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φ-Curvatures, Harmonic-Einstein Manifolds and Einstein-Type Structures

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Introduction

The aim of this thesis is to study the geometry of connected, complete, possibly compact, Riemannian manifolds $(M, \langle \, , \, \rangle)$ endowed with with a (gradient) Einstein-type structure of the form

$$\begin{cases} \mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) - \mu df \otimes df = \lambda \langle \, , \, \rangle \\ \tau(\varphi) = d\varphi(\nabla f), \end{cases} \tag{1}$$

where the *φ-Ricci tensor* is defined as

$$\mathrm{Ric}^{\varphi} := \mathrm{Ric} - \alpha \varphi^{*} \langle \, , \, \rangle_N \tag{2}$$

for some $\alpha \in \mathbb{R} \setminus \{0\}$, $\varphi : M \to (N, \langle \, , \, \rangle_N)$ a smooth map with tension field $\tau(\varphi)$ and target a Riemannian manifold $(N, \langle \, , \, \rangle_N)$ and $f, \mu, \lambda \in \mathcal{C}^{\infty}(M)$. We often consider μ, and sometimes also λ, to be constant.

The structure described by (1) generalizes some well known particular cases that have been intensively studied by researchers in the last decade. Indeed, for $\mu \equiv 0$, $\lambda \in \mathbb{R}$ and φ constant, (1) characterizes gradient Ricci solitons

$$\mathrm{Ric} + \mathrm{Hess}(f) = \lambda \langle \, , \, \rangle. \tag{3}$$

In case in (3) we allow $\lambda \in \mathcal{C}^{\infty}(M)$ we obtain the Ricci almost soliton equation introduced in [PRRiS]. Note that when $\lambda(x) = a + bS(x)$ for some constants $a, b \in \mathbb{R}$ and $S(x)$ the scalar curvature of $(M, \langle \, , \, \rangle)$, for $x \in M$, the soliton corresponding to (3) is called a Ricci-Bourguignon soliton after the recent work of G. Catino, L. Cremaschi, Z. Djadli, C. Mantegazza, and L. Mazzieri [CCDMM]. For a "flow"derivation of the gradient Ricci almost solitons equation in the general case see the work of [GWX].

In case $\mu = 0$, $\lambda \in \mathbb{R}$ and $\alpha > 0$ the system (1) represents Ricci-harmonic solitons introduced by R. Müller, [M]. As expected the concept comes from the study of a combination of the Ricci and harmonic maps flows. We refer to [M] for details and interesting analytic motivations.

For φ and μ constants, with $\mu = \frac{1}{\tau}$ for some $\tau > 0$, and $\lambda \in \mathbb{R}$, (1) describes quasi-Einstein manifolds

$$\mathrm{Ric} + \mathrm{Hess}(f) - \frac{1}{\tau} df \otimes df = \lambda \langle \, , \, \rangle \tag{4}$$

Letting $\mu, \lambda \in \mathcal{C}^{\infty}(M)$ we obtain the generalized quasi-Einstein condition

$$\mathrm{Ric} + \mathrm{Hess}(f) - \mu df \otimes df = \lambda \langle \, , \, \rangle. \tag{5}$$

See, for instance, [Ca] and [AG]. Obviously (5) extends the quasi-Einstein requirement (4).

To approach the study mentioned above, that is the argument of Part II of the thesis, we introduce some new curvature tensors that take into account the curvature of a Riemannian manifold endowed with a smooth map φ. Furthermore, since Ricci solitons and quasi-Einstein manifolds are usually seen as a perturbation of Einstein manifolds (the choice of a constant potential in (3) and in (4) led to an Einstein metric), we recall the concept of harmonic-Einstein manifolds so that the Einstein-type structures will be seen as a perturbation of harmonic-Einstein manifolds.

The thesis is divided in two parts. Part I is not just preliminary for Part II but it is interesting also on its own. It is composed by the first two Chapters of the thesis.

In Chapter 1 we introduce the new curvature tensors mentioned above, called the φ-curvature tensors. Formally almost all of them are defined in the same way as the standard curvatures using the φ-Ricci tensor, defined in (2), instead of the Ricci tensor. More precisely: the φ-scalar curvature, denoted by S^{φ}, is defined as the trace of the φ-Ricci tensor; the φ-Schouten tensor is defined as

$$A^{\varphi} = \mathrm{Ric}^{\varphi} - \frac{S^{\varphi}}{2(m-1)}\langle\, ,\, \rangle,$$

where $m \geq 2$ is the dimension of M; the φ-Cotton tensor C^{φ} represents the obstruction to the commutation of the covariant derivatives of the φ-Schouten tensor and so on. The only tensor whose definition is different from the one probably expected is the φ-Bach tensor B^{φ}.

When φ is a constant map all the φ-curvatures reduce to the standard curvature tensors.

Their properties are almost the same as the properties of the tensors that they generalize. For instance, the φ-Weyl tensor W^{φ} has the same symmetries of the Riemann tensor and its $(1,3)$-version is a conformal invariant. The only relevant difference is that the φ-Cotton, the φ-Weyl and the φ-Bach tensor are not, in general, totally traceless. Their traces are related to the map φ and, clearly, they vanish in case φ is a constant map. We can say more: the φ-Weyl, the φ-Cotton and the φ-Bach tensors are totally traceless if and only if, respectively, φ is constant, is conservative (that is, the energy stress tensor related to the map φ is divergence free) and is harmonic (with the exceptional case $m = 4$ where φ-Bach il always traceless). As a consequence the role of the map φ is not negligible, hence in this Chapter we also recall some properties for smooth maps, such as weakly conformality and homothety, that will be met also in the sequel.

The fact that the φ-curvature are not, in general, totally traceless have consequences especially in the computations. Even thought when φ is conservative we are able to recover a generalization of Schur's identity, that relates the divergence of φ-Ricci to the gradient of the φ-scalar curvature, the divergence of φ-Weyl is not related with the φ-Cotton as in the case of their standard counterparts. As a consequence, in order to have that φ-Weyl is harmonic it is not sufficient that W^{φ} is divergence free.

In Chapter 1 we also determine the transformation laws for the φ-curvatures under a conformal change of the metric. We show that on a four-dimensional manifold the φ-Bach tensor is a conformal invariant, that is one of the motivation that justify its definition. The other motivations are contained in Chapter 2, where we study harmonic-Einstein manifolds and their fundamental properties. A Riemannian manifold $(M, \langle\, ,\, \rangle)$ is said to be harmonic-Einstein if the traceless part of the φ-Ricci tensor vanishes for some harmonic map $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ and if the φ-Ricci tensor has constant trace, that is, if it carries a structure of the type

$$\begin{cases} \mathrm{Ric}^{\varphi} = \Lambda\langle\, ,\, \rangle \\ \tau(\varphi) = 0, \end{cases} \tag{6}$$

for some $\Lambda \in \mathbb{R}$. We shall see that when $m \geq 3$, the requirement of constant φ-scalar curvature is unnecessary, generalizing Schur's Lemma for Einstein manifolds. Its proof follows easily from the generalization of Schur's identity, since a harmonic map is conservative. The only relevant curvatures properties of harmonic-EInstein manifolds are encoded in W^{φ} and the sign of S^{φ}, since the other φ-curvatures are trivial.

System (6) is a starting point in our investigation in the sense that it justifies, in a geometric contest, the interest of studying a structure of the type (1). Indeed if we perform a conformal deformation of the metric $\langle\, ,\, \rangle$ of M, then from (6) we obtain a solution of (1) for $m \geq 3$ with $\mu = -\frac{1}{m-2}$ and viceversa, where the function λ satisfies

$$\Delta_f f + (m-2)\lambda = (m-2)\Lambda e^{-\frac{2f}{m-2}}. \tag{7}$$

Here Δ_f is the symmetric diffusion operator (or weighted Laplacian)

$$\Delta_f = \Delta - \langle \nabla f, \nabla \rangle.$$

Thus we can think of the study of

$$\begin{cases} \mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) + \dfrac{1}{m-2} df \otimes df = \lambda\langle\, ,\, \rangle \\ \tau(\varphi) = d\varphi(\nabla f) \end{cases} \tag{8}$$

as of that of (6) under conformal deformations of the original metric $\langle\,,\,\rangle$ of M. This parallels what happens in the study of Einstein and conformally Einstein metrics.

Knowing the transformation laws under a conformal change of metric and the φ-curvatures of harmonic-Einstein manifolds we will be able to prove that a conformally harmonic-Einstein manifolds of dimension $m \geq 3$ satisfy

$$\begin{cases} C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk} = 0 \\ (m-2)B^{\varphi}_{ij} + \dfrac{m-4}{m-2} W^{\varphi}_{tijk} f_t f_k = 0 \end{cases} \tag{9}$$

where f is related to the conformal factor in the change of the metric and f_i,C^{φ}_{ijk}, W^{φ}_{tijk} and B^{φ}_{ij} are, respectively, the components of ∇f, φ-Cotton, the φ-Weyl and the φ-Bach tensors in a local orthonormal coframe. In case φ is constant the above integrability conditions become the integrability condition for a conformally Einstein metric, that have been proved to be sufficient, under a further mild assumption of genericity of the metric, to guarantee the existence of a conformally Einstein metric on M by R. Gover and P. Nurowski, [GN]. We extend this result to the case of (9) showing that, under a corresponding mild additional assumption of genericity of the metric and on the map φ (related to the injectivity of a curvature operator $\mathcal{W}^{\varphi}$, defined in terms of φ-Weyl, and of the singular points of φ), they are sufficient conditions to generate a conformally harmonic-Einstein structure on M.

The two integrability conditions (9) are not a special feature of the system (8). An analogous of them holds also for the Einstein-type structure (1). In case φ is a constant map the analogous for (3) of the integrability conditions in (9) have been used to study the local geometry of Bach flat gradient Ricci solitons by H.-D. Cao and Q. Chen in [CC]. Their results has been extended by G. Catino, P. Mastrolia, D. D. Monticelli and M. Rigoli to gradient Einstein-type manifolds in Theorem 1.2 of [CMMR]. The latter are structure of the type (1) with φ a constant map, $\mu \in \mathbb{R}$ and $\lambda(x) = \rho S(x) + \lambda$ for some real constants ρ and λ. These results suggest to study (1) from the same point of view and in Chapter 6 we are able to characterize, when $\mu \neq -\frac{1}{m-2}$ (the equality case pertaining to conformally harmonic-Einstein manifolds) and $\alpha > 0$, from the adequate integrability conditions and the properness of the function f, the local geometry of a complete Riemannian manifold with a non trivial gradient Einstein-type structure and φ-Bach tensor that vanishes along the direction of ∇f. Notice that for conformally harmonic-Einstein manifolds the latter requirement is always satisfied, as one can immediately deduce contracting the second equation of (9) against ∇f. The main result of Chapter 6 is that, in a neighborhood of every regular level set of f, the manifold $(M, \langle\,,\,\rangle)$ is a warped product with $(m-1)$-dimensional harmonic-Einstein fibers, given by the level sets of f. Moreover the map is uniquely determined by its restriction on a single leave of the foliation. Assuming further a genericity condition and the constancy of λ we are able to prove that the manifold is harmonic-Einstein. This Chapter can be seen as the core of this thesis and the problem of characterize the local structure of Einstein-type structure as (1) is the one that led us to define the φ-curvature and justify their definition, especially for φ-Bach.

A justification for the study of harmonic-Einstein manifolds is given by General Relativity. Indeed a four dimensional Lorentzian harmonic-Einstein manifold is a solution of the Einstein field equations, for a proper choice of the constant α, with as energy-stress tensor the one of a wave map (that is, a harmonic map with source a Lorentzian manifold). Investigating standard static spacetimes (that are, Lorentzian manifold given by the warped product of a three dimensional Riemannian manifold with an open real interval) that are harmonic-Einstein manifolds with respect to a wave map that does not depend on the "time"we realize that the spatial part supports a structure of the type (1) and the warping factor u satisfies $\Delta u + \lambda u = 0$, for some $\lambda \in \mathbb{R}$. As we shall see a warped product $M \times_u F$, where $u = e^{-\frac{f}{d}}$, is a harmonic-Einstein manifold with respect to a map Φ given by the lifting to $M \times F$ of a smooth map $\varphi : M \to (N, \langle\,,\,\rangle_N)$ if and only if F is Einstein with scalar curvature $d\Lambda$, where d is the dimension of F and $\Lambda \in \mathbb{R}$ and

$$\begin{cases} \mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) - \dfrac{1}{d} df \otimes df = \lambda \langle\,,\,\rangle \\ \tau(\varphi) = d\varphi(\nabla f), \end{cases}$$

where the constant λ satisfies

$$\Delta_f f = d\lambda - d\Lambda e^{\frac{2}{d}f}. \tag{10}$$

In particular the study of (1) with $\mu = 1$ and $m = 3$ has repercussions to the study of the standard static spacetimes mentioned above. Notice that this can be seen as an extension of some results of J. Corvino, see [Co], that deals with the vacuum case. More generally, the study of (1) with $\mu = \frac{1}{d}$ has application to the study of warped product harmonic-Einstein manifolds. The possibility of constructing examples of Einstein manifolds realized as warped product metrics is an old interesting question considered in A. Besse's book, [B], so we may expect that also the more general problem of finding harmonic-Einstein manifolds realized as warped products can be interesting.

It is not a case that (7) and (10) holds, respectively, for conformally harmonic-Einstein manifolds and for harmonic-Einstein warped products; this is a consequence of the validity of (1). Indeed, it is well known, from the work of D. S. Kim and Y. H. Kim, [KK], that the validity of (4) on M yields, via a non-trivial consequence of the second Bianchi identities, the validity of the equation

$$\Delta_f f - \tau\lambda = -\beta e^{\frac{2}{\tau}f} \tag{11}$$

for some constant $\beta \in \mathbb{R}$. We extend the validity of this equation to the structure (1) for every μ, obtaining the so called *Hamilton-type identities*. It is interesting that in these equations the map φ and the constant α does not appear. This observation let us extend some results for (4) that rely on (11) to the more general structures (1).

We also evaluate the Laplacian of the square norm of the traceless part T^φ of the φ-Ricci tensor and, as a consequence, we prove a "gap"property that shows that whenever $|T^\varphi|$ is sufficiently small, a stochastically complete manifold carries a harmonic-Einstein type structure, if some necessary conditions are satisfied. This compares and generalize some previous results, see [MMR].

It is important to observe that in all the results discussed up to now the target manifold $(N, \langle\, ,\, \rangle_N)$ can be any Riemannian manifold. We show that, when we put some restraints on the curvature of the target manifold (and we assume that the density of energy is sufficiently small, in case of negative φ-scalar curvature), for a complete manifold the concept of being harmonic-Einstein collapse to one of being Einstein. This result is achieved showing that φ is constant via the classical Bochner formula for smooth maps and the assumption on the curvature of the target manifold is an appropriate upper bound on the largest eigenvalue of the curvature operator. Notice that a harmonic-Einstein manifolds can be a Einstein manifold even thought φ is not a constant map: this happens if and only if φ is homothetic.

Einstein manifolds in low dimension have been characterized: a Riemannian manifold of dimension $m \in \{2, 3\}$ is Einstein if and only if it has constant sectional curvature. In higher dimension a Einstein manifold has constant sectional curvature if and only if it is locally conformally flat.

For surfaces the Ricci tensor is always proportional to the metric hence the problem of finding a Einstein metric on a surface reduces to the one of finding a metric of constant scalar curvature on it. The uniformization of Riemann surfaces provides a way to select a complete metric of constant scalar curvature in every conformal class of metrics according to the topology of the surface. Observe that choosing a conformal class of metrics on a surface is equivalent to choose a complex structure on it. For harmonic-Einstein manifold the situation is different. The Ricci tensor is always proportional to the metric but, in order to obtain that the φ-Ricci tensor is proportional to the metric the map φ must be weakly conformal. The fact of being weakly conformal depends only on the complex structure, exactly as for the fact of being harmonic. A weakly conformal and harmonic map on a Riemann surface is a minimal branched immersion. Then the problem of finding a harmonic-Einstein metric on a Riemann surface reduces to the problem of finding a metric of constant φ-scalar curvature for a minimal branched immersion. We will not go further into this study.

In higher dimension we shall see that a harmonic-Einstein manifold has constant sectional curvature if and only if it is Einstein, since this requirement forces the map φ to be homothetic. An analogous phenomenon happens also when we consider local symmetry and harmonic curvature: for a Einstein manifold they are equivalent to conformal local symmetry and harmonic Weyl curvature, respectively. For harmonic-Einstein manifold the conditions above imply the same restriction on the geometry of the manifold together with some conditions on the map φ, that we shall investigate.

In Part II, together with (1), we also consider the more general Einstein-type structure

$$\begin{cases} \mathrm{Ric}^{\varphi} + \frac{1}{2}\mathcal{L}_X\langle\,,\,\rangle = \mu X^{\flat}\otimes X^{\flat} + \lambda\langle\,,\,\rangle \\ \tau(\varphi) = d\varphi(X), \end{cases} \tag{12}$$

for some $X \in \mathfrak{X}(M)$ and with $X^{\flat}$ denoting the 1-form dual to X via the musical isomorphism $^{\flat}$. Notice that (12) reduces to (1) when $X = \nabla f$. Interesting results for the structure (12) are obtained when $\mu = 0$ and X is non-Killing.

The compact case is quite rigid once we require constancy of the φ-scalar curvature. Indeed, when $\mu \neq 0$, $\alpha > 0$ and $\lambda, f \in \mathcal{C}^{\infty}(M)$ with f non-constant a Riemannian manifold with constant φ-scalar curvature that supports an Einstein-type structure as in (1) is always isometric to a Euclidean sphere and φ is a constant map. When $\mu = 0$ the same happens under the same hypothesis for the general structure (12), when X is not a Killing vector field. Our results extend the ones of [BBR] and [BG] to the case when, a priori, φ is not constant.

In proving the mentioned results we extend the well known fact, due to M. Obata, see [O], that a compact Einstein manifold endowed with a non-Killing conformal vector field is isometric to a Euclidean sphere, obtaining that if a compact harmonic-Einstein manifold with $\alpha > 0$ is endowed with a vertical (i.e., annihilated by the differential of φ), non-Killing conformal vector field then φ is constant and the Riemannian manifold is isometric to the Euclidean sphere.

The study of particular vector fields on a harmonic-Einstein manifold is treated in Chapter 4. The motivation is that dealing with harmonic-Einstein manifolds that supports a non trivial Einstein-type structure as (12) is equivalent to dealing with harmonic-Einstein manifolds that posses a vector field that satisfies

$$\begin{cases} \frac{1}{2}\mathcal{L}_X\langle\,,\,\rangle - \mu X^{\flat}\otimes X^{\flat} = \left(\lambda - \frac{S^{\varphi}}{m}\right)\langle\,,\,\rangle \\ d\varphi(X) = 0. \end{cases}$$

The aim of Chapter 4 is to show that, essentially, eventually under some assumptions on the critical points of the potential function f, the only complete manifolds that supports a non-trivial (that is, with non-constant potential) Einstein-type structure as (1) are space forms. When $\mu = 0$ we are also able to obtain some results in this direction in the generic case (12).

In the compact case we are able to obtain rigidity results also in case the φ-Schouten tensor is a Codazzi tensor field and one of its normalized higher order symmetric functions in its eigenvalues is a positive constant (necessary conditions to have the isometry with the Euclidean sphere and the constancy of φ). The φ-scalar curvature is constant if and only the first symmetric function of the eigenvalues of the φ-Schouten tensor is constant, hence we can see this as a generalization of the previous results obtained assuming the constancy of the φ-scalar curvature. The rigidity in the compact case is the subject of Chapter 5.

As one can expect, assuming λ constant in (12), we are able to prove several interesting results in the complete case; that is the aim of Chapter 7. Above all we mention the estimates on the infumum of the φ-scalar curvature S^{φ}_{*}, that are obtained as a consequence of a general formula for the Laplacian of the φ-scalar curvature and the validity of the weak maximum principle for the weighted Laplacian, that, in turns, is guaranteed by appropriate estimates on the volume growth of geodesic balls. In contrast to the results obtained in the other Chapters we are not able to obtain the estimates on S^{φ}_{*} for every $\mu \in \mathbb{R}$, indeed we shall restrict to the case $\mu \in [0, 1]$. Moreover, if $\mu \neq 0$ we restrict to the gradient Einstein-type structure (1) and we also require some additional properties for the potential function. For φ constant our estimates have been obtained in Theorem 3 of [R].

Finally we also deal with some non-existence results. Firstly, if $\mu \neq 0$, setting $u = e^{-\mu f}$ and tracing the first equation in (1) we obtain

$$Lu := \Delta u + \mu(m\lambda - S^{\varphi}) = 0. \tag{13}$$

Since $u > 0$, by a well known result of [FCS] and [MP], the operator L is stable or, in other words, its spectral radius $\lambda^1_L(M)$ is non-negative. Thus, instability of L yields a non-existence result for (1) at least in case μ is non-zero constant. Toward this aim we detect appropriate conditions on the coefficient of the linear term in (13).

Secondly, with the aid of a Bochner-type formula for the square norm of X for complete Einstein-type structures as (12), we provide non-existence results assuming an upper bound on the φ-scalar curvature and for $\mu > \frac{1}{2}$. In the gradient case (1) we are able to obtain the same result also for $\mu \leq 0$, assuming eventually a suitable integrability condition. It is interesting that the only structures arising from a harmonic-Einstein warped product, as explained above, to which we are able to apply the non-existence result is the one where the dimension of the fibre is $d = 1$. As a consequence we obtain that the existence of a complete φ-static metric, that is a metric such that (1) holds with $\mu = 1$, $f \in \mathcal{C}^{\infty}(M)$ and $\Delta_f f = -\lambda \in \mathbb{R}$, forces M to be non-compact and $\lambda < 0$.

Contents

Notations and conventions

All the manifolds are assumed to be smooth and connected. In what follows we shall freely use the method of the moving frame, as illustrated in Chapter 1 of [AMR], fixing two orthonormal coframes on the Riemannian manifolds $(M, \langle\, ,\, \rangle)$ and $(N, \langle\, ,\, \rangle)$ of dimension, respectively, m and n. We fix the indexes ranges

$$1 \le i, j, k, t, \ldots \le m, \quad 1 \le a, b, c, d \ldots \le n.$$

With $\{e_i\}$, $\{\theta^i\}$, $\{\theta^i_j\}$, $\{\Theta^i_j\}$ and $\{E_a\}$, $\{\omega^a\}$, $\{\omega^a_b\}$, $\{\Omega^a_b\}$ we shall respectively denote local orthonormal frames, coframes, the respectively Levi-Civita connection forms and curvature forms on the open subsets $\mathcal{U}$ of M and $\mathcal{V}$ on N. Throughout this thesis we adopt the Einstein summation convention over repeated indexes. Locally the metric $\langle\, ,\, \rangle$ is given by

$$\langle\, ,\, \rangle = \delta_{ij}\theta^i \otimes \theta^j,$$

and the dual frame $\{e_i\}$ is defined by the relations

$$\theta^j(e_i) = \delta^j_i,$$

The Levi-Civita connection forms $\{\theta^i_j\}$ are characterized, from Proposition 1.1 of [AMR], from the skew-symmetry property

$$\theta^i_j + \theta^j_i = 0,$$

and the validity of the first structure equations

$$d\theta^i + \theta^i_j \wedge \theta^j = 0.$$

The curvature forms $\{\Theta^i_j\}$ are defined by the second structure equations

$$d\theta^i_j + \theta^i_k \wedge \theta^k_j = \Theta^i_j$$

and they are skew-symmetric, that is,

$$\Theta^i_j + \Theta^j_i = 0.$$

The components in the basis $\{\theta^i \otimes \theta^j : 1 \le i < j \le m\}$ of the space of the skew-symmetric 2-forms on $\mathcal{U}$ are given by the components of the Riemann curvature tensor of $(M, \langle\, ,\, \rangle)$, that is,

$$\Theta^i_j = \frac{1}{2} R^i_{jkt}\theta^k \wedge \theta^t$$

where, denoting by R the $(1, 3)$ version of the curvature tensor of $(M, \langle\, ,\, \rangle)$,

$$R = R^i_{jkt}\theta^k \otimes \theta^t \otimes \theta^j \otimes e_i.$$

Recall that, for every $X, Y, Z \in \mathfrak{X}(M)$, where $\mathfrak{X}(M)$ denotes the $\mathcal{C}^\infty(M)$-module of smooth vector fields on M,

$$R(X, Y)Z = \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z,$$

where $[\,,\,]$ is the Lie bracket. The $(0,4)$ version of the curvature tensor of $(M,\langle\,,\,\rangle)$ is denoted by Riem and is defined by, for every $X,Y,Z,W\in\mathfrak{X}(M)$, by

$$\mathrm{Riem}(W,Z,X,Y)=\langle R(X,Y)Z,W\rangle,$$

locally

$$\mathrm{Riem}=R_{ijkt}\theta^i\otimes\theta^j\otimes\theta^k\otimes\theta^t,$$

where

$$R_{ijkt}=R^i_{jkt}.$$

The Ricci tensor is defined as the trace of Riemann, that is,

$$\mathrm{Ric}=R_{ij}\theta^i\otimes\theta^j \quad \text{where} \quad R_{ij}=R^k_{ikj}.$$

The Riemann tensor has the following symmetries

$$R_{ijkt}+R_{ijtk}=0,\quad R_{ijkt}+R_{jikt}=0,\quad R_{ijkt}=R_{ktij}$$

and satisfies the first Bianchi identity

$$R_{ijkt}+R_{iktj}+R_{itjk}=0$$

and the second Bianchi identity

$$R_{ijkt,l}+R_{ijtl,k}+R_{ijlk,t}=0,$$

where, for an arbitrary tensor field of type (r,s)

$$T=T^{i_1\ldots i_r}_{j_1\ldots j_s}\theta^{j_1}\otimes\ldots\otimes\theta^{j_s}\otimes e_{i_1}\otimes\ldots\otimes e_{i_r},$$

its covariant derivative is defined as the tensor field of type $(r,s+1)$

$$\nabla T=T^{i_1\ldots i_r}_{j_1\ldots j_s,k}\theta^k\otimes\theta^{j_1}\otimes\ldots\otimes\theta^{j_s}\otimes e_{i_1}\otimes\ldots\otimes e_{i_r},$$

by the relation

$$T^{i_1\ldots i_r}_{j_1\ldots j_s,k}\theta^k=dT^{i_1\ldots i_r}_{j_1\ldots j_s}-\sum_{t=1}^{s}T^{i_1\ldots i_r}_{j_1\ldots j_{t-1}hj_{t+1}\ldots j_s}\theta^h_{j_t}+\sum_{t=1}^{r}T^{i_1\ldots i_{t-1}hi_{t+1}\ldots i_r}_{j_1\ldots j_s}\theta^{i_t}_h.$$

The following commutation relation holds

$$T^{i_1\ldots i_r}_{j_1\ldots j_s,kt}=T^{i_1\ldots i_r}_{j_1\ldots j_s,tk}+\sum_{t=1}^{s}R^h_{j_tkt}T^{i_1\ldots i_r}_{j_1\ldots j_{t-1}hj_{t+1}\ldots j_s}-\sum_{t=1}^{r}R^{i_t}_{hkt}T^{i_1\ldots i_{t-1}hi_{t+1}\ldots i_r}_{j_1\ldots j_s}. \tag{14}$$

The formula above can be proved in general but, for simplicity of notations, we prove it for a tensor of type $(1,1)$. With the same argument one can prove it for general tensor fields.

Proposition 15. *Let T be a tensor of type $(1,1)$ on the Riemannian manifold $(M,\langle\,,\,\rangle)$, locally given by*

$$T=T^i_j\theta^j\otimes e_i.$$

Then

$$T^i_{j,kt}=T^i_{j,tk}+R^s_{jkt}T^i_s-R^i_{skt}T^s_j. \tag{16}$$

Proof. By definition of covariant derivative
$$T^i_{j,k}\theta^k = dT^i_j - T^i_s\theta^s_j + T^s_j\theta^i_s.$$
Taking the differential of the relation above we get
$$dT^i_{j,k}\wedge\theta^k + T^i_{j,k}d\theta^k = -dT^i_s\wedge\theta^s_j - T^i_s d\theta^s_j + dT^s_j\wedge\theta^i_s + T^s_j d\theta^i_s. \tag{17}$$
Once again, from the definition of covariant derivative
$$T^i_{j,kt}\theta^t = dT^i_{j,k} - T^i_{t,k}\theta^t_j - T^i_{j,t}\theta^t_k + T^t_{j,k}\theta^i_t.$$
Inserting the relation above into (17), using the first and the second structure equation we obtain
$$T^i_{j,kt}\theta^k\wedge\theta^t = T^i_s\Theta^s_j - T^s_j\Theta^i_s.$$
Recalling that
$$\Theta^i_j = \frac{1}{2}R^i_{jkt}\theta^k\wedge\theta^t,$$
skew-symmetrizing the above we conclude the validity of (16). ☐

Let $\varphi : M \to N$ be a smooth map and suppose, from now on, to have chosen the local coframes so that $\varphi^{-1}(\mathcal{V}) \subseteq \mathcal{U}$. We set
$$\varphi^*\omega^a = \varphi^a_i\theta^i$$
so that the differential $d\varphi$ of φ, a section of $T^*M\otimes\varphi^{-1}TN$, where $\varphi^{-1}TN$ is the pullback bundle, can be written as
$$d\varphi = \varphi^a_i\theta^i\otimes E_a.$$
The energy density $e(\varphi)$ of the map φ is defined as
$$e(\varphi) = \frac{|d\varphi|^2}{2},$$
where $|d\varphi|^2$ is the square of the Hilbert-Schmidt norm of $d\varphi$, that is,
$$|d\varphi|^2 = \varphi^a_i\varphi^a_i.$$
Observe that
$$|d\varphi|^2 = \mathrm{tr}(\varphi^*\langle\,,\,\rangle).$$
The generalized second fundamental tensor of the map φ is given by $\nabla d\varphi$, locally
$$\nabla d\varphi = \varphi^a_{ij}\theta^j\otimes\theta^i\otimes E_a,$$
where its coefficient are defined according to the rule
$$\varphi^a_{ij}\theta^j = d\varphi^a_i - \varphi^a_k\theta^k_i + \varphi^b_i\omega^a_b.$$
The tension field $\tau(\varphi)$ of the map φ is the section of $\varphi^{-1}TN$ defined by
$$\tau(\varphi) = \mathrm{tr}(\nabla d\varphi)$$
and it is locally given by
$$\tau(\varphi) = \varphi^a_{ii}E_a.$$
Let $\Omega\subseteq M$ be a relatively compact domain and let E_Ω be the energy functional on Ω, that is,
$$E_\Omega(\varphi) := \int_\Omega e(\varphi).$$
Recall that a smooth map $\varphi : (M, \langle\,,\,\rangle)$ is harmonic if for each relatively compact domain $\Omega\subseteq M$ it is a stationary point of the energy functional $E_\Omega : \mathcal{C}^\infty(M,N)\to\mathbb{R}$ with respect to variations preserving φ on $\partial\Omega$. It can be verified that φ is harmonic if and only if its tension field vanishes.

Part I

φ-curvatures and harmonic-Einstein manifolds

Chapter 1

φ-curvature tensors

In this Chapter we introduce some new curvature tensor fields and we describe their fundamental properties. Those tensor fields shall be called φ-curvatures and they take into account the geometry of a Riemannian manifold $(M, \langle\, ,\, \rangle)$ equipped with a smooth map $\varphi : M \to (N, \langle\, ,\, \rangle_N)$.

In Section 1.1 we fix the terminology for some properties that may be satisfied from the smooth map φ and that appears quite frequently in the sequel. Precisely we recall the definition of (weakly) conformal, homothetic, conservative, affine and relatively affine maps. We also define almost relatively affine maps. Further we state some known results relating those properties, that shall be useful.

In Section 1.2, the core of this Chapter, we define the φ-curvatures, comparing them with the classic curvature tensors (that can be seen as the φ-curvatures when φ is taken as a constant map). Those curvature tensors are the φ-Ricci Ric^φ, the φ-scalar S^φ, the φ-Schouten A^φ, the φ-Weyl W^φ, the φ-Cotton C^φ, the φ-Bach B^φ and the φ-traceless part T^φ of φ-Ricci. We also describe their symmetries and evaluate their traces and divergences.

In Section 1.3 we provide the transformation laws for the φ-curvatures under a conformal change of the metric. As major consequence we prove that in the four-dimensional case the φ-Bach tensor is a conformal invariant.

In the last Section of the Chapter, Section 1.4, we investigate the consequence on the vanishing of some tensors related to the φ-Weyl tensor and its derivatives on the geometry of the manifold and the smooth map φ. The consequences on the geometry of $(M, \langle\, ,\, \rangle)$ include and generalize the classic notions of locally conformally flat, harmonic Weyl curvature and conformally symmetric manifolds while the consequences on the map φ are related to the properties recalled in Section 1.1.

1.1 Smooth maps and conservation laws

Let $\varphi : (M, \langle\, ,\, \rangle) \to (N, \langle\, ,\, \rangle_N)$ be a smooth map between Riemannian manifolds of dimension, respectively, m and n.

Definition 1.1.1. The map φ is *weakly conformal* if there exists $\zeta \in \mathcal{C}^\infty(M)$ such that

$$\varphi^*\langle\, ,\, \rangle_N = \zeta \langle\, ,\, \rangle. \tag{1.1.2}$$

Remark 1.1.3. If φ is weakly conformal then, taking the trace of (1.1.2), we get

$$\zeta = \frac{|d\varphi|^2}{m}. \tag{1.1.4}$$

In particular $\zeta \geq 0$ on M.

Definition 1.1.5. Let φ be weakly conformal and $x \in M$. If $\zeta(x) = 0$ then x is called *branching point* of φ, otherwise x is called *regular point* of φ.

Remark 1.1.6. Assume φ is weakly conformal and $x \in M$ is a regular point for φ. Then φ is an immersion (that is, $d\varphi$ is injective) in a neighbourhood of x. Indeed, if $X \in T_xM$, evaluating (1.1.2) at X yields

$$|d\varphi(X)|_N^2 = \zeta(x)|X|^2,$$

and since $\zeta(x) \neq 0$, $d\varphi(X) = 0$ if and only if $X = 0$.
As a consequence, if φ is weakly conformal and $m > n$ then φ is constant. Indeed, assume by contradiction that φ is non-constant. Then there exists $x \in M$ regular point and thus, since φ is an immersion in a neighbourhood of x, $m \leq n$, that is a contradiction.

Definition 1.1.7. The map φ is *conformal* if (1.1.2) holds for some positive function ζ on M, that is, φ is weakly conformal with no branching points.

Remark 1.1.8. If φ is conformal then, by Remark 1.1.6, φ is an immersion of M into N.

Definition 1.1.9. The map φ is *homothetic* if (1.1.2) holds for some constant $\zeta \in \mathbb{R}$.

Remark 1.1.10. If φ is homothetic, from (1.1.4), we deduce that $|d\varphi|^2$ is constant.

Remark 1.1.11. If φ is a non-constant homothety, that is, if ζ is a positive constant, then the following is an isometric immersion

$$\varphi : (M, \zeta\langle\, ,\, \rangle) \to (N, \langle\, ,\, \rangle_N).$$

Definition 1.1.12. The *stress-energy tensor of* φ is given by

$$\mathbb{S} := \varphi^*\langle\, ,\, \rangle_N - e(\varphi)\langle\, ,\, \rangle, \tag{1.1.13}$$

where

$$e(\varphi) := \frac{1}{2}|d\varphi|^2$$

is the *density of energy* of φ. The map φ is called *conservative* if $\mathbb{S}$ is divergence free.

Remark 1.1.14. The stress-energy tensor (of harmonic maps) had been first defined by Baird and Eells in [BaE], with a different sign convention. Notice that, its vanishing and the vanishing of its divergence are independent from the sign convention.

Remark 1.1.15. The following are some trivial examples of conservative maps:

i) Constant maps;

ii) Weakly conformal maps, if $m = 2$;

iii) Homothetic maps.

To prove *ii*) and *iii*) let φ be a weakly conformal map. From (1.1.2), (1.1.4) and the definition (1.1.13) of $\mathbb{S}$ we deduce

$$\mathbb{S} = -\frac{m-2}{2m}|d\varphi|^2\langle\, ,\, \rangle. \tag{1.1.16}$$

If $m = 2$ then $\mathbb{S} = 0$, in particular $\operatorname{div}(\mathbb{S}) = 0$. If φ is homothetic then $|d\varphi|^2$ is constant, hence $\mathbb{S}$ is parallel and, in particular, $\operatorname{div}(\mathbb{S}) = 0$.

Proposition 1.1.17. *Let $\mathbb{S}$ be the stress-energy tensor of the smooth map $\varphi : (M, g) \to (N, h)$. Then*

$$div(\mathbb{S}) = \langle \tau(\varphi), d\varphi \rangle_N,$$

that is, in a local orthonormal coframe,

$$\mathbb{S}_{ij,j} = \varphi^a_{jj}\varphi^a_i. \tag{1.1.18}$$

Proof. In a local orthonormal coframe the components of $\mathbb{S}$ are given by

$$\mathbb{S}_{ij} = \varphi^a_i \varphi^a_j - \frac{|d\varphi|^2}{2}\delta_{ij}.$$

Taking the divergence of the above, using the symmetry of $\nabla d\varphi$, we get

$$\begin{aligned}\mathbb{S}_{ij,j} =&(\varphi^a_i \varphi^a_j)_j - \frac{|d\varphi|^2_j}{2}\delta_{ij}\\ =&\varphi^a_i\varphi^a_{jj} + \varphi^a_{ij}\varphi^a_j - \frac{|d\varphi|^2_i}{2}\\ =&\varphi^a_i\varphi^a_{jj} + \varphi^a_{ji}\varphi^a_j - \varphi^a_{ji}\varphi^a_j\\ =&\varphi^a_{jj}\varphi^a_i,\end{aligned}$$

that is (1.1.18). □

As an immediate consequence we have

Corollary 1.1.19. *If φ is harmonic then φ is conservative.*

As a partial converse of the above Corollary we have the following Proposition, whose proof is contained in [BaE].

Proposition 1.1.20. *If φ is a differentiable submersion almost everywhere and it is conservative then φ is harmonic.*

Remark 1.1.21. In the Proposition above the hypothesis that φ is a differentiable submersion almost everywhere cannot be removed. Indeed, there are situations in which φ is conservative even though it is not harmonic. For instance, let φ be a isometric immersion. Since $\tau(\varphi) = m\mathrm{H}$, where H is the mean curvature field of the immersion, see (1.170) of [AMR], φ is harmonic if and only if it is a minimal immersion. Since isometric immersion are clearly homothetic maps, from *iii*) of Remark 1.1.15 they are always conservative even thought they can be not minimal.

In the next Proposition we characterize the situations where $\mathbb{S}$ vanishes on M, that are the critical points of the energy for variation of the domain metric (rather then variations of the map), see [S].

Proposition 1.1.22. *Let φ be a non-constant map, then $\mathbb{S} = 0$ if and only if $m = 2$ and φ is weakly conformal.*

Proof. If $\mathbb{S} = 0$ then

$$\varphi^*\langle\, ,\,\rangle_N = \frac{|d\varphi|^2}{2}\langle\, ,\,\rangle, \tag{1.1.23}$$

thus φ is weakly conformal. Taking the trace of (1.1.23) we deduce

$$|d\varphi|^2 = \frac{m}{2}|d\varphi|^2,$$

that is,

$$\frac{m-2}{2}|d\varphi|^2 = 0.$$

Then either $|d\varphi|^2 = 0$ on M or $m = 2$, but since φ is non-constant we must have $m = 2$. The converse follows immediately from (1.1.16). □

The next Proposition is a sort of analogous of the above Proposition when $m \geq 3$.

Proposition 1.1.24. *If $m \geq 3$, φ is conservative and weakly conformal then φ is homothetic.*

Proof. Since φ is weakly conformal (1.1.16) holds, hence we may take its divergence to infer

$$\mathrm{div}(\mathbb{S}) = -\frac{m-2}{2m} d|d\varphi|^2.$$

Since φ is conservative $\mathrm{div}(\mathbb{S}) = 0$ and, using $m > 2$, from the above we infer $d|d\varphi|^2 = 0$ on M. Then, since M is connected, $|d\varphi|^2$ must be constant on M and thus φ is a homothetic map. □

The next definitions are contained in [IY].

Definition 1.1.25. The map φ is *affine* if $d\varphi$ the generalized second fundamental tensor of φ vanishes, that is,

$$\nabla d\varphi = 0. \tag{1.1.26}$$

Remark 1.1.27. Affine maps are totally geodesic, that is, they maps geodesic into geodesics. Moreover, affine maps are harmonic since the tension of a smooth map is the trace of its generalized second fundamental tensor.

Definition 1.1.28. The map φ is *relatively affine* if $\varphi^*\langle\, ,\, \rangle_N$ is parallel, that is,

$$\nabla \varphi^*\langle\, ,\, \rangle_N = 0$$

Remark 1.1.29. If φ is affine the it is also relatively affine.

Remark 1.1.30. One can see that φ is a relative affine map if and only if, in a local orthonormal coframe,

$$\varphi^a_{ij}\varphi^a_k = 0. \tag{1.1.31}$$

Indeed, a computation shows

$$(\varphi^a_i\varphi^a_j)_k = \varphi^a_{ik}\varphi^a_j + \varphi^a_i\varphi^a_{jk}.$$

If (1.1.31) holds then, from the above, $\nabla\varphi^*\langle\, ,\, \rangle_N = 0$. On the other hand, if $\nabla\varphi^*\langle\, ,\, \rangle_N = 0$, using the symmetry of $\nabla d\varphi$ and the above we easily conclude that (1.1.31) holds.

Remark 1.1.32. If φ is relatively affine then, summing (1.1.31) on i and j and on i and k, respectively, one gets that φ is conservative and $|d\varphi|^2$ is constant on M. On the other hand relatively affine maps can be not harmonic (and, as a consequence, non affine), see page 41 of [X] and references therein for examples.

Recall that, as defined in [P], a symmetric 2-times covariant tensor field is harmonic if it is a Codazzi tensor, that is, his covariant derivative is totally symmetric, and it is divergence free (or equivalently, if it is a Codazzi tensor with constant trace).

Definition 1.1.33. The map φ is *almost relatively affine* if $\varphi^*\langle\, ,\, \rangle_N$ is harmonic.

Remark 1.1.34. The author has not find in the literature the definition of smooth maps φ such that $\varphi^*\langle\, ,\, \rangle_N$ is a Codazzi tensor nor such that $\varphi^*\langle\, ,\, \rangle_N$ is harmonic, but since in our study we ran into the latter situation he find reasonable to give the definition above.

Remark 1.1.35. Relatively affine (and thus also affine) maps are almost relatively affine. It is easy to see that $\varphi^*\langle\, ,\, \rangle_N$ is Codazzi if and only if, in a local orthonormal coframe

$$\varphi^a_{ij}\varphi^a_k = \varphi^a_{ik}\varphi^a_j. \tag{1.1.36}$$

If φ is almost relatively affine, tracing (1.1.36), we get

$$\mathrm{div}(\mathbb{S}) = \frac{1}{2} d|d\varphi|^2,$$

where $\mathbb{S}$ is the energy-stress tensor of the map φ, defined by (1.1.13). As a consequence the almost relatively affine map φ, since $|d\varphi|^2$ is constant, is also conservative.

The *vertical distribution* of φ is determined by the vector fields $X \in \mathfrak{X}(M)$ such that $d\varphi(X) = 0$. From Proposition 2.1 of [IY] a relatively affine map has constant rank on M equal to q and, in case $0 < q < n$, the vertical distribution has dimension $q - n$ and it is parallel, that is, if X, Y are such that $d\varphi(X) = 0$ and $d\varphi(Y) = 0$, then $d\varphi(\nabla_X Y) = 0$.

1.2 Definition of φ-curvatures and properties

Let $(M, \langle\,,\,\rangle)$ be Riemannian manifold of dimension m and $\varphi : (M, \langle\,,\,\rangle) \to (N, \langle\,,\,\rangle_N)$ a smooth map, where the target $(N, \langle\,,\,\rangle_N)$ is a Riemannian manifold of dimension n. We fix $\alpha \in \mathcal{C}(M)$, $\alpha \not\equiv 0$ on M.

Definition 1.2.1. Indicating with Ric the usual Ricci tensor of $(M, \langle\,,\,\rangle)$ we define the *φ-Ricci tensor* by setting

$$\mathrm{Ric}^{\varphi} := \mathrm{Ric} - \alpha\varphi^*\langle\,,\,\rangle_N. \tag{1.2.2}$$

In a local orthonormal coframe

$$R^{\varphi}_{ij} = R_{ij} - \alpha\varphi^a_i\varphi^a_j$$

where

$$\mathrm{Ric}^{\varphi} = R^{\varphi}_{ij}\theta^i \otimes \theta^j, \quad \mathrm{Ric} = R_{ij}\theta^i \otimes \theta^j \quad \text{and} \quad d\varphi = \varphi^a_i\theta^i \otimes E_a.$$

Remark 1.2.3. The φ-Ricci curvature and the Ricci curvature coincide if and only if φ is locally costant on $\{x \in M : \alpha(x) \neq 0\}$. Indeed, $\mathrm{Ric}^{\varphi} = \mathrm{Ric}$ if and only if $\alpha\varphi^*\langle\,,\,\rangle_N = 0$, that is, $\varphi^*\langle\,,\,\rangle_N = 0$ on the open subset $\{x \in M : \alpha(x) \neq 0\}$ of M. Since $\langle\,,\,\rangle_N$ is a Riemannian metric on N and, for every $X \in T_{x_0}M$, where $x_0 \in M$,

$$(\varphi^*\langle\,,\,\rangle_N)(X, X) = |d\varphi(X)|^2_N, \tag{1.2.4}$$

we deduce that $\varphi^*\langle\,,\,\rangle_N = 0$ at a point $x_0 \in M$ if and only if $d\varphi = 0$ at x_0. Then $\varphi^*\langle\,,\,\rangle_N = 0$ on $\{x \in M : \alpha(x) \neq 0\}$ if and only if $d\varphi = 0$ on $\{x \in M : \alpha(x) \neq 0\}$, that is, φ is locally costant on $\{x \in M : \alpha(x) \neq 0\}$.

Definition 1.2.5. The *φ-scalar curvature* S^{φ} is defined as

$$S^{\varphi} = \mathrm{tr}(\mathrm{Ric}^{\varphi}).$$

Using (1.2.2) we get

$$S^{\varphi} := S - \alpha|d\varphi|^2, \tag{1.2.6}$$

where S is the usual scalar curvature of $(M, \langle\,,\,\rangle)$ and $|d\varphi|^2$ is the square of the Hilbert-Schmidt norm of the section $d\varphi$ of the vector bundle φ^*TN.

Remark 1.2.7. Observe that $S^{\varphi} = S$ if and only if $\alpha|d\varphi|^2 = 0$, that is, $|d\varphi|^2 = 0$ on $\{x \in M : \alpha(x) \neq 0\}$. Then the φ-scalar curvature and the usual scalar curvature coincide if and only if φ is locally constant on $\{x \in M : \alpha(x) \neq 0\}$.

Remark 1.2.8. The φ-Ricci tensor and the φ-scalar first appeared in the work [M] of R. Müller and the notation adopted here have also been used by L. F. Wang in [W].

Definition 1.2.9. When $m \geq 2$ we introduce the *φ-Schouten tensor* A^{φ} in analogy with the standard case

$$A^{\varphi} := \mathrm{Ric}^{\varphi} - \frac{S^{\varphi}}{2(m-1)}\langle\,,\,\rangle. \tag{1.2.10}$$

In a local orthonormal coframe

$$A^{\varphi}_{ij} = R^{\varphi}_{ij} - \frac{S^{\varphi}}{2(m-1)}\delta_{ij},$$

where

$$A^{\varphi} = A^{\varphi}_{ij}\theta^i \otimes \theta^j.$$

An immediate computation using (1.2.2) and (1.2.6) gives the relation of A^{φ} with the usual Schouten tensor A, that is,

$$A^{\varphi} = A - \alpha\left(\varphi^*\langle\,,\,\rangle_N - \frac{|d\varphi|^2}{2(m-1)}\langle\,,\,\rangle\right), \tag{1.2.11}$$

where

$$A = \mathrm{Ric} - \frac{S}{2(m-1)}\langle\,,\,\rangle.$$

Remark 1.2.12. Assume $m = 2$. Since in this situation Ric is proportional to the metric $\langle\, , \rangle$, the Schouten tensor A vanishes. As a consequence, from (1.2.11) we get

$$A^{\varphi} = -\alpha \mathbb{S}.$$

where $\mathbb{S}$ is the stress-energy tensor defined by (1.1.13). In particular $A^{\varphi} = A$ if and only if $\mathbb{S} = 0$ on $\{x \in M : \alpha(x) \neq 0\}$, that is equivalent in case φ is non-constant on $\{x \in M : \alpha(x) \neq 0\}$, in view of Proposition 1.1.22, to the fact that φ is weakly conformal on $\{x \in M : \alpha(x) \neq 0\}$.
Assume $m \geq 3$. Observe that $A^{\varphi} = A$ if and only if

$$\varphi^{*}\langle\, , \rangle_N = \frac{|d\varphi|^2}{2(m-1)}\langle\, , \rangle \quad \text{on } \{x \in M : \alpha(x) \neq 0\}. \tag{1.2.13}$$

In particular φ is weakly conformal, when restricted to $\{x \in M : \alpha(x) \neq 0\}$. By taking the trace of (1.2.13) we infer

$$\frac{m-2}{2(m-1)}|d\varphi|^2 = 0 \quad \text{on } \{x \in M : \alpha(x) \neq 0\}.$$

In conclusion, when $m \geq 3$, $A^{\varphi} = A$ if and only if φ is locally constant on $\{x \in M : \alpha(x) \neq 0\}$.

