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Quantum-fluid description of the free-electron laser

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Using the Madelung transformation we show that in a quantum free-electron laser the beam obeys the equations of a quantum fluid in which the potential is the classical potential plus a quantum potential. The classical limit is shown explicitly.

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I. INTRODUCTION

In the quantum free-electron laser (FEL) model, the electron beam is described as a macroscopic matter wave [1–4]. When slippage due to the difference between the light and electron velocities is neglected, the electron-beam–wave interaction is described by the following equations for the dimensionless radiation amplitude $A(\bar{z})$ and the matter-wave field $\Psi(\theta,\bar{z})$ [5]:

$$i\frac{\partial\Psi(\theta,\bar{z})}{\partial\bar{z}} = -\frac{1}{2\bar{\rho}}\frac{\partial^2}{\partial\theta^2}\Psi(\theta,\bar{z}) - i\bar{\rho}[A(\bar{z})e^{i\theta} - \text{c.c.}]\Psi(\theta,\bar{z}), \tag{1}$$

$$\frac{dA(\bar{z})}{d\bar{z}} = \int_0^{2\pi} d\theta|\Psi(\theta,\bar{z})|^2 e^{-i\theta} + i\delta A(\bar{z}). \tag{2}$$

The electron beam is therefore described by a Schrödinger equation for a matter-wave field Ψ in a self-consistent pendulum potential proportional to A, where $|A|^2=|a|^2/(N\bar{\rho})$, $|a|^2$ is the average number of photons in the interaction volume V, and $|\Psi|^2$ is the space-time-dependent electron density, normalized to unity. In Eqs. (1) and (2) we have adopted the universal scaling used in the classical FEL theory [6–8], i.e., $\theta=(k+k_w)z-ckt$ is the electron phase, where $k_w=2\pi/\lambda_w$ and $k=\omega/c=2\pi/\lambda$ are the wiggler and radiation wave numbers, $\bar{z}=z/L_g$ is the dimensionless wiggler length, $L_g=\lambda_w/4\pi\rho$ is the gain length, $\rho=\gamma_r^{-1}(a_w/4ck_w)^{2/3}(e^2n/m\epsilon_0)^{1/3}$ is the classical FEL parameter, $\gamma_r=\sqrt{(\lambda/2\lambda_w)(1+a_w^2)}$ is the resonant energy in mc^2 units, a_w is the wiggler parameter, and n is the electron density. Finally, $\bar{p}=(\gamma-\gamma_0)/\rho\gamma_0$ is the dimensionless electron momentum and $\delta=(\gamma_0-\gamma_r)/\rho\gamma_0$ is the detuning parameter, where $\gamma_0\approx\gamma_r$ is the initial electron energy in mc^2 units.

Whereas the classical FEL equations in the above universal scaling do not contain any explicit parameter (see Ref. [8]), the quantum FEL equations (1) and (2) depend on the quantum FEL parameter

$$\bar{\rho} = \left(\frac{mc\gamma_r}{\hbar k}\right)\rho. \tag{3}$$

From the definition of A, it follows that $\bar{\rho}|A|^2=|a|^2/N$ is the average number of photons emitted per electron. Hence, since in the classical steady-state high-gain FEL A reaches a maximum value of the order of unity, $\bar{\rho}$ represents the maximum number of photons emitted per electron, and the classical regime occurs for $\bar{\rho}\gg 1$. Note also that in Eq. (1) $\bar{\rho}$ appears as a "mass" term, so one expects a classical limit when the mass is large. As we shall see, when $\bar{\rho}<1$ the dynamical behavior of the system changes substantially from a classical to a quantum regime.

II. QUANTUM FLUID DESCRIPTION

We now perform a Madelung-like transformation [9,10], writing the wave function as

$$\Psi = R\exp(i\bar{\rho}S),$$

which allows us to rewrite the Maxwell-Schrodinger equations, Eqs. (1) and (2), as a system of quantum fluid equations

$$\frac{\partial R}{\partial\bar{z}} = -\frac{\partial R}{\partial\theta}\frac{\partial S}{\partial\theta} - \frac{R}{2}\frac{\partial^2 S}{\partial\theta^2}, \tag{4}$$

$$\frac{\partial S}{\partial\bar{z}} = -\frac{1}{2}\left(\frac{\partial S}{\partial\theta}\right)^2 - V(\theta,\bar{z}), \tag{5}$$

$$\frac{dA}{d\bar{z}} = \int_0^{2\pi} R^2 e^{-i\theta}d\theta + i\delta A, \tag{6}$$

where the potential V in Eq. (5) is defined as the sum of a classical term and a quantum term, i.e.,

$$V(\theta,\bar{z}) = V_C + V_Q,$$

where

$$V_C = -i(Ae^{i\theta} - \text{c.c.}) \tag{7}$$

is the classical component of the potential and

$$V_Q = -\frac{1}{2\bar{\rho}^2 R}\frac{\partial^2 R}{\partial \theta^2} \tag{8}$$

is the quantum component of the potential, which becomes negligible as $\bar{\rho}\to\infty$.

Defining fluid density and velocity variables

$$n = R^2 = |\Psi|^2, \quad u = \frac{\partial S}{\partial \theta},$$

we can also rewrite Eqs. (4)–(6) in an alternative fluid form as

$$\frac{\partial n}{\partial \bar{z}} + \frac{\partial}{\partial \theta}(nu) = 0, \tag{9}$$

$$\frac{\partial u}{\partial \bar{z}} + u\frac{\partial u}{\partial \theta} = -\frac{\partial V}{\partial \theta}, \tag{10}$$

$$\frac{dA}{d\bar{z}} = \int_0^{2\pi} ne^{-i\theta}d\theta + i\delta A. \tag{11}$$

It can be seen that Eq. (9) is a continuity equation and Eq. (10) is a Newton-like equation for a fluid. Note that when Eq. (9) is integrated with respect to θ, then the normalization condition becomes

$$\int_0^{2\pi} n(\theta,\bar{z})d\theta = 1,$$

which is satisfied if n and u are periodic functions of θ between 0 and 2π.

A straightforward calculation shows that Eqs. (9)–(11) admit two constants of motion,

$$\langle \bar{p}\rangle + |A|^2 = \mathcal{C}_1 \tag{12}$$

and

$$\frac{\langle \bar{p}^2\rangle}{2} - i(Ab^* - \text{c.c.}) - \delta|A|^2 = \mathcal{C}_2, \tag{13}$$

where $\langle \bar{p}\rangle = \langle u\rangle = \int_0^{2\pi} d\theta\, nu$ is the average momentum, $\langle \bar{p}^2\rangle = \langle u^2 + 2V_Q\rangle = \int_0^{2\pi} d\theta\, n(u^2+2V_Q)$ is the momentum variance, and

$$b = \int_0^{2\pi} ne^{-i\theta}d\theta$$

is the bunching. These constants of motion are well known in the classical FEL model [8] and describe energy conservation and a gain-spread relation. Notice the quantum contribution to the momentum variance proportional to the average quantum potential.

III. FOURIER EXPANSION AND LINEAR ANALYSIS

If R and S are periodic functions of θ, they can be expanded in a Fourier series:

$$R(\theta,\bar{z}) = \sum_m r_m(\bar{z})e^{im\theta}, \tag{14}$$

$$S(\theta,\bar{z}) = \sum_n s_n(\bar{z})e^{in\theta}, \tag{15}$$

with $r_{-m} = s_m^*$ and $s_{-m} = r_m^*$, since R and S are real variables. Multiplying Eq. (5) by R and using (14) and (15) in Eqs. (4)–(6), we obtain

$$\sum_m r_{k-m}\frac{ds_m}{d\bar{z}} = -\frac{1}{2}\sum_{m,n} n(n-m)r_{k-m}s_n s_{n-m}^* + i(Ar_{k-1} - A^* r_{k+1}) - \frac{k^2}{2\bar{\rho}^2}r_k, \tag{16}$$

$$\frac{dr_k}{d\bar{z}} = \frac{1}{2}\sum_m (k^2 - m^2)r_m s_{m-k}^*, \tag{17}$$

$$\frac{dA}{d\bar{z}} = \sum_m r_m r_{m-1}^* + i\delta A. \tag{18}$$

Equations (16)–(18) are our working equations which can be numerically solved, as will be shown elsewhere.

Equations (16)–(18) admit an equilibrium solution with no field ($A=0$) and unbunched electron beam ($n=1/2\pi$, i.e., $a_n = \delta_{n0}$ and $s_n = 0$). Linearizing Eqs. (16)–(18) around this equilibrium in the first order of the variables A, r_1, and s_1, we obtain

$$\frac{dA}{d\bar{z}} = 2r_1 + i\delta A, \tag{19}$$

$$\frac{dr_1}{d\bar{z}} = \frac{s_1}{2}, \tag{20}$$

$$\frac{ds_1}{d\bar{z}} = iA - \frac{1}{2\bar{\rho}^2}r_1. \tag{21}$$

Looking for solutions proportional to $\exp(i\lambda\bar{z})$, we obtain the well-known cubic equation of the quantum FEL [1],

$$(\lambda - \delta)\left(\lambda^2 - \frac{1}{4\bar{\rho}^2}\right) + 1 = 0, \tag{22}$$

which reduces to the classical dispersion relation in the limit $\bar{\rho} \gg 1$.

IV. CONCLUSIONS

It has been shown that the quantum FEL model can be rewritten in a form where the electron beam is described as a quantum fluid coupled to the electromagnetic field. The evolution of the quantum fluid is determined by a self-consistent potential which consists of a classical and a quantum contribution. In the limit where $\bar{\rho} \gg 1$, the quantum contribution to the potential becomes negligible and the force equation reduces to that of a Newtonian fluid. Using a Fourier expansion, linear stability analysis of these quantum fluid equations produced a dispersion relation identical to that derived from the Schrödinger equation. These results show that there are interesting connections between the quantum FEL and quantum plasma instabilities.

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