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BAR CODE VS JANET TREE

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Abstract. In this paper we study how to compute Janet-multiplicative variables for the elements of a given finite set of terms. A comparison between Bar Codes and the Janet tree defined by Gerdt-Blinkov-Yanovich and reformulated by Seiler is given.

1. Introduction

The concept of *involutive division* dates back to the work by Janet (1920, 1924, 1927, 1929). Given the polynomial ring $\mathcal{P} := \mathbf{k}[x_1, ..., x_n]$, in n variables, and considered a semigroup/monomial ideal $J \triangleleft \mathcal{P}$, and its minimal set of generators $\mathsf{G}(J)$, he introduced the notion of *multiplicative variable* for a term $u \in \mathsf{G}(J)$. All multiples of u of the form ut, where t is a product of powers of multiplicative variables for u constitute the *cone* of u. Janet introduced also the *completion*, a procedure whose aim is to enlarge $\mathsf{G}(J)$ to a set $\mathsf{G}'(J)$ so that the cones of all its elements turn out to cover the whole J. This way, J is the (disjoint) union of the cones of the generators in $\mathsf{G}'(J)$. Now, if J is the initial ideal of some ideal $I \triangleleft \mathcal{P}$, the elements of $\mathsf{G}(J)$ (and then of $\mathsf{G}'(J)$) are the leading terms of some generating polynomials of I. While reducing a term $w \in \mathcal{T}$ with respect to I, reduction of it is allowed only by the polynomial whose leading term contains w in its cone.

The notion and formal definition of involutive division has been provided by Gerdt and Blinkov (1998a,b, 2011), who employed it for fast computation of Groebner bases and for solving partial differential equations.

Bar Codes (Ceria 2019d,e) are diagrams representing finite sets of terms; in particular, if the set is the finite Groebner escalier of a zerodimensional ideal, the Bar Code allows to desume many properties of the aforementioned ideal. For example, Ceria (2019d) used Bar Codes to state a bijection between zerodimensional (strongly) stable ideals in two or three variables and some particular partitions of their (constant) affine Hilbert polynomial. On the other hand, Ceria and Mora (2018) defined an efficient iterative algorithm to compute the finite Groebner escalier of the vanishing ideal of a finite set of points by means of Bar Codes. Ceria (2019a) also showed that Bar Codes allow to compute Pommaret bases and their "Axis of Evil" factorization (Ceria 2014) for zerodimensional radical ideals, represented by their (finite) variety.

In this paper, we use Bar Codes in the involutive framework. In particular, given a finite set of terms U, we see how to find the multiplicative variables for the elements in U. A comparison between Bar Codes and the *Janet tree* (Gerdt *et al.* 2001) defined by Gerdt-Blinkov-Yanovich and reformulated by Seiler (2010) is given.

2. Notation

Following the notation of Mora (2003, 2005, 2015, 2016), we indicate with $\mathcal{P} := \mathbf{k}[x_1,...,x_n]$ the ring of polynomials in n variables with coefficients over the field $\mathbf{k}$. The *semigroup of terms*, generated by the set of variables $\{x_1,...,x_n\}$, is defined as

$$\mathcal{T} := \{x^\gamma := x_1^{\gamma_1}\cdots x_n^{\gamma_n} | \gamma := (\gamma_1,...,\gamma_n) \in \mathbb{N}^n\}.$$

Given a term $t = x_1^{\gamma_1}\cdots x_n^{\gamma_n}$ its degree is $\deg(t) = \sum_{i=1}^n \gamma_i$ while, for each $h \in \{1,...,n\}$ $\deg_h(t) := \gamma_h$ is its *h-degree*.

We call *semigroup ordering* on $\mathcal{T}$ a total ordering $<$ such that we have

$$t_1 < t_2 \Rightarrow st_1 < st_2,\ \forall s, t_1, t_2 \in \mathcal{T}.$$

A semigroup ordering that is also a well ordering is called *term ordering*. Given a term ordering $<$ and a polynomial $g \in \mathcal{P}$, we denote by $\mathsf{T}(g)$ its *leading term*, namely its maximal term with respect to the assigned term ordering.

To construct Bar Codes we will use the *lexicographical term ordering* (Lex, for short) induced by the variable ordering $x_1 < ... < x_n$, namely we will set:

$$x_1^{\gamma_1}\cdots x_n^{\gamma_n} < x_1^{\delta_1}\cdots x_n^{\delta_n} \Leftrightarrow \exists j | \gamma_j < \delta_j,\ \gamma_i = \delta_i,\ \forall i > j.$$

We say that a subset $J \subseteq \mathcal{T}$ such that if $t \in J$ then $st \in J$, for each $s \in \mathcal{T}$, is a *semigroup ideal*. A subset $\mathsf{N} \subseteq \mathcal{T}$ such that if $t \in \mathsf{N}$ then $s \in \mathsf{N}$, for each $s|t$, is instead an *order ideal*. It is clear that a subset $\mathsf{N} \subseteq \mathcal{T}$ is an order ideal if and only if the complement $\mathcal{T} \setminus \mathsf{N} = J$ is a semigroup ideal.