Remark 1.2.14. An easy computation shows

$$\operatorname{tr}(A^{\varphi}) = \frac{m-2}{2(m-1)} S^{\varphi}. \tag{1.2.15}$$

Indeed, using (1.2.10) and (1.2.4) we infer

$$\operatorname{tr}(A^{\varphi}) = \operatorname{tr}(\mathrm{Ric}^{\varphi}) - \frac{m}{2(m-1)} S^{\varphi} = \frac{m-2}{2(m-1)} S^{\varphi}.$$

We recall the Kulkarni-Nomizu product, that we shall indicate with the "parrot"operator $\owedge$, of two symmetric 2-covariant tensors. It gives rise to a 4-covariant tensor with the same symmetries of Riem, the Riemann curvature tensor. In components, with respect to a local orthonormal coframe, given the 2-covariant symmetric tensors T and V we have

$$(V \owedge T)_{ijkt} := V_{ik}T_{jt} - V_{it}T_{jk} + V_{jt}T_{ik} - V_{jk}T_{it}. \tag{1.2.16}$$

Definition 1.2.17. For $m \geq 3$, the φ-*Weyl tensor* is defined by

$$W^{\varphi} := \mathrm{Riem} - \frac{1}{m-2} A^{\varphi} \owedge \langle\, , \rangle, \tag{1.2.18}$$

where Riem is the Riemann tensor of $(M, \langle\, , \rangle)$.

In a local orthonormal coframe

$$W^{\varphi}_{tijk} = R_{tijk} - \frac{1}{m-2}(A^{\varphi}_{tj}\delta_{ik} - A^{\varphi}_{tk}\delta_{ij} + A^{\varphi}_{ik}\delta_{tj} - A^{\varphi}_{ij}\delta_{tk}),$$

where

$$W^{\varphi} = W^{\varphi}_{tijk}\theta^t \otimes \theta^i \otimes \theta^j \otimes \theta^k \quad \text{and} \quad \mathrm{Riem} = R_{tijk}\theta^t \otimes \theta^i \otimes \theta^j \otimes \theta^k.$$

From the standard decomposition of the Riemann curvature tensor we know that, for $m \geq 3$,

$$\mathrm{Riem} = W + \frac{1}{m-2} A \owedge \langle\, , \rangle,$$

where W is the standard Weyl tensor of $(M, \langle\, , \rangle)$. From the distributivity of $\owedge$ with respect to sums, together with (1.2.11), we deduce the expression of W^{φ} in terms of W:

$$W^{\varphi} = W + \frac{\alpha}{m-2}\left(\varphi^{*}\langle\, , \rangle_N - \frac{|d\varphi|^2}{2(m-1)}\langle\, , \rangle\right) \owedge \langle\, , \rangle. \tag{1.2.19}$$

Remark 1.2.20. Notice that $W^\varphi = W$ if and only if (1.2.13) holds, this is due to the fact that $\cdot \owedge \langle \, , \, \rangle$ is injective. Then, since $m \geq 3$, W^φ coincide with W if and only if φ is locally constant on $\{x \in M : \alpha(x) \neq 0\}$ on M, as seen in Remark 1.2.12.

If a four times covariant tensor field K has the same symmetries of Riem then all his traces can be determined by K_{kikj}, hence it is convenient to denote

$$\mathrm{tr}(K)_{ij} := K_{kikj}.$$

Observe that $\mathrm{tr}(K)$ is a two times covariant tensor field and $\mathrm{tr}(\mathrm{Riem}) = \mathrm{Ric}$.

Proposition 1.2.21. *The φ-Weyl tensor field has the same symmetries of Riem and*

$$tr(W^\varphi) = \alpha\varphi^*\langle \, , \, \rangle_N. \tag{1.2.22}$$

Proof. The φ-Weyl tensor field has the same symmetries of Riem because, as mentioned above, the Kulkarni-Nomizu product of two times covariant tensor fields have the same symmetries of Riem. Observe that, using (1.2.18), (1.2.10), (1.2.15) and (1.2.2),

$$\begin{aligned} W^\varphi_{jijk} =& R_{jijk} - \frac{1}{m-2}(A^\varphi_{jj} - A^\varphi_{jk}\delta_{ij} + A^\varphi_{ik}\delta_{jj} - A^\varphi_{ij}\delta_{jk}) \\ =& R_{ik} - A^\varphi_{ik} - \frac{1}{m-2}A^\varphi_{jj}\delta_{ik} \\ =& R_{ik} - R^\varphi_{ik} + \frac{S^\varphi}{2(m-1)}\delta_{ik} - \frac{1}{m-2}\frac{m-2}{2(m-1)}S^\varphi\delta_{ik} \\ =& \alpha\varphi^a_i\varphi^a_k, \end{aligned}$$

that is, (1.2.22). □

Remark 1.2.23. Combining the above Proposition with Remark 1.2.20, the φ-Weyl tensor field is totally traceless if and only if it coincide with the Weyl tensor.

Remark 1.2.24. Assume $m = 3$. Is well known that $W = 0$, hence from (1.2.19),

$$W^\varphi = \alpha\left(\varphi^*\langle \, , \, \rangle_N - \frac{|d\varphi|^2}{4}\langle \, , \, \rangle\right) \owedge \langle \, , \, \rangle.$$

For the rest of the section we consider α to be a non-null constant. The next result, analogous to Schur's identity, typically shows how the geometry of φ enters into the picture.

Proposition 1.2.25. *In a local orthonormal coframe*

$$R^\varphi_{ij,i} = \frac{1}{2}S^\varphi_j - \alpha\varphi^a_{ii}\varphi^a_j, \tag{1.2.26}$$

where φ^a_{ii} are the components of the tension field $\tau(\varphi)$ of the map φ.

Proof. By taking the covariant derivative of (1.2.6) we get

$$\frac{1}{2}S_j = \frac{1}{2}S^\varphi_j + \alpha\varphi^a_{ij}\varphi^a_i$$

and by the usual Schur's identity we obtain

$$R_{ij,i} = \frac{1}{2}S^\varphi_j + \alpha\varphi^a_{ij}\varphi^a_i. \tag{1.2.27}$$

Using (1.2.2) we infer

$$R^\varphi_{ij,i} = R_{ij,i} - \alpha\varphi^a_{ii}\varphi^a_j - \alpha\varphi^a_i\varphi^a_{ji}.$$

Therefore, from the symmetries of $\nabla d\varphi$,

$$R^\varphi_{ij,i} = R_{ij,i} - \alpha\varphi^a_{ij}\varphi^a_i - \alpha\varphi^a_{ii}\varphi^a_j$$

and, using (1.2.27), from the above we conclude the validity of (1.2.26). □

Remark 1.2.28. In global notation (1.2.26) becomes

$$\mathrm{div}(\mathrm{Ric}^{\varphi}) = \frac{1}{2}dS^{\varphi} - \alpha\mathrm{div}(\mathbb{S}),$$

where $\mathbb{S}$ is the stress-energy tensor of the map φ, defined in (1.1.13). Since α is constant it can also be written as

$$\mathrm{div}(\mathrm{Ric}^{\varphi} + \alpha\mathbb{S}) = \frac{1}{2}dS^{\varphi}. \tag{1.2.29}$$

A trivial computation, similar to the one performed in the proof of the above Proposition, shows that (1.2.29) holds even in case we consider α to be a non-constant differentiable function. We stated the Proposition above with α constant because in that case, using (1.2.26), the following analogous of the usual Schur's identity holds

$$R^{\varphi}_{ij,i} = \frac{1}{2}S^{\varphi}_{j}$$

if φ is conservative (actually, also the converse implication holds). When dealing with harmonic-Einstein manifolds in Chapter 2 a key fact will be the validity of the formula above.

Definition 1.2.30. Analogously to the standard case, when $m \geq 2$, we define the *φ-Cotton tensor* C^{φ} as the obstruction to the commutativity of the covariant derivative of A^{φ}, that is, in a local orthonormal coframe,

$$C^{\varphi}_{ijk} := A^{\varphi}_{ij,k} - A^{\varphi}_{ik,j}. \tag{1.2.31}$$

Using definition (1.2.10) of A^{φ} we compute the indicated covariant derivatives in (1.2.31) to obtain C^{φ} expressed in terms of the usual Cotton tensor C of $(M, \langle\, , \,\rangle)$ in the following

Proposition 1.2.32. *If α is constant then the φ-Cotton tensor field and the Cotton tensor field C of $(M, \langle\, , \,\rangle)$ are related by*

$$C^{\varphi}_{ijk} = C_{ijk} - \alpha\left[\varphi^a_{ik}\varphi^a_j - \varphi^a_{ij}\varphi^a_k - \frac{\varphi^a_t}{m-1}(\varphi^a_{tk}\delta_{ij} - \varphi^a_{tj}\delta_{ik})\right]. \tag{1.2.33}$$

Proof. An easy computation using (1.2.31), (1.2.10), (1.2.2) and(1.2.6) shows that

$$\begin{aligned}
C^{\varphi}_{ijk} =& A^{\varphi}_{ij,k} - A^{\varphi}_{ik,j}\\
=& R^{\varphi}_{ij,k} - \frac{S^{\varphi}_k}{2(m-1)}\delta_{ij} - R^{\varphi}_{ik,j} + \frac{S^{\varphi}_j}{2(m-1)}\delta_{ik}\\
=& R_{ij,k} - \alpha(\varphi^a_i\varphi^a_j)_k - \frac{S_k}{2(m-1)}\delta_{ij} + \alpha\frac{|d\varphi|^2_k}{2(m-1)}\delta_{ij}\\
&- R_{ik,j} + \alpha(\varphi^a_i\varphi^a_k)_j + \frac{S_j}{2(m-1)}\delta_{ij} - \alpha\frac{|d\varphi|^2_j}{2(m-1)}\delta_{ik}\\
=& A_{ij,k} - A_{ik,j} - \alpha\left[\varphi^a_{ik}\varphi^a_j + \varphi^a_i\varphi^a_{jk} - \frac{\varphi^a_t\varphi^a_{tk}}{m-1}\delta_{ij} - \varphi^a_{ij}\varphi^a_k - \varphi^a_i\varphi^a_{kj} + \frac{\varphi^a_t\varphi^a_{tj}}{m-1}\delta_{ik}\right],
\end{aligned}$$

that is, since $C_{ijk} = A_{ij,k} - A_{ik,j}$, (1.2.33). ☐

The relations in the Proposition below are obtained by computation.

Proposition 1.2.34. *The φ-Cotton tensor field satisfies the following properties:*

$$C^{\varphi}_{ikj} = -C^{\varphi}_{ijk} \quad \textit{and therefore} \quad C^{\varphi}_{ijj} = 0; \tag{1.2.35}$$

$$C^{\varphi}_{jji} = \alpha\varphi^a_{jj}\varphi^a_i = -C^{\varphi}_{jij}; \tag{1.2.36}$$

$$C^{\varphi}_{ijk} + C^{\varphi}_{jki} + C^{\varphi}_{kij} = 0. \tag{1.2.37}$$

Proof. We prove only (1.2.36) because (1.2.35) and (1.2.37) are trivially satisfied. Using (1.2.10), (1.2.15) and (1.2.26) we deduce

$$\begin{aligned} C^{\varphi}_{iik} &= A^{\varphi}_{ii,k} - A^{\varphi}_{ik,i} \\ &= \left(\frac{m-2}{2(m-1)} S^{\varphi}\right)_k - R^{\varphi}_{ik,i} + \frac{S^{\varphi}_k}{2(m-1)} \\ &= \frac{m-2}{2(m-1)} S^{\varphi}_k - \left(\frac{S^{\varphi}_k}{2} - \alpha\varphi^a_{ii}\varphi^a_k\right) + \frac{1}{2(m-1)} S^{\varphi}_k \\ &= \alpha\varphi^a_{ii}\varphi^a_k, \end{aligned}$$

that is, (1.2.36). □

Remark 1.2.38. Observe that, since $\alpha \neq 0$, $C^{\varphi} = C$ if and only if the tensor field

$$\varphi^*\langle\,,\,\rangle_N - \frac{|d\varphi|^2}{2(m-1)}\langle\,,\,\rangle \tag{1.2.39}$$

is a Codazzi tensor. Natural examples of situations in which (1.2.39) is a Codazzi tensor are when $m = 2$ and φ is weakly conformal or when $m \geq 3$ and φ is homothetic. Indeed, in both cases,

$$\varphi^*\langle\,,\,\rangle_N - \frac{|d\varphi|^2}{2(m-1)}\langle\,,\,\rangle = \frac{m-2}{2m(m-1)}|d\varphi|^2\langle\,,\,\rangle. \tag{1.2.40}$$

If $m = 2$ the right hand side of (1.2.40) vanishes while, if $m \geq 3$ and $|d\varphi|^2$ is constant, then the right hand side of (1.2.40) is parallel. Another situation in which $C^{\varphi} = C$ is when φ is almost relatively affine, see Definition 1.1.33. Notice that in all the examples above the map φ is conservative, as one can expected since C is totally traceless.

Remark 1.2.41. For a three times covariant tensor field C on M that is skew symmetric in the last two indices, that is $C_{ikj} = -C_{ikj}$, all its traces are determined by C_{ijj}, hence it is convenient to denote:

$$\operatorname{tr}(C)_i := C_{ijj}.$$

Then $\operatorname{tr}(C)$ is a 1-form on M. Observe that (1.2.36) gives

$$\operatorname{tr}(C^{\varphi}) = \alpha \operatorname{div}(\mathbb{S}).$$

Explicitating (1.2.31) in terms of $R^{\varphi}_{ij,k}$ we obtain the commutation relations

$$R^{\varphi}_{ij,k} = R^{\varphi}_{ik,j} + C^{\varphi}_{ijk} + \frac{1}{2(m-1)}(S^{\varphi}_k\delta_{ij} - S^{\varphi}_j\delta_{ik}), \tag{1.2.42}$$

that shall be useful later on.

Remark 1.2.43. If $m = 2$, from the symmetries of C^{φ}, the only non-vanishing components of C^{φ} can be determined by C^{φ}_{iik} (no sum on i) for $\{i, k\} = \{1, 2\}$. It is immediate to see that (no sum on i)

$$C^{\varphi}_{iik} = \alpha \operatorname{div}(\mathbb{S})_k,$$

indeed using (1.2.36),

$$\alpha \operatorname{div}(\mathbb{S})_k = \operatorname{tr}(C^{\varphi})_k = C^{\varphi}_{iik} + C^{\varphi}_{kkk} = C^{\varphi}_{iik}.$$

Then $C^{\varphi} = 0$ if and only if φ is conservative, for $m = 2$.

In the next Proposition we evaluate the divergence of W^{φ} in terms of C^{φ}.

Proposition 1.2.44. *For $m \geq 3$, in a local orthonormal coframe,*

$$W^{\varphi}_{tijk,t} = \frac{m-3}{m-2}C^{\varphi}_{ikj} + \alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j) + \frac{\alpha}{m-2}\varphi^a_{tt}(\varphi^a_j\delta_{ik} - \varphi^a_k\delta_{ij}). \tag{1.2.45}$$

Proof. Observe that from (1.2.19) we can express W^{φ}_{tijk} componentwise in the form

$$W^{\varphi}_{tijk} = W_{tijk} + \frac{\alpha}{m-2}(\varphi^a_t\varphi^a_j\delta_{ik} - \varphi^a_t\varphi^a_k\delta_{ij} + \varphi^a_i\varphi^a_k\delta_{tj} - \varphi^a_i\varphi^a_j\delta_{tk})$$
$$-\alpha\frac{|d\varphi|^2}{(m-1)(m-2)}(\delta_{tj}\delta_{ik} - \delta_{tk}\delta_{ij}).$$

Taking covariant derivatives, tracing, using the well known formula (see for instance equation (1.87) of [AMR])

$$W_{tijk,t} = -\frac{m-3}{m-2}C_{ijk}, \tag{1.2.46}$$

and (1.2.33) we obtain

$$\begin{aligned}
W^{\varphi}_{tijk,t} =& W_{tijk,t} + \frac{\alpha}{m-2}(\varphi^a_{tt}\varphi^a_j\delta_{ik} + \varphi^a_t\varphi^a_{jt}\delta_{ik} - \varphi^a_{tt}\varphi^a_k\delta_{ij} - \varphi^a_t\varphi^a_{kt}\delta_{ij}) \\
&+ \frac{\alpha}{m-2}(\varphi^a_{ij}\varphi^a_k + \varphi^a_i\varphi^a_{kj} - \varphi^a_{ik}\varphi^a_j - \varphi^a_i\varphi^a_{jk}) + \frac{\alpha}{m-2}\left[-\frac{2\varphi^a_s\varphi^a_{st}}{m-1}(\delta_{tj}\delta_{ik} - \delta_{tk}\delta_{ij})\right] \\
=&\frac{m-3}{m-2}C_{ikj} + \frac{\alpha}{m-2}\left[\varphi^a_{tt}(\varphi^a_j\delta_{ik} - \varphi^a_k\delta_{ij}) + \varphi^a_t(\varphi^a_{jt}\delta_{ij} - \varphi^a_{kt}\delta_{ij}) + \varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j\right] \\
&+ \frac{\alpha}{m-2}\left[-\frac{2}{m-1}\varphi^a_s(\varphi^a_{sj}\delta_{ik} - \varphi^a_{sk}\delta_{ij})\right] \\
=&\frac{m-3}{m-2}C^{\varphi}_{ikj} + \alpha\frac{m-3}{m-2}\left[\varphi_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j - \frac{\varphi^a_t}{m-1}(\varphi^a_{tj}\delta_{ik} - \varphi^a_{tk}\delta_{ij})\right] \\
&+ \frac{\alpha}{m-2}\left[\varphi^a_{tt}(\varphi^a_j\delta_{ik} - \varphi^a_k\delta_{ij}) + \frac{m-3}{m-1}\varphi^a_t(\varphi^a_{jt}\delta_{ij} - \varphi^a_{kt}\delta_{ij}) + \varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j\right] \\
=&\frac{m-3}{m-2}C^{\varphi}_{ikj} + \alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j) + \frac{\alpha}{m-2}\varphi^a_{tt}(\varphi^a_j\delta_{ik} - \varphi^a_k\delta_{ij}),
\end{aligned}$$

that is, (1.2.45). □

The following proposition contains the 'fake Bianchi identity"for W^{φ}.

Proposition 1.2.47. *In a local orthonormal frame*

$$W^{\varphi}_{tijk,l} + W^{\varphi}_{tikl,j} + W^{\varphi}_{tilj,k} = \frac{1}{m-2}(C^{\varphi}_{tjk}\delta_{il} + C^{\varphi}_{tkl}\delta_{ij} + C^{\varphi}_{tlj}\delta_{ik} - C^{\varphi}_{ijk}\delta_{tl} - C^{\varphi}_{ikl}\delta_{tj} - C^{\varphi}_{ilj}\delta_{tk}).$$

Proof. It follows from a computation using the decomposition (1.2.18), the second Bianchi identity for Riem and the definition (1.2.31) of C^{φ}. □

Remark 1.2.48. Formula (1.2.45) can also be deduced taking the trace of the fake Bianchi identity above, using (1.2.22) and (1.2.36).

Definition 1.2.49. We introduce, for $m \geq 3$, the φ-Bach tensor B^{φ} by setting, in a local orthonormal coframe

$$(m-2)B^{\varphi}_{ij} = C^{\varphi}_{ijk,k} + R^{\varphi}_{tk}(W^{\varphi}_{tikj} - \alpha\varphi^a_t\varphi^a_i\delta_{jk}) + \alpha\left(\varphi^a_{ij}\varphi^a_{kk} - \varphi^a_{kkj}\varphi^a_i - \frac{1}{m-2}|\tau(\varphi)|^2\delta_{ij}\right). \tag{1.2.50}$$

Remark 1.2.51. If φ is a constant map then the φ-Bach tensor reduces to the usual Bach tensor B, whose components in a local orthonormal coframe are given by

$$(m-2)B_{ij} = C_{ijk,k} + R_{tk}W_{tikj}.$$

Proposition 1.2.52. *Let $m \geq 3$, the φ-Bach tensor is symmetric and*

$$tr(B^{\varphi}) = \alpha\frac{m-4}{(m-2)^2}|\tau(\varphi)|^2. \tag{1.2.53}$$

Proof. We rewrite B^{φ} in the form

$$(m-2)B^{\varphi} = V + Z$$

where:

$$V_{ij} := C^{\varphi}_{ijk,k} - \alpha R^{\varphi}_{kj}\varphi^a_k\varphi^a_i - \alpha\varphi^a_{kkj}\varphi^a_i, \quad Z_{ij} := R^{\varphi}_{tk}W^{\varphi}_{tikj} + \alpha\varphi^a_{ij}\varphi^a_{kk} - \frac{\alpha}{m-2}|\tau(\varphi)|^2\delta_{ij}.$$

Since Z is symmetric it remains to show that V shares the same property. To verify this fact, in other words that $V_{ij} = V_{ji}$, we see that, explicitating both sides of the equality, it turns out to be equivalent to show that

$$\alpha[\varphi^a_k(R^{\varphi}_{ik}\varphi^a_j - R^{\varphi}_{kj}\varphi^a_i) + \varphi^a_{kki}\varphi^a_j - \varphi^a_{kkj}\varphi^a_i] = C^{\varphi}_{jik,k} - C^{\varphi}_{ijk,k} = -(C^{\varphi}_{ijk} - C^{\varphi}_{jik})_k.$$

By using (1.2.35) and (1.2.37) we have

$$-(C^{\varphi}_{ijk} - C^{\varphi}_{jik})_k = -(C^{\varphi}_{ijk} + C^{\varphi}_{jki})_k = C^{\varphi}_{kij,k},$$

hence the above equality is equivalent to

$$C^{\varphi}_{kij,k} = \alpha[\varphi^a_k(R^{\varphi}_{ik}\varphi^a_j - R^{\varphi}_{kj}\varphi^a_i) + \varphi^a_{kki}\varphi^a_j - \varphi^a_{kkj}\varphi^a_i]. \tag{1.2.54}$$

It remains to compute $C^{\varphi}_{kij,k}$ to verify (1.2.54). From the general formula (14) we get

$$A^{\varphi}_{ik,jk} = A^{\varphi}_{ki,kj} + R_{kj}A^{\varphi}_{ki} + R^t_{ijk}A^{\varphi}_{tk}. \tag{1.2.55}$$

Using (1.2.31) and (1.2.55) twice we have

$$C^{\varphi}_{kij,k} = A^{\varphi}_{ki,jk} - A^{\varphi}_{kj,ik} = (A^{\varphi}_{ki,kj} + R_{kj}A^{\varphi}_{ki} + R^t_{ijk}A^{\varphi}_{kt}) - (A^{\varphi}_{kj,ki} + R_{ki}A^{\varphi}_{kj} + R^t_{jik}A^{\varphi}_{kt}).$$

Hence, with the aid of (1.2.10), we deduce

$$\begin{aligned} C^{\varphi}_{kij,k} = &\left(R^{\varphi}_{ki,k} - \frac{S^{\varphi}_k}{2(m-1)}\delta_{ki}\right)_j + R_{kj}A^{\varphi}_{ki} + R^t_{ijk}A^{\varphi}_{kt} \\ &- \left(R^{\varphi}_{kj,k} - \frac{S^{\varphi}_k}{2(m-1)}\delta_{kj}\right)_i - R_{ki}A^{\varphi}_{kj} - R^t_{jik}A^{\varphi}_{kt}. \end{aligned}$$

From (1.2.26) and the symmetries of Riem we obtain

$$\begin{aligned} C^{\varphi}_{kij,k} &= \left(\frac{1}{2}S^{\varphi}_i - \alpha\varphi^a_{kk}\varphi^a_i - \frac{S^{\varphi}_i}{2(m-1)}\right)_j + R_{kj}A^{\varphi}_{ki} + R^t_{ijk}A^{\varphi}_{kt} \\ &\quad - \left(\frac{1}{2}S^{\varphi}_j - \alpha\varphi^a_{kk}\varphi^a_j - \frac{S^{\varphi}_j}{2(m-1)}\right)_i - R_{ki}A^{\varphi}_{kj} - R^t_{ijk}A^{\varphi}_{kt} \\ &= \left(\frac{m-2}{2(m-1)}S^{\varphi}_i - \alpha\varphi^a_{kk}\varphi^a_i\right)_j + R_{kj}A^{\varphi}_{ki} - \left(\frac{m-2}{2(m-1)}S^{\varphi}_j - \alpha\varphi^a_{kk}\varphi^a_j\right)_i - R_{ki}A^{\varphi}_{kj}. \end{aligned}$$

Since $\mathrm{Hess}(S^\varphi)$ is symmetric we deduce

$$C^\varphi_{kij,k} = \alpha(\varphi^a_{kk}\varphi^a_j)_i - \alpha(\varphi^a_{kk}\varphi^a_i)_j + R_{kj}A^\varphi_{ki} - R_{ki}A^\varphi_{kj}.$$

Using once again (1.2.10) and the symmetry of $\nabla d\varphi$

$$\begin{aligned} C^\varphi_{kij,k} =& \alpha(\varphi^a_{kki}\varphi^a_j + \varphi^a_{kk}\varphi^a_{ji} - \varphi^a_{kkj}\varphi^a_i - \varphi^a_{kk}\varphi^a_{ij}) \\ &+ R_{kj}\left(R^\varphi_{ki} - \frac{S^\varphi}{2(m-1)}\delta_{ki}\right) - R_{ki}\left(R^\varphi_{kj} - \frac{S^\varphi}{2(m-1)}\delta_{kj}\right) \\ =& \alpha(\varphi^a_{kki}\varphi^a_j - \varphi^a_{kkj}\varphi^a_i) + R_{kj}R^\varphi_{ki} - \frac{S^\varphi}{2(m-1)}R_{ij} - R_{ki}R^\varphi_{kj} + \frac{S^\varphi}{2(m-1)}R_{ji}. \end{aligned}$$

By plugging (1.2.2) into the above we finally conclude

$$\begin{aligned} C^\varphi_{kij,k} =& \alpha(\varphi^a_{kki}\varphi^a_j - \varphi^a_{kkj}\varphi^a_i) + (R^\varphi_{kj} + \alpha\varphi^a_k\varphi^a_j)R^\varphi_{ki} - (R^\varphi_{ki} + \alpha\varphi^a_k\varphi^a_i)R^\varphi_{kj} \\ =& \alpha(\varphi^a_{kki}\varphi^a_j - \varphi^a_{kkj}\varphi^a_i) + \alpha\varphi^a_k\varphi^a_j R^\varphi_{ki} - \alpha\varphi^a_k\varphi^a_i R^\varphi_{kj} \\ =& \alpha[\varphi^a_{kki}\varphi^a_j - \varphi^a_{kkj}\varphi^a_i + \varphi^a_k(R^\varphi_{ki}\varphi^a_j - R^\varphi_{kj}\varphi^a_i)], \end{aligned}$$

and this proves the validity of (1.2.54).

We now compute $\mathrm{tr}(B^\varphi)$. From (1.2.50) we have

$$(m-2)B^\varphi_{ii} = C^\varphi_{iik,k} + R^\varphi_{tk}W^\varphi_{tiki} - \alpha R^\varphi_{ik}\varphi^a_k\varphi^a_i + \alpha\left(|\tau(\varphi)|^2 - \varphi^a_{kki}\varphi^a_i - \frac{m}{m-2}|\tau(\varphi)|^2\right).$$

Then with the aid of (1.2.36) and (1.2.22) we infer

$$\begin{aligned} (m-2)B^\varphi_{ii} =& \alpha(\varphi^a_{ii}\varphi^a_k)_k + \alpha R^\varphi_{tk}\varphi^a_t\varphi^a_k - \alpha R^\varphi_{ik}\varphi^a_k\varphi^a_i - \alpha\frac{2}{m-2}|\tau(\varphi)|^2 - \alpha\varphi^a_{kki}\varphi^a_i \\ =& \alpha\varphi^a_{iik}\varphi^a_k + \alpha|\tau(\varphi)|^2 - \alpha\frac{2}{m-2}|\tau(\varphi)|^2 - \alpha\varphi^a_{kki}\varphi^a_i \\ =& \alpha\frac{m-4}{m-2}|\tau(\varphi)|^2, \end{aligned}$$

which is equivalent to (1.2.53). □

We conclude with

Definition 1.2.56. We define the *traceless part of the φ-Ricci tensor* by

$$T^\varphi = \mathrm{Ric}^\varphi - \frac{S^\varphi}{m}\langle\,,\,\rangle. \tag{1.2.57}$$

Denoting by T the traceless part of the Ricci tensor, using (1.2.2) and (1.2.6),

$$T^\varphi = T - \alpha\left(\varphi^*\langle\,,\,\rangle_N - \frac{|d\varphi|^2}{m}\langle\,,\,\rangle\right). \tag{1.2.58}$$

Remark 1.2.59. Using (1.2.58) it is immediate to obtain that the traceless part of the φ-Ricci tensor coincide with the traceless part of Ricci if and only φ is weakly conformal (on $\{x \in M : \alpha(x) \neq 0\}$, when α is assumed to be a function)

Remark 1.2.60. If $m = 2$ then

$$T^\varphi = A^\varphi,$$

and thus, form Remark 1.2.12,

$$T^\varphi = -\alpha\mathbb{S}.$$

Remark 1.2.61. All the φ-curvatures Ric^φ, S^φ, A^φ, W^φ, C^φ, B^φ and T^φ agrees with the original curvatures in case either $\alpha \equiv 0$ on M or φ is a constant map. The case where φ is constant will be referred in the sequel as the standard case.

Remark 1.2.62. In low dimension some standard curvature tensor are trivial. On the contrary their modified counterparts detect the geometry not only of $(M, \langle\, ,\, \rangle)$ but of $\varphi : (M, \langle\, ,\, \rangle) \to (N, \langle\, ,\, \rangle_N)$ and thus they can be non trivial.

1.3 Transformation laws under a conformal change of the metric

Let $(M, \langle\, ,\, \rangle)$ be a Riemannian manifold of dimension $m \geq 3$, let $f \in \mathcal{C}^\infty(M)$ and let

$$\widetilde{\langle\, ,\, \rangle} := e^{-\frac{2}{m-2}f} \langle\, ,\, \rangle. \tag{1.3.1}$$

Let $\{e_i\}$ be a local orthonormal frame for $(M, \langle\, ,\, \rangle)$ defined on an open set $\mathcal{U} \subseteq M$. Let $\{\theta^i\}$, $\{\theta^i_j\}$ and $\{\Theta^i_j\}$ be, respectively, the dual coframe, the Levi-Civita connection forms and the curvature forms associated to $\{\theta^i\}$. In the next well known Proposition we collect the transformation laws under the conformal change of the metric (1.3.1) for these objects, and as a consequence, also for the Riemann tensor.

Proposition 1.3.2. *Set*

$$\widetilde{e}_i := e^{\frac{1}{m-2}f} e_i, \tag{1.3.3}$$

then $\{\widetilde{e}_i\}$ is a orthonormal frame for $(M, \widetilde{\langle\, ,\, \rangle})$ on $\mathcal{U}$. Denote by $\{\widetilde{\theta}^i\}$, $\{\widetilde{\theta}^i_j\}$ and $\{\widetilde{\Theta}^i_j\}$ the associated coframe, Levi-Civita connection forms and curvature forms on $\mathcal{U}$. Then

$$\widetilde{\theta}^i = e^{-\frac{1}{m-2}f} \theta^i, \tag{1.3.4}$$

$$\widetilde{\theta}^i_j = \theta^i_j + \frac{1}{m-2}(f_i\theta^j - f_j\theta^i), \tag{1.3.5}$$

$$\widetilde{\Theta}^i_j = \Theta^i_j + \frac{1}{m-2}\left[f_{ik}\delta_{tj} - f_{jk}\delta_{it} + \frac{1}{m-2}(f_if_k\delta_{tj} - f_kf_j\delta_{it} - |\nabla f|^2\delta_{ik}\delta_{tj})\right]\theta^k \wedge \theta^t. \tag{1.3.6}$$

Moreover, denoting by $\widetilde{Riem}$ the Riemann tensor of $(M, \widetilde{\langle\, ,\, \rangle})$,

$$\widetilde{Riem} = Riem + \frac{1}{m-2}\left[Hess(f) + \frac{1}{m-2}\left(df \otimes df - \frac{|\nabla f|^2}{2}\langle\, ,\, \rangle\right)\right] \owedge \langle\, ,\, \rangle,$$

that is,

$$\begin{aligned} e^{-\frac{2}{m-2}f}\widetilde{R}^i_{jkt} =& R^i_{jkt} + \frac{1}{m-2}(f_{ik}\delta_{jt} - f_{it}\delta_{jk} + f_{jt}\delta_{ik} - f_{jk}\delta_{it}) \\ &+ \frac{1}{(m-2)^2}(f_if_k\delta_{jt} - f_if_t\delta_{jk} + f_jf_t\delta_{ik} - f_jf_k\delta_{it}) \\ &- \frac{|\nabla f|^2}{(m-2)^2}(\delta_{ik}\delta_{jt} - \delta_{it}\delta_{jk}), \end{aligned} \tag{1.3.7}$$

where

$$Riem = R^i_{jkt}\theta^k \otimes \theta^t \otimes \theta^j \otimes e_i, \quad \widetilde{Riem} = \widetilde{R}^i_{jkt}\widetilde{\theta}^k \otimes \widetilde{\theta}^t \otimes \widetilde{\theta}^j \otimes \widetilde{e}_i.$$

Proof. Clearly (1.3.3) is a local orthonormal frame for $(M, \widetilde{\langle\, ,\, \rangle})$, indeed

$$\widetilde{\langle \widetilde{e}_i, \widetilde{e}_j \rangle} = \delta_{ij}.$$

Clearly, using (1.3.3) and (1.3.4),

$$\widetilde{\theta}^i(\widetilde{e}_j) = \delta^i_j,$$

hence $\{\widetilde{\theta}^i\}$ is the dual coframe corresponding to $\{\widetilde{e}_i\}$. The Levi-Civita connection forms are given by (1.3.5). Indeed they are skew symmetric and they satisfy the first structure equation and those properties characterize them. Using (1.3.4) and the first structure equations

$$d\theta^i = -\theta^i_j \wedge \theta^j \tag{1.3.8}$$

we obtain

$$\begin{aligned} d\widetilde{\theta}^i =& d(e^{-\frac{1}{m-2}f}\theta^i) \\ =& -\frac{1}{m-2}e^{-\frac{1}{m-2}f}df \wedge \theta^i + e^{-\frac{1}{m-2}f}d\theta^i \\ =& -e^{-\frac{1}{m-2}f}\left(\frac{1}{m-2}df \wedge \theta^i + \theta^i_j \wedge \theta^j\right). \end{aligned}$$

From (1.3.5) and (1.3.4) we get

$$\begin{aligned} \widetilde{\theta}^i_j \wedge \widetilde{\theta}^j =& e^{-\frac{1}{m-2}f}\left(\theta^i_j - \frac{f_j}{m-2}\theta^i + \frac{f_i}{m-2}\theta^j\right) \wedge \theta^j \\ =& e^{-\frac{1}{m-2}f}\left(\theta^i_j \wedge \theta^j - \frac{f_j}{m-2}\theta^i \wedge \theta^j\right) \\ =& e^{-\frac{1}{m-2}f}\left(\frac{1}{m-2}df \wedge \theta^i + \theta^i_j \wedge \theta^j\right). \end{aligned}$$

By comparing with the above we obtain the validity of the structure equations

$$d\widetilde{\theta}^i = -\widetilde{\theta}^i_j \wedge \widetilde{\theta}^j,$$

as claimed. Recall the second structure equations

$$d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \Theta^i_j. \tag{1.3.9}$$

From the second structure equations with respect to the metric $\widetilde{\langle\, ,\, \rangle}$, using (1.3.5) we obtain

$$\begin{aligned} \widetilde{\Theta}^i_j =& d\widetilde{\theta}^i_j + \widetilde{\theta}^i_k \wedge \widetilde{\theta}^k_j \\ =& d\left[\theta^i_j + \frac{1}{m-2}(f_i\theta^j - f_j\theta^i)\right] + \left[\theta^i_k + \frac{1}{m-2}(f_i\theta^k - f_k\theta^i)\right] \wedge \left[\theta^k_j + \frac{1}{m-2}(f_k\theta^j - f_j\theta^k)\right] \\ =& d\theta^i_j + \theta^i_k \wedge \theta^k_j + \frac{1}{m-2}(df_i \wedge \theta^j + f_i d\theta^j - df_j \wedge \theta^i - f_j d\theta^i) \\ &+ \frac{1}{m-2}[(f_i\theta^k - f_k\theta^i) \wedge \theta^k_j + \theta^i_k \wedge (f_k\theta^j - f_j\theta^k)] + \frac{1}{(m-2)^2}(f_i\theta^k - f_k\theta^i) \wedge (f_k\theta^j - f_j\theta^k). \end{aligned}$$

From the above, using (1.3.8) and (1.3.9) we deduce

$$\begin{aligned} \widetilde{\Theta}^i_j =& \Theta^i_j + \frac{1}{m-2}(df_i \wedge \theta^j - f_i\theta^j_k \wedge \theta^k - df_j \wedge \theta^i + f_j\theta^i_k \wedge \theta^k) \\ &+ \frac{1}{m-2}[f_i\theta^k \wedge \theta^k_j - f_k\theta^i \wedge \theta^k_j + f_k\theta^i_k \wedge \theta^j - f_j\theta^i_k \wedge \theta^k] \\ &+ \frac{1}{(m-2)^2}(f_kf_i\theta^k \wedge \theta^j - f_jf_i\theta^k \wedge \theta^k - f_kf_k\theta^i \wedge \theta^j + f_kf_j\theta^i \wedge \theta^k), \end{aligned}$$

that is, using the skew-symmetry of the Levi Civita connection forms and the alternating property of the wedge product

$$\begin{aligned} \widetilde{\Theta}^i_j =& \Theta^i_j + \frac{1}{m-2}[(df_i - f_k\theta^k_i) \wedge \theta^j - (df_j - f_k\theta^k_j) \wedge \theta^i] \\ &+ \frac{1}{(m-2)^2}(f_kf_i\theta^k \wedge \theta^j - |\nabla f|^2\theta^i \wedge \theta^j + f_kf_j\theta^i \wedge \theta^k). \end{aligned}$$

From the definition of covariant derivative

$$f_{ij}\theta^j = df_i - f_j\theta^j_i,$$

by plugging into the above we get

$$\widetilde{\Theta}^i_j = \Theta^i_j + \frac{1}{m-2}[f_{ik}\theta^k\wedge\theta^j - f_{jk}\theta^k\wedge\theta^i] + \frac{1}{(m-2)^2}(f_kf_i\theta^k\wedge\theta^j + f_kf_j\theta^i\wedge\theta^k - |\nabla f|^2\theta^i\wedge\theta^j),$$

that is (1.3.6). Recall

$$\Theta^i_j = \frac{1}{2}R^i_{jkt}\theta^k\wedge\theta^t,$$

hence, from (1.3.6) we get

$$\begin{aligned}\frac{1}{2}\widetilde{R}^i_{jkt}\widetilde{\theta}^k\wedge\widetilde{\theta}^t =& \frac{1}{2}R^i_{jkt}\theta^k\wedge\theta^t \\ &+ \frac{1}{m-2}\left[f_{ik}\delta_{jt} - f_{jk}\delta_{it} + \frac{1}{m-2}(f_if_k\delta_{jt} - f_kf_j\delta_{it} - |\nabla f|^2\delta_{ik}\delta_{jt})\right]\theta^k\wedge\theta^t.\end{aligned}$$

By skew-symmetrizing the above we obtain

$$\begin{aligned}e^{-\frac{2}{m-2}f}\widetilde{R}^i_{jkt} =& R^i_{jkt} + \frac{1}{m-2}(f_{ik}\delta_{jt} - f_{it}\delta_{jk} + f_{jt}\delta_{ik} - f_{jk}\delta_{it}) \\ &+ \frac{1}{(m-2)^2}(f_if_k\delta_{jt} - f_if_t\delta_{jk} + f_jf_t\delta_{ik} - f_jf_k\delta_{it}) - \frac{|\nabla f|^2}{(m-2)^2}(\delta_{ik}\delta_{jt} - \delta_{it}\delta_{jk}),\end{aligned}$$

that is (1.3.7). □

Remark 1.3.10. Recall that the Weyl tensor W is a conformal invariant, when we consider its $(1,3)$-version (see for instance Section 1.4 of [AMR]), that is, in a local orthonormal coframe,

$$e^{-\frac{2}{m-2}f}\widetilde{W}^i_{jkt} = W^i_{jkt}. \tag{1.3.11}$$

Let $(N, \langle\,,\,\rangle_N)$ be a Riemannian manifold of dimension n, we denote by $\{E_a\}$, $\{\omega^a\}$ and $\{\omega^a_b\}$ the local orthonormal frame, coframe and the corresponding Levi-Civita connection forms on an open set $\mathcal{V}$ such that $\varphi^{-1}(\mathcal{V}) \subseteq \mathcal{U}$. Clearly $d\varphi$ is independent on the choice of the metric on M, it means,

$$\widetilde{\varphi}^a_i = e^{\frac{1}{m-2}f}\varphi^a_i, \tag{1.3.12}$$

where

$$\varphi^a_i\theta^i\otimes E_a = d\varphi = \widetilde{\varphi}^a_i\widetilde{\theta}^i\otimes E_a.$$

As an immediate consequence we get

$$\widetilde{|d\varphi|}^2 = e^{\frac{2}{m-2}f}|d\varphi|^2. \tag{1.3.13}$$

By definition

$$\nabla d\varphi = \varphi^a_{ij}\theta^j\otimes\theta^i\otimes E_a, \quad \varphi^a_{ij}\theta^j = d\varphi^a_i - \varphi^a_j\theta^j_i + \varphi^b_i\omega^a_b$$

and

$$\widetilde{\nabla} d\varphi = \widetilde{\varphi}^a_{ij}\widetilde{\theta}^j\otimes\widetilde{\theta}^i\otimes E_a, \quad \widetilde{\varphi}^a_{ij}\widetilde{\theta}^j = d\widetilde{\varphi}^a_i - \widetilde{\varphi}^a_j\widetilde{\theta}^j_i + \widetilde{\varphi}^b_i\omega^a_b.$$

We denote by $\widetilde{\tau}(\varphi)$ the tension of the map

$$\varphi : (M, \widetilde{\langle\,,\,\rangle}) \to (N, \langle\,,\,\rangle_N),$$

in components

$$\widetilde{\tau}(\varphi)^a = \widetilde{\varphi}^a_{ii}.$$

In the next Proposition we determine the transformation laws for the quantities of our interest related to the smooth map φ, under the conformal change of the metric (1.3.1).

Proposition 1.3.14. *In a local orthonormal coframe*

$$\widetilde{\varphi}^a_{ij} = e^{\frac{2}{m-2}f}\left[\varphi^a_{ij} + \frac{1}{m-2}(\varphi^a_i f_j + \varphi^a_j f_i - \varphi^a_k f_k \delta_{ij})\right], \tag{1.3.15}$$

in particular

$$\tau(\widetilde{\varphi}) = e^{\frac{2}{m-2}f}\left[\tau(\varphi) - d\varphi(\nabla f)\right]. \tag{1.3.16}$$

Moreover, in a local orthonormal coframe,

$$\widetilde{\varphi}^a_{iik} = e^{\frac{3}{m-2}f}\left[\varphi^a_{iik} - \varphi^a_{ik}f_i - \varphi^a_i f_{ik} + \frac{2}{m-2}(\varphi^a_{ii}f_k - \varphi^a_i f_i f_k)\right]. \tag{1.3.17}$$

Proof. The validity of (1.3.15) follows easily using (1.3.4), the definition of $\widetilde{\varphi}^a_{ij}$, (1.3.12), (1.3.5) and the definition of φ^a_{ij} as follows:

$$\begin{aligned}
\widetilde{\varphi}^a_{ij}e^{-\frac{1}{m-2}f}\theta^j =& \widetilde{\varphi}^a_{ij}\widetilde{\theta}^j \\
=& d\widetilde{\varphi}^a_i - \widetilde{\varphi}^a_j\widetilde{\theta}^j_i + \widetilde{\varphi}^b_i\omega^a_b \\
=& d(e^{\frac{1}{m-2}f}\varphi^a_i) - e^{\frac{1}{m-2}f}\varphi^a_j\left[\theta^j_i + \frac{1}{m-2}(f_j\theta^i - f_i\theta^j)\right] + e^{\frac{1}{m-2}f}\varphi^b_i\omega^a_b \\
=& e^{\frac{1}{m-2}f}(d\varphi^a_i - \varphi^a_j\theta^j_i + \varphi^b_i\omega^a_b) + \frac{1}{m-2}e^{\frac{1}{m-2}f}\varphi^a_i df - \frac{1}{m-2}e^{\frac{1}{m-2}f}\varphi^a_j(f_j\theta^i - f_i\theta^j) \\
=& e^{\frac{1}{m-2}f}\left[\varphi^a_{ij} + \frac{1}{m-2}(\varphi^a_i f_j + \varphi^a_j f_i - \varphi^a_k f_k\delta_{ij})\right]\theta^j.
\end{aligned}$$

Taking the trace of (1.3.15) we immediately get (1.3.16). For convenience we denote

$$T^a_{ij} = \varphi^a_{ij} + \frac{1}{m-2}(\varphi^a_i f_j + \varphi^a_j f_i - \varphi^a_t f_t \delta_{ij}),$$

then, with the aid of (1.3.5),

$$\begin{aligned}
\widetilde{\varphi}^a_{ijk}\widetilde{\theta}^k =& d\widetilde{\varphi}^a_{ij} - \widetilde{\varphi}^a_{kj}\widetilde{\theta}^k_i - \widetilde{\varphi}^a_{ik}\widetilde{\theta}^k_j + \widetilde{\varphi}^b_{ij}\omega^a_b \\
=& d(e^{\frac{2}{m-2}f}T^a_{ij}) - e^{\frac{2}{m-2}}T^a_{kj}\left[\theta^k_i + \frac{1}{m-2}(f_k\theta^i - f_i\theta^k)\right] \\
& - e^{\frac{2}{m-2}f}T^a_{ik}\left[\theta^k_j + \frac{1}{m-2}(f_k\theta^j - f_j\theta^k)\right] + e^{\frac{2}{m-2}f}T^b_{ij}\omega^a_b.
\end{aligned}$$

Thus, using also (1.3.4) and the definition of T,

$$\begin{aligned}
e^{-\frac{3}{m-2}f}\widetilde{\varphi}^a_{ijk}\theta^k =& \frac{2}{m-2}T^a_{ij}f_k\theta^k + T^a_{ijk}\theta^k - \frac{1}{m-2}T^a_{kj}(f_k\theta^i - f_i\theta^k) - \frac{1}{m-2}T^a_{ik}(f_k\theta^j - f_j\theta^k) \\
=& \left[T^a_{ijk} + \frac{2}{m-2}T^a_{ij}f_k + \frac{1}{m-2}(T^a_{kj}f_i - T^a_{tj}f_t\delta_{ik} + T^a_{ik}f_j - T^a_{it}f_t\delta_{jk})\right]\theta^k,
\end{aligned}$$

that is,

$$e^{-\frac{3}{m-2}f}\widetilde{\varphi}^a_{ijk} = T^a_{ijk} + \frac{2}{m-2}T^a_{ij}f_k + \frac{1}{m-2}(T^a_{kj}f_i - T^a_{tj}f_t\delta_{ik} + T^a_{ik}f_j - T^a_{it}f_t\delta_{jk}).$$

Summing on $i = j$ and using the relations

$$T^a_{ii} = \varphi^a_{ii} - \varphi^a_i f_i, \quad T^a_{iik} = \varphi^a_{iik} - \varphi^a_{ik}f_i - \varphi^a_i f_{ik},$$

(the first follows immediately from the definition of T while the second is obtained taking covariant derivative of the first), we get from the above

$$
\begin{aligned}
e^{-\frac{3}{m-2}f}\widetilde{\varphi}^a_{iik} =& T^a_{iik} + \frac{2}{m-2}T^a_{ii}f_k + \frac{2}{m-2}(T^a_{ki}f_i - T^a_{ik}f_i + T^a_{ik}f_i - T^a_{ki}f_i)\\
=& T^a_{iik} + \frac{2}{m-2}T^a_{ii}f_k\\
=& \varphi^a_{iik} - \varphi^a_{ik}f_i - \varphi^a_i f_{ik} + \frac{2}{m-2}(\varphi^a_{ii}f_k - \varphi^a_i f_i f_k),
\end{aligned}
$$

that is (1.3.17). □

Our aim is to determine the transformation laws of the φ-curvatures under the conformal change of the metric (1.3.1). We fix $\alpha \in \mathbb{R}\setminus\{0\}$ and we denote by $\widetilde{\mathrm{Ric}}^{\varphi}$ the φ-Ricci tensor related to the map $\varphi : (M, \widetilde{\langle\,,\,\rangle}) \to (N, \langle\,,\,\rangle_N)$, that is,

$$
\widetilde{\mathrm{Ric}}^{\varphi} = \widetilde{\mathrm{Ric}} - \alpha\varphi^*\langle\,,\,\rangle_N. \tag{1.3.18}
$$

We denote by $\widetilde{S}^{\varphi}$ the φ-scalar curvature associated to $\varphi : (M, \widetilde{\langle\,,\,\rangle}) \to (N, \langle\,,\,\rangle_N)$, that is, $\widetilde{S}^{\varphi} = \widetilde{S} - \alpha|\widetilde{d\varphi}|^2$. The same applies for all the other φ-curvatures. In the following Proposition we deal with the transformation laws for the φ-Ricci curvature, the φ-scalar curvature and, as a consequence, for the φ-Schouten tensor. We denote by

$$
\Delta_f f := \Delta - \langle\nabla f, \nabla\,\rangle
$$

the f-Laplacian.

