Bar code: a visual representation for finite sets of terms and its applications

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Mathematics Subject Classification (2010). Primary 05E40, Secondary 13P10.

Keywords. Bar Code, monomial ideals, Groebner escalier.

Abstract. The Bar Code is a bidimensional diagram representing a finite set of terms in any number of variables. In particular, one can represent the (lexicographical) Groebner escalier of a zerodimensional monomial ideal and use this representation to desume many of its properties. The aim of this paper is to give a general description of the Bar Code and it construction, giving then an overview of all the applications studied so far.

1. Introduction

Representing in diagrams zerodimensional monomial ideals is a useful tool to increase the intuition on the properties of that ideals. The first paper presenting such kind of representation is due to Galligo [18]. In the book [35], Miller and Sturmfels introduce the *staircase diagram* for the case of 2 and 3 variables, as a tool to subsume the structure of the considered ideals. In particular, they are able to compute resolutions "by pictures". Such a representation, which is very useful for the case of 2 and 3 variables, cannot be applied to "bigger dimensions" (see section 3), since no-one can draw pictures in dimension four or even bigger.

In this paper we show how to extend such kind of representation to any number of variables (section 4) with the Bar Code, a *bidimensional diagram* which encodes the properties of zerodimensional ideals in $n \geq 1$ variables. From a Bar Code, one can deduce the star set (see section 5), which is the Pommaret basis of the represented zerodimensional ideal. Finally in section 6, we will see some applications of Bar Codes, both in the involutive (section 6.2) and in the non-involutive context (section 6.1).

2. Notation

Throughout this paper we mainly follow the notation of [37], for what concerns monomial ideals. We denote by $\mathcal{P} := \mathbf{k}[x_1, ..., x_n]$ the graded ring of polynomials in n variables with coefficients in the field $\mathbf{k}$.

The *semigroup of terms*, generated by $\{x_1, ..., x_n\}$ is $\mathcal{T} := \{x^\gamma := x_1^{\gamma_1} \cdots x_n^{\gamma_n} | \gamma := (\gamma_1, ..., \gamma_n) \in \mathbb{N}^n\}$. If $\tau = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, then $\deg(\tau) = \sum_{i=1}^n \gamma_i$ is the *degree* of τ and, for each $h \in \{1, ..., n\}$ $\deg_h(\tau) := \gamma_h$ is the *h-degree* of τ. The symbol $\mathcal{T}_{\leq d}$ denotes the degree $\leq d$ part of $\mathcal{T}$, namely $\mathcal{T}_{\leq d} = \{x^\gamma \in \mathcal{T} | \deg(x^\gamma) \leq d\}$. Analogously, $\mathcal{P}_{\leq d}$ denotes the degree $\leq d$ part of $\mathcal{P}$ and, given an ideal I of $\mathcal{P}$, $I_{\leq d}$ is its degree $\leq d$ part, i.e. $I_{\leq d} = I \cap \mathcal{P}_{\leq d}$. For each term $\tau \in \mathcal{T}$ and $x_j | \tau$, the only $\upsilon \in \mathcal{T}$ such that $\tau = x_j \upsilon$ is called j-th *predecessor* of τ.

A *semigroup ordering* $<$ on $\mathcal{T}$ is a total ordering such that $\tau_1 < \tau_2 \Rightarrow \tau\tau_1 < \tau\tau_2, \forall \tau, \tau_1, \tau_2 \in \mathcal{T}$. For each semigroup ordering $<$ on $\mathcal{T}$, we can represent a polynomial $f \in \mathcal{P}$ as a linear combination of terms arranged w.r.t. $<$, with coefficients in the base field $\mathbf{k}$: $f = \sum_{\tau \in \mathcal{T}} c(f, \tau)\tau = \sum_{i=1}^s c(f, \tau_i)\tau_i : c(f, \tau_i) \in \mathbf{k}^*, \tau_i \in \mathcal{T}, \tau_1 > ... > \tau_s$, with $\mathsf{T}(f) := \tau_1$ the *leading term* of f. A *term ordering* is a semigroup ordering such that 1 is lower than every variable or, equivalently, it is a *well ordering*. Unless otherwise specified, we consider the *lexicographical ordering* induced by $x_1 < ... < x_n$, i.e: $x_1^{\gamma_1} \cdots x_n^{\gamma_n} <_{Lex} x_1^{\delta_1} \cdots x_n^{\delta_n} \Leftrightarrow \exists j | \gamma_j < \delta_j, \gamma_i = \delta_i, \forall i > j$, which is a term

ordering. Given a term $\tau \in \mathcal{T}$, $\min(\tau)$ is the smallest variable dividing τ. For $M \subset \mathcal{T}$, $\overline{M}$ is the list obtained by ordering the elements of M increasingly w.r.t. Lex. For example, if $M = \{x_2, x_1^2\} \subset \mathbf{k}[x_1, x_2]$, $x_1 < x_2$, $\overline{M} = [x_1^2, x_2]$.
A subset $J \subseteq \mathcal{T}$ is a *semigroup ideal* if $\tau \in J \Rightarrow \sigma\tau \in J, \forall \sigma \in \mathcal{T}$; a subset $\mathsf{N} \subseteq \mathcal{T}$ is an *order ideal* if $\tau \in \mathsf{N} \Rightarrow \sigma \in \mathsf{N}\, \forall \sigma | \tau$. We have that $\mathsf{N} \subseteq \mathcal{T}$ is an order ideal if and only if $\mathcal{T} \setminus \mathsf{N} = J$ is a semigroup ideal. Given a semigroup ideal $J \subset \mathcal{T}$ we define $\mathsf{N}(J) := \mathcal{T} \setminus J$. The minimal set of generators $\mathsf{G}(J)$ of J, called the *monomial basis* of J, satisfies the conditions below

$$\mathsf{G}(J) := \{\tau \in J \,|\, \text{ each predecessor of } \tau \in \mathsf{N}(J)\} = \{\tau \in \mathcal{T} \,|\, \mathsf{N}(J) \cup \{\tau\} \text{is an order ideal}, \tau \notin \mathsf{N}(J)\}.$$

For all subsets $G \subset \mathcal{P}$, we have $\mathsf{T}\{G\} := \{\mathsf{T}(g),\ g \in G\}$ and $\mathsf{T}(G) := \{\tau\mathsf{T}(g),\ \tau \in \mathcal{T}, g \in G\}$. Fixed a term order $<$, for any ideal $I \triangleleft \mathcal{P}$ the monomial basis of the semigroup ideal $\mathsf{T}(I) = \mathsf{T}\{I\}$ is called *monomial basis* of I and denoted again by $\mathsf{G}(I)$, whereas the order ideal $\mathsf{N}(I) := \mathcal{T} \setminus \mathsf{T}(I)$ is called *Groebner escalier* of I.

Definition 1. [37, Vol. 2] *A set $G \subset I$ is a* Groebner basis *of I wrt $<$ if $T(G) = T\{I\}$.*

If $I \triangleleft \mathcal{P}$ is an ideal, we define its associated *variety* as $V(I) = \{P \in \overline{\mathbf{k}}^n,\ f(P) = 0,\ \forall f \in \mathcal{I}\}$, where $\overline{\mathbf{k}}$ is the algebraic closure of $\mathbf{k}$.

Definition 2. *The* affine Hilbert function *of an ideal $I \triangleleft \mathcal{P}$ is the function $HF_I : \mathbb{N} \to \mathbb{N};\ d \mapsto dim(\mathcal{P}_{\leq d}/I_{\leq d})$.*

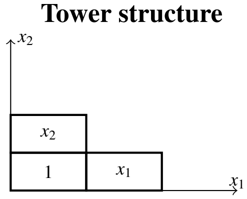
For d sufficiently large, the affine Hilbert function of I can be written as:$HF_I(d) = \sum_{i=0}^{l} b_i \binom{d}{l-i}$, where l is the Krull dimension of $V(I)$, b_i are integers and b_0 is positive.

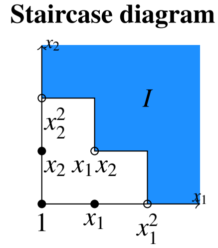
Definition 3. *The polynomial which is equal to $HF_I(d)$, for d sufficiently large, is called the* affine Hilbert polynomial *of I and denoted $H_I(d)$.*

3. Representing zerodimensional monomial ideals

In this section, we see how zerodimensional monomial ideals are represented in literature, before giving, in the next section, the definition and the construction of Bar Codes, where we encode finite sets of terms. In particular, we usually represent the Groebner escalier $\mathsf{N}(I)$ of a zerodimensional monomial ideal I, automatically desuming some of its properties. First of all, we point out that, since $\mathcal{T} \cong \mathbb{N}^n$, a term $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ can be regarded as the point $(\gamma_1, ..., \gamma_n)$ in the n-dimensional space. For the case of two and three variables, Miller and Sturmfels [35], introduced the *staircase diagram*, in which each element of $\mathcal{T}$ is drawn as a point in the n-dimensional space ($n = 2, 3$) and much emphasis is given to the escalier. They have been able to prove results on irreducible decompositions, minimal free resolutions and Hilbert series, by means of this picture. An equivalent reformulation of the staircase diagram has been given in [30, 13, 14, 15, 31] and called *tower structure* of I in [5].

Example 4. Consider the radical ideal $I = (x_1^2 - x_1, x_1x_2, x_2^2 - 2x_2) \triangleleft \mathbf{k}[x_1, x_2]$, defined by its lexicographical reduced Groebner basis. Since w.r.t. Lex[1], we have $\mathsf{T}(x_1^2 - x_1) = x_1^2$, $\mathsf{T}(x_1x_2) = x_1x_2$, $\mathsf{T}(x_2^2 - 2x_2) = x_2^2$, we can conclude that the lexicographical Groebner escalier of I is $\mathsf{N}(I) = \{1, x_1, x_2\}$, so it can be represented by the following picture:





◇

[1]Actually, it can be easily observed that $\mathsf{T}(x_1^2 - x_1) = x_1^2$, $\mathsf{T}(x_1x_2) = x_1x_2$, $\mathsf{T}(x_2^2 - 2x_2) = x_2^2$ trivially holds for each term order.

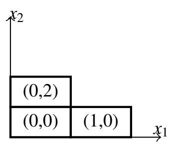
For a radical ideal I, notice that if $|\mathsf{N}(I)| < \infty$, it holds $|V(I)| = |\mathsf{N}(I)| < \infty$, so the associated variety consists of a finite set of points. It has been proved by Cerlienco-Mureddu [13] that, in this case, any ordering on the points in $V(I)$ gives a precise $1-1$ correspondence between the terms in $\mathsf{N}(I)$ and the points in $V(I)$, so it is also possible to label each point in the tower structure with the corresponding point of the (ordered) variety $V(I)$.