The minimal set of generators $\mathsf{G}(J)$ of a semigroup ideal J is called *monomial basis* of J. We define also the following set, associated to J: $\mathsf{N}(J) := \mathcal{T} \setminus J$. For any $G \subset \mathcal{P}$, $\mathsf{T}\{G\} := \{\mathsf{T}(g), g \in G\}$ and $\mathsf{T}(G)$ is the semigroup ideal of leading terms defined as $\mathsf{T}(G) := \{t\mathsf{T}(g) | t \in \mathcal{T}, g \in G\}$.

Fixed a term order $<$, and an ideal $I \triangleleft \mathcal{P}$ the monomial basis of $\mathsf{T}(I) = \mathsf{T}\{I\}$ is named *monomial basis* of I and we denote it again by $\mathsf{G}(I)$. The ideal $In(I) := (\mathsf{T}(I))$ is called *initial ideal* of I, and the order ideal $\mathsf{N}(I) := \mathcal{T} \setminus \mathsf{T}(I)$ is the *Groebner escalier* of I.

3. Recap on Bar Codes

In this section we give an outline of the main definitions and facts on Bar Codes, strongly depending on Ceria (2019d,e). Let us start with the definition of Bar Code.

Definition 1. *A Bar Code* B *is a picture composed by segments, called* bars*, superimposed in horizontal rows, which satisfies conditions a.,b. below. Denote by*

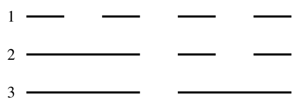
- $\mathsf{B}_j^{(i)}$ *the j-th bar (from left to right) of the i-th row (from top to bottom),* $1 \le i \le n$*, i.e. the* j-th i-bar*;*
- $\mu(i)$ *the number of bars of the i-th row*
- $l_1(\mathsf{B}_j^{(1)}) := 1,\ \forall j \in \{1,2,...,\mu(1)\}$ *the* $1-$length *of the* 1*-bars;*

- $l_i(\mathsf{B}_j^{(k)})$, $2 \leq k \leq n$, $1 \leq i \leq k-1$, $1 \leq j \leq \mu(k)$ *the* i-length *of* $\mathsf{B}_j^{(k)}$*, i.e. the number of i-bars lying over* $\mathsf{B}_j^{(k)}$

a. $\forall i, j,\ 1 \leq i \leq n-1,\ 1 \leq j \leq \mu(i),\ \exists! \overline{j} \in \{1,...,\mu(i+1)\}$ *s.t.* $\mathsf{B}_{\overline{j}}^{(i+1)}$ *lies under* $\mathsf{B}_j^{(i)}$

b. $\forall i_1, i_2 \in \{1,...,n\}$, $\sum_{j_1=1}^{\mu(i_1)} l_1(\mathsf{B}_{j_1}^{(i_1)}) = \sum_{j_2=1}^{\mu(i_2)} l_1(\mathsf{B}_{j_2}^{(i_2)})$*; we will then say that* all the rows have the same length.

Example 2. An example of Bar Code B is



The 1-bars have unitary length. For what concerns the other rows, $l_1(\mathsf{B}_1^{(2)}) = 2$, $l_1(\mathsf{B}_2^{(2)}) = l_1(\mathsf{B}_3^{(2)}) = 1$, $l_2(\mathsf{B}_1^{(3)}) = 1, l_1(\mathsf{B}_1^{(3)}) = 2$ and $l_2(\mathsf{B}_2^{(3)}) = l_1(\mathsf{B}_2^{(3)}) = 2$. Then we have $\sum_{j_1=1}^{\mu(1)} l_1(\mathsf{B}_{j_1}^{(1)}) = \sum_{j_2=1}^{\mu(2)} l_1(\mathsf{B}_{j_2}^{(2)}) = \sum_{j_3=1}^{\mu(3)} l_1(\mathsf{B}_{j_3}^{(3)}) = 4$. ◊

It is possible to associate a Bar code to a finite set of terms, by means of the procedure we describe below (for more details, see Ceria 2019e); an alternative construction is provided by Ceria (2019d).

Given a term $t = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \in \mathcal{T} \subset \mathbf{k}[x_1,...,x_n]$, for each $i \in \{1,...,n\}$, we set $\pi^i(t) := x_i^{\gamma_i} \cdots x_n^{\gamma_n} \in \mathcal{T}$. For a finite set of terms $M \subset \mathcal{T}$, for each $i \in \{1,...,n\}$, we define $M^{[i]} := \pi^i(M) := \{\pi^i(t) | t \in M\}$. We consider $M \subseteq \mathcal{T}$, with $|M| = m < \infty$ and we order its elements in increasing order with respect to the lexicographical ordering, obtaining the list $\overline{M} = [t_1,...,t_m]$. Then, we construct the sets $M^{[i]}$, and the corresponding lists[1] $\overline{M}^{[i]}$, for $i = 1,...,n$, ordered w.r.t. Lex.

We define the $n \times m$ matrix of terms $\mathcal{M}$ such that, for $i = 1,...,n$, $\overline{M}^{[i]}$ is its i-th row, namely the matrix $\mathcal{M} = (\pi_i(t_j))_{1 \leq i \leq n, 1 \leq j \leq m}$. We are ready to define the Bar Code diagram associated to M, which is a Bar Code in the sense of Definition 1.

Definition 3. *The* Bar Code diagram B *associated to M (or, equivalently, to* $\overline{M}$*) is a* $n \times m$ *diagram, made by segments s.t. the i-th row of* B, $1 \leq i \leq n$ *is constructed as follows:*

(1) *take the i-th row of* $\mathcal{M}$*, i.e.* $\overline{M}^{[i]}$
(2) *consider all the sublists of repeated terms, i.e.* $[\pi^i(t_{j_1}), \pi^i(t_{j_1+1}), ..., \pi^i(t_{j_1+h})]$ *s.t.* $\pi^i(t_{j_1}) = \pi^i(t_{j_1+1}) = ... = \pi^i(t_{j_1+h})$*, noticing that*[2] $0 \leq h < m$
(3) *underline each sublist with a segment*
(4) *delete the terms of* $\overline{M}^{[i]}$*, leaving only the segments (i.e. the* i-bars*).*

We usually label each 1*-bar* $\mathsf{B}_j^{(1)}$*,* $j \in \{1,...,\mu(1)\}$ *with the term* $t_j \in \overline{M}$*.*

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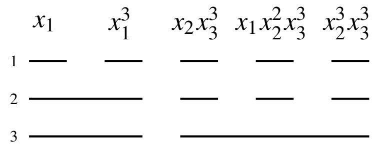
[1] $\overline{M}$ contains only distinct elements, while there may be repetitions in the sets $\overline{M}^{[i]}$, for $1 < i \leq n$. In case some repeated terms occur in $\overline{M}^{[i]}$, $1 < i \leq n$, they clearly need to be consecutive in the list, due to the imposed lexicographical ordering.

[2] Clearly if a term $\pi^i(t_{\overline{j}})$ is not repeated in the list $\overline{M}^{[i]}$, the sublist containing it will be $[\pi_i(t_{\overline{j}})]$, namely we will have $h = 0$.

Example 4. Given $M = \{x_1, x_1^3, x_2x_3^3, x_1x_2^2x_3^3, x_2^3x_3^3\} \subset \mathbf{k}[x_1, x_2, x_3]$, we display the table on the left and the Bar Code on the right:

x_1	x_1^3	$x_2x_3^3$	$x_1x_2^2x_3^3$	$x_2^3x_3^3$
1	1	$x_2x_3^3$	$x_2^2x_3^3$	$x_2^3x_3^3$
1	1	x_3^3	x_3^3	x_3^3



$\diamond$

Now we see a procedure to associate a finite set $M_{\mathsf{B}} \subset \mathcal{T}$ to a Bar Code B. We already gave a more general procedure to do so (Ceria 2019d) and now we specialize it in order to have a *unique* set of terms for each Bar Code. Here we give only the specialized version, namely we follow the two steps $\mathfrak{B}1$ and $\mathfrak{B}2$ below:

$\mathfrak{B}1$ take the n-th row, composed by the bars $B_1^{(n)}, ..., B_{\mu(n)}^{(n)}$. Let $l_1(B_j^{(n)}) = \ell_j^{(n)}$, for $j \in \{1, ..., \mu(n)\}$. Label each bar $B_j^{(n)}$ with $\ell_j^{(n)}$ copies of x_n^{j-1}.

$\mathfrak{B}2$ For each $i = 1, ..., n-1$, $1 \leq j \leq \mu(n-i+1)$ consider the bar $B_j^{(n-i+1)}$ and suppose that it has been labelled by $\ell_j^{(n-i+1)}$ copies of a term t. Consider all the $(n-i)$-bars $B_{\overline{j}}^{(n-i)}, ..., B_{\overline{j}+h}^{(n-i)}$ lying immediately above $B_j^{(n-i+1)}$; note that h satisfies $0 \leq h \leq \mu(n-i) - \overline{j}$. Denote the 1-lengths of $B_{\overline{j}}^{(n-i)}, ..., B_{\overline{j}+h}^{(n-i)}$ by $l_1(B_{\overline{j}}^{(n-i)}) = \ell_{\overline{j}}^{(n-i)}$,..., $l_1(B_{\overline{j}+h}^{(n-i)}) = \ell_{\overline{j}+h}^{(n-i)}$. For each $0 \leq k \leq h$, label $B_{\overline{j}+k}^{(n-i)}$ with $\ell_{\overline{j}+k}^{(n-i)}$ copies of tx_{n-i}^k.

Definition 5. *A Bar Code* B *is called* admissible *if the set M obtained by applying* $\mathfrak{B}1$ *and* $\mathfrak{B}2$ *to* B *is an order ideal.*

Using $\mathfrak{B}1$ and $\mathfrak{B}2$ is the only way to associate an order ideal to an admissible Bar Code, by definition of order ideal.

Definition 6. *Given a Bar Code* B*, let us consider a* 1*-bar* $B_{j_1}^{(1)}$*, with* $j_1 \in \{1, ..., \mu(1)\}$*. The* e-list *associated to* $B_{j_1}^{(1)}$ *is the n-tuple* $e(B_{j_1}^{(1)}) := (b_{j_1,n}, ..., b_{j_1,1})$*, defined as follows:*

- *consider the n-bar* $B_{j_n}^{(n)}$*, lying under* $B_{j_1}^{(1)}$*. The number of n-bars on the left of* $B_{j_n}^{(n)}$ *is* $b_{j_1,n}$*.*
- *for each* $i = 1, ..., n-1$*, let* $B_{j_{n-i+1}}^{(n-i+1)}$ *and* $B_{j_{n-i}}^{(n-i)}$ *be the* $(n-i+1)$*-bar and the* $(n-i)$*-bar lying under* $B_{j_1}^{(1)}$*. Consider the* $(n-i+1)$*-block associated to* $B_{j_{n-i+1}}^{(n-i+1)}$*, i.e.* $B_{j_{n-i+1}}^{(n-i+1)}$ *and all the bars lying over it. The number of* $(n-i)$*-bars of the block, which lie on the left of* $B_{j_{n-i}}^{(n-i)}$ *is* $b_{j_1,n-i}$*.*

Remark 7. Given a Bar Code B, we take a 1-bar $B_j^{(1)}$, with $j \in \{1, ..., \mu(1)\}$.
Looking at Definition 6 and at the two steps $\mathfrak{B}1$ and $\mathfrak{B}2$, we can see that the entries of the e-list $e(B_j^{(1)}) := (b_{j,n}, ..., b_{j,1})$ are equal to the exponents of the term labelling $B_j^{(1)}$, obtained by means of $\mathfrak{B}1$ and $\mathfrak{B}2$ applied to B (compare Example 4).

Proposition 8 (Admissibility criterion)**.** *(see Ceria 2019d, Proposition 6) A Bar Code* B *is admissible if and only if, for each* 1*-bar* $\mathsf{B}_j^{(1)}$*,* $j \in \{1, ..., \mu(1)\}$*, the e-list* $e(\mathsf{B}_j^{(1)}) = (b_{j,n}, ..., b_{j,1})$

satisfies the following condition:

$$\forall k \in \{1,\ldots,n\} \text{ s.t. } b_{j,k} > 0, \exists \overline{j} \in \{1,\ldots,\mu(1)\} \setminus \{j\} \text{ s.t. }$$
$$e(\mathsf{B}^{(1)}_{\overline{j}}) = (b_{j,n},\ldots,b_{j,k+1},(b_{j,k})-1,b_{j,k-1},\ldots,b_{j,1}).$$

□

Until now, we focused on the correspondence between Bar Codes and Groebner escaliers of monomial ideals. We show now that, given an admissible Bar Code B and the associated order ideal N, a particular generating set for the monomial ideal I s.t. $\mathsf{N}(I) = \mathsf{N}$ can be deduced.

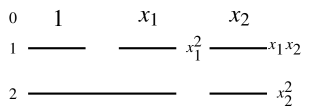
Definition 9. *The* star set *of an order ideal* N *and of its associated Bar Code* B *is a set* $\mathcal{F}_{\mathsf{N}}$ *constructed as follows:*

a) $\forall 1 \le i \le n$, *let* t_i *be a term which labels a* 1*-bar lying over* $\mathsf{B}^{(i)}_{\mu(i)}$*, then* $x_i\pi^i(t_i) \in \mathcal{F}_{\mathsf{N}}$*;*

b) $\forall 1 \le i \le n-1$, $\forall 1 \le j \le \mu(i)-1$ *let* $\mathsf{B}^{(i)}_j$ *and* $\mathsf{B}^{(i)}_{j+1}$ *be two consecutive bars not lying over the same* $(i+1)$*-bar and let* $t^{(i)}_j$ *be a term which labels a* 1*-bar lying over* $\mathsf{B}^{(i)}_j$*, then* $x_i\pi^i(t^{(i)}_j) \in \mathcal{F}_{\mathsf{N}}$.