Proposition 1.3.19. *In the notations above*

$$
\widetilde{Ric}^{\varphi} = Ric^{\varphi} + Hess(f) + \frac{1}{m-2}(df\otimes df + \Delta_f f\langle\,,\,\rangle), \tag{1.3.20}
$$

that is, in local orthonormal coframe,

$$
e^{-\frac{2}{m-2}f}\widetilde{R}^{\varphi}_{ik} = R^{\varphi}_{ik} + f_{ik} + \frac{1}{m-2}(f_i f_k + \Delta_f f\delta_{ik}). \tag{1.3.21}
$$

Moreover

$$
e^{-\frac{2}{m-2}f}\widetilde{S}^{\varphi} = S^{\varphi} + \frac{m-1}{m-2}(2\Delta f - |\nabla f|^2) \tag{1.3.22}
$$

and

$$
\widetilde{A}^{\varphi} = A^{\varphi} + Hess(f) + \frac{1}{m-2}\left(df\otimes df - \frac{|\nabla f|^2}{2}\langle\,,\,\rangle\right)
$$

hold. The latter in local orthonrmal coframe is given by

$$
e^{-\frac{2}{m-2}f}\widetilde{A}^{\varphi}_{ij} = A^{\varphi}_{ij} + f_{ij} + \frac{1}{m-2}\left(f_i f_j - \frac{|\nabla f|^2}{2}\delta_{ij}\right). \tag{1.3.23}
$$

Proof. To obtain (1.3.21), that is,

$$
e^{-\frac{2}{m-2}f}(\widetilde{R}_{ik} - \alpha\widetilde{\varphi}^a_i\widetilde{\varphi}^a_k) = R_{ik} - \alpha\varphi^a_i\varphi^a_k + f_{ik} + \frac{1}{m-2}(f_i f_k + \Delta_f f\delta_{ik}),
$$

since (1.3.12) holds, it is sufficient to take the trace of (1.3.7). Indeed

$$
\begin{aligned}
e^{-\frac{2}{m-2}f}\widetilde{R}_{ik} =& e^{-\frac{2}{m-2}f}\widetilde{R}^i_{jkj}\\
=& R^i_{jkj} + \frac{1}{m-2}[(m-2)f_{ik} + \Delta f\delta_{ik}]\\
&+ \frac{1}{(m-2)^2}[(m-2)f_i f_k + |\nabla f|^2\delta_{ik}] - \frac{|\nabla f|^2}{(m-2)^2}(m-1)\delta_{ik}\\
=& R_{ik} + f_{ik} + \frac{1}{m-2}[f_i f_k + (\Delta f - |\nabla f|^2)\delta_{ik}].
\end{aligned}
$$

To obtain (1.3.22) it is sufficient to take the trace of the above. Indeed

$$e^{-\frac{2}{m-2}f}\widetilde{S} = S + \Delta f + \frac{|\nabla f|^2}{m-2} + \frac{m}{m-2}\Delta_f f = S + \frac{m-1}{m-2}(2\Delta f - |\nabla f|^2),$$

so that, using also (1.3.13),

$$e^{-\frac{2}{m-2}f}\widetilde{S}^\varphi = e^{-\frac{2}{m-2}f}(\widetilde{S} - \alpha\widetilde{|d\varphi|}^2) = S + \frac{m-1}{m-2}(2\Delta f - |\nabla f|^2) - \alpha|d\varphi|^2 = S^\varphi + \frac{m-1}{m-2}(2\Delta f - |\nabla f|^2).$$

Now (1.3.23) follows from the definition (1.2.10) of the φ-Schouten tensor and the formulas (1.3.21) and (1.3.22), indeed

$$\begin{aligned}
e^{-\frac{2}{m-2}f}\widetilde{A}^\varphi_{ij} =& e^{-\frac{2}{m-2}f}\widetilde{R}^\varphi_{ij} - \frac{e^{-\frac{2}{m-2}f}\widetilde{S}^\varphi}{2(m-1)}\delta_{ij} \\
=& R^\varphi_{ij} + f_{ij} + \frac{f_i f_j}{m-2} + \frac{1}{m-2}\Delta_f f\delta_{ij} - \frac{1}{2(m-1)}\left[S^\varphi + \frac{m-1}{m-2}(2\Delta f - |\nabla f|^2)\right]\delta_{ij} \\
=& A^\varphi_{ij} + f_{ij} + \frac{f_i f_j}{m-2} + \frac{1}{m-2}\Delta_f f\delta_{ij} - \frac{1}{m-2}\Delta f\delta_{ij} + \frac{1}{2(m-2)}|\nabla f|^2\delta_{ij} \\
=& A^\varphi_{ij} + f_{ij} + \frac{1}{m-2}\left(f_i f_j - \frac{|\nabla f|^2}{2}\delta_{ij}\right).
\end{aligned}$$

□

Remark 1.3.24. If we set

$$u := e^{-\frac{f}{2}}, \tag{1.3.25}$$

an immediate computation using (1.3.22) implies the validity of the Yamabe equation

$$\frac{4(m-1)}{m-2}\Delta u - S^\varphi u + \widetilde{S}^\varphi u^{\frac{m+2}{m-2}} = 0. \tag{1.3.26}$$

Then the problem of finding metrics in a fixed conformal class with prescribed φ-scalar curvature can be tackled with the same techniques used in the standard case (where φ is constant). See, for instance, Section 2.1 of [MaMR].

In the next Proposition we deal with the transformation laws for the φ-Cotton tensor.

Proposition 1.3.27. *In a local orthonormal coframe*

$$e^{-\frac{3}{m-2}f}\widetilde{C}^\varphi_{ijk} = C^\varphi_{ijk} + W^\varphi_{tijk}f_t. \tag{1.3.28}$$

Proof. For simplicity of notation we set

$$\widetilde{A}^\varphi_{ij} = e^{\frac{2}{m-2}f}T_{ij}, \quad T_{ij} := A^\varphi_{ij} + f_{ij} + \frac{1}{m-2}\left(f_i f_j - \frac{|\nabla f|^2}{2}\delta_{ij}\right). \tag{1.3.29}$$

To obtain the transformation law for the φ-Cotton tensor we first need the transformation law for the covariant derivative of $\widetilde{A}^\varphi$. First of all we express the coefficients of $\widetilde{\nabla}\widetilde{A}^\varphi$ in terms of T and ∇T. The formula is the following:

$$e^{-\frac{3}{m-2}f}\widetilde{A}^\varphi_{ij,k} = \frac{2}{m-2}T_{ij}f_k + T_{ij,k} + \frac{1}{m-2}(T_{kj}f_i - T_{tj}f_t\delta_{ki} + T_{ik}f_j - T_{it}f_t\delta_{jk}). \tag{1.3.30}$$

To obtain the above we use the definition of covariant derivative and (1.3.5) to get

$$\begin{aligned}
\widetilde{A}^\varphi_{ij,k}\widetilde{\theta}^k =& d\widetilde{A}^\varphi_{ij} - \widetilde{A}^\varphi_{kj}\widetilde{\theta}^k_i - \widetilde{A}^\varphi_{ik}\widetilde{\theta}^k_j \\
=& d(e^{\frac{2}{m-2}f}T_{ij}) - e^{\frac{2}{m-2}f}T_{kj}\left(\theta^k_i - \frac{f_i}{m-2}\theta^k + \frac{f_k}{m-2}\theta^i\right) \\
& - e^{\frac{2}{m-2}f}T_{ik}\left(\theta^k_j - \frac{f_j}{m-2}\theta^k + \frac{f_k}{m-2}\theta^j\right),
\end{aligned}$$

that is,

$$
\begin{aligned}
e^{-\frac{2}{m-2}f}\widetilde{A}^{\varphi}_{ij,k}\widetilde{\theta}^k =& \frac{2}{m-2}T_{ij}df + dT_{ij} \\
& - T_{kj}\left(\theta^k_i - \frac{f_i}{m-2}\theta^k + \frac{f_k}{m-2}\theta^i\right) - T_{ik}\left(\theta^k_j - \frac{f_j}{m-2}\theta^k + \frac{f_k}{m-2}\theta^j\right) \\
=& \frac{2}{m-2}T_{ij}df + (dT_{ij} - T_{kj}\theta^k_i - T_{ik}\theta^k_j) + \frac{1}{m-2}[T_{kj}(f_i\theta^k - f_k\theta^i) + T_{ik}(f_j\theta^k - f_k\theta^j)].
\end{aligned}
$$

Using (1.3.4) and the definition of $T_{ij,k}$ we infer

$$
\begin{aligned}
e^{-\frac{3}{m-2}f}\widetilde{A}^{\varphi}_{ij,k}\theta^k =& \frac{2}{m-2}T_{ij}f_k\theta^k + T_{ij,k}\theta^k + \frac{1}{m-2}[T_{kj}(f_i\theta^k - f_k\theta^i) + T_{ik}(f_j\theta^k - f_k\theta^j)] \\
=& \left[\frac{2}{m-2}T_{ij}f_k + T_{ij,k} + \frac{1}{m-2}(T_{kj}f_i - T_{tj}f_t\delta_{ki} + T_{ik}f_j - T_{it}f_t\delta_{jk})\right]\theta^k,
\end{aligned}
$$

that implies (1.3.30). Now, using the definition of the φ-Cotton tensor, (1.3.30) twice and the symmetry of T we get

$$
\begin{aligned}
e^{-\frac{3}{m-2}f}\widetilde{C}^{\varphi}_{ijk} =& e^{-\frac{3}{m-2}f}(\widetilde{A}^{\varphi}_{ij,k} - \widetilde{A}^{\varphi}_{ik,j}) \\
=& \frac{2}{m-2}T_{ij}f_k + T_{ij,k} + \frac{1}{m-2}(T_{kj}f_i - T_{tj}f_t\delta_{ki} + T_{ik}f_j - T_{it}f_t\delta_{jk}) \\
& - \frac{2}{m-2}T_{ik}f_j - T_{ik,j} - \frac{1}{m-2}(T_{jk}f_i - T_{tk}f_t\delta_{ji} + T_{ij}f_k - T_{it}f_t\delta_{kj}) \\
=& T_{ij,k} - T_{ik,j} + \frac{2}{m-2}(T_{ij}f_k - T_{ik}f_j) + \frac{1}{m-2}[T_{ik}f_j - T_{ij}f_k + (T_{tk}\delta_{ji} - T_{tj}\delta_{ki})f_t],
\end{aligned}
$$

that is,

$$
e^{-\frac{3}{m-2}f}\widetilde{C}^{\varphi}_{ijk} = T_{ij,k} - T_{ik,j} + \frac{1}{m-2}(T_{ij}\delta_{kt} - T_{ik}\delta_{jt} + T_{tk}\delta_{ji} - T_{tj}\delta_{ki})f_t. \tag{1.3.31}
$$

To express the right hand side of the above in terms of C^{φ} we first observe that, from the definition (1.3.29) of T,

$$
T_{ij,k} = A^{\varphi}_{ij,k} + f_{ijk} + \frac{1}{m-2}(f_{ik}f_j + f_if_{jk} - f_tf_{tk}\delta_{ij}),
$$

so that, using the commutation rule (see (14))

$$
f_{ijk} = f_{ikj} + R^t_{ijk}f_t,
$$

we get

$$
\begin{aligned}
T_{ij,k} - T_{ik,j} =& A^{\varphi}_{ij,k} + f_{ijk} + \frac{1}{m-2}(f_{ik}f_j + f_if_{jk} - f_tf_{tk}\delta_{ij}) \\
& - \left[A^{\varphi}_{ik,j} + f_{ikj} + \frac{1}{m-2}(f_{ij}f_k + f_if_{kj} - f_tf_{tj}\delta_{ik})\right] \\
=& C^{\varphi}_{ijk} + R^t_{ijk}f_t + \frac{1}{m-2}[f_{ik}f_j - f_{ij}f_k + f_t(f_{tj}\delta_{ik} - f_{tk}\delta_{ij})].
\end{aligned}
$$

Moreover an easy computation using (1.3.29) shows that

$$
\begin{aligned}
(T_{ij}\delta_{kt} - T_{ik}\delta_{jt} + T_{tk}\delta_{ji} - T_{tj}\delta_{ki})f_t =& A^{\varphi}_{ij}f_k - A^{\varphi}_{ik}f_j + A^{\varphi}_{tk}f_t\delta_{ji} - A^{\varphi}_{tj}f_t\delta_{ki} \\
& + f_{ij}f_k - f_{ik}f_j + f_{tk}f_t\delta_{ji} - f_{tj}f_t\delta_{ki},
\end{aligned}
$$

indeed

$$
\begin{aligned}
(T_{ij}\delta_{kt}-&T_{ik}\delta_{jt}+T_{tk}\delta_{ji}-T_{tj}\delta_{ki})f_t\\
=&(A^{\varphi}_{ij}\delta_{kt}-A^{\varphi}_{ik}\delta_{jt}+A^{\varphi}_{tk}\delta_{ji}-A^{\varphi}_{tj}\delta_{ki})f_t+(f_{ij}\delta_{kt}-f_{ik}\delta_{jt}+f_{tk}\delta_{ji}-f_{tj}\delta_{ki})f_t\\
&+\frac{1}{m-2}(f_if_j\delta_{kt}-f_if_k\delta_{jt}+f_tf_k\delta_{ji}-f_tf_j\delta_{ki})f_t-\frac{|\nabla f|^2}{2(m-2)}(\delta_{ij}\delta_{kt}-\delta_{ik}\delta_{jt}+\delta_{tk}\delta_{ji}-\delta_{tj}\delta_{ki})f_t\\
=&A^{\varphi}_{ij}f_k-A^{\varphi}_{ik}f_j+A^{\varphi}_{tk}f_t\delta_{ji}-A^{\varphi}_{tj}f_t\delta_{ki}+f_{ij}f_k-f_{ik}f_j+f_{tk}f_t\delta_{ji}-f_{tj}f_t\delta_{ki}\\
&+\frac{1}{m-2}(f_if_jf_k-f_if_kf_j+|\nabla f|^2f_k\delta_{ji}-|\nabla f|^2f_j\delta_{ki})-\frac{|\nabla f|^2}{2(m-2)}(\delta_{ij}f_k-\delta_{ik}f_j+f_k\delta_{ji}-f_j\delta_{ki})\\
=&A^{\varphi}_{ij}f_k-A^{\varphi}_{ik}f_j+A^{\varphi}_{tk}f_t\delta_{ji}-A^{\varphi}_{tj}f_t\delta_{ki}+f_{ij}f_k-f_{ik}f_j+f_{tk}f_t\delta_{ji}-f_{tj}f_t\delta_{ki}\\
&+\frac{|\nabla f|^2}{m-2}(f_k\delta_{ji}-f_j\delta_{ki})-\frac{|\nabla f|^2}{m-2}(f_k\delta_{ji}-f_j\delta_{ki}).
\end{aligned}
$$

Plugging the two relations above into (1.3.31) we finally conclude

$$
\begin{aligned}
e^{-\frac{3}{m-2}f}\widetilde{C}^{\varphi}_{ijk}=&C^{\varphi}_{ijk}+R^t_{ijk}f_t+\frac{1}{m-2}[f_{ik}f_j-f_{ij}f_k+f_t(f_{tj}\delta_{ik}-f_{tk}\delta_{ij})]\\
&+\frac{1}{m-2}(A^{\varphi}_{ij}f_k-A^{\varphi}_{ik}f_j+A^{\varphi}_{tk}f_t\delta_{ji}-A^{\varphi}_{tj}f_t\delta_{ki})\\
&+\frac{1}{m-2}(f_{ij}f_k-f_{ik}f_j+f_{tk}f_t\delta_{ji}-f_{tj}f_t\delta_{ki})\\
=&C^{\varphi}_{ijk}+R^t_{ijk}f_t-\frac{1}{m-2}(A^{\varphi}_{tj}\delta_{ki}-A^{\varphi}_{tk}\delta_{ij}+A^{\varphi}_{ik}\delta_{tj}-A^{\varphi}_{ij}\delta_{tk})f_t.
\end{aligned}
$$

Thus follows (1.3.28), in view of the decomposition (1.2.18). □

Remark 1.3.32. Using (1.3.11), (1.3.12) and (1.3.13), from the relation between the φ-Weyl and the Weyl tensor (1.2.19) we deduce that the $(1,3)$ version of the φ-Weyl tensor is a conformal invariant, that is,

$$e^{-\frac{2}{m-2}f}\widetilde{W}^{\varphi}_{ijkt}=W^{\varphi}_{ijkt}. \tag{1.3.33}$$

The last transformation law we are going to illustrate is the one for the φ-Bach tensor B^{φ} and is the hardest to obtain. In order to determine it we first need to evaluate the transformation law for the tensor

$$V_{ij}:=C^{\varphi}_{ijk,k}-\alpha(R^{\varphi}_{jk}\varphi^a_k+\varphi^a_{kkj})\varphi^a_i, \tag{1.3.34}$$

that is the content of

Lemma 1.3.35. *In the above notations, in a local orthonormal coframe,*

$$
\begin{aligned}
e^{-\frac{4}{m-2}f}\widetilde{V}_{ij}=&V_{ij}+f_{tk}W^{\varphi}_{tijk}-\frac{m-5}{m-2}f_tf_kW^{\varphi}_{tijk}+\frac{m-4}{m-2}(C^{\varphi}_{jki}+C^{\varphi}_{ikj})f_k\\
&+\alpha\left\{\varphi^a_{ij}\varphi^a_kf_k+\frac{1}{m-2}[(\varphi^a_kf_k-\varphi^a_{kk})(\varphi^a_if_j+\varphi^a_jf_i)-\varphi^a_{tt}\varphi^a_kf_k\delta_{ij}-\Delta_ff\varphi^a_i\varphi^a_j]\right\}
\end{aligned}
\tag{1.3.36}
$$

Proof. We procede exactly as in the proof of the Proposition above. We set

$$\widetilde{C}^{\varphi}_{ijk}=e^{\frac{3}{m-2}f}T_{ijk},\quad T_{ijk}=C^{\varphi}_{ijk}+f_tW^{\varphi}_{tijk}.$$

From the definition of covariant derivative and using (1.3.5)

$$\begin{aligned}\widetilde{C}^{\varphi}_{ijk,s}\widetilde{\theta}^s =& d\widetilde{C}^{\varphi}_{ijk} - \widetilde{C}^{\varphi}_{sjk}\widetilde{\theta}^s_i - \widetilde{C}^{\varphi}_{isk}\widetilde{\theta}^s_j - \widetilde{C}^{\varphi}_{ijs}\widetilde{\theta}^s_k \\ =& d(e^{\frac{3}{m-2}f}T_{ijk}) \\ &- e^{\frac{3}{m-2}f}T_{sjk}\left(\theta^s_i - \frac{f_i}{m-2}\theta^s + \frac{f_s}{m-2}\theta^i\right) \\ &- e^{\frac{3}{m-2}f}T_{isk}\left(\theta^s_j - \frac{f_j}{m-2}\theta^s + \frac{f_s}{m-2}\theta^j\right) \\ &- e^{\frac{3}{m-2}f}T_{ijs}\left(\theta^s_k - \frac{f_k}{m-2}\theta^s + \frac{f_s}{m-2}\theta^k\right),\end{aligned}$$

hence

$$\begin{aligned}e^{-\frac{4}{m-2}f}\widetilde{C}^{\varphi}_{ijk,s}\theta^s =& \frac{3}{m-2}T_{ijk}df + T_{ijk,s}\theta^s \\ &- \frac{1}{m-2}T_{sjk}(-f_i\theta^s + f_s\theta^i) \\ &- \frac{1}{m-2}T_{isk}(-f_j\theta^s + f_s\theta^j) \\ &- \frac{1}{m-2}T_{ijs}(-f_k\theta^s + f_s\theta^k).\end{aligned}$$

Then we deduce

$$\begin{aligned}e^{-\frac{4}{m-2}f}\widetilde{C}^{\varphi}_{ijk,s} =& T_{ijk,s} + \frac{3}{m-2}T_{ijk}f_s + \frac{1}{m-2}(f_iT_{sjk} + f_jT_{isk} + f_kT_{ijs}) \\ &- \frac{f_t}{m-2}(T_{tjk}\delta_{is} + T_{itk}\delta_{js} + T_{ijt}\delta_{ks}).\end{aligned}$$

Summing on $s = k$ an easy calculation shows that

$$e^{-\frac{4}{m-2}f}\widetilde{C}^{\varphi}_{ijk,k} = T_{ijk,k} - \frac{m-4}{m-2}T_{ijk}f_k + \frac{1}{m-2}(T_{kjk}f_i + T_{ikk}f_j) - \frac{f_k}{m-2}(T_{kji} + T_{ikj}). \tag{1.3.37}$$

Using (1.2.45), that we report here for the reader convenience,

$$W^{\varphi}_{tijk,t} = \frac{m-3}{m-2}C^{\varphi}_{ikj} + \alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j) + \frac{\alpha}{m-2}\varphi^a_{tt}(\varphi^a_j\delta_{ik} - \varphi^a_k\delta_{ij}),$$

we infer

$$T_{ijk,k} = C^{\varphi}_{ijk,k} + f_{tk}W^{\varphi}_{tijk} + \frac{m-3}{m-2}C^{\varphi}_{jki}f_k + \alpha(\varphi^a_{ji}\varphi^a_kf_k - \varphi^a_{jk}f_k\varphi^a_i) + \frac{\alpha}{m-2}\varphi^a_{tt}(\varphi^a_if_j - \varphi^a_kf_k\delta_{ij}). \tag{1.3.38}$$

Indeed, by taking the divergence of the relation that defines T,

$$\begin{aligned}T_{ijk,k} =& (C^{\varphi}_{ijk} + f_tW^{\varphi}_{tijk})_k \\ =& C^{\varphi}_{ijk,k} + f_{tk}W^{\varphi}_{tijk} + f_tW^{\varphi}_{tijk,k} \\ =& C^{\varphi}_{ijk,k} + f_{tk}W^{\varphi}_{tijk} + f_kW^{\varphi}_{tjik,t} \\ =& C^{\varphi}_{ijk,k} + f_{tk}W^{\varphi}_{tijk} + \frac{m-3}{m-2}C^{\varphi}_{jki}f_k + \alpha(\varphi^a_{ji}\varphi^a_kf_k - \varphi^a_{jk}f_k\varphi^a_i) + \frac{\alpha}{m-2}\varphi^a_{tt}(\varphi^a_if_j - \varphi^a_kf_k\delta_{ji}).\end{aligned}$$

Clearly

$$T_{ijk}f_k = C^{\varphi}_{ijk}f_k + f_tf_kW^{\varphi}_{tijk}. \tag{1.3.39}$$

The traces of T are given by, using (1.2.36), (1.2.22) and the symmetries of tensors involved,

$$T_{kjk} = C^{\varphi}_{kjk} + f_t W^{\varphi}_{tkjk} = -\alpha\varphi^a_{kk}\varphi^a_j + \alpha\varphi^a_t\varphi^a_j f_t = \alpha(\varphi^a_k f_k - \varphi^a_{kk})\varphi^a_j$$

and

$$T_{ikk} = 0,$$

then we easily get

$$T_{kjk}f_i + T_{ikk}f_j = \alpha(\varphi^a_k f_k - \varphi^a_{kk})\varphi^a_j f_i. \tag{1.3.40}$$

Using the definition of T, the skew symmetry in the first two indices of W^{φ} and the identity (1.2.37) for C^{φ} we evaluate

$$\begin{aligned} f_k T_{kji} + f_k T_{ikj} =& f_k(C^{\varphi}_{kji} + W^{\varphi}_{tkji}f_t) + f_k(C^{\varphi}_{ikj} + W^{\varphi}_{tikj}f_t)\\ =& f_k(C^{\varphi}_{kji} + C^{\varphi}_{ikj}) + f_t f_k W^{\varphi}_{tikj}\\ =& - f_k C^{\varphi}_{jik} + f_t f_k W^{\varphi}_{tikj}\\ =& f_k C^{\varphi}_{jki} + f_t f_k W^{\varphi}_{tikj}. \end{aligned}$$

Plugging the above together with (1.3.38), (1.3.39) and (1.3.40) in (1.3.37) we finally get

$$\begin{aligned} e^{-\frac{4}{m-2}f}\widetilde{C}^{\varphi}_{ijk,k} =& C^{\varphi}_{ijk,k} + f_{tk}W^{\varphi}_{tijk} - \frac{m-5}{m-2}f_t f_k W^{\varphi}_{tijk} + \frac{m-4}{m-2}(C^{\varphi}_{jki} + C^{\varphi}_{ikj})f_k\\ &+ \alpha\left\{\varphi^a_{ij}\varphi^a_k f_k - \varphi^a_{jk}f_k\varphi^a_i + \frac{1}{m-2}[\varphi^a_{kk}(\varphi^a_i f_j - \varphi^a_j f_i) + \varphi^a_k f_k \varphi^a_j f_i - \varphi^a_{tt}\varphi^a_k f_k \delta_{ij}]\right\}. \end{aligned}$$

To conclude the proof notice that, with the aid of (1.3.21) and (1.3.17),

$$\begin{aligned} e^{-\frac{4}{m-2}f}(\widetilde{R}^{\varphi}_{kj}\widetilde{\varphi}^a_k\widetilde{\varphi}^a_i + \widetilde{\varphi}^a_{kkj}\widetilde{\varphi}^a_i) =& \left[R^{\varphi}_{kj} + f_{kj} + \frac{1}{m-2}(f_k f_j + \Delta_f f\delta_{kj})\right]\varphi^a_k\varphi^a_i\\ &+ \left[\varphi^a_{kkj} - \varphi^a_{kj}f_k - \varphi^a_k f_{jk} + \frac{2}{m-2}(\varphi^a_{kk}f_j - \varphi^a_k f_k f_j)\right]\varphi^a_i\\ =& R^{\varphi}_{kj}\varphi^a_k\varphi^a_i + f_{kj}\varphi^a_k\varphi^a_i + \frac{1}{m-2}(\varphi^a_k f_k f_j\varphi^a_i + \Delta_f f\varphi^a_i\varphi^a_j)\\ &+ \varphi^a_{kkj}\varphi^a_i - \varphi^a_{kj}f_k\varphi^a_i - \varphi^a_k f_{jk}\varphi^a_i + \frac{2}{m-2}(\varphi^a_{kk}\varphi^a_i f_j - \varphi^a_k f_k\varphi^a_i f_j), \end{aligned}$$

that is,

$$\begin{aligned} e^{-\frac{4}{m-2}f}(\widetilde{R}^{\varphi}_{kj}\widetilde{\varphi}^a_k\widetilde{\varphi}^a_i + \widetilde{\varphi}^a_{kkj}\widetilde{\varphi}^a_i) =& R^{\varphi}_{kj}\varphi^a_k\varphi^a_i + \varphi^a_{kkj}\varphi^a_i\\ &- \varphi^a_{kj}f_k\varphi^a_i + \frac{1}{m-2}(\Delta_f f\varphi^a_i\varphi^a_j - \varphi^a_k f_k\varphi^a_i f_j + 2\varphi^a_{kk}\varphi^a_i f_j). \end{aligned}$$

Inserting the relation obtained so far into definition (1.3.34) of V we obtain the validity of (1.3.36). □

Now we are ready to prove

Theorem 1.3.41. *In the above notations, we have*

$$e^{-\frac{4}{m-2}f}(m-2)\widetilde{B}^{\varphi}_{ij} = (m-2)B^{\varphi}_{ij} - \frac{m-4}{m-2}f_k(C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk} - C^{\varphi}_{jki}). \tag{1.3.42}$$

Proof. From the definition of φ-Bach (1.2.50) and (1.3.34)

$$(m-2)B^{\varphi}_{ij} = V_{ij} + W^{\varphi}_{tikj}R^{\varphi}_{tk} + \alpha\varphi^a_{tt}\left(\varphi^a_{ij} - \frac{1}{m-2}\varphi^a_{kk}\delta_{ij}\right) \tag{1.3.43}$$

Using (1.3.21), (1.3.33) and (1.2.22) we obtain

$$\begin{aligned}
e^{-\frac{4}{m-2}f}\widetilde{W}^{\varphi}_{tikj}\widetilde{R}^{\varphi}_{tk} =& W^{\varphi}_{tikj}\left(R^{\varphi}_{tk}+f_{tk}+\frac{f_tf_k}{m-2}+\frac{\Delta_f f}{m-2}\delta_{tk}\right)\\
=& W^{\varphi}_{tikj}R^{\varphi}_{tk}+W^{\varphi}_{tikj}f_{tk}+\frac{1}{m-2}W^{\varphi}_{tikj}f_tf_k+\alpha\frac{\Delta_f f}{m-2}\varphi^a_i\varphi^a_j\\
=& W^{\varphi}_{tikj}R^{\varphi}_{tk}-W^{\varphi}_{tijk}f_{tk}-\frac{1}{m-2}W^{\varphi}_{tijk}f_tf_k+\alpha\frac{\Delta_f f}{m-2}\varphi^a_i\varphi^a_j.
\end{aligned}$$

Using (1.3.15) three times a computation yields

$$\begin{aligned}
e^{-\frac{4}{m-2}f}\widetilde{\varphi}^a_{tt}\left(\widetilde{\varphi}^a_{ij}-\frac{1}{m-2}\widetilde{\varphi}^a_{kk}\delta_{ij}\right) =& \varphi^a_{tt}\left(\varphi^a_{ij}-\frac{1}{m-2}\varphi^a_{kk}\delta_{ij}\right)\\
&+\frac{1}{m-2}\varphi^a_{tt}\varphi^a_kf_k\delta_{ij}-\varphi^a_kf_k\varphi^a_{ij}+\frac{1}{m-2}(\varphi^a_{kk}-\varphi^a_kf_k)(\varphi^a_if_j+\varphi^a_jf_i).
\end{aligned}$$

Combining these two relations with (1.3.36) and (1.3.43) we deduce the validity of (1.3.42). □

Remark 1.3.44. If φ is a constant map then (1.3.42) reduces to the well known (see, for instance, equation (3.36) of [CMMR16])

$$e^{-\frac{4}{m-2}f}(m-2)\widetilde{B}_{ij}=(m-2)B_{ij}-\frac{m-4}{m-2}f_k(C_{ijk}+f_tW_{tijk}-C_{jki}).$$

As a consequence, for $m=4$, the Bach tensor is a conformal invariant.

As an immediate consequence of the transformation law for φ-Bach we generalize the conformal invariance in the four dimensional case.

Corollary 1.3.45. *If $m=4$ then B^{φ} is a conformal invariant, that is,*

$$e^{-2f}\widetilde{B}^{\varphi}_{ij}=B^{\varphi}_{ij}.$$

1.4 Vanishing conditions on φ-Weyl and its derivatives

Let $(M,\langle\,,\,\rangle)$ be a Riemannian manifold of dimension $m\geq 4$. Recall the following classic definitions:

(i) The Riemannian manifold $(M,\langle\,,\,\rangle)$ is locally conformally flat if

$$W=0.$$

(ii) The Riemannian manifold $(M,\langle\,,\,\rangle)$ has *harmonic Weyl curvature* if W is divergence free, that is, in a local orthonormal coframe

$$W_{tijk,t}=0.$$

(iii) The Riemannian manifold $(M,\langle\,,\,\rangle)$ is called *conformally symmetric* if

$$\nabla W=0.$$

Recall that a 4-times covariant tensor K that has the same symmetries of the Riemann tensor is harmonic if the induced two forms on $\wedge^2 M$ is harmonic, that is, K satisfies the second Bianchi identity and is divergence free. Observe that Riem and W are harmonic if and only if they are divergence free. Indeed, Riem always satisfies the second Bianchi identity while, in case W is divergence free, $C=0$ and thus W satisfies also the second Bianchi identity (see Lemma 1.2 of [AMR], that is, Proposition 1.2.47 with φ constant). For W^{φ} the situation is different, we need to require both the conditions above and not just that it is divergence free to obtain that it is harmonic.

We give the following

Definition 1.4.1. Let $\varphi : (M, \langle\, , \rangle) \to (N, \langle\, , \rangle_N)$ be a smooth map, where $m \geq 4$, and $\alpha \in \mathbb{R} \setminus \{0\}$.

(i) The Riemannian manifold $(M, \langle\, , \rangle)$ has *harmonic φ-Weyl curvature* if W^φ is harmonic, that is, W^φ is divergence free

$$W^\varphi_{tijk,t} = 0 \tag{1.4.2}$$

and satisfies the second Bianchi identity

$$W^\varphi_{tijk,l} + W^\varphi_{tikl,j} + W^\varphi_{tilj,k} = 0. \tag{1.4.3}$$

(ii) The Riemannian manifold $(M, \langle\, , \rangle)$ is called *φ-conformally symmetric* if

$$\nabla W^\varphi = 0. \tag{1.4.4}$$

Remark 1.4.5. Since W is totally traceless, if W is proportional to $\langle\, , \rangle \owedge \langle\, , \rangle$, that is,

$$W = \frac{\xi}{2}\langle\, , \rangle \owedge \langle\, , \rangle$$

for some $\xi \in \mathcal{C}^\infty(M)$, then it is easy to see that $\xi = 0$ and thus $W = 0$, that is, $(M, \langle\, , \rangle)$ is locally conformally flat. It is not unusual that when a tensor with the same symmetries of Riem is proportional to $\langle\, , \rangle \owedge \langle\, , \rangle$ we get some strong rigidity results. For instance, if Riem is proportional to $\langle\, , \rangle \owedge \langle\, , \rangle$ then $(M, \langle\, , \rangle)$ has constant sectional curvature.

When dealing with the φ-Weyl curvature instead of the usual Weyl curvature we have

Proposition 1.4.6. *Let $\varphi : (M, \langle\, , \rangle) \to (N, \langle\, , \rangle_N)$ be a smooth map, where $m \geq 4$, and $\alpha \in \mathbb{R}\setminus\{0\}$. Assume that, for some $\xi \in \mathcal{C}^\infty(M)$,*

$$W^\varphi = \frac{\xi}{2}\langle\, , \rangle \owedge \langle\, , \rangle. \tag{1.4.7}$$

Then

$$\xi = \frac{\alpha|d\varphi|^2}{m(m-1)}, \tag{1.4.8}$$

φ is weakly conformal, that is,

$$\varphi^*\langle\, , \rangle_N = \frac{|d\varphi|^2}{m}\langle\, , \rangle,$$

and $(M, \langle\, , \rangle)$ is locally conformally flat. Moreover, if $\xi \in \mathbb{R}$, then φ is a homothetic map and, if $\xi = 0$, then φ is a constant map.

As a consequence,

$$\mathring{W}^\varphi = 0 \tag{1.4.9}$$

if and only if φ is weakly conformal and $(M, \langle\, , \rangle)$ is locally conformally flat, where

$$\mathring{W}^\varphi := W^\varphi - \frac{\alpha|d\varphi|^2}{2m(m-1)}\langle\, , \rangle \owedge \langle\, , \rangle \tag{1.4.10}$$

is the traceless part of W^φ.

Proof. Locally (1.4.7) reads

$$W^\varphi_{tijk} = \xi(\delta_{tj}\delta_{ik} - \delta_{tk}\delta_{ij}). \tag{1.4.11}$$

Summing (1.4.11) on $t = j$, using (1.2.22), we obtain

$$\alpha\varphi^a_i\varphi^a_k = (m-1)\xi\delta_{ik},$$

that is, since $\alpha \neq 0$,

$$\varphi^*\langle\, , \rangle_N = \frac{(m-1)\xi}{\alpha}\langle\, , \rangle.$$

Taking the trace of the above we get (1.4.8) and that φ is weakly conformal. Then, from (1.2.19) we have

$$W^{\varphi} = W + \frac{\alpha|d\varphi|^2}{2m(m-1)}\langle\,,\,\rangle \owedge \langle\,,\,\rangle.$$

But then, using (1.4.7) and (1.4.8), $W = 0$, that is, $(M, \langle\,,\,\rangle)$ is locally conformally flat. If $\xi \in \mathbb{R}$ then, from (1.4.8), $|d\varphi|^2$ is constant and thus φ is a homothetic map.

If φ is weakly conformal and $(M, \langle\,,\,\rangle)$ is locally conformally flat then (1.4.9) holds trivially, using (1.2.19). □

The Proposition above shows that a vanishing condition on φ-Weyl affects both the geometry of $(M, \langle\,,\,\rangle)$ and of the smooth map $\varphi : M \to (N, \langle\,,\,\rangle_N)$. In the next two Propositions we deal with the cases where $(M, \langle\,,\,\rangle)$ is φ-conformally symmetric and the φ-Weyl curvature is harmonic, obtaining the same twofold effect.

Proposition 1.4.12. *Let $\varphi : (M, \langle\,,\,\rangle) \to (N, \langle\,,\,\rangle_N)$ be a smooth map, where $m \geq 4$, and $\alpha \in \mathbb{R} \setminus \{0\}$. Then $(M, \langle\,,\,\rangle)$ is φ-conformally symmetric if and only if φ is relatively affine and $(M, \langle\,,\,\rangle)$ is conformally symmetric.*

Proof. Assume that $(M, \langle\,,\,\rangle)$ is φ-conformally symmetric, that is, (1.4.4) holds. By setting

$$U := \varphi^*\langle\,,\,\rangle_N - \frac{|d\varphi|^2}{2(m-1)}\langle\,,\,\rangle, \tag{1.4.13}$$

we get, using (1.4.4) and (1.2.19),

$$W_{tijk,l} + \frac{\alpha}{m-2}(U_{tj,l}\delta_{ik} - U_{tk,l}\delta_{ij} + U_{ik,l}\delta_{tj} - U_{ij,l}\delta_{tk}) = 0. \tag{1.4.14}$$

Summing the above for $t = j$, using the traceless property of the Weyl tensor, we deduce

$$\frac{\alpha}{m-2}(U_{jj,l}\delta_{ik} - U_{ik,l} + mU_{ik,l} - U_{ik,l}) = 0,$$

that is, since $\alpha \neq 0$,

$$(m-2)U_{ik,l} + U_{jj,l}\delta_{ik} = 0.$$

Recalling the definition of U, from the above we conclude

$$(m-2)\left(\varphi^a_i\varphi^a_k - \frac{|d\varphi|^2}{2(m-1)}\delta_{ik}\right)_l + \frac{m-2}{2(m-1)}|d\varphi|^2_l\delta_{ik} = 0,$$

that is, $\varphi^*\langle\,,\,\rangle_N$ is parallel. Then φ is relatively affine, by Definition 1.1.28, and $|d\varphi|^2$ is constant on M, by Remark 1.1.30, hence also U is parallel. As a consequence (1.4.14) immediately gives $\nabla W = 0$.

Assume that φ is relatively affine and that $(M, \langle\,,\,\rangle)$ is conformally symmetric. Since φ is relatively affine then $\varphi^*\langle\,,\,\rangle_N$ is parallel and $|d\varphi|^2$ is constant, hence also U defined in (1.4.13) is parallel. Then, taking the covariant derivative of (1.2.19) and using also that W is parallel, we obtain that W^{φ} is parallel. □

During the proof of the proof second Proposition we need the following

Lemma 1.4.15. *If $m \geq 4$ then A^{φ} is Codazzi if and only if*

$$W^{\varphi}_{tijk,t} = \alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j). \tag{1.4.16}$$

Proof. If A^{φ} is Codazzi then $C^{\varphi} = 0$. In particular it is traceless, that is,

$$0 = C^{\varphi}_{iij} = \alpha\varphi^a_{ii}\varphi^a_j.$$

Then, using (1.2.45), we immediately get (1.4.16). Conversely, if (1.4.16) holds, summing on $k = i$ we obtain

$$W^{\varphi}_{tiji,t} = \alpha(\varphi^a_{ij}\varphi^a_i - \varphi^a_{ii}\varphi^a_j).$$

On the other hand, using (1.2.22),

$$W^{\varphi}_{tiji,t} = \alpha(\varphi^a_{tt}\varphi^a_j + \varphi^a_t\varphi^a_{jt}).$$

Then, comparing with the above

$$\alpha(\varphi^a_{ij}\varphi^a_i - \varphi^a_{ii}\varphi^a_j) = \alpha(\varphi^a_{tt}\varphi^a_j + \varphi^a_t\varphi^a_{jt}),$$

that implies,

$$\varphi^a_{ii}\varphi^a_j = 0.$$

Then, using (1.4.16) and (1.2.45)

$$\begin{aligned}\alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j) =& W^{\varphi}_{tijk,t}\\ =& \frac{m-3}{m-2}C^{\varphi}_{ikj} + \alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j) + \frac{\alpha}{m-2}\varphi^a_{tt}(\varphi^a_j\delta_{ik} - \varphi^a_k\delta_{ij})\\ =& \frac{m-3}{m-2}C^{\varphi}_{ikj} + \alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j),\end{aligned}$$

that implies $C^{\varphi} = 0$, that is, A^{φ} is a Codazzi tensor. ☐

Proposition 1.4.17. *Let $\varphi : (M, \langle\, ,\, \rangle) \to (N, \langle\, ,\, \rangle_N)$ be a smooth map, where $m \geq 4$, and $\alpha \in \mathbb{R} \setminus \{0\}$.*

i) If W^{φ} is divergence free, that is, (1.4.2) *holds, then $\varphi^*\langle\, ,\, \rangle_N$ is divergence free, φ is conservative and $|d\varphi|^2$ is constant. Moreover*

$$W_{tijk,t} = \frac{\alpha}{m-2}(\varphi^a_{ik}\varphi^a_j - \varphi^a_{ij}\varphi^a_k). \tag{1.4.18}$$

ii) If W^{φ} satisfies the second Bianchi identity, that is, (1.4.3) *holds, then*

$$W^{\varphi}_{tijk,t} = \alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j) \tag{1.4.19}$$

and $C^{\varphi} = 0$.

As a consequence, $(M, \langle\, ,\, \rangle)$ has harmonic φ-Weyl curvature if and only if φ is almost relatively affine and $(M, \langle\, ,\, \rangle)$ has harmonic Weyl curvature.

Proof. Assume that (1.4.2) holds. From (1.2.45) we have

$$\frac{m-3}{m-2}C^{\varphi}_{ijk} = \alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j) + \frac{\alpha}{m-2}\varphi^a_{tt}(\varphi^a_j\delta_{ik} - \varphi^a_k\delta_{ij}). \tag{1.4.20}$$

Summing the above on $j = i$ and using (1.2.36) we obtain

$$\alpha\frac{m-3}{m-2}\varphi^a_{ii}\varphi^a_k = \alpha(\varphi^a_{ii}\varphi^a_k - \varphi^a_{ik}\varphi^a_i) + \frac{\alpha}{m-2}\varphi^a_{tt}(\varphi^a_k - m\varphi^a_k),$$

that is,

$$\alpha(\varphi^a_{ii}\varphi^a_k + \varphi^a_{ik}\varphi^a_i) = 0.$$

Since $\alpha \neq 0$ we conclude

$$(\varphi^a_i\varphi^a_k)_i = 0.$$

Taking the divergence of (1.2.19), using that W^{φ} is divergence free, we deduce

$$W_{tijk,t} + \frac{\alpha}{m-2}(U_{tj,t}\delta_{ik} - U_{tk,t}\delta_{ij} + U_{ik,j} - U_{ij,k}) = 0,$$

where U is defined as in (1.4.13), that is, since $\varphi^*\langle\, ,\,\rangle_N$ is divergence free

$$W_{tijk,t} + \frac{\alpha}{m-2}\left(-\frac{|d\varphi|^2_j}{m-1}\delta_{ik} + \frac{|d\varphi|^2_k}{m-1}\delta_{ij} + \varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j\right) = 0. \tag{1.4.21}$$

Summing the above on $k = i$, using the traceless property of W, we get

$$\frac{\alpha}{m-2}\left(-\frac{1}{2}|d\varphi|^2_j - \varphi^a_{ii}\varphi^a_j\right) = 0,$$

that implies

$$\varphi^a_{ii}\varphi^a_j = \frac{1}{2}|d\varphi|^2_j.$$

Observe that, since $\varphi^*\langle\, ,\,\rangle_N$ is divergence free,

$$\varphi^a_{ii}\varphi^a_j = -\varphi^a_i\varphi^a_{ji} = -\frac{1}{2}|d\varphi|^2_j.$$

Combining it with the above we conclude that $|d\varphi|^2$ is constant and then, once again from the above, φ is conservative. Now from (1.4.21) we deduce, since $|d\varphi|^2$ is constant, the validity of (1.4.18).

Assume that (1.4.3) holds. Then, using Proposition 1.2.47,

$$C^\varphi_{tjk}\delta_{il} + C^\varphi_{tkl}\delta_{ij} + C^\varphi_{tlj}\delta_{ik} - C^\varphi_{ijk}\delta_{tl} - C^\varphi_{ikl}\delta_{tj} - C^\varphi_{ilj}\delta_{tk} = 0.$$

Summing the above on $t = l$ and using (1.2.36) we get

$$(m-3)C^\varphi_{ijk} = \alpha\varphi^a_{tt}(\varphi^a_j\delta_{ik} - \varphi^a_k\delta_{ij}). \tag{1.4.22}$$

From (1.2.45), using the above we obtain (1.4.19). From the Lemma above (1.4.19) is equivalent to $C^\varphi = 0$.

Assume that $(M, \langle\, ,\,\rangle)$ has harmonic φ-Weyl curvature. Combining (1.4.19) with the fact that W^φ is divergence free and $\alpha \neq 0$ we get

$$\varphi^a_{ij}\varphi^a_k = \varphi^a_{ik}\varphi^a_j,$$

that is, $\varphi^*\langle\, ,\,\rangle_N$ is a Codazzi tensor. Inserting the above into (1.4.18) we conclude that also W is divergence free.

Assume that φ is almost relatively affine and that $(M, \langle\, ,\,\rangle)$ has harmonic Weyl curvature. Since $(M, \langle\, ,\,\rangle)$ has harmonic Weyl curvature then $(M, \langle\, ,\,\rangle)$ is Cotton flat, that is, the Schouten tensor is a Codazzi tensor. Since $\varphi^*\langle\, ,\,\rangle_N$ is harmonic then $\varphi^*\langle\, ,\,\rangle_N$ is a Codazzi tensor with constant trace, hence also U defined in (1.4.13) is a Codazzi tensor. As a consequence also

$$A^\varphi = A + \alpha U$$

is a Codazzi tensor, that is, $(M, \langle\, ,\,\rangle)$ is φ-Cotton flat. Then φ is conservative and from (1.2.45) we obtain that W^φ is divergence free while from Proposition 1.2.47 we get that W^φ satisfies the second Bianchi identity. In conclusion, W^φ is harmonic. □

Remark 1.4.23. If $m = 3$ the vanishing conditions of φ-Weyl considered above have repercussions only the map φ and not on the geometry of $(M, \langle\, ,\,\rangle)$. Indeed it is immediate, using (1.2.19) and that $W = 0$, to deduce

$$\mathring{W}^\varphi = \alpha\left(\varphi^*\langle\, ,\,\rangle_N - \frac{|d\varphi|^2}{3}\langle\, ,\,\rangle\right) \owedge \langle\, ,\,\rangle,$$

where $\mathring{W}^\varphi$ is defined as in (1.4.10). Since $\alpha \neq 0$ and $\cdot \owedge \langle\, ,\,\rangle$ is injective, we obtain that $\mathring{W}^\varphi = 0$ if and only if φ is weakly conformal. Similarly one can prove that W^φ is harmonic or parallel if and only if, respectively, $\varphi^*\langle\, ,\,\rangle_N$ is harmonic or parallel.

Chapter 2

Harmonic-Einstein manifolds

In this Chapter we define harmonic-Einstein manifolds and we prove some results regarding them, generalizing some classic and also some new results on Einstein manifolds.

In Section 2.1 we give the definition of harmonic-Einstein manifolds and we generalize the classic Schur's lemma for Einstein manifolds. Then we show that to determine the geometry of a harmonic-Einstein manifold the only φ-curvature needed are the φ-Weyl and the sign of the φ-scalar curvature. Finally we characterize the harmonic-Einstein manifolds that are also Einstein in terms of φ, it must be a homothetic map. In Subsection 2.1.1 we relate the condition of harmonic curvature and local symmetry of a harmonic-Einstein manifold to the concept introduced in Section 1.4 above. Moreover we characterize harmonic-Einstein manifolds with constant sectional curvature, they are locally conformally flat and Einstein. In Subsection 2.1.2 we consider Riemann surfaces that are harmonic-Einstein manifolds. As for the Einstein manifolds, the bidimensional case presents some remarkable differences with respect to the higher dimensional case. This Section is just a beginning for the study of such surfaces, we will not proceed further in this direction in this thesis.

In Section 2.2 we prove that under some restrictions on the curvature of the target manifold (and an upper bound for the density of energy of φ, in case of negative φ-scalar curvature) the map φ is constant and thus the concept of harmonic-Einstein manifold collapse to the concept of Einstein manifold. The restriction is on the largest eigenvalue of the curvature operator of the pullback bundle $\varphi^{-1}TN$, it must be lower than the constant α.

In Section 2.3 we define conformally harmonic-Einstein manifolds, that are manifolds which are harmonic-Einstein after a conformal change of the metric. We characterize them in Theorem 2.3.5, providing the first important motivation for the study of Einstein-type structures. Further we prove that conformally harmonic-Einstein manifolds satisfy two integrability conditions and, in Subsection 2.3.1, we show that the validity of these two integrability conditions is also a sufficient condition for being conformally harmonic-Einstein manifold, if we assume a genericity condition on the metric, that is related to the injectivity of a curvature operator, denoted by $\mathcal{W}^{\varphi}$, and on the smooth map φ.

In Section 2.4 we compute the Laplacian of square norm of the traceless part of the φ-Ricci tensor and, as a consequence, we prove that a stochastically complete Riemannian manifold is harmonic-Einstein in case some necessary conditions hold, provided the norm of the traceless part of the φ-Ricci tensor is sufficiently small. To give a quantitative estimate on the threshold for the norm of the traceless part of the φ-Ricci tensor we state and prove an estimate on the biggest eigenvalue of the curvature operator $\mathcal{W}^{\varphi}$, the same operator appearing in Subsection 2.3.1.