Example 5. Consider again the radical ideal $I = (x_1^2 - x_1, x_1x_2, x_2^2 - 2x_2) \triangleleft \mathbf{k}[x_1, x_2]$ of example 4. The corresponding variety can be easily computed and, actually, it is finite: $V(I) = \{(0,0), (0,2), (1,0)\}$. We can also note that, exactly as expected, $|\mathsf{N}(I)| = |V(I)| = 3$. The correspondence given by Cerlienco-Mureddu (see [13] for more details on how the correspondence is constructed) is displayed below; the corresponding reorderings of $V(I)$ are indicated in square brackets:

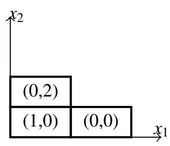
$$\begin{array}{c} \Phi_1 : \mathsf{N}(I) \to V(I) \\ 1 \mapsto (0,0) \\ x_2 \mapsto (0,2) \\ x_1 \mapsto (1,0). \\ [(0,0),(0,2),(1,0)]; \\ [(0,0),(1,0),(0,2)]. \end{array} \quad \begin{array}{c} \Phi_2 : \mathsf{N}(I) \to V(I) \\ 1 \mapsto (1,0) \\ x_2 \mapsto (0,2) \\ x_1 \mapsto (0,0). \\ [(1,0),(0,0),(0,2)]. \end{array} \quad \begin{array}{c} \Phi_3 : \mathsf{N}(I) \to V(I) \\ 1 \mapsto (1,0) \\ x_2 \mapsto (0,0) \\ x_1 \mapsto (0,2). \\ [(1,0),(0,2),(0,0)]. \end{array} \quad \begin{array}{c} \Phi_4 : \mathsf{N}(I) \to V(I) \\ 1 \mapsto (0,2) \\ x_2 \mapsto (0,0) \\ x_1 \mapsto (1,0). \\ [(0,2),(0,0),(1,0)]; \\ [(0,2),(1,0),(0,0)]. \end{array}$$

Now, we can label the points in the tower structure with the corresponding point of $V(I)$, as it can be seen in the pictures below.

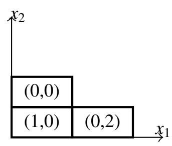
For Φ_1:



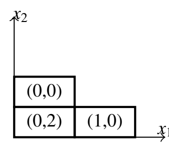
For Φ_2:



For Φ_3:



For Φ_4:

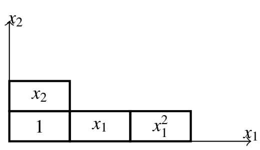


◇

The construction of Examples 4 and 5 is a sort of "inverse" of Macaulay's construction (see [34] p.548) in which from a finite order ideal N, a finite set of point $\mathbf{X}$ and a Groebner basis of $I(\mathbf{X})$ are produced so that the lexicographical Groebner escalier of $I(\mathbf{X})$ is exactly N.

Example 6. For the case in which we have two variables, the tower structure of a zerodimensional radical ideal I s.t. $V(I) = \{P_1, ..., P_s\}$ is represented by h towers, where h is the number of different values appearing as first coordinate of the points in $V(I)$, so that each tower corresponds to a "first coordinate". For each $1 \leq i \leq h$, the i-th tower contains as many elements as the number of occurrences of the associated first coordinate. Displaying these towers in nonincreasing order by height, one obtains a tower structure for I (see the one obtained in example 5 via the maps Φ_1, Φ_4), and so also the escalier $\mathsf{N}(I)$. This is not the case for three or more variables, since some shifts in the towers' planes are needed. For example, given the zerodimensional radical ideal $I = (x_1^2 - x_1, x_1x_2, x_2^2 - x_2, x_1x_3 - x_3, x_2x_3, x_3^2 - x_3) \triangleleft \mathbf{k}[x_1, x_2, x_3]$, whose variety is $V(I) = \{(0,0,0), (0,1,0), (1,0,0), (1,0,1)\}$, we have $\mathsf{N}(I) = \{1, x_1, x_2, x_3\}$, which cannot be represented with a natural extension to three variables of the procedure explained above. In such an extension, the towers are in the x_2 direction if the points have only the same first coordinate and in the x_3 direction if both the first and the second coordinate are the same. ◇

Example 7. Let us consider the zerodimensional radical ideal $I = (x_1^3 - 3x_1^2 + 2x_1, x_1x_2, x_2^2 - 2x_2) \triangleleft \mathbf{k}[x_1, x_2]$, defined by its lexicographical reduced Groebner basis. Since, w.r.t. Lex, $\mathsf{T}(x_1^3 - 3x_1^2 + 2x_1) = x_1^3$, $\mathsf{T}(x_1x_2) = x_1x_2$, $\mathsf{T}(x_2^2 - 2x_2) = x_2^2$, we can conclude that the lexicographical Groebner escalier of I is $\mathsf{N}(I) = \{1, x_1, x_1^2, x_2\}$, so it can be represented with the picture on the left.



Consider now the zerodimensional radical ideal $I' = (x_1^3 - x_1, x_1x_2, x_2^2 - 2x_2, x_3 + x_1^2 - x_1) \triangleleft \mathbf{k}[x_1, x_2, x_3]$, defined via its reduced lexicographical Groebner basis. Since w.r.t. Lex, we have $\mathsf{T}(x_1^3 - x_1) = x_1^3$, $\mathsf{T}(x_1x_2) = x_1x_2$, $\mathsf{T}(x_2^2 - 2x_2) = x_2^2$, $\mathsf{T}(x_3 + x_1^2 - x_1) = x_3$, we can conclude that the lexicographical Groebner escalier of I' is $\mathsf{N}(I') = \{1, x_1, x_1^2, x_2\}$, so it can be represented with the same picture as above and then I and I' have the same tower structure, even if $I' \triangleleft \mathcal{P} = \mathbf{k}[x_1, x_2, x_3]$ and $I \triangleleft \mathbf{k}[x_1, x_2]$. The reason of this fact is that $x_3 \notin \mathsf{N}(I')$; indeed, x_3 is the leading term of $x_3 + x_1^2 - x_1$. In general, the reason is that there is a polynomial $(x_3 - \sum_{t \in \mathsf{N}(I')} c_t t) \in I'$.

In a slightly different situation (i.e. in solving equations) the ability of detecting linear relations $\mod I'$ among the elements of $\{1, x_1, x_2, x_3\}$ and, equivalently, producing a basis of the vector space generated by $\{1, x_1, x_2, x_3\}$, $\mathbf{Span}(1, x_1, x_2, x_3) \mod I'$, is crucial (see [3, 32]). This is the case, for instance of $I'' = (x_1^3 - x_1, x_1x_2, x_2^2 - 2x_2, x_3 - x_1) \triangleleft \mathbf{k}[x_1, x_2, x_3]$, where $\mathbf{Span}(1, x_1, x_2, x_3) = \mathbf{Span}(1, x_1, x_2) \mod I''$ $\diamond$

Unfortunately, as one can easily understand, the tower structure (as well as the staircase diagram) becomes rather complicated when we have an high number of terms in $\mathsf{N}(I)$ and/or of linearly *independent* variables in $\mathcal{P}$, i.e. when we deal with a large number of points, and/or we have really to draw the structure for high-dimensional spaces[2]. Moreover, as shown in example 7, from the tower structure it is impossible to understand the ring in which the Groebner escalier has been computed, since linearly dependent variables are discarded (see [32]). For these reasons, we introduce now the *Bar Code diagram*, namely a (compact) *bidimensional picture* which keeps track of all the information contained in the tower structure, making them simple to extract.

4. Bar Code associated to a finite set of terms

We define now, in general, what is a Bar Code. After that, we see how to associate to a finite set of terms a Bar Code and, vice versa, how to associate a finite set of terms to a given Bar Code.

Definition 8. *A Bar Code* B *is a picture composed by segments, called* bars*, superimposed in horizontal rows, which satisfies conditions a., b. below. Denote by*

- $\mathsf{B}_j^{(i)}$ *the j-th bar (from left to right) of the i-th row (from top to bottom), i.e. the* j-th i-bar*;*
- $\mu(i)$ *the number of bars of the i-th row*
- $l_1(\mathsf{B}_j^{(1)}) := 1, \forall j \in \{1, 2, ..., \mu(1)\}$ *the* $(1-)$length *of the* 1*-bars;*
- $l_i(\mathsf{B}_j^{(k)})$*,* $2 \le k \le n$*,* $1 \le i \le k-1$*,* $1 \le j \le \mu(k)$ *the* i-length *of* $\mathsf{B}_j^{(k)}$*, i.e. the number of i-bars lying over* $\mathsf{B}_j^{(k)}$

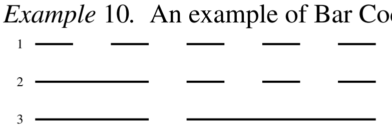
a. $\forall i, j$, $1 \le i \le n-1$, $1 \le j \le \mu(i)$, $\exists! \bar{j} \in \{1, ..., \mu(i+1)\}$ *s.t.* $\mathsf{B}_{\bar{j}}^{(i+1)}$ *lies under* $\mathsf{B}_j^{(i)}$

b. $\forall i_1, i_2 \in \{1, ..., n\}$, $\sum_{j_1=1}^{\mu(i_1)} l_1(\mathsf{B}_{j_1}^{(i_1)}) = \sum_{j_2=1}^{\mu(i_2)} l_1(\mathsf{B}_{j_2}^{(i_2)})$*; we will then say that* all the rows have the same length.

We denote by $\mathcal{B}_n$ the set of all Bar Codes composed by n rows. Note that if $1 \le i_1 < i_2 \le n$, $1 \le j_1 \le \mu(i_1)$, $1 \le j_2 \le \mu(i_2)$ and $\mathsf{B}_{j_2}^{(i_2)}$ lies below $\mathsf{B}_{j_1}^{(i_1)}$, then $l_1(\mathsf{B}_{j_2}^{(i_2)}) \ge l_1(\mathsf{B}_{j_1}^{(i_1)})$.

Definition 9. *We call* bar list *of a Bar Code* B*, composed by n rows, the list* $\mathsf{L}_\mathsf{B} := (\mu(1), ..., \mu(n))$.

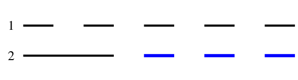
Example 10. An example of Bar Code B is



The 1-bars have length 1. As regards the other rows, $l_1(\mathsf{B}_1^{(2)}) = 2$, $l_1(\mathsf{B}_2^{(2)}) = l_1(\mathsf{B}_3^{(2)}) = l_1(\mathsf{B}_4^{(2)}) = 1$, so $\sum_{j_1=1}^{\mu(1)} l_1(\mathsf{B}_{j_1}^{(1)}) = l_2(\mathsf{B}_1^{(3)}) = 1, l_1(\mathsf{B}_1^{(3)}) = 2$ and $l_2(\mathsf{B}_2^{(3)}) = l_1(\mathsf{B}_2^{(3)}) = 3$, $\sum_{j_2=1}^{\mu(2)} l_1(\mathsf{B}_{j_2}^{(2)}) = \sum_{j_3=1}^{\mu(3)} l_1(\mathsf{B}_{j_3}^{(3)}) = 5$. The bar list is $\mathsf{L}_\mathsf{B} := (5, 4, 2)$. $\diamond$

Definition 11. *Given a Bar Code* B*, for each* $1 \le l \le n$*,* $l \le i \le n$*,* $1 \le j \le \mu(i)$*, an* l-block *associated to a bar* $B_j^{(i)}$ *of* B *is the set containing* $B_j^{(i)}$ *itself and all the bars of the* $(l-1)$ *rows lying immediately above* $B_j^{(i)}$*.*

Example 12. Take again the Bar Code B of example 10. Consider the bar $B_2^{(3)}$ (so $i = n = 3$, $j = 2 = \mu(3)$) and set $l = 2$. The 2-block associated to $B_2^{(3)}$ consists of $B_2^{(3)}$ itself and of the bars $B_2^{(2)}, B_3^{(2)}, B_4^{(2)}$, as shown by the thick (blue) lines in the picture below:



$\diamond$

[2]Actually, in this context, "high-dimensional" means "of dimension greater than or equal to" 4.

We outline now the construction of the Bar Code associated to a finite set of terms. In order to do it, we need to introduce the operators π^i, $i = 1, ..., n$ on the terms. First of all, we associate to each term $\tau = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \in \mathcal{T} \subset \mathbf{k}[x_1, ..., x_n]$, n terms (one for each variable in $\mathcal{P}$). More precisely, for each $i \in \{1, ..., n\}$, we let

$$\pi^i(\tau) := x_i^{\gamma_i} \cdots x_n^{\gamma_n} \in \mathcal{T}, \text{ i.e. } \pi^i(\tau) = \frac{\tau}{x_1^{\gamma_1} \cdots x_{i-1}^{\gamma_{i-1}}}.$$

We can extend this procedure to a finite set of terms $M \subset \mathcal{T}$, defining, for each $i \in \{1, ..., n\}$, $M^{[i]} := \pi^i(M) := \{\pi^i(\tau) : \tau \in M\}$. The terms in $M^{[i]}$ will play a fundamental role for the construction of the Bar Code diagram. Here we list some features of the operators π^i, that will be useful in what follows.

1. For each $\tau \in \mathcal{T}$, $\pi^1(\tau) = \tau$.
2. If $\tau = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, $\gamma_i = deg_i(\tau) = 0$ then $\pi^i(\tau) = x_{i+1}^{\gamma_{i+1}} \cdots x_n^{\gamma_n} = \pi^{i+1}(\tau)$.
3. It holds $\tau <_{Lex} \sigma \Rightarrow \pi^i(\tau) \leq_{Lex} \pi^i(\sigma)$, $\forall i \in \{1, ..., n\}$.
4. For each term τ and for any pair of indices i, j, say $1 \leq i < j \leq n$, we have that, since $x_i < x_j$, $\pi^j(\pi^i(\tau)) = \pi^i(\pi^j(\tau)) = \pi^j(\tau)$.
5. For each $\sigma, \tau \in \mathcal{T}$, $\forall 1 \leq i < n$, it holds $\pi^i(\tau) = \pi^i(\sigma) \Rightarrow \pi^{i+1}(\tau) = \pi^{i+1}(\sigma)$.

Example 13. Take $\tau = x_1 x_2^3 x_3^4 \in \mathbf{k}[x_1, x_2, x_3]$. Clearly $\pi^1(\tau) = x_1 x_2^3 x_3^4$, while $\pi^2(\tau) = x_2^3 x_3^4$ and $\pi^3(\tau) = x_3^4$. For $\sigma_1 := x_2 x_3^5 >_{Lex} \tau$, $\pi^2(\tau) = x_2^3 x_3^4 <_{Lex} \pi^2(\sigma_1) = x_2 x_3^5$ and $\pi^3(\tau) = x_3^4 <_{Lex} \pi^3(\sigma_1) = x_3^5$; for $\sigma_2 := x_1^5 x_2^3 x_3^4 >_{Lex} \tau$, $\pi^2(\tau) = x_2^3 x_3^4 = \pi^2(\sigma_2)$ and $\pi^3(\tau) = \pi^3(\sigma_2) = x_3^4$. Moreover, $\pi^3(\pi^2(\tau)) = \pi^3(x_2^3 x_3^4) = x_3^4 = \pi^2(\pi^3(\tau))$. ◊

Now we take $M \subseteq \mathcal{T}$, with $|M| = m < \infty$ and we order its elements increasingly w.r.t. Lex, getting the list $\overline{M} = [\tau_1, ..., \tau_m]$. Then, we construct the sets $M^{[i]}$, and the corresponding lexicographically ordered lists $\overline{M}^{[i]}$, for $i = 1, ..., n$. We notice that $\overline{M}$ cannot contain repeated terms, while the $\overline{M}^{[i]}$, for $1 < i \leq n$, can. In case some repeated terms occur in $\overline{M}^{[i]}$, $1 < i \leq n$, they clearly have to be adjacent in the list, due to the lexicographical ordering. We can now define the $n \times m$ matrix of terms $\mathcal{M}$ as the matrix s.t. its i-th row is $\overline{M}^{[i]}$, $i = 1, ..., n$, i.e. $\mathcal{M} = (\pi_i(\tau_j))_{1 \leq i \leq n, 1 \leq j \leq m}$.

Definition 14. *The* Bar Code diagram B *associated to M (or, equivalently, to $\overline{M}$) is a $n \times m$ diagram, made by segments s.t. the i-th row of* B, $1 \leq i \leq n$ *is constructed as follows:*

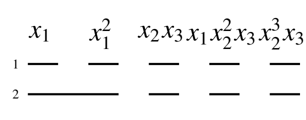
1. *take the i-th row of $\mathcal{M}$, i.e. $\overline{M}^{[i]}$*
2. *consider all the sublists of repeated terms, i.e. $[\pi^i(\tau_{j_1}), \pi^i(\tau_{j_1+1}), ..., \pi^i(\tau_{j_1+h})]$ s.t. $\pi^i(\tau_{j_1}) = \pi^i(\tau_{j_1+1}) = ... = \pi^i(\tau_{j_1+h})$, noticing that*[3] $0 \leq h < m$
3. *underline each sublist with a segment*
4. *delete the terms of $\overline{M}^{[i]}$, leaving only the segments (i.e. the i-*bars*).*

We usually label each 1*-bar $\mathsf{B}_j^{(1)}$, $j \in \{1, ..., \mu(1)\}$ with the term $\tau_j \in \overline{M}$.*

By property 5. of the operators π^i and, since for each $1 \leq i \leq n$, $|\overline{M}^{[i]}| = \sum_{j=1}^{\mu(i)} l_1(\mathsf{B}_j^{(i)})$, a Bar Code diagram is a Bar Code in the sense of Definition 8.

Example 15. Given $M = \{x_1, x_1^2, x_2x_3, x_1x_2^2x_3, x_2^3x_3\} \subset \mathbf{k}[x_1, x_2, x_3]$, we have $\overline{M}^{[1]} = [x_1, x_1^2, x_2x_3, x_1x_2^2x_3, x_2^3x_3]$ $\overline{M}^{[2]} = [1, 1, x_2x_3, x_2^2x_3, x_2^3x_3]$, $\overline{M}^{[3]} = [1, 1, x_3, x_3, x_3]$, leading to the 3×5 table on the left and then to the Bar Code on the right:

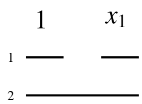
x_1	x_1^2	x_2x_3	$x_1x_2^2x_3$	$x_2^3x_3$
1	1	x_2x_3	$x_2^2x_3$	$x_2^3x_3$
1	1	x_3	x_3	x_3

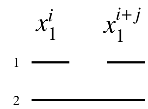


◊

[3] Clearly if a term $\pi^i(\tau_{\overline{j}})$ is not repeated in $\overline{M}^{[i]}$, the sublist containing it will be only $[\pi^i(\tau_{\overline{j}})]$, i.e. $h = 0$.

Remark 16. *We can easily observe that Bar Codes associated to different sets of terms,* need not *to be different. For example, if* $M := \{1, x_1\}$, $M' := \{x_1^i, x_1^{i+j}\} \subset \mathbf{k}[x_1, x_2]$, $i, j \in \mathbb{N}$, $j \neq 0$, *both the Bar Code* B *associated to* M *and the Bar Code* B' *associated to* M' *are*





We will see soon that this cannot happen for order ideals.

Now we explain how to associate a finite set of terms M_{B} to a given Bar Code B. In order to do it, we have to follow the steps below:

BC1 consider the n-th row, composed by the bars $B_1^{(n)}, ..., B_{\mu(n)}^{(n)}$. Let $l_1(B_j^{(n)}) = \ell_j^{(n)}$, for $j \in \{1, ..., \mu(n)\}$ and $a_1, ..., a_{\mu(n)} \in \mathbb{N}$, s.t. $a_k < a_h$ if $k < h$. Label each bar $B_j^{(n)}$ with $\ell_j^{(n)}$ copies of $x_n^{a_j}$.

BC2 For each $i = 1, ..., n-1$, $1 \leq j \leq \mu(n-i+1)$ take the bar $B_j^{(n-i+1)}$ and suppose it has been labelled by $\ell_j^{(n-i+1)}$ copies of a term τ. Construct the 2-block associated to $B_j^{(n-i+1)}$ which, by definition, is composed by $B_j^{(n-i+1)}$ and by all the $(n-i)$-bars $B_{\overline{j}}^{(n-i)}, ..., B_{\overline{j}+h}^{(n-i)}$, lying immediately above $B_j^{(n-i+1)}$; note that h satisfies $0 \leq h \leq \mu(n-i) - \overline{j}$. Denote the 1-lenghts of $B_{\overline{j}}^{(n-i)} ... B_{\overline{j}+h}^{(n-i)}$ by $l_1(B_{\overline{j}}^{(n-i)}) = \ell_{\overline{j}}^{(n-i)}$,..., $l_1(B_{\overline{j}+h}^{(n-i)}) = \ell_{\overline{j}+h}^{(n-i)}$ and fix $h+1$ natural numbers $a_{\overline{j}} < a_{\overline{j}+1} < ... < a_{\overline{j}+h}$. For each $0 \leq k \leq h$, label $B_{\overline{j}+k}^{(n-i)}$ with $\ell_{\overline{j}+k}^{(n-i)}$ copies of $\tau x_{n-i}^{a_{\overline{j}+k}}$.

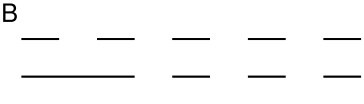
Clearly, if, given a Bar Code B, we apply BC1 and BC2 to get a set $M \subset \mathcal{T}$, and then we construct the Bar Code associated to M, we get back B. Indeed, BC1 and BC2 exactly construct the elements of the ordered lists $\overline{M}^{[i]}$, $i = 1, ..., n$.
Given a Bar Code B, applying steps BC1 and BC2, we can generate an *infinite* number of sets $M \subset \mathcal{T}$. We modify the steps BC1 and BC2 getting $\mathfrak{BC}1$ and $\mathfrak{BC}2$ so that, for each Bar Code B, the set of terms generated by applying them turns out to be *unique*:

$\mathfrak{BC}1$ consider the n-th row, composed by the bars $B_1^{(n)}, ..., B_{\mu(n)}^{(n)}$. Let $l_1(B_j^{(n)}) = \ell_j^{(n)}$, for $j \in \{1, ..., \mu(n)\}$. Label each bar $B_j^{(n)}$ with $\ell_j^{(n)}$ copies of x_n^{j-1}.

$\mathfrak{BC}2$ For each $i = 1, ..., n-1$, $1 \leq j \leq \mu(n-i+1)$ consider the bar $B_j^{(n-i+1)}$ and suppose that it has been labelled by $\ell_j^{(n-i+1)}$ copies of a term τ. Construct the 2-block associated to $B_j^{(n-i+1)}$ which, by definition, is composed by $B_j^{(n-i+1)}$ and by all the $(n-i)$-bars $B_{\overline{j}}^{(n-i)}, ..., B_{\overline{j}+h}^{(n-i)}$ lying immediately above $B_j^{(n-i+1)}$; note that h satisfies $0 \leq h \leq \mu(n-i) - \overline{j}$. Denote the 1-lenghts of $B_{\overline{j}}^{(n-i)}, ..., B_{\overline{j}+h}^{(n-i)}$ by $l_1(B_{\overline{j}}^{(n-i)}) = \ell_{\overline{j}}^{(n-i)}$,..., $l_1(B_{\overline{j}+h}^{(n-i)}) = \ell_{\overline{j}+h}^{(n-i)}$. For each $0 \leq k \leq h$, label $B_{\overline{j}+k}^{(n-i)}$ with $\ell_{\overline{j}+k}^{(n-i)}$ copies of τx_{n-i}^k.

It is important to notice that not all Bar Codes can be associated to order ideals, as easily shown by the example below.

Example 17. Consider the Bar Code B



We cannot associate any order ideal to it.
Indeed, using either BC1, BC2 or $\mathfrak{BC}1$,$\mathfrak{BC}2$, we obtain terms of the form

$$\begin{array}{ccccc} x_1^{\alpha_1}x_2^{\beta_1}x_3^{\gamma_1} & x_1^{\alpha_2}x_2^{\beta_1}x_3^{\gamma_1} & x_1^{\alpha_3}x_2^{\delta_1}x_3^{\gamma_2} & x_1^{\alpha_4}x_2^{\delta_2}x_3^{\gamma_2} & x_1^{\alpha_5}x_2^{\delta_3}x_3^{\gamma_2} \\ x_2^{\beta_1}x_3^{\gamma_1} & x_2^{\beta_1}x_3^{\gamma_1} & x_2^{\delta_1}x_3^{\gamma_2} & x_2^{\delta_2}x_3^{\gamma_2} & x_2^{\delta_3}x_3^{\gamma_2} \\ x_3^{\gamma_1} & x_3^{\gamma_1} & x_3^{\gamma_2} & x_3^{\gamma_2} & x_3^{\gamma_2} \end{array},$$

with $\gamma_1 < \gamma_2$, $\delta_1 < \delta_2 < \delta_3$, $\alpha_1 < \alpha_2$ and so the associated set of terms M turns out to be
$M = \{x_1^{\alpha_1}x_2^{\beta_1}x_3^{\gamma_1}, x_1^{\alpha_2}x_2^{\beta_1}x_3^{\gamma_1}, x_1^{\alpha_3}x_2^{\delta_1}x_3^{\gamma_2}, x_1^{\alpha_4}x_2^{\delta_2}x_3^{\gamma_2}, x_1^{\alpha_5}x_2^{\delta_3}x_3^{\gamma_2}\}$. To be an order ideal, M must contain all the divisors

of its elements: $\forall \tau \in M$, if $\sigma \mid \tau$ then $\sigma \in M$, so we have to lay down some conditions on the exponents. Let us start examining $x_1^{\alpha_1} x_2^{\beta_1} x_3^{\gamma_1}$ and $x_1^{\alpha_2} x_2^{\beta_1} x_3^{\gamma_1}$. Knowing that $\alpha_1 < \alpha_2$, we need to take $\alpha_1 = 0$ and $\alpha_2 = 1$. Indeed, otherwise, M should contain at least another term of the form $x_1^{\alpha_0} x_2^{\beta_1} x_3^{\gamma_1}$, $\alpha_0 \neq \alpha_1, \alpha_2$ and $\alpha_0 < \max(\alpha_1, \alpha_2)$. The exponent β_1 must be equal to zero, otherwise at least $x_1^{\alpha_1} x_2^{\beta_1 - 1} x_3^{\gamma_1}$ and $x_1^{\alpha_2} x_2^{\beta_1 - 1} x_3^{\gamma_1}$ would belong to M. For analogous reasons, we have to choose $\gamma_1 = 0, \gamma_2 = 1$ and $\alpha_3 = \alpha_4 = \alpha_5 = 0$. We get $M = \{1, x_1, x_2^{\delta_1} x_3, x_2^{\delta_2} x_3, x_2^{\delta_3} x_3\}$. But let us examine $\delta_1 < \delta_2 < \delta_3$. Similarly to what said for the other exponents, we have only one possible choice for them, i.e. $\delta_1 = 0,\ \delta_2 = 1\ \delta_3 = 2$[4], but then also x_2 and x_2^2 should belong to M, and this is impossible: there is only one possible power of x_2 for $\gamma_1 = 0$ and this contradiction proves that B cannot be associated to any order ideal. ◇

Inspired by example 17, we define *admissible Bar Codes* as follows:

Definition 18. *A Bar Code* B *is* admissible *if the set M obtained by applying* $\mathfrak{BC}$ *and* $\mathfrak{BC}2$ *to* B *is an order ideal.*

Remark 19. *By definition of order ideal, using* $\mathfrak{BC}1$ *and* $\mathfrak{BC}2$ *is the only way an order ideal can be associated to an admissible Bar Code. Indeed, if we label two consecutive bars with two terms $\tau x_i^{a_i}$, $\tau x_i^{a_i+h}$, $h > 1$, then also the terms σ with $\pi^i(\sigma) = \tau x_i^{a_i+1}$ would belong to M and it would have to label a bar between those labelled by $\tau x_i^{a_i}$ and $\tau x_i^{a_i+h}$, giving a contradiction.*

We need now an *admissibility criterion* for Bar Codes. In order to be able to state it, we start with the following trivial lemma.

Lemma 20. *Given a set $M \subset \mathcal{T}$, the following conditions are equivalent*

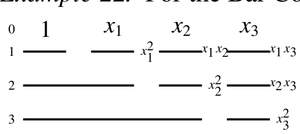
1. *M is an order ideal.*
2. *$\forall \tau \in M$, if $\sigma \mid \tau$, then $\sigma \in M$.*
3. *$\forall \tau \in M$ each predecessor of τ belongs to M.*

We give then the definition of *e-list*, associated to each 1-bar of a given Bar Code.

Definition 21. *Given a Bar Code* B*, let us consider a* 1*-bar $B_{j_1}^{(1)}$, with $j_1 \in \{1, ..., \mu(1)\}$. The* e-list *associated to $B_{j_1}^{(1)}$ is the n-tuple $e(B_{j_1}^{(1)}) := (b_{j_1,1}, ..., b_{j_1,n})$, defined as follows:*

- *consider the n-bar $B_{j_n}^{(n)}$, lying under $B_{j_1}^{(1)}$. The number of n-bars on the left of $B_{j_n}^{(n)}$ is $b_{j_1,n}$.*
- *for each $i = 1, ..., n-1$, let $B_{j_{n-i+1}}^{(n-i+1)}$ and $B_{j_{n-i}}^{(n-i)}$ be the $(n-i+1)$-bar and the $(n-i)$-bar lying under $B_{j_1}^{(1)}$. Consider the $(n-i+1)$-block associated to $B_{j_{n-i+1}}^{(n-i+1)}$. The number of $(n-i)$-bars of the block, which lie on the left of $B_{j_{n-i}}^{(n-i)}$ is $b_{j_1,n-i}$.*

Example 22. For the Bar Code B



the e-lists are $e(B_1^{(1)}) := (0,0,0)$; $e(B_2^{(1)}) := (1,0,0)$; $e(B_3^{(1)}) := (0,1,0)$ and $e(B_4^{(1)}) := (0,0,1)$. ◇

Remark 23. *Given a Bar Code* B*, fix a* 1*-bar $B_j^{(1)}$, with $j \in \{1, ..., \mu(1)\}$. Comparing definition* 21 *and the steps* $\mathfrak{BC}1$ *and* $\mathfrak{BC}2$ *described above, we can observe that the values of the e-list $e(B_j^{(1)}) := (b_{j,1}, ..., b_{j,n})$ are exactly the exponents of the term labelling $B_j^{(1)}$, obtained applying* $\mathfrak{BC}1$ *and* $\mathfrak{BC}2$ *to* B.

Proposition 24 (Admissibility criterion). *A Bar Code* B *is admissible if and only if, for each* 1*-bar $\mathsf{B}_j^{(1)}$, $j \in \{1, ..., \mu(1)\}$, the e-list $e(\mathsf{B}_j^{(1)}) = (b_{j,1}, ..., b_{j,n})$ satisfies the following condition: $\forall k \in \{1, ..., n\}$ s.t. $b_{j,k} > 0$, $\exists \bar{j} \in \{1, ..., \mu(1)\} \setminus \{j\}$ s.t. $e(\mathsf{B}_{\bar{j}}^{(1)}) = (b_{j,1}, ..., b_{j,k-1}, (b_{j,k}) - 1, b_{j,k+1}, ..., b_{j,n})$.*

Proof. It is a trivial consequence of Lemma 20 and Remark 23. □

[4]Notice that these assignments are those given by $\mathfrak{BC}1$ and $\mathfrak{BC}2$.

Consider the sets $\mathcal{A}_n := \{\mathsf{B} \in \mathcal{B}_n \text{ s.t. } \mathsf{B} \text{ admissible}\}$ and $\mathcal{N}_n := \{\mathsf{N} \subset \mathcal{T},\ |\mathsf{N}| < \infty \text{ s.t. } \mathsf{N} \text{ order ideal}\}$. We can define the map $\eta : \mathcal{A}_n \to \mathcal{N}_n;\ \mathsf{B} \mapsto \mathsf{N}$, where N is the order ideal obtained applying $\mathfrak{BC}1$ and $\mathfrak{BC}2$ to B.
By $\mathfrak{BC}1$ and $\mathfrak{BC}2$, η is a function; it is trivially surjective. Moreover, it is injective since, if $\mathsf{B}, \mathsf{B}' \in \mathcal{A}_n$ and $\mathsf{B} \neq \mathsf{B}'$ they have at least one pair of indices i, j s.t. $l_1(\mathsf{B}_j^{(i)}) \neq l_1(\mathsf{B'}_j^{(i)})$ and this changes the result of the application of $\mathfrak{BC}1/\mathfrak{BC}2$. From the arguments above, we can then deduce that there is a bijection between admissible n-Bar Codes and finite order ideals of $\mathcal{T} \subset \mathbf{k}[x_1, ..., x_n]$.
In the Lemma below we state some properties of admissible Bar Codes related to lengths.

Lemma 25. *If B is an admissible Bar Code, the following two conditions hold:*

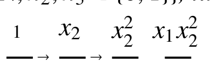
a) $l_{n-1}(\mathsf{B}_1^{(n)}) \geq ... \geq l_{n-1}(\mathsf{B}_{\mu(n)}^{(n)})$

b) $\forall 1 \leq i \leq n-2$, $\forall 1 \leq j \leq \mu(i+2)$ *take the* $(i+2)$*-bar* $\mathsf{B}_j^{(i+2)}$ *and let* $\mathsf{B}_{j_1}^{(i+1)}, ..., \mathsf{B}_{j_1+h}^{(i+1)}$ *(where* h *satisfies* $h \in \{0, ..., \mu(i+1) - j_1\}$*) be the* $(i+1)$*-bars lying over* $\mathsf{B}_j^{(i+2)}$.
Then $l_i(\mathsf{B}_{j_1}^{(i+1)}) \geq ... \geq l_i(\mathsf{B}_{j_1+h}^{(i+1)})$.

Proof. Let us start proving a). If for some $1 \leq l \leq \mu(n) - 1$ it holds $l_{n-1}(\mathsf{B}_l^{(n)}) < l_{n-1}(\mathsf{B}_{l+1}^{(n)})$ the Bar Code would be not admissible. Indeed, let $\mathsf{B}_k^{(1)}$ be the rightmost 1-bar over $\mathsf{B}_{l+1}^{(n)}$ and $e(\mathsf{B}_k^{(1)}) = (b_{k,1}, ..., b_{k,n})$ be its e-list. By construction (see Definition 21), $b_{k,n-1} = l_{n-1}(\mathsf{B}_{l+1}^{(n)}) - 1$. Now, this proves that there cannot exist a 1-bar labelling $(b_{k,1}, ..., b_{k,n-1}, b_{k,n} - 1)$, since $l_{n-1}(\mathsf{B}_l^{(n)}) < l_{n-1}(\mathsf{B}_{l+1}^{(n)})$ and so the 1-bars $\mathsf{B}_{\overline{k}}^{(1)}$ over $\mathsf{B}_l^{(n)}$ have $b_{\overline{k},n-1} \leq l_{n-1}(\mathsf{B}_l^{(n)}) - 1 < l_{n-1}(\mathsf{B}_{l+1}^{(n)}) - 1 = b_{k,n-1}$, contradicting the assumption of admissibility (see Proposition 24). An analogous argument proves that if for some $\forall 1 \leq i \leq n-2$, $\forall 1 \leq j \leq \mu(i+2)$ we take the $(i+2)$-bar $\mathsf{B}_j^{(i+2)}$ and $\mathsf{B}_{j_1+h}^{(i+2)}$ s.t. h satisfies $h \in \{0, ..., \mu(i+1) - j_1\}$ is the $(i+1)$-bars lying over $\mathsf{B}_j^{(i+2)}$, it happens that for a fixed $l \in \{1, ..., \mu(i+1) - 1 - j_1\}$ $l_i(\mathsf{B}_{j_1+l}^{(i+1)}) < l_i(\mathsf{B}_{j_1+l+1}^{(i+1)})$, B is not admissible and so also b) is true. □

In what follows, unless differently specified, we always consider admissible Bar Codes, so, in general, we will omit the word "admissible".

Remark 26. *In principle, it is possible to represent with a Bar Code also infinite order ideals, by means of a simple modification, i.e. the introduction of the symbol "$\rightarrow$" immediately after a l-bar for some $1 \leq l \leq n$, meaning that there should actually be infinitely many l-blocks equal to that containing that bar. For example, the Bar Code of* $I = (x_1^2 x_2^2) \triangleleft \mathbf{k}[x_1, x_2]$, *whose lexicographical Groebner escalier is* $\mathsf{N}(I) = \{x_1^{h_1} x_2^{h_2}, x_1^{h_3} x_2^{h_4},\ h_1, h_4 \in \mathbb{N}, h_2, h_3 \in \{0, 1\}\}$, *turns out to be*



In particular, the arrow on the right of 1 *represents the terms of the form* $x_1^{h_1}$, $h_1 \in \mathbb{N} \setminus \{0\}$, *the one on the right of* x_2 *represents the terms of the form* $x_1^{h_1} x_2$, $h_1 \in \mathbb{N} \setminus \{0\}$*; finally the bottom arrow represents the terms of the form* $x_2^{h_4}, x_1 x_2^{h_4}$, $h_4 \in \mathbb{N}$, $h_4 > 2$.

Remark 27. *It is possible to give an alternative construction for the Bar Code (this remark deeply depends on* [4]*), that is deduced from the point trie* [17, 33] *by applying the correlation between monomials and points introduced by* [34, p. 548], [36, 33]. *The main tools for our alternative construction are the Σ-algorithm and the point trie, deeply examined in* [33], *of which we borrow the notation.*
First of all, we endow a given set Ω with an equivalence relation = and we extend it to Ω^n: $A := (a_1, ..., a_n), B := (b_1, .., , b_n) \in \Omega^n$, $A = B \Leftrightarrow \forall 1 \leq i \leq n$, $a_i = b_i$. For $1 \leq i \leq n$, we define $\pi : \Omega^n \to \Omega^i$ the projection map s.t. $\pi_i((a_1, ..., a_n)) = (a_1, ..., a_i)$. Now, the witness *of $A, B \in \Omega^n$ is the smallest value i such that $\pi_i(A) \neq \pi_i(B)$. If $\{p_1, ..., p_N\} \subset \Omega^n$, let $\Sigma_0 = \{\{1, ..., N\}\}$ and Σ_i, $1 \leq i \leq n$, the set of equivalence classes of $\pi_i(p_1), ..., \pi_i(p_N)$. Clearly $|\Sigma_n| = N$. The point trie is a tree representation for the elements in Ω^n, constructed using the Σ_i's: the vertices are labelled by the elements in the Σ_i's and there is an edge from a vertex labelled by $\Sigma_{i,k} \in \Sigma_i$ to a vertex with label $\Sigma_{i+1,h} \in \Sigma_{i+1}$ exactly when $\Sigma_{i+1,h} \subset \Sigma_{i,k}$. Applying the correlation between monomials and points introduced by* [34] *p. 548,* [36, 33] *we can now give an alternative construction of the Bar Code, deduced by the point trie. Let $\Omega = \mathbb{N}$; then $\Omega^n = \mathbb{N}^n \cong \mathcal{T}$ can be seen as the set of the exponents' lists of each term in n variables: if $\tau = x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$ it can be identified by the list $(\gamma_n, ..., \gamma_1)$. This allows to build the Bar Code of a finite set $M \subset \mathcal{T}$ (or, equivalently, of the lex-ordered list $\overline{M}$) as follows:*

- *let* $\mathfrak{M} = \{(\gamma_n, ..., \gamma_1) \in \mathbb{N}^n := x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n} \in \overline{M}\}$*;*
- *compute the* Σ_i*'s w.r.t.* $\mathfrak{M}$ *and return them as* ${}^T\left(\Sigma_n, \Sigma_{n-1}, ..., \Sigma_1\right)$*;*
- *substitute each* $\Sigma_{i,k} \in \Sigma_i$*,* $1 \le i \le n$ *with a bar, whose length is* $|\Sigma_{i,k}|$*.*

The obtained diagram is a Bar Code.

5. The star set

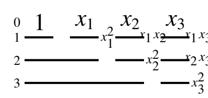
Up to this point, we have discussed the link between Bar Codes and order ideals, i.e. we focused on the link between Bar Codes and Groebner escaliers of monomial ideals. In this section, we show that, given a Bar Code B and the order ideal $\mathsf{N} = \eta(\mathsf{B})$ it is possible to deduce a very specific generating set for the monomial ideal I s.t. $\mathsf{N}(I) = \mathsf{N}$.

Definition 28. *The* star set *of an order ideal* N *and of its associated Bar Code* $\mathsf{B} = \eta^{-1}(\mathsf{N})$ *is a set* $\mathcal{F}_{\mathsf{N}}$ *constructed as follows:*

a) $\forall 1 \le i \le n$*, let* τ_i *be a term which labels a* 1*-bar lying over* $\mathsf{B}^{(i)}_{\mu(i)}$*, then* $x_i\pi^i(\tau_i) \in \mathcal{F}_{\mathsf{N}}$*;*
b) $\forall 1 \le i \le n-1$*,* $\forall 1 \le j \le \mu(i) - 1$ *let* $\mathsf{B}^{(i)}_j$ *and* $\mathsf{B}^{(i)}_{j+1}$ *be two consecutive bars not lying over the same* $(i+1)$*-bar and let* $\tau^{(i)}_j$ *be a term which labels a* 1*-bar lying over* $\mathsf{B}^{(i)}_j$*, then* $x_i\pi^i(\tau^{(i)}_j) \in \mathcal{F}_{\mathsf{N}}$*.*

We usually represent $\mathcal{F}_{\mathsf{N}}$ within the associated Bar Code B, inserting each $\tau \in \mathcal{F}_{\mathsf{N}}$ on the right of the bar from which it is deduced. Reading the terms from left to right and from the top to the bottom, $\mathcal{F}_{\mathsf{N}}$ is ordered w.r.t. Lex.

Example 29.
For $\mathsf{N} = \{1, x_1, x_2, x_3\} \subset \mathbf{k}[x_1, x_2, x_3]$, associated to the Bar Code of example 22, we have $\mathcal{F}_{\mathsf{N}} = \{x_1^2, x_1x_2, x_2^2, x_1x_3, x_2x_3, x_3^2\}$; looking at Definition 28, we can see that the terms x_1x_3, x_2x_3, x_3^2 come from a), whereas the terms x_1^2, x_1x_2, x_2^2 come from b).



◇

In [10], given a monomial ideal I, the authors define the following set, calling it *star set*:
$\mathcal{F}(I) = \left\{x^\gamma \in \mathcal{T} \setminus \mathsf{N}(I) \;\middle|\; \frac{x^\gamma}{\min(x^\gamma)} \in \mathsf{N}(I)\right\}$. We can prove the following proposition, which connects the definition above to our construction.

Proposition 30. *With the above notation* $\mathcal{F}_{\mathsf{N}} = \mathcal{F}(I)$*.*

Proof. We start proving $\mathcal{F}_{\mathsf{N}} \subseteq \mathcal{F}(I)$. Consider $\sigma \in \mathcal{F}_{\mathsf{N}}$; by definition of $\mathcal{F}_{\mathsf{N}}$ there are two possibilities

a) $\sigma = x_i\pi^i(\tau_i)$, with $1 \le i \le n$ and τ_i a term which labels a 1-bar lying over $\mathsf{B}^{(i)}_{\mu(i)}$;
b) $\sigma = x_i\pi^i(\tau^{(i)}_j)$, with $1 \le i \le n-1$, $1 \le j \le \mu(i) - 1$, where $\tau^{(i)}_j$ is a term which labels a 1-bar lying over $\mathsf{B}^{(i)}_j$ i.e. the rightmost bar over some $\mathsf{B}^{(i+1)}_u$, whereas $\mathsf{B}^{(i)}_{j+1}$ is the leftmost bar over $\mathsf{B}^{(i+1)}_{u+1}$. Notice that we can chooose one $\tau^{(i)}_j$ indifferently over $\mathsf{B}^{(i)}_j$, since for each term over $\mathsf{B}^{(i)}_j$ the operator π^i gives the same result.

Let us examine a) and b) separately.

a) By definition, $\sigma > \tau_i$; indeed $\deg_h(\sigma) = \deg_h(\tau_i)$ for $i+1 \le h \le n$ and $\deg_i(\sigma) > \deg_i(\tau_i)$. Clearly, $\sigma \notin \mathsf{N}$, because if it was in the Groebner escalier, applying the steps described in Definition 14, $\pi^i(\sigma) = \sigma = x_i\pi^i(\tau_i)$ would be put in a list that is subsequent to the one containing $\pi^i(\tau_i)$, but, in this case, there would be $\mu(i)+1$ i-bars instead of $\mu(i)$, contradicting the definition of $\mu(i)$. Since $\min(\sigma) = x_i$, $\frac{\sigma}{\min(\sigma)} = \pi^i(\tau_i) \mid \tau_i$, so $\frac{\sigma}{\min(\sigma)} \in \mathsf{N}$ and $\sigma \in \mathcal{F}(I)$.
b) Analogously to case a), $\sigma > \tau^{(i)}_j$. Let us prove that $\sigma \notin \mathsf{N}$. If $\sigma \in \mathsf{N}$ then σ would label a 1-bar lying over $\mathsf{B}^{(i)}_{j+1}$ but, since $\pi^{i+1}(\sigma) = \pi^{i+1}(\tau^{(i)}_j)$, $\mathsf{B}^{(i)}_j$ $\mathsf{B}^{(i)}_{j+1}$ would lie over the same $(i+1)$-bar, contradicting the hypothesis. As above, since $\min(\sigma) = x_i$, $\frac{\sigma}{\min(\sigma)} = \pi^i(\tau^{(i)}_j) \mid \tau^{(i)}_j$, so $\frac{\sigma}{\min(\sigma)} \in \mathsf{N}$ and $\sigma \in \mathcal{F}(I)$.

We prove now that $\mathcal{F}_{\mathsf{N}} \supseteq \mathcal{F}(I)$. Let us consider $\sigma \in \mathcal{F}(I)$ and let $\min(\sigma) = x_i$, $1 \le i \le n$. By definition of $\mathcal{F}(I)$, $\sigma \notin \mathsf{N}$ and $\widetilde{\sigma} := \frac{\sigma}{x_i} \in \mathsf{N}$, so it labels a 1-bar lying over some i-bar $\mathsf{B}^{(i)}_j$. Denote by $\mathsf{B}^{(1)}_{\overline{j}}, ..., \mathsf{B}^{(1)}_{\overline{j}+h}$ (where h satisfies $0 \le h \le \mu(i) - \overline{j}$) the 1-bars lying over $\mathsf{B}^{(i)}_j$. Two possibilities may occur:

a) $\bar{j}+h=\mu(i)$; in this case $x_i\pi^i(\widetilde{\sigma})=\sigma\in\mathcal{F}_{\mathsf{N}}$ by Definition 28.
b) otherwise consider the term $\tau_{\bar{j}+h}$, which labels $\mathsf{B}^{(1)}_{\bar{j}+h}$, and the subsequent term $\tau_{\bar{j}+h+1}$, labelling $\mathsf{B}^{(1)}_{\bar{j}+h+1}$. Notice that $\pi^i(\tau_{\bar{j}+h})=\pi^i(\widetilde{\sigma})$. By Definition 14, $\tau_{\bar{j}+h}<_{Lex}\tau_{\bar{j}+h+1}$. If $\pi^i(\tau_{\bar{j}+h})=\pi^i(\tau_{\bar{j}+h+1})$ this would contradict the maximality of h, so, by property 3. of the operators π^i, it must be $\pi^i(\tau_{\bar{j}+h})<_{Lex}\pi^i(\tau_{\bar{j}+h+1})$. But, if $\pi^{i+1}\tau_{\bar{j}+h})=\pi^{i+1}(\tau_{\bar{j}+h+1})$, then $\sigma\mid\tau_{\bar{j}+h+1}$ and so $\sigma\in\mathsf{N}$, that is impossible since $\sigma\in\mathcal{F}(I)$. This means then that $\pi^{i+1}(\tau_{\bar{j}+h})<_{Lex}\pi^{i+1}(\tau_{\bar{j}+h+1})$, so we can deduce that $\mathsf{B}^{(1)}_{\bar{j}+h}$ and $\mathsf{B}^{(1)}_{\bar{j}+h+1}$ lie over two consecutive i-bars not lying over the same $(i+1)$-bar, so $\sigma=x_i\pi^i(\widetilde{\sigma})=x_i\pi^i(\tau_{\bar{j}+h})\in\mathcal{F}_{\mathsf{N}}$.

□

Remark 31. *By Proposition* 30*, being $\mathcal{F}_{\mathsf{N}}=\mathcal{F}(I)$, it trivially holds $\mathsf{G}(I)\subseteq\mathcal{F}_{\mathsf{N}}\subseteq\mathsf{B}(I)$. In general, the inclusions may be strict; if $\mathcal{F}_{\mathsf{N}}=\mathsf{G}(I)$, we say that $\mathsf{B}_{\mathsf{N}}:=\eta^{-1}(\mathsf{N})$ is a* full *Bar Code.*

The star set $\mathcal{F}(I)$ of a monomial ideal I is strongly connected to Janet's theory [24, 25, 26, 27] and to the notion of Pommaret basis [39, 40, 43], as explicitly pointed out in [10]. For completeness sake, we recall it below.

Definition 32. [24, ppg.75-9] *Let $M\subset\mathcal{T}$ be a set of terms and $\tau=x_1^{\gamma_1}\cdots x_n^{\gamma_n}$ be an element of M. A variable x_j is called* Janet-multiplicative *for τ w.r.t. M if there is no term in M of the form $\tau'=x_1^{\delta_1}\cdots x_j^{\delta_j}x_{j+1}^{\gamma_{j+1}}\cdots x_n^{\gamma_n}$ with $\delta_j>\gamma_j$. We will denote by $M_J(\tau,M)$ the set of multiplicative variables for τ w.r.t. M.*

Definition 33. *With the previous notation, the* cone *of τ w.r.t. M is the set $C_J(\tau,M):=\{\tau x_1^{\lambda_1}\cdots x_n^{\lambda_n}\mid$ where $\lambda_j\neq 0$ only if x_j is multiplicative for τ w.r.t. $M\}$.*

Definition 34. [24, ppg.75-9] *A set of terms $M\subset\mathcal{T}$ is called* complete *if for every $\tau\in M$ and $x_j\notin M_J(\tau,M)$, there exists $\tau'\in M$ such that $x_j\tau\in C_J(\tau',M)$. Moreover, M is* stably complete [43, 10] *if it is complete and for every $\tau\in M$ it holds $M_J(\tau,M)=\{x_i\mid x_i\le\min(\tau)\}$. If a set M is stably complete and finite, then it is the* Pommaret basis *of $I=(M)$.*

Theorem 35. *For every monomial ideal I, the star set $\mathcal{F}(I)$ is the unique stably complete system of generators of I. Hence, if M is stably complete, $M=\mathcal{F}((M))$.*

By Proposition 30, the Bar Code gives a simple way to deduce the star set from the Groebner escalier of a zerodimensional monomial ideal.

6. Applications of Bar Code

In this section, we give an overview on possible applications of Bar Codes to the study of zerodimensional monomial ideals. Many of them are related to involutive divisions, but there are also some of them which have nothing to do with this context. We first see the applications that are not linked to the involutive framework, i.e. enumerative combinatorics on (strongly) stable ideals (see 6.1.1) and an iterative Lex Game algorithm for zerodimensional ideals defined by sets of points (see 6.1.2). Then we switch to the involutive applications (6.2).

6.1. Non-involutive applications

6.1.1. Counting 0-dimensional (strongly) stable ideals. In the paper [4], Bar Codes are employed as a tool for counting zerodimensional stable and strongly stable ideals in 2, 3 variables, given their constant affine Hilbert polynomial p. These ideals are defined in a purely combinatorial way

Definition 36. ([25, pg.41], [27], c.f.[37, Vol. 4, p. 673,679]) *A monomial ideal $J\triangleleft\mathcal{P}=\mathbf{k}[x_1,...,x_n]$ is called* stable [16]*] if it holds $\tau\in J,\ x_j>\min(\tau)\Longrightarrow\frac{x_j\tau}{\min(\tau)}\in J$.*

Definition 37 ([41, 42, 22, 23, 18, 38])**.** *A monomial ideal $I\triangleleft\mathcal{P}=\mathbf{k}[x_1,...,x_n]$ is called* strongly stable *(*[2, 1]*) if, for every term $\tau\in I$ and pair of variables $x_i,\ x_j$ with $x_i|\tau$ and $x_i<x_j$, then also $\frac{\tau x_j}{x_i}\in I$ or, equivalently, for every $\sigma\in\mathsf{N}(I)$, and pair of variables $x_i,\ x_j$ with $x_i|\sigma$ and $x_i>x_j$, then also $\frac{\sigma x_j}{x_i}\in\mathsf{N}(I)$.*

It is possible to prove that there is a bijection between (strongly) stable ideals in 2, 3 variables and some partitions of the affine Hilbert polynomial p.

Definition 38 ([44])**.** *An* integer partition *of $p \in \mathbb{N}$ is a k-tuple $(\lambda_1, ..., \lambda_k) \in \mathbb{N}^k$ s.t. $\sum_{i=1}^{k} \lambda_i = p$ and $\lambda_1 \geq ... \geq \lambda_k$.*

For $p, k \in \mathbb{N}$, we denote $I_{(p,k)} := \{(\lambda_1, ..., \lambda_k) \in \mathbb{N}^k,\ \lambda_1 > ... > \lambda_k > 0$ and $\sum_{j=1}^{k} \lambda_j = p\}$ the set of partitions of p in k distinct parts.

Definition 39 ([28])**.** *A* plane partition *π of a positive integer $p \in \mathbb{N}$, is a partition of p in which the parts have been arranged in a 2-dimensional array, weakly decreasing across rows and down columns. If the inequality is strict across rows (resp. columns), we say that the partition is* row-strict *(resp* column-strict*). Different configurations are regarded as different plane partitions. The* norm *of π is the sum $n(\pi) := \sum_{i,j} \pi_{i,j}$ of all its parts, i.e. p.*

Definition 40 ([28])**.** *Let D_r denote the set of all r-tuples $\lambda = (\lambda_1, ..., \lambda_r)$ of integers with $\lambda_1 \geq ... \geq \lambda_r$. For $\lambda, \mu \in D_r$, we write $\lambda \geq \mu$ if $\lambda_i \geq \mu_i$ for all $i = 1, 2, ..., r$. Let c, d arbitrary integers and $\lambda, \mu \in D_r$, with $\lambda \geq \mu$. We call an array ρ of integers of the form*

$$\begin{array}{ccccccccccc} & & & \rho_{1,\mu_1+1} & \rho_{1,\mu_1+2} & \cdots & \cdots & \cdots & \rho_{1,\lambda_1} \\ & \rho_{2,\mu_2+1} & \cdots & \cdots & \cdots & \cdots & \cdots & \rho_{2,\lambda_2} & \\ & & & \cdots & \cdots & \cdots & \cdots & & \\ \rho_{r,\mu_r+1} & \cdots & \cdots & \rho_{r,\lambda_r} & & & & & \end{array}$$

a (c, d)-plane partition *of shape λ/μ if $\rho_{i,j} \geq \rho_{i,j+1} + c$ for $1 \leq i \leq r$, $\mu_i < j < \lambda_i$, and $\rho_{i,j} \geq \rho_{i+1,j} + d$ for $1 \leq i \leq r-1$, $\mu_i < j \leq \lambda_{i+1}$. In the case $\mu = 0$, we shortly say that ρ is of shape λ.*

Definition 41 ([29])**.** *Let c, d be arbitrary integers and λ be a partition with $\lambda_r \geq r$. We call "*shifted (c, d)-plane partition *of* shape λ*" an array π of integers of the form*

$$\begin{array}{ccccccccc} \pi_{1,1} & \pi_{1,2} & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \pi_{1,\lambda_1} \\ & \pi_{2,2} & \cdots & \cdots & \cdots & \cdots & \cdots & \pi_{2,\lambda_2} & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & \pi_{r,r} & \cdots & \cdots & \pi_{r,\lambda_r} & & \end{array}$$

s.t. $\pi_{i,j} \geq \pi_{i,j+1} + c$ for $1 \leq i \leq r$, $i \leq j < \lambda_i$, and $\pi_{i,j} \geq \pi_{i+1,j} + d$ for $1 \leq i \leq r-1$, $i < j \leq \lambda_{i+1}$.

The symbol $\mathcal{P}_\lambda(c, d)$ denotes the set of (c, d)-plane partitions of shape λ, while $\mathcal{S}_\lambda(c, d)$ denotes the set of shifted (c, d)-plane partitions of shape λ.
The main results of [4] are summarized in what follows. The case of (strongly) stable ideals in two variables is the simplest one. For stable ideals we have

Proposition 42. *The number of Bar Codes $\mathsf{B} \in \mathcal{B}_2$ with bar list (p, h) and s.t. $\eta(B) = \mathsf{N} \subset \mathbf{k}[x_1, x_2]$ is the Groebner escalier of a stable ideal $J \triangleleft \mathbf{k}[x_1, x_2]$ equals the number of integer partitions of p in h distinct parts.*

and since

Proposition 43. *Denoting by B a Bar Code associated to a stable ideal $I \triangleleft \mathbf{k}[x_1, x_2]$ with affine Hilbert polynomial $H_I(d) = p \in \mathbb{N}$ and by $\mathsf{L}_\mathsf{B} = (p, h)$ its bar list, the maximal value that h can assume is $h := \left\lfloor \frac{-1+\sqrt{1+8p}}{2} \right\rfloor$.*

we can desume that

Proposition 44. *The number of stable ideals $J \triangleleft \mathbf{k}[x_1, x_2]$ with $H\_(t, J) = p$ is $\sum_{i=1}^{h} Q(p, i)$, where $h := \left\lfloor \frac{-1+\sqrt{1+8p}}{2} \right\rfloor$ and $Q(p, i)$ is the number of integer partitions of p into i distinct parts.*

and this result holds also for strongly stable ideals, since

Lemma 45. *An ideal $I \triangleleft \mathbf{k}[x_1, x_2]$ is stable if and only if it is strongly stable.*

For the case of three variables, we can start by observing that

Corollary 46. *The number of Bar Codes associated to stable ideals in $\mathbf{k}[x_1, ..., x_n]$, $n > 2$, whose bar list is $(p, h, \underbrace{1, ..., 1}_{3,...,n})$, $p, h \in \mathbb{N}$, $p \geq h$ equals the number of integer partitions of p in h distinct parts, i.e. $p = \alpha_1 + ... + \alpha_h$, $\alpha_1 > ... > \alpha_h > 0$. Moreover, the maximal value that h can take in the bar list $(p, h, 1, ..., 1)$ is $h := \left\lfloor \frac{-1+\sqrt{1+8p}}{2} \right\rfloor$.*

so we only have to deal with the bar lists of the form (p,h,k), $k>1$.

Lemma 47. *With the previous notation, it holds:*

1. $k \in \{1,...,l\}$, *where* $l := \max_{i\in\mathbb{N}}\{i^3+3i^2+2i \le 6p\}$;
2. $h \in \{\frac{k(k+1)}{2},...,m\}$, *where* $m = \max_{r\ge \frac{k(k+1)}{2}} \{r \mid \exists \lambda \in I_{(r,k)}, Sm(\lambda) \le p\}$, *and* $Sm(\lambda) := Sm([\lambda_1,...,\lambda_k]) = \sum_{i=1}^{k} \frac{\lambda_i(\lambda_i+1)}{2}$.

Denoting $\mathcal{P}_{(p,h,k)} = \{\rho \in \mathcal{P}_{\overline{\beta}}(1,1)$ for some $\overline{\beta} \in I_{(h,k)}$ and s.t. $n(\rho) = p\}$

Theorem 48. *There is a bijection between* $\mathcal{P}_{(p,h,k)}$ *and the set* $\mathsf{B}^{(S)}_{(p,h,k)} = \{\mathsf{B} \in \mathcal{A}_3$ *s.t.* $\mathsf{L}_\mathsf{B} = (p,h,k)$, $\eta(\mathsf{B}) = \mathsf{N}(J)$, *$J$ stable*$\}$.

In conclusion, counting stable ideals becomes an easy application of Krattenthaler's formulas [28] for counting the elements in $\mathcal{P}_{(p,h,k)}$. The situation is analogous for strongly stable ideals.
Indeed, denoting $\mathcal{S}_{(p,h,k)} = \{\pi \in \mathcal{S}_\lambda(1,0),\ n(\pi) = p,\ \lambda_i = i + \overline{\alpha_i} - 1,\ 1 \le i \le k,$ for some $\overline{\alpha} \in I_{(h,k)}\}$, we have

Theorem 49. *There is a bijection between* $\mathcal{S}_{(p,h,k)}$ *and the set* $\mathsf{B}_{(p,h,k)} = \{\mathsf{B} \in \mathcal{A}_3$ *s.t.* $\mathsf{L}_\mathsf{B} = (p,h,k)$, $\eta(\mathsf{B}) = \mathsf{N}(J)$, *$J$ strongly stable*$\}$.

and again determinantal formulas by Krattenthaler [29] solve the problem of counting. The paper [4] terminates with a conjecture about the number of (strongly) stable ideals with fixed affine Hilbert polynomial for the case of $n \ge 4$ variables. The idea consists in introducing (shifted) partitions in dimension $n-1$ and establishing analogous bijections with (strongly) stable 0-dimensional ideals in n variables. In the case of $n-1$-dimensional partitions ($n \ge 4$), anyway, there are no determinantal formulas and counting them is still an open problem.

6.1.2. An iterative Lex Game. In the paper [9], we employ the Bar Code in order to give an iterative version of the Lex Game algorithm [17]. In 1990 Cerlienco and Mureddu [13, 14, 15] gave a combinatorial algorithm which, given an ordered set of points $\underline{\mathbf{X}} = [P_1,...,P_N] \subset \mathbf{k}^n$, $\mathbf{k}$ a field, returns the lexicographical Gröbner escalier $\underline{\mathsf{N}(I(\mathbf{X}))} \subset \mathcal{T} := \{x^\gamma := x_1^{\gamma_1}\cdots x_n^{\gamma_n} \mid \gamma := (\gamma_1,...,\gamma_n) \in \mathbb{N}^n\}$ of the vanishing ideal $I(\mathbf{X}) := \{f \in \mathcal{P} : f(P_i) = 0, \forall i \in \{1,...,N\}\} \subset \mathcal{P} := \mathbf{k}[x_1,...,x_n]$, by giving a bijection between $\mathbf{X}$ and $\mathsf{N}(I(\mathbf{X}))$. This algorithm performs iteration on the points (so it gives the aforementioned bijection for all the ideals in the *Macaulay chain* $I_i := I(\{P_1,...,P_i\})$ $1 \le i \le N$) and recursion on the variables, with complexity $O\left(n^2N^2\right)$. The Lex Game has been developed as an improvement of Cerlienco-Mureddu algorithm. It employs two tries (rooted trees with both nodes and edges with their own label), called point trie (see remark 27) and lex trie, and reaches a very better complexity, $O(nN + N\min(N,nr))$, where $r < n$ is the maximal number of edges from a vertex in the point trie, *dropping iterativity for the sake of efficiency.* Using a combination of the point trie, which represents the reciprocal relations among the points and the Bar Code in order to store the monomials, it is possible to give an iterative algorithm with complexity $O(N^2n\log(N))$, so very near to that of the Lex Game algorithm. Indeed, our algorithm mimicks Cerlienco-Mureddu's one, but it stores in the Bar Code the information that is (lost and) computed by Cerlienco-Mureddu *more than once*. The construction of the point trie is identical to that of the Lex Game, with complexity $nN + N\min(N,nr) \sim O(nNr)$ and for each of the N points, while constructing the new branch of the point trie corresponding to the new point, we obtain for free the level h in which the new point forks from the previous ones and the node in the $(h-1)$-level in which the point is still not splitted from the previous ones (so in particular the σ-antecedent,i.e. the maximally-indexed point with the same first $h-1$ coordinates as the newly inserted point). Then, the algorithm updates the Bar Code, reading and writing each bar only once. In particular, it takes - if it exists- the h-block following that of the σ-antecedent, lengthens the $h+1,...,n$-bar under this block and keeps track of the corresponding exponents of the monomials (i.e. 0). Then it lengthens also the h-bar of this block and keep track of the h-exponent of the monomials over it and walks in the path of the trie corresponding to the new point, from level $h-1$ to level 1, repeating the procedure. Instead, if such following h-block in the Bar Code does not exist yet the algorithm inserts it by adding the $1,...,h$-bars (whose length is one). The cost of detecting the following h block is the same as identifying the last element belonging to the current h-block and in the $(h-1)$-node of the trie in which the new point appears. Since the number of points both in the ordered h-block and in the $(h-1)$-ordered node are bounded by N, the complexity of this problem is $N\log(N)$. Therefore, this procedure costs $N^2n\log(N)$ As an example of the results provided by our algorithm, one can see that the four maps and pictures given in example 5, are exactly the results one gets by applying either Cerlienco-Mureddu algorithm or our iterative Lex game, in correspondence with the ordering given to the set of input point[5] In the same paper, an efficient algorithm for computing a family of squarefree separator polynomial for $\mathbf{X}$ is given[6], as well as a fast algorithm to get Auzinger-Stetter matrices.

6.2. Involutive applications.

We see now some applications of Bar Code to involutive divisions. The idea of involutive divisions dates back to the works by Janet and Riquier [24, 25, 26, 27] and has been formalized by Gerdt and Blinkov

Definition 50 (Gerdt-Blinkov, [19]). *An* involutive division *L or L*-division *on T is a relation $|_L$ defined, for each finite set $U \subset T$, on the set $U \times T$ in such a way that the following holds for each $u, u_1 \in U$ and $t, t_1 \in T$*
(i) $u |_L t \Rightarrow u \mid t$;
(ii) $u |_L u$;
(iii) $u |_L ut, u |_L ut_1 \Leftrightarrow u |_L utt_1$;
(iv) $u |_L t, u_1 |_L t \Rightarrow$ either $u |_L u_1$ or $u_1 |_L u$;
(v) $u |_L u_1, u_1 |_L t \Rightarrow u |_L t$;
(vi) if $V \subseteq U$ and $u \in V$ then $u |_L t$ w.r.t. $U \Rightarrow u |_L t$ w.r.t. V.
If $u |_L t = uw$, u is called an involutive divisor *of t, t is called an* involutive multiple *of u and w is said to be* multiplicative *for u. If $u \nmid_L t = uw$, w is said to be* non-multiplicative *for u.* □

In particular, Janet division is assigned on a finite set of terms, by equipping each term with a set of multiplicative variables as seen at the end of section 5 (see definitions 32, 33). In [6, 8], we focus on applications of Bar Codes related to Janet division. In particular, [8] focuses on determining Janet multiplicative variables for the terms in a finite set $U \subset \mathsf{T}$ using the Bar Code associated to U and the star set construction (i.e. the construction with stars explained in section 5). The relation with multiplicative variables is stated by the proposition below:

Proposition 51. *Let $U \subseteq \mathcal{T}$ be a finite set of terms and let us denote by B_U its Bar Code. For each $t \in U$ x_i, $1 \leq i \leq n$ is multiplicative for t if and only if, in B_U, the i-bar $\mathsf{B}_j^{(i)}$, over which t lies, is followed by a star.*

The above proposition implies that a mere Bar Code and star set construction is enough to determine Janet multiplicative variables for a finite set of terms. Then, the Bar Code can be seen as an alternative to Gerdt's Janet tree [20] and it is actually rather similar to the (equivalent) presentation given by Seiler in [43]. We point out that with an analogous of Proposition 51, Bar Code allows also to deal with *Janet-like division* [21], identifying non-multiplicative powers, despite the fact that actually Janet-like division is *not* an involutive division. Moreover, in [6], we deal with completeness of terms' sets w.r.t. Janet division (Definition 34). The first result in [6] is that

Proposition 52. *Let $U \subseteq \mathcal{T}$ be a finite set of terms and B be its Bar Code. Let $t \in U$, $x_i \in NM_J(t, U)$ and $\mathsf{B}_j^{(i)}$ the i-bar under t. Let $s \in U$; it holds $s |_J x_i t$ if and only if*

1. *$s \mid x_i t$*
2. *s lies over $\mathsf{B}_{j+1}^{(i)}$ and*
3. *$\forall j'$ appearing with nonzero exponent in $\frac{x_i t}{s}$ there is a star after the j'-bar under s.*

Moreover, basing on this proposition, we can construct an algorithm that given a finite set of terms $U = \{t_1, ..., t_m\} \subseteq \mathcal{T}$, finds whether there exists an *ordering on the variables* $x_1, ..., x_n$ such that U is *complete*. The idea consists in constructing the Bar Code B of the set $U = \{t_1, ..., t_m\} \subset \mathcal{T}$ from the maximal variable to the minimal one, checking if, with the choice made up to the current point on the variables' ordering, the conditions of proposition 52 hold for each term in U, and going back retracting our steps in case of failure, so modifying previously made choices. With this *backtracking technique*, we have to check less ordering with respect to the $n!$ orderings one would have in general to try.

In [7], Bar Code is employed in the context of *polynomial interpolation*. Considered a finite set of distinct points $\mathbf{X} = \{P_1, ..., P_N\} \subset \mathbf{k}^n$, the zerodimensional ideal $\mathsf{T}(I(\mathbf{X}))$ is quasi-stable by definition, so it has a Pommaret basis, which is exactly given by the terms of the star set of the Bar Code associated to $\mathsf{N}(I((\mathbf{X}))$. By the iterative Lex Game, we can compute $\mathsf{N}(I((\mathbf{X}))$ point by point; then again exploiting the Bar Code, we can compute the factors of an Axis of Evil factorization for a Pommaret basis of $I(\mathbf{X})$ point by point, so we can do it for each ideal in the Macaulay chain. In particular, we compute one factor for each bar, according to the points

[5] The Lex game gives the same escalier, but it is not so focused on the bijection between $\mathbf{X}$ and $\mathsf{N}(I(\mathbf{X}))$.
[6] For an efficient implementation, see [11, 12].

corresponding to the terms lying over that bar. That factors, chosen in a suitable way, according to the star set element we want to deal with, constitute the Axis of Evil factorization for a Pommaret basis of $I(\mathbf{X})$.

Example 53. For the set $\mathbf{X} = \{(0,0,0),(1,2,3),(1,4,5),(0,1,4)\}$, we have that the lexicographical Groebner escalier is $\mathsf{N}(\mathbf{X}) = \{1, x_1, x_2, x_1x_2\}$, corresponding to the following Bar Code.

$$\begin{array}{ccccc} & 1 & x_1 & x_2 & x_1x_2 \\ x_1 & \rule{1cm}{0.4pt} & \rule{1cm}{0.4pt} & \rule{1cm}{0.4pt} & \rule{1cm}{0.4pt} \\ x_2 & \multicolumn{2}{c}{\rule{2.5cm}{0.4pt}} & \multicolumn{2}{c}{\rule{2.5cm}{0.4pt}} \\ x_3 & \multicolumn{4}{c}{\rule{5cm}{0.4pt}} \end{array}$$

The polynomials in the three variables x_1, x_2, x_3 corresponding to the bars, in left-to-right order are: $\mathfrak{X}_1 = \{x_1, x_1 - 1, x_1 - 1, x_1\}$, for 1-bars, $\mathfrak{X}_2 = \{x_2 - 2x_1, x_2 - 4 - 3(x_1 - 1)\}$, for 2-bars and $\mathfrak{X}_3 = \{x_3 - x_2 - x_1 + 3(x_2 - 2x_1)(x_1 - 1)\}$ for the only 3 bar. The factorized Pommaret basis is then

$$\begin{aligned} \mathcal{G} &= \{x_1(x_1-1), (x_2-2x_1)(x_1-1)x_1, (x_2-2x_1)(x_2-4-3(x_1-1)), x_3-x_2-x_1+3(x_2-2x_1)(x_1-1)\} \\ &= \{x_1^2 - x_1, x_1^2x_2 - x_1x_2 - 2x_1^3 + 2x_1^2, x_2^2 - 5x_1x_2 - x_2 + 6 * x_1^2 + 2x_1, x_3 + 3x_1x_2 - 4x_2 - 6x_1^2 + 5x_1\} \end{aligned}$$

◊

In the same paper we show also that, with some small modifications to Moeller algorithm, it is possible to compute a Pommaret basis of $I(\mathbf{X})$ (and actually of any ideal in the Macaulay chain).

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