We can display $\mathcal{F}_{\mathsf{N}}$ within the associated Bar Code B; it is enough to insert every $t \in \mathcal{F}_{\mathsf{N}}$ on the right of the bar from which it is desumed. Reading the terms from left to right and from the top to the bottom, means reading $\mathcal{F}_{\mathsf{N}}$ ordered with respect to Lex.

Example 10.
For $\mathsf{N} = \{1, x_1, x_2\} \subset \mathbf{k}[x_1, x_2]$, we have $\mathcal{F}_{\mathsf{N}} = \{x_1^2, x_1x_2, x_2^2\}$; in relation with Definition 9, we can observe that the terms x_1x_2, x_2^2 come from part a), while the term x_1^2 comes from part b).



◇

Given a monomial ideal I, Ceria *et al.* (2015) define the *star set*:

$$\mathcal{F}(I) = \left\{ x^\gamma \in \mathcal{T} \setminus \mathsf{N}(I) \,\middle|\, \frac{x^\gamma}{\min(x^\gamma)} \in \mathsf{N}(I) \right\},$$

where $\min(x^\gamma)$ is the minimal variable appearing with nonzero exponent in x^γ

Proposition 11. *(Ceria 2019d, Proposition 21) With the above notation* $\mathcal{F}_{\mathsf{N}} = \mathcal{F}(I)$.

There is a very strong connection between the star set $\mathcal{F}(I)$ of a monomial ideal I and Janet's theory (Janet 1920, 1924, 1927, 1929), and to the notion of Pommaret basis (Pommaret 1978; Pommaret and Haddak 1991; Seiler 2010). Such a relation was explicitly addressed by Ceria *et al.* (2015). In particular, for some monomial ideals called quasi-stable ideals, the star set is finite and coincides with the Pommaret basis.

4. Janet decomposition

Given a monomial/semigroup ideal $J \subset \mathcal{T}$ and its monomial basis $\mathsf{G}(J)$, Janet (1920) introduced both the notion of *multiplicative variables* and the connected decomposition of J into disjoint *cones*. In accordance to definition of involutive division (Gerdt and Blinkov 1998a), the involutive cones can be either disjoined or embedded.

Definition 12. *(Janet 1920, pp.75-9) Let $U \subset \mathcal{T}$ be a set of terms and $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ be an element of U. A variable x_j is called* multiplicative *for t with respect to U if there is no term in U of the form $t' = x_1^{\beta_1} \cdots x_j^{\beta_j} x_{j+1}^{\alpha_{j+1}} \cdots x_n^{\alpha_n}$ with $\beta_j > \alpha_j$. We denote by $M(t, U)$ the set of Janet multiplicative variables for t with respect to U.*
The variables that are not multiplicative for t w.r.t. U are called non-multiplicative *and we denote by $NM(t, U)$ the set containing them.*

It is clear that the above definition depends on the order set on the variables.

Example 13. Consider the set $U = \{x_1, x_2\} \subset \mathbf{k}[x_1, x_2]$. If $x_1 < x_2$, then $M(x_1, U) = \{x_1\}$, $NM(x_1, U) = \{x_2\}$, $M(x_2, U) = \{x_1, x_2\}$, $NM(x_2, U) = \emptyset$. If, instead $x_2 < x_1$, then $M(x_1, U) = \{x_1, x_2\}$, $NM(x_1, U) = \emptyset$, $M(x_2, U) = \{x_2\}$, $NM(x_2, U) = \{x_1\}$. ◊

Definition 14. *With the previous notation, the* cone *of t with respect to U is defined as the set*

$$C_J(t, U) := \{t x_1^{\lambda_1} \cdots x_n^{\lambda_n} \,|\, \textit{where } \lambda_j \neq 0 \textit{ only if } x_j \textit{ is multiplicative for } t \textit{ w.r.t. } U\}.$$

Example 15. Consider the set $J = \{x_1^3, x_2^3, x_1^4 x_2 x_3, x_3^2\} \subseteq \mathbf{k}[x_1, x_2, x_3]$; suppose $x_1 < x_2 < x_3$. Let $t = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} = x_1^3$, so $\alpha_1 = 3$, $\alpha_2 = \alpha_3 = 0$. The variable x_1 is multiplicative for t w.r.t J since there are no terms $t' = x_1^{\beta_1} x_2^{\beta_2} x_3^{\beta_3} \in J$ satisfying both conditions:

- $\beta_1 > 3$;
- $\beta_2 = \beta_3 = 0$.