In the last Section of the Chapter, Section 2.5, we prove with the formalism of the moving frame the classic formulas for the Riemann curvature of a warped product Riemannian manifold and we apply them in order to characterize warped product harmonic-Einstein manifolds with respect to a map that is constant on the leaves of the canonical fibration of the warped product. This is the subject of Theorem 2.5.26, providing the second important motication for the study of Einstein-type structures. Finally, in Subsection 2.5.1, we discuss some applications in General Relativity. We show that four dimensional Lorentzian harmonic-Einstein warped products, for an appropriate constant α, are solutions of the Einstein field equations and as energy stress tensor the one of the harmonic-map. Furthermore, we introduce φ-static metric in order to characterize the standard static spacetimes that are harmonic-Einstein, when the map that is constant on the leaves of the canonical fibration.

2.1 Definition and properties

Next definition is analogous to that of an Einstein manifold.

Definition 2.1.1. A *harmonic-Einstein structure* on a smooth manifold M of dimension $m \geq 2$ is the data of:

(i) A Riemannian metric $\langle\, ,\, \rangle$ on M;

(ii) A smooth map $\varphi : M \to (N, \langle\, ,\, \rangle_N)$, where the target $(N, \langle\, ,\, \rangle_N)$ is a Riemannian manifold;

(iii) A constant $\alpha \in \mathbb{R} \setminus \{0\}$

such that, for some $\lambda \in \mathcal{C}^\infty(M)$,

$$\begin{cases} \mathrm{Ric}^\varphi = \lambda \langle\, ,\, \rangle \\ \tau(\varphi) = 0, \end{cases} \tag{2.1.2}$$

where Ric^φ is defined by (1.2.2). For brevity in the following we say that $(M, \langle\, ,\, \rangle)$ is a *harmonic-Einstein manifold* (with respect to φ and α, if it is not clear from the context). Further, if $\lambda = 0$ we say that $(M, \langle\, ,\, \rangle)$ is *harmonic-Ricci flat* (with respect to φ and α) or also *φ-Ricci flat*.

To have a strict parallelism with the notion of Einstein manifold, in case $m = 2$ we require in addiction λ to be constant. Note that for $m \geq 3$ this is automatic because of the following version of Schur's lemma.

Proposition 2.1.3. *Let $(M, \langle\, ,\, \rangle)$ be a Riemannian manifold of dimension $m \geq 2$, $\alpha \in \mathbb{R} \setminus \{0\}$, $\lambda \in \mathcal{C}^\infty(M)$ and suppose that for some $\varphi : M \to (N, \langle\, ,\, \rangle_N)$*

$$Ric^\varphi = \lambda \langle\, ,\, \rangle. \tag{2.1.4}$$

Then

$$\frac{m-2}{2} d\lambda = \alpha\, div(\mathbb{S}),$$

where $\mathbb{S}$ is the energy-stress tensor of the map φ, in a local orthonormal coframe

$$\frac{m-2}{2} \lambda_j = \alpha \varphi^a_{ii} \varphi^a_j. \tag{2.1.5}$$

In particular, if $m \geq 3$ and φ is conservative then λ is constant.

Proof. We trace (2.1.4) to obtain $S^\varphi = m\lambda$ and then

$$S^\varphi_j = m\lambda_j. \tag{2.1.6}$$

On the other hand, taking covariant derivative of (2.1.4) we have

$$R^\varphi_{ij,k} = \lambda_k \delta_{ij}.$$

Tracing with respect to i and k

$$R^\varphi_{ij,i} = \lambda_j.$$

We then use (1.2.26) to obtain (2.1.5). □

It is well known that, essentially, the only non trivial curvature on an Einstein manifold is the Weyl tensor. Indeed, if $(M, \langle\, ,\, \rangle)$ is an Einstein manifold then S is constant,

$$\mathrm{Ric} = \frac{S}{m}\langle\, ,\, \rangle \quad \text{that implies} \quad T = 0 \quad \text{and} \quad A = \frac{m-2}{2m(m-1)} S\langle\, ,\, \rangle.$$

If the dimension $m \geq 3$ then $C = 0$, $B = 0$ and

$$\overset{\circ}{\mathrm{Riem}} = W,$$

where

$$\overset{\circ}{\mathrm{Riem}} := \mathrm{Riem} - \frac{S}{2m(m-1)}\langle\, ,\, \rangle \owedge \langle\, ,\, \rangle.$$

In the present setting the analogous results are given by the following

Proposition 2.1.7. *Let $(M, \langle\, ,\, \rangle)$ be a harmonic-Einstein manifold for some $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Then S^φ is constant,*

$$Ric^\varphi = \frac{S^\varphi}{m}\langle\, ,\, \rangle \quad \textit{that implies} \quad T^\varphi = 0 \quad \textit{and} \quad A^\varphi = \frac{m-2}{2m(m-1)} S^\varphi\langle\, ,\, \rangle.$$

If the dimension $m \geq 3$ then $C^\varphi = 0$, $B^\varphi = 0$ and

$$\overset{\circ}{Riem} = \overset{\circ}{W}{}^\varphi, \tag{2.1.8}$$

where all the φ-curvatures are defined in Section 1.2.

Proof. We already proved above the constancy of S^φ. Observe that the validity of the first equation of (2.1.2) is equivalent to

$$T^\varphi = 0,$$

where T^φ is the traceless part of the φ-Ricci tensor and is defined by (1.2.57). Using the definition (2.1.2) of harmonic-Einstein manifold we deduce

$$A^\varphi = \mathrm{Ric}^\varphi - \frac{S^\varphi}{2(m-1)}\langle\, ,\, \rangle = \frac{m-2}{2m(m-1)} S^\varphi\langle\, ,\, \rangle.$$

Inserting the above into the decomposition (1.2.18) we immediately get

$$\mathrm{Riem} = W^\varphi + \frac{S^\varphi}{2m(m-1)}\langle\, ,\, \rangle \owedge \langle\, ,\, \rangle,$$

that is equivalent to, using the definition of φ-scalar curvature,

$$\mathrm{Riem} - \frac{S}{2m(m-1)}\langle\, ,\, \rangle \owedge \langle\, ,\, \rangle = W^\varphi - \frac{\alpha|d\varphi|^2}{2m(m-1)}\langle\, ,\, \rangle \owedge \langle\, ,\, \rangle,$$

that is (2.1.8). Furthermore, since S^φ is constant it follows that A^φ is parallel, hence is a Codazzi tensor field, and then $C^\varphi = 0$. Using (1.2.22) and once again (2.1.2) we have

$$R^\varphi_{tk}(W^\varphi_{tikj} - \alpha\varphi^a_i\varphi^a_t\delta_{jk}) = \frac{S^\varphi}{m}(W^\varphi_{kikj} - \alpha\varphi^a_i\varphi^a_j) = 0.$$

From the above, $C^\varphi = 0$ and the fact that φ is harmonic, we deduce

$$(m-2)B^\varphi_{ij} = C^\varphi_{ijk,k} + R^\varphi_{tk}(W^\varphi_{tikj} - \alpha\varphi^a_t\varphi^a_i\delta_{jk}) + \alpha\left(\varphi^a_{ij}\varphi^a_{kk} - \varphi^a_{kkj}\varphi^a_i - \frac{1}{m-2}|\tau(\varphi)|^2\delta_{ij}\right) = 0$$

thus $(M, \langle\, ,\, \rangle)$ is φ-Bach flat. ☐

Clearly when φ is constant a harmonic-Einstein manifold is an Einstein manifold. This is not the only situation in which these two notions collapse, as shown in the next

Proposition 2.1.9. *Let $(M, \langle\, ,\, \rangle)$ be a harmonic-Einstein manifold of dimension $m \geq 2$ for some $\alpha \in \mathbb{R}\setminus\{0\}$ and $\varphi : (M, \langle\, ,\, \rangle) \to (N, \langle\, ,\, \rangle_N)$, that is, the following system holds:*

$$\begin{cases} Ric^{\varphi} = \dfrac{S^{\varphi}}{m}\langle\, ,\, \rangle \\ \tau(\varphi) = 0. \end{cases} \tag{2.1.10}$$

Then $(M, \langle\, ,\, \rangle)$ is Einstein if and only if φ is homothetic.

Proof. Assume $(M, \langle\, ,\, \rangle)$ is Einstein, that is,

$$\mathrm{Ric} = \frac{S}{m}\langle\, ,\, \rangle.$$

Plugging the above into the first equation of (2.1.10) and using (1.2.6) we deduce

$$\alpha\left(\varphi^*\langle\, ,\, \rangle_N - \frac{|d\varphi|^2}{m}\langle\, ,\, \rangle\right) = 0.$$

Since $\alpha \neq 0$ the above implies that φ is weakly conformal. Moreover, since both S and S^{φ} are constant, from the definition of φ-scalar curvature (1.2.6) and $\alpha \neq 0$, we infer that $|d\varphi|^2$ is constant. Then φ is homothetic.

The converse is trivial. □

Remark 2.1.11. Let $(M, \langle\, ,\, \rangle)$ be a harmonic-Einstein manifold for some $\alpha \in \mathbb{R} \setminus \{0\}$ and a non-constant smooth map $\varphi : (M, \langle\, ,\, \rangle) \to (N, \langle\, ,\, \rangle_N)$, that is, the following system holds

$$\begin{cases} \mathrm{Ric}^{\varphi} = \lambda\langle\, ,\, \rangle \\ \tau(\varphi) = 0. \end{cases} \tag{2.1.12}$$

for some $\lambda \in \mathbb{R}$. Assume $(M, \langle\, ,\, \rangle)$ is Einstein. Then, from the Proposition above φ is homothetic and, since it is also non-constant, from Remark 1.1.11 we have that $\varphi : (M, \zeta\langle\, ,\, \rangle) \to (N, \langle\, ,\, \rangle_N)$ is an isometric immersion, where $\zeta = |d\varphi|^2 \in \mathbb{R}$. We set

$$\widetilde{\langle\, ,\, \rangle} := \zeta\langle\, ,\, \rangle.$$

Then $(M, \widetilde{\langle\, ,\, \rangle})$ is an Einstein manifold minimally immersed in $(N, \langle\, ,\, \rangle_N)$ via $\varphi : (M, \widetilde{\langle\, ,\, \rangle}) \to (N, \langle\, ,\, \rangle_N)$. Indeed, using (1.3.20) (with f constant) and the first equation of (2.1.12) we immediately get

$$\widetilde{\mathrm{Ric}}^{\varphi} = \frac{\lambda}{\zeta}\widetilde{\langle\, ,\, \rangle},$$

that is, since φ is homothetic,

$$\widetilde{\mathrm{Ric}} = \left(\frac{\lambda}{\zeta} + \alpha\right)\widetilde{\langle\, ,\, \rangle}$$

The fact that $\varphi : (M, \zeta\langle\, ,\, \rangle) \to (N, \langle\, ,\, \rangle_N)$ is a minimal immersion is due to the fact that $\varphi : (M, \widetilde{\langle\, ,\, \rangle}) \to (N, \langle\, ,\, \rangle_N)$ is a isometric immersion such that $\tilde{\tau}(\varphi) = 0$ (it can be easily seen using (1.3.16) with f constant). Notice that

$$S = m\left(\frac{\lambda}{\zeta} + \alpha\right)$$

does not have necessary the same sign of λ.

Remark 2.1.13. The Remark above shows why it can be interesting the study of harmonic-Einstein manifold when φ is an isometric immersion, that is, the study of Einstein manifolds minimally immersed. Another interesting study may be the one when φ is a Riemannian submersion (that probably will have some interesting applications in Physics for non-linear σ-models). A Riemannian submersion $\varphi : (M, \langle\, ,\, \rangle) \to (N, \langle\, ,\, \rangle_N)$ is horizontaly homothetic and the following are equivalent: φ is harmonic; φ is a harmonic morphism; the foliation of M consisting of the fibres of φ is minimal, that is, every fiber of φ is a minimal submanifold of $(M, \langle\, ,\, \rangle)$. For the definitions and more details see [BW] and [FPI].

2.1.1 Symmetries and sectional curvatures

Let $(M, \langle\, ,\, \rangle)$ be a harmonic-Einstein manifold of dimension $m \geq 3$. From Proposition 2.1.7 we have

$$\mathring{\mathrm{Riem}} = \mathring{W}^{\varphi}$$

that is equivalent to, using the definition of φ-scalar curvature,

$$\mathrm{Riem} = W^{\varphi} + \frac{S^{\varphi}}{2m(m-1)}\langle\, ,\, \rangle \owedge \langle\, ,\, \rangle, \tag{2.1.14}$$

Notice that $\mathring{\mathrm{Riem}}$ vanishes if and only if $(M, \langle\, ,\, \rangle)$ has constant sectional curvature. If $m \geq 4$, $\mathring{W}^{\varphi}$ vanishes if and only $(M, \langle\, ,\, \rangle)$ is locally conformally flat and φ is weakly conformal, see Proposition 1.4.6 (if $m = 3$, from Remark 1.4.23, $\mathring{W}^{\varphi}$ vanishes if and only if φ is weakly conformal). Recall that, if $m \geq 3$, from Proposition 1.1.24, a harmonic map that is also weakly conformal is homothetic. Recall moreover that, from Proposition 2.1.9, a harmonic-Einstein manifold is Einstein if and only if φ is a homothetic map.

The discussion above implies the validity of

Proposition 2.1.15. *Let $(M, \langle\, ,\, \rangle)$ be a harmonic-Einstein manifold of dimension $m \geq 3$. The following are equivalent:*

(i) $(M, \langle\, ,\, \rangle)$ *has constant sectional curvature;*

(ii) φ *is homothetic and, if $m \geq 4$,* $(M, \langle\, ,\, \rangle)$ *is locally conformally flat.*

(iii) $(M, \langle\, ,\, \rangle)$ *is Einstein and, if $m \geq 4$, locally conformally flat.*

(iv) $(M, \langle\, ,\, \rangle)$ *is Einstein and it has constant sectional curvature.*

Another interesting feature of the φ-Weyl curvature of harmonic-Einstein manifold of dimension $m \geq 3$ is that, since S^{φ} is constant, from (2.1.14), we have

$$\nabla \mathrm{Riem} = \nabla W^{\varphi}. \tag{2.1.16}$$

Since Riem satisfies the second Bianchi identity then also W^{φ} does (this can also be easily seen combining Proposition 1.2.47 with $C^{\varphi} = 0$, that is given by Proposition 2.1.7).

Recall the following classic definitions:

(i) A Riemannian manifold of dimension $m \geq 3$ has *harmonic curvature tensor* if Riem is divergence free, or equivalently, Ric is Codazzi (and it has constant sectional curvature), or equivalently it has harmonic Weyl curvature and Ric is harmonic (that is, is Codazzi with constant trace);

(ii) A Riemannian manifold of dimension $m \geq 3$ is *locally symmetric* if the Riemann tensor if parallel. Locally symmetric Riemannian manifolds have parallel Ricci tensor.

We then have, combining (2.1.16) with Proposition 1.4.12, Proposition 1.4.17 and Remark 1.4.23,

Proposition 2.1.17. *Let $(M, \langle\, ,\, \rangle)$ be a harmonic-Einstein manifold of dimension $m \geq 3$. Then:*

(i) $(M, \langle\, ,\, \rangle)$ *has harmonic curvature if and only if φ is almost relatively affine and, if $m \geq 4$,* $(M, \langle\, ,\, \rangle)$ *has harmonic Weyl curvature. If is this the case, then Ric and $\varphi^*\langle\, ,\, \rangle_N$ are both harmonic tensor.*

(ii) $(M, \langle\, ,\, \rangle)$ *is locally symmetric if and only if φ is relatively affine and, if $m \geq 4$,* $(M, \langle\, ,\, \rangle)$ *is conformally symmetric. If is this the case, then Ric and $\varphi^*\langle\, ,\, \rangle_N$ are both parallel tensor with constant trace.*

2.1.2 Some remarks for Riemann surfaces

Let M be a surface, that is, a smooth manifold of dimension 2. By fixing a Riemannian metric $\langle\,,\rangle$ on M then

$$\mathrm{Riem} = \frac{S}{2}\langle\,,\rangle \owedge \langle\,,\rangle.$$

As a consequence the Ricci tensor Ric of $(M, \langle\,,\rangle)$ is always proportional to the metric tensor $\langle\,,\rangle$. In particular $(M, \langle\,,\rangle)$ is Einstein if S is constant, that is, if and only if $(M, \langle\,,\rangle)$ has constant sectional curvature.

Recall that a Riemann surface (M, J) is given by a complex manifold M of dimension 1 with complex structure J, or equivalently, a oriented smooth manifold M of dimension 2 endowed with an almost complex structure J. In the two dimensional case giving a complex structure is equivalent to choose a conformal class of metric on M, where the conformal class of a Riemannian metric $\langle\,,\rangle$ on M is defined as

$$[\langle\,,\rangle] = \{e^{-2f}\langle\,,\rangle : f \in \mathcal{C}^\infty(M)\}.$$

The famous uniformization theorem states that on a surface M, in any fixed conformal class of metrics $[\langle\,,\rangle]$ there exists a complete Riemannian metric of constant curvature. In particular, every Riemann surface (M, J) can be endowed with a Riemannian metric $\langle\,,\rangle$ such that $(M, \langle\,,\rangle)$ is Einstein and J is determined by $[\langle\,,\rangle]$. In case M is compact, the sign of the curvature depends on the topology of the surface, indeed when S is positive the universal covering of M is given by a sphere immersed in $\mathbb{R}^3$, when $S = 0$ by the Eucludean plane and when $S < 0$ by the hyperbolic plane.

Let (M, J) be a Riemann surface and let $\varphi : M \to (N, \langle\,,\rangle_N)$ be a smooth map. We fix a Riemannian metric $\langle\,,\rangle$ on M such that J is determined by $[\langle\,,\rangle]$. Then $\varphi : (M, \langle\,,\rangle) \to (N, \langle\,,\rangle_N)$ is weakly conformal if

$$\varphi^*\langle\,,\rangle_N = \frac{|d\varphi|^2}{m}\langle\,,\rangle.$$

Observe that conformality of φ depends on $[\langle\,,\rangle]$, that is, on the complex structure induced by $\langle\,,\rangle$, and not on the choice of $\langle\,,\rangle$. Hence we can say that $\varphi : (M, J) \to (N, \langle\,,\rangle_N)$ is conformal, without fixing a Riemannian metric on M. The same applies for the harmonicity of φ, see Section 10 of the first report in [EL].

Fix a Riemannian metric $\langle\,,\rangle$ on M. Since Ric is always proportional to $\langle\,,\rangle$ and $\alpha \neq 0$ we immediately deduce that

$$\mathrm{Ric}^\varphi = \mathrm{Ric} - \alpha\varphi^*\langle\,,\rangle_N$$

is proportional to $\langle\,,\rangle$ if and only if $\varphi : (M, J) \to (N, \langle\,,\rangle_N)$ is weakly conformal. As a consequence, if $(M, \langle\,,\rangle)$ is harmonic-Einstein then $\varphi : (M, J) \to (N, \langle\,,\rangle_N)$ is weakly-conformal and harmonic. In [BW], see Section 3.5, the weakly-conformal and harmonic maps from a surface are called *minimal branched immersions*.

Observe that, if $\varphi : (M, J) \to (N, \langle\,,\rangle_N)$ is a minimal branched immersion then there exists a Riemannian metric $\langle\,,\rangle$ on M such that $(M, \langle\,,\rangle)$ is harmonic-Einstein if and only if

$$S^\varphi = S - \alpha|d\varphi|^2$$

is constant. If $[\langle\,,\rangle]$ corresponds to J, to find such a metric it is sufficient to find a solution $f \in \mathcal{C}^\infty(M)$ of

$$2\Delta f + S^\varphi - \widetilde{S}^\varphi e^{-2f} = 0, \tag{2.1.18}$$

if we denote by $\widetilde{S}^\varphi$ the φ-scalar curvature of the metric $\widetilde{\langle\,,\rangle} = e^{-2f}\langle\,,\rangle$ and the map $\varphi : (M, \langle\,,\rangle) \to (N, \langle\,,\rangle_N)$, that is,

$$\widetilde{S}^\varphi = \widetilde{S} - \alpha e^{2f}|d\varphi|^2.$$

It follows from (2.11) of [**?**] with the choice of $u = -f$, and the fact that

$$\widetilde{|d\varphi|^2} = e^{2f}|d\varphi|^2.$$

With the following Proposition we show that, independently on the topology of the surface, when we are given a compact non-constant minimal immersion from a Riemann surface we can always find a Riemannian metric inducing the complex structure with vanishing φ-scalar curvature, by choosing α adequately. This is in contrast with the standard case, where the only compact Riemannian surfaces that admits a metric of zero sectional curvature are the one with genus $g = 1$. The topology of M is relevant for the sign of the constant α.

Proposition 2.1.19. *Let (M, J) be a compact Riemann surface and let $\varphi : M \to (N, \langle\, , \rangle_N)$ be a non-constant minimal branched immersion. Set*

$$\alpha = \frac{\pi\chi(M)}{E(\varphi)}, \tag{2.1.20}$$

where $\chi(M)$ is the Euler characteristic of M and $E(\varphi)$ is the energy of φ. Then there exists a metric in the conformal class $[\langle\, , \rangle]$ corresponding to J with vanishing φ-scalar curvature.

Proof. As seen above, to find a metric $\widetilde{\langle\, , \rangle}$ in the conformal class $[\langle\, , \rangle]$ with φ-scalar $\widetilde{S}^\varphi = 0$, from (2.1.18), it is sufficient to find $f \in \mathcal{C}^\infty(M)$ such that

$$\Delta f + \frac{S^\varphi}{2} = 0,$$

Since M is compact the equation above, that is a Poisson equation, admits a solution $f \in \mathcal{C}^\infty(M)$ if and only if

$$\int_M S^\varphi = 0. \tag{2.1.21}$$

Recall that, from Gauss-Bonnet formula,

$$\int_M S = 2\pi\chi(M),$$

where $\chi(M)$ is the Euler characteristic of M. Moreover, by definition,

$$E(\varphi) = \frac{1}{2}\int_M |d\varphi|^2.$$

Combining the two equations above we conclude with the definition of S^φ we conclude

$$\frac{1}{2}\int_M S^\varphi = \pi\chi(M) - \alpha E(\varphi).$$

As a consequence, in order to obtain (2.1.21) we must have

$$\pi\chi(M) = \alpha E(\varphi).$$

Since φ is non-constant the above amounts requiring (2.1.20). □

Remark 2.1.22. If M is a compact oriented surface then

$$\chi(M) = 2 - 2g,$$

where g is the genus of M. In particular α, given by (2.1.20), is positive if and only if $g = 0$.

2.2 The role of the curvature of the target manifold

Let $(M, \langle\, , \rangle)$ be a Riemannian manifold and $\varphi : M \to (N, \langle\, , \rangle_N)$ a smooth map. Recall the definition of the curvature operator $\mathfrak{R}$ acting on $S^2(\varphi^{-1}TN)$, the space of symmetric 2-covariant tensor fields on $\varphi^{-1}TN$: let ${}^N R_{acbd}$ denote the components of the curvature tensor of N in a local orthonormal coframe $\{\omega^a\}$, for $1 \leq a, b, \ldots \leq n$, where n is the dimension of N, let $\beta = \beta_{ab}\omega^a \otimes \omega^b$ be an element of $S^2(\varphi^{-1}TN)$ and define

$$\mathfrak{R}(\beta) := {}^N R_{acbd}\beta_{cd}\omega^a \otimes \omega^b.$$

It is not difficult to see that, introduced in $S^2(\varphi^{-1}TN)$ the natural inner product $(\, ,)$, induced by $\langle\, , \rangle_N$, the operator $\mathfrak{R} : S^2(\varphi^{-1}TN) \to S^2(\varphi^{-1}TN)$ is self-adjoint and thus diagonalizable. We let $\Lambda(x)$ to denote its largest eigenvalue at $x \in M$. We have

Theorem 2.2.1. *Let $(M, \langle\, , \rangle)$ be a complete m-dimensional manifold with $m \geq 2$ which is hamonic-Einstein, that is, such that*

$$\begin{cases} Ric^{\varphi} = \dfrac{S^{\varphi}}{m}\langle\, , \rangle \\ \tau(\varphi) = 0 \end{cases} \tag{2.2.2}$$

for some $\varphi : M \to (N, \langle\, , \rangle_N)$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Assume that

$$\Lambda^* := \sup_M \Lambda < \alpha, \tag{2.2.3}$$

and, if $\alpha < 0$,

$$e(\varphi)^* := \sup_M e(\varphi) < +\infty, \tag{2.2.4}$$

where $e(\varphi)$ is the density of energy of φ. Depending on the sign of the constant S^{φ}, we have

i) if $S^{\varphi} \geq 0$, then φ is constant and $(M, \langle\, , \rangle)$ is Einstein with scalar curvature $S = S^{\varphi}$;

ii) if $S^{\varphi} < 0$, then

$$0 \leq e(\varphi)^* \leq -\frac{S^{\varphi}}{2(\alpha - \Lambda^*)}.$$

Remark 2.2.5. If $(M, \langle\, , \rangle)$ is harmonic-Einstein with $\alpha > 0$ then, since $\varphi^*\langle\, , \rangle_N \geq 0$, we have

$$\mathrm{Ric} \geq \frac{S^{\varphi}}{m}\langle\, , \rangle.$$

If $S^{\varphi} > 0$ then, by Myers' theorem, M is compact and thus Λ^* and $e(\varphi)^*$ are both finite. This shows how for $\alpha, S^{\varphi} > 0$ request (2.2.4) is not needed. It is interesting that, even though M is non necessarily compact when $\alpha > 0$ and $S^{\varphi} = 0$, we do not need (2.2.4) the same. This is pointed out in

Corollary 2.2.6. *In the assumption of the Theorem above suppose that the manifold is harmonic-Ricci flat. Then φ is constant and $(M, \langle\, , \rangle)$ is Ricci flat.*

Remark 2.2.7. Since $\alpha > 0$, $\mathrm{Ric}^{\varphi} = 0$ immediately implies that $\mathrm{Ric} \geq 0$ on the complete manifold $(M, \langle\, , \rangle)$. In case the harmonic map φ has bounded image and N is simply connected with non-positive sectional curvature by a Theorem of S. Y. Cheng [Ch] we know that φ is constant and as a consequence $(M, \langle\, , \rangle)$ is Ricci flat. The setting of Corollary 2.2.6 is more general and, in any case, different.

Proof (of Theorem 2.2.1). Since φ is harmonic Weitzenböck-Bochner formula reads

$$\frac{1}{2}\Delta|d\varphi|^2 = |\nabla d\varphi|^2 + {}^N R_{abcd}\varphi^a_i\varphi^b_j\varphi^c_j\varphi^d_i + R_{ij}\varphi^a_i\varphi^a_j, \tag{2.2.8}$$

for a proof of the above see Proposition 1.5 of [AMR]. Having set

$$\beta := \varphi_i^a \varphi_i^b \omega^a \otimes \omega^b,$$

we have

$$^N R_{abcd} \varphi_i^a \varphi_j^b \varphi_j^c \varphi_i^d = -(\mathfrak{R}(\beta), \beta) \geq -\Lambda |\beta|_N^2. \tag{2.2.9}$$

Plugging into (2.2.9) and using (1.2.2) we conclude

$$\frac{1}{2}\Delta |d\varphi|^2 \geq |\nabla d\varphi|^2 - \Lambda |\beta|_N^2 + R_{ij}^{\varphi} \varphi_i^a \varphi_j^a + \alpha |\beta|_N^2,$$

that is, since $(M, \langle \, , \, \rangle)$ is harmonic-Einstein and, by (2.2.3), $\Lambda^* < +\infty$,

$$\frac{1}{2}\Delta |d\varphi|^2 \geq (\alpha - \Lambda^*)|\beta|_N^2 + \frac{S^{\varphi}}{m}|d\varphi|^2, \tag{2.2.10}$$

Notice that

$$|\beta|_N^2 = |\varphi^* \langle \, , \, \rangle_N|^2$$

and from Newton's inequality

$$|\varphi^* \langle \, , \, \rangle_N|^2 \geq \frac{|d\varphi|^4}{m},$$

hence the above implies

$$|\beta|_N^2 \geq \frac{|d\varphi|^4}{m}.$$

Plugging into (2.2.10), since (2.2.3) holds, we get

$$\frac{1}{2}\Delta |d\varphi|^2 \geq \frac{\alpha - \Lambda^*}{m}|d\varphi|^4 + \frac{S^{\varphi}}{m}|d\varphi|^2$$

By setting

$$u := |d\varphi|^2$$

the above yields

$$\frac{m}{2}\Delta u \geq (\alpha - \Lambda^*)u^2 + S^{\varphi} u, \tag{2.2.11}$$

where the constant $\alpha - \Lambda^*$ is strictly positive because of (2.2.3).

We observe that the first equation of (2.2.2) in case $\alpha > 0$ imply

$$\mathrm{Ric} \geq \frac{S^{\varphi}}{m}\langle \, , \, \rangle$$

where S^{φ} is constant and therefore completeness of $(M, \langle \, , \, \rangle)$ yields the validity of the Omori-Yau maximum principle for the Laplace-Beltrami operator Δ. In case $\alpha < 0$ we obtain the same result, since the fact that $\varphi^* \langle \, , \, \rangle \geq 0$ implies

$$\varphi^* \langle \, , \, \rangle \leq |d\varphi|^2 \langle \, , \, \rangle$$

and thus, using (2.2.4),

$$\varphi^* \langle \, , \, \rangle \leq 2e(\varphi)^* \langle \, , \, \rangle.$$

Hence

$$\mathrm{Ric} \geq \left(\frac{S^{\varphi}}{m} + 2\alpha e(\varphi)^* \right) \langle \, , \, \rangle.$$

We then apply Theorem 3.6 of [AMR] to deduce

$$u^* := \sup_M u < +\infty$$

and the Omori-Yau maximum principle again to conclude, from (2.2.11), that

$$u^* \left[(\alpha - \Lambda^*)u^* + S^{\varphi} \right] \leq 0. \tag{2.2.12}$$

From (2.2.12) and the definition of u we immediately deduce conclusions *i)* and *ii)*. □

2.3 Conformally harmonic-Einstein manifolds

Next result is one of the important motivations for the general structure we shall introduce in Chapter 3. We begin with the following

Definition 2.3.1. A Riemannian manifold $(M, \langle \, , \, \rangle)$ of dimension $m \geq 3$ is said to be *conformally harmonic-Einstein* if there exists $\psi \in \mathcal{C}^\infty(M)$, $\psi > 0$ on M such that, having defined

$$\widetilde{\langle \, , \, \rangle} := \psi^2 \langle \, , \, \rangle,$$

the Riemannian manifold $(M, \widetilde{\langle \, , \, \rangle})$ is harmonic-Einstein.

Proposition 2.3.2. *Let $(M, \langle \, , \, \rangle)$ be a Riemannian manifold of dimension $m \geq 3$ such that, by setting*

$$\widetilde{\langle \, , \, \rangle} = e^{-\frac{2}{m-2}f} \langle \, , \, \rangle$$

for some $f \in \mathcal{C}^\infty(M)$, we have that $(M, \widetilde{\langle \, , \, \rangle})$ is harmonic-Einstein. Then

$$C^\varphi_{ijk} + f_t W^\varphi_{tijk} = 0 \tag{2.3.3}$$

and

$$(m-2)B^\varphi_{ij} + \frac{m-4}{m-2} W^\varphi_{tijk} f_t f_k = 0 \tag{2.3.4}$$

hold.

Proof. A harmonic-Einstein manifold is φ-Cotton flat and φ-Bach flat, see Proposition 2.1.7. As a consequence, $\widetilde{C}^\varphi = 0$ and $\widetilde{B}^\varphi = 0$. Using (1.3.28) and $\widetilde{C}^\varphi = 0$ we immediately get (2.3.3). Using (1.3.42), $\widetilde{B}^\varphi = 0$ and (2.3.3) twice we infer the validity of (2.3.4). □

Theorem 2.3.5. *Let $(M, \langle \, , \, \rangle)$ be a Riemannian manifold of dimension $m \geq 3$, let $\varphi : M \to (N, \langle \, , \, \rangle_N)$ be a smooth map and let $\alpha \in \mathbb{R} \setminus \{0\}$. Then there exists $\psi \in \mathcal{C}^\infty(M)$, $\psi > 0$ on M and $\Lambda \in \mathcal{C}^\infty(M)$ such that, having defined $\widetilde{\langle \, , \, \rangle} := \psi^2 \langle \, , \, \rangle$,*

$$\begin{cases} \widetilde{Ric} - \alpha \varphi^* \langle \, , \, \rangle_N = \Lambda \widetilde{\langle \, , \, \rangle} \\ \widetilde{\tau}(\varphi) = 0, \end{cases} \tag{2.3.6}$$

if and only if for some $f, \lambda \in \mathcal{C}^\infty(M)$

$$\begin{cases} Ric - \alpha \varphi^* \langle \, , \, \rangle_N + Hess(f) + \dfrac{1}{m-2} df \otimes df = \lambda \langle \, , \, \rangle \\ \tau(\varphi) = d\varphi(\nabla f). \end{cases} \tag{2.3.7}$$

In this case f and ψ are related by

$$\psi = e^{-\frac{f}{m-2}} \tag{2.3.8}$$

while Λ and λ satisfy

$$\Delta_f f + (m-2)\lambda = (m-2)\Lambda e^{\frac{-2}{m-2}f}. \tag{2.3.9}$$

Here Δ_f is the symmetric diffusion operator $\Delta - \langle \nabla f, \nabla \, \rangle$.

Remark 2.3.10. Note that, since $m \geq 3$, Λ is constant by Proposition 2.1.3.

Remark 2.3.11. We shall see later, see Remark 6.1.17, that the system (2.3.7) satisfies the integrability conditions (2.3.3) and (2.3.4). Using the Theorem above we get as a consequence of Remark 6.1.17 a different proof of Proposition 2.3.2, a proof that does not rely on the transformation laws (1.3.28) and (1.3.42).

Remark 2.3.12. It is worth to observe that (2.3.4) implies that if $(M, \langle\, ,\,\rangle)$ is a four dimensional conformally harmonic-Einstein manifold then it is φ-Bach flat. This partly motivates the definition of B^{φ} given in (1.2.50). Indeed, in this way the situation parallels that of four dimensional conformally Einstein manifolds that are always Bach flat (another way to see that a four dimensional conformally harmonic-Einstein manifold is Bach flat is to combine Corollary 1.3.45 with Proposition 2.1.7).

In order to prove the Theorem above we shall need (1.3.20) and (1.3.16), that we report here for the sake of the reader:

$$\widetilde{\mathrm{Ric}}^{\varphi} = \mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) + \frac{1}{m-2} df \otimes df + \frac{\Delta_f f}{m-2}\langle\, ,\,\rangle, \tag{2.3.13}$$

$$\tau(\widetilde{\varphi}) = e^{\frac{2}{m-2}f}(\tau(\varphi) - d\varphi(\nabla f)). \tag{2.3.14}$$

Proof (of Thereom 2.3.5). By (2.3.14) we deduce that $\tau(\varphi) = d\varphi(\nabla f)$ if and only if $\tau(\widetilde{\varphi}) = 0$. Suppose (2.3.6) holds, for some $\Lambda \in \mathbb{R}$, where $\widetilde{\langle\, ,\,\rangle} = \psi^2\langle\, ,\,\rangle$ with ψ given by (2.3.8). Using (2.3.13) we obtain

$$\mathrm{Ric} - \alpha\varphi^*\langle\, ,\,\rangle_N + \mathrm{Hess}(f) + \frac{1}{m-2} df \otimes df + \frac{\Delta_f f}{m-2}\langle\, ,\,\rangle = \Lambda\widetilde{\langle\, ,\,\rangle},$$

that is,

$$\mathrm{Ric} + \mathrm{Hess}(f) + \frac{1}{m-2} df \otimes df - \alpha\varphi^*\langle\, ,\,\rangle_N = \left(e^{-\frac{2}{m-2}f}\Lambda - \frac{\Delta_f f}{m-2}\right)\langle\, ,\,\rangle,$$

that gives the first equation of (2.3.7) once we define λ as in (2.3.9). Conversely suppose that (2.3.7) holds for some $f, \lambda \in \mathcal{C}^{\infty}(M)$. Define ψ as in (2.3.8) and $\widetilde{\langle\, ,\,\rangle} = \psi^2\langle\, ,\,\rangle$. From (2.3.13) and (2.3.7) we obtain

$$\widetilde{\mathrm{Ric}} - \alpha\varphi^*\langle\, ,\,\rangle_N = \lambda\langle\, ,\,\rangle + \frac{\Delta_f f}{m-2}\langle\, ,\,\rangle = e^{\frac{2}{m-2}f}\left(\lambda + \frac{\Delta_f f}{m-2}\right)\widetilde{\langle\, ,\,\rangle},$$

that is the first equation of (2.3.6) with Λ given by (2.3.9). □

2.3.1 A sufficient condition for being conformally harmonic-Einstein

From Proposition 2.3.2 a conformally harmonic-Einstein manifold $(M, \langle\, ,\,\rangle)$ satisfies the two integrability conditions (2.3.3) and (2.3.4). Suppose now we are given $f \in \mathcal{C}^{\infty}(M)$, $\alpha \in \mathbb{R} \setminus \{0\}$ and a smooth map $\varphi : M \to (N, \langle\, ,\,\rangle_N)$ such that (2.3.3) and (2.3.4) are satisfied. Does it follow that $(M, \langle\, ,\,\rangle)$ is conformally harmonic-Einstein? To answer the question we need to introduce the next genericity condition.

Definition 2.3.15. Let $(M, \langle\, ,\,\rangle)$ be a Riemannian manifold of dimension $m \geq 3$ and denote by $S^2_0(M)$ the bundle of the 2-times covariant, symmetric, traceless tensor fields on M. We define, for a smooth map $\varphi : M \to (N, \langle\, ,\,\rangle_N)$,

$$\mathcal{W}^{\varphi} : S^2_0(M) \to S^2_0(M)$$

by setting for $\beta \in S^2_0(M)$, $\beta = \beta_{ij}\theta^i \otimes \theta^j$,

$$\mathcal{W}^{\varphi}(\beta) = \left[W^{\varphi}_{tikj} - \frac{\alpha}{2}\varphi^a_t(\varphi^a_i\delta_{kj} + \varphi^a_j\delta_{ki})\right]\beta_{tk}\theta^i \otimes \theta^j. \tag{2.3.16}$$

It is easy using the properties of φ-Weyl to verify that $\mathcal{W}^{\varphi}$ is well defined, that is, $\mathcal{W}^{\varphi}(\beta)$ is 2-times covariant, symmetric and traceless for every $\beta \in S^2_0(M)$, and that it is self-adjoint with respect to the standard extension of $\langle\, ,\,\rangle$ to $S^2_0(M)$, that we denote with the same symbol. Thus $\mathcal{W}^{\varphi}$ is diagonalizable.

Definition 2.3.17. Let M be a smooth manifold. We say that the pair $(\langle\, ,\,\rangle, \varphi)$ is *generic*, where $\langle\, ,\,\rangle$ is a Riemannian metric and $\varphi : M \to (M, \langle\, ,\,\rangle_N)$ a smooth map, if φ is possibly singular (that is, $d\varphi$ is zero) only at isolated points and if $\mathcal{W}^{\varphi}$ is injective, in other words if all its eigenvalues are non null everywhere on M.

We are now ready to state the following Proposition, that extends a result of A. R. Gover and P. Nurowski [GN], Section 2.4, that deals with the conformally Einstein case and that we can consider as the degenerate case $d\varphi \equiv 0$.

Proposition 2.3.18. *Let $(M, \langle\, ,\, \rangle)$ be a Riemannian manifold of dimension $m \geq 3$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\varphi : M \to (N, \langle\, ,\, \rangle_N)$. Suppose that $(\langle\, ,\, \rangle, \varphi)$ is generic and that the integrability conditions* (2.3.3) *and* (2.3.4) *are satisfied for some $f \in \mathcal{C}^\infty(M)$. Then, defining*

$$\widetilde{\langle\, ,\, \rangle} := e^{-\frac{2}{m-2}f} \langle\, ,\, \rangle,$$

the Riemannian manifold $(M, \widetilde{\langle\, ,\, \rangle})$ is harmonic-Einstein, that is, $(M, \langle\, ,\, \rangle)$ is conformally harmonic-Einstein.

Proof. We trace (2.3.3) with respect to i and j and we use (1.2.22) and (1.2.36) to obtain, for each $k = 1, \ldots, m$,

$$\alpha\varphi^a_k(\varphi^a_{ii} - \varphi^a_i f_i) = 0.$$

Fix $x \in M$. If there exists k such that $\varphi^a_k(x) \neq 0$ then we have the validity of the following equality at x:

$$\tau(\varphi) = d\varphi(\nabla f). \tag{2.3.19}$$

Otherwise the same holds by continuity, because by assumption the points where $\varphi^a_k = 0$ for every k are isolated. In conclusion (2.3.19) holds on M. Next, taking the covariant derivative of (2.3.3), using (1.2.45) and (2.3.19) we obtain

$$\begin{aligned}
C^\varphi_{ijk,k} =& -(f_t W^\varphi_{tijk})_k \\
=& - f_{tk}W^\varphi_{tijk} - f_t W^\varphi_{tijk,k} \\
=& - f_{tk}W^\varphi_{tijk} - f_k W^\varphi_{tjik,t} \\
=& - f_{tk}W^\varphi_{tijk} - f_k \left(\frac{m-3}{m-2}C^\varphi_{jki} + \alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{jk}\varphi^a_i) + \frac{\alpha}{m-2}\varphi^a_{tt}(\varphi^a_i\delta_{jk} - \varphi^a_k\delta_{ij})\right) \\
=& - f_{tk}W^\varphi_{tijk} - f_k\frac{m-3}{m-2}C^\varphi_{jki} + \alpha(-\varphi^a_{ij}f_k\varphi^a_k + \varphi^a_{jk}f_k\varphi^a_i) + \frac{\alpha}{m-2}\varphi^a_{tt}(-\varphi^a_i f_k\delta_{jk} + \varphi^a_k f_k\delta_{ij}) \\
=& f_{tk}W^\varphi_{tikj} - \frac{m-3}{m-2}f_k C^\varphi_{jki} + \alpha(\varphi^a_{jk}f_k\varphi^a_i - \varphi^a_{ij}\varphi^a_{kk}) + \frac{\alpha}{m-2}(|\tau(\varphi)|^2\delta_{ij} - \varphi^a_{tt}\varphi^a_i f_j).
\end{aligned}$$

The last formula enables us to express $(m-2)B^\varphi_{ij}$, defined in (1.2.50), in the form

$$\begin{aligned}
(m-2)B^\varphi_{ij} =& f_{tk}W^\varphi_{tikj} - \frac{m-3}{m-2}f_k C^\varphi_{jki} + \alpha(\varphi^a_{jk}f_k\varphi^a_i - \varphi^a_{ij}\varphi^a_{kk}) + \frac{\alpha}{m-2}(|\tau(\varphi)|^2\delta_{ij} - \varphi^a_{tt}\varphi^a_i f_j) \\
&+ R^\varphi_{tk}W^\varphi_{tikj} - \alpha R^\varphi_{kj}\varphi^a_k\varphi^a_i + \alpha\left(\varphi^a_{ij}\varphi^a_{kk} - \varphi^a_{kkj}\varphi^a_i - \frac{1}{m-2}|\tau(\varphi)|^2\delta_{ij}\right) \\
=& (R^\varphi_{tk} + f_{tk})W^\varphi_{tikj} - \frac{m-3}{m-2}f_k C^\varphi_{jki} + \alpha\varphi^a_{jk}f_k\varphi^a_i - \frac{\alpha}{m-2}\varphi^a_{tt}\varphi^a_i f_j - \alpha R^\varphi_{kj}\varphi^a_k\varphi^a_i - \alpha\varphi^a_{kkj}\varphi^a_i,
\end{aligned}$$

and using once again (2.3.19)

$$\begin{aligned}
(m-2)B^\varphi_{ij} =& (R^\varphi_{tk} + f_{tk})W^\varphi_{tikj} - \frac{m-3}{m-2}f_k C^\varphi_{jki} - \frac{\alpha}{m-2}\varphi^a_{tt}\varphi^a_i f_j - \alpha R^\varphi_{kj}\varphi^a_k\varphi^a_i - \alpha\varphi^a_k f_{kj}\varphi^a_i \\
=& (R^\varphi_{tk} + f_{tk})(W^\varphi_{tikj} - \alpha\varphi^a_k\varphi^a_i\delta_{jt}) - \frac{m-3}{m-2}f_k C^\varphi_{jki} - \frac{\alpha}{m-2}\varphi^a_{tt}\varphi^a_i f_j.
\end{aligned}$$

Thus the second integrability condition (2.3.4) can be expressed as

$$(R^\varphi_{tk} + f_{tk})(W^\varphi_{tikj} - \alpha\varphi^a_k\varphi^a_i\delta_{jt}) - \frac{m-3}{m-2}f_k C^\varphi_{jki} - \frac{\alpha}{m-2}\varphi^a_{tt}\varphi^a_i f_j + \frac{m-4}{m-2}W^\varphi_{tijk}f_t f_k = 0.$$

Inserting (2.3.19) and (2.3.3) into the above we get

$$\begin{aligned}0 =&(R^\varphi_{tk} + f_{tk})(W^\varphi_{tikj} - \alpha\varphi^a_k\varphi^a_i\delta_{jt}) + \frac{m-3}{m-2}f_kf_tW^\varphi_{tjki} - \frac{\alpha}{m-2}\varphi^a_tf_t\varphi^a_if_j + \frac{m-4}{m-2}W^\varphi_{tijk}f_tf_k\\ =&\left(R^\varphi_{tk} + f_{tk} + \frac{1}{m-2}f_tf_k\right)(W^\varphi_{tikj} - \alpha\varphi^a_k\varphi^a_i\delta_{jt}).\end{aligned}$$

Next we define

$$\lambda := \frac{1}{m}\left(S^\varphi + \Delta f + \frac{|\nabla f|^2}{m-2}\right),$$

so that the symmetric 2-times covariant tensor field

$$\beta := \mathrm{Ric}^\varphi + \mathrm{Hess}(f) + \frac{1}{m-2}df \otimes df - \lambda\langle\, ,\,\rangle$$

is traceless. From the above identity and from (1.2.22) we then have

$$\begin{aligned}(W^\varphi_{tikj} - \alpha\varphi^a_k\varphi^a_i\delta_{jt})\beta_{tk} =&(W^\varphi_{tikj} - \alpha\varphi^a_k\varphi^a_i\delta_{jt})\left(R^\varphi_{tk} + f_{tk} + \frac{1}{m-2}f_tf_k - \lambda\delta_{tk}\right)\\ =&(W^\varphi_{tikj} - \alpha\varphi^a_k\varphi^a_i\delta_{jt})\left(R^\varphi_{tk} + f_{tk} + \frac{1}{m-2}f_tf_k\right) - \lambda(W^\varphi_{kikj} - \alpha\varphi^a_i\varphi^a_j) = 0.\end{aligned}$$

Interchanging the role of i and j in the above equation we get

$$0 = (W^\varphi_{tjki} - \alpha\varphi^a_j\varphi^a_t\delta_{ik})\beta_{tk} = (W^\varphi_{tikj} - \alpha\varphi^a_j\varphi^a_t\delta_{ik})\beta_{tk}.$$

Summing up the last two formulas

$$\begin{aligned}0 =&(W^\varphi_{tikj} - \alpha\varphi^a_i\varphi^a_t\delta_{jk})\beta_{tk} + (W^\varphi_{tikj} - \alpha\varphi^a_j\varphi^a_t\delta_{ik})\beta_{tk}\\ =&[2W^\varphi_{tikj} - \alpha\varphi^a_t(\varphi^a_i\delta_{kj} + \varphi^a_j\delta_{ki})]\beta_{tk}\\ =&2\left(W^\varphi_{tikj} - \frac{1}{2}\alpha\varphi^a_t(\varphi^a_i\delta_{kj} + \varphi^a_j\delta_{ki})\right)\beta_{tk}.\end{aligned}$$

Hence,

$$\mathcal{W}^\varphi(\beta) = \left(W^\varphi_{tikj} - \frac{1}{2}\alpha\varphi^a_t(\varphi^a_i\delta_{kj} + \varphi^a_j\delta_{ki})\right)\beta_{tk}\theta^i \otimes \theta^j = 0.$$

Thus, since $\mathcal{W}^\varphi$ is injective, $\beta = 0$, that is,

$$\mathrm{Ric}^\varphi + \mathrm{Hess}(f) + \frac{1}{m-2}df \otimes df = \lambda\langle\, ,\,\rangle.$$

The latter together with (2.3.19) and Theorem 2.3.5 show that $(M, \widetilde{\langle\, ,\,\rangle})$ is harmonic-Einstein. □

2.4 A gap result for harmonic-Einstein manifolds

Let $(M, \langle\, ,\,\rangle)$ be a Riemannian manifold of dimension m, $\varphi : M \to (N, \langle\, ,\,\rangle_N)$ a smooth map, $\alpha \in \mathbb{R} \setminus \{0\}$ and set T^φ to denote the traceless part of the φ-Ricci tensor, defined as in (1.2.57), that is,

$$T^\varphi = \mathrm{Ric}^\varphi - \frac{S^\varphi}{m}\langle\, ,\,\rangle. \tag{2.4.1}$$

Let the operator $\mathcal{W}^\varphi$ be defined as in (2.3.16). Notice that for every $\beta \in S^2_0(M)$

$$\langle\mathcal{W}^\varphi(\beta), \beta\rangle = W^\varphi_{tikj}\beta_{tk}\beta_{ij} - \alpha\varphi^a_i\varphi^a_j\beta_{ik}\beta_{kj}, \tag{2.4.2}$$

where $\beta = \beta_{ij}\theta^i \otimes \theta^j$. We also set

$$\mathrm{div}(C^\varphi) := C^\varphi_{ijk,k}\theta^i \otimes \theta^j. \tag{2.4.3}$$

and

$$\mathrm{tr}(C^\varphi) = C^\varphi_{kki}\theta^i. \tag{2.4.4}$$

Observe that, from (1.2.36),

$$\mathrm{tr}(C^\varphi)_{i,j} = \alpha(\varphi^a_{kk}\varphi^a_i)_j = \alpha(\varphi^a_{kki}\varphi^a_j + \varphi^a_{kk}\varphi^a_{ij}). \tag{2.4.5}$$

Next result is computational but not trivial.

Theorem 2.4.6. *In the above setting and for $m \geq 3$ we have*

$$\begin{aligned}\frac{1}{2}\Delta|T^\varphi|^2 =&|\nabla T^\varphi|^2 + \frac{m-2}{2(m-1)}tr(T^\varphi \circ Hess(S^\varphi)) + \frac{m}{m-2}tr[(T^\varphi)^3] + \frac{S^\varphi}{m-1}|T^\varphi|^2\\ &+ tr(div(C^\varphi)\circ T^\varphi) - \langle \mathcal{W}^\varphi(T^\varphi), T^\varphi\rangle - tr(T^\varphi \circ \nabla tr(C^\varphi))\end{aligned} \tag{2.4.7}$$

Proof. A simple calculation shows the validity of

$$\frac{1}{2}\Delta|T^\varphi|^2 = |\nabla T^\varphi|^2 + T^\varphi_{ij,kk}T^\varphi_{ij}.$$

From (2.4.1),

$$T^\varphi_{ij,kk} = R^\varphi_{ij,kk} - \frac{\Delta S^\varphi}{m}\delta_{ij},$$

and since T^φ is traceless the formula above can be rewritten as

$$\frac{1}{2}\Delta|T^\varphi|^2 = |\nabla T^\varphi|^2 + R^\varphi_{ij,kk}T^\varphi_{ij}. \tag{2.4.8}$$

Now we want to evaluate $R^\varphi_{ij,kk}$. First we derive the following commutation relation, alternative to (1.2.42),

$$R^\varphi_{ij,k} = R^\varphi_{ik,j} + R^t_{ikj,t} + \alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j). \tag{2.4.9}$$

To prove it we use the second Bianchi identity and the definition (1.2.2) of the φ-Ricci tensor

$$\begin{aligned}R^t_{ijk,t} =& - R^t_{ikt,j} - R^t_{itj,k} = R_{ik,j} - R_{ij,k}\\ =&R^\varphi_{ik,j} + \alpha(\varphi^a_i\varphi^a_k)_j - R^\varphi_{ij,k} - \alpha(\varphi^a_i\varphi^a_j)_k\\ =&R^\varphi_{ik,j} - R^\varphi_{ij,k} + \alpha(\varphi^a_{ij}\varphi^a_k + \varphi^a_i\varphi^a_{kj}) - \alpha(\varphi^a_{ik}\varphi^a_j + \varphi^a_i\varphi^a_{jk})\\ =&R^\varphi_{ik,j} - R^\varphi_{ij,k} + \alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j).\end{aligned}$$

To compute the coefficients of $\Delta\mathrm{Ric}^\varphi$ we then use (2.4.9), together with (14), (1.2.26) and (1.2.2) to get:

$$\begin{aligned}R^\varphi_{ij,kk} =&[R^\varphi_{ik,j} + R^t_{ikj,t} + \alpha(\varphi^a_{ij}\varphi^a_k - \varphi^a_{ik}\varphi^a_j)]_k\\ =&R^\varphi_{ik,jk} + R^t_{ikj,tk} + \alpha(\varphi^a_{ijk}\varphi^a_k + \varphi^a_{ij}\varphi^a_{kk} - \varphi^a_{ikk}\varphi^a_j - \varphi^a_{ik}\varphi^a_{jk})\\ =&R^\varphi_{ik,kj} + R^t_{ijk}R^\varphi_{tk} + R^t_{kjk}R^\varphi_{it} + R^t_{ikj,tk} + \alpha(\varphi^a_{ijk}\varphi^a_k + \varphi^a_{ij}\varphi^a_{kk} - \varphi^a_{ikk}\varphi^a_j - \varphi^a_{ik}\varphi^a_{jk})\\ =&\left(\frac{1}{2}S^\varphi_i - \alpha\varphi^a_{kk}\varphi^a_i\right)_j + R^t_{ijk}R^\varphi_{tk} + R_{tj}R^\varphi_{it} + R^t_{ikj,tk} + \alpha(\varphi^a_{ijk}\varphi^a_k + \varphi^a_{ij}\varphi^a_{kk} - \varphi^a_{ikk}\varphi^a_j - \varphi^a_{ik}\varphi^a_{jk})\\ =&\frac{1}{2}S^\varphi_{ij} - \alpha(\varphi^a_{kkj}\varphi^a_i + \varphi^a_{kk}\varphi^a_{ij}) + R^t_{ijk}R^\varphi_{tk} + R^\varphi_{kj}R^\varphi_{ik} + \alpha R^\varphi_{ik}\varphi^a_k\varphi^a_j + R^t_{ikj,tk}\\ &+ \alpha(\varphi^a_{ijk}\varphi^a_k + \varphi^a_{ij}\varphi^a_{kk} - \varphi^a_{ikk}\varphi^a_j - \varphi^a_{ik}\varphi^a_{jk})\\ =&\frac{1}{2}S^\varphi_{ij} + R^t_{ijk}R^\varphi_{tk} + R^\varphi_{kj}R^\varphi_{ik} + \alpha R^\varphi_{ik}\varphi^a_k\varphi^a_j + R^t_{ikj,tk} + \alpha(-\varphi^a_{kkj}\varphi^a_i + \varphi^a_{ijk}\varphi^a_k - \varphi^a_{ikk}\varphi^a_j - \varphi^a_{ik}\varphi^a_{jk}).\end{aligned}$$

Exploiting (2.4.9) and the commutation relation (1.2.42)

$$
\begin{aligned}
R^t_{ikj,tk} =& [R^\varphi_{ij,k} - R^\varphi_{ik,j} + \alpha(\varphi^a_{ik}\varphi^a_j - \varphi^a_{ij}\varphi^a_k)]_k \\
=& \left[C^\varphi_{ijk} + \frac{1}{2(m-1)}(S^\varphi_k\delta_{ij} - S^\varphi_j\delta_{ik})\right]_k + \alpha(\varphi^a_{ikk}\varphi^a_j + \varphi^a_{ik}\varphi^a_{jk} - \varphi^a_{ijk}\varphi^a_k - \varphi^a_{ij}\varphi^a_{kk}) \\
=& C^\varphi_{ijk,k} + \frac{1}{2(m-1)}(\Delta S^\varphi\delta_{ij} - S^\varphi_{ij}) + \alpha(\varphi^a_{ikk}\varphi^a_j + \varphi^a_{ik}\varphi^a_{jk} - \varphi^a_{ijk}\varphi^a_k - \varphi^a_{ij}\varphi^a_{kk}),
\end{aligned}
$$

and inserting into the above we obtain

$$
\begin{aligned}
R^\varphi_{ij,kk} =& \frac{m-2}{2(m-1)}S^\varphi_{ij} + R^t_{ijk}R^\varphi_{tk} + R^\varphi_{kj}R^\varphi_{ik} + C^\varphi_{ijk,k} + \frac{\Delta S^\varphi}{2(m-1)}\delta_{ij} \\
&+ \alpha(R^\varphi_{ik}\varphi^a_k\varphi^a_j - \varphi^a_{kkj}\varphi^a_i - \varphi^a_{ij}\varphi^a_{kk}).
\end{aligned}
\tag{2.4.10}
$$

Indeed,

$$
\begin{aligned}
R^\varphi_{ij,kk} =& \frac{1}{2}S^\varphi_{ij} + R^t_{ijk}R^\varphi_{tk} + R^\varphi_{kj}R^\varphi_{ik} + \alpha R^\varphi_{ik}\varphi^a_k\varphi^a_j + \alpha(-\varphi^a_{kkj}\varphi^a_i + \varphi^a_{ijk}\varphi^a_k - \varphi^a_{ikk}\varphi^a_j - \varphi^a_{ik}\varphi^a_{jk}) \\
&+ C^\varphi_{ijk,k} + \frac{1}{2(m-1)}(\Delta S^\varphi\delta_{ij} - S^\varphi_{ij}) + \alpha(\varphi^a_{ikk}\varphi^a_j + \varphi^a_{ik}\varphi^a_{jk} - \varphi^a_{ijk}\varphi^a_k - \varphi^a_{ij}\varphi^a_{kk}) \\
=& \frac{1}{2}S^\varphi_{ij} + R^t_{ijk}R^\varphi_{tk} + R^\varphi_{kj}R^\varphi_{ik} + \alpha R^\varphi_{ik}\varphi^a_k\varphi^a_j - \alpha\varphi^a_{kkj}\varphi^a_i \\
&+ C^\varphi_{ijk,k} + \frac{1}{2(m-1)}(\Delta S^\varphi\delta_{ij} - S^\varphi_{ij}) - \alpha\varphi^a_{ij}\varphi^a_{kk} \\
=& \frac{m-2}{2(m-1)}S^\varphi_{ij} + R^t_{ijk}R^\varphi_{tk} + R^\varphi_{kj}R^\varphi_{ik} + C^\varphi_{ijk,k} + \frac{\Delta S^\varphi}{2(m-1)}\delta_{ij} + \alpha(R^\varphi_{ik}\varphi^a_k\varphi^a_j - \varphi^a_{kkj}\varphi^a_i - \varphi^a_{ij}\varphi^a_{kk}).
\end{aligned}
$$

Using the decomposition (1.2.18), that in components reads

$$
R_{tijk} = W^\varphi_{tijk} + \frac{1}{m-2}(R^\varphi_{tj}\delta_{ik} - R^\varphi_{tk}\delta_{ij} + R^\varphi_{ik}\delta_{tj} - R^\varphi_{ij}\delta_{tk}) - \frac{S^\varphi}{(m-1)(m-2)}(\delta_{tj}\delta_{ik} - \delta_{tk}\delta_{ij}),
$$

we obtain

$$
\begin{aligned}
R^t_{ijk}R^\varphi_{tk} =& W^\varphi_{tijk}R^\varphi_{tk} + \frac{1}{m-2}(R^\varphi_{tj}\delta_{ik} - R^\varphi_{tk}\delta_{ij} + R^\varphi_{ik}\delta_{tj} - R^\varphi_{ij}\delta_{tk})R^\varphi_{tk} \\
&- \frac{S^\varphi}{(m-1)(m-2)}(\delta_{tj}\delta_{ik} - \delta_{tk}\delta_{ij})R^\varphi_{tk} \\
=& W^\varphi_{tijk}R^\varphi_{tk} + \frac{1}{m-2}(R^\varphi_{tj}R^\varphi_{ti} - |\mathrm{Ric}^\varphi|^2\delta_{ij} + R^\varphi_{ik}R^\varphi_{jk} - R^\varphi_{ij}S^\varphi) \\
&- \frac{S^\varphi}{(m-1)(m-2)}(R^\varphi_{ij} - S^\varphi\delta_{ij}) \\
=& W^\varphi_{tijk}R^\varphi_{tk} + \frac{1}{m-2}(2R^\varphi_{kj}R^\varphi_{ki} - |\mathrm{Ric}^\varphi|^2\delta_{ij} - R^\varphi_{ij}S^\varphi) - \frac{S^\varphi}{(m-1)(m-2)}(R^\varphi_{ij} - S^\varphi\delta_{ij}) \\
=& W^\varphi_{tijk}R^\varphi_{tk} + \frac{2}{m-2}R^\varphi_{ik}R^\varphi_{kj} - \frac{1}{m-2}|\mathrm{Ric}^\varphi|^2\delta_{ij} \\
&+ \frac{(S^\varphi)^2}{(m-1)(m-2)}\delta_{ij} - \frac{m}{(m-1)(m-2)}S^\varphi R^\varphi_{ij}.
\end{aligned}
$$

Inserting the last formula in (2.4.10) we obtain

$$
\begin{aligned}
R^\varphi_{ij,kk} =& \frac{m-2}{2(m-1)}S^\varphi_{ij} + \frac{m}{m-2}R^\varphi_{kj}R^\varphi_{ik} + C^\varphi_{ijk,k} + W^\varphi_{tijk}R^\varphi_{tk} - \frac{m}{(m-1)(m-2)}S^\varphi R^\varphi_{ij} \\
&+ \alpha(R^\varphi_{ik}\varphi^a_k\varphi^a_j - \varphi^a_{kkj}\varphi^a_i - \varphi^a_{ij}\varphi^a_{kk}) \\
&+ \left[\frac{(S^\varphi)^2}{(m-1)(m-2)} + \frac{1}{2(m-1)}\Delta S^\varphi - \frac{1}{m-2}|\mathrm{Ric}^\varphi|^2\right]\delta_{ij}.
\end{aligned}
\tag{2.4.11}
$$

Using the fact that T^φ is traceless, from (2.4.11), we infer

$$\begin{aligned}R^\varphi_{ij,kk}T^\varphi_{ij} =&\frac{m-2}{2(m-1)}T^\varphi_{ij}S^\varphi_{ij}+\frac{m}{m-2}T^\varphi_{ij}R^\varphi_{kj}R^\varphi_{ik}+T^\varphi_{ij}C^\varphi_{ijk,k}+W^\varphi_{tijk}T^\varphi_{ij}R^\varphi_{tk}\\&-\frac{m}{(m-1)(m-2)}S^\varphi T^\varphi_{ij}R^\varphi_{ij}+\alpha T^\varphi_{ij}(R^\varphi_{ik}\varphi^a_k\varphi^a_j-\varphi^a_{kkj}\varphi^a_i-\varphi^a_{ij}\varphi^a_{kk}).\end{aligned}\tag{2.4.12}$$

The following relations can be easily deduced from (2.4.1) (and (1.2.22) for the last one)

$$R^\varphi_{kj}R^\varphi_{ik}T^\varphi_{ij}=T^\varphi_{kj}T^\varphi_{ik}T^\varphi_{ij}+\frac{2S^\varphi}{m}|T^\varphi|^2,$$

$$R^\varphi_{ik}\varphi^a_k\varphi^a_jT^\varphi_{ij}=T^\varphi_{ik}\varphi^a_k\varphi^a_jT^\varphi_{ij}+\frac{S^\varphi}{m}T^\varphi_{ij}\varphi^a_i\varphi^a_j,$$

$$T^\varphi_{ij}R^\varphi_{tk}W^\varphi_{tijk}=T^\varphi_{ij}T^\varphi_{tk}W^\varphi_{tijk}-\alpha\frac{S^\varphi}{m}T^\varphi_{ij}\varphi^a_i\varphi^a_j.$$

Using them all in (2.4.12) we conclude that

$$\begin{aligned}R^\varphi_{ij,kk}T^\varphi_{ij} =&\frac{m-2}{2(m-1)}T^\varphi_{ij}S^\varphi_{ij}+\frac{m}{m-2}T^\varphi_{kj}T^\varphi_{ik}T^\varphi_{ij}+\frac{1}{m-1}S^\varphi|T^\varphi|^2+T^\varphi_{ij}C^\varphi_{ijk,k}\\&+T^\varphi_{ij}T^\varphi_{tk}W^\varphi_{tijk}+\alpha T^\varphi_{ik}\varphi^a_k\varphi^a_jT^\varphi_{ij}-\alpha T^\varphi_{ij}(\varphi^a_{kkj}\varphi^a_i+\varphi^a_{ij}\varphi^a_{kk}).\end{aligned}$$

Inserting the last formula in (2.4.8) we finally obtain

$$\begin{aligned}\frac{1}{2}\Delta|T^\varphi|^2 =&|\nabla T^\varphi|^2+\frac{m-2}{2(m-1)}T^\varphi_{ij}S^\varphi_{ij}+\frac{m}{m-2}T^\varphi_{kj}T^\varphi_{ik}T^\varphi_{ij}+\frac{1}{m-1}S^\varphi|T^\varphi|^2+T^\varphi_{ij}C^\varphi_{ijk,k}\\&+T^\varphi_{ij}T^\varphi_{tk}W^\varphi_{tijk}+\alpha T^\varphi_{ik}\varphi^a_k\varphi^a_jT^\varphi_{ij}-\alpha T^\varphi_{ij}(\varphi^a_{kkj}\varphi^a_i+\varphi^a_{ij}\varphi^a_{kk}),\end{aligned}$$

that is (2.4.7), using (2.4.2) with $\beta=T^\varphi$ and (2.4.5). □

We let $\eta(x)$ denote the largest eigenvalue of $\mathcal{W}^\varphi : S^2_0(M)\to S^2_0(M)$ at $x\in M$ and we set

$$\eta^*:=\sup_M\eta.$$

We are now ready to prove the following

Theorem 2.4.13. *Let $(M,\langle\, ,\,\rangle)$ be a stochastically complete Riemannian manifold of dimension $m\geq 3$ and let $\varphi : M\to(N,\langle\, ,\,\rangle_N)$ be a smooth map, $\alpha\in\mathbb{R}$, $\alpha>0$. Assume*

i) S^φ is constant.

ii) φ is harmonic.

iii) $div(C^\varphi)=0$.

Then, either $(M,\langle\, ,\,\rangle)$ is harmonic-Einstein or

$$\sup_M|T^\varphi|\geq\sqrt{\frac{m-1}{m}}\left(\frac{S^\varphi}{m-1}-\eta^*\right).\tag{2.4.14}$$

Remark 2.4.15. Note that by Proposition 2.1.3, Definition 2.1.1 and Proposition 2.1.7, conditions *i*), *ii*) and *iii*) are necessary for $(M,\langle\, ,\,\rangle)$ to be harmonic-Einstein. Furthermore (2.4.14) is not empty only if $|T^\varphi|$ is bounded and

$$S^\varphi>(m-1)\eta^*.\tag{2.4.16}$$

Proof. First of all note that if $\eta^* = +\infty$ then (2.4.14) holds true. Thus we can suppose $\eta^* < +\infty$. In the assumptions of the Theorem $\operatorname{div}(C^\varphi) = 0$ and, since φ is harmonic, by (2.4.4) and (1.2.36), $\operatorname{tr}(C^\varphi) = 0$. Thus equation (2.4.7) becomes

$$\frac{1}{2}\Delta|T^\varphi|^2 = |\nabla T^\varphi|^2 + \frac{m}{m-2}\operatorname{tr}[(T^\varphi)^3] + \frac{S^\varphi}{m-1}|T^\varphi|^2 - \langle \mathcal{W}^\varphi(T^\varphi), T^\varphi\rangle. \tag{2.4.17}$$

Since T^φ is traceless, Okumura's inequality, [Ok], (see also Lemma 6.2 of [AMR]) gives the validity of

$$\operatorname{tr}[(T^\varphi)^3] \geq -\frac{m-2}{\sqrt{m(m-1)}}|T^\varphi|^3.$$

Furthermore, from the estimates on the largest eigenvalue of $\mathcal{W}^\varphi$

$$\langle \mathcal{W}^\varphi(T^\varphi), T^\varphi\rangle \leq \eta^*|T^\varphi|^2.$$

Inserting these informations in (2.4.17) and setting $u := |T^\varphi|^2$ we deduce the validity of the differential inequality

$$\frac{1}{2}\Delta u \geq \left(\frac{S^\varphi}{m-1} - \eta^* - \frac{m}{\sqrt{m(m-1)}}\sqrt{u}\right)u. \tag{2.4.18}$$

If $u^* := \sup_M u = +\infty$ then (2.4.14) is obviously satisfied. Thus let $u^* < +\infty$. Since stochastically completeness is equivalent to the validity of the weak maximum principle for the Laplace-Beltrami operator, see [PRS03], applying the latter to (2.4.18) we obtain

$$0 \geq \left(\frac{S^\varphi}{m-1} - \eta^* - \frac{m}{\sqrt{m(m-1)}}\sqrt{u^*}\right)u^*.$$

Thus either $u^* = 0$, that is, $T^\varphi = 0$ on M and $(M, \langle\,,\,\rangle)$ is harmonic-Einstein or

$$\frac{S^\varphi}{m-1} - \eta^* - \sqrt{\frac{mu^*}{m-1}} \leq 0.$$

The latter inequality implies (2.4.14). ☐

As a consequence we obtain the following "gap"result for $|T^\varphi|^2$.

Corollary 2.4.19. *Under the assumptions of Theorem 2.4.13 let*

$$\sup_M |T^\varphi| < \sqrt{\frac{m-1}{m}}\left(\frac{S^\varphi}{m-1} - \eta^*\right), \tag{2.4.20}$$

then $(M, \langle\,,\,\rangle)$ is harmonic-Einstein.

Remark 2.4.21. Notice that (2.4.20) implies $\eta^* < +\infty$, otherwise we would have a contradiction, and (2.4.16).

To conclude this Section we provide an estimate, even though is non sharp, for η^*.

Proposition 2.4.22. *Let $(M, \langle\,,\,\rangle)$ be a Riemannian manifold of dimension $m \geq 3$, $\varphi : M \to (N, \langle\,,\,\rangle_N)$ a smooth map and $\alpha > 0$. Assume*

$$e(\varphi)^* := \sup_M e(\varphi) < +\infty, \tag{2.4.23}$$

where $e(\varphi)$ is the density of energy of φ, and, for $m \geq 4$,

$$|W^\varphi|^* := \sup_M |W^\varphi| < +\infty. \tag{2.4.24}$$

Then, if $m = 3$

$$\eta^* \leq \alpha e(\varphi)^* \tag{2.4.25}$$

and if $m \geq 4$

$$\eta^* \leq \sqrt{\frac{m-2}{2(m-1)}}|W^\varphi|^* + \frac{2\alpha}{m-2}e(\varphi)^*. \tag{2.4.26}$$

Proof. We set: for every $\beta \in S^2_0(M)$, $\beta = \beta_{ij}\theta^i \otimes \theta^j$,

$$\mathcal{W}(\beta) := W_{tikj}\beta_{tk}\theta^i \otimes \theta^j.$$

Then $\mathcal{W} : S^2_0(M) \to S^2_0(M)$ is well defined and self-adjoint with respect to the standard extension of $\langle\, ,\, \rangle$ to $S^2_0(M)$ (it can be seen as $\mathcal{W}^\varphi$ for φ constant). Moreover from Huisken's inequality (see Lemma 2.9 in [H] or also Proposition 8.8 in [AMR], whose proof can be extended to the case where $T \in S^2_0(M)$)

$$|\langle \mathcal{W}(\beta), \beta\rangle| \leq \sqrt{\frac{m-2}{2(m-1)}|W|^2|\beta|^2}. \tag{2.4.27}$$

From (2.4.2) and (1.2.19) we get

$$\langle \mathcal{W}^\varphi(\beta), \beta\rangle = \langle \mathcal{W}(\beta), \beta\rangle - \alpha\frac{2}{m-2}|d\varphi(\beta)|^2 + \frac{\alpha}{(m-1)(m-2)}|d\varphi|^2|\beta|^2, \tag{2.4.28}$$

where, in local coordinates,

$$d\varphi(\beta) = \varphi^a_j\beta_{ij}\theta^i \otimes E_a.$$

From (2.4.28) we deduce

$$\langle \mathcal{W}^\varphi(\beta), \beta\rangle \leq \langle \mathcal{W}(\beta), \beta\rangle + \frac{\alpha}{(m-1)(m-2)}|d\varphi|^2|\beta|^2$$

and using (2.4.27) we have

$$|\langle \mathcal{W}^\varphi(\beta), \beta\rangle| \leq \left(\sqrt{\frac{m-2}{2(m-1)}|W|^2} + \frac{\alpha}{(m-1)(m-2)}|d\varphi|^2\right)|\beta|^2. \tag{2.4.29}$$

If $m = 3$ the above reads, since $W = 0$,

$$|\langle \mathcal{W}^\varphi(\beta), \beta\rangle| \leq \frac{\alpha}{2}|d\varphi|^2|\beta|^2,$$

that is, using (2.4.23), (2.4.25). To obtain (2.4.26) for $m \geq 4$ we need the following relation between $|W|^2$ and $|W^\varphi|^2$:

$$|W^\varphi|^2 = |W|^2 + \frac{4\alpha^2}{m-2}|\varphi^*\langle\, ,\, \rangle_N|^2 - \frac{2\alpha^2}{(m-1)(m-2)}|d\varphi|^4. \tag{2.4.30}$$

To prove (2.4.30) we use (1.2.19) and the symmetries of W^φ to get

$$\begin{aligned}|W^\varphi|^2 =& W^\varphi_{tikj}W^\varphi_{tikj}\\ =& W^\varphi_{tikj}\left[W_{tikj} + \frac{\alpha}{m-2}(\varphi^a_t\varphi^a_k\delta_{ij} - \varphi^a_t\varphi^a_j\delta_{ik} + \varphi^a_i\varphi^a_j\delta_{tk} - \varphi^a_i\varphi^a_k\delta_{tj}) - \frac{\alpha}{(m-1)(m-2)}|d\varphi|^2(\delta_{tk}\delta_{ij} - \delta_{tj}\delta_{ik})\right]\\ =& W^\varphi_{tikj}W_{tikj} + \frac{4\alpha}{m-2}W^\varphi_{tiki}\varphi^a_t\varphi^a_k - \frac{2\alpha}{(m-1)(m-2)}|d\varphi|^2W^\varphi_{kiki},\end{aligned}$$

and we conclude using (1.2.19), the fact that W is totally trace free and (1.2.22). From (2.4.30) we obtain

$$|W|^2 \leq |W^\varphi|^2 + \frac{2\alpha^2}{(m-1)(m-2)}|d\varphi|^4,$$

so that

$$\sqrt{\frac{m-2}{2(m-1)}|W|^2} \leq \sqrt{\frac{m-2}{2(m-1)}|W^\varphi|^2 + \left(\frac{\alpha}{m-1}|d\varphi|^2\right)^2} \leq \sqrt{\frac{m-2}{2(m-1)}}|W^\varphi| + \frac{\alpha}{m-1}|d\varphi|^2. \tag{2.4.31}$$

Plugging the above into (2.4.29) we get

$$|\langle \mathcal{W}^\varphi(\beta), \beta\rangle| \leq \left(\sqrt{\frac{m-2}{2(m-1)}}|W^\varphi| + \frac{\alpha}{m-2}|d\varphi|^2\right)|\beta|^2,$$

and then (2.4.26) holds, using (2.4.24) and (2.4.23). □

Remark 2.4.32. The estimate above is non sharp, indeed assume $\mathring{W}^\varphi = 0$, that is, via Proposition 1.4.6, φ is weakly conformal and $(M, \langle\, , \,\rangle)$ is locally conformally flat. Then it is easy to see that

$$\eta = -\alpha\frac{|d\varphi|^2}{m-1}.$$

Indeed, using (2.4.2),

$$W^\varphi_{tikj} = \frac{\alpha|d\varphi|^2}{m(m-1)}(\delta_{tk}\delta_{ij} - \delta_{tj}\delta_{ik})$$

and that φ is weakly conformal, for every $\beta \in S^2_0(M)$, $\beta = \beta_{ij}\theta^i \otimes \theta^j$

$$\langle \mathcal{W}^\varphi(\beta), \beta\rangle = \frac{\alpha|d\varphi|^2}{m(m-1)}(\delta_{tk}\delta_{ij} - \delta_{tj}\delta_{ik})\beta_{tk}\beta_{ij} - \alpha\frac{|d\varphi|^2}{m}\delta_{ij}\beta_{ik}\beta_{kj} = -\frac{\alpha|d\varphi|^2}{m(m-1)}|\beta|^2 - \alpha\frac{|d\varphi|^2}{m}|\beta|^2,$$

that is,

$$\langle \mathcal{W}^\varphi(\beta), \beta\rangle = -\alpha\frac{|d\varphi|^2}{m-1}|\beta|^2.$$

Notice that in this case, since

$$|W^\varphi|^2 = \frac{2\alpha^2|d\varphi|^4}{m(m-1)},$$

we can equivalently say that

$$\eta = -\sqrt{\frac{m}{2(m-1)}}|W^\varphi|.$$

2.5 Harmonic-Einstein warped products

Let $(M, \langle\, , \,\rangle)$ and $(F, \langle\, , \,\rangle_F)$ be two Riemannian manifolds of dimension m and d respectively. Let $u \in \mathcal{C}^\infty(M)$, $u > 0$ on M.

Definition 2.5.1. we denote by $\bar{M} = M \times F$ the product manifold, by

$$\overline{\langle\, , \,\rangle} := \pi_M^*\langle\, , \,\rangle + (u \circ \pi_M)^2\pi_F^*\langle\, , \,\rangle_F,$$

where $\pi_M : \bar{M} \to M$ and $\pi_F : \bar{M} \to F$ are the canonical projections, and by

$$M \times_u F := (\bar{M}, \overline{\langle\, , \,\rangle})$$

the *warped product* with base $(M, \langle\, , \,\rangle)$, fibre $(F, \langle\, , \,\rangle_F)$ and warping function u.

We are going to identify $T(M \times F)$ with $TM \otimes TF$, so that

$$\overline{\langle\, , \,\rangle} \equiv \langle\, , \,\rangle + u^2\langle\, , \,\rangle_F.$$

We use the following indexes conventions

$$1 \leq i, j, \ldots \leq m, \quad 1 \leq \alpha, \beta, \ldots \leq d, \quad 1 \leq A, B, \ldots \leq m + d.$$

Let $\{e_i\}$, $\{\theta^i\}$, $\{\theta^i_j\}$, $\{\Theta^i_j\}$ be, respectively, a local orthonormal frame, the dual coframe, the relative connection and curvature forms on an open subset $\mathcal{U}$ of M and let $\{\varepsilon_\alpha\}$, $\{\psi^\alpha\}$, $\{\psi^\alpha_\beta\}$, $\{\Psi^\alpha_\beta\}$ be the same quantities on an open subset $\mathcal{W}$ of F.

In the next well known Proposition we determine the local orthonormal frame, the dual coframe, the relative connection and curvature forms on $\overline{\mathcal{U}} := \mathcal{U} \times \mathcal{W}$ induced by the choices above, that we denote by $\{\overline{e}_A\}$, $\{\overline{\theta}^A\}$, $\{\overline{\theta}^A_B\}$, $\{\overline{\Theta}^A_B\}$, respectively.

Proposition 2.5.2. *In the notations above*

$$\overline{e}_i = \pi^*_M(e_i) \equiv e_i, \quad \overline{e}_{m+\alpha} = \frac{1}{u \circ \pi_M}\pi^*_F(\varepsilon_\alpha) \equiv \frac{1}{u}\varepsilon_\alpha, \tag{2.5.3}$$

$$\overline{\theta}^i = \pi^*_M(\theta^i) \equiv \theta^i, \quad \overline{\theta}^{m+\alpha} = u \circ \pi_M \cdot \pi^*_F \psi^\alpha \equiv u\psi^\alpha, \tag{2.5.4}$$

$$\overline{\theta}^i_j = \theta^i_j, \quad \overline{\theta}^{m+\alpha}_{m+\beta} = \psi^\alpha_\beta, \quad \overline{\theta}^{m+\alpha}_i = u_i\psi^\alpha = -\overline{\theta}^i_{m+\alpha}, \tag{2.5.5}$$

$$\overline{\Theta}^i_j = \Theta^i_j, \quad \overline{\Theta}^{m+\alpha}_{m+\beta} = \Psi^\alpha_\beta - |\nabla u|^2\psi^\alpha \wedge \psi^\beta, \quad \overline{\Theta}^{m+\alpha}_i = u_{ij}\theta^j \wedge \psi^\alpha = -\overline{\Theta}^i_{m+\alpha}. \tag{2.5.6}$$

The non-vanishing components of $\overline{Riem}$ are determined by

$$\bar{R}^i_{jkt} = R^i_{jkt}, \quad \bar{R}_{i\,m+\alpha\,j\,m+\beta} = -\frac{u_{ij}}{u}\delta_{\alpha\beta}, \quad \bar{R}^{m+\alpha}_{m+\beta\,m+\gamma\,m+\delta} = \frac{1}{u^2}{}^F R^\alpha_{\beta\gamma\delta} - \frac{|\nabla u|^2}{u^2}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}), \tag{2.5.7}$$

where R^i_{jkt} and ${}^F R^\alpha_{\beta\gamma\delta}$ are the components of the Riemann tensors of $(M, \langle\, ,\, \rangle)$ and $(F, \langle\, ,\, \rangle_F)$, respectively.

Proof. It is clear that $\{\overline{e}_A\}$ defined as in (2.5.3) is a local orthonormal frame, indeed

$$\overline{\langle \overline{e}_i, \overline{e}_j \rangle} = \langle e_i, e_j \rangle = \delta_{ij}, \quad \overline{\langle \overline{e}_i, \overline{e}_{m+\alpha} \rangle} = 0$$

and

$$\overline{\langle \overline{e}_{m+\alpha}, \overline{e}_{m+\beta} \rangle} = u^2 \left\langle \frac{\varepsilon_\alpha}{u}, \frac{\varepsilon_\beta}{u} \right\rangle_F = \langle \varepsilon_\alpha, \varepsilon_\beta \rangle_F = \delta_{\alpha\beta}.$$

The relations (2.5.4) follows immediately from (2.5.3).

To show the validity of (2.5.5) recall that the first structure equation on $M \times_u F$ are given by

$$d\overline{\theta}^A = -\overline{\theta}^A_B \wedge \overline{\theta}^B.$$

For $A = i$ we obtain, using (2.5.4),

$$\begin{aligned} d\overline{\theta}^i &= -\,\overline{\theta}^i_B \wedge \overline{\theta}^B \\ &= -\,\overline{\theta}^i_j \wedge \overline{\theta}^j - \overline{\theta}^i_{m+\alpha} \wedge \overline{\theta}^{m+\alpha} \\ &= -\,\overline{\theta}^i_j \wedge \theta^j - u\overline{\theta}^i_{m+\alpha} \wedge \psi^\alpha \end{aligned}$$

and since, from the first structure equation on M,

$$d\overline{\theta}^i = d\theta^i = -\theta^i_j \wedge \theta^j,$$

we conclude from the above

$$(\overline{\theta}^i_j - \theta^i_j) \wedge \theta^j + u\overline{\theta}^i_{m+\alpha} \wedge \psi^\alpha = 0. \tag{2.5.8}$$

For $A = m + \alpha$ we obtain, using (2.5.4),

$$\begin{aligned} d\overline{\theta}^{m+\alpha} &= -\,\overline{\theta}^{m+\alpha}_B \wedge \overline{\theta}^B \\ &= -\,\overline{\theta}^{m+\alpha}_i \wedge \overline{\theta}^i - \overline{\theta}^{m+\alpha}_{m+\beta} \wedge \overline{\theta}^{m+\beta} \\ &= -\,\overline{\theta}^{m+\alpha}_i \wedge \theta^i - u\overline{\theta}^{m+\alpha}_{m+\beta} \wedge \psi^\beta \end{aligned}$$

and since, from the first structure equation for F,

$$\begin{aligned}d\overline{\theta}^{m+\alpha} =& d(u\psi^\alpha)\\ =& du \wedge \psi^\alpha + ud\psi^\alpha\\ =& u_i\theta^i \wedge \psi^\alpha - u\psi^\alpha_\beta \wedge \psi^\beta\\ =& -u_i\psi^\alpha \wedge \theta^i - u\psi^\alpha_\beta \wedge \psi^\beta\end{aligned}$$

we conclude from the above

$$u(\overline{\theta}^{m+\alpha}_{m+\beta} - \psi^\alpha_\beta) \wedge \psi^\beta + (\overline{\theta}^{m+\alpha}_i - u_i\psi^\alpha) \wedge \theta^i = 0. \tag{2.5.9}$$

It is immediate to verify that $\{\overline{\theta}^A_B\}$ given by (2.5.5) are skew-symmetric, that is, $\overline{\theta}^A_B = -\overline{\theta}^B_A$, and satisfies (2.5.8) and (2.5.9), hence they are the connection forms associated to the coframe $\{\overline{\theta}^A\}$.

The second structure equation reads as

$$d\overline{\theta}^A_B = -\overline{\theta}^A_C \wedge \overline{\theta}^C_B + \overline{\Theta}^A_B.$$

For $A = i$ and $B = j$, using (2.5.5),

$$\begin{aligned}d\overline{\theta}^i_j =& -\overline{\theta}^i_C \wedge \overline{\theta}^C_j + \overline{\Theta}^i_j\\ =& -\overline{\theta}^i_k \wedge \overline{\theta}^k_j - \overline{\theta}^i_{m+\alpha} \wedge \overline{\theta}^{m+\alpha}_j + \overline{\Theta}^i_j\\ =& -\theta^i_k \wedge \theta^k_j + u_iu_j\psi^\alpha \wedge \psi^\alpha + \overline{\Theta}^i_j\\ =& -\theta^i_k \wedge \theta^k_j + \overline{\Theta}^i_j.\end{aligned}$$

From the second structure equation on M we have

$$d\overline{\theta}^i_j = d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \Theta^i_j,$$

hence from the above we infer

$$\overline{\Theta}^i_j = \Theta^i_j. \tag{2.5.10}$$

For $A = m+\alpha$ and $B = m+\beta$, using (2.5.5),

$$\begin{aligned}d\overline{\theta}^{m+\alpha}_{m+\beta} =& -\overline{\theta}^{m+\alpha}_C \wedge \overline{\theta}^C_{m+\beta} + \overline{\Theta}^{m+\alpha}_{m+\beta}\\ =& -\overline{\theta}^{m+\alpha}_i \wedge \overline{\theta}^i_{m+\beta} - \overline{\theta}^{m+\alpha}_{m+\gamma} \wedge \overline{\theta}^{m+\gamma}_{m+\beta} + \overline{\Theta}^{m+\alpha}_{m+\beta}\\ =& u_iu_i\psi^\alpha \wedge \psi^\beta - \psi^\alpha_\gamma \wedge \psi^\gamma_\beta + \overline{\Theta}^{m+\alpha}_{m+\beta}\\ =& |\nabla u|^2\psi^\alpha \wedge \psi^\beta - \psi^\alpha_\gamma \wedge \psi^\gamma_\beta + \overline{\Theta}^{m+\alpha}_{m+\beta}.\end{aligned}$$

From the second structure equation on F we have

$$d\overline{\theta}^{m+\alpha}_{m+\beta} = d\psi^\alpha_\beta = -\psi^\alpha_\gamma \wedge \psi^\gamma_\beta + \Psi^\alpha_\beta,$$

hence from the above we obtain

$$\overline{\Theta}^{m+\alpha}_{m+\beta} = \Psi^\alpha_\beta - |\nabla u|^2\psi^\alpha \wedge \psi^\beta. \tag{2.5.11}$$

For $A = m+\alpha$ and $B = i$, using (2.5.5),

$$\begin{aligned}d\overline{\theta}^{m+\alpha}_i =& -\overline{\theta}^{m+\alpha}_C \wedge \overline{\theta}^C_i + \overline{\Theta}^{m+\alpha}_i\\ =& -\overline{\theta}^{m+\alpha}_j \wedge \overline{\theta}^j_i - \overline{\theta}^{m+\alpha}_{m+\beta} \wedge \overline{\theta}^{m+\beta}_i + \overline{\Theta}^{m+\alpha}_i\\ =& -u_j\psi^\alpha \wedge \theta^j_i - u_i\psi^\alpha_\beta \wedge \psi^\beta + \overline{\Theta}^{m+\alpha}_i\\ =& u_j\theta^j_i \wedge \psi^\alpha - u_i\psi^\alpha_\beta \wedge \psi^\beta + \overline{\Theta}^{m+\alpha}_i.\end{aligned}$$

From the definition of the Hessian of u we have

$$u_{ij}\theta^j = du_i - u_j\theta_i^j,$$

thus we infer, using (2.5.4),

$$\begin{aligned}d\overline{\theta}_i^{m+\alpha} =& d(u_i\psi^\alpha)\\ =& du_i\wedge\psi^\alpha + u_i d\psi^\alpha\\ =& u_{ij}\theta^j\wedge\psi^\alpha + u_j\theta_i^j\wedge\psi^\alpha - u_i\psi_\beta^\alpha\wedge\psi^\beta.\end{aligned}$$

Comparing it with the above we obtain

$$\overline{\Theta}_i^{m+\alpha} = u_{ij}\theta^j\wedge\psi^\alpha. \tag{2.5.12}$$

Combining (2.5.10), (2.5.11) and (2.5.12) we deduce the validity of (2.5.6).

By definition of $\overline{\mathrm{Riem}}$, the Riemann tensor of $M\times_u F$, we have

$$\overline{\Theta}_B^A = \frac{1}{2}\bar{R}_{BCD}^A\overline{\theta}^C\wedge\overline{\theta}^D.$$

For $A = m+\alpha$ and $B = i$, with the aid of (2.5.4),

$$\begin{aligned}\overline{\Theta}_i^{m+\alpha} =& \frac{1}{2}\bar{R}_{iCD}^{m+\alpha}\overline{\theta}^C\wedge\overline{\theta}^D\\ =& \frac{1}{2}\bar{R}_{ijk}^{m+\alpha}\overline{\theta}^j\wedge\overline{\theta}^k + \frac{1}{2}\bar{R}_{i\,m+\beta\,m+\gamma}^{m+\alpha}\overline{\theta}^{m+\beta}\wedge\overline{\theta}^{m+\gamma} + \frac{1}{2}\bar{R}_{ij\,m+\beta}^{m+\alpha}\overline{\theta}^j\wedge\overline{\theta}^{m+\beta} + \frac{1}{2}\bar{R}_{i\,m+\beta\,j}^{m+\alpha}\overline{\theta}^{m+\beta}\wedge\overline{\theta}^j\\ =& \frac{1}{2}\bar{R}_{ijk}^{m+\alpha}\theta^j\wedge\theta^k + \frac{u^2}{2}\bar{R}_{i\,m+\beta\,m+\gamma}^{m+\alpha}\psi^\beta\wedge\psi^\gamma - u\bar{R}_{i\,m+\beta\,j}^{m+\alpha}\theta^j\wedge\psi^\beta,\end{aligned}$$

and using (2.5.12) we obtain

$$\bar{R}_{m+\alpha\,ijk} = 0, \quad \bar{R}_{i\,m+\alpha\,m+\beta\,m+\gamma} = 0, \quad \bar{R}_{i\,m+\alpha\,j\,m+\beta} = -\frac{u_{ij}}{u}\delta_{\alpha\beta}. \tag{2.5.13}$$

For $A = i$ and $B = j$, using (2.5.4), (2.5.13) and the symmetries of $\overline{\mathrm{Riem}}$,

$$\begin{aligned}\overline{\Theta}_j^i =& \frac{1}{2}\bar{R}_{jCD}^i\overline{\theta}^C\wedge\overline{\theta}^D\\ =& \frac{1}{2}\bar{R}_{jkt}^i\theta^k\wedge\theta^t + \frac{u^2}{2}\bar{R}_{j\,m+\alpha\,m+\beta}^i\psi^\alpha\wedge\psi^\beta + u\bar{R}_{jk\,m+\alpha}^i\theta^k\wedge\psi^\alpha\\ =& \frac{1}{2}\bar{R}_{jkt}^i\theta^k\wedge\theta^t + \frac{u^2}{2}\bar{R}_{j\,m+\alpha\,m+\beta}^i\psi^\alpha\wedge\psi^\beta,\end{aligned}$$

but since, from (2.5.10) and the definition of Riem

$$\overline{\Theta}_j^i = \Theta_j^i = \frac{1}{2}R_{jkt}^i\theta^k\wedge\theta^t$$

we obtain

$$\bar{R}_{jkt}^i = R_{jkt}^i, \quad \bar{R}_{j\,m+\alpha\,m+\beta}^i = 0. \tag{2.5.14}$$

For $A = m+\alpha$ and $B = m+\beta$, using (2.5.4), (2.5.13) and (2.5.14),

$$\overline{\Theta}_{m+\beta}^{m+\alpha} = \frac{1}{2}\bar{R}_{m+\beta\,CD}^{m+\alpha}\overline{\theta}^C\wedge\overline{\theta}^D = \frac{u^2}{2}\bar{R}_{m+\beta\,m+\gamma\,m+\delta}^{m+\alpha}\psi^\gamma\wedge\psi^\delta,$$

inserting (2.5.11) we deduce

$$\frac{1}{2}({}^F R^{\alpha}_{\beta\gamma\delta} - u^2 \bar{R}^{m+\alpha}_{m+\beta\, m+\gamma\, m+\delta})\psi^{\gamma} \wedge \psi^{\delta} = |\nabla u|^2 \psi^{\alpha} \wedge \psi^{\beta}.$$

Skew-symmetrizing the above we obtain

$${}^F R^{\alpha}_{\beta\gamma\delta} - u^2 \bar{R}^{m+\alpha}_{m+\beta\, m+\gamma\, m+\delta} = |\nabla u|^2(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}),$$

that is,

$$\bar{R}^{m+\alpha}_{m+\beta\, m+\gamma\, m+\delta} = \frac{1}{u^2}{}^F R^{\alpha}_{\beta\gamma\delta} - \frac{|\nabla u|^2}{u^2}(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}).$$

Hence (2.5.7) holds. □

Since $u > 0$ on M there exists $f \in \mathcal{C}^{\infty}(M)$ such that

$$u = e^{-\frac{f}{d}}. \tag{2.5.15}$$

As a consequence of the above Proposition we have

Corollary 2.5.16. *In the notations above, the non-vanishing components of $\overline{Ric}$, the Ricci tensor of $(\bar{M}, \overline{\langle\, ,\, \rangle})$, are given by*

$$\bar{R}_{ij} = R_{ij} - d\frac{u_{ij}}{u}, \quad \bar{R}_{m+\alpha\, m+\beta} = -\left(\frac{\Delta u}{u} + (d-1)\frac{|\nabla u|^2}{u^2}\right)\delta_{\alpha\beta} + \frac{1}{u^2}{}^F R_{\alpha\beta}, \tag{2.5.17}$$

where R_{ij} and ${}^F R_{\alpha\beta}$ are the components of the Ricci tensors of $(M, \langle\, ,\, \rangle)$ and $(F, \langle\, ,\, \rangle_F)$, respectively. Equivalently, in terms of f, where f is defined by (2.5.15)*,*

$$\bar{R}_{ij} = R_{ij} + f_{ij} - \frac{1}{d}f_i f_j, \quad \bar{R}_{m+\alpha\, m+\beta} = \frac{\Delta_f f}{d}\delta_{\alpha\beta} + e^{\frac{2f}{d}}\, {}^F R_{\alpha\beta}. \tag{2.5.18}$$

Proof. Tracing the relations (2.5.7) we obtain

$$\bar{R}_{ij} = \bar{R}_{iAjA} = \bar{R}_{ikjk} + \bar{R}_{i\, m+\alpha\, j\, m+\alpha} = R_{ij} - \frac{u_{ij}}{u}\delta_{\alpha\alpha} = R_{ij} - d\frac{u_{ij}}{u},$$

$$\begin{aligned}
\bar{R}_{m+\alpha\, m+\beta} =& \bar{R}_{m+\alpha\, A\, m+\beta\, A} \\
=& \bar{R}_{m+\alpha\, i\, m+\beta\, i} + R_{m+\alpha\, m+\gamma\, m+\beta\, m+\gamma} \\
=& -\frac{u_{ii}}{u}\delta_{\alpha\beta} + \frac{1}{u^2}{}^F R_{\alpha\beta} - (d-1)\frac{|\nabla u|^2}{u^2}\delta_{\alpha\beta} \\
=& -\left(\frac{\Delta u}{u} + (d-1)\frac{|\nabla u|^2}{u^2}\right)\delta_{\alpha\beta} + \frac{1}{u^2}{}^F R_{\alpha\beta}
\end{aligned}$$

and

$$\bar{R}_{i\, m+\alpha} = \bar{R}_{iA\, m+\alpha\, A} = \bar{R}_{ij\, m+\alpha\, j} + \bar{R}_{i\, m+\beta\, m+\alpha\, m+\beta} = 0.$$

Hence (2.5.17) holds. From the definition (2.5.15) of f

$$u_i = -\frac{1}{d}e^{-\frac{f}{d}}f_i = -\frac{u}{d}f_i, \tag{2.5.19}$$

in particular

$$|\nabla u|^2 = \frac{u^2}{d^2}|\nabla f|^2,$$

and also

$$u_{ij} = -\frac{u}{d}\left(f_{ij} - \frac{1}{d}f_i f_j\right),$$

in particular

$$\Delta u = -\frac{u}{d}\left(\Delta f - \frac{1}{d}|\nabla f|^2\right).$$

By plugging the above relations into (2.5.17) we conclude the validity of (2.5.18). □

Let $\varphi : (M, \langle\, ,\, \rangle) \to (N, \langle\, ,\, \rangle_N)$ be a smooth map and denote

$$\Phi := \varphi \circ \pi_M : \bar{M} \to (N, \langle\, ,\, \rangle_N). \tag{2.5.20}$$

We use the indexes convention

$$1 \le a, b, \ldots \le n,$$

where n is the dimension of N. Let $\{E_a\}$, $\{\omega^a\}$, $\{\omega^a_b\}$, $\{\Omega^a_b\}$ be, respectively, an orthonormal frame, orthonormal coframe, connection forms and curvatures form on a open subset $\mathcal{V}$ of N such that $\varphi^{-1}(\mathcal{V}) \subseteq \overline{\mathcal{U}}$.

Proposition 2.5.21. *In the assumptions and the notations above*

$$\bar{\tau}(\Phi) = \tau(\varphi) - d\varphi(\nabla f), \tag{2.5.22}$$

where $\bar{\tau}(\Phi)$ is the tension of $\Phi : M \times_u F \to (N, \langle\, ,\, \rangle_N)$.

Proof. Let $\bar{d}$ be the differential on $\bar{M}$. Then, using (2.5.4),

$$\bar{d}\Phi = \Phi^a_A \overline{\theta}^A \otimes E_a = \Phi^a_i \theta^i \otimes E_a + u\Phi^a_{m+\alpha}\psi^\alpha \otimes E_a.$$

Due to the definition (2.5.20) of Φ we have

$$\bar{d}\Phi = \pi^*_M d\varphi \equiv d\varphi = \varphi^a_i \theta^i \otimes E_a,$$

hence, by comparison with the above,

$$\Phi^a_i = \varphi^a_i, \quad \Phi^a_{m+\alpha} = 0. \tag{2.5.23}$$

In particular

$$\overline{|\bar{d}\Phi|}^2 = \Phi^a_A \Phi^a_A = \varphi^a_i \varphi^a_i = |d\varphi|^2.$$

Moreover

$$\Phi^a_{AB}\overline{\theta}^B = \bar{d}\Phi^a_A - \Phi^a_B \overline{\theta}^B_A + \Phi^b_A \omega^a_b,$$

that is, using (2.5.5),

$$\Phi^a_{Aj}\theta^j + u\Phi^a_{A\,m+\alpha}\psi^\alpha = \bar{d}\Phi^a_A - \Phi^a_j \overline{\theta}^j_A - \Phi^a_{m+\alpha}\overline{\theta}^{m+\alpha}_A + \Phi^b_A \omega^a_b,$$

For $A = i$ we obtain, using (2.5.23) and (2.5.5),

$$\Phi^a_{ij}\theta^j + u\Phi^a_{i\,m+\alpha}\psi^\alpha = d\varphi^a_i - \varphi^a_j \theta^j_i + \varphi^b_i \omega^a_b = \varphi^a_{ij}\theta^j,$$

hence

$$\Phi^a_{ij} = \varphi^a_{ij}, \quad \Phi^a_{i\,m+\alpha} = 0. \tag{2.5.24}$$

For $A = m + \beta$ we obtain using (2.5.23) and (2.5.5),

$$\Phi^a_{m+\beta\,j}\theta^j + u\Phi^a_{m+\beta\,m+\alpha}\psi^\alpha = u_j \varphi^a_j \psi^\beta,$$

hence, for $f \in \mathcal{C}^\infty(M)$ given by (2.5.15), using (2.5.19),

$$\Phi^a_{m+\alpha\, j} = 0, \quad \Phi^a_{m+\alpha\, m+\beta} = \varphi^a_j \frac{u_j}{u} \delta_{\alpha\beta} = -\frac{1}{d}\varphi^a_j f_j \delta_{\alpha\beta}. \tag{2.5.25}$$

Then, using (2.5.24) and (2.5.25), we infer

$$\bar{\tau}(\Phi)^a = \Phi^a_{AA} = \Phi^a_{ii} + \Phi^a_{m+\alpha\, m+\alpha} = \tau(\varphi)^a - \varphi^a_j f_j,$$

that is, (2.5.22). □

We are now able to prove the following theorem, the main result of this Section. This result is another important motivation for the introduction of the general structure of Chapter 3, together with Theorem 2.3.5.

Theorem 2.5.26. *Let $(M, \langle\, ,\, \rangle)$ and $(F, \langle\, ,\, \rangle_F)$ be Riemannian manifolds of dimension m and d respectively. Let $f \in \mathcal{C}^\infty(M)$ and $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ smooth. Set u as in (2.5.15) and $\Phi : \bar{M} \to (N, \langle\, ,\, \rangle_N)$ as in (2.5.20). Then $M \times_u F$ is harmonic-Einstein, that is, satisfies, for some $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R} \setminus \{0\}$,*

$$\begin{cases} \overline{Ric} - \alpha \Phi^* \langle\, ,\, \rangle_N = \lambda \overline{\langle\, ,\, \rangle} \\ \tau(\Phi) = 0 \end{cases} \tag{2.5.27}$$

if and only if $(M, \langle\, ,\, \rangle)$ satisfies

$$\begin{cases} Ric - \alpha \varphi^* \langle\, ,\, \rangle_N + Hess(f) - \frac{1}{d} df \otimes df = \lambda \langle\, ,\, \rangle \\ \tau(\varphi) = d\varphi(\nabla f) \end{cases} \tag{2.5.28}$$

and $(F, \langle\, ,\, \rangle_F)$ satisfies

$$^F Ric = \Lambda \langle\, ,\, \rangle_F, \tag{2.5.29}$$

where Λ is constant and is given by

$$\Lambda = \frac{1}{d}(d\lambda - \Delta_f f) e^{-\frac{2f}{d}}. \tag{2.5.30}$$

Proof. Assume that $M \times_u F$ satisfies (2.5.27). From Corollary 2.5.16 we infer

$$R_{ij} + f_{ij} - \frac{1}{d} f_i f_j = \alpha \varphi^a_i \varphi^a_j + \lambda \delta_{ij}, \tag{2.5.31}$$

and

$$^F R_{\alpha\beta} = \frac{1}{d}\left(d\lambda - \Delta_f f\right) e^{-\frac{2f}{d}} \delta_{\alpha\beta}. \tag{2.5.32}$$

From Proposition 2.5.21 we deduce

$$\tau(\varphi) = d\varphi(\nabla f). \tag{2.5.33}$$

Combining (2.5.31) and (2.5.33) we obtain (2.5.28). As we shall see in Proposition 7.1.5, with the choice of $\mu = \frac{1}{d}$, the validity of (2.5.28) implies the existence of some constant Λ such that

$$\Delta_f f - d\lambda = -d\Lambda e^{\frac{2f}{d}},$$

that is, (2.5.30). Plugging (2.5.30) into (2.5.32) we deduce the validity of (2.5.29).

Conversely, suppose that $(M, \langle\, ,\, \rangle)$ and $(F, \langle\, ,\, \rangle_F)$ satisfy, respectively, (2.5.28) and (2.5.29), where Λ is defined by (2.5.30). Since (2.5.28) holds then, using once again Proposition 7.1.5, we deduce that Λ is constant. From Corollary 2.5.16, using the first equation of (2.5.28), we infer

$$\bar{R}_{ij} = R_{ij} + f_{ij} - \frac{1}{d} f_i f_j = \alpha \varphi^a_i \varphi^a_j + \lambda \delta_{ij},$$

and from (2.5.29) and (2.5.30),

$$\bar{R}_{m+\alpha\, m+\beta} = \frac{\Delta_f f}{d}\delta_{\alpha\beta} + e^{\frac{2d}{f}}\,{}^F R_{\alpha\beta} = \lambda\delta_{\alpha\beta}.$$

Combining it with the above and recalling the validity of (2.5.23), since from the second equation of (2.5.28) and Proposition 2.5.21 the map $\Phi : \bar{M} \to (N, \langle\,,\,\rangle_N)$ is harmonic, we conclude that (2.5.27) holds. □

Now we deal the "dual "case. Let $\gamma : (F, \langle\,,\,\rangle_F) \to (N, \langle\,,\,\rangle_N)$ be a smooth map and denote

$$\Gamma := \gamma \circ \pi_F : \bar{M} \to (N, \langle\,,\,\rangle_N). \tag{2.5.34}$$

Proposition 2.5.35. *In the assumptions and the notations above*

$$\bar{\tau}(\Gamma) = \frac{1}{u^2}\,{}^F\tau(\gamma). \tag{2.5.36}$$

Proof. Using (2.5.4),

$$\bar{d}\Gamma = \Gamma^a_A \bar{\theta}^A \otimes E_a = \Gamma^a_i \theta^i \otimes E_a + u\Gamma^a_{m+\alpha}\psi^\alpha \otimes E_a.$$

Due to the definition (2.5.34) of Γ we have

$$\bar{d}\Gamma = \pi_F^* d\gamma \equiv d\gamma = \gamma^a_\alpha \psi^\alpha \otimes E_a,$$

hence

$$\Gamma^a_i = 0, \quad \Gamma^a_{m+\alpha} = \frac{\gamma^a_\alpha}{u}. \tag{2.5.37}$$

In particular

$$\overline{|\bar{d}\Gamma|}^2 = \Gamma^a_A \Gamma^a_A = \frac{1}{u^2}\gamma^a_i\gamma^a_i = \frac{1}{u^2}|d\gamma|^2_F.$$

Moreover

$$\Gamma^a_{AB}\bar{\theta}^B = \bar{d}\Gamma^a_A - \Gamma^a_B \bar{\theta}^B_A + \Gamma^b_A \omega^a_b,$$

that is, using (2.5.5),

$$\Gamma^a_{Aj}\theta^j + u\Gamma^a_{A\,m+\alpha}\psi^\alpha = \bar{d}\Gamma^a_A - \Gamma^a_j \bar{\theta}^j_A - \Gamma^a_{m+\alpha}\bar{\theta}^{m+\alpha}_A + \Gamma^b_A\omega^a_b,$$

For $A = i$ we obtain, using (2.5.37) and (2.5.5),

$$\Gamma^a_{ij}\theta^j + u\Gamma^a_{i\,m+\alpha}\psi^\alpha = -\frac{u_i}{u}\gamma^a_\alpha\psi^\alpha,$$

hence

$$\Gamma^a_{ij} = 0, \quad \Gamma^a_{i\,m+\alpha} = -\frac{u_i}{u^2}\gamma^a_\alpha. \tag{2.5.38}$$

For $A = m + \beta$ we obtain using (2.5.23) and (2.5.5),

$$\begin{aligned}\Gamma^a_{m+\beta\, j}\theta^j + u\Gamma^a_{m+\beta\, m+\alpha}\psi^\alpha =& d\left(\frac{\gamma^a_\beta}{u}\right) - \frac{\gamma^a_\alpha}{u}\psi^\alpha_\beta + \frac{\gamma^b_\beta}{u}\omega^a_b \\ =& \frac{1}{u}\left(d\gamma^a_\beta - \gamma^a_\alpha\psi^\alpha_\beta + \gamma^b_\beta\omega^a_b\right) - \frac{1}{u^2}\gamma^a_\beta u_j\theta^j,\end{aligned}$$

hence, from the definition of covariant derivative of $d\gamma$,

$$\Gamma^a_{m+\beta\, j} = -\frac{1}{u^2}\gamma^a_\beta u_j, \quad \Gamma^a_{m+\beta\, m+\alpha} = \frac{1}{u^2}\gamma^a_{\beta\alpha}. \tag{2.5.39}$$

Then, using (2.5.38) and (2.5.39), we infer

$$\bar{\tau}(\Gamma)^a = \Gamma^a_{AA} = \Gamma^a_{ii} + \Gamma^a_{m+\alpha\, m+\alpha} = \frac{1}{u^2}\gamma^a_{\alpha\alpha} = \frac{1}{u^2}\,{}^F\tau(\gamma)^a,$$

that is, (2.5.36). □

Assume $M = I \ni 0$ is an open interval on $\mathbb{R}$ with the Euclidean metric, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\gamma : (F, \langle\, , \rangle_F) \to (N, \langle\, , \rangle_N)$ is a smooth map. We set Γ as in (2.5.34). From (2.5.36) we get that Γ is harmonic if and only if γ is harmonic. In our setting (2.5.17) yields

$$\bar{R}_{11} = -d\frac{u''}{u}, \quad \bar{R}_{1+\alpha\, 1+\beta} = -\left(\frac{u''}{u} + (d-1)\frac{(u')^2}{u^2}\right)\delta_{\alpha\beta} + \frac{1}{u^2}{}^F R_{\alpha\beta}.$$

Using (2.5.37) we get, from the above

$$\bar{R}^{\Gamma}_{11} = -d\frac{u''}{u}, \quad \bar{R}^{\Gamma}_{1+\alpha\, 1+\beta} = -\left(\frac{u''}{u} + (d-1)\frac{(u')^2}{u^2}\right)\delta_{\alpha\beta} + \frac{1}{u^2}{}^F R^{\gamma}_{\alpha\beta}. \tag{2.5.40}$$

Assume $I \times_u F$ is harmonic-Einstein with respect to Γ and α, that is,

$$\begin{cases} \overline{\mathrm{Ric}} - \alpha\Gamma^*\langle\, , \rangle_N = \lambda\overline{\langle\, , \rangle} \\ \bar{\tau}(\Gamma) = 0. \end{cases}$$

Then γ is harmonic and from (2.5.40) we obtain

$$u'' = -\frac{\lambda}{d}u \tag{2.5.41}$$

and

$${}^F R_{\alpha\beta} = \left(\lambda + \frac{u''}{u} + (d-1)\left(\frac{u'}{u}\right)^2\right)u^2\delta_{\alpha\beta}.$$

By plugging (2.5.41) into the above we get

$${}^F R_{\alpha\beta} = (d-1)\left((u')^2 + \frac{\lambda}{d}u^2\right)\delta_{\alpha\beta}. \tag{2.5.42}$$

Observe that

$$(u')^2 + \frac{\lambda}{d}u^2$$

is constant on I, indeed

$$\left((u')^2 + \frac{\lambda}{d}u^2\right)' = 2u'\left(u'' + \frac{\lambda}{d}u\right),$$

hence we conclude using (2.5.41). Hence can be rewritten as (assuming $0 \in I$)

$${}^F R_{\alpha\beta} = (d-1)\left(u'(0)^2 + \frac{\lambda}{d}u(0)^2\right)\delta_{\alpha\beta}.$$

In conclusion the following hold

$$\begin{cases} {}^F\mathrm{Ric}^{\gamma} = (d-1)\left(u'(0)^2 + \frac{\lambda}{d}u(0)^2\right)\langle\, , \rangle_F \\ {}^F\tau(\gamma) = 0, \end{cases} \tag{2.5.43}$$

hence $(F, \langle\, , \rangle_F)$ is harmonic-Einstein with respect to γ and α and moreover u solves (2.5.41). Observe that the solutions of (2.5.41) are given by

$$u = u'(0)\mathrm{sn}_{\frac{\lambda}{d}} + u(0)\mathrm{cn}_{\frac{\lambda}{d}}, \tag{2.5.44}$$

where

$$\mathrm{sn}_\kappa(t) := \begin{cases} \frac{1}{\sqrt{-\kappa}}\sinh(\sqrt{-\kappa}t) & \text{for } \kappa < 0 \\ t & \text{for } \kappa = 0 \\ \frac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}t) & \text{for } \kappa > 0 \end{cases} \quad \text{for every } t \in \mathbb{R}.$$

and

$$\mathrm{cn}_\kappa := \mathrm{sn}'_\kappa.$$

We obtained, since also the converse implication holds,

Proposition 2.5.45. *Let $I \ni 0$ be an open interval on $\mathbb{R}$ with the Euclidean metric, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\gamma : (F, \langle\,,\,\rangle_F) \to (N, \langle\,,\,\rangle_N)$ be a smooth map. Let $u \in \mathcal{C}^\infty(I)$ and set Γ as in* (2.5.34). *Then $I \times_u F$ is harmonic-Einstein with respect to Γ and α if and only if $(F, \langle\,,\,\rangle_F)$ is harmonic-Einstein with respect to γ and α and u is given by* (2.5.44). *In this case the following relation holds:*

$${}^F S^\gamma = m(d-1)\left(u'(0)^2 + \frac{\bar{S}^\Gamma}{d(m+1)}u(0)^2\right).$$

Remark 2.5.46. In particular, in the assumption of the Proposition above, in case u is constant (that is the case where the warped product is a Riemannian product), we get from (2.5.41), since $u > 0$, that $\lambda = 0$. As a consequence the following are equivalent:

(i) $I \times_u F$ is harmonic-Einstein with respect to Γ and α;

(ii) $I \times_u F$ is Γ-Ricci flat for α;

(iii) $(F, \langle\,,\,\rangle_F)$ is γ-Ricci flat for α.

2.5.1 Lorentzian setting

Remark 2.5.47. We say that a Lorentzian manifold $(\bar{M}, \overline{\langle\,,\,\rangle})$ of dimension m, for $m \geq 3$, is harmonic-Einstein if there exists $\Phi : \bar{M} \to (N, \langle\,,\,\rangle_N)$, where $(N, \langle\,,\,\rangle_N)$ is a Riemannian manifold, and $\alpha \in \mathbb{R} \setminus \{0\}$ such that

$$\begin{cases} \overline{\mathrm{Ric}} - \alpha\Phi^*\langle\,,\,\rangle_N = \lambda\overline{\langle\,,\,\rangle} \\ \bar{\tau}(\Phi) = 0, \end{cases} \tag{2.5.48}$$

for some constant λ. The situation when a Lorentzian manifold is harmonic-Einstein, as one can expects due to the analogy with Lorentzian Einstein manifolds, is interesting in view of applications to General Relativity. Indeed, taking the trace of the first equation of (2.5.48), we infer, since $m \geq 2$,

$$\overline{\mathrm{Ric}} - \frac{\overline{S}}{2}\overline{\langle\,,\,\rangle} + \Lambda\overline{\langle\,,\,\rangle} = \alpha\left(\Phi^*\langle\,,\,\rangle_N - \frac{\overline{|\bar{d}\Phi|}^2}{2}\overline{\langle\,,\,\rangle}\right),$$

where

$$\Lambda = \frac{m-2}{2m}\overline{S}^\Phi.$$

Hence, for $m = 4$ and

$$\alpha = \frac{8\pi G}{c^4}, \tag{2.5.49}$$

where G is Newton's gravitational constant and c is the speed of light in vacuum, the above yields

$$\overline{\mathrm{Ric}} - \frac{\overline{S}^\Phi}{2}\overline{\langle\,,\,\rangle} + \Lambda\overline{\langle\,,\,\rangle} = \frac{8\pi G}{c^4}\bar{\mathbb{S}},$$

where $\bar{\mathbb{S}}$ is the stress energy tensor of the wave map (harmonic map, with as source a Lorentzian manifold) Φ, as defined in (1.1.13), and it is divergence free because Φ is wave map. Hence the spacetime $(\bar{M}, \overline{\langle\,,\,\rangle})$ is a solution of the Einstein field equations with zero constant

$$\Lambda = \frac{\overline{S}^\Phi}{4}.$$

Now let $(M, \langle\,,\,\rangle)$ be a m-dimensional Riemannian manifold, with $m \geq 2$. Denote

$$\bar{M} := M \times I,$$

where $I \subseteq \mathbb{R}$ is an open interval. Let $f \in \mathcal{C}^{\infty}(M)$ and

$$u := e^{-f},$$

we consider on $\bar{M}$ the Lorentzian metric

$$\overline{\langle\,,\,\rangle} := \langle\,,\,\rangle - u^2 dt \otimes dt,$$

where t denotes the coordinate on I, and we denote by $M \times_u I := (\bar{M}, \overline{\langle\,,\,\rangle})$ the Lorentzian warped product.

Let $\varphi : M \to (N, \langle\,,\,\rangle_N)$ be a smooth map, set Φ as in (2.5.20). One can prove the analogous of Corollary 2.5.16 and Proposition 2.5.21 also in the Lorentzian setting. We have, see Corollary 43 of [On], for every X, Y tangent vector to the base M and V, W tangent vector to the fibre F,

$$\begin{aligned}
&\overline{\mathrm{Ric}}(X,Y) = \mathrm{Ric}(X,Y) - \frac{1}{u}\mathrm{Hess}(u)(X,Y)\\
&\overline{\mathrm{Ric}}(X,V) = 0\\
&\overline{\mathrm{Ric}}(V,W) = -\frac{\Delta u}{u}\overline{\langle V,W\rangle}.
\end{aligned}$$

Moreover, since $\Phi := \varphi \circ \pi_M$,

$$\Phi^*\langle\,,\,\rangle_N = \pi_M^* \varphi^* \langle\,,\,\rangle_N,$$

and using $u = e^{-f}$ the above relations imply

$$\begin{aligned}
&(\overline{\mathrm{Ric}} - \alpha\Phi^*\langle\,,\,\rangle_N)(X,Y) = (\mathrm{Ric} - \alpha\varphi^*\langle\,,\,\rangle_N)(X,Y) + \mathrm{Hess}(f)(X,Y) - df \otimes df(X,Y)\\
&(\overline{\mathrm{Ric}} - \alpha\Phi^*\langle\,,\,\rangle_N)(X,V)^{\Phi} = 0\\
&(\overline{\mathrm{Ric}} - \alpha\Phi^*\langle\,,\,\rangle_N)(V,W) = \Delta_f f\overline{\langle V,W\rangle}.
\end{aligned} \tag{2.5.50}$$

It is also easy to prove the validity of

$$\bar{\tau}(\Phi) = \tau(\varphi) + \frac{1}{u}d\varphi(\nabla u),$$

that is,

$$\bar{\tau}(\Phi) = \tau(\varphi) - d\varphi(\nabla f). \tag{2.5.51}$$

Assume $M \times_u I$ is harmonic-Einstein for some constant $\alpha \in \mathbb{R} \setminus \{0\}$ and Φ as above, that is, (2.5.48) holds for some $\lambda \in \mathbb{R}$. Using (2.5.50) and the first equation of (2.5.48) we deduce the validity of

$$\mathrm{Ric} - \alpha\varphi^*\langle\,,\,\rangle_N + \mathrm{Hess}(f) - df \otimes df = \lambda\langle\,,\,\rangle$$

with

$$\Delta_f f = \lambda, \tag{2.5.52}$$

or equivalently

$$\Delta u + \lambda u = 0.$$

Furthermore, from (2.5.51) we immediately get

$$\tau(\varphi) = d\varphi(\nabla f).$$

In conclusion:

$$\begin{cases}
\mathrm{Ric} - \alpha\varphi^*\langle\,,\,\rangle_N + \mathrm{Hess}(f) - df \otimes df = \lambda\langle\,,\,\rangle\\
\tau(\varphi) = d\varphi(\nabla f)\\
\Delta_f f = \lambda
\end{cases} \tag{2.5.53}$$

On the other hand, assume (2.5.53) holds. From (2.5.51) we immediately get $\bar\tau(\Phi) = 0$. Moreover, using (2.5.50) we deduce

$$\overline{\mathrm{Ric}} - \alpha\Phi^*\langle\,,\,\rangle_N = \Delta_f f\overline{\langle\,,\,\rangle},$$

that is, using (2.5.52),

$$\overline{\mathrm{Ric}} - \alpha\Phi^*\langle\,,\,\rangle_N = \lambda\overline{\langle\,,\,\rangle}.$$

Definition 2.5.54. Let $(M, \langle\,,\,\rangle)$ be a Riemannian manifold of dimension m and let $\varphi : M \to (N, \langle\,,\,\rangle_N)$ a smooth map. If there exists $\alpha \in \mathbb{R}\setminus\{0\}$, $f \in \mathcal{C}^\infty(M)$ and $\lambda \in \mathbb{R}$ such that (2.5.53) holds we say that the Riemannian metric $\langle\,,\,\rangle$ on M is *φ-static*.

We summarize the discussion above in the next

Proposition 2.5.55. *Let $(M, \langle\,,\,\rangle)$ be a m-dimensional Riemannian manifold and let $I \subseteq \mathbb{R}$ be an open interval. Let $\alpha \in \mathbb{R}\setminus\{0\}$, $\varphi : M \to (N, \langle\,,\,\rangle_N)$ a smooth map and $f \in \mathcal{C}^\infty(M)$. We set Φ as in* (2.5.20)*. The Lorentzian warped product manifold $M \times_{e^{-f}} I$ is harmonic-Einstein for some constant $\alpha \in \mathbb{R}\setminus\{0\}$ and Φ as above if and only if the Riemannian metric $\langle\,,\,\rangle$ on M is φ-static.*

Remark 2.5.56. In case φ is constant we recover Proposition 2.7 of [Co] and the classic concept of static metrics.

Remark 2.5.57. Assume the Lorentzian warped product is harmonic-Einstein, that is (2.5.48) holds for some constant $\alpha \in \mathbb{R}\setminus\{0\}$, Φ as above and $\lambda \in \mathbb{R}$. Clearly, tracing the harmonic-Einstein equation,

$$\lambda = \frac{\bar S^\Phi}{m+1}.$$

Moreover, as showed above, (2.5.53) holds. Taking the trace of the first equation of (2.5.53) and using also the third equation of it we infer

$$S^\varphi = (m-1)\lambda.$$

Combining with the above we conclude

$$S^\varphi = (m-1)\lambda = \frac{m-1}{m+1}\bar S^\Phi.$$

Notice that, if M is compact, then the third equation of (2.5.53) easily implies $\lambda = 0$ and f constant, in particular $M \times_u I$ is a Riemannian product. Then to obtain examples of non-trivial warped products we need to consider M to be non-compact (and also $\bar S^\Phi < 0$, as we shall see later on in Remark 7.4.15).

The situation when the Lorentzian warped product $M \times_u I$ is harmonic-Einstein (for some constant $\alpha \in \mathbb{R}\setminus\{0\}$ and Φ as above) is interesting in view of Remark 2.5.47, especially for $m = 3$ and α given by (2.5.49). Indeed in this case the Lorentzian warped product $M \times_u I$ is a standard static spacetime that is a solution of the Einstein equation (with zero cosmological constant) where the stress-energy tensor is given by

$$\bar{\mathbb{S}} = \Phi^*\langle\,,\,\rangle_N - \bar e(\Phi)\overline{\langle\,,\,\rangle},$$

where $\bar e(\Phi)$ is the density of energy of Φ. Since $\Phi = \varphi\circ\pi_M$ we have $\bar e(\Phi) = e(\varphi)$ and thus,

$$\bar{\mathbb{S}} = \pi_M^*\mathbb{S} + e(\varphi)e^{-2f}dt\otimes dt, \tag{2.5.58}$$

where $\mathbb{S}$ is the energy-stress tensor of φ, that is,

$$\mathbb{S} = \varphi^*\langle\,,\,\rangle - e(\varphi)\langle\,,\,\rangle.$$

We summarize in the next

Proposition 2.5.59. *Let $M\times_u I$ be a four dimensional Lorentzian warped product that is harmonic-Einstein with respect α given by* (2.5.49) *and Φ given by* (2.5.20)*, for a smooth map $\varphi : M \to (N, \langle\,,\,\rangle_N)$. Then $M\times_u I$ is a solution of the Einstein field equations with cosmological constant $\Lambda = \frac{S^\varphi}{2}$ and with as stress-energy tensor the one of the wave map Φ, that is given by* (2.5.58)*.*

Part II

Einstein-type structures

Chapter 3

Definition of Einstein-type structures and basic formulas

In what follow $\mathfrak{X}(M)$ will denote the $\mathcal{C}^\infty(M)$-module of the vector fields on M.

Definition 3.0.1. We say that the Riemannian manifold $(M, \langle\, , \,\rangle)$ carries an *Einstein-type structure* if there exist $X \in \mathfrak{X}(M)$, $\varphi : M \to (N, \langle\, , \,\rangle_N)$ for some Riemannian manifold $(N, \langle\, , \,\rangle_N)$, and functions $\alpha, \lambda, \mu \in \mathcal{C}^\infty(M)$ such that

$$\begin{cases} \mathrm{Ric} + \frac{1}{2}\mathcal{L}_X\langle\, , \,\rangle - \mu X^\flat \otimes X^\flat - \alpha\varphi^*\langle\, , \,\rangle_N = \lambda\langle\, , \,\rangle \\ \tau(\varphi) = d\varphi(X), \end{cases} \tag{3.0.2}$$

where $^\flat : \mathfrak{X}(M) \to \bigwedge^1(M)$ is the musical isomorphism and $\mathcal{L}_X\langle\, , \,\rangle$ denotes the Lie derivative of the metric along the vector field X.

In case $X = \nabla f$ for some $f \in \mathcal{C}^\infty(M)$ we say that $(M, \langle\, , \,\rangle)$ carries a *gradient Einstein-type structure*. In case the Einstein-type structure is gradient (3.0.2) takes the form

$$\begin{cases} \mathrm{Ric} + \mathrm{Hess}(f) - \mu df \otimes df - \alpha\varphi^*\langle\, , \,\rangle_N = \lambda\langle\, , \,\rangle \\ \tau(\varphi) = d\varphi(\nabla f). \end{cases} \tag{3.0.3}$$

Remark 3.0.4. The gradient Einstein-type structures as (3.0.3) for $\mu = -\frac{1}{m-2}$, in view of Theorem 2.3.5, coincide with conformally harmonic-Einstein manifolds, while, for $\mu = \frac{1}{d}$ for some positive integer d, are the base for some harmonic-Einstein warped product, see Theorem 2.5.26 for the precise statement. These situations motivate the study of gradient Einstein-type structures. Furthermore, the structure described by (3.0.2) generalizes some well known particular cases that have been intensively studied by researchers in the last decade. Indeed, (3.0.2) characterizes:

i) Ricci solitons for φ constant, $\mu = 0$ and $\lambda \in \mathbb{R}$, that is,

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X\langle\, , \,\rangle = \lambda\langle\, , \,\rangle.$$

Letting $\lambda \in \mathcal{C}^\infty(M)$ we obtain Almost Ricci solitons, whose gradient version has been introduced in [PRRiS]. Note that when $\lambda = a + bS$ for some constants $a, b \in \mathbb{R}$ and S the scalar curvature of $(M, \langle\, , \,\rangle)$, the corresponding soliton is called a Ricci-Bourguignon soliton after the recent work of G. Catino, L. Cremaschi, Z. Djadli, C. Mantegazza, and L. Mazzieri [CCDMM]. For a flow derivation of the gradient Ricci almost soliton equation in the general case see the work [GWX];

ii) Quasi-Einstein manifold for $X=\nabla f$, φ constant, $\mu=\frac{1}{d}$ for some positive integer d and $\lambda\in\mathbb{R}$, that is,

$$\mathrm{Ric}+\mathrm{Hess}(f)-\frac{1}{d}df\otimes df=\lambda\langle\,,\,\rangle.$$

In case X is not necessarily a gradient vector field we have generalized n-quasi Einstein manifolds, introduced in [BR]. In literature the generalized quasi-Einstein condition is given by

$$\mathrm{Ric}+\mathrm{Hess}(f)-\mu df\otimes df=\lambda\langle\,,\,\rangle$$

with $\mu,\lambda\in\mathcal{C}^\infty(M)$. See, for instance, [C] and [AG].

iii) Ricci-harmonic solitons for $\mu=0$, $\lambda\in\mathbb{R}$ and $\alpha>0$, whose gradient version has been introduced by R. Müller in [M]. As expected the concept comes from the study of a combination of the Ricci and harmonic maps flows. We refer to [M] for details and interesting analytic motivations.

iv) τ-quasi Ricci-harmonic metrics for $X=\nabla f$ and $\mu=\frac{1}{\tau}$, where τ is a positive constant, introduced in [W].

In what follows the term $\mathrm{Ric}-\alpha\varphi^*\langle\,,\,\rangle_N$ will be simply written as Ric^φ, when there is no risk of confusion, following the notation introduced in Chapter 1.

In Section 3.1 we compute the gradient and the Laplacian of the φ-scalar curvature for Riemannian manifolds supporting an Einstein-type structure, these formulas shall be used frequently in the following Chapters.

In Section 3.2 we present a general non-existence result, related to the existence of a first positive zero for the solution of a Cauchy problem, for gradient Einstein-type structure with μ constant and different from zero.

3.1 Basic formulas

The following commutation relations, valid for every $Y\in\mathfrak{X}(M)$, follows from the general commutation relation (14):

$$Y^i_{jk}-Y^i_{kj}=Y^tR_{tijk}, \tag{3.1.1}$$

$$Y^i_{jkl}-Y^i_{jlk}=Y^t_jR_{tikl}+Y^i_tR^t_{jkl}. \tag{3.1.2}$$

We shall need them in the proof of the following

Proposition 3.1.3. *Let $(M,\langle\,,\,\rangle)$ be a Riemannian manifold of dimension m with an Einstein-type structure as in (3.0.2), with $\alpha\in\mathbb{R}\setminus\{0\}$, $X\in\mathfrak{X}(M)$, $\lambda\in\mathcal{C}^\infty(M)$, $\mu\in\mathbb{R}$ and $\varphi:M\to(N,\langle\,,\,\rangle_N)$ smooth. Then in a local orthonormal coframe the following hold,*

$$R^\varphi_{ij,k}+R^\varphi_{ik,j}-R_{tijk}X^t+\frac{1}{2}(X^j_k-X^k_j)_i=\mu[X^i_kX^j-X^i_jX^k+X^i(X^j_k-X^k_j)]+\lambda_k\delta_{ij}-\lambda_j\delta_{ik}, \tag{3.1.4}$$

$$\frac{1}{2}S^\varphi_k-R^\varphi_{ik}X^i+\frac{1}{2}(X^i_k-X^k_i)_i=\mu\left[\frac{1}{2}(X^i_k+X^k_i)X^i+\frac{3}{2}(X^i_k-X^k_i)X^i-X^i_iX^k\right]+(m-1)\lambda_k, \tag{3.1.5}$$

$$\begin{aligned}\frac{1}{2}\Delta_{(1+2\mu)X}S^\varphi+(1-\mu)(|T^\varphi|^2+\alpha|\tau(\varphi)|^2)+\left[\frac{(m-1)\mu+1}{m}S^\varphi-\mu(m-1)\lambda\right](S^\varphi-m\lambda)&\\=(m-1)\Delta_{2\mu X}\lambda+\frac{\mu}{2}D,&\end{aligned} \tag{3.1.6}$$

where

$$D:=2[(X^i_k-X^k_i)X^i]_k+(X^i_k-X^k_i)X^i_k. \tag{3.1.7}$$

Here Δ_Y, for $Y\in\mathfrak{X}(M)$, stands for the operator $\Delta-\langle Y,\nabla\ \rangle$.

Proof. In a local orthonormal coframe (3.0.2) is given by

$$\begin{cases} R^{\varphi}_{ij} + \frac{1}{2}(X^i_j + X^j_i) = \mu X^i X^j + \lambda\delta_{ij} \\ \varphi^a_{ii} = \varphi^a_i X^i. \end{cases} \tag{3.1.8}$$

Taking the covariant derivative of the first equation in (3.1.8) yields

$$R^{\varphi}_{ij,k} + \frac{1}{2}(X^i_{jk} + X^j_{ik}) = \mu(X^i_k X^j + X^i X^j_k) + \lambda_k\delta_{ij}.$$

Inverting the role of j and k and by subtraction we obtain

$$R^{\varphi}_{ij,k} - R^{\varphi}_{ik,j} + \frac{1}{2}(X^i_{jk} - X^i_{kj} + X^j_{ik} - X^k_{ij}) = \mu(X^i_k X^j - X^i_j X^k + X^i X^j_k - X^i X^k_j) + \lambda_k\delta_{ij} - \lambda_j\delta_{ik}.$$

Using three times (3.1.1) and the first Bianchi identity we deduce

$$\frac{1}{2}(X^i_{jk} - X^i_{kj} + X^j_{ik} - X^k_{ij}) = R_{tijk}X^t + \frac{1}{2}(X^j_k - X^k_j)_i.$$

Plugging into the above we have

$$R^{\varphi}_{ij,k} - R^{\varphi}_{ik,j} + R_{tijk}X^t + \frac{1}{2}(X^j_k - X^k_j)_i = \mu[X^i_k X^j - X^i_j X^k + X^i(X^j_k - X^k_j)] + \lambda_k\delta_{ij} - \lambda_j\delta_{ik},$$

that is (3.1.4). Summing on $i = j$ in (3.1.4) we get

$$S^{\varphi}_k - R^{\varphi}_{ik,i} - R_{ik}X^i + \frac{1}{2}(X^i_k - X^k_i)_i = \mu(2X^i_k X^i - X^i_i X^k - X^i X^k_i) + (m-1)\lambda_k.$$

Using (1.2.26), the second equation of (3.1.8) and the definition (1.2.2) of Ric^{φ} we infer

$$R^{\varphi}_{ik,i} + R_{ik}X^i = \frac{1}{2}S^{\varphi}_k - \alpha\varphi^a_{ii}\varphi^a_k + R_{ik}X^i = \frac{1}{2}S^{\varphi}_k + R^{\varphi}_{ik}X^i$$

and inserting into the above we obtain (3.1.5).

Tracing the first equation of (3.1.8) we deduce

$$S^{\varphi} + X^i_i = \mu|X|^2 + m\lambda, \tag{3.1.9}$$

using it together with the first equation of (3.1.8) in (3.1.5) we get

$$\begin{aligned} \frac{1}{2}S^{\varphi}_k - R^{\varphi}_{ik}X^i + \frac{1}{2}(X^i_k - X^k_i)_i =& \mu\left[(-R^{\varphi}_{ik} + \mu X^i X^k + \lambda\delta_{ik})X^i + (S^{\varphi} - \mu|X|^2 - m\lambda)X^k\right] \\ &+ \mu\frac{3}{2}(X^i_k - X^k_i)X^i + (m-1)\lambda_k, \end{aligned}$$

or equivalently

$$\frac{1}{2}S^{\varphi}_k + \frac{1}{2}(X^i_k - X^k_i)_i = (1-\mu)R^{\varphi}_{ik}X^i + \mu(S^{\varphi} - (m-1)\lambda)X^k + \mu\frac{3}{2}(X^i_k - X^k_i)X^i + (m-1)\lambda_k, \tag{3.1.10}$$

Contracting (3.1.10) against X we obtain

$$\frac{1}{2}S^{\varphi}_k X^k + \frac{1}{2}(X^i_k - X^k_i)_i X^k = (1-\mu)R^{\varphi}_{ik}X^i X^k + \mu(S^{\varphi} - (m-1)\lambda)|X|^2 + (m-1)\lambda_k X^k,$$

thus

$$(1-\mu)\mu R^{\varphi}_{ik}X^i X^k = \frac{\mu}{2}S^{\varphi}_k X^k + \frac{\mu}{2}(X^i_k - X^k_i)_i X^k - \mu(S^{\varphi} - (m-1)\lambda)\mu|X|^2 - (m-1)\mu\lambda_k X^k. \tag{3.1.11}$$

From (3.1.2) easily follows

$$X^i_{kik} = X^k_{iik},$$

then taking the divergence of (3.1.10) and inserting the commutation relation above we get

$$\begin{aligned}\frac{1}{2}S^\varphi_{kk} =& (1-\mu)(R^\varphi_{ik,k}X^i + R^\varphi_{ik}X^i_k) + \mu(S^\varphi_k - (m-1)\lambda_k)X^k + \mu(S^\varphi - (m-1)\lambda)X^k_k \\ &+ \mu\frac{3}{2}(X^i_k - X^k_i)_k X^i + \mu\frac{3}{2}(X^i_k - X^k_i)X^i_k + (m-1)\lambda_{kk}.\end{aligned} \tag{3.1.12}$$

Contracting the first equation of (3.1.8) against Ric^φ we infer

$$|\mathrm{Ric}^\varphi|^2 + R^\varphi_{ik}X^i_k = \mu R^\varphi_{ij}X^iX^k + \lambda S^\varphi,$$

that using the definition (2.4.1) it is equivalent to

$$|T^\varphi|^2 + \frac{(S^\varphi)^2}{m} + R^\varphi_{ik}X^i_k = \mu R^\varphi_{ij}X^iX^k + \lambda S^\varphi,$$

that is,

$$R^\varphi_{ik}X^i_k = -|T^\varphi|^2 - \frac{S^\varphi}{m}(S^\varphi - m\lambda) + \mu R^\varphi_{ij}X^iX^k.$$

Using the above and (1.2.26) we deduce

$$\begin{aligned}(1-\mu)(R^\varphi_{ik,k}X^i + R^\varphi_{ik}X^i_k) =& \left(\frac{1}{2} - \frac{\mu}{2}\right)S^\varphi_i X^i - (1-\mu)\alpha\varphi^a_{kk}\varphi^a_i X^i - (1-\mu)|T^\varphi|^2 \\ &- (1-\mu)\frac{S^\varphi}{m}(S^\varphi - m\lambda) + \mu(1-\mu)R^\varphi_{ij}X^iX^k\end{aligned}$$

and from the second equation of (3.1.8) and (3.1.11) it follows

$$\begin{aligned}(1-\mu)(R^\varphi_{ik,k}X^i + R^\varphi_{ik}X^i_k) =& \frac{1}{2}S^\varphi_i X^i - (1-\mu)(|T^\varphi|^2 + \alpha|\tau(\varphi)|^2) - (1-\mu)\frac{S^\varphi}{m}(S^\varphi - m\lambda) \\ &+ \frac{\mu}{2}(X^i_k - X^k_i)_i X^k - \mu(S^\varphi - (m-1)\lambda)\mu|X|^2 - (m-1)\mu\lambda_k X^k.\end{aligned}$$

Inserting the above into (3.1.12) we obtain

$$\begin{aligned}\frac{1}{2}S^\varphi_{kk} =& \frac{1+2\mu}{2}S^\varphi_i X^i - (1-\mu)(|T^\varphi|^2 + \alpha|\tau(\varphi)|^2) - (1-\mu)\frac{S^\varphi}{m}(S^\varphi - m\lambda) \\ &+ \frac{\mu}{2}(X^i_k - X^k_i)_i X^k - \mu(S^\varphi - (m-1)\lambda)(-X^k_k + \mu|X|^2) - 2(m-1)\mu\lambda_k X^k \\ &+ \mu\frac{3}{2}(X^i_k - X^k_i)_k X^i + \mu\frac{3}{2}(X^i_k - X^k_i)X^i_k + (m-1)\lambda_{kk},\end{aligned}$$

that, using (3.1.9), can be written as

$$\begin{aligned}\frac{1}{2}S^\varphi_{kk} =& \frac{1+2\mu}{2}S^\varphi_i X^i - (1-\mu)(|T^\varphi|^2 + \alpha|\tau(\varphi)|^2) - \left[(1-\mu)\frac{S^\varphi}{m} + \mu S^\varphi - \mu(m-1)\lambda\right](S^\varphi - m\lambda) \\ &+ \frac{\mu}{2}(X^i_k - X^k_i)_i X^k + \mu\frac{3}{2}(X^i_k - X^k_i)_k X^i + \mu\frac{3}{2}(X^i_k - X^k_i)X^i_k + (m-1)(\lambda_{kk} - 2\mu\lambda_k X^k),\end{aligned}$$

that is,

$$\begin{aligned}&\frac{1}{2}\Delta_{(1+2\mu)X}S^\varphi + (1-\mu)(|T^\varphi|^2 + \alpha|\tau(\varphi)|^2) + \left[\frac{(m-1)\mu+1}{m}S^\varphi - \mu(m-1)\lambda\right](S^\varphi - m\lambda) \\ &\qquad = (m-1)\Delta_{2\mu X}\lambda + \frac{\mu}{2}[(X^i_k - X^k_i)_i X^k + 3(X^i_k - X^k_i)_k X^i + 3(X^i_k - X^k_i)X^i_k].\end{aligned}$$

We then conclude the validity of (3.1.6), since

$$\begin{aligned}(X^i_k - X^k_i)_i X^k + 3(X^i_k - X^k_i)_k X^i + 3(X^i_k - X^k_i)X^i_k =& 2(X^i_k - X^k_i)_k X^i + 3(X^i_k - X^k_i)X^i_k \\ =& 2[(X^i_k - X^k_i)X^i]_k + (X^i_k - X^k_i)X^i_k \\ =& D.\end{aligned}$$

□

Remark 3.1.13. In case $\mu = 0$ equation (3.1.6) can be rewritten as

$$\frac{1}{2}\Delta_X S^\varphi + \alpha|\tau(\varphi)|^2 + |T^\varphi|^2 + \frac{S^\varphi}{m}(S^\varphi - m\lambda) = (m-1)\Delta\lambda. \tag{3.1.14}$$

Observe that in case $X = \nabla f$ (or, more generally, in case ∇X is symmetric), equation (3.1.4) and (3.1.5) become, respectively,

$$R^\varphi_{ij,k} - R^\varphi_{ik,j} - R_{tijk}f_t = \mu(f_{ik}f_j - f_{ij}f_k) + \lambda_k\delta_{ij} - \lambda_j\delta_{ik}, \tag{3.1.15}$$

$$\frac{1}{2}S^\varphi_k - R^\varphi_{ik}f_i = \mu(f_{ik}f_i - \Delta f f_k) + (m-1)\lambda_k. \tag{3.1.16}$$

Moreover D defined in (3.1.7) vanishes identically and thus (3.1.6) takes the form

$$\begin{aligned}&\frac{1}{2}\Delta_{(1+2\mu)f}S^\varphi + (1-\mu)(|T^\varphi|^2 + \alpha|\tau(\varphi)|^2) \\ &\qquad + \left[\frac{(m-1)\mu+1}{m}S^\varphi - \mu(m-1)\lambda\right](S^\varphi - m\lambda) = (m-1)\Delta_{2\mu f}\lambda.\end{aligned} \tag{3.1.17}$$

Remark 3.1.18. Formula (3.1.17) when λ is constant reduces to (2.1) of [W].

3.2 A non-existence result

The results of this Section are part of a joint work with Marco Rigoli. We now present a general non-existence result for gradient Einstein-type structure with $\mu \neq 0$.

Proposition 3.2.1. *Let $(M, \langle\,,\,\rangle)$ be a complete Riemannian manifold of dimension m. For $r \in \mathbb{R}^+$, let*

$$v(r) := vol(\partial B_r), \quad A(r) := \frac{\mu}{v(r)}\int_{\partial B_r}(m\lambda - S^\varphi),$$

where B_r is the geodesic ball of radius r with centre in $o \in M$. Let $z \in Lip_{loc}(\mathbb{R}^+_0)$ be a solution of the Cauchy problem

$$\begin{cases}(vz')' + Avz = 0 & on\ \mathbb{R}^+\\ z(0^+) = z_0 > 0, & (vz')(0^+) = 0.\end{cases} \tag{3.2.2}$$

Suppose that z admits a first zero $R_0 \in \mathbb{R}^+$. Then there exist no $f, \lambda \in \mathcal{C}^\infty(M)$ and $\alpha, \mu \in \mathbb{R}\setminus\{0\}$, such that

$$Ric^\varphi + Hess(f) - \mu df \otimes df = \lambda\langle\,,\,\rangle. \tag{3.2.3}$$

Proof. By contradiction assume the existence of $f, \lambda \in \mathcal{C}^\infty(M)$ and $\alpha, \mu \in \mathbb{R}\setminus\{0\}$, such that (3.2.3) holds. Since $\mu \neq 0$ the positive function $u := e^{-\mu f}$ satisfies

$$\mathrm{Hess}(f) - \mu df \otimes df = -\frac{\mathrm{Hess}(u)}{\mu u},$$

and (3.2.3) can be rewritten as

$$\mathrm{Ric}^\varphi - \frac{\mathrm{Hess}(u)}{\mu u} = \lambda\langle\,,\,\rangle.$$

Taking the trace of the above we obtain $Lu = 0$, where

$$Lu := \Delta u + q(x)u, \quad q := \mu(m\lambda - S^{\varphi}).$$

Since $u > 0$, by a well known result of [FCS] and [MP], the operator L is stable or, in other words, its spectral radius $\lambda_1^L(M)$ is non-negative.

Now we prove that under our assumptions $\lambda_L^1(M) < 0$, obtaining the desired contradiction. Observe that $v \in L^{\infty}_{loc}(\mathbb{R}^+_0)$, $v > 0$ on $\mathbb{R}^+$ and $v^{-1} \in L^{\infty}_{loc})(\mathbb{R}^+)$ by (iii) of Proposition 1.6 of [BMR]. By Proposition 3.2 and Proposition 3.6 of [BMR] there exists a solution of (3.2.2) is in $Lip_{loc}(\mathbb{R}^+_0)$ and its possible zeroes are isolated. Suppose that z admits a first zero $R_0 \in \mathbb{R}^+$. We define

$$\psi := z \circ r,$$

where r is the distance function from the fixed origin $o \in M$. We consider the Rayleigh quotient

$$Q(\psi) := \left(\int_{B_{R_0}} \psi^2 \right)^{-1} \int_{B_{R_0}} (|\nabla \psi|^2 - q\psi^2).$$

From the co-area formula and Gauss lemma we get

$$Q(\psi) = \left(\int_0^{R_0} z^2 v \right)^{-1} \int_0^{R_0} [(z')^2 v - Avz^2].$$

Integrating by parts and using (3.2.2) we obtain

$$\int_0^{R_0} (z')^2 v = zz'v|_0^{R_0} - \int_0^{R_0} z(vz') = \int_0^{R_0} Avz^2,$$

so that $Q(\psi) = 0$. Then by the min-max characterization $\lambda_1^L(B_{R_0}) \leq 0$ and by monotonicity of the eigenvalues of L we infer $\lambda_1^L(M) < 0$. □

Remark 3.2.4. It remains to determine some sufficient conditions under which a solution z of (3.2.2), always existing by Proposition 3.2 of [BMR], admits a first zero. From Corollary 5.2 of [BMR], if $A \geq 0$ on $\mathbb{R}^+$, $A \not\equiv 0$ and either $g^{-1} \notin L^1(+\infty)$ or otherwise there exist $r > R > 0$ such that $A \not\equiv 0$ on $[0, R]$ and

$$\int_R^r (\sqrt{A} - \sqrt{\chi_g}) > -\frac{1}{2} \left(\log \int_0^R Av + \log \int_R^{+\infty} \frac{1}{g} \right), \tag{3.2.5}$$

z has a first zero. Here $g \in L^{\infty}_{loc}(\mathbb{R}^+_0)$ is such that $g^{-1} \in L^{\infty}_{loc}(\mathbb{R}^+)$ and $0 \leq v \leq g$ on $\mathbb{R}^+_0$, while χ_g is the critical curve relative to g defined by

$$\chi_g(r) = \left\{ 2g(r) \int_r^{+\infty} \frac{1}{g} \right\}^{-2}.$$

Note that (3.2.5) can be rewritten as

$$\int_R^r (\sqrt{A} - \sqrt{\chi_g}) > -\frac{1}{2} \left(\log \int_{B_R} \mu(m\lambda - S^{\varphi}) + \log \int_R^{+\infty} \frac{1}{g} \right).$$

Observe that the existence of a first zero can be guarantee from an oscillatory condition. For instance, from Corollary 2.9 of [MaMR], if for some $r_0 \in \mathbb{R}^+$

$$\mu \lim_{r \to +\infty} \int_{B_r \setminus B_{r_0}} (m\lambda - S^{\varphi}) = +\infty, \tag{3.2.6}$$

then every solution of (3.2.2) is oscillatory.

By way of example, we have

Proposition 3.2.7. *Suppose* $\mu(S^{\varphi} - m\lambda) \le 0$ *on* M,

$$v(r) \le Cr^{\theta}, \tag{3.2.8}$$

for some constants $C > 0$ *and* $\theta \in \mathbb{R}$ *and, in case* $\theta > 1$, *that for some* $R \in \mathbb{R}^{+}$ *and for some constant* $D > \frac{\theta-1}{2}$

$$\int_{\partial B_r} \mu(m\lambda - S^{\varphi}) \ge \frac{D^2}{r^2} v(r) \quad \text{for } r \ge R, \tag{3.2.9}$$

Then a solution z *of* (3.2.2) *admits a first zero.*

Proof. From (3.2.8) we can choose

$$g(r) = Cr^{\theta}.$$

Clearly $g^{-1} \notin L^1(+\infty)$ if and only if $\theta \le 1$. In case $\theta > 1$

$$\chi_g(r) = \left(\frac{\theta - 1}{2r}\right)^2.$$

Hence (3.2.5) can be rewritten as

$$\int_R^r \sqrt{A} - \frac{\theta - 1}{2}(\log r - \log R) > -\frac{1}{2}\log \int_{B_R} \mu(m\lambda - S^{\varphi}) - \frac{1}{2}\log \frac{R^{1-\theta}}{C(\theta - 1)},$$

that is,

$$\int_R^r \sqrt{A} - \frac{\theta - 1}{2}\log r > \frac{1}{2}\log C + \frac{1}{2}\log(\theta - 1) - \frac{1}{2}\log \int_{B_R} \mu(m\lambda - S^{\varphi}).$$

From (3.2.9) and the definition of A we immediately see that

$$\sqrt{A(r)} \ge \frac{D}{r} \quad \text{for } r \ge R. \tag{3.2.10}$$

Using (3.2.10), to obtain the validity of (3.2.5) for some $r \ge R$ it is sufficient that

$$D\int_R^r \frac{ds}{s} - \frac{\theta - 1}{2}\log r > \frac{1}{2}\log C + \frac{1}{2}\log(\theta - 1) - \frac{1}{2}\log \int_{B_R} \mu(m\lambda - S^{\varphi}),$$

that is,

$$D\,(\log r - \log R) - \frac{\theta - 1}{2}\log r > \log(\sqrt{C(\theta - 1)}) - \frac{1}{2}\log \int_{B_R} \mu(m\lambda - S^{\varphi}),$$

or equivalently,

$$\left(D - \frac{\theta - 1}{2}\right)\log r > \log(R^D\sqrt{C(\theta - 1)}) - \frac{1}{2}\log \int_{B_R} \mu(m\lambda - S^{\varphi}). \tag{3.2.11}$$

Since $D > \frac{\theta-1}{2}$ there exists r large enough such that (3.2.11) holds. Then, from Remark 3.2.4, we can conclude the proof. Observe that $A \not\equiv 0$ on $[0, R]$ is guaranteed by the fact that, since by assumption $\mu(S^{\varphi} - m\lambda) \le 0$ on M on M, $S^{\varphi} \not\equiv m\lambda$ on B_R that, in turns, is guaranteed by (3.2.9). ☐

Remark 3.2.12. We consider, in case $\theta > 1$, the limiting case $v(r) = Cr^{\theta}$. Inserting this information into (3.2.9) we obtain

$$\int_{\partial B_r} \mu(m\lambda - S^{\varphi}) \ge CD^2 r^{\theta - 2} \quad \text{for } r \ge R.$$

An immediate computation and the fact that $\theta > 1$, shows that

$$\int_{B_r \setminus B_R} \mu(m\lambda - S^\varphi) \geq \frac{CD^2}{\theta - 1}(r^{\theta - +1} - R^{\theta-1})$$

and therefore the integral diverges as $r \to +\infty$. This means that condition (3.2.6) is satisfied and the solution is even oscillatory.

Observe that instead of (3.2.9) we may have assumed the strongest condition

$$\int_{\partial B_r} \mu(m\lambda - S^\varphi) \geq CD^2 r^{\theta-2} \quad \text{for } r \geq R. \tag{3.2.13}$$

Indeed, using (3.2.8),

$$\int_{\partial B_r} \mu(m\lambda - S^\varphi) \geq CD^2 r^{\theta-2} = cr^\theta \frac{D^2}{r^2} \geq \frac{D^2}{r^2} v(r) \quad \text{for } r \geq R,$$

hence (3.2.13) implies (3.2.9). But (3.2.13) implies also (3.2.6) for $r_0 = R$, so not only admits a zero but is even oscillatory. Indeed

$$\int_{B_r \setminus B_R} \mu(m\lambda - S^\varphi) = \int_R^r \left(\int_{\partial B_s} \mu(m\lambda - S^\varphi) \right) ds \geq CD^2 \int_R^r s^{\theta-2} ds = \frac{CD^2}{\theta-1}(r^{\theta-1} - R^{\theta-1})$$

so that, since $\theta > 1$,

$$\lim_{r \to +\infty} \int_{B_r \setminus B_R} \mu(m\lambda - S^\varphi) = +\infty \tag{3.2.14}$$

Notice that, in general, (3.2.9) does not imply (3.2.6), hence this condition guarantee the existence of a first zero and not the oscillation for the solution z. For instance, assume for some constant $\gamma < 1$ and $B > 0$,

$$v(r) = Br^\gamma \quad \text{for every } r \geq R.$$

Then, from (3.2.9),

$$\int_{\partial B_r} \mu(m\lambda - S^\varphi) \geq \frac{BD^2}{r^{2-\gamma}} \quad \text{for } r \geq R. \tag{3.2.15}$$

Since $\gamma < 1$

$$\int_{B_r \setminus B_R} \mu(m\lambda - S^\varphi) \geq BD^2 \int_R^r \frac{ds}{s^{2-\gamma}} = \frac{BD^2}{1-\gamma}\left(\frac{1}{R^{1-\gamma}} - \frac{1}{r^{1-\gamma}} \right) \to \frac{BD^2}{(1-\gamma)R^{1-\gamma}} \quad \text{for } r \to +\infty,$$

hence

$$\lim_{r \to +\infty} \int_{B_r \setminus B_R} \mu(m\lambda - S^\varphi)$$

is not forced to be infinite.

As another example we give

Proposition 3.2.16. *Suppose $S^\varphi \geq m\lambda$ on M,*

$$v(r) \leq \Lambda \exp\{ar^\alpha \log^\beta r\}, \tag{3.2.17}$$

for some constants $\Lambda, a, \alpha > 0$ and $\beta \geq 0$ and

$$\int_{\partial B_r} \mu(m\lambda - S^\varphi) \geq \frac{9a^2}{4}(\alpha \log r + \beta r)^2 r^{2(\alpha-1)} \log^{2(\beta-1)} r v(r). \tag{3.2.18}$$

Then a solution z of (3.2.2) *admits a first zero.*

Proof. The proof is similar to that of Proposition 3.2.7. From (3.2.17) we can choose

$$g(r) = \Lambda \exp\{ar^\alpha \log^\beta r\}.$$

Clearly $g^{-1} \notin L^1(+\infty)$. We claim that the validity of, for some r and R large enough,

$$\int_R^r \sqrt{A} - ar^\alpha \log^\beta r > -\frac{1}{2}\log \int_{B_R} \mu(m\lambda - S^\varphi) + \frac{1}{2} - \frac{3a}{2} R^\alpha \log^\beta R \tag{3.2.19}$$

implies the validity of (3.2.5). Indeed, if we define

$$\widetilde{\chi}_g(t) := \left(\frac{g'(t)}{2g(t)}\right)^2,$$

then

$$\sqrt{\widetilde{\chi}_g(t)} \sim \sqrt{\chi_g(t)} \quad \text{ for } t \to +\infty,$$

see (4.4) of [BMR09]. In particular, if R is large enough, for every $t \geq R$,

$$\sqrt{\chi_g(t)} < 2\sqrt{\widetilde{\chi}_g(t)}.$$

Then we deduce

$$\int_R^r \sqrt{\chi_g} < 2\int_R^r \sqrt{\widetilde{\chi}_g} = \log g(r) - \log g(R) = ar^\alpha \log^\beta r - aR^\alpha \log^\beta R,$$

so that

$$\int_R^r \sqrt{A} - \int_R^r \sqrt{\chi_g} > \int_R^r \sqrt{A} - ar^\alpha \log^\beta r + aR^\alpha \log^\beta R. \tag{3.2.20}$$

Moreover

$$-\frac{1}{2}\log \int_t^{+\infty} \frac{1}{g} \sim -\frac{a}{2} t^\alpha \log^\beta t \quad \text{ for } t \to +\infty,$$

hence for R large enough we have

$$-\frac{1}{2}\log \int_R^{+\infty} \frac{1}{g} < \frac{1}{2}(1 - aR^\alpha \log^\beta R). \tag{3.2.21}$$

Using (3.2.20) and (3.2.21) we deduce the validity of the claim.

Clearly (3.2.17) implies

$$\sqrt{A(t)} \geq \frac{3a}{2}(\alpha \log t + \beta t) t^{\alpha-1} \log^{\beta-1} t = \frac{3a}{2}(t^\alpha \log^\beta t)'.$$

Using the above, the validity of (3.2.19) is implied by the validity of

$$\frac{3a}{2} r^\alpha \log^\beta r > -\frac{1}{2}\log \int_{B_R} \mu(m\lambda - S^\varphi) + \frac{1}{2}. \tag{3.2.22}$$

The right hand side of (3.2.22) the above is monotone decreasing in R, then it is sufficient that (3.2.22) holds for some $R = R_0$ to obtain that it holds also for all $R \geq R_0$. Then we may fix R such that $A \not\equiv 0$ on $[0, R]$, clearly for r large enough we obtain the validity of (3.2.22). Then we can conclude the proof, as in Proposition 3.2.7. □

Chapter 4

Non trivial Einstein-type structures on harmonic-Einstein manifolds

Let $(M, \langle\,,\,\rangle)$ be a Riemannian manifold of dimension $m \geq 2$ that supports a Einstein-type structure, that is,

$$\begin{cases} \mathrm{Ric}^{\varphi} + \frac{1}{2}\mathcal{L}_X\langle\,,\,\rangle - \mu X^\flat \otimes X^\flat = \lambda\langle\,,\,\rangle \\ \tau(\varphi) = d\varphi(X), \end{cases} \tag{4.0.1}$$

for some $\alpha \in \mathbb{R}\setminus\{0\}$, $\lambda, \mu \in \mathcal{C}^\infty(M)$ and $X \in \mathfrak{X}(M)\setminus\{0\}$.

Definition 4.0.2. We say that the Einstein-type structure (4.0.1) is *non-trivial* if $X \neq 0$.

Remark 4.0.3. If (4.0.1) is trivial then

$$\begin{cases} \mathrm{Ric}^{\varphi} = \lambda\langle\,,\,\rangle \\ \tau(\varphi) = 0. \end{cases}$$

In particular

$$\lambda = \frac{S^\varphi}{m}$$

and, assuming λ constant in case $m = 2$ (if $m \geq 3$ it is automatic in view of Proposition 2.1.3), $(M, \langle\,,\,\rangle)$ is harmonic-Einstein.

We shall see that the converse is not true, that is, there exists harmonic-Einstein manifolds that supports non trivial Einstein-type structures.

Definition 4.0.4. We say that the Einstein-type structure (4.0.1) *reduces to a harmonic-Einstein structure* if $(M, \langle\,,\,\rangle)$ is harmonic-Einstein with respect to the map φ and α, that is, if

$$\begin{cases} \mathrm{Ric}^{\varphi} = \frac{S^\varphi}{m}\langle\,,\,\rangle \\ \tau(\varphi) = 0 \end{cases} \tag{4.0.5}$$

holds and further S^φ is constant if $m = 2$.

Notice that the Einstein-type structure (4.0.1) reduces to a harmonic-Einstein structure if and only if

$$\begin{cases} \frac{1}{2}\mathcal{L}_X\langle\,,\,\rangle - \mu X^\flat \otimes X^\flat = \left(\lambda - \frac{S^\varphi}{m}\right)\langle\,,\,\rangle \\ d\varphi(X) = 0, \end{cases} \tag{4.0.6}$$

with S^φ constant if $m = 2$. In literature the first equation of the above is called the almost quasi-Yamabe soliton equation. In the following we say that the vector field X is *vertical* in case the second equation of the above holds, that is, in case X is annihilated by the differential of φ. This terminology is motivated by the case of of submersions.

The aim of this Chapter is to study complete Riemannian manifolds that admits a non-trivial Einstein-type structure that reduces to a harmonic-Einstein type structure. For this purpose we study harmonic-Einstein manifolds endowed with a vector field X such that (4.0.6) holds, for some $\lambda \in \mathcal{C}^\infty(M)$ and $\mu \in \mathbb{R}$. When $\mu = 0$ we are able to study the generic case, sometimes requiring the compactness of M and that X is non-Killing, while when $\mu \neq 0$ our results deal only with the gradient case.

In Section 4.1 we the consequence of the presence of a conformal vertical vector field on the geometry of a harmonic-Einstein manifolds. We begin by generalize the classical result of M. Obata, that says that a compact Einstein manifold that admits a conformal non-homothetic vector fields must be isometric to the sphere, see Proposition 4.1.14, where we assume $\alpha > 0$. Then we investigate the remaining cases, that is, when X is Killing and when X is homothetic non-Killing. In the latter we are able to study the complete case.

In Subsection 4.1.1, applying the result obtained in Section 4.1, we shows triviality results for generic Einstein-type structures. We show that if a compact Riemannian manifold supports a Einstein-type structure with X non Killing and $\alpha > 0$ that reduces to a harmonic-Einstein structure then it is isometric to the sphere, the map φ is constant and λ is non-constant, see Corollary 4.1.37. Moreover we prove that if a complete Riemannian manifold of dimension supports a Einstein-type structure with λ constant and X non-Killing that reduces to a harmonic-Einstein structure is flat and φ is constant, see Proposition 4.1.35.

In Section 4.2 we study non-trivial gradient Einstein-type structures that reduces to harmonic-Einstein structures. Our main results are Theorem 4.2.19 when $\mu = 0$ and Theorem 4.2.25 when $\mu \neq 0$. Those results shows that (eventually assuming that the potential function f has exactly one the critical point), essentially, the only complete Riemannian manifolds admitting a non-trivial gradient Einstein-trype structure that reduces to a harmonic-Einstein structure are space forms. We conclude with Corollary 4.2.32 and Corollary 4.2.33, showing that the only compact harmonic-Einstein manifold that supports a non-trivial Einstein-type structure is the sphere.

4.1 Harmonic-Einstein manifolds and vertical conformal vector fields

In the next Lemma we provide a formula for the Laplacian of the conformal factor of a vertical conformal vector field.

Lemma 4.1.1. *Let $\varphi : (M, \langle\, , \,\rangle) \to (N, \langle\, , \,\rangle_N)$ be a smooth map. Let $X \in \mathfrak{X}(M)$ be a vertical vector field that is conformal, that is,*

$$\begin{cases} \frac{1}{2}\mathcal{L}_X\langle\, , \,\rangle = \eta\langle\, , \,\rangle \\ d\varphi(X) = 0, \end{cases} \tag{4.1.2}$$

for some $\eta \in \mathcal{C}^\infty(M)$. Then

$$\Delta\eta + \frac{S^\varphi}{m-1}\eta + \frac{1}{2(m-1)}\langle\nabla S^\varphi, X\rangle = 0. \tag{4.1.3}$$

Proof. We rewrite the first equation of (4.1.2) in local form with respect to an orthonormal coframe as

$$X^i_j + X^j_i = 2\eta\delta_{ij}. \tag{4.1.4}$$

Observe that, contracting (4.1.4) against Ric^φ we get

$$R^\varphi_{ij}X^i_j + R^\varphi_{ij}X^j_i = 2S^\varphi\eta,$$

that is, since Ric^φ is symmetric

$$R^\varphi_{ij} X^i_j = \eta S^\varphi. \tag{4.1.5}$$

Moreover from Schur's identity (1.2.26) and the second equation of (4.1.2), that in local form is given by $\varphi^a_i X^i = 0$,

$$R^\varphi_{ji,j} X^i = \frac{1}{2} S^\varphi_i X^i. \tag{4.1.6}$$

Observe that, from the second equation of (4.1.2) and the definition (1.2.2) we have

$$R_{ij} X^j = R^\varphi_{ij} X^j.$$

Using the commutation relation (3.1.1) and the above we get

$$\begin{aligned}\Delta(\mathrm{div}(X)) =&(X^i_i)_{jj} = (X^i_{ij})_j = (X^i_{ji} + R_{kiij} X^k)_j\\ =&X^i_{jij} - (R_{ij} X^i)_j\\ =&X^i_{jij} - (R^\varphi_{ij} X^i)_j\\ =&(X^i_j)_{ij} - R^\varphi_{ij,j} X^i - R^\varphi_{ij} X^i_j.\end{aligned}$$

With the aid of (4.1.4), (4.1.6) and (4.1.5) the latter can be rewritten in the form

$$\begin{aligned}\Delta(\mathrm{div}(X)) =&(-X^j_i + 2\eta\delta_{ij})_{ij} - \frac{1}{2} S^\varphi_i X^i - S^\varphi \eta\\ =&- X^j_{iij} + 2\Delta\eta - \frac{1}{2} S^\varphi_i X^i - S^\varphi \eta.\end{aligned}$$

Using the commutation relation (3.1.2) we obtain

$$X^j_{iij} = X^j_{iji} + R_{kjij} X^k_i + R^k_{iij} X^j_k = X^j_{iji} + R_{ki} X^k_i - R_{kj} X^j_k = X^j_{iji},$$

and inserting into the last above we infer

$$\Delta(\mathrm{div}(X)) = -(X^j_{ij})_i + 2\Delta\eta - \frac{1}{2} S^\varphi_i X^i - S^\varphi \eta.$$

Using once again (3.1.1), (4.1.4), (4.1.6) and (4.1.5) and proceeding as above

$$\begin{aligned}\Delta(\mathrm{div}(X)) =&- (X^j_{ji} + R_{kjij} X^k)_i + 2\Delta\eta - \frac{1}{2} S^\varphi_i X^i - S^\varphi \eta\\ =&- X^j_{jii} - (R_{ki} X^k)_i + 2\Delta\eta - \frac{1}{2} S^\varphi_i X^i - S^\varphi \eta\\ =&- X^j_{jii} - (R^\varphi_{ki} X^k)_i + 2\Delta\eta - \frac{1}{2} S^\varphi_i X^i - S^\varphi \eta\\ =&- \Delta\mathrm{div}(X) - R^\varphi_{ki,i} X^k - R^\varphi_{ki} X^k_i + 2\Delta\eta - \frac{1}{2} S^\varphi_i X^i - S^\varphi \eta\\ =&- \Delta\mathrm{div}(X) - \frac{1}{2} S^\varphi_i X^i - S^\varphi \eta + 2\Delta\eta - \frac{1}{2} S^\varphi_i X^i - S^\varphi \eta\\ =&- \Delta\mathrm{div}(X) - S^\varphi_i X^i - 2S^\varphi \eta + 2\Delta\eta,\end{aligned}$$

that is,

$$\Delta(\mathrm{div}(X)) = \Delta\eta - \frac{1}{2} S^\varphi_i X^i - S^\varphi \eta.$$

Observe that, taking the trace of the first equation of (4.1.2),

$$\mathrm{div}(X) = m\eta,$$

so that from the above we obtain

$$\Delta\eta = \frac{1}{m}\left(\Delta\eta - \frac{1}{2}\langle\nabla S^{\varphi}, X\rangle - S^{\varphi}\eta\right),$$

that immediately gives (4.1.3). ☐

Our aim now is to extend the well known fact, due to M. Obata, that a compact Einstein manifold that admits a non-Killing conformal vector field is isometric to a Euclidean sphere. To do so we first recall and prove the next, well known,

Theorem 4.1.7 (Licherowicz-Obata). *Let $(M, \langle\, ,\, \rangle)$ be a compact Riemannian manifold of dimension m satisfying for some $\kappa \in \mathbb{R}$*

$$Ric \geq (m-1)\kappa\langle\, ,\, \rangle. \tag{4.1.8}$$

Let $u \in \mathcal{C}^{\infty}(M)$ be a non-constant eigenfunction of $-\Delta$ relative to the eigenvalue $\lambda \in \mathbb{R}$, that is,

$$\Delta u + \lambda u = 0. \tag{4.1.9}$$

Then

$$\lambda \geq m\kappa, \tag{4.1.10}$$

equality holding if and only if $(M, \langle\, ,\, \rangle)$ is isometric to a Euclidean sphere $\mathbb{S}^m$ of $\mathbb{R}^{m+1}$ of constant sectional curvature $\kappa > 0$ (hence the equality holds in (4.1.8)).

Proof. From Newton's inequality and (4.1.9)

$$|\mathrm{Hess}(u)|^2 \geq \frac{1}{m}(\Delta u)^2 = -\frac{\lambda}{m}u\Delta u,$$

the equality holds on M if and only if $\mathrm{Hess}(u) = \eta\langle\, ,\, \rangle$ for some $\eta \in \mathcal{C}^{\infty}(M)$. Recall Bochner formula

$$\frac{1}{2}\Delta(|\nabla u|^2) = |\mathrm{Hess}(u)|^2 + \langle\nabla(\Delta u), \nabla u\rangle + \mathrm{Ric}(\nabla u, \nabla u),$$

so that, using the above, (4.1.9) and (4.1.8) we get

$$\frac{1}{2}\Delta(|\nabla u|^2) \geq -\frac{\lambda}{m}u\Delta u - \lambda|\nabla u|^2 + (m-1)\kappa|\nabla u|^2.$$

Observe that

$$\mathrm{div}(u\nabla u) = u\Delta u + |\nabla u|^2, \tag{4.1.11}$$

hence from the above we conclude the validity of

$$\frac{1}{2}\Delta(|\nabla u|^2) \geq -\frac{\lambda}{m}\mathrm{div}(u\nabla u) + (m-1)\left(\kappa - \frac{\lambda}{m}\right)|\nabla u|^2. \tag{4.1.12}$$

Integrating the above on M, using the divergence theorem,

$$0 \geq (m-1)\left(\kappa - \frac{\lambda}{m}\right)\int_M |\nabla u|^2$$

and since u is non constant we infer

$$\kappa - \frac{\lambda}{m} \leq 0,$$

that is, (4.1.10).

Suppose that the equality holds in (4.1.10), then from (4.1.12) we get

$$\frac{1}{2}\Delta(|\nabla u|^2) \geq -\kappa\mathrm{div}(u\nabla u) = -\frac{\kappa}{2}\Delta(u^2).$$

Thus the function

$$|\nabla u|^2 + \kappa u^2$$

is subharmonic on M so that, since M is compact, is constant. In particular

$$\frac{1}{2}\Delta(|\nabla u|^2) = -\frac{\kappa}{2}\Delta(u^2) = -\kappa \mathrm{div}(u\nabla u),$$

and by plugging into Bochner formula, using (4.1.9) with $\lambda = m\kappa$, (4.1.8) and (4.1.11), we conclude

$$\begin{aligned}|\mathrm{Hess}(u)|^2 =& \frac{1}{2}\Delta(|\nabla u|^2) - \langle \nabla(\Delta u), \nabla u\rangle - \mathrm{Ric}(\nabla u, \nabla u)\\ =& -\kappa \mathrm{div}(u\nabla u) + m\kappa|\nabla u|^2 - \mathrm{Ric}(\nabla u, \nabla u)\\ \leq& -\kappa \mathrm{div}(u\nabla u) + m\kappa|\nabla u|^2 - (m-1)\kappa|\nabla u|^2\\ =& -\kappa u\Delta u.\end{aligned}$$

But Newton's inequality is given by, since $\lambda = m\kappa$,

$$|\mathrm{Hess}(u)|^2 \geq \frac{\lambda}{m}u\Delta u = -\kappa u\Delta u,$$

thus it is saturated on M as a consequence of the above inequality. Then

$$\mathrm{Hess}(u) = \eta\langle\, ,\, \rangle$$

for some $\eta \in \mathcal{C}^\infty(M)$. Since $\Delta u = -m\kappa u$, taking the trace of the above we obtain

$$\eta = -\kappa u.$$

But then u is a solution of

$$\mathrm{Hess}(u) + \kappa u\langle\, ,\, \rangle = 0.$$

Since M is compact $\kappa > 0$, Riemannian manifolds that admits non trivial solutions to this equation has been characterized by Obata in [O]: $(M, \langle\, ,\, \rangle)$ is isometric to the sphere immersed in $\mathbb{R}^{m+1}$ of constant sectional curvature given by κ. As a consequence the equality holds in (4.1.8). ☐

Remark 4.1.13. On the m-dimensional sphere of constant sectional curvature κ functions η (except the identically zero function) such that

$$\Delta\eta + m\kappa\eta = 0$$

are called *first order spherical harmonics*. Together with the zero function they form the eigenspace relative to the first positive eigenvalue of $-\Delta$, which has dimension $m+1$.

With the aid of formula (4.1.3) and Lichnerowicz-Obata Theorem we are able to prove

Proposition 4.1.14. *Let $(M, \langle\, ,\, \rangle)$ be a compact, harmonic-Einstein manifold of dimension $m \geq 2$ with respect to some $\alpha > 0$ and $\varphi : M \to (N, \langle\, ,\, \rangle_N)$, that is,*

$$\begin{cases} Ric^\varphi = \dfrac{S^\varphi}{m}\langle\, ,\, \rangle \\ \tau(\varphi) = 0. \end{cases} \tag{4.1.15}$$

If there exists a conformal and non-Killing, vector field $X \in \mathfrak{X}(M)$ such that

$$d\varphi(X) = 0, \tag{4.1.16}$$

then φ is constant and $(M, \langle\, ,\, \rangle)$ is isometric to a Euclidean sphere $\mathbb{S}^m$ in $\mathbb{R}^{m+1}$ of constant sectional curvature

$$\kappa := \frac{S^\varphi}{m(m-1)} > 0. \tag{4.1.17}$$

Moreover, the conformal factor η of X is a first order spherical harmonic.

Proof. Let $X \in \mathfrak{X}(M)$ be a non-Killing conformal vector field with conformal factor $\eta \in \mathcal{C}^\infty(M)$, $\eta \not\equiv 0$, that is,

$$\mathcal{L}_X \langle\, , \,\rangle = 2\eta \langle\, , \,\rangle. \tag{4.1.18}$$

Since S^φ is constant, by Proposition 2.1.3 for $m \geq 3$ and by definition of harmonic-Einstein manifold for $m = 2$, formula (4.1.3) becomes

$$\Delta\eta + \frac{S^\varphi}{m-1}\eta = 0. \tag{4.1.19}$$

Multiplying by η the above and integrating by parts we obtain

$$\int_M |\nabla\eta|^2 = \frac{S^\varphi}{m-1}\int_M \eta^2,$$

and thus, since η is non-constant, $S^\varphi \geq 0$. Suppose by contradiction that $S^\varphi = 0$, then η is harmonic on the compact Riemannian manifold $(M, \langle\, , \,\rangle)$, hence it is constant. Taking the trace of (4.1.18) we get

$$\mathrm{div}(X) = m\eta$$

and since η is constant, integrating over M, with the aid of the divergence theorem, we deduce also that $\eta = 0$, contradiction. We have therefore proved that $S^\varphi > 0$. From the first equation in (4.1.15), $\alpha > 0$ and the fact that $\langle\, , \,\rangle_N$ is a Riemannian metric on N we obtain

$$\mathrm{Ric} \geq \frac{S^\varphi}{m}\langle\, , \,\rangle. \tag{4.1.20}$$

Since X is not Killing, η does not vanish identically on M and from (4.1.19) and $S^\varphi > 0$ we deduce that η cannot be a constant. The validity of (4.1.19) and (4.1.20) allows us to apply Lichnerowicz-Obata Theorem (see Theorem 4.1.7) to deduce that $(M, \langle\, , \,\rangle)$ is isometric to a Euclidean sphere $\mathbb{S}^m$ of $\mathbb{R}^{m+1}$ of constant sectional curvature κ given by (4.1.17). We now observe that, from the first equation in (4.1.15) and the fact that we have now equality in (4.1.20), because of (4.1.17) and the isometry, we have

$$\frac{S^\varphi}{m}\langle\, , \,\rangle = \mathrm{Ric}^\varphi = \mathrm{Ric} - \alpha\varphi^*\langle\, , \,\rangle_N = \frac{S^\varphi}{m}\langle\, , \,\rangle - \alpha\varphi^*\langle\, , \,\rangle_N,$$

and since $\alpha \neq 0$

$$\varphi^*\langle\, , \,\rangle_N = 0.$$

Thus φ is constant. Notice that now (4.1.19) can be rewritten as

$$\Delta\eta + m\kappa\eta = 0,$$

hence η is a first order spherical harmonic. □

Now we deal with harmonic-Einstein manifolds admitting a vertical Killing vector field. The connection between vertical Killing vector fields and the sign of φ-Ricci is, essentially, the same as the one between Killing vector fields and the sign of Ricci. To motivate our assertion, in the next Proposition we extend the classic result of Bochner that a compact Riemannian manifold with negative Ricci curvature does not admit a non-trivial Killing vector field.

Proposition 4.1.21. *Let $(M, \langle\, , \,\rangle)$ be a compact Riemannian manifold, $\varphi : M \to (N, \langle\, , \,\rangle_N)$ be a smooth map and $\alpha \in \mathbb{R} \setminus \{0\}$. Let X be a vertical Killing vector field, that is,* (4.1.33) *holds. If $\mathrm{Ric}^\varphi \leq 0$ then X is parallel. Further, if Ric^φ is strictly negative at a point $x_0 \in M$, then $X = 0$.*

Proof. The proof is a trivial application of the Bochner formula for vector fields

$$\frac{1}{2}\Delta|X|^2 = |\nabla X|^2 + \mathrm{div}(\mathcal{L}_X\langle\, , \,\rangle)(X) - \langle\nabla \mathrm{div}(X), X\rangle - \mathrm{Ric}(X, X), \tag{4.1.22}$$

see for instance Lemma 8.1 of [AMR]. Since X is Killing it is also divergence free. Moreover

$$\mathrm{Ric}(X,X) = \mathrm{Ric}^{\varphi}(X,X) + \alpha|d\varphi(X)|^2,$$

using also that X is vertical from the Bochner formula above we get

$$\frac{1}{2}\Delta|X|^2 = |\nabla X|^2 - \mathrm{Ric}^{\varphi}(X,X). \tag{4.1.23}$$

Integrating on M, using the divergence theorem we obtain

$$\int_M |\nabla X|^2 = \int_M \mathrm{Ric}^{\varphi}(X,X).$$

Since $\mathrm{Ric}^{\varphi} \leq 0$ we deduce that X is parallel. As a consequence $|X|^2$ is constant. Then (4.1.23) reads

$$\mathrm{Ric}^{\varphi}(X,X) = 0. \tag{4.1.24}$$

Assume $\mathrm{Ric}^{\varphi} < 0$ at x_0, then $X = 0$ at x_0. Since $|X|^2$ is constant then $X = 0$ on M. ☐

Remark 4.1.25. The thesis of the Proposition above holds also if either X is a vertical homothetic vector field or, in case $m = 2$, X is a vertical conformal vector field. Indeed, let $X \in \mathfrak{X}(M)$ be such that

$$\begin{cases} \frac{1}{2}\mathcal{L}_X\langle\, , \,\rangle = \eta\langle\, , \,\rangle \\ d\varphi(X) = 0 \end{cases}$$

for some $\eta \in \mathcal{C}^{\infty}(M)$. Then, using the Bochner formula (4.1.22) we obtain

$$\frac{1}{2}\Delta|X|^2 = |\nabla X|^2 - (m-2)\langle\nabla\eta, X\rangle - \mathrm{Ric}^{\varphi}(X,X),$$

that is (4.1.23) under the assumptions above.

From Proposition 4.1.21 we immediately get

Corollary 4.1.26. *Let $(M, \langle\, , \,\rangle)$ be a compact, harmonic-Einstein manifold of dimension $m \geq 2$ with respect to $\alpha \in \mathbb{R} \setminus \{0\}$ and $\varphi : M \to (N, \langle\, , \,\rangle_N)$, that is, (4.1.15) holds. If there exists a vertical Killing vector field $X \in \mathfrak{X}(M) \setminus \{0\}$ then $S^{\varphi} \geq 0$. If $S^{\varphi} = 0$ then X is parallel.*

Proof. Assume by contradiction $S^{\varphi} < 0$. From Proposition 4.1.21 if $S^{\varphi} < 0$ then $X = 0$, that is a contradiction. If $S^{\varphi} = 0$, Proposition 4.1.21, then X is parallel. ☐

4.1.1 Generic Einstein-type structures

In this subsection we apply the results on vertical conformal vector fields obtained above in Section 4.1 to study non trivial Einstein-type structures that reduces to harmonic-Einstein structures.

Let $(M, \langle\, , \,\rangle)$ be a complete Riemannian manifold of dimension $m \geq 2$ that supports a non trivial Einstein-type structure as (4.0.1) for some $\lambda \in \mathcal{C}^{\infty}(M)$, $\mu \in \mathbb{R}$, $X \in \mathfrak{X}(M)$, $\varphi : M \to (N, \langle\, , \,\rangle_N)$ smooth and $\alpha \in \mathbb{R} \setminus \{0\}$. Recall that, if (4.0.1) reduces to a harmonic-Einstein structure, then $(M, \langle\, , \,\rangle)$ is a harmonic-Einstein manifold and X satisfies (4.0.6). In order to produce interesting results we shall restrict to the case $\mu = 0$. The motivation is illustrated in the next Remark.

Remark 4.1.27. Let $(M, \langle\, , \,\rangle)$ be a Riemannian manifold that supports a non trivial Einstein-type structure as (4.0.1) for some $\alpha \in \mathbb{R} \setminus \{0\}$, $\lambda, \mu \in \mathcal{C}^{\infty}(M)$ and $X \in \mathfrak{X}(M) \setminus \{0\}$. Assume that the structure (4.0.1) reduces to a harmonic-Einstein structure. If X is conformal then

$$X = 0 \quad \text{on } \{x \in M : \mu(x) \neq 0\}. \tag{4.1.28}$$

Indeed, since (4.0.1) reduces to a harmonic-Einstein structure, (4.0.6) holds and combining its first equation with the fact that, since X is conformal, there exists $\eta \in \mathcal{C}^\infty(M)$ such that

$$\frac{1}{2}\mathcal{L}_X\langle\, ,\, \rangle = \eta\langle\, ,\, \rangle,$$

we get

$$-\mu X^\flat \otimes X^\flat = \left(\lambda - \eta - \frac{S^\varphi}{m}\right)\langle\, ,\, \rangle. \tag{4.1.29}$$

Taking the trace of the above we have

$$-\mu|X|^2 = m\lambda - m\eta - S^\varphi. \tag{4.1.30}$$

From (4.1.29) we also get

$$-\mu X^\flat \otimes X^\flat(X,X) = \left(\lambda - \eta - \frac{S^\varphi}{m}\right)\langle X, X\rangle,$$

that is,

$$-\mu|X|^4 = \left(\lambda - \eta - \frac{S^\varphi}{m}\right)|X|^2.$$

Combining the above with (4.1.30) we conclude

$$(m-1)\mu|X|^4 = 0,$$

that implies (4.1.28). Notice that, if μ is constant then $\mu = 0$, since the Einstein-type structure is non trivial.

Assume then that $\mu = 0$. Hence (4.0.6) reduces to

$$\begin{cases} \frac{1}{2}\mathcal{L}_X\langle\, ,\, \rangle = \left(\lambda - \frac{S^\varphi}{m}\right)\langle\, ,\, \rangle \\ d\varphi(X) = 0, \end{cases} \tag{4.1.31}$$

that is, X is a vertical conformal vector field.

Remark 4.1.32. In the assumptions above, X is a vertical Killing vector field, that is,

$$\begin{cases} \frac{1}{2}\mathcal{L}_X\langle\, ,\, \rangle = 0 \\ d\varphi(X) = 0. \end{cases} \tag{4.1.33}$$

if and only if

$$\lambda = \frac{S^\varphi}{m}. \tag{4.1.34}$$

This may happen, since $(M, \langle\, ,\, \rangle)$ has constant φ-scalar curvature, only if λ is constant and, if M is compact, using Corollary 4.1.26, only in case $\lambda \geq 0$.

When λ is constant and X is non-Killing we have that X is a vertical homothetic vector field. In the complete case, we have

Proposition 4.1.35. *Let $(M, \langle\, ,\, \rangle)$ be a complete Riemannian manifold of dimension $m \geq 2$ that supports a Einstein-type structure as* (4.0.1) *for some $\lambda \in \mathbb{R}$, $\mu = 0$, $X \in \mathfrak{X}(M)$ non-Killing, where $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ and $\alpha \in \mathbb{R}\setminus\{0\}$. Assume that* (4.0.1) *reduces to a harmonic-Einstein structure. Then $(M, \langle\, ,\, \rangle)$ is flat, φ is constant and*

$$\lambda \neq \frac{S^\varphi}{m}. \tag{4.1.36}$$

Proof. Our hypothesis implies that (4.1.31) holds. Since X is non-Killing we have, from the first equation of (4.1.31), the validity of (4.1.36). Then X is a vertical homothetic, non-Killing vector field. By a result known to Tashiro, Theorem 4.1 of [T], if a complete Riemannian manifold $(M, \langle\, ,\, \rangle)$ admits a homothetic non-Killing vector field then $(M, \langle\, ,\, \rangle)$ is flat. Then $(M, \langle\, ,\, \rangle)$ is flat and thus, since $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein, φ is a weakly conformal map. The φ-scalar curvature of $(M, \langle\, ,\, \rangle)$ is constant, because $(M, \langle\, ,\, \rangle)$ harmonic-Einstein, but since $(M, \langle\, ,\, \rangle)$ is flat $S = 0$ and thus $S^\varphi = -\alpha|d\varphi|^2$. In conclusion $|d\varphi|^2$ and thus φ is homothetic. Assume by contradiction that φ is non-constant. Then, since φ is homothetic, there exists $a \in \mathbb{R}^+$ such that

$$\varphi^*\langle\, ,\, \rangle = \frac{a}{m}\langle\, ,\, \rangle.$$

Evaluating the above along X we get

$$|d\varphi(X)|^2 = \frac{a}{m}|X|^2$$

and from the second equation of (4.1.31) we obtain that $|X|^2 = 0$ and thus $X = 0$, that is a contradiction, since X is not Killing. □

When λ is a generic function, as an easy application of Proposition 4.1.14, we can deal with the compact case.

Corollary 4.1.37. *Let $(M, \langle\, ,\, \rangle)$ be a compact Riemannian manifold of dimension $m \geq 2$ that supports a Einstein-type structure as* (4.0.1) *for some $\lambda \in \mathcal{C}^\infty(M)$, $\mu = 0$, $X \in \mathfrak{X}(M)$ non-Killing, $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ smooth and $\alpha > 0$. Assume that* (4.0.1) *reduces to a harmonic-Einstein structure. Then φ is constant and $(M, \langle\, ,\, \rangle)$ is isometric to the sphere of constant sectional curvature*

$$\kappa = \frac{S^\varphi}{m(m-1)}$$

immersed in $\mathbb{R}^{m+1}$. Moreover, up to a translation, λ is a first order spherical harmonic (in particular, λ is non-constant).

Proof. Our hypothesis implies X is a vertical conformal, non-Killing vector field, that is, (4.1.31) holds. As we seen in Proposition 4.1.14, if a compact harmonic-Einstein manifold of dimension $m \geq 2$ with $\alpha > 0$ supports a vertical conformal, non-Killing vector field then it is isometric to a sphere immersed in $\mathbb{R}^{m+1}$ and φ is constant. As a consequence $S^\varphi = S$ is a positive constant. Moreover the conformal factor

$$\eta = \lambda - \frac{S^\varphi}{m}$$

of X is a first order spherical harmonic. □

4.2 Gradient Einstein-type structures

In the gradient case the Einstein-type structure (4.0.1) is given by

$$\begin{cases} \mathrm{Ric}^\varphi + \mathrm{Hess}(f) - \mu df \otimes df = \lambda\langle\, ,\, \rangle \\ \tau(\varphi) = d\varphi(\nabla f), \end{cases} \tag{4.2.1}$$

where $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ is smooth, $f, \lambda \in \mathcal{C}^\infty(M)$, $\mu \in \mathbb{R}$ and $\alpha \in \mathbb{R} \setminus \{0\}$. We know that the structure (4.2.1) reduces to a harmonic-Einstein structure if and only if (4.0.6) holds, that in our setting is given by,

$$\begin{cases} \mathrm{Hess}(f) - \mu df \otimes df = \left(\lambda - \dfrac{S^\varphi}{m}\right)\langle\, ,\, \rangle \\ d\varphi(\nabla f) = 0, \end{cases} \tag{4.2.2}$$

Moreover, the structure is non trivial provided f non-constant.

Remark 4.2.3. For $\mu = 0$ (4.2.2) reads

$$\begin{cases} \mathrm{Hess}(f) = \left(\lambda - \frac{S^\varphi}{m}\right)\langle\,,\,\rangle \\ d\varphi(\nabla f) = 0, \end{cases} \tag{4.2.4}$$

that is, ∇f is a vertical conformal vector field. For $\mu \neq 0$ we set

$$u := e^{-\mu f} \tag{4.2.5}$$

Clearly $u > 0$ on M,

$$\nabla u = -\mu u \nabla f,$$

$$\mathrm{Hess}(u) = -\mu u(\mathrm{Hess}(f) - \mu df \otimes df),$$

and f is non-constant if and only if u is non-constant. Then (4.2.2) is equivalent to

$$\begin{cases} \mathrm{Hess}(u) = -\mu u \left(\lambda - \frac{S^\varphi}{m}\right)\langle\,,\,\rangle \\ d\varphi(\nabla u) = 0, \end{cases} \tag{4.2.6}$$

that is, ∇u is a vertical conformal vector field.

In both the cases above we have the existence of a function $v \in \mathcal{C}^\infty(M)$ such that ∇v is vertical and conformal. In the following Proposition we show that the conformal factor of a gradient vertical conformal vector field on a harmonic-Einstein manifold assumes a particular form.

Proposition 4.2.7. *Let $(M, \langle\,,\,\rangle)$ be a Riemannian manifold of dimension $m \geq 2$ such that T^φ is zero, where T^φ is the traceless part of the φ-Ricci tensor, and S^φ is constant. Let $v \in \mathcal{C}^\infty(M)$ be such that ∇v is a vertical conformal vector field, that is,*

$$\begin{cases} \mathit{Hess}(v) = \eta\langle\,,\,\rangle \\ d\varphi(\nabla v) = 0 \end{cases} \tag{4.2.8}$$

for some $\eta \in \mathcal{C}^\infty(M)$. Then there exists $\zeta \in \mathbb{R}$ such that

$$\mathit{Hess}(v) + \frac{S^\varphi}{m(m-1)} v\langle\,,\,\rangle = \zeta\langle\,,\,\rangle. \tag{4.2.9}$$

If $S^\varphi \neq 0$ then

$$v = \frac{m(m-1)}{S^\varphi}(\zeta - \eta). \tag{4.2.10}$$

As a consequence if ∇v is homothetic and non-Killing, that is, if $\eta \in \mathbb{R} \setminus \{0\}$, then $S^\varphi = 0$. If M is compact and v is non-constant then $S^\varphi > 0$.

Proof. Tracing the first equation of (4.2.8) we deduce

$$\frac{\Delta v}{m} = \eta, \tag{4.2.11}$$

Taking the divergence of the first equation of (4.2.8) and using (4.2.11) we deduce

$$v_{ijj} = (\eta\delta_{ij})_j = \eta_i = \left(\frac{\Delta v}{m}\right)_i. \tag{4.2.12}$$

On the other hand, commutating the last two indexes of v_{jij}, using the definition of Ric^{φ}, the second equation of (4.2.8), that $T^{\varphi}=0$ and that S^{φ} is constant we get

$$\begin{aligned} v_{jij} =& v_{jji} + R^{k}_{jij} v_k \\ =& (\Delta v)_i + R_{ij} v_j \\ =& (\Delta v)_i + R^{\varphi}_{ij} v_j + \alpha \varphi^a_i \varphi^a_j v_j \\ =& (\Delta v)_i + \frac{S^{\varphi}}{m} v_i \\ =& \left(\Delta v + \frac{S^{\varphi}}{m} v\right)_i , \end{aligned}$$

Comparing it with (4.2.12) we deduce, since M is connected, there exists $\zeta \in \mathbb{R}$ such that

$$\Delta v + \frac{S^{\varphi}}{m} v = \frac{\Delta v}{m} + (m-1)\zeta,$$

or equivalently,

$$\frac{\Delta v}{m} = \zeta - \frac{S^{\varphi}}{m(m-1)} v.$$

Since the first equation of (4.2.8) can be written as

$$\mathrm{Hess}(v) = \frac{\Delta v}{m} \langle\, , \,\rangle,$$

from the above we deduce the validity of (4.2.9).

If we suppose that $S^{\varphi} \neq 0$ then we can define

$$\bar{v} := v - \frac{m(m-1)\zeta}{S^{\varphi}}.$$

Notice that, using (4.2.9),

$$\mathrm{Hess}(\bar{v}) + \frac{S^{\varphi}}{m(m-1)} \bar{v} \langle\, , \,\rangle = 0. \tag{4.2.13}$$

Moreover, from the first equation of (4.2.8) and the above,

$$\eta \langle\, , \,\rangle = \mathrm{Hess}(v) = \mathrm{Hess}(\bar{v}) = -\frac{S^{\varphi}}{m(m-1)} \bar{v} \langle\, , \,\rangle,$$

that implies

$$-\frac{S^{\varphi}}{m(m-1)} \bar{v} = \eta.$$

Hence, recalling the definition of $\bar{v}$,

$$v - \frac{m(m-1)\zeta}{S^{\varphi}} = \bar{v} = -\frac{m(m-1)\eta}{S^{\varphi}},$$

that is (4.2.10).

Suppose that η is constant. Assume by contradiction $S^{\varphi} \neq 0$. From (4.2.10), v is constant, that is a contradiction since ∇v is non-Killing.

Assume M is compact and v is non-constant. If, by contradiction, $S^{\varphi} = 0$ by tracing (4.2.9) we obtain $\Delta v = m\zeta$, hence v is constant, that is a contradiction. Then $S^{\varphi} \neq 0$. Taking the trace of (4.2.13) and multiplying it by $\bar{v}$, integrating and using the divergence theorem we get

$$\int_M |\nabla \bar{v}|^2 = \frac{S^{\varphi}}{m-1} \int_M \bar{v}^2.$$

Since v is non-constant also $\bar{v}$ is non constant and thus $S^{\varphi} > 0$. □

Complete Riemannian manifolds that admits a non trivial solution $v \in \mathcal{C}^\infty(M)$ of

$$\mathrm{Hess}(v) + \kappa v \langle\, , \,\rangle = \zeta \langle\, , \,\rangle, \tag{4.2.14}$$

for some $\kappa \in \mathbb{R}$ and some $\zeta \in \mathbb{R} \setminus \{0\}$, have been studied from the Japanese school between the 60's and the 80's. Notice that, if $\kappa \neq 0$, by setting

$$\bar{v} = v + \frac{\zeta}{\kappa},$$

then $\bar{v}$ solves

$$\mathrm{Hess}(\bar{v}) + \kappa \bar{v} \langle\, , \,\rangle = 0.$$

Unifying in a single statement the results of M. Obata, Y. Tashiro and M. Kanai obtained, respectively, in [O], [T] and [K] (see also Theorem 2.10 of [MRS] and Theorem 8.5 of [AMR] for more modern and readable proof), we obtain

Theorem 4.2.15. *Let $(M, \langle\, , \,\rangle)$ be a complete Riemannian manifold of dimension m. There exists a non-trivial solution $v \in \mathcal{C}^\infty(M)$ of (4.2.14) for some $\kappa \in \mathbb{R}$ and some $\zeta \in \mathbb{R} \setminus \{0\}$ if and only if, according to the sign of κ,*

i) $(M, \langle\, , \,\rangle)$ is isometric to the sphere of constant sectional curvature κ immersed in $\mathbb{R}^{m+1}$, in case $\kappa > 0$. Moreover, up to a translation, v is a first order spherical harmonic.

ii) $(M, \langle\, , \,\rangle)$ is isometric to the Euclidean space of dimension m, in case $\kappa = 0$. Moreover

$$v(x) = \frac{\zeta}{2}|x|^2 + \langle b, x\rangle + c$$

for some $b \in \mathbb{R}^m$ and $c \in \mathbb{R}$.

iii) $(M, \langle\, , \,\rangle)$ is isometric to the hyperbolic space of constant sectional curvature κ and of dimension m, in case $\kappa < 0$ and v has precisely one critical point. Moreover, up to a translation, v solves $\Delta v + m\kappa v = 0$.

In order to obtain a complete viewpoint on gradient vertical conformal vector fields the only remaining case to deal is the one where the vector field is Killing.

Remark 4.2.16. Let $(M, \langle\, , \,\rangle)$ be a complete Riemannian manifold and $\varphi : (M, \langle\, , \,\rangle) \to (N, \langle\, , \,\rangle_N)$ a smooth map. Assume $f \in \mathcal{C}^\infty(M)$ is a non-trivial affine function such that ∇f is vertical, that is, f is non constant and satisfies

$$\begin{cases} \mathrm{Hess}(f) = 0 \\ d\varphi(\nabla f) = 0. \end{cases} \tag{4.2.17}$$

The first equation of (4.2.17), via the Innami splitting theorem [I], yields that $(M, \langle\, , \,\rangle)$ splits as a Riemannian product $\mathbb{R} \times \Sigma$, where Σ is any level set of f, that is a totally geodesic hypersurface of $(M, \langle\, , \,\rangle)$ when endowed with the induced metric $\langle\, , \,\rangle_\Sigma := \imath^* \langle\, , \,\rangle$, where $\imath : \Sigma \to M$ is the inclusion.

Moreover, identifying M with $\mathbb{R} \times \Sigma$, the second equation of (4.2.17) implies that

$$\varphi = \psi \circ \pi_\Sigma, \tag{4.2.18}$$

where $\psi := \varphi|_\Sigma = \varphi \circ \imath$ and $\pi_\Sigma : \mathbb{R} \times \Sigma \to \Sigma$ is the canonical projection.

In order to obtain (4.2.18) we quickly review how the isometry of the Innami splitting theorem is constructed. From the first equation of (4.2.17) we have that $|\nabla f|^2$ is constant on M, and since f is non-constant, $|\nabla f| = a > 0$. We set $\Sigma := f^{-1}(\{b\})$ for some $b \in \mathbb{R}$ (such that $\Sigma \neq \varnothing$). Since $\nabla f \neq 0$ on Σ, then Σ is a smooth hypersurface of M. We set

$$Y := \frac{\nabla f}{a},$$

then Y is a complete vector field defined on Σ that is normal to Σ and that defines an orientation. Moreover, the flow of the vector field Y

$$\phi : \mathbb{R} \times \Sigma \to M$$

coincide with the normal exponential map to Σ and is bijective. Finally, the signed distance function from Σ is given by

$$\frac{f}{a},$$

hence identifying M with $\mathbb{R}\times\Sigma$ we get

$$f(t,x)=at+b.$$

To obtain that ϕ is an isometry is sufficient to endow Σ with the induced metric $\langle\, ,\,\rangle_\Sigma$, $\mathbb{R}$ with the Euclidean metric $dt\otimes dt$ (so that the gradient of f has norm a) and consider the product metric on $\mathbb{R}\times\Sigma$.

Finally, identifying M with $\mathbb{R}\times\Sigma$ we have, via the second equation of (4.2.17), that

$$a d\varphi\left(\frac{d}{dt}\right)=0,$$

hence φ is independent from $t\in\mathbb{R}$, that is, (4.2.18) holds. This means that the value of φ is conserved along the flow of Y.

In conclusion, up to isometry, the only complete Riemannian manifolds $(M,\langle\, ,\,\rangle)$ endowed with a smooth map $\varphi : M\to(N,\langle\, ,\,\rangle_N)$ and a non constant function $f\in\mathcal{C}^\infty(M)$ such that (4.2.17) holds are given by Riemannian products $\mathbb{R}\times\Sigma$ and φ corresponds to $\psi\circ\pi_\Sigma$, where $\psi:\Sigma\to(N,\langle\, ,\,\rangle_N)$ is a smooth map and $\pi_\Sigma:\mathbb{R}\times\Sigma\to\Sigma$ is the canonical projection.

In the next Theorem we deal with the complete case, when $\mu=0$.

Theorem 4.2.19. *Let $(M,\langle\, ,\,\rangle)$ be a complete Riemannian manifold of dimension $m\geq 2$ that supports a non-trivial gradient Einstein-type structure as* (4.2.1) *for $\mu=0$, $\varphi:M\to(N,\langle\, ,\,\rangle_N)$ smooth, $\alpha\in\mathbb{R}\setminus\{0\}$ and some $f,\lambda\in\mathcal{C}^\infty(M)$. Assume that* (4.2.1) *reduces to a harmonic-Einstein structure. We set*

$$\kappa:=\frac{S^\varphi}{m(m-1)}. \tag{4.2.20}$$

Then κ is constant and

i) if $\kappa>0$ then φ is constant and $(M,\langle\, ,\,\rangle)$ is isometric to the sphere of constant sectional curvature κ immersed in $\mathbb{R}^{m+1}$. Moreover, up to a translation, f is a first order spherical harmonic and $\lambda+\kappa f$ is constant. In particular λ is non constant;

ii) if $\kappa<0$ then φ is constant $(M,\langle\, ,\,\rangle)$ is isometric to the hyperbolic space of constant sectional curvature κ and of dimension m, in case f has precisely one critical point. Moreover, up to a translation, f solves $\Delta f+m\kappa f=0$ and $\lambda+\kappa f$ is constant. In particular λ is non constant;

iii) if $\kappa=0$ then λ is constant and

a) if $\lambda\neq 0$ then φ is constant and $(M,\langle\, ,\,\rangle)$ is isometric to the Euclidean space $\mathbb{R}^m$ and

$$f(x)=\frac{\lambda}{2}|x|^2+\langle x,b\rangle+c \quad \textit{for every } x\in\mathbb{R}^m, \tag{4.2.21}$$

for some $b\in\mathbb{R}^m$ and $c\in\mathbb{R}$;

b) If $\lambda=0$ then $(M,\langle\, ,\,\rangle)$ splits as the Riemannian product of $\mathbb{R}$ with a totally geodesic ψ-Ricci flat hypersurface Σ, where $\psi:=\varphi|_\Sigma$. Moreover φ is given by $\psi\circ\pi_\Sigma$ on $\mathbb{R}\times\Sigma$, where $\pi_\Sigma:\mathbb{R}\times\Sigma\to\Sigma$ is the canonical projection and the function f can be expressed on $\mathbb{R}\times\Sigma$ as

$$f(t,x)=at+b \quad \textit{for every } t\in\mathbb{R} \textit{ and } x\in\Sigma, \tag{4.2.22}$$

for some $a>0$ and $b\in\mathbb{R}$ such that $\Sigma=f^{-1}(\{b\})$.

Proof. First of all, notice that to prove that φ is constant it is sufficient to show that $(M, \langle\, , \,\rangle)$ has constant sectional curvature equal to κ given by (4.2.20). Indeed, if this is the case, we easily get $S = S^{\varphi}$ and thus φ is constant. Our assumptions implies the validity of (4.2.4). From Proposition 4.2.7, we deduce that

$$\mathrm{Hess}(f) + \frac{S^{\varphi}}{m(m-1)} f \langle\, , \,\rangle = \zeta \langle\, , \,\rangle \tag{4.2.23}$$

for some constant ζ. If $S^{\varphi} \neq 0$ the isometry and the fact that $\bar{f} = f - \frac{\zeta}{\kappa}$ solves

$$\Delta \bar{f} + m\kappa \bar{f} = 0 \tag{4.2.24}$$

follow from Theorem 4.2.15, using (4.2.23). Moreover, from (4.2.10), we immediately deduce

$$\lambda + \kappa f = \frac{S^{\varphi}}{m} + \zeta.$$

If $S^{\varphi} = 0$, combining (4.2.23) and (4.2.4), we deduce $\lambda = \zeta$ is constant. If $\lambda \neq 0$ we conclude, once again, by Theorem 4.2.15. If $\lambda = 0$ then f is a non trivial affine function, hence we conclude that $(M, \langle\, , \,\rangle)$ splits by Remark 4.2.16. It remains to prove only that $(\Sigma, \langle\, , \,\rangle_{\Sigma})$ is ψ-Ricci flat. This fact follows easily from Remark 2.5.46, since $(M, \langle\, , \,\rangle)$ is φ-Ricci flat and φ is given by $\psi \circ \pi_{\Sigma}$. ☐

The next Theorem deals with the case $\mu \neq 0$. We sketch the main points of the proof because, essentially, is the same as the proof of the Theorem above.

Theorem 4.2.25. *Let $(M, \langle\, , \,\rangle)$ be a complete Riemannian manifold of dimension $m \geq 2$ that supports a non trivial gradient Einstein-type structure as* (4.2.1) *for $\mu \neq 0$, $\varphi : M \to (N, \langle\, , \,\rangle_N)$, $\alpha \in \mathbb{R} \setminus \{0\}$ and some $f, \lambda \in \mathcal{C}^{\infty}(M)$. Assume that* (4.2.1) *reduces to a harmonic-Einstein structure. We set κ as in* (4.2.20) *and u as in* (4.2.5)*. Then κ is constant and there exists a constant ζ such that*

$$\mu u \left(\lambda - \kappa \frac{(m-1)\mu + 1}{\mu} \right) + \zeta = 0. \tag{4.2.26}$$

i) If $\kappa > 0$ then φ is constant and $(M, \langle\, , \,\rangle)$ is isometric to the sphere of constant sectional curvature κ immersed in $\mathbb{R}^{m+1}$. We set

$$\bar{u} := u - \frac{\zeta}{\kappa}. \tag{4.2.27}$$

Then $\bar{u}$ is a first order spherical harmonic. Moreover, $\zeta \neq 0$, that is, λ is non-constant.

ii) If $\kappa < 0$ then φ is constant $(M, \langle\, , \,\rangle)$ is isometric to the hyperbolic space of constant sectional curvature κ and of dimension m, in case f has precisely one critical point. We set

$$\bar{u} := u - \frac{\zeta}{\kappa}. \tag{4.2.28}$$

Then $\bar{u}$ solves $\Delta \bar{u} + m\kappa \bar{u} = 0$. Moreover, λ is constant if and only if $\zeta = 0$ and, if this is the case, then

$$\lambda = \kappa \frac{(m-1)\mu + 1}{\mu}. \tag{4.2.29}$$

iii) If $\kappa = 0$ then $(M, \langle\, , \,\rangle)$ is isometric to the Euclidean space $\mathbb{R}^m$ and

$$f(x) = -\frac{1}{\mu} \log \left(\frac{\zeta}{2} |x|^2 + \langle x, b \rangle + c \right) \qquad \text{for every } x \in \mathbb{R}^m, \tag{4.2.30}$$

for some $b \in \mathbb{R}^m$ and $c \in \mathbb{R}$. In particular, $\lambda = -\frac{\zeta}{\mu u}$ is non-constant.

Proof. Our assumptions and Remark 4.2.3 easily gives the validity of (4.2.6). From Proposition 4.2.7, we deduce the validity of

$$\mathrm{Hess}(u) + \frac{S^\varphi}{m(m-1)} u \langle \, , \, \rangle = \zeta \langle \, , \, \rangle \tag{4.2.31}$$

for some constant ζ. If $S^\varphi \neq 0$ we proceed as in the proof of the Theorem above, with the difference that from (4.2.10) we get

$$\kappa u = \zeta + \mu u \left(\lambda - \frac{S^\varphi}{m} \right),$$

that is, (4.2.26). Observe that if λ is constant, since u is non-constant, from (4.2.26) we infer $\zeta = 0$ and (4.2.29) holds. The converse is trivial, hence λ is constant if and only if $\zeta = 0$ and, in this case, (4.2.29) holds. From the first equation of (4.2.1) we infer

$$\mathrm{Hess}(f) - \mu df \otimes df = \left(\lambda - \frac{S^\varphi}{m} \right) \langle \, , \, \rangle,$$

that is, using (4.2.29),

$$\mathrm{Hess}(f) - \mu df \otimes df = \frac{\kappa}{\mu} \langle \, , \, \rangle.$$

If $\kappa > 0$ then, since we already know that $(M, \langle \, , \, \rangle)$ is isometric to the sphere, M is compact and the above gives a contradiction at the points of maximum of f if $\mu > 0$ and at the point of minimum of f if $\mu < 0$. To conclude the proof of the cases where $S^\varphi \neq 0$ notice also that the critical points of f and u coincides.

Assume $S^\varphi = 0$. Observe that $\zeta \neq 0$. Indeed, if by contradiction $\zeta = 0$, from (4.2.31) u is an affine function. On a complete Riemannian manifold there are no non-constant and positive affine function, see Remark 4.2.16. Moreover u is constant if and only if f is constant, hence we get a contradiction. Then, from (4.2.31), we deduce the isometry with the Euclidean space and that

$$u(x) = \frac{\zeta}{2}|x|^2 + \langle x, b \rangle + c \quad \text{for every } x \in \mathbb{R}^m,$$

that implies (4.2.30). Moreover, from (4.2.26) we get $\mu \lambda u = -\zeta$. ☐

In the compact case we get the following results, that shall be useful later. To prove them is sufficient to show that $S^\varphi > 0$ and then apply Theorem 4.2.19 and Theorem 4.2.25, respectively. Notice that $S^\varphi > 0$ is a consequence of Proposition 4.2.7.

Corollary 4.2.32. *Let $(M, \langle \, , \, \rangle)$ be a compact Riemannian manifold of dimension $m \geq 2$ that supports a non trivial gradient Einstein-type structure as* (4.2.1) *for $\mu = 0$ and some $f, \lambda \in \mathcal{C}^\infty(M)$, $\varphi : M \to (N, \langle \, , \, \rangle_N)$ smooth and $\alpha \in \mathbb{R} \setminus \{0\}$. Assume that* (4.2.1) *reduces to a harmonic-Einstein structure. Then φ is constant and $(M, \langle \, , \, \rangle)$ is isometric to the sphere of constant sectional curvature*

$$\kappa = \frac{S^\varphi}{m(m-1)}$$

immersed in $\mathbb{R}^{m+1}$. Moreover, up to a translation, λ is a first order spherical harmonic (in particular, λ is non-constant).

Corollary 4.2.33. *Let $(M, \langle \, , \, \rangle)$ be a compact Riemannian manifold of dimension $m \geq 2$ that support a gradient Einstein-type structure as in* (4.2.1) *for $\mu \neq 0$ and some $f, \lambda \in \mathcal{C}^\infty(M)$ with f non-constant, where $\varphi : M \to (N, \langle \, , \, \rangle_N)$ and $\alpha \in \mathbb{R} \setminus \{0\}$. Assume that* (4.2.1) *reduces to a harmonic-Einstein structure. We set κ as in* (4.2.20) *and u as in* (4.2.5)*. Then φ is constant and $(M, \langle \, , \, \rangle)$ is isometric to the sphere of constant sectional curvature κ immersed in $\mathbb{R}^{m+1}$. Moreover, up to a translation, u is a first order spherical harmonic and* (4.2.26) *holds for some $\zeta \in \mathbb{R} \setminus \{0\}$. In particular, λ is non-constant.*

Chapter 5

Rigidity results in the compact case

In this Chapter we provide rigidity results for compact non-trivial Einstein-type structures. The general procedure is to show that in case the φ-scalar curvature is constant the Einstein-type structure reduces to a harmonic-Einstein structure. As a consequence of the results of Chapter 4 we conclude the isometry with the sphere and the constancy of φ, that is the rigidity mentioned above. This procedure can be adapted also in case one of the higher order symmetric function of the eigenvalues of the φ-Schouten tensor is a positive constant, assuming that the Riemannian manifold is φ-Cotton flat (or equivalently, that the φ-Schouten tensor is Codazzi). Observe that the constancy of the φ-scalar curvature is equivalent to the constancy of the first order symmetric function of the eigenvalues of the φ-Schouten tensor σ_1^{φ}, see Remark 5.2.33, hence this assumption generalize the previous one tp higher order curvatures.

In Section 5.1 we prove rigidity in case the φ-scalar curvature is constant. When $\mu = 0$ we deal with the generic case, obtaining rigidity when X is non-Killing, while when $\mu \neq 0$ we deal only with the gradient case.

We begin Section 5.2 with Subsection 5.2.1, recalling the fundamental properties of Codazzi tensors, such as Newton's and Garding's inequalities, and where we define the Newton endomorphisms and the higher order curvatures. Those properties shall be useful in Subsection 5.2.2, where we prove rigidity results in case one of the higher order symmetric function of the eigenvalues of the φ-Schouten tensor is a positive constant and the Riemannian manifold is φ-Cotton flat (when $\mu = 0$ we deal with the generic case, obtaining rigidity when X is non-Killing, while when $\mu \neq 0$ we deal only with the gradient case, as in Section 5.1).

5.1 Rigidity with constant φ-scalar curvature

We prove two rigidity results, that distinguish between the cases $\mu = 0$ and $\mu \neq 0$. We begin with the case $\mu = 0$ where we are able to study a generic Einstein-type structure.

Theorem 5.1.1. *Let $(M, \langle \, , \, \rangle)$ be a compact Riemannian manifold of dimension $m \geq 2$ with an Einstein-type structure of the form*

$$\begin{cases} Ric^{\varphi} + \dfrac{1}{2}\mathcal{L}_X \langle \, , \, \rangle = \lambda \langle \, , \, \rangle \\ \tau(\varphi) = d\varphi(X), \end{cases} \tag{5.1.2}$$

for some $X \in \mathfrak{X}(M)$, $\lambda \in \mathcal{C}^{\infty}(M)$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\varphi : M \to (N, \langle \, , \, \rangle_N)$ smooth. Assume that $\alpha > 0$ and that S^{φ} is constant. Then the structure (5.1.2) *reduces to a harmonic-Einstein structure, that is,*

$$\begin{cases} Ric^{\varphi} = \dfrac{S^{\varphi}}{m} \langle \, , \, \rangle \\ \tau(\varphi) = 0. \end{cases} \tag{5.1.3}$$

Proof. Recall that we have the validity of (3.1.14), that is,

$$\frac{1}{2}\Delta_X S^\varphi = -\alpha|\tau(\varphi)|^2 - |T^\varphi|^2 - (S^\varphi - m\lambda)\frac{S^\varphi}{m} + (m-1)\Delta\lambda,$$

where T^φ is the traceless φ-Ricci tensor, defined in (2.4.1). Tracing the first equation of (5.1.2) we obtain

$$S^\varphi - m\lambda = -\mathrm{div}(\mathrm{X}), \tag{5.1.4}$$

thus inserting into the above we get

$$\frac{1}{2}\Delta S^\varphi = \frac{1}{2}\langle X, \nabla S^\varphi\rangle - \alpha|\tau(\varphi)|^2 - |T^\varphi|^2 + \frac{S^\varphi}{m}\mathrm{div}(X) + (m-1)\Delta\lambda.$$

Integrating over M, using the divergence theorem and integrating by parts we infer

$$\frac{m-2}{2m}\int_M \langle X, \nabla S^\varphi\rangle = \int_M (|T^\varphi|^2 + \alpha|\tau(\varphi)|^2).$$

Since $\alpha > 0$ and S^φ is constant we get $T^\varphi = 0$ and $\tau(\varphi) = 0$, hence (5.1.3) holds. Then $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein. □

Combining the above Theorem with Corollary 4.1.37 and Corollary 4.2.32 we easily obtain

Corollary 5.1.5. *Let $(M, \langle\, ,\, \rangle)$ be a compact Riemannian manifold of dimension $m \geq 2$ with an Einstein-type structure of the form* (5.1.2)*, for some $X \in \mathfrak{X}(M)$, $\lambda \in \mathcal{C}^\infty(M)$, $\alpha \in \mathbb{R}\setminus\{0\}$ and $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ smooth. Assume that S^φ is constant. Then φ is constant and $(M, \langle\, ,\, \rangle)$ is isometric to the sphere of constant sectional curvature κ immersed in $\mathbb{R}^{m+1}$, where κ is given by*

$$\kappa = \frac{S^\varphi}{m(m-1)}, \tag{5.1.6}$$

provided one of the following holds:

i) X is non-Killing and $\alpha > 0$.

ii) $X = \nabla f$ for some non-constant $f \in \mathcal{C}^\infty(M)$.

Moreover, up to a translation, λ is a first order spherical harmonic.

The following Theorem is the result analogous to Theorem 5.1.1 for gradient Einstein-type structures with $\mu \neq 0$. However its proof is not based on equation (3.1.17), as probably expected, but on the powerful identity (5.1.14) below.

Theorem 5.1.7. *Let $(M, \langle\, ,\, \rangle)$ be a compact manifold of dimension $m \geq 2$ with a gradient Einstein-type structure of the form*

$$\begin{cases} Ric^\varphi + Hess(f) - \mu df \otimes df = \lambda\langle\, ,\, \rangle \\ \tau(\varphi) = d\varphi(\nabla f), \end{cases} \tag{5.1.8}$$

for some $f, \lambda \in \mathcal{C}^\infty(M)$, $\alpha, \mu \in \mathbb{R}\setminus\{0\}$ and $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ smooth. Assume that S^φ is constant and $\alpha > 0$. Then the structure (5.1.8) *reduces to a harmonic-Einstein structure, that is,* (5.1.3) *holds.*

Proof. Let

$$u := e^{-\mu f} \tag{5.1.9}$$

We compute $\mathrm{div}(T^\varphi(\nabla u, \cdot)^\sharp)$. Exploiting the definition of T^φ, in a local orthonormal coframe, we have

$$(T^\varphi_{ij}u_i)_j = T^\varphi_{ij,j}u_i + T^\varphi_{ij}u_{ij} = R^\varphi_{ij,j}u_i - \frac{S^\varphi_i}{m}u_i + T^\varphi_{ij}u_{ij}. \tag{5.1.10}$$

Using (5.1.9) a computation yields

$$u_i = -\mu u f_i, \quad u_{ij} = -\mu u(f_{ij} - \mu f_i f_j), \tag{5.1.11}$$

so that, using the first equation of (5.1.8)

$$u_{ij} = \mu u(R^{\varphi}_{ij} - \lambda\delta_{ij}). \tag{5.1.12}$$

Moreover from (1.2.26), the first equation of (5.1.11) and the second equation of (5.1.8)

$$R^{\varphi}_{ij,j}u_i = \frac{1}{2}S^{\varphi}_i u_i - \alpha\varphi^a_{jj}\varphi^a_i u_i = \frac{1}{2}S^{\varphi}_i u_i + \mu u\alpha\varphi^a_{jj}\varphi^a_i f_i = \frac{1}{2}S^{\varphi}_i u_i + \mu u\alpha\varphi^a_{ii}\varphi^a_{jj}. \tag{5.1.13}$$

Inserting (5.1.12) and (5.1.13) into (5.1.10), since T^{φ} is traceless, we obtain

$$\begin{aligned}(T^{\varphi}_{ij}u_i)_j =& \frac{1}{2}S^{\varphi}_i u_i + \mu u\alpha\varphi^a_{ii}\varphi^a_{jj} - \frac{S^{\varphi}_i}{m}u_i + \mu T^{\varphi}_{ij}(R^{\varphi}_{ij} - \lambda\delta_{ij})u \\ =& \frac{m-2}{2m}S^{\varphi}_i u_i + \mu(\alpha\varphi^a_{ii}\varphi^a_{jj} + T^{\varphi}_{ij}T^{\varphi}_{ij})u,\end{aligned}$$

that is, in global notation

$$\operatorname{div}(T^{\varphi}(\nabla u, \cdot)^{\sharp}) = \frac{m-2}{2m}\langle\nabla S^{\varphi}, \nabla u\rangle + \mu(\alpha|\tau(\varphi)|^2 + |T^{\varphi}|^2)u. \tag{5.1.14}$$

Since S^{φ} is constant, integrating over M and using the divergence theorem we deduce

$$\mu\int_M(\alpha|\tau(\varphi)|^2 + |T^{\varphi}|^2)u = 0.$$

From $\mu \neq 0$, $\alpha > 0$ and $u > 0$ on M we obtain $T^{\varphi} = 0$, that is the first equation of (5.1.3), and $\tau(\varphi) = 0$, that is the second equation of (5.1.3). □

Combining the Theorem above with Corollary 4.2.33 we immediately get

Corollary 5.1.15. *Let $(M, \langle\,,\,\rangle)$ be a compact manifold of dimension $m \geq 2$ with a non trivial gradient Einstein-type structure of the form* (5.1.8) *for some $f, \lambda \in \mathcal{C}^{\infty}(M)$, $\alpha, \mu \in \mathbb{R}\setminus\{0\}$ and $\varphi : M \to (N, \langle\,,\,\rangle_N)$ smooth. Assume that S^{φ} is constant and $\alpha > 0$. Then φ is constant and $(M, \langle\,,\,\rangle)$ is isometric to the sphere of constant sectional curvature κ immersed in $\mathbb{R}^{m+1}$, where κ is given by* (5.1.6)*. Finally, up to a translation, u is a first order spherical harmonic, where u is defined by* (5.1.9)*, and for some $\zeta \in \mathbb{R}\setminus\{0\}$,*

$$\lambda = -\frac{\zeta}{\mu u} + \frac{1 + (m-1)\mu}{\mu}\kappa.$$

In particular λ is non constant.

5.2 Rigidity for φ-Cotton flat manifolds

Next we present two more rigidity results, again distinguishing between the cases $\mu = 0$ and $\mu \neq 0$. In both the results we assume that the manifold is φ-Cotton flat. We begin with some remarks on Codazzi tensor fields, since a manifold is φ-Cotton flat if and only if the φ-Schouten tensor is a Codazzi tensor field.

5.2.1 Codazzi tensor fields and useful formulas

In this section we present a general formula for a 2-times covariant, symmetric tensor field T on a Riemannian manifold $(M, \langle\, ,\, \rangle)$ of dimension m. For $x \in M$ fixed, we set

$$\lambda_1 \leq \ldots \leq \lambda_m,$$

to denote the (possibly coinciding) eigenvalues of T at x and we consider the *elementary symmetric functions of the eigenvalues of T*

$$S_0 := 1, \quad S_k := \sum_{1 \leq i_1 < \ldots < i_k \leq m} \lambda_{i_1} \ldots \lambda_{i_k} \quad \text{for } 1 \leq k \leq m. \tag{5.2.1}$$

In other words the S_k's are the coefficients of the polynomial expansion

$$\det(I + \lambda T) = \sum_{k=0}^{m} S_k \lambda^k, \tag{5.2.2}$$

where I is the identity. As usual we normalize the S_k's by setting

$$S_k = \binom{m}{k} \sigma_k,$$

obtaining the *normalized symmetric function of the eigenvalues of T*. In this way we obtain the validity of Newton's inequalities in the form

$$\sigma_{k-1}\sigma_{k+1} \leq \sigma_k^2 \quad \text{for } 1 \leq k \leq m-1. \tag{5.2.3}$$

Furthermore, if $\sigma_{k-1} \neq 0$ at x, equality holds in (5.2.3) if and only if all the eigenvalues of T at x are equal. The σ_k's give rise to continuous functions on M and, from the classic results of [G], we deduce that if for some k, $1 \leq k \leq m$, we have $\sigma_k > 0$ everywhere on M then, for $1 \leq i \leq k$, $\sigma_i > 0$ on M and furthermore, Gårding's inequalities hold,

$$\sigma_1 \geq \sigma_2^{\frac{1}{2}} \geq \ldots \geq \sigma_k^{\frac{1}{k}}, \tag{5.2.4}$$

with equality at a point $x \in M$ at some stage of the chain if and only if T has equal eigenvalues at x. The next Lemma follows directly by (5.2.4) and will be used later.

Lemma 5.2.5. *In the notations above suppose that $\sigma_k > 0$ on M for some $2 \leq k \leq m-1$, where $m \geq 3$ is the dimension of M. Then*

$$\sigma_1\sigma_k - \sigma_{k+1} \geq 0 \tag{5.2.6}$$

with equality holding at a point $x \in M$ if and only if T is proportional to the metric at x.

Proof. Since $\sigma_k > 0$ on M, by Gårding's inequalities

$$\sigma_1 \geq \ldots \geq \sigma_{k-1}^{\frac{1}{k-1}} \geq \sigma_k^{\frac{1}{k}} > 0.$$

From $\sigma_{k-1} > 0$ on M and Newton's inequalities (5.2.3)

$$\sigma_{k+1} = \frac{\sigma_{k+1}\sigma_{k-1}}{\sigma_{k-1}} \leq \frac{\sigma_k^2}{\sigma_{k-1}} = \sigma_k \frac{\sigma_k}{\sigma_{k-1}}.$$

We claim

$$\frac{\sigma_k}{\sigma_{k-1}} \leq \sigma_1,$$

and since $\sigma_k > 0$, from the above we obtain

$$\sigma_{k+1} \leq \sigma_k \sigma_1,$$

that is (5.2.6). It remains to prove the claim. We use Gårding's inequalities twice and $\sigma_1, \sigma_k > 0$ to deduce

$$\sigma_k = \sigma_k^{\frac{1}{k}} \sigma_k^{\frac{k-1}{k}} \leq \sigma_1 \sigma_k^{\frac{k-1}{k}} \leq \sigma_1 \sigma_{k-1}.$$

Since $\sigma_{k-1} > 0$ this implies the claim. Observe that the equality in (5.2.6) holds at a point if and only if T is proportional to the metric at that point since this forces Newton's inequality and Gårding's inequalities, used to prove the validity of (5.2.6), to be equalities at that point. $\square$

Associated with T one considers the *Newton endomorphisms*

$$P_k = P_k(T) : \mathfrak{X}(M) \to \mathfrak{X}(M) \quad \text{for } 0 \leq k \leq m,$$

inductively defined by

$$P_0 := I, \quad P_k := S_k I - t \circ P_{k-1} \quad \text{for } 1 \leq k \leq m, \tag{5.2.7}$$

where $t : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is the endomorphism induced by T. Notice that

$$P_k = \sum_{i=0}^{k} (-1)^1 \binom{m}{k-i} \sigma_{k-1} t^i$$

and, from Cayley-Hamilton theorem and (5.2.2), $P_m = 0$ on M. Moreover, having set

$$c_k := (m-k)\binom{m}{k}, \tag{5.2.8}$$

we have

$$\operatorname{tr}(P_k) = (m-k)S_k = c_k \sigma_k, \quad \operatorname{tr}(t \circ P_{k-1}) = kS_k = c_{k-1}\sigma_k. \tag{5.2.9}$$

The Newton's endomorphisms give rise to a family of second order differential operators L_k defined as follows. Setting hess(u) for the endomorphism induced by Hess(u), where $u \in \mathcal{C}^2(M)$,

$$L_k u := \operatorname{tr}(P_k \circ \operatorname{hess}(u)). \tag{5.2.10}$$

A computation shows that L_k can be written in the form:

$$L_k u = \operatorname{div}(P_k(\nabla u)) - \langle \operatorname{div}(P_k), \nabla u \rangle. \tag{5.2.11}$$

Obviously,

$$\operatorname{div}(P_0) = 0 = \operatorname{div}(P_m). \tag{5.2.12}$$

To compute div(P_k) for the remaining values of k we introduce the 3-times covariant tensor field C of components

$$C_{ijk} := T_{ij,k} - T_{ik,j}. \tag{5.2.13}$$

Using the definition of P_k, for $1 \leq k \leq m-1$,

$$\operatorname{div}(P_k)_j = -\operatorname{div}(P_{k-1})_i T_{ij} - C_{ijs}(P_k)_{is}. \tag{5.2.14}$$

In particular when T is a Codazzi tensor field all the Newton's endomorphisms are divergence free. Hence if T is Codazzi equation (5.2.11) becomes

$$\operatorname{tr}(P_k \circ \operatorname{hess}(u)) = L_k u = \operatorname{div}(P_k(\nabla u)), \tag{5.2.15}$$

We remark that, having fixed the 2-times covariant tensor field T, we can define an operator

$$\widetilde{L}_k : \mathfrak{X}(M) \to \mathfrak{X}(M) \quad \text{for } 0 \leq k \leq m,$$

by setting, for every $Z \in \mathfrak{X}(M)$

$$\widetilde{L}_k(Z) := \frac{1}{2}\mathrm{tr}(P_k \circ l_Z), \tag{5.2.16}$$

where $l_Z : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is the endomorphism associated to the Lie derivative of the metric in the direction of Z

$$\mathcal{L}_Z\langle\, ,\, \rangle.$$

A computation yields

$$\widetilde{L}_k(Z) = \mathrm{div}(P_k(Z)) - \langle \mathrm{div}(P_k), Z\rangle,$$

hence if T is Codazzi

$$\widetilde{L}_k(Z) = \mathrm{div}(P_k(Z)).$$

We then obtain the following generalization of (5.2.15)

$$\mathrm{div}(P_k(Z)) = \frac{1}{2}\mathrm{tr}(P_k \circ l_Z). \tag{5.2.17}$$

5.2.2 Rigidity with constant higher order φ-scalar curvature

In order to obtain the next rigidity results we shall make use of

Lemma 5.2.18. *Let $(M, \langle\, ,\, \rangle)$ and $(N, \langle\, ,\, \rangle_N)$ be Riemannian manifolds, $\varphi : M \to (N, \langle\, ,\, \rangle_N)$, $X \in \mathfrak{X}(M)$, $\alpha \in \mathbb{R} \setminus \{0\}$ and suppose that the following compatibility condition holds*

$$\tau(\varphi) = d\varphi(X). \tag{5.2.19}$$

Let $tr(C^\varphi)$ be the 1-form defined in (2.4.4)*, then*

$$tr(C^\varphi)(X) = \alpha|\tau(\varphi)|^2. \tag{5.2.20}$$

In particular if the φ-Schouten tensor, defined in (1.2.10)*, is a Codazzi tensor then φ must be harmonic.*

Proof. In a local orthonormal coframe (5.2.19) reads

$$\varphi^a_{ii} = \varphi^a_i X^i$$

and from (1.2.36)

$$\mathrm{tr}(C^\varphi)_i = \alpha\varphi^a_{kk}\varphi^a_i,$$

hence we easily conclude

$$\mathrm{tr}(C^\varphi)(X) = \mathrm{tr}(C^\varphi)_i X^i = \alpha\varphi^a_{kk}\varphi^a_i X^i = \alpha\varphi^a_{kk}\varphi^a_{ii} = \alpha|\tau(\varphi)|^2,$$

that is, (5.2.20). Observe that, by definition (1.2.31), if A^φ is a Codazzi tensor then $C^\varphi = 0$. If this is the case, from $\alpha \neq 0$ and (5.2.20) we deduce $\tau(\varphi) = 0$. □

In the following we will denote by σ^φ_k and P^φ_k the normalized k^{th} symmetric function of the eigenvalues of the φ-Schouten tensor, for brevity, the k^{th} φ-scalar curvature and the Newton endomorphism corresponding to the φ-Schouten tensor, respectively.

Theorem 5.2.21. *Let $(M, \langle\, ,\, \rangle)$ be a compact Riemannian manifold of dimension $m \geq 3$ with an Einstein-type structure of the form* (5.1.2) *with $X \in \mathfrak{X}(M)$, $\lambda \in \mathcal{C}^\infty(M)$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ smooth. Suppose that $(M, \langle\, ,\, \rangle)$ is φ-Cotton flat, that is,*

$$C^\varphi = 0 \tag{5.2.22}$$

and that σ^φ_k is a positive constant for some $k = 2, \ldots, m-1$. Then (5.1.2) *reduces to a harmonic-Einstein structure.*

Remark 5.2.23. If $m = 2$ then $A^\varphi = T^\varphi$, hence $\sigma_1^\varphi = 0$ and thus, from Newton's inequality $\sigma_2^\varphi \leq 0$. This motivates the hypothesis $m \geq 3$.

Proof. Since (5.2.22) holds the φ-Schouten tensor A^φ is a Codazzi tensor. Then (5.2.17) holds $T = A^\varphi$, that is, for $Z = X$,

$$\operatorname{div}(P_k^\varphi(X)) = \frac{1}{2}\operatorname{tr}(P_k^\varphi \circ l_X). \tag{5.2.24}$$

Expressing the first equation of (5.1.2) in terms of A^φ we obtain

$$\frac{1}{2}\mathcal{L}_X\langle\,,\,\rangle = -\frac{S^\varphi}{2(m-1)}\langle\,,\,\rangle - A^\varphi + \lambda\langle\,,\,\rangle,$$

so that

$$\frac{1}{2}l_X = \left(\lambda - \frac{S^\varphi}{2(m-1)}\right)I - a^\varphi, \tag{5.2.25}$$

where l_X and a^φ denotes the endomorphisms of $\mathfrak{X}(M)$ induced by $\mathcal{L}_X\langle\,,\,\rangle$ and A^φ, respectively. Inserting (5.2.25) in (5.2.24) a computation using (5.2.9) yields

$$\operatorname{div}(P_k^\varphi(X)) = c_k\left[\left(\lambda - \frac{S^\varphi}{2(m-1)}\right)\sigma_k^\varphi - \sigma_{k+1}^\varphi\right], \tag{5.2.26}$$

where c_k is defined in (5.2.8). Since we are assuming that $\sigma_k^\varphi > 0$, from Lemma 5.2.5 we deduce the validity of

$$\sigma_1^\varphi\sigma_k^\varphi - \sigma_{k+1}^\varphi \geq 0, \tag{5.2.27}$$

equality holding at a point if and only if at that point A^φ, and therefore Ric^φ, is proportional to the metric. Since M is compact by the Hodge-de Rham decomposition (see, for instance, [ABR])

$$X = \nabla h + Y,$$

for some $h \in \mathcal{C}^\infty(M)$ and $Y \in \mathfrak{X}(M)$ with $\operatorname{div}(Y) = 0$. Thus, $\operatorname{div}(X) = \Delta h$ and tracing the first equation of (5.1.2)

$$S^\varphi + \Delta h = m\lambda,$$

that can be rewritten in the following way:

$$\sigma_1^\varphi + \frac{\Delta h}{m} = \lambda - \frac{S^\varphi}{2(m-1)},$$

where we used that, tracing (1.2.10),

$$\sigma_1^\varphi = \frac{\operatorname{tr}(A^\varphi)}{m} = \frac{S^\varphi}{m} - \frac{S^\varphi}{2(m-1)}. \tag{5.2.28}$$

Plugging into (5.2.26) we have

$$\operatorname{div}(P_k^\varphi)(X) = c_k\left(\sigma_1^\varphi\sigma_k^\varphi - \sigma_{k+1}^\varphi + \frac{\sigma_k^\varphi}{m}\Delta h\right).$$

Integrating on M, since σ_k^φ is constant, we infer:

$$\int_M(\sigma_1^\varphi\sigma_k^\varphi - \sigma_{k+1}^\varphi) = 0.$$

By (5.2.27) and the above we deduce that the actually equality holds in (5.2.27) all on M. It follows that A^φ is a trivial Codazzi tensor field, that is, is a constant multiple of the metric $\langle\,,\,\rangle$. In particular S^φ is constant and also Ric^φ is proportional to the metric on all M. Combining it with Lemma 5.2.18 we conclude that $(M, \langle\,,\,\rangle)$ is harmonic-Einstein. □

Combining the above Theorem with Corollary 4.1.37 and Corollary 4.2.32 we easily obtain

Corollary 5.2.29. *Let $(M, \langle\, , \,\rangle)$ be a compact Riemannian manifold of dimension $m \geq 3$ with an Einstein-type structure of the form* (5.1.2) *with $X \in \mathfrak{X}(M)$, $\lambda \in \mathcal{C}^\infty(M)$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\varphi : M \to (N, \langle\, , \,\rangle_N)$ smooth. Suppose that $(M, \langle\, , \,\rangle)$ is φ-Cotton flat and that σ_k^φ is a positive constant for some $k = 2, \ldots, m-1$. Then φ is constant and $(M, \langle\, , \,\rangle)$ is isometric to the sphere of constant sectional curvature κ immersed in $\mathbb{R}^{m+1}$, where κ is given by*

$$\kappa = \frac{2(\sigma_k^\varphi)^{\frac{1}{k}}}{m-2}, \tag{5.2.30}$$

provided one of the following holds:

i) X is non-Killing and $\alpha > 0$.

ii) $X = \nabla f$ for some non-constant $f \in \mathcal{C}^\infty(M)$.

Moreover, up to a translation, λ is a first order spherical harmonic.

Proof. The only thing we need to prove is the validity of (5.2.30). Notice that, since A^φ is proportional to the metric, using (5.2.28) and the constancy of φ

$$\frac{m-2}{2m(m-1)} S = \sigma_1^\varphi = (\sigma_2^\varphi)^{\frac{1}{2}} = \ldots = (\sigma_m^\varphi)^{\frac{1}{m}}.$$

Thus we have

$$\kappa = \frac{S}{m(m-1)} = \frac{2(\sigma_k^\varphi)^{\frac{1}{k}}}{m-2},$$

as in (5.2.30). □

In Theorem 5.2.21 we dealt with the case $\mu = 0$ with a general vector field X. Now we consider the case $\mu \neq 0$ but we restrict ourselves to the gradient case, $X = \nabla f$ for some $f \in \mathcal{C}^\infty(M)$. We have

Theorem 5.2.31. *Let $(M, \langle\, , \,\rangle)$ be a compact Riemannian manifold of dimension $m \geq 3$ with a non trivial gradient Einstein-type structure of the form* (5.1.8)*, with $\varphi : M \to (N, \langle\, , \,\rangle_N)$ smooth, $f, \lambda \in \mathcal{C}^\infty(M)$ and $\mu, \alpha \in \mathbb{R} \setminus \{0\}$. Suppose that $(M, \langle\, , \,\rangle)$ is φ-Cotton flat, that is* (5.2.22) *holds, that f is non-constant and that σ_k^φ is a positive constant for some $k = 2, \ldots, m-1$. Then* (5.1.8) *reduces to a harmonic-Einstein structure.*

Proof. We set

$$u := e^{-\mu f}.$$

Then

$$\mathrm{Ric}^\varphi - \frac{1}{\mu u} \mathrm{Hess}(u) = \lambda \langle\, , \,\rangle,$$

that is equivalent, using the definition of A^φ, to

$$\mathrm{Hess}(u) = \mu u \left[A^\varphi - \left(\lambda - \frac{S^\varphi}{2(m-1)} \langle\, , \,\rangle \right) \right].$$

Then, as in the proof of Theorem 5.2.21 but using (5.2.15), we obtain

$$\mathrm{div}(P_k^\varphi(\nabla f)) = \mu c_k \left[u(\sigma_{k+1}^\varphi - \sigma_1^\varphi \sigma_k^\varphi) + \frac{\sigma_k^\varphi}{m\mu} \Delta u \right].$$

Using constancy of σ_k and integrating on M we get

$$\mu c_k \int_M u(\sigma_{k+1}^\varphi - \sigma_1^\varphi \sigma_k^\varphi) = 0$$

and since $u > 0$ and $\mu \neq 0$,

$$\sigma_{k+1}^\varphi - \sigma_1^\varphi \sigma_k^\varphi = 0, \quad \text{on } M.$$

We now conclude as in Theorem 5.2.21. □

Combining the Theorem above with Corollary 4.2.33 we obtain

Corollary 5.2.32. *Let $(M, \langle\, , \,\rangle)$ be a compact Riemannian manifold of dimension $m \geq 3$ with a non trivial gradient Einstein-type structure as* (5.1.8) *with $f, \lambda \in \mathcal{C}^\infty(M)$, $\varphi : M \to (N, \langle\, , \,\rangle_N)$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\mu \in \mathbb{R} \setminus \{0\}$. Suppose that $(M, \langle\, , \,\rangle)$ is φ-Cotton flat, that is* (5.2.22) *holds, and that σ_k^φ is a positive constant for some $k = 2, \ldots, m-1$. Then φ is constant and $(M, \langle\, , \,\rangle)$ is isometric to a Euclidean sphere $\mathbb{S}^m$ in $\mathbb{R}^{m+1}$ of constant sectional curvature κ given by* (5.2.30).

Remark 5.2.33. Observe that, since

$$\sigma_1^\varphi = \frac{m-2}{2(m-1)} S^\varphi,$$

Theorem 5.1.1 and Theorem 5.1.7 can be interpreted as the case $k = 1$ of Theorem 5.2.21 and Theorem 5.2.31, respectively. In those Theorems the assumptions of φ-Cotton flatness and on the sign of the curvature are unnecessary.

Chapter 6

Gradient Einstein-type structures with vanishing conditions on φ-Bach

In this Chapter we shall consider a Riemannian manifold $(M, \langle\, , \,\rangle)$ with a non trivial gradient Einstein-type structure of the form

$$\begin{cases} \mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) - \mu df \otimes df = \lambda \langle\, , \,\rangle \\ \tau(\varphi) = d\varphi(\nabla f), \end{cases} \tag{6.0.1}$$

for some $\alpha \in \mathbb{R} \setminus \{0\}$, $\mu \in \mathbb{R}$, $\lambda, f \in \mathcal{C}^{\infty}(M)$ and $\varphi : M \to (N, \langle\, , \,\rangle_N)$. Our aim is to prove the structure Theorem 6.4.1 below, generalizing Theorem 1.2 of [CMMR], and Theorem 6.4.3, that is new even in the standard case where φ is constant.

We begin with Section 6.1, defining the tensor D^{φ} and computing the first two integrability conditions related to the system (6.0.1), see (6.1.11) and (6.1.16). In case $\mu = -\frac{1}{m-2}$, as we seen in Section 2.3, the Riemannian manifold is conformally-Einstein and in the two integrability conditions, that are given by (2.3.3) and (2.3.4), does not appear the tensor D^{φ}. This is justify from the fact that, as we shall see in Remark 6.1.14, the vanishing of the tensor D^{φ} is related to the fact that the φ-Schouten tensor is Codazzi in a conformal metric.

In Section 6.2 we show that $D^{\varphi} = 0$ and φ is harmonic provided that $\alpha > 0$, $\mu \neq -\frac{1}{m-2}$, the potential function f is proper and non constant and the φ-Bach tensor vanishes in the direction of ∇f. See Remark 6.2.13 for more details on those assumptions.

In Section 6.3, we draw the consequences on the local structure of $(M, \langle\, , \,\rangle)$ of the vanishing of D^{φ} and $\tau(\varphi)$. We show in Proposition 6.3.15 and Proposition 6.3.45 that the level set corresponding to a regular value of f is a totally umbilical hypersurface with constant mean curvature that is harmonic-Eistein with respect to the induced metric, α and the restriction of φ. Then in Proposition 6.3.29 we prove that $C^{\varphi} = 0$ on $\{x \in M : \nabla f(x) \neq 0\}$.

In the last Section of the Chapter, that is Section 6.4, we combine the results of the previous Sections in order to state and prove Theorem 6.4.1 and Theorem 6.4.3. In both those theorems we assume that (6.0.1) satisfies the hypothesis mentioned above. In the first we prove that in a neighbourghood of every regular level set of f, the manifold $(M, \langle, , \rangle)$ is isometric to a warped product with $(m-1)$-dimensional totally umbilical and with constant mean curvature harmonic-Einstein leaves (with respect to the induced metric, α and the restriction of φ) and that φ can be recovered by its value on a single leaf. In the latter one we prove that not only D^{φ} and $\tau(\varphi)$ must vanish but also C^{φ} and B^{φ}, when λ is constant of the foliation. Moreover we prove that the traceless part of the φ-Ricci tensor T^{φ} belongs to the kernel of the curvature operator $\mathcal{W}^{\varphi}$, introduced in in Chapter 2. Assuming thus a genericity condition we get that $(M, \langle\, , \,\rangle)$ is harmonic-Einstein.

6.1 The tensor D^φ and the first two integrability conditions

In the following we shall use (3.1.15) and (3.1.16), which we report here for the reader's convenience

$$R^\varphi_{ij,k} - R^\varphi_{ik,j} = f_t R_{tikj} + \mu(f_{ik}f_j - f_{ij}f_k) + \lambda_k\delta_{ij} - \lambda_j\delta_{ik}, \tag{6.1.1}$$

$$\frac{1}{2}S^\varphi_i = R^\varphi_{ki}f_k + \mu(f_{ki}f_k - \Delta f f_i) + (m-1)\lambda_i. \tag{6.1.2}$$

We now come to the definition of the tensor D^φ that shall reveal essential in our study.

Definition 6.1.3. Let $m \geq 3$. In a local orthonormal coframe we let the components of D^φ be given by

$$D^\varphi_{ijk} := \frac{1}{m-2}\left[R^\varphi_{ij}f_k - R^\varphi_{ik}f_j + \frac{1}{m-1}f_t(R^\varphi_{tk}\delta_{ij} - R^\varphi_{tj}\delta_{ik}) - \frac{S^\varphi}{m-1}(f_k\delta_{ij} - f_j\delta_{ik})\right]. \tag{6.1.4}$$

We observe that if φ is a constant map then D^φ coincides with the tensor D defined in [CC], with a different sign convention. The following properties are easily verified by computation.

Proposition 6.1.5. *The tensor D^φ is skew-symmetric in the last two indices and it is totally trace free, that is, in a local orthonormal coframe,*

$$D^\varphi_{ikj} = -D^\varphi_{ijk}, \tag{6.1.6}$$

$$D^\varphi_{kii} = D^\varphi_{iki} = D^\varphi_{iik} = 0. \tag{6.1.7}$$

An essential feature of D^φ is that it can be expressed purely in terms of the potential function f. Indeed, we have the following

Proposition 6.1.8. *In the present setting, with $m \geq 3$, in a local orthonormal coframe we have*

$$D^\varphi_{ijk} = \frac{1}{m-2}\left[f_{ik}f_j - f_{ij}f_k + \frac{1}{m-1}f_t(f_{tj}\delta_{ik} - f_{tk}\delta_{ij}) - \frac{\Delta f}{m-1}(f_j\delta_{ik} - f_k\delta_{ij})\right]. \tag{6.1.9}$$

Proof. The proof is computational, using the first equation of (6.0.1). Taking its trace

$$S^\varphi + \Delta f = \mu|\nabla f|^2 + m\lambda,$$

hence using the above in the definition (6.1.4), together with the first equation of (6.0.1), we obtain

$$\begin{aligned}
D^\varphi_{ijk} =& \frac{1}{m-2}\left[(-f_{ij} + \mu f_i f_j + \lambda\delta_{ij})f_k - (-f_{ik} + \mu f_i f_k + \lambda\delta_{ik})f_j\right] \\
&+ \frac{1}{(m-1)(m-2)}f_t[(-f_{tk} + \mu f_t f_k + \lambda\delta_{tk})\delta_{ij} - (-f_{tj} + \mu f_t f_j + \lambda\delta_{tj})\delta_{ik}] \\
&- \frac{-\Delta f + \mu|\nabla f|^2 + m\lambda}{(m-1)(m-2)}(f_k\delta_{ij} - f_j\delta_{ik}) \\
=& \frac{1}{m-2}\left[f_{ik}f_j - f_{ij}f_k + \frac{1}{m-1}f_t(f_{tj}\delta_{ik} - f_{tk}\delta_{ij}) - \frac{\Delta f}{m-1}(f_j\delta_{ik} - f_k\delta_{ij})\right],
\end{aligned}$$

that is (6.1.9). □

Now we prove the first integrability condition of the system (6.0.1).

Proposition 6.1.10. *In the present setting, with $m \geq 3$, in a local orthonormal coframe we have*

$$C^\varphi_{ijk} + f_t W^\varphi_{tijk} = [1 + (m-2)\mu]D^\varphi_{ijk}. \tag{6.1.11}$$

Proof. Using (6.1.1) in (1.2.42) we obtain

$$C^{\varphi}_{ijk} + \frac{1}{2(m-1)}\left(S^{\varphi}_k\delta_{ij} - S^{\varphi}_j\delta_{ik}\right) + f_t R^t_{ijk} - \mu(f_{ik}f_j - f_{ij}f_k) - \lambda_k\delta_{ij} + \lambda_j\delta_{ik} = 0. