, \qquad \varphi(0) = \varphi_0. \tag{2.10}$$

Remark 2. As a consequence, the total concentration is conserved. Indeed, take $\psi = 1$ in (2.8) so that $\langle \varphi_t, 1 \rangle = 0$ and $(\varphi(t), 1) = (\varphi_0, 1)$ for all $t \in [0, T]$.

Remark 3. The initial conditions (2.10) are meant in the weak sense. Indeed we have $u \in C_w([0, T]; G_{div})$ and $\varphi \in C_w([0, T]; H)$.

Assumptions (A1)–(A5) are enough to establish the existence of a global weak solution [17]. However, to prove the results of this paper, we shall need to replace (A4) with the following stronger assumption (compare with [9, (A2)]).

(A6) $F \in C^2(\mathbb{R})$ and there exist $c_7 > 0$, $c_8 > 0$ and $q > 0$ such that

$$F''(s) + a(x) \geq c_7 |s|^{2q} - c_8, \qquad \forall s \in \mathbb{R}, \quad a.e. \ x \in \Omega.$$

6

Thanks to (A6) further regularity properties for φ, φ_t, u_t can be established and, in particular, the energy identity in two dimensions can be obtained. For this reason, in the case assumption (A6) holds, it is convenient to introduce the following

Definition 2. *Suppose (A6) holds and let $T > 0$, $h \in L^2(0, T; V'_{div})$, $u_0 \in G_{div}$, $\varphi_0 \in H$ with $F(\varphi_0) \in L^1(\Omega)$ be given. A couple $[u, \varphi]$ is a weak solution to (1.1)-(1.4) on $(0, T)$ satisfying (1.6)-(1.7) if $[u, \varphi]$ is a weak solution in the sense of Definition 1 satisfying the further regularity property*

$$\varphi \in L^\infty(0, T; L^{2+2q}(\Omega)). \tag{2.11}$$

Summing up, the main results of [17] are contained in the following

Theorem 1. *Let $h \in L^2_{loc}([0, \infty); V'_{div})$, $u_0 \in G_{div}$, $\varphi_0 \in H$ such that $F(\varphi_0) \in L^1(\Omega)$ and suppose that (A1)-(A5) are satisfied. Then, for every given $T > 0$, there exists a weak solution $[u, \varphi]$ (in the sense of Definition 1) which satisfies the following energy inequality for almost all $t > 0$*

$$\mathcal{E}(u(t), \varphi(t)) + \int_0^t \Big(2\|\sqrt{\nu(\varphi)}Du\|^2 + \|\nabla\mu\|^2 \Big) d\tau \leq \mathcal{E}(u_0, \varphi_0) + \int_0^t \langle h(\tau), u \rangle d\tau, \tag{2.12}$$

where we have set

$$\mathcal{E}(u(t), \varphi(t)) = \frac{1}{2}\|u(t)\|^2 + \frac{1}{4}\int_\Omega \int_\Omega J(x-y)(\varphi(x, t) - \varphi(y, t))^2 dx dy + \int_\Omega F(\varphi(t)).$$

If (A6) holds in place of (A4) then we also have

- *there exists a weak solution $[u, \varphi]$ (in the sense of Definition 2) such that*

$$\varphi_t \in L^2(0, T; V'), \quad if \quad d = 2 \quad or \quad d = 3 \text{ and } q \geq 1/2, \tag{2.13}$$

$$u_t \in L^2(0, T; V'_{div}), \quad if \quad d = 2, \tag{2.14}$$

which still satisfies the energy inequality (2.12) for almost all $t > 0$;

- *if $d = 2$ then any weak solution (in the sense of Definition 2) is such that*

$$u \in C([0, \infty); G_{div}), \qquad \varphi \in C([0, \infty); H),$$

and

$$\frac{d}{dt}\mathcal{E}(u, \varphi) + 2\|\sqrt{\nu(\varphi)}Du\|^2 + \|\nabla\mu\|^2 = \langle h(t), u \rangle, \tag{2.15}$$

i.e., (2.12) with the equal sign holds for every $t \geq 0$; in addition, if

$$\|h\|_{L^2_{tb}(0, \infty; V'_{div})} := \Big(\sup_{t \geq 0} \int_t^{t+1} \|h(\tau)\|^2_{V'_{div}} d\tau \Big)^{1/2} < \infty$$

7

then the following dissipative estimate is satisfied

$$\mathcal{E}(u(t), \varphi(t)) \leq \mathcal{E}(u_0, \varphi_0)e^{-kt} + F(m_0)|\Omega| + K, \qquad \forall t \geq 0, \qquad (2.16)$$

where $m_0 = (\varphi_0, 1)$ and k, K are two positive constants which are independent of the initial data, with K depending on Ω, ν_1, J, F, $\|h\|_{L^2_{tb}(0,\infty;V'_{div})}$.

Remark 4. If $u \in C_w([0,T]; G_{div})$ and $\varphi \in C_w([0,T]; H)$ are the weakly continuous representatives of the global weak solution $z = [u, \varphi]$ given by Theorem 1, then the energy inequality (2.12) holds also for *all $t \geq 0$* (see Lemma 2 below).

We conclude by observing that it is straightforward to deduce from Theorem 1 the following result for the convective nonlocal Cahn-Hilliard equation with a given velocity field.

Corollary 1. *Let $u \in L^2_{loc}([0,\infty); V_{div} \cap L^\infty(\Omega)^d)$ be given and let $\varphi_0 \in H$ be such that $F(\varphi_0) \in L^1(\Omega)$. Suppose that (A1), (A3), (A5) and (A6) (with $q \geq \frac{1}{2}$ if $d = 3$) are satisfied. Then, for every $T > 0$, there exists a weak solution $\varphi \in L^2(0, T; V) \cap H^1(0, T; V')$ to (2.7)-(2.8) on $[0, T]$ such that $\varphi(0) = \varphi_0$ and $(\varphi(t), 1) = (\varphi_0, 1)$ for all $t \in [0, T]$. In addition, the following energy identity holds for all $t \geq 0$*

$$\frac{d}{dt}\left(\frac{1}{4}\int_\Omega \int_\Omega J(x-y)(\varphi(x,t) - \varphi(y,t))^2 dx dy + \int_\Omega F(\varphi(t))\right) + \|\nabla\mu\|^2 = (u\varphi, \nabla\mu).$$
$$(2.17)$$

3 Global attractor in 2D

We first report for the reader's convenience some basic definitions and results from the theory of generalized semiflows (see [7]).

Let $\mathcal{X}$ be a metric space (not necessarily complete) with metric $\mathbf{d}$. For any $A, B \subset \mathcal{X}$ the Hausdorff semidistance between A and B is $\text{dist}(A, B) := \sup_{a \in A} \inf_{b \in B} \mathbf{d}(a, b)$.

Definition 3. *A generalized semiflow $\mathcal{G}$ on $\mathcal{X}$ is a family of maps $z : [0, \infty) \to \mathcal{X}$ satisfying the following hypothesis*

(H1) *(Existence) For each $z_0 \in \mathcal{X}$ there exists at least one $z \in \mathcal{G}$ with $z(0) = z_0$.*

(H2) *(Translates of solutions are solutions) If $z \in \mathcal{G}$ and $\tau \geq 0$, then $z^\tau \in \mathcal{G}$, where $z^\tau(t) := z(t + \tau)$, for every $t \geq 0$.*

(H3) *(Concatenation) If $z_1, z_2 \in \mathcal{G}$ and $\tau \geq 0$, with $z_1(\tau) = z_2(0)$, setting*

$$z(t) := \begin{cases} z_1(t) & if \quad 0 \leq t \leq \tau, \\ z_2(t) & if \quad t > \tau, \end{cases}$$

then $z \in \mathcal{G}$.

(H4) *(Upper semicontinuity with respect to initial data) If $z_j \in \mathcal{G}$ with $z_j(0) \to z_0$, then there exist a subsequence $\{z_{j_k}\}$ of $\{z_j\}$ and $z \in \mathcal{G}$ with $z(0) = z_0$ such that $z_{j_k}(t) \to z(t)$ for each $t \geq 0$.*

If $\mathcal{G}$ is a generalized semiflow and $E \subset \mathcal{X}$, we define for every $t \geq 0$

$$T(t)E = \{z(t) : z \in \mathcal{G} \text{ with } z(0) \in E\}.$$

The *positive orbit* of $z \in \mathcal{G}$ is the set $\gamma^+(z) = \{z(t) : t \geq 0\}$. If $E \subset \mathcal{X}$, then the *positive orbit* of E is the set $\gamma^+(E) = \cup_{t \geq 0} T(t)E$. For $\tau \geq 0$ we also set

$$\gamma^\tau(E) = \bigcup_{t \geq \tau} T(t)E = \gamma^+(T(\tau)E).$$

The $\omega-limit$ of $z \in \mathcal{G}$ is the set

$$\omega(z) := \{w \in \mathcal{X} : z(t_j) \to w \text{ for some sequence } t_j \to \infty\}.$$

If $E \subset \mathcal{X}$ the $\omega-limit$ of E is the set

$$\omega(E) := \{w \in \mathcal{X} : \exists z_j \in \mathcal{G}, z_j(0) \in E, z_j(0) \text{ bounded, and } \exists t_j \to \infty \text{ s.t. } z_j(t_j) \to w\}.$$

The subset $\mathcal{A}$ is a *global attractor* for the generalized semiflow $\mathcal{G}$ if $\mathcal{A}$ is compact, invariant, i.e. $T(t)\mathcal{A} = \mathcal{A}$ for all $t \geq 0$, and attracts all bounded subsets of $\mathcal{X}$, i.e. $\text{dist}(T(t)B, \mathcal{A}) \to 0$ as $t \to \infty$, for every bounded set $B \subset \mathcal{X}$.

The generalized semiflow $\mathcal{G}$ is *eventually bounded* if, given any bounded set $B \subset \mathcal{G}$, there exists $\tau \geq 0$ such that $\gamma^\tau(B)$ is bounded.

$\mathcal{G}$ is *point dissipative* if there is a bounded set $\mathcal{B}_0$ such that for any $z \in \mathcal{G}$ there exists $t_0 = t_0(z) \geq 0$ such that $z(t) \in \mathcal{B}_0$ for all $t \geq t_0$.

$\mathcal{G}$ is *asymptotically compact* if for any sequence $z_j \in \mathcal{G}$ with $z_j(0)$ bounded, and any sequence $t_j \to \infty$, the sequence $z_j(t_j)$ is precompact.

$\mathcal{G}$ is *compact* if for any sequence $z_j \in \mathcal{G}$ with $z_j(0)$ bounded there exists a subsequence z_{j_k} such that $z_{j_k}(t)$ converges for every $t > 0$.

Proposition 1. *Let $\mathcal{G}$ be asymptotically compact. Then $\mathcal{G}$ is eventually bounded.*

9

Proposition 2. *Let $\mathcal{G}$ be eventually bounded and compact. Then $\mathcal{G}$ is asymptotically compact.*

Theorem 2. *A generalized semiflow $\mathcal{G}$ has a global attractor if and only if $\mathcal{G}$ is point dissipative and asymptotically compact. The global attractor $\mathcal{A}$ is unique and given by*

$$\mathcal{A} = \bigcup \{\omega(B) : B \text{ is a bounded subset of } \mathcal{X}\} = \omega(\mathcal{X}).$$

Furthermore $\mathcal{A}$ is the maximal compact invariant subset of $\mathcal{X}$.

We now turn to our system (1.1)-(1.4) endowed with (1.6) in the case $d = 2$. Also, we suppose that h is time independent, i.e.,

$$h \in V'_{div}. \tag{3.1}$$

We first have to choose a suitable metric space where the weak solutions can be defined in order to construct the associated generalized semiflow.

We therefore fix $m \geq 0$ and introduce the metric space

$$\mathcal{X}_m := G_{div} \times \mathcal{Y}_m,$$

where

$$\mathcal{Y}_m := \{\varphi \in H \ : \ F(\varphi) \in L^1(\Omega), \ |(\varphi, 1)| \leq m\}, \tag{3.2}$$

endowed with the metric

$$\mathbf{d}(z_1, z_2) = \|u_1 - u_2\| + \|\varphi_1 - \varphi_2\| + \left| \int_\Omega F(\varphi_1) - \int_\Omega F(\varphi_2) \right|^{1/2},$$

for every $z_1 := [u_1, \varphi_1]$ and $z_2 := [u_2, \varphi_2]$ in $\mathcal{X}_m$.

On account of Theorem 1, let us now denote by $\mathcal{G}$ the set of all weak solutions in the sense of Definition 2 (we shall assume (A6)) corresponding to all initial data $z_0 := [u_0, \varphi_0] \in \mathcal{X}_m$. Our aim is to prove that $\mathcal{G}$ is a generalized semiflow on $\mathcal{X}_m$.

Proposition 3. *Let $d = 2$. Suppose that (A1)-(A3), (A5), (A6) and (3.1) hold. Then $\mathcal{G}$ is a generalized semiflow on $\mathcal{X}_m$.*

Proof. It is immediate to see that $\mathcal{G}$ satisfies (H1)-(H3) of Definition 3. The only property which is not trivial to prove is (H4). We therefore consider a sequence $\{z_j\}$, with $z_j := [u_j, \varphi_j]$, of weak solutions (cf. Definition 2) such that $z_j(0) := [u_{j0}, \varphi_{j0}] \to z_0 := [u_0, \varphi_0]$ in $\mathcal{X}_m$. Since every weak solution satisfies the energy identity, for each $j \in \mathbb{N}$ and for every $t \geq 0$ we can write

$$\mathcal{E}(z_j(t)) + \int_0^t \left(2\|\sqrt{\nu(\varphi_j)}Du_j\|^2 + \|\nabla\mu_j\|^2 \right)d\tau = \mathcal{E}(z_{j0}) + \int_0^t \langle h, u_j\rangle d\tau, \tag{3.3}$$

10

where $z_{j0} := z_j(0)$. From this identity, by recalling the definition of the energy functional $\mathcal{E}$ and using (A1)-(A3), (A5)-(A6) we deduce that $\{u_j\}$ is bounded in $L^\infty(0,T;G_{div}) \cap L^2(0,T;V_{div})$ for every $T > 0$, $\{\varphi_j\}$ is bounded in $L^\infty(0,T;H) \cap L^2(0,T;V)$ for every $T > 0$ and $\{\mu_j\}$ is bounded in $L^2(0,T;V)$ for every $T > 0$ (cf. [17] for details). From equations (2.8) and (2.9), written for each weak solution $[u_j, \varphi_j]$, and arguing as in [17] we also show that $\{u_j'\}$ is bounded in $L^2(0,T;V_{div}')$ for every $T > 0$ and that $\{\varphi_j'\}$ is bounded in $L^2(0,T;V')$ for every $T > 0$. Therefore, we deduce that there exist $u \in L^\infty(0,T;G_{div}) \cap L^2(0,T;V_{div})$ for every $T > 0$, $\varphi \in L^\infty(0,T;H) \cap L^2(0,T;V)$ for every $T > 0$ and $\mu \in L^2(0,T;V)$ for every $T > 0$ such that, for a subsequence that we do not relabel, we have

$$u_j \rightharpoonup u \quad \text{weakly}^* \text{ in } L^\infty(0,T;G_{div}) \text{ and weakly in } L^2(0,T;V_{div}), \tag{3.4}$$

$$u_j' \rightharpoonup u' \quad \text{weakly in } L^2(0,T;V_{div}'), \tag{3.5}$$

$$u_j \to u \quad \text{strongly in } L^2(0,T;G_{div}), \tag{3.6}$$

$$\varphi_j \rightharpoonup \varphi \quad \text{weakly}^* \text{ in } L^\infty(0,T;H) \text{ and weakly in } L^2(0,T;V), \tag{3.7}$$

$$\varphi_j' \rightharpoonup \varphi' \quad \text{weakly in } L^2(0,T;V'), \tag{3.8}$$

$$\varphi_j \to \varphi \quad \text{strongly in } L^2(0,T;H), \tag{3.9}$$

$$\mu_j \rightharpoonup \mu \quad \text{weakly in } L^2(0,T;V). \tag{3.10}$$

From (3.5) and (3.8) we obtain

$$u_j(t) \rightharpoonup u(t) \quad \text{weakly in } G_{div}, \quad \forall t \geq 0, \tag{3.11}$$

$$\varphi_j(t) \rightharpoonup \varphi(t) \quad \text{weakly in } H, \quad \forall t \geq 0. \tag{3.12}$$

Indeed, for every $v \in V_{div}$ and every $t \geq 0$, we have

$$\int_0^T \langle u_j'(\tau) - u'(\tau), v \rangle \chi_{[0,t]}(\tau) d\tau = (u_j(t) - u(t), v) - (u_{j0} - u_0, v) \to 0$$

as $j \to \infty$. Hence $(u_j(t), v) \to (u(t), v)$, for every $v \in V_{div}$ and every $t \geq 0$ so that (3.11) follows from the density of V_{div} in G_{div} and from the boundedness of the sequence of u_j in $L^\infty(0,T;H)$ for every $T > 0$. By the same argument we get (3.12). By means of the convergences above and of the fact that each z_j is a weak solution, by passing to the limit in the variational formulation for $z_j = [u_j, \varphi_j]$ we infer that $z = [u, \varphi]$ is a weak solution as well. Furthermore, from (3.11) and (3.12) we get $z(0) = z_0$. We are now left to prove the convergence in $\mathcal{X}_m$ for each time $t \geq 0$. In order to do that, let us represent the potential F in the following form

$$F(s) = G(x,s) - \left(a(x) - \frac{c_0}{2}\right)\frac{s^2}{2}, \tag{3.13}$$

where, due to (A3), function $G(x, \cdot)$ is strictly convex in $\mathbb{R}$ for almost every $x \in \Omega$. By means of (3.13) the energy $\mathcal{E}$ can be rewritten in the form

$$\mathcal{E}(z) = \frac{1}{2}\|u\|^2 + \frac{c_0}{4}\|\varphi\|^2 - \frac{1}{2}(\varphi, J * \varphi) + \int_\Omega G(x, \varphi(x))dx, \qquad (3.14)$$

for every $z = [u, \varphi] \in \mathcal{X}_m$. As a consequence of the weak convergences (3.11) and (3.12) we see that we have

$$\liminf_{j\to\infty} \mathcal{E}(z_j(t)) \geq \mathcal{E}(z(t)), \quad \forall t \geq 0. \qquad (3.15)$$

Indeed (3.15) follows from the weak lower semicontinuity in H of the L^2-norm and of the convex integral functional in G, and from the compactness of the convolution operator $J * \cdot : H \to H$ (cf. (A1)). Recall that if we consider the functional $\mathcal{L} : H \to \mathbb{R} \cup \{+\infty\}$, where

$$\mathcal{L}(\varphi) := \int_\Omega G(x, \varphi(x))dx,$$

for every $\varphi \in H$ such that $G(\cdot, \varphi(\cdot)) \in L^1(\Omega)$ ($\mathcal{L}(\varphi) = +\infty$ otherwise), due to the convexity of $G(x, \cdot)$ for a.e. $x \in \Omega$ and to the lower bound $G(x, s) \geq -\alpha s^2 - \beta$, for every $s \in \mathbb{R}$ and for some $\alpha, \beta \geq 0$, then $\mathcal{L}$ is weakly lower-semicontinuous in H.

Since each weak solution satisfy the energy equation (3.3), we have

$$\limsup_{j\to\infty} \mathcal{E}(z_j(t)) = \mathcal{E}(z_0) - \liminf_{j\to\infty} \int_0^t \left(2\|\sqrt{\nu(\varphi_j)}Du_j\|^2 + \|\nabla\mu_j\|^2\right)d\tau + \int_0^t \langle h, u\rangle d\tau$$

$$\leq \mathcal{E}(z(0)) - \int_0^t \left(2\|\sqrt{\nu(\varphi)}Du\|^2 + \|\nabla\mu\|^2\right)d\tau + \int_0^t \langle h, u\rangle d\tau$$

$$= \mathcal{E}(z(t)), \qquad \forall t \geq 0, \qquad (3.16)$$

due to (3.4), (3.10) and on account of the fact that, since $z_j(0) \to z_0 = z(0)$ in $\mathcal{X}_m$, then $u_j(0) \to u_0 = u(0)$ in G_{div}, $\varphi_j(0) \to \varphi_0 = \varphi(0)$ in H and $\int_\Omega F(\varphi_j(0)) \to \int_\Omega F(\varphi_0) = \int_\Omega F(\varphi(0))$. We have also used the fact that

$$\sqrt{\nu(\varphi_j)}Du_j \rightharpoonup \sqrt{\nu(\varphi)}Du \qquad \text{weakly in } L^2(H). \qquad (3.17)$$

This convergence easily follows from the uniform bound $\|\sqrt{\nu(\varphi_j)}\|_\infty \leq \sqrt{\nu_2}$, the strong convergence $\sqrt{\nu(\varphi_j)} \to \sqrt{\nu(\varphi)}$ in $L^2(H)$ and the weak convergence (3.4) (see [17] for details). Therefore $\mathcal{E}(z_j(0)) \to \mathcal{E}(z(0))$ as $j \to \infty$. Hence, from (3.15) and (3.16) we get

$$\mathcal{E}(z_j(t)) \to \mathcal{E}(z(t)) \quad \text{as } j \to \infty, \quad \forall t \geq 0. \qquad (3.18)$$

We know that $(\varphi_j(t), J * \varphi_j(t)) \to (\varphi(t), J * \varphi(t))$ for every $t \geq 0$. Then (3.18) yield

$$\frac{1}{2}\|u_j(t)\|^2 + \frac{c_0}{4}\|\varphi_j(t)\|^2 + \int_\Omega G(x, \varphi_j(x, t))dx \to \frac{1}{2}\|u(t)\|^2$$

12

$$+ \frac{c_0}{4}\|\varphi(t)\|^2 + \int_\Omega G(x, \varphi(x, t))dx, \quad \text{as } j \to \infty, \quad \forall t \geq 0.$$

Therefore, for all $t \geq 0$, we have (cf. also (3.11)- (3.12))

$$\frac{1}{2}\|u_j(t) - u(t)\|^2 + \frac{c_0}{4}\|\varphi_j(t) - \varphi(t)\|^2 + \int_\Omega G(x, \varphi_j(x, t))dx - \int_\Omega G(x, \varphi(x, t))dx$$

$$= \frac{1}{2}\|u_j(t)\|^2 + \frac{c_0}{4}\|\varphi_j(t)\|^2 + \int_\Omega G(x, \varphi_j(x, t))dx - (u_j(t), u(t))$$

$$- \frac{c_0}{2}(\varphi_j(t), \varphi(t)) + \frac{1}{2}\|u(t)\|^2 + \frac{c_0}{4}\|\varphi(t)\|^2 - \int_\Omega G(x, \varphi(x, t))dx \to 0, \tag{3.19}$$

as $j \to \infty$. Therefore we infer that

$$\limsup_{j \to \infty} \int_\Omega G(x, \varphi_j(x, t))dx \leq \int_\Omega G(x, \varphi(x, t))dx, \quad \forall t \geq 0,$$

and, due to the $H-$weak lower semicontinuity of the integral functional $\mathcal{L}$, we obtain

$$\int_\Omega G(x, \varphi_j(x, t))dx \to \int_\Omega G(x, \varphi(x, t))dx, \quad \forall t \geq 0. \tag{3.20}$$

From (3.19) and (3.20) we finally get

$$u_j(t) \to u(t) \quad \text{strongly in } G_{div}, \quad \forall t \geq 0,$$

$$\varphi_j(t) \to \varphi(t) \quad \text{strongly in } H, \quad \forall t \geq 0,$$

and, on account of (3.13) and (3.20), we also have

$$\int_\Omega F(\varphi_j(t)) \to \int_\Omega F(\varphi(t)), \quad \forall t \geq 0.$$

Hence $z_j(t) \to z(t)$ in $\mathcal{X}_m$, for every $t \geq 0$. We thus conclude that (H4) holds. $\quad\square$

As a consequence of (2.16) we have the following

Proposition 4. *Let the hypotheses of Proposition 3 hold. Then $\mathcal{G}$ is point dissipative and eventually bounded.*

Proof. Due to (A6) there exists $\gamma = \gamma(c_7, c_8, J, |\Omega|) \geq 0$ such that $\mathcal{E}(z) \geq -\gamma$ for every $z \in \mathcal{X}_m$. Therefore, setting $\overline{\mathcal{E}}(z) := \mathcal{E}(z) + \gamma \geq 0$, from (2.16) we deduce

$$\frac{1}{2}\|u(t)\|^2 + \frac{1}{2}\|\sqrt{a}\varphi(t)\|^2 - \frac{1}{2}(J * \varphi(t), \varphi(t)) + \int_\Omega F(\varphi(t))$$

$$\leq \overline{\mathcal{E}}(z_0)e^{-kt} + L_m, \quad \forall t \geq 0, \tag{3.21}$$

where $z_0 := [u_0, \varphi_0]$ and $L_m = F(m)|\Omega| + K$. Now, by using (A6) again we have

$$\frac{1}{2}\|\sqrt{a}\varphi\|^2 - \frac{1}{2}(J * \varphi, \varphi) + \int_\Omega F(\varphi) \geq c_9\|\varphi\|^2 - \gamma,$$

and therefore

$$\frac{1}{2}\|u(t)\|^2 + c_9\|\varphi(t)\|^2 \le \overline{\mathcal{E}}(z_0)e^{-kt} + L_m + \gamma. \tag{3.22}$$

From (3.21) we infer

$$\frac{1}{2}\|u(t)\|^2 + \int_\Omega F(\varphi(t)) \le \overline{\mathcal{E}}(z_0)e^{-kt} + L_m + c_{10}\|\varphi(t)\|^2 \le c_{11}\overline{\mathcal{E}}(z_0)e^{-kt} + c_{11}L_m + c_{12}, \tag{3.23}$$

where $c_{10} = \frac{1}{2}\|J\|_{L^1}$, $c_{11} = 1 + c_{10}/c_9$ and $c_{12} = \gamma c_{10}/c_9$. Therefore, (3.22) and (3.23) entail

$$\|u(t)\|^2 + \|\varphi(t)\|^2 + \left|\int_\Omega F(\varphi(t)) - \int_\Omega F(0)\right| \le c_{13}\overline{\mathcal{E}}(z_0)e^{-kt} + c_{13}|L_m| + c_{14}, \tag{3.24}$$

for all $t \ge 0$. Here the expressions of the positive constants c_{13} and c_{14} in terms of the previous constants are omitted for the sake of simplicity. Setting $z(t) := [u(t), \varphi(t)]$, (3.24) can be rewritten as follows

$$\mathbf{d}^2(z(t), 0) \le c_{13}\overline{\mathcal{E}}(z_0)e^{-kt} + c_{13}|L_m| + c_{14}, \qquad \forall t \ge 0. \tag{3.25}$$

Choosing therefore R_0 such that $R_0^2 > c_{13}|L_m| + c_{14}$, from (3.25) we deduce that

$$\mathbf{d}(z(t), 0) \le R_0,$$

for every $t \ge t_0(z_0)$, where

$$t_0 = \frac{1}{k}\log\frac{c_{13}\overline{\mathcal{E}}(z_0)}{R_0^2 - (c_{13}|L_m| + c_{14})},$$

which means that $\mathcal{G}$ is point dissipative. By using a similar argument, (3.24) implies that $\mathcal{G}$ is also eventually bounded. $\square$

We can now prove our main result.

Theorem 3. *Let the hypotheses of Proposition 3 hold. Then $\mathcal{G}$ possesses a global attractor.*

Proof. By Proposition 4 we know that the generalized semiflow $\mathcal{G}$ is point dissipative. Since, again by Proposition 4, $\mathcal{G}$ is also eventually bounded, according with Theorem 2, we only need to show that $\mathcal{G}$ is compact (see also Proposition 2). Let us first observe that the compact embedding $V \hookrightarrow\hookrightarrow L^{p'}(\Omega)$ and the Aubin-Lions lemma imply

$$L^2(0, T; V) \cap H^1(0, T, V') \hookrightarrow\hookrightarrow L^2(0, T; L^{p'}(\Omega)). \tag{3.26}$$

Therefore, from (3.7) and (3.8) we deduce that, for a subsequence that we do not relabel, we have

$$\varphi_j \to \varphi, \qquad \text{strongly in } L^2(0, T; L^{p'}(\Omega)),$$

14

and hence, for a further subsequence, $\varphi_j(t) \to \varphi(t)$ strongly in $L^{p'}(\Omega)$ for a.e. $t \in (0, T)$. Since F has polynomial growth of order p' (cf. Remark 1), then by Lebesgue's theorem we deduce

$$\int_\Omega F(\varphi_j(t)) \to \int_\Omega F(\varphi(t)), \qquad \text{a.e. } t > 0. \tag{3.27}$$

Hence, the strong convergences (3.6), (3.9), which imply that for a subsequence we have

$$u_j(t) \to u(t) \quad \text{strongly in } G_{div}, \quad \text{a.e. } t > 0, \tag{3.28}$$

$$\varphi_j(t) \to \varphi(t) \quad \text{strongly in } H, \quad \text{a.e. } t > 0, \tag{3.29}$$

and (3.27) allow to deduce that $\mathcal{E}(z_j(t)) \to \mathcal{E}(z(t))$ for almost all $t > 0$. Now, setting

$$\widetilde{\mathcal{E}}(z(t)) := \mathcal{E}(z(t)) - \int_0^t \langle h, u(\tau) \rangle d\tau,$$

we still have $\widetilde{\mathcal{E}}(z_j(t)) \to \widetilde{\mathcal{E}}(z(t))$ for almost all $t > 0$. Since for each j the function $\widetilde{\mathcal{E}}(z_j(\cdot))$ is decreasing on $[0, \infty)$ and $\widetilde{\mathcal{E}}(z(\cdot))$ is continuous on $[0, \infty)$, then $\widetilde{\mathcal{E}}(z_j(t)) \to \widetilde{\mathcal{E}}(z(t))$ for *all* $t > 0$. Hence

$$\mathcal{E}(z_j(t)) \to \mathcal{E}(z(t)), \qquad \forall t > 0. \tag{3.30}$$

Now, by means of the same argument used to deduce (H4), from (3.30) we infer that $z_j(t) \to z(t)$ in $\mathcal{X}_m$, for all $t > 0$. Thus $\mathcal{G}$ is compact. $\qquad\square$

Remark 5. In the nonautonomous case (say, h depending on time) it would be interesting to establish the existence of a pullback attractor along the lines of [42] (see also its references), where uniqueness also fails but energy identity holds.

4 The convective nonlocal Cahn-Hilliard equation

Here we show that the existence of the global attractor for (1.1)-(1.2), assuming that $u \in L^\infty(\Omega)^d$ is given and independent of time for $d = 2, 3$, can be proven arguing as in the previous section.

First, recalling Corollary 1, we prove a uniqueness result.

Proposition 5. *Let $u \in L^2(0, T; L^\infty(\Omega)^d \cap V_{div})$ be given and let $\varphi_0 \in H$ be such that $F(\varphi_0) \in L^1(\Omega)$. Suppose also that (A1), (A3), (A5) and (A6) (with $q \geq \frac{1}{2}$ if $d = 3$) are satisfied. Then, there exists a unique weak solution $\varphi \in L^2(0, T; V) \cap H^1(0, T; V')$ to (2.7)-(2.8) on $(0, T)$ such that $\varphi(0) = \varphi_0$.*

Proof. Suppose that φ_i, $i = 1, 2$, are two weak solutions and set $\varphi = \varphi_1 - \varphi_2$. Then we have

$$\langle \varphi_t, \psi \rangle + (\nabla \mu, \nabla \psi) = (u, \varphi \nabla \psi), \qquad \forall \psi \in V, \tag{4.1}$$

15

where (cf. 2.7)

$$\mu = a\varphi - J * \varphi + F'(\varphi_1) - F'(\varphi_2).$$

Note that $(\varphi, 1) = 0$. Then consider the operator $B_N := -\Delta$ with domain

$$D(B_N) = \left\{ \phi \in H^2(\Omega) \ : \ \frac{\partial \phi}{\partial n} = 0 \text{ on } \partial\Omega \right\}.$$

and take $\psi = B_N^{-1}\varphi(t) \in D(B_N)$ as test function in (4.1). Thus we obtain

$$\frac{d}{dt}\|B_N^{-1/2}\varphi\|^2 + 2(a\varphi - J * \varphi + F'(\varphi_1) - F'(\varphi_2), \varphi) = (u, \varphi\nabla B_N^{-1}\varphi)$$

and, thanks to (A3), we get

$$\frac{d}{dt}\|B_N^{-1/2}\varphi\|^2 + 2c_0\|\varphi\|^2 \leq 2|(J * \varphi, \varphi)| + C_1\|u\|_{L^\infty(\Omega)^d}\|\varphi\|\|B_N^{-1/2}\varphi\|. \tag{4.2}$$

On the other hand, recalling (A1) and using Young's inequality, we have

$$|(J * \varphi, \varphi)| \leq \|B_N^{1/2}(J * \varphi)\|\|B_N^{-1/2}\varphi\| \leq \frac{c_0}{4}\|\varphi\|^2 + C_2\|B_N^{-1/2}\varphi\|^2, \tag{4.3}$$

where $C_2 > 0$ depends on c_0 and on J. Then, combining (4.2) with (4.3) and using once more the Young inequality, the standard Gronwall lemma entails that $\varphi \equiv 0$. □

A consequence of Corollary 1 and Proposition 5 is that we can define a semiflow $S(t)$ on $\mathcal{Y}_m$ (cf. (3.2)) endowed the metric

$$\bar{\mathbf{d}}(\varphi_1, \varphi_2) = \|\varphi_1 - \varphi_2\| + \left| \int_\Omega F(\varphi_1) - \int_\Omega F(\varphi_2) \right|^{1/2}, \quad \forall \varphi_1, \varphi_2 \in \mathcal{Y}_m,$$

where $m \geq 0$ is given.

We can now prove

Theorem 4. *Suppose that (A1), (A3), (A5) and (A6) (with $q \geq \frac{1}{2}$ if $d = 3$) are satisfied and assume $u \in L^\infty(\Omega)^d$ is given and independent of time. In addition, if $d = 3$, suppose that $p' \in [2, 6)$ in (2.3). Then the dynamical system $(\mathcal{Y}_m, S(t))$ possesses a connected global attractor.*

Proof. Observe that the energy identity (2.17) entails

$$\mathcal{E}(\varphi(t)) + \int_0^t \|\nabla\mu\|^2 d\tau = \mathcal{E}(\varphi_0) + \int_0^t (u\varphi, \nabla\mu)d\tau,$$

from which we have

$$\mathcal{E}(\varphi(t)) + \frac{1}{2}\int_0^t \|\nabla\mu\|^2 d\tau \leq \mathcal{E}(\varphi_0) + \frac{1}{2}u_*^2\int_0^t \|\varphi\|^2 d\tau,$$

16

where the energy functional $\mathcal{E}$ is now given by

$$\mathcal{E}(\varphi) := \frac{c_0}{2}\|\varphi\|^2 - \frac{1}{2}(\varphi, J*\varphi) + \int_\Omega G(x, \varphi(x))dx,$$

and $u_* := \|u\|_{L^\infty(\Omega)^d}$. Therefore, the argument used in the previous section can be adapted to this case. Indeed, in order to prove the compactness of the semiflow $S(t)$ we note that, if $d = 3$ and $p' \in (2, 6]$, the compact injection (3.26) is valid and still implies (3.27). Hence, by using the strong convergence $\varphi_j(t) \to \varphi(t)$ in H for a.e $t > 0$ we have $\mathcal{E}(\varphi_j(t)) \to \mathcal{E}(\varphi(t))$ for a.e. $t > 0$. Setting now

$$\widetilde{\mathcal{E}}(\varphi(t)) := \mathcal{E}(\varphi(t)) - \int_0^t (u\varphi, \nabla\mu)d\tau,$$

the strong convergence to φ in $L^2(H)$ for the sequence $\{\varphi_j\}$ and the weak convergence to μ in $L^2(V)$ for the sequence $\{\mu_j\}$ imply that $\widetilde{\mathcal{E}}(\varphi_j(t)) \to \widetilde{\mathcal{E}}(\varphi(t))$ for a.e $t > 0$. Since, due to (2.17) the function $\widetilde{\mathcal{E}}(\varphi_j(\cdot))$ is decreasing on $[0, \infty)$ and $\widetilde{\mathcal{E}}(\varphi(\cdot))$ in continuous on $[0, \infty)$, then $\widetilde{\mathcal{E}}(\varphi_j(t)) \to \widetilde{\mathcal{E}}(\varphi(t))$ for *all* $t > 0$. Hence $\mathcal{E}(\varphi_j(t)) \to \mathcal{E}(\varphi(t))$ for all $t > 0$ and arguing as in the previous section we get $\varphi_j(t) \to \varphi(t)$ in $\mathcal{Y}_m$ for all $t > 0$. Therefore, the semiflow $S(t)$ is compact. In addition, the uniqueness of solution trivially implies the Kneser property (see, e.g., [8, 38]) so that the global attractor is also connected. $\qquad\square$

Remark 6. The connectedness of the global attractor for the full system remains an open issue.

5 A dissipative estimate in 3D

In dimension three, of course we are not able to prove an energy identity like (2.15). Actually, one could argue in the spirit of [7], under the (unproven) assumption that the weak solution $z = [u, \varphi]$ is strongly continuous from $[0, \infty)$ to $\mathcal{X}_m$. However, we shall not consider this possibility, but we shall construct a (generalized) notion of attractor (see next section). Nevertheless, we are able to prove that a dissipative estimate like (2.16) can still be recovered in the three dimensional case. This the main aim of the present section. We observe that, since such dissipative estimate relies on the validity of the energy inequality (2.12) only, then it holds for any weak solution in the sense of Definition 2.

We need the following basic lemma, which is obtained by suitably modifying [7, Lemma 7.2].

Lemma 1. *Let $\theta \in L^1(0, T)$ for every $T > 0$ and suppose that*

$$\theta(t) + k\int_0^t \theta(\tau)\tau \leq \int_s^t f(\tau)d\tau + \theta(s) + k\int_0^s \theta(\tau)d\tau \qquad (5.1)$$

holds for a.e. t, $s \in (0, \infty)$, with $t \geq s$, where $f \in L^1(0, T)$ for every $T > 0$ and the constant $k \geq 0$ are given. Then we have

$$\theta(t) \leq \theta(s)e^{-k(t-s)} + \int_s^t e^{-k(t-\tau)}f(\tau)d\tau, \tag{5.2}$$

for a.e. t, $s \in (0, \infty)$, with $t \geq s$. Furthermore, suppose $\theta : [0, \infty) \to \mathbb{R}$ is a l.s.c. representative satisfing (5.1) for a.e. $t, s \in (0, \infty)$, with $t \geq s$. Then (5.1) and (5.2) also hold for every $t \geq s$ and for a.e. $s > 0$, and if, in addition, (5.1) holds for $s = 0$, then we have

$$\theta(t) \leq \theta(0)e^{-kt} + \int_0^t e^{-k(t-\tau)}f(\tau)d\tau, \tag{5.3}$$

for all $t \in [0, \infty)$. In particular, suppose $f(t) = l + g(t)$, where $l \in \mathbb{R}$ is a given constant and $g \in L^1_{tb}(0, \infty)$, i.e., g belongs to $L^1_{loc}([0, \infty))$ and is translation bounded, that is,

$$\|g\|_{L^1_{tb}} := \sup_{t \geq 0} \int_t^{t+1} |g(\tau)|d\tau < \infty.$$

Then we have

$$\theta(t) \leq \theta(0)e^{-kt} + \frac{l}{k} + \frac{\|g\|_{L^1_b}}{1 - e^{-k}}, \tag{5.4}$$

for all $t \in [0, \infty)$.

Proof. Setting

$$\rho(t) = \theta(t) + k\int_0^t \theta(\tau)d\tau - \int_0^t f(\tau)d\tau,$$

from (5.1) we have $\rho(t) \leq \rho(s)$ for a.e. t, $s \in (0, \infty)$, with $t \geq s$. We therefore deduce that

$$\dot{\rho} \leq 0 \quad \text{in } \mathcal{D}'(0, \infty) \tag{5.5}$$

Indeed, take $\varphi \in \mathcal{D}(0, \infty)$, $\varphi \geq 0$. We have

$$0 \leq \int_0^\infty \frac{\rho(t) - \rho(t+h)}{h}\varphi(t)dt = \int_0^\infty \rho(t)\frac{\varphi(t) - \varphi(t-h)}{h}dt,$$

for all $h > 0$. Letting $h \to 0$, from the previous relation and by means of Lebesgue's theorem we get (5.5). From (5.5) we now get $\dot{\theta} + k\theta \leq f$ in $\mathcal{D}'(0, \infty)$ and hence

$$\frac{d}{dt}\left(e^{kt}\left(\theta - \frac{l}{k}\right) - \int_0^t e^{k\tau}f(\tau)d\tau\right) \leq 0 \quad \text{in } \mathcal{D}'(0, \infty).$$

Setting

$$\omega = e^{kt}(\theta - \frac{l}{k}) - \int_0^t e^{k\tau}f(\tau)d\tau,$$

we therefore have

$$\dot{\omega} \leq 0 \quad \text{in } \mathcal{D}'(0, \infty), \tag{5.6}$$

18

from which we now show that

$$\omega(t) \leq \omega(s), \tag{5.7}$$

for a.e. t, $s \in (0, \infty)$, with $t \geq s$. Indeed, let $\{\chi_\epsilon\}_{\epsilon>0}$, $\chi_\epsilon \geq 0$, be a sequence of mollifiers belonging to $\mathcal{D}(\mathbb{R})$ and consider the convolution $\omega_\epsilon = \chi_\epsilon * \overline{\omega}$, where $\overline{\omega}$ is the trivial extention of ω to the whole real line. Since $\omega_\epsilon \in C^\infty(\mathbb{R})$, we have, for every $\varphi \in \mathcal{D}(0, \infty)$, $\varphi \geq 0$

$$
\begin{aligned}
\int_0^\infty \dot{\omega}_\epsilon \varphi &= -\int_0^\infty \omega_\epsilon \dot{\varphi} = -\int_0^\infty \dot{\varphi}(t)dt \int_{\mathbb{R}} \chi_\epsilon(t-\tau)\overline{\omega}(\tau)d\tau \\
&= -\int_{\mathbb{R}} \overline{\omega}(\tau)d\tau \int_{\mathbb{R}} \chi_\epsilon(t-\tau)\dot{\varphi}(t)dt = -\int_{\mathbb{R}} \overline{\omega}(\tau)(\chi_\epsilon * \dot{\varphi})(\tau)d\tau \\
&= -\int_0^\infty \omega(\tau)\frac{d}{d\tau}(\chi_\epsilon * \varphi)(\tau)d\tau \leq 0
\end{aligned}
\tag{5.8}
$$

for $\epsilon > 0$ small enough (i.e., such that $\chi_\epsilon * \varphi \in \mathcal{D}(0, \infty)$, that occurs when $\epsilon < \min(\mathrm{supp}\varphi)$), due to (5.6). Hence $\dot{\omega}_\epsilon(\tau) \leq 0$ for every $\tau \in (0, \infty)$, from which we deduce $\omega_\epsilon(t) \leq \omega_\epsilon(s)$, for every t, $s \in (0, \infty)$, with $t \geq s$. Letting $\epsilon \to 0$ and using the fact that $\omega_\epsilon \to \omega$ a.e. in $(0, \infty)$, we get (5.7). Thus, on account of the definition of ω, from (5.7) we deduce (5.2).

Suppose now that $\theta : [0, \infty) \to \mathbb{R}$ is a l.s.c. representative and that (5.1) holds for a.e. $t, s \in (0, \infty)$ with $t \geq s$. Let N_1 be a null set such that (5.1) holds for every t, $s \in (0, \infty) - N_1$, with $t \geq s$. Let $t \in [0, \infty)$, $s \in (0, \infty) - N_1$ and take a sequence $t_j \in (0, \infty) - N_1$ such that $t_j \to t$. Write (5.1) for s and t_j. By virtue of the lower semicontinuity of θ we see that (5.1) holds also for all $t \geq s$ and a.e $s \in (0, \infty)$. The same argument can be applied to (5.2). Suppose in addition that the l.s.c. representative θ satisfies (5.1) also for $s = 0$ and for all $t \in [0, \infty)$. Take a sequence $t_j \in (0, \infty) - N_1$ such that $t_j \to 0$ and write (5.1) for $s = 0$ and $t = t_j$. By virtue of the lower semicontinuity of θ we get $\theta(t_j) \to \theta(0)$. Now, let N_2 be a null set such that (5.2) holds for for every $s \in (0, \infty) - N_2$ and every $t \geq s$ and take a sequence $s_k \in (0, \infty) - N$, where $N = N_1 \cup N_2$, such that $s_k \to 0$. Write (5.2) for $s = s_k$ and for $t \in (0, \infty)$. Since $\theta(s_k) \to \theta(0)$, by letting $k \to \infty$ in (5.2) we get (5.3).

Finally, suppose that f has the form $f(t) = l + g(t)$, with $l \in \mathbb{R}$ a given constant and g translation bounded in $L^1_{loc}([0, \infty))$. By observing that (see, e.g., [14, Chap. XV, Cor. 1.7])

$$\int_0^t e^{-k(t-\tau)}g(\tau)d\tau \leq \frac{\|g\|_{L^1_{tb}}}{1-e^{-k}},$$

we immediately get (5.4). $\qquad\square$

Henceforth we shall denote by $u \in C_w([0, \infty); G_{div})$ and $\varphi \in C_w([0, \infty); H)$ weakly continuous representatives of u and φ, where $[u, \varphi] =: z$ is the weak solution corresponding to u_0 and φ_0 given by Theorem 1.

The following lemma, which will be used to prove the dissipative estimate in 3D, ensures the lower semicontinuity of the energy $\mathcal{E}(z(\cdot))$ from $[0,\infty)$ to $\mathbb{R}$.

Lemma 2. *Let $z := [u, \varphi]$ be the weak solution corresponding to u_0 and φ_0 and given by Theorem 1. Then, the function $\mathcal{E}(z(\cdot)) : [0,\infty) \to \mathbb{R}$ is lower semicontinuous.*

Proof. Let us represent the potential F as

$$F(s) = \widetilde{G}(x, s) - a(x)\frac{s^2}{2}, \quad \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega, \tag{5.9}$$

where $\widetilde{G}(x, \cdot)$ is strictly convex for a.e. $x \in \Omega$, owing to (A3). Then, the energy $\mathcal{E}(z(\cdot))$ takes the form

$$\mathcal{E}(z(t)) = \frac{1}{2}\|u(t)\|^2 - \frac{1}{2}(\varphi(t), J * \varphi(t)) + \int_\Omega \widetilde{G}(x, \varphi(x, t))dx.$$

Therefore, the lower semicontinuity of $\mathcal{E}(z(\cdot)) : [0,\infty) \to \mathbb{R}$ is a consequence of the weak lower semicontinuity in G_{div} of the L^2-norm, of the compactness of the convolution operator $J * \cdot : H \to H$ and of the convexity of the integral funcional in $\widetilde{G}$ given by the last term in the relation above. $\qquad\square$

In dimension three, we can prove that the same global weak solution constructed in Theorem 1 also satisfies energy inequality (2.12) between two arbitrary times s an t (i.e., for a.e. $s \geq 0$, including $s = 0$ and for all $t \geq s$), provided that a further growth assumption on F is fulfilled (not needed in dimension two). This is stated in the following

Lemma 3. *Assume (A1)-(A4) hold. In addition, suppose that (A5) holds with $p \in (6/5, 2]$ when $d = 3$. Let $z := [u, \varphi]$ be the weak solution (in the sense of Definition 1) corresponding to u_0 and φ_0 and given by Theorem 1. Then, the following energy inequality is satisfied*

$$\mathcal{E}(z(t)) + \int_s^t (2\|\sqrt{\nu(\varphi)}Du\|^2 + \|\nabla\mu\|^2)d\tau \leq \mathcal{E}(z(s)) + \int_s^t \langle h(\tau), u(\tau)\rangle d\tau, \tag{5.10}$$

for a.e. $s \geq 0$, including $s = 0$, and for every $t \geq s$, where $\mu = a\varphi - J * \varphi + F'(\varphi)$.

Proof. We can argue as in the proof of (2.12) (see [17, Theorem 1]) and integrate the energy identity satisfied by the approximate solutions $z_n := [u_n, \varphi_n]$ of the Faedo-Galerkin scheme between s and t, with $0 \leq s \leq t$. When we pass to the limit as $n \to \infty$ in the integrated identity we have to consider the functional integral term $\int_\Omega F(\varphi_n(s))$ on the right hand side. Recalling now the bounds for the sequences $\{u_n\}$, $\{\varphi_n\}$ and $\{\varphi'_n\}$, in particular (see [17])

$$\|\varphi_n\|_{L^2(0,T;V)} \leq c, \qquad \|\varphi'_n\|_{L^{4/3}(0,T;V')} \leq c, \qquad \forall T > 0,$$

20

and using the Aubin-Lions lemma which ensures the compact embedding

$$L^2(0,T;V) \cap W^{1,4/3}(0,T;V') \hookrightarrow\hookrightarrow L^2(0,T;L^{p'}(\Omega)),$$

with $p' \in [2,6)$ (since $p \in (6/5,2]$), at least for a subsequence we have

$$\varphi_n(s) \to \varphi(s), \qquad \text{strongly in } L^{p'}(\Omega),$$

for a.e. $s > 0$. Since F has a polynomial growth of order p' (cf. Remark 1), then by Lebesgue's theorem we have

$$\int_\Omega F(\varphi_n(s)) \to \int_\Omega F(\varphi(s)),$$

for a.e. $s > 0$. Using now the lower semicontinuity of the norm we therefore get (5.10) for a.e s and a.e. t, with $0 \le s \le t$. By means of a suitable approximation of the initial datum φ_0 and of the fact that F is a quadratic perturbation of a convex function we deduce, as in the proof of [17, Theorem 1], that (5.10) holds also for $s = 0$ and for a.e $t > 0$. Finally, due to the lower semicontinuity of $\mathcal{E}(z(\cdot)) : [0,\infty) \to \mathbb{R}$ (see Lemma 2), we deduce that (5.10) holds also for every $t \ge s$. $\qquad\square$

Remark 7. If the growth restriction on F does not hold, then we can only say that for every $s \ge 0$ there exists a global weak solution (with initial data given at time s by the solution constructed in Theorem 1 with initial data given at time $s = 0$ and considered at time s) satisfying (5.10) for all $t \ge s$ (such global weak solution not necessarily coincides, between s and t, with the global weak solution constructed in Theorem 1 with initial data given at time $s = 0$ and generally depends on s).

We can now prove the following

Theorem 5. *Suppose (A1)-(A3) and (A5)-(A6) hold. Also, let $h \in L^2_{tb}(0,\infty,V'_{div})$ be given. Then every weak solution $z = [u,\varphi]$ (in the sense of Definition 2) fulfilling the energy inequality (5.10) for a.e. $s \ge 0$, including $s = 0$, and every $t \ge s$, satisfies the dissipative inequality*

$$\mathcal{E}(z(t)) \le \mathcal{E}(z_0)e^{-kt} + F(m_0)|\Omega| + K, \tag{5.11}$$

for all $t \ge 0$, where $m_0 = (\varphi_0, 1)$, and k, K are two positive constants that are independent of the initial data with K depending on Ω, ν_1, J, F and on $\|h\|_{L^2_{tb}(0,\infty;V'_{div})}$.

Remark 8. Since, under the growth restriction $p \in (6/5, 2]$ the weak solution of Theorem 1, which is constructed via a Faedo-Galerkin method, satisfies the energy inequality (5.10), then for such weak solution the dissipative estimate (5.11) holds. Nevertheless, the validity of (5.11) does not depend neither on the fact that the weak solution is constructed as in Theorem 1 nor on the growth restriction, but it relies on the validity of the energy inequality (5.10) only.

Proof. Let us first suppose that $(\varphi_0, 1) = 0$ and multiply equation $\mu = a\varphi - J * \varphi + F'(\varphi)$ by φ in $L^2(\Omega)$. We obtain

$$(\mu, \varphi) = \frac{1}{2}\int_\Omega \int_\Omega J(x-y)(\varphi(x) - \varphi(y))^2 dxdy + (F'(\varphi), \varphi). \tag{5.12}$$

Observe now that, by writing the potential F as in (5.9) and using the convexity of $\widetilde{G}(x, \cdot)$, then, for every $s \in \mathbb{R}$ and a.e. $x \in \Omega$ we have

$$(F'(s) + a(x)s)s \geq F(s) + \frac{a(x)}{2}s^2 - F(0),$$

and hence

$$F'(s)s \geq F(s) - \frac{a(x)}{2}s^2 - F(0).$$

Thus, from (5.12) we get

$$(\mu, \varphi) \geq \frac{1}{2}\int_\Omega \int_\Omega J(x-y)(\varphi(x) - \varphi(y))^2 dxdy + \int_\Omega F(\varphi(t)) - \frac{1}{2}\|\sqrt{a}\varphi\|^2 - F(0)|\Omega|. \tag{5.13}$$

On the other hand, we have

$$(\mu, \varphi) = (\mu - \overline{\mu}, \varphi) \leq C_P \|\nabla\mu\|\|\varphi\|,$$

where C_P is the Poincaré-Wirtinger constant and $\overline{\mu} := \frac{1}{|\Omega|}(\mu, 1)$. Furthermore, (A6) implies that there exist $C_9 > 0$ and $C_{10} > 0$ such that $F(s) \geq C_9|s|^{2+2q} - C_{10}$ for all $s \in \mathbb{R}$, and therefore from (5.13) we get

$$\frac{1}{8}\int_\Omega \int_\Omega J(x-y)(\varphi(x) - \varphi(y))^2 dxdy + \frac{1}{2}\int_\Omega F(\varphi) + \frac{C_9}{2}\int_\Omega |\varphi|^{2+2q} - \frac{C_{10}}{2}|\Omega|$$
$$\leq C_{11}\|\varphi\|^2 + \|\nabla\mu\|^2 + F(0)|\Omega|,$$

with $C_{11} = \frac{1}{4}(3\|J\|_{L^1} + C_P^2)$. We deduce

$$\frac{1}{8}\int_\Omega \int_\Omega J(x-y)(\varphi(x) - \varphi(y))^2 dxdy + \frac{1}{2}\int_\Omega F(\varphi) \leq \|\nabla\mu\|^2 + C_{12}, \tag{5.14}$$

and therefore

$$\frac{1}{2}\mathcal{E}(z(t)) \leq C_{13}\left(\frac{\nu_1}{2}\|\nabla u(t)\|^2 + \|\nabla\mu(t)\|^2\right) + C_{12}, \tag{5.15}$$

where $C_{13} = \max(1, 1/2\lambda_1\nu_1)$, λ_1 being the first eigenvalue of the Stokes operator A. We point out that all constants only depend on the parameters of the problem and are independent of the initial data.

Observe that the energy inequality (5.10) yields

$$\mathcal{E}(z(t)) + \int_s^t \left(\frac{\nu_1}{2}\|\nabla u\|^2 + \|\nabla\mu\|^2\right)d\tau \leq \mathcal{E}(z(s)) + \frac{1}{2\nu_1}\int_s^t \|h(\tau)\|_{V'_{div}}^2 d\tau,$$

22

for a.e $s \geq 0$, including $s = 0$, and every $t \geq s$. Therefore, on account of (5.15), we obtain the integral inequality

$$\mathcal{E}(z(t)) + k \int_0^t \mathcal{E}(z(\tau))d\tau \leq l(t-s) + \frac{1}{2\nu_1} \int_s^t \|h(\tau)\|_{V_{div}'}^2 d\tau + \mathcal{E}(z(s)) + k \int_0^s \mathcal{E}(z(\tau))d\tau,$$
(5.16)

for a.e. $s \geq 0$, including $s = 0$, and every $t \geq s$, where $k = 1/2C_{13}$ and $l = C_{12}/C_{13}$. Since, by Lemma 2, $\mathcal{E}(z(\cdot)) : [0, \infty) \to \mathbb{R}$ is lower semicontinuous, then by applying Lemma 1 we deduce that

$$\mathcal{E}(z(t)) \leq \mathcal{E}(z_0)e^{-kt} + K,$$
(5.17)

for all $t \geq 0$, where the constant $K > 0$ is given by

$$K = \frac{1}{2\nu_1} \frac{1}{1 - e^{-k}} \|h\|_{L_{tb}^2(0,\infty;V_{div}')}^2 + \frac{l}{k}.$$
(5.18)

Now, let $z := [u, \varphi]$ be a weak solution corresponding to data $z_0 := [u_0, \varphi_0]$ with $m_0 := (\varphi_0, 1) \neq 0$ for the problem with potential F and fulfilling the energy inequality (5.10) for a.e. $s \geq 0$, including $s = 0$, and for every $t \geq s$. Then $\widetilde{z} := [u, \widetilde{\varphi}]$, where $\widetilde{\varphi} = \varphi - m_0$, is a weak solution with data $\widetilde{z}_0 := [u_0, \varphi_0 - m_0]$ for the same problem with potential $\widetilde{F}$ and viscosity $\widetilde{\nu}$ given by

$$\widetilde{F}(s) := F(s + m_0) - F(m_0), \qquad \widetilde{\nu}(s) := \nu(s + m_0).$$

The weak solution $\widetilde{z}$ fulfills $(\widetilde{\varphi}, 1) = 0$ and it can be easily checked that (5.10) holds for $\widetilde{z}$, namely that we have

$$\widetilde{\mathcal{E}}(\widetilde{z}(t)) + \int_s^t (2\|\sqrt{\widetilde{\nu}(\widetilde{\varphi})}Du\|^2 + \|\nabla\widetilde{\mu}\|^2)d\tau \leq \widetilde{\mathcal{E}}(\widetilde{z}(s)) + \int_s^t \langle h(\tau), u\rangle d\tau, \qquad (5.19)$$

for a.e. $s \geq 0$, including $s = 0$, and for every $t \geq s$, where

$$\widetilde{\mathcal{E}}(\widetilde{z}(t)) := \frac{1}{2}\|u(t)\|^2 + \frac{1}{4}\int_\Omega \int_\Omega J(x-y)(\widetilde{\varphi}(x,t) - \widetilde{\varphi}(y,t))^2 dx dy + \int_\Omega \widetilde{F}(\widetilde{\varphi}(t))$$

and $\widetilde{\mu} := a\widetilde{\varphi} - J * \widetilde{\varphi} + \widetilde{F}'(\widetilde{\varphi}) = a\varphi - J * \varphi + F'(\varphi) = \mu$. Indeed (5.19) is an immediate consequence of the following identity

$$\widetilde{\mathcal{E}}(\widetilde{z}(t)) = \mathcal{E}(z(t)) - F(m_0)|\Omega|,$$
(5.20)

and of the fact that z satisfies (5.10). By applying the argument above we therefore deduce that the weak solution $\widetilde{z}$ satisfies (5.17) and by combining this inequality with (5.20) we get (5.11). $\qquad \square$

Remark 9. Assumption (A6) in Theorem 5 can be replaced by (A4) provided that either
(i) $c_1 > \frac{3}{2}\|J\|_{L^1}$ or (ii) $C_P < \frac{c_0}{2\|\nabla J\|_{L^1}}$ holds. Indeed, using (A4) from (5.13) we have

$$\frac{1}{8}\int_\Omega \int_\Omega J(x-y)(\varphi(x)-\varphi(y))^2 dxdy + \frac{1}{2}\int_\Omega F(\varphi) + \frac{c_1}{2}\|\varphi\|^2 - \frac{c_2}{2}|\Omega|$$
$$\leq \frac{3}{4}(\varphi, J*\varphi) + C_P\|\nabla\mu\|\|\varphi\| \leq \frac{3}{4}\|J\|_{L^1}\|\varphi\|^2 + C_P\|\nabla\mu\|\|\varphi\|. \tag{5.21}$$

From (5.21) it clear that, if (i) holds an inequality like (5.14) can be obtained again. On the other hand, if (ii) holds then it can be proved that (see [17, Remark 9]) $\|\nabla\mu\|^2 \geq \beta\|\nabla\varphi\|^2$ for $\overline{\varphi} = 0$, where $\beta = (c_0 - 2C_P\|\nabla J\|_{L^1})^2$. Then an inequality like (5.14) can still be recovered from (5.21). Therefore, inequality (5.15) (with different values of C_{12}, C_{13}) holds in both cases and the same argument used in the proof can be used to show that every weak solution $z = [u, \varphi]$ (in the sense of Definition 1) which fulfills the energy inequality (5.10) also satisfies (5.11).

6 Existence of a trajectory attractor

There exist various methods to define a generalized notion of global attractor for the Navier-Stokes equations in dimension three. Here we will follow the so-called trajectory approach presented in [14] (see also [15, 21, 47]). For alternative approaches, the reader is referred to, e.g., [16, 18, 35, 46] and references therein. In this section our assumption (A6) will be slightly strengthened. We shall deal mainly with the case $d = 3$, though the case $d = 2$ will also be considered.

We begin to define, for any given $m \geq 0$ and $M > 0$, the functional space

$$\mathcal{F}_M = \Big\{ [v, \psi] \in L^\infty(0, M; G_{div} \times L^{p'}(\Omega)) \cap L^2(0, M; V_{div} \times V) :$$
$$v_t \in L^{4/3}(0, M; V'_{div}), \ \psi_t \in L^2(0, M; V'), \ |(\psi(t), 1)| \leq m, \ t \in [0, M] \Big\}.$$

which is a complete metric space with respect to the metric induced by the norm

$$\|[v, \psi]\|_{\mathcal{F}_M} = \|[v, \psi]\|_{L^\infty(0, M; G_{div} \times L^{p'}(\Omega))} + \|[\nabla v, \nabla\psi]\|_{L^2(0, M; H \times H)}$$
$$+ \|v_t\|_{L^{4/3}(0, M; V'_{div})} + \|\psi_t\|_{L^2(0, M; V')}.$$

Then we introduce the spaces

$$\mathcal{F}_{loc}^+ = \Big\{ [v, \psi] \in L_{loc}^\infty([0, \infty); G_{div} \times L^{p'}(\Omega)) \cap L_{loc}^2([0, \infty); V_{div} \times V) :$$
$$v_t \in L_{loc}^{4/3}([0, \infty); V'_{div}), \ \psi_t \in L_{loc}^2([0, \infty); V'), \ |(\psi(t), 1)| \leq m, \ t \geq 0 \Big\},$$
$$\mathcal{F}_b^+ = \Big\{ [v, \psi] \in L^\infty(0, \infty; G_{div} \times L^{p'}(\Omega)) \cap L_{tb}^2(0, \infty; V_{div} \times V) :$$

$$v_t \in L_{tb}^{4/3}(0, \infty; V'_{div}), \ \psi_t \in L_{tb}^2(0, \infty; V'), \ |(\psi(t), 1)| \le m, \ t \ge 0 \Big\}.$$

We recall that $\mathcal{F}_b^+$ can be viewed as a complete metric space as $\mathcal{F}_M$ by endowing it with the metric induced by the norm

$$\|[v, \psi]\|_{\mathcal{F}_b^+} = \|[v, \psi]\|_{L^\infty(0, \infty; G_{div} \times L^{p'}(\Omega))} + \|[\nabla v, \nabla \psi]\|_{L_{tb}^2(0, \infty; H \times H)}$$
$$+ \|v_t\|_{L_{tb}^{4/3}(0, \infty; V'_{div})} + \|\psi_t\|_{L_{tb}^2(0, \infty; V')}.$$

We will indicate by Θ_M the space $\mathcal{F}_M$ endowed with the following sequential topology

Definition 4. $\{[v_n, \psi_n]\} \subset \mathcal{F}_M$ *converges to* $[v, \psi] \in \mathcal{F}_M$ *as* $n \to \infty$ *in* Θ_M *if*

$$v_n \rightharpoonup v \quad \text{weakly}^* \text{ in } L^\infty(0, M; G_{div}) \text{ and weakly in } L^2(0, M; V_{div}),$$
$$(v_n)_t \rightharpoonup v_t \quad \text{weakly in } L^{4/3}(0, M; V'_{div}),$$
$$\psi_n \rightharpoonup \psi \quad \text{weakly}^* \text{ in } L^\infty(0, M; L^{p'}(\Omega)) \text{ and weakly in } L^2(0, M; V),$$
$$(\psi_n)_t \rightharpoonup \psi_t \quad \text{weakly in } L^2(0, M; V').$$

Then the inductive limit of $\{\Theta_M\}_{M>0}$ will be denoted by Θ_{loc}^+ (see [14, Chap. XII, Def. 1.3]). We recall that Θ_M and Θ_{loc}^+ have countable topological bases.

Remark 10. We have given all the definitions above with reference to the case $d = 3$. If $d = 2$, in the definitions of functional spaces $\mathcal{F}_M, \mathcal{F}_{loc}^+, \mathcal{F}_b^+$ (and in the corresponding norms) we replace the regularity assumptions $L^{4/3}, L_{loc}^{4/3}, L_{tb}^{4/3}$ for v_t with L^2, L_{loc}^2, L_{tb}^2, respectively, while in Definition 4 the second condition is replaced by the convergence $(v_n)_t \rightharpoonup v_t$ weakly in $L^2(0, M; V'_{div})$.

We now consider the union of all weak solutions with external force h (in the sense of Definition 2 with $T = M$) satisfying inequality (5.10) on $[0, M]$ and we denote it by $\mathcal{K}_h^M$, while $\mathcal{K}_h^+$ will be the union of all weak solutions in $\mathcal{F}_{loc}^+$ with external force h satisfying (5.10) on $[0, \infty)$.

The first result concerns with the $(\Theta_M, L^2(0, M; V'_{div}))$-closure of the family $\{\mathcal{K}_h^M, h \in L^2(0, M; V'_{div})\}$. More precisely, we prove that the graph set

$$\bigcup_{h \in \mathcal{L}_M} \mathcal{K}_h^M \times \{h\}$$

is closed in the topological space $\Theta_M \times \mathcal{L}_M$, where $\mathcal{L}_M$ is $L^2(0, M; V'_{div})$ or $L_w^2(0, M; G_{div})$ if $d = 3$, and $\mathcal{L}_M$ is $L_w^2(0, M; V'_{div})$ if $d = 2$ (cf., for instance, [14, Chap. XV, Prop. 1.1]).

Proposition 6. *Let (A1)-(A3) hold. In addition, suppose that (A5) holds with $p \in (\frac{6}{5}, \frac{3}{2}]$ if $d = 3$ and with $p \in (1, 2)$ if $d = 2$ and that (A6) holds with $2q + 2 = p'$. Let $h_m \in$*

$L^2(0, M, V'_{div})$ *and consider* $[v_m, \psi_m] \in \mathcal{K}^M_{h_m}$. *Let* $\{[v_m, \psi_m]\}$ *converge to* $[v, \psi]$ *according to Definition 4. If*

$$h_m \rightharpoonup h \quad \text{weakly in } L^2(0, M; V'_{div}), \quad d = 2,$$

$$h_m \to h \quad \text{strongly in } L^2(0, M; V'_{div}) \text{ or } h_m \rightharpoonup h \text{ weakly in } L^2(0, M; G_{div}), \quad d = 3,$$

then $[v, \psi] \in \mathcal{K}^M_h$.

Proof. Consider first the case $d = 3$. Since $[v_m, \psi_m] \in \mathcal{K}^M_{h_m}$, each weak solution $z_m := [v_m, \psi_m]$ is such that: (i) $v_m \in L^\infty(0, M; G_{div}) \cap L^2(0, M; V_{div})$, $(v_m)_t \in L^{4/3}(0, M; V'_{div})$, $\psi_m \in L^\infty(0, M; L^{2+2q}(\Omega)) \cap L^2(0, M; V)$, $(\psi_m)_t \in L^2(0, M; V')$ (we are assuming $q \geq 1/2$), $\mu_m \in L^2(0, M; V)$, where $\mu_m = a\psi_m - J * \psi_m + F'(\psi_m)$; (ii) the weak formulation (2.8), (2.9), (2.7) for $[v_m, \psi_m]$ and μ_m holds with external force h_m, and (iii) the energy inequality

$$\mathcal{E}(z_m(t)) + \int_s^t (2\|\sqrt{\nu(\psi_m)}Dv_m\|^2 + \|\nabla\mu_m\|^2) d\tau \leq \mathcal{E}(z_m(s)) + \int_s^t \langle h_m(\tau), u_m \rangle d\tau \quad (6.1)$$

is satisfied for every $m \in \mathbb{N}$, for a.e. $s \in [0, M]$, including $s = 0$, and for every $t \in [0, M]$ with $t \geq s$. Observe that the third convergence condition in Definition 3 is compatible with the regularity property $\psi_m \in L^\infty(0, M; L^{2+2q}(\Omega))$ due to the constraint $2q + 2 = p'$. Note that $q \geq 1/2$ since $p \leq 3/2$.

Thanks to the convergences listed in Definition 4 and to the polynomial control (2.3) on F it is easy to see that there exists $c > 0$ such that

$$|\mathcal{E}(z_m(s))| \leq c, \qquad \forall m, \quad \text{a.e. } s \in (0, M) \quad (6.2)$$

Hence, (6.1) and the convergence assumption on the sequence $\{h_m\}$ entail the control $\|\nabla\mu_m\|_{L^2(0, M; H)} \leq c$. Furthermore, since $(\mu_m, 1) = (F'(\psi_m), 1)$ and since from (A4) we have $|F'(\psi_m)|^p \leq c|F(\psi_m)| + c \leq c|\psi_m|^{p'} + c$ which, along with the third convergence of Definition 4, implies that

$$\|F'(\psi_m)\|_{L^\infty(0, M; L^p(\Omega))} \leq c, \quad (6.3)$$

we deduce that $|(\mu_m, 1)| \leq c$ and therefore that $\|\mu_m\|_{L^2(0, M; V)} \leq c$. Observe that we also have the estimate $\|F'(\psi_m)\|_{L^2(0, M; V)} \leq c$. As a consequence, there exists $\mu \in L^2(0, M; V)$ such that for a subsequence we have

$$\mu_m \rightharpoonup \mu, \qquad \text{weakly in } L^2(0, M; V). \quad (6.4)$$

Definition 4 also implies that, up to subsequences, $\{[v_m, \psi_m]\}$ strongly converges to $[v, \psi]$ in $L^2(0, M; G_{div} \times H)$ and thus $\{\psi_m\}$ also converges to ψ almost everywhere in $\Omega \times (0, M)$. We hence get that $\mu = a\psi - J * \psi + F'(\psi)$. Now, on account of Definition 4, of the strong convergences obtained above and of (6.4), we can now pass to the limit in the variational

formulation (2.8), (2.9), (2.7) for the weak solution $[v_m, \psi_m]$ with external force h_m and thus deduce that $[v, \psi]$ is a weak solution with external force h.

It remains to prove that the weak solution $[v, \psi]$ satisfies the energy inequality (5.10) on $[0, M]$ with external force h. To this aim we pass to the limit in (6.1) as $m \to \infty$. We exploit the convergence $h_m \to h$, strongly in $L^2(0, M; V'_{div})$ and, in order to pass to the limit in the nonlinear functional term $\int_\Omega F(\psi_m(s))$ on the right hand side of (6.1) we notice that, due to the third and fourth convergences assumed in Definition 4 and to (3.26), we have that $\psi_n(s) \to \psi(s)$ strongly in $L^{p'}(\Omega)$ for a.e. $s > 0$ and hence, since $p > 6/5$, we get $\int_\Omega F(\psi_m(s)) \to \int_\Omega F(\psi(s))$ for a.e. $s > 0$. By also using (6.4), the convergence

$$\sqrt{\nu(\psi_m)} Dv_m \rightharpoonup \sqrt{\nu(\psi)} Dv \qquad \text{weakly in } L^2(0, M; H),$$

(cf. (3.17)) and the lower semicontinuity of the $L^2(0, M; H)-$norm we thus get that $[v, \psi]$ with external force h satisfies (5.10) for a.e. $s \in [0, M]$, including $s = 0$, and for every $t \in [0, M]$ with $t \geq s$. Hence $[v, \psi] \in \mathcal{K}_h^M$. The same conclusion holds if we suppose that $h_m \rightharpoonup h$ weakly in $L^2(0, M; G_{div})$. Indeed, arguing as in [14, Chap. XV, Prop. 1.1] and relying on the strong convergence $u_m \to u$ in $L^2(0, M; G_{div})$ we have that

$$\int_s^t \langle h_m(\tau), u_m(\tau)\rangle d\tau \to \int_s^t \langle h(\tau), u(\tau)\rangle d\tau, \qquad \text{as } m \to \infty.$$

If $d = 2$, the situation is easier since the energy identity can be deduced from the weak formulation (see also [14, Chap. XV, proof of Prop. 1.1]). $\qquad \square$

Remark 11. The main reason for assuming that $\psi_n \rightharpoonup \psi$ weakly* in $L^\infty(0, M; L^{p'}(\Omega))$ in Definition 3, rather then the apparently more natural convergence condition $\psi_n \rightharpoonup \psi$ weakly* in $L^\infty(0, M; L^{2+2q}(\Omega))$, is in order to ensure (6.2). Obviously, as pointed out above, the relation $p' = 2 + 2q$ is needed.

Consider now $h_0 \in L^2_{tb}(0, \infty; V'_{div})$ so that h_0 is translation compact in $L^2_{loc,w}([0, \infty); V'_{div})$. Then set

$$\mathcal{H}_+(h_0) := [\{h_0(\cdot + \tau) \,|\, \tau \geq 0\}]_{L^2_{loc,w}([0,\infty); V'_{div})},$$

where $[\cdot]_X$ denotes the closure in the space X. The following property will be useful in the next proposition: if $h_0 \in L^2_{tb}(0, \infty; V'_{div})$ and $h \in \mathcal{H}_+(h_0)$, then $h \in L^2_{tb}(0, \infty; V'_{div})$ as well and

$$\|h\|_{L^2_{tb}(0,\infty; V'_{div})} \leq \|h_0\|_{L^2_{tb}(0,\infty; V'_{div})}. \tag{6.5}$$

We recall that the translation semigroup $T(t)$ is continuous on $\mathcal{H}_+(h_0)$ and $T(t)\mathcal{H}_+(h_0) = \mathcal{H}_+(h_0)$ for all $t \geq 0$. This translation semigroup can also be defined on $\mathcal{K}_h^+$ for any $h \in \mathcal{H}_+(h_0)$. Indeed, if $[v, \psi] \in \mathcal{K}_h^M$ then $T(\tau)[v, \psi] \in \mathcal{K}_{T(\tau)h}^{M-\tau}$, i.e. $T(\tau)\mathcal{K}_h^M \subseteq \mathcal{K}_{T(\tau)h}^{M-\tau}$ for all $M \geq \tau \geq 0$. Thus, recalling [14, Chap. XIV, Props. 1.1 and 1.2], we have, for all $t \geq 0$,

$$T(t)\mathcal{K}_h^+ \subseteq \mathcal{K}_{T(t)h}^+, \qquad T(t)\mathcal{K}_{\mathcal{H}_+(h_0)}^+ \subseteq \mathcal{K}_{\mathcal{H}_+(h_0)}^+.$$

where $\mathcal{K}^+_{\mathcal{H}_+(h_0)} := \bigcup_{h \in \mathcal{H}_+(h_0)} \mathcal{K}^+_h$ is the so-called *united trajectory space* (see [14, Chap. XIV, Def. 1.2]).

We can now prove the following (see [14, Chap. XV, Prop. 1.2])

Proposition 7. *Let (A1)-(A3) hold. In addition, suppose that (A5) holds with $p \in (1, \frac{3}{2}]$ if $d = 3$ and with $p \in (1, 2)$ if $d = 2$ and that (A6) holds with $2q = p' - 2$. If $h_0 \in L^2_{tb}(0, \infty; V'_{div})$ then, for all $h \in \mathcal{H}_+(h_0)$, we have $\mathcal{K}^+_h \subset \mathcal{F}^+_b$ and the following dissipative estimate holds*

$$\|T(t)[v, \psi]\|_{\mathcal{F}^+_b} \leq \Lambda_0 \|[v, \psi]\|_{L^\infty(0,1;G_{div} \times L^{p'}(\Omega))} e^{-\kappa t} + \Lambda_1, \qquad (6.6)$$

for all $t \geq 1$ and all $[v, \psi] \in \mathcal{K}^+_h$. Here Λ_0, κ and Λ_1 are positive constants with $k = \min(1/2, \lambda_1 \nu)$ and Λ_0, Λ_1 depending on $\nu_1, \nu_2, \lambda_1, F, J, |\Omega|$, with Λ_1 depending also on $\|h_0\|_{L^2_{tb}(0,\infty;V'_{div})}$.

Proof. Take $[v, \psi] \in \mathcal{K}^+_h$. Then, by definition $z := [v, \psi]$ is a weak solution corresponding to the external force h satisfying (5.10) and hence (5.16) on $[0, \infty)$. By applying Lemma 1 we get

$$\mathcal{E}(z(t)) \leq \mathcal{E}(z(s))e^{-k(t-s)} + \frac{1}{2\nu_1} \int_s^t e^{-k(t-\tau)} \left(\|h(\tau)\|^2_{V'_{div}} + 2\nu_1 l \right) d\tau,$$

for a.e. $s \geq 0$, including $s = 0$ and for every $t \geq s$. The constants k, l are the same as in the proof of Theorem 5. In particular we have $k = \min(1/2, \lambda_1 \nu_1)$. Thus, we deduce

$$\mathcal{E}(z(t)) \leq e^k \sup_{s \in (0,1)} \mathcal{E}(z(s))e^{-kt} + \frac{1}{2\nu_1} \int_0^t e^{-k(t-\tau)} \left(\|h(\tau)\|^2_{V'_{div}} + 2\nu_1 l \right) d\tau, \qquad (6.7)$$

for all $t \geq 1$. Now, notice that, due to (2.3), (A5) and to the assumption $p' = 2 + 2q$ there exist two constants $k_1, k_2 > 0$ depending on F and J such that

$$k_1(\|\varphi(s)\|^{p'}_{L^{p'}(\Omega)} + \|u(s)\|^2 - 1) \leq \mathcal{E}(z(s)) \leq k_2(\|\varphi(s)\|^{p'}_{L^{p'}(\Omega)} + \|u(s)\|^2 + 1). \quad (6.8)$$

Henceforth c will stand for a positive constant, that may vary from line to line, that depends on ν_1, λ_1, F, J and $|\Omega|$. From (6.7) we obtain

$$\|\varphi(t)\|^{p'}_{L^{p'}(\Omega)} + \|u(t)\|^2 \leq c(\|\varphi\|^{p'}_{L^\infty(0,1;L^{p'}(\Omega))} + \|u\|^2_{L^\infty(0,1;G_{div})})e^{-kt}$$

$$+ \frac{1}{2\nu_1} \int_0^t e^{-k(t-\tau)} \|h(\tau)\|^2_{V'_{div}} d\tau + \frac{l}{k} + c, \qquad \forall t \geq 1, \qquad (6.9)$$

which immediately leads to

$$\|T(t)\varphi\|^{p'}_{L^\infty(0,\infty;L^{p'}(\Omega))} + \|T(t)u\|^2_{L^\infty(0,\infty;G_{div})} \leq c(\|\varphi\|^{p'}_{L^\infty} + \|u\|^2_{L^\infty})e^{-kt} + K + c, \quad (6.10)$$

for all $t \geq 1$, where we have set $\|\varphi\|_{L^\infty} := \|\varphi\|_{L^\infty(0,1;L^{p'}(\Omega))}$ and $\|u\|_{L^\infty} := \|u\|_{L^\infty(0,1;G_{div})}$.
The constant K also depends on h_0 and has the following form (see (5.18) and (6.5))

$$K = \frac{1}{2\nu_1} \frac{1}{1-e^{-k}} \|h_0\|^2_{L^2_{tb}(0,\infty;V'_{div})} + \frac{l}{k}.$$

By (5.10) we have, for a.e. $t \geq 0$ (including $t = 0$),

$$\int_t^{t+1} \left(\frac{\nu_1}{2} \|\nabla u\|^2 + \|\nabla \mu\|^2 \right) d\tau \leq \mathcal{E}(z(t)) - \mathcal{E}(z(t+1)) + \frac{1}{2\nu_1} \int_t^{t+1} \|h(\tau)\|^2_{V'_{div}} d\tau. \quad (6.11)$$

Furthermore, by means of (A2) and by multiplying the gradient of (2.7) by $\nabla \varphi$, it can be shown that (see [17])

$$\|\nabla \mu\|^2 \geq k_3 \|\nabla \varphi\|^2 - k_4 \|\varphi\|^2, \quad (6.12)$$

where $k_3 = c_0^2/4$ and $k_4 = 2\|\nabla J\|^2_{L^1}$. Therefore, combining (6.11) and (6.12) with (6.8) and (6.9) we get

$$\int_t^{t+1} \left(\frac{\nu_1}{2} \|\nabla u\|^2 + k_3 \|\nabla \varphi\|^2 \right) d\tau \leq c(\|\varphi\|^{p'}_{L^\infty} + \|u\|^2_{L^\infty})e^{-kt} + cK + c, \quad (6.13)$$

from which we deduce

$$\|T(t)\varphi\|^2_{L^2_{tb}(0,\infty;V)} + \|T(t)u\|^2_{L^2_{tb}(0,\infty;V_{div})} \leq c(\|\varphi\|^{p'}_{L^\infty} + \|u\|^2_{L^\infty})e^{-kt} + cK + c, \quad (6.14)$$

for all $t \geq 1$. Let us come to the terms in (6.6) containing the time derivatives. As far as the contribution arising from the Korteweg force term is concerned, since, if $d = 3$ we have

$$\|\varphi \nabla \mu\|_{V'_{div}} \leq c\|\varphi\|_{L^3(\Omega)}\|\nabla \mu\| \leq c\|\varphi\|^{1/2}\|\varphi\|^{1/2}_{L^6(\Omega)}\|\nabla \mu\| \leq c\|\varphi\|^{1/2}\|\varphi\|^{1/2}_V\|\nabla \mu\|,$$

then, on account of (6.9), of the fact that $\|\varphi\| \leq \|\varphi\|^{p'/2}_{L^{p'}(\Omega)} + c$, and of (6.11), (6.13) we get

$$\left(\int_t^{t+1} \|\varphi \nabla \mu\|^{4/3}_{V'_{div}} d\tau \right)^{3/4} \leq c\|\varphi\|^{1/2}_{L^\infty(t,t+1;H)} \left(\int_t^{t+1} \|\varphi\|^2_V d\tau \right)^{1/4} \left(\int_t^{t+1} \|\nabla \mu\|^2 d\tau \right)^{1/2}$$

$$\leq \left(c(\|\varphi\|^{p'}_{L^\infty} + \|u\|^2_{L^\infty})e^{-kt} + K + c \right)^{1/4} \left(c(\|\varphi\|^{p'}_{L^\infty} + \|u\|^2_{L^\infty})e^{-kt} + cK + c \right)^{1/4}$$

$$\left(c(\|\varphi\|^{p'}_{L^\infty} + \|u\|^2_{L^\infty})e^{-kt} + cK + c \right)^{1/2} \leq c(\|\varphi\|^{p'}_{L^\infty} + \|u\|^2_{L^\infty})e^{-kt} + cK + c, \quad (6.15)$$

for all $t \geq 1$. If $d = 2$ then we have

$$\|\varphi \nabla \mu\|_{V'_{div}} \leq c\|\varphi\|_{L^{2+2q}(\Omega)}\|\nabla \mu\| = c\|\varphi\|_{L^{p'}(\Omega)}\|\nabla \mu\|.$$

Therefore, recalling (6.9), (6.13) and (6.11), we obtain

$$\left(\int_t^{t+1} \|\varphi \nabla \mu\|^2_{V'_{div}} d\tau \right)^{1/2} \leq c\|\varphi\|_{L^\infty(t,t+1;L^{p'}(\Omega))} \left(\int_t^{t+1} \|\nabla \mu\|^2 d\tau \right)^{1/2}$$

29

$$\leq \Big(c(\|\varphi\|_{L^\infty}^{p'} + \|u\|_{L^\infty}^2)e^{-kt} + K + c\Big)^{1/2}\Big(c(\|\varphi\|_{L^\infty}^{p'} + \|u\|_{L^\infty}^2)e^{-kt} + cK + c\Big)^{1/2}$$

$$\leq c(\|\varphi\|_{L^\infty}^{p'} + \|u\|_{L^\infty}^2)e^{-kt} + cK + c, \qquad \forall t \geq 1. \tag{6.16}$$

Therefore (6.15) and (6.16) for $d = 3, 2$ entail

$$\|T(t)(\varphi\nabla\mu)\|_{L_{tb}^{4/d}(0,\infty;V'_{div})} \leq c(\|\varphi\|_{L^\infty}^{p'} + \|u\|_{L^\infty}^2)e^{-kt} + cK + c, \qquad \forall t \geq 1. \tag{6.17}$$

Furthermore, for $d = 3$, recalling (2.1) we have

$$\Big(\int_t^{t+1}\|\mathcal{B}(u,u)\|_{V'_{div}}^{4/3}\,d\tau\Big)^{3/4} \leq c\|u\|_{L^\infty(t,t+1;G_{div})}^{1/2}\Big(\int_t^{t+1}\|\nabla u\|^2 d\tau\Big)^{3/4}$$

$$\leq \Big(c(\|\varphi\|_{L^\infty}^{p'} + \|u\|_{L^\infty}^2)e^{-kt} + K + c\Big)^{1/4}\Big(c(\|\varphi\|_{L^\infty}^{p'} + \|u\|_{L^\infty}^2)e^{-kt} + cK + c\Big)^{3/4}$$

$$\leq c(\|\varphi\|_{L^\infty}^{p'} + \|u\|_{L^\infty}^2)e^{-kt} + cK + c, \qquad \forall t \geq 1,$$

while, for $d = 2$, recalling (2.2) we obtain

$$\Big(\int_t^{t+1}\|\mathcal{B}(u,u)\|_{V'_{div}}^2\,d\tau\Big)^{1/2} \leq c\|u\|_{L^\infty(t,t+1;G_{div})}\Big(\int_t^{t+1}\|\nabla u\|^2 d\tau\Big)^{1/2}$$

$$\leq c(\|\varphi\|_{L^\infty}^{p'} + \|u\|_{L^\infty}^2)e^{-kt} + cK + c, \qquad \forall t \geq 1.$$

Hence, for $d = 3, 2$ we get

$$\|T(t)\mathcal{B}(u,u)\|_{L_{tb}^{4/d}(0,\infty;V'_{div})} \leq c(\|\varphi\|_{L^\infty}^{p'} + \|u\|_{L^\infty}^2)e^{-kt} + cK + c, \qquad \forall t \geq 1. \tag{6.18}$$

Recalling equation (1.3) which can be written as

$$u_t + \mathcal{A}(u,\varphi) + \mathcal{B}(u,u) = -\varphi\nabla\mu + h \qquad \text{in } V'_{div}, \text{ a.e. in } (0,\infty),$$

we deduce by comparison that

$$\|u_t\|_{L^{4/d}(t,t+1;V'_{div})} \leq \nu_2\|u\|_{L^2(t,t+1;V_{div})} + \|\mathcal{B}(u,u)\|_{L^{4/d}(t,t+1;V'_{div})}$$
$$+ \|\varphi\nabla\mu\|_{L^{4/d}(t,t+1;V'_{div})} + \|h\|_{L^2(t,t+1;V'_{div})}.$$

Therefore, using (6.17) and (6.18), we obtain

$$\|T(t)u_t\|_{L_{tb}^{4/d}(0,\infty;V'_{div})} \leq c(\|\varphi\|_{L^\infty}^{p'} + \|u\|_{L^\infty}^2)e^{-kt} + cK + c, \qquad \forall t \geq 1. \tag{6.19}$$

Now, from (1.1), for $d = 3$ we can write

$$\|\varphi_t\|_{V'} \leq \|\nabla\mu\| + c\|u\|_{L^{2(1+\frac{1}{q})}(\Omega)}\|\varphi\|_{L^{2+2q}(\Omega)} \leq \|\nabla\mu\| + c\|\nabla u\|\|\varphi\|_{L^{p'}(\Omega)},$$

while, for $d = 2$ we have

$$\|\varphi_t\|_{V'} \leq \|\nabla\mu\| + c\|\nabla u\|\|\varphi\|_{L^{2+2q}(\Omega)} = \|\nabla\mu\| + c\|\nabla u\|\|\varphi\|_{L^{p'}(\Omega)}.$$

The contribution from the transport term gives

$$\left(\int_t^{t+1} \|\nabla u\|^2 \|\varphi\|_{L^{p'}(\Omega)}^2 d\tau\right)^{1/2} \leq \|\varphi\|_{L^\infty(t,t+1;L^{p'}(\Omega))}\left(\int_t^{t+1} \|\nabla u\|^2 d\tau\right)^{1/2}$$
$$\leq c(\|\varphi\|_{L^\infty}^{p'} + \|u\|_{L^\infty}^2)e^{-kt} + cK + c, \qquad \forall t \geq 1.$$

Thus, in both cases $d = 2$ and $d = 3$, we find

$$\|T(t)\varphi_t\|_{L_{tb}^2(0,\infty;V')} \leq c(\|\varphi\|_{L^\infty}^{p'} + \|u\|_{L^\infty}^2)e^{-kt} + cK + c, \qquad \forall t \geq 1. \tag{6.20}$$

Finally, collecting (6.10), (6.14), (6.19) and (6.20), we get (6.6) with $\Lambda_0 = c$ and $\Lambda_1 = cK + c$. $\qquad\square$

Propositions 6 and 7 are the basic ingredients to establish next theorem, which is the main result of this section. We denote by $Z(h_0)$ the set of all complete symbols in $\mathcal{H}_+(h_0)$, i.e., the set of functions $\zeta : \mathbb{R} \to V'_{div}$, $\zeta \in L^2_{loc}(\mathbb{R}; V'_{div})$ such that $\Pi_+ T(t)\zeta \in \omega(\mathcal{H}_+(h_0))$, for all $t \in \mathbb{R}$, where Π_+ is the restriction operator to the semiaxis $[0, \infty)$. To any complete symbol $\zeta \in Z(h_0)$ there corresponds, by [14, Chap XIV, Definition 2.5] (see also [15, Definition 4.4]), the kernel $\mathcal{K}_\zeta$ which consists of all weak solutions $z : \mathbb{R} \to G_{div} \times H$ with external force ζ (in the sense of Definition 2 with $T \in \mathbb{R}$) satisfying inequality (5.10) on $\mathbb{R}$ and that are bounded in the space $\mathcal{F}_b$ (the space $\mathcal{F}_b$ is defined as $\mathcal{F}_b^+$ with the time semiaxis $[0, \infty)$ replaced with $\mathbb{R}$ in the definition of $\mathcal{F}_b^+$; in the same way $\mathcal{F}_{loc}$ and Θ_{loc} can be defined). Then, we set

$$\mathcal{K}_{Z(h_0)} := \bigcup_{\zeta \in Z(h_0)} \mathcal{K}_\zeta.$$

Theorem 6. *Let (A1)-(A3) hold. In addition, suppose that (A5) holds with $p \in (\frac{6}{5}, \frac{3}{2}]$ if $d = 3$ and with $p \in (1, 2)$ if $d = 2$ and that (A6) holds with $2q + 2 = p'$. If*

$$h_0 \in L^2_{tb}(0, \infty; V'_{div}), \quad d = 2,$$
$$h_0 \in L^2_{tb}(0, \infty; G_{div}), \quad d = 3,$$

then $\{T(t)\}$ acting on $\mathcal{K}^+_{\mathcal{H}(h_0)}$ possesses the uniform (with respect to $h \in \mathcal{H}(h_0)$) trajectory attractor $\mathcal{U}_{\mathcal{H}(h_0)}$. This set is bounded in $\mathcal{F}_b^+$ and compact in Θ^+_{loc}. Moreover, we have

$$\mathcal{U}_{\mathcal{H}(h_0)} = \mathcal{U}_{\omega(\mathcal{H}_+(h_0))} = \mathcal{K}_{Z(h_0)},$$

where $\mathcal{U}_{\omega(\mathcal{H}_+(h_0))}$ is the uniform (with respect to $h \in \omega(\mathcal{H}_+(h_0))$) trajectory attractor of the family $\{\mathcal{K}^+_h : h \in \omega(\mathcal{H}_+(h_0))\}$, $\mathcal{U}_{\omega(\mathcal{H}_+(h_0))} \subset \mathcal{K}^+_{\omega(\mathcal{H}_+(h_0))}$. The kernel $\mathcal{K}_\zeta$ is not empty for any $\zeta \in Z(h_0)$; the set $\mathcal{K}_{Z(h_0)}$ is bounded in $\mathcal{F}_b$ and compact in Θ_{loc}.

31

Proof. The family of trajectory spaces $\{\mathcal{K}_h^+ : h \in \mathcal{H}_+(h_0)\}$ is $(\Theta_{loc}^+, \mathcal{H}_+(h_0))$−closed due to Proposition 6. Notice that the assumption on h_0 ensures that the symbol space $\Sigma := \mathcal{H}_+(h_0)$ is a compact metric space. Thanks to (6.6) it is easy to see that the ball

$$B_{\mathcal{F}_b^+}(2\Lambda_1) := \{[v, \psi] \in \mathcal{F}_b^+ : \|[v, \psi]\|_{\mathcal{F}_b^+} \leq 2\Lambda_1\}$$

is a uniformly (w.r.t. $h \in \mathcal{H}_+(h_0)$) absorbing set for the family $\{\mathcal{K}_h^+ : h \in \mathcal{H}_+(h_0)\}$. The ball $B_{\mathcal{F}_b^+}(2\Lambda_1)$ is compact in Θ_{loc}^+ and bounded in $\mathcal{F}_b^+$. The conditions of [14, Chap. XIV, Thm 2.1 and Thm. 3.1]) are thus satisfied and the thesis follows. $\qquad\square$

References

[1] H. Abels, *On a diffusive interface model for two-phase flows of viscous, incompressible fluids with matched densities*, Arch. Ration. Mech. Anal. **194** (2009), 463-506.

[2] H. Abels, *Existence of weak solutions for a diffuse interface model for viscous, incompressible fluids with general densities*, Comm. Math. Phys. **289** (2009), 45-73.

[3] H. Abels, *Longtime behavior of solutions of a Navier-Stokes/Cahn-Hilliard system*, Proceedings of the Conference "Nonlocal and Abstract Parabolic Equations and their Applications", Bedlewo, Banach Center Publ. **86** (2009), 9-19.

[4] H. Abels, E. Feireisl, *On a diffuse interface model for a two-phase flow of compressible viscous fluids*, Indiana Univ. Math. J. **57** (2008), 659-698.

[5] D.M. Anderson, G.B. McFadden, A.A. Wheeler, *Diffuse-interface methods in fluid mechanics*, Annu. Rev. Fluid Mech. **30**, Annual Reviews, Palo Alto, CA, 1998, 139-165.

[6] V.E. Badalassi, H. Ceniceros, S. Banerjee, *Computation of multiphase systems with phase field models*, J. Comput. Phys. **190** (2003), 371-397.

[7] J.M. Ball, *Continuity properties and global attractors of generalized semiflows and the Navier-Stokes equation*, J. Nonlinear Sci. **7** (1997), 475-502 (Erratum, J. Nonlinear Sci. **8** (1998), 233).

[8] J. M. Ball, *Global attractors for damped semilinear wave equations*, Discrete Contin. Dyn. Syst. **10** (2004) 31-52.

[9] P.W. Bates, J. Han, *The Neumann boundary problem for a nonlocal Cahn-Hilliard equation*, J. Differential Equations **212** (2005), 235-277.

[10] P.W. Bates, J. Han, *The Dirichlet boundary problem for a nonlocal Cahn-Hilliard equation*, J. Math. Anal. Appl. **311** (2005), 289-312.

[11] F. Boyer, *Mathematical study of multi-phase flow under shear through order parameter formulation*, Asymptot. Anal. **20** (1999), 175-212.

[12] F. Boyer, *Nonhomogeneous Cahn-Hilliard fluids*, Ann. Inst. H. Poincaré Anal. Non Linéaire **18** (2001), 225-259.

[13] F. Boyer, *A theoretical and numerical model for the study of incompressible mixture flows*, Comput. & Fluids **31** (2002), 41-68.

[14] V.V. Chepyzhov, M. Vishik, *Attractors for Equations of Mathematical Physics*, Amer. Math. Soc. Colloq. Publ., vol. **49**, American Mathematical Society, Providence, RI, 2002.

[15] V.V. Chepyzhov, M. Vishik, *Evolution equations and their trajectory attractors*, J. Math. Pures Appl. **76** (1997), 913-964.

[16] A. Cheskidov, C. Foias, *On global attractors of the 3D-Navier-Stokes equations*, J. Differential Equations **231** (2006), 714-754.

[17] P. Colli, S. Frigeri, M. Grasselli, *Global existence of weak solutions to a nonlocal Cahn-Hilliard-Navier-Stokes system*, submitted.

[18] N.J. Cutland, *Global attractors for small samples and germs of 3D Navier-Stokes equations*, Nonlinear Anal. **62** (2005), 265-281.

[19] M. Doi, *Dynamics of domains and textures*, Theoretical Challenges in the Dynamics of Complex Fluids (T.C. McLeish Ed.), NATO-ASI Ser. **339**, Kluwer Academic, Dordrecht, 1997, 293-314.

[20] X. Feng, *Fully discrete finite element approximation of the Navier-Stokes-Cahn-Hilliard diffuse interface model for two-phase flows*, SIAM J. Numer. Anal. **44** (2006), 1049-1072.

[21] F. Flandoli, B. Schmalfuss, *Weak solutions and attractors for the 3-dimensional Navier-Stokes equations with nonregular force,* J. Dynam. Differential Equations **11** (1999), 355-398.

[22] H. Gajewski, *On a nonlocal model of non-isothermal phase separation*, Adv. Math. Sci. Appl. **12** (2002), 569-586.

[23] H. Gajewski, K. Zacharias, *On a nonlocal phase separation model*, J. Math. Anal. Appl. **286** (2003), 11-31.

[24] C.G. Gal, M. Grasselli, *Asymptotic behavior of a Cahn-Hilliard-Navier-Stokes system in 2D*, Ann. Inst. H. Poincaré Anal. Non Linéaire **27** (2010), 401-436.

[25] C.G. Gal, M. Grasselli, *Trajectory attractors for binary fluid mixtures in 3D*, Chinese Ann. Math. Ser. B **31** (2010), 655-678.

[26] C.G. Gal, M. Grasselli, *Instability of two-phase flows: a lower bound on the dimension of the global attractor of the Cahn-Hilliard-Navier-Stokes system,* Phys. D **240** (2011), 629-635.

[27] M. Grasselli, D. Pražák, *Longtime behavior of a diffuse interface model for binary fluid mixtures with shear dependent viscosity,* submitted.

[28] G. Giacomin, J.L. Lebowitz, *Phase segregation dynamics in particle systems with long range interactions. I. Macroscopic limits*, J. Statist. Phys. **87** (1997), 37-61.

[29] G. Giacomin, J.L. Lebowitz, *Phase segregation dynamics in particle systems with long range interactions. II. Phase motion*, SIAM J. Appl. Math. **58** (1998), 1707-1729.

[30] M.E. Gurtin, D. Polignone, J. Viñals, *Two-phase binary fluids and immiscible fluids described by an order parameter*, Math. Models Meth. Appl. Sci. **6** (1996), 8-15.

[31] J. Han, *The Cauchy problem and steady state solutions for a nonlocal Cahn-Hilliard equation*, Electron. J. Differential Equations **113** (2004), 9 pp.

[32] B. Haspot, *Existence of global weak solution for compressible fluid models with a capillary tensor for discontinuous interfaces*, Differential Integral Equations **23** (2010), 899-934.

[33] P.C. Hohenberg, B.I. Halperin, *Theory of dynamical critical phenomena*, Rev. Mod. Phys. **49** (1977), 435-479.

[34] D. Jasnow, J. Viñals, *Coarse-grained description of thermo-capillary flow*, Phys. Fluids **8** (1996), 660-669.

[35] A.V. Kapustyan, J. Valero, *Weak and strong attractors for the 3D Navier-Stokes system*, J. Differential Equations **240** (2007), 249-278.

[36] D. Kay, V. Styles, R. Welford, *Finite element approximation of a Cahn-Hilliard-Navier-Stokes system*, Interfaces Free Bound. **10** (2008), 5-43.

[37] J. Kim, K. Kang, J. Lowengrub, *Conservative multigrid methods for Cahn-Hilliard fluids*, J. Comput. Phys. **193** (2004), 511-543.

[38] P.E. Kloeden, J. Valero, *The Kneser property of the weak solutions of the three dimensional Navier-Stokes equations*, Discrete Contin. Dyn. Syst. **28** (2010), 161-179.

[39] C. Liu, J. Shen, *A phase field model for the mixture of two incompressible fluids and its approximation by a Fourier-spectral method*, Phys. D **179** (2003), 211-228.

[40] S.-O. Londen, H. Petzeltová, *Convergence of solutions of a non-local phase-field system*, Discrete Contin. Dyn. Syst. Ser. S **4** (2011), 653-670.

[41] J. Lowengrub, L. Truskinovsky, *Quasi-incompressible Cahn-Hilliard fluids and topological transitions*, Proc. R. Soc. London A **454** (1998), 2617-2654.

[42] P. Marín-Rubio, J. Real, *Pullback attractors for 2D-Navier-Stokes equations with delays in continuous and sub-linear operators*, Discrete Contin. Dyn. Syst. **26** (2010), 989-1006.

[43] A. Morro, *Phase-field models of Cahn-Hilliard Fluids and extra fluxes*, Adv. Theor. Appl. Mech. **3** (2010), 409-424.

[44] C. Rhode, *On local and non-local Navier-Stokes-Korteweg systems for liquid-vapour phase transitions*, Z. Angew. Math. Mech. **85** (2005), 839-857.

[45] C. Rhode, *A Local and Low-Order Navier-Stokes-Korteweg System*, Nonlinear partial differential equations and hyperbolic wave phenomena (H. Holden and K.H. Karlsen Eds.), 315-337, Contemp. Math., **526**, Amer. Math. Soc., Providence, RI, 2010.

[46] R.M.S. Rosa, *Asymptotic regularity conditions for the strong convergence towards weak limit sets and weak attractors of the 3D Navier-Stokes equations*, J. Differential Equations **229** (2006), 257-269.

[47] G.R. Sell, *Global attractors for the three-dimensional Navier-Stokes equations*, J. Dynam. Differential Equations **8** (1996), 1-33.

[48] J. Shen, X. Yiang, *Energy stable schemes for Cahn-Hilliard phase-field model of two-phase incompressible flows*, Chinese Ann. Math. Ser. B **31** (2010), 743-758.

[49] V.N. Starovoitov, *The dynamics of a two-component fluid in the presence of capillary forces*, Math. Notes **62** (1997), 244-254.

[50] R. Temam, *Navier-Stokes equations and nonlinear functional analysis*, Second edition, CBMS-NSF Reg. Conf. Ser. Appl. Math., **66**, SIAM, Philadelphia, PA, 1995.

[51] L. Zhao, H. Wu, H. Huang, *Convergence to equilibrium for a phase-field model for the mixture of two viscous incompressible fluids*, Commun. Math. Sci. **7** (2009), 939-962.