On the other hand, x_2 is not multiplicative for t since $t'' = x_2^3 \in U$ satisfies $t'' = x_1^{\gamma_1} x_2^{\gamma_2} x_3^{\gamma_3}$ with $\gamma_2 = 3 > 0 = \alpha_2$, $\gamma_3 = \alpha_3 = 0$. Similarly, x_3 is not multiplicative since $x_3^2 \in U$. In conclusion, we have $M(t, U) = \{x_1\}$, $NM(t, U) = \{x_2, x_3\}$; $C_J(t, U) = \{x_1^h | h \in \mathbb{N}, h \geq 3\}$. ◊

Remark 16. Notice that, by definition of multiplicative variable, the only element in the intersection $C_J(t, U) \cap U$ is t itself. Indeed, if $t \in U$ and also $ts \in U$ for a non constant term s, then $\max(s)$ cannot be multiplicative for t, hence. $ts \notin C_J(t, U)$.

Janet introduced then the concept of *complete system* and gave a procedure, called *completion*, to find the decomposition in cones.

Definition 17. *(Janet 1920, pp.75-9) A set of terms $U \subset \mathcal{T}$ is called* complete *if for every $t \in U$ and $x_j \in NM(t, U)$, there exists $t' \in U$ such that $x_j t \in C_J(t', U)$. The term t' is called* involutive divisor *of $x_j t$ w.r.t. Janet division.*

Since the notion of completeness depends on that of multiplicative variable, both of them depend on the variables' ordering.

Remark 18. If $U = \{t\} \subseteq \mathbf{k}[x_1, ..., x_n]$ has cardinality 1, then it is complete, since $M(t, u) = \{x_1, ..., x_n\}$.

In the same paper (Janet 1920), with the aim of describing Riquier's formulation (Riquier 1910) of the description for the general solutions of a PDE problem, Janet gave a similar decomposition in terms of disjoint cones, generated by multiplicative variables, also for the related normal set/order ideal/*escalier* $\mathbf{N}(J)$.

For each term t of a finite set $U \subset \mathcal{T}$ it is easy to assign its Janet multiplicative variables (see Definition 12) by means of the Bar Code associated to U.

Suppose $x_1 < x_2 < ... < x_n$ and consider a finite set $U \subset \mathcal{T} \subset \mathbf{k}[x_1, ..., x_n]$. It is always possible to associate a Bar Code B to U. Once B is constructed (even if it is not necessary that B is an admissible Bar Code) we can mimick on it the set up we generally perform to construct the star set. In particular:

a) $\forall 1 \leq i \leq n$, put a star symbol $*$ on the right of the bar $\mathsf{B}^{(i)}_{\mu(i)}$;
b) $\forall 1 \leq i \leq n-1$, $\forall 1 \leq j \leq \mu(i)-1$ let $\mathsf{B}^{(i)}_j$ and $\mathsf{B}^{(i)}_{j+1}$ be two consecutive bars not lying over the same $(i+1)$-bar; put a star symbol $*$ between these two bars.

Now, given a term $t \in U$, to detect its multiplicative variables it is enough to check the bars over which it lies, as stated in the following proposition.

Proposition 19. *Let $U \subseteq \mathcal{T}$ be a finite set of terms and let us denote by B_U its Bar Code. For each $t \in U$ x_i, $1 \leq i \leq n$ is multiplicative for t if and only if the i-bar $\mathsf{B}^{(i)}_j$ of B_U, over which t lies, is followed by a star.*

Proof. "$\Leftarrow$"
Let $t = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in U$ and $\mathsf{B}^{(i)}_j$ the i-bar of B_U under t.
Suppose that there is a star placed just on the right of $\mathsf{B}^{(i)}_j$: we have to prove that $x_i \in M(t, U)$.
Suppose first $i = n$; if there is $s \in U$, with $\deg_n(s) > \alpha_n$, then s should lie over $\mathsf{B}^{(n)}_k$, for some integer $k > j$, and so there could be no stars after $\mathsf{B}^{(n)}_j$, contradicting the hypothesis. Then, such an s cannot exist and we have $x_n \in M(t, U)$.
Let now $i < n$: if $j = \mu(i)$ and if there is $s \in U$, $\deg_i(s) > \alpha_i$, $\deg_h(s) = \alpha_h$ for $i+1 \leq h \leq n$, then this would lie over an i-bar on the right of $\mathsf{B}^{(i)}_j = \mathsf{B}^{(i)}_{\mu(i)}$. This contradicts the maximality of $\mu(i)$, so there cannot exist such a term and $x_i \in M(t, U)$.
Let otherwise $j < \mu(i)$ and let t be the term that lies over $\mathsf{B}^{(i)}_j$ and we denote by $\mathsf{B}^{(i+1)}_{j'}$ the $(i+1)$-bar under it. After $\mathsf{B}^{(i)}_j$ there is a star, so $\mathsf{B}^{(i)}_{j+1}$ must be over $\mathsf{B}^{(i+1)}_{j'+1}$.
Now, if $x_i \in NM(t, U)$, then there is a term $s = x_1^{\beta_1} \cdots x_i^{\beta_i} x_{i+1}^{\alpha_{i+1}} \cdots x_n^{\alpha_n} \in U$ such that $\beta_i > \alpha_i$. Since $\deg_l(s) = \deg_l(t)$ for $l = i+1, ..., n$, then s would have to lie over $\mathsf{B}^{(i+1)}_{j'}$, but since $\deg_i(s) > \deg_i(t)$ s should also lie over an i-bar on the right of $\mathsf{B}^{(i)}_j$, which is impossible. We can conclude that $x_i \in M(t, U)$.
"$\Rightarrow$"
Let $t \in U$, $x_i \in M(t, U)$ and $\mathsf{B}^{(i)}_j$ the i-bar under t. We prove that there must be necessarily a star after $\mathsf{B}^{(i)}_j$.
If $i = n$, being $x_n \in M(t, U)$, there are no terms $s \in U$ s.t. $\deg_n(s) > \deg_n(t)$, i.e. $\alpha_n = \max\{\deg_n(u) : u \in U\}$. This implies $j = \mu(n)$, so after $\mathsf{B}^{(n)}_j$ there is a star.
If $i < n$, let $\mathsf{B}^{(i+1)}_{j'}$ be the $(i+1)$-bar under $\mathsf{B}^{(i)}_j$; if, by contradiction, $\mathsf{B}^{(i)}_j$ is not followed by

a star, also $\mathsf{B}^{(i)}_{j+1}$ would be over $\mathsf{B}^{(i+1)}_{j'}$. Now, any term $s \in U$, lying over $\mathsf{B}^{(i)}_{j+1}$, would be s.t. $\deg_{i+1}(s) = \alpha_{i+1}$,..., $\deg_n(s) = \alpha_n$ and $\deg_i(s) > \alpha_i$, so the existence of s would make x_i non-multiplicative for t, which is impossible. □

Example 20. For the set $U = \{x_1^3, x_2^3, x_1^4x_2x_3, x_3^2\} \subseteq \mathbf{k}[x_1, x_2, x_3]$, $x_1 < x_2 < x_3$, of example 15, we have the following Bar Code

$$\begin{array}{lcccc}
0 & x_1^3 & x_2^3 & x_1^4x_2x_3 & x_3^2 \\
1 & \text{—— } * & \text{—— } * & \text{—— } * & \text{—— } * \\
2 & \text{——} & \text{—— } * & \text{—— } * & \text{—— } * \\
3 & \multicolumn{2}{c}{\text{————————}} & \text{——} & \text{—— } *
\end{array}$$

Then, looking at the stars, we can desume that:

- $M(x_1^3, U) = \{x_1\}$, $NM(x_1^3, U) = \{x_2, x_3\}$;
- $M(x_2^3, U) = \{x_1x_2\}$, $NM(x_2^3, U) = \{x_3\}$;
- $M(x_1^4x_2x_3, U) = \{x_1, x_2\}$, $NM(x_1^4x_2x_3, U) = \{x_3\}$;
- $M(x_3^2, U) = \{x_1, x_2, x_3\}$, $NM(x_3^2, U) = \emptyset$.

and actually this configuration is in accordance with Janet's definition. Indeed

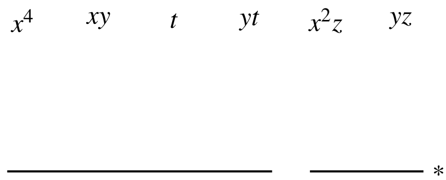
- $M(x_1^3, U) = \{x_1\}$: no terms with $x_2^0x_3^0$ have x_1-degree greater than 3. Since $x_2^3, x_3^2 \in U$, $x_2, x_3 \in NM(x_1^3, U)$;
- $M(x_2^3, U) = \{x_1, x_2\}$: no terms with $x_2^3x_3^0$ have x_1-degree greater than 0, nor terms with x_3^0 have x_2-degree greater than 3. Since $x_3^2 \in U$, $x_3 \in NM(x_2^3, U)$;
- $M(x_1^4x_2x_3, U) = \{x_1, x_2\}$: no terms with x_2x_3 have x_1-degree greater than 4, nor terms with x_3 have x_2-degree greater than 1. Since $x_3^2 \in U$, $x_3 \in NM(x_1^4x_2x_3, U)$;
- $M(x_3^2, U) = \{x_1, x_2, x_3\}$: neither terms with $x_2^0x_3^2$ have x_1-degree greater than 0, nor terms with x_3^2 have x_2-degree greater than 0. There are no terms with x_3-degree greater than 2.

◇

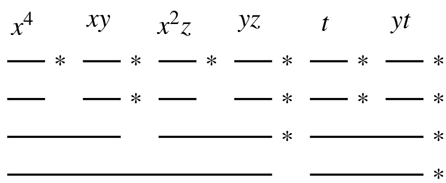
Proposition 19, first proved in this paper, has been used in two papers, where it is stated without proof. This proposition was used by Ceria (2019b), together with the definition of complete set, to desume an algorithm that, given a finite set $U \subset \mathcal{T}$, computes - if it exists - an ordering on the variables $x_1, ..., x_n$ such that U turns out to be Janet complete. If such an ordering does not exist, then it returns an error. The algorithm is defined as *greedy* since it uses *backtracking techniques* to avoid the trial and error procedure, which would have needed to compute all $n!$ Bar Codes. This proposition was also stated and applied by Ceria (2019c) for studying a particular division, called Janet-like division.

Example 21. For the set $U = \{x^4, xy, x^2z, yz, t, yt\} \subset \mathbf{k}[x, y, z, t]$, our greedy algorithm first observes that x cannot be the maximal variable since it appears with non-consecutive exponents in U (i.e. $0, 1, 2, 4$), so x would be a nonmultiplicative variable for x^2z, and there would no potential involutive divisor in U for x^3z.

The other variables are good candidates for being the maximal variable, so we try with z, getting:



In this case, z is nonmultiplicative for $t \in U$ and there is no divisor of zt among x^2z, yz, so z is a bad choice for the maximal variable. Choosing t and then continuing with z, y, x, we get the following Bar Code, which proves that U is complete w.r.t. $x < y < z < t$:



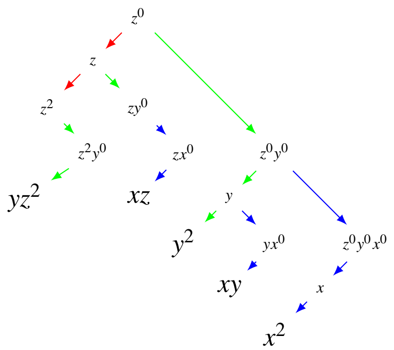
◇

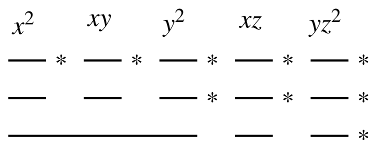
5. Bar Code and Janet tree

The Bar Code we are using to detect multiplicative variables is a reformulation of Gerdt-Blinkov-Yanovich *Janet tree* (Gerdt *et al.* 2001), but in the (equivalent) presentation given by Seiler (2010). However, given a finite set of terms, the algorithms for producing its Janet decomposition which can be deduced from both the formulations above of the Janet tree, are different from the algorithm naturally arising from the previous Proposition 19.

The Gerdt-Blinkov-Yanovich *Janet tree* (Gerdt *et al.* 2001) is a *binary tree* representing the structure of a finite set of terms $U = \{t_1, ..., t_m\}$. The root represents the term 1, whereas the leaves represent the terms $t_1, ..., t_m$. The term $t_j x_i$ (increased by one degree of the current variable x_i) is assigned to the left child whereas the right child points at the next variable with respect to chosen ordering. In this representation, assigning multiplicative variables is done by walking on the tree from the root to the leaves. In particular, consider the path corresponding to the term $t_i \in U$. When walking on it, every time we move from a node w to the direction a new variable x_j, we have to ask us whether in the graph there are still arrows from w in the previous variable x_{j+1}, which do not belong to the path of t_i. If so, $x_{j+1} \in NM(t_i, U)$; otherwise $x_{j+1} \in M(t_i, U)$.

Example 22. For the set $U = \{z^2y, xz, y^2, xy, x^2\} \subset \mathcal{T}$, $x < y < z$, we display the Janet tree, the Bar Code and the table of multiplicative variables:

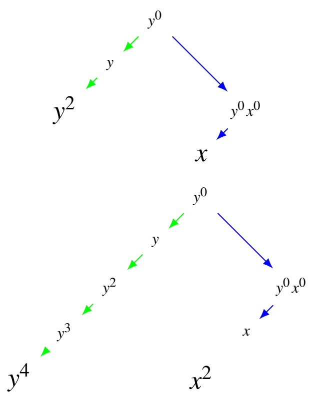


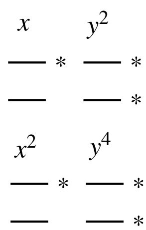


t	$M_L(t,U)$
x^2	x
xy	x
y^2	x,y
xz	x,y
yz^2	x,y,z

$\diamond$

We need to remark that there is a big difference between the Janet tree and the Bar Code representation defined here, namely that the Bar Code is *independent on the degree of the monomials*, in the sense that, as an example, for $M_1 = \{x, y^2\}$, $M_2 = \{x^2, y^4\}$ the Bar Code is the same, while the Janet tree increases its size with the degree of the terms in the given set.





It is true that in practical cases Janet-like divisions (Gerdt and Blinkov 2005a,b) are used in the case of high-degree sets M, but we remark that Bar Codes can simply deal also with that case (Ceria 2019c).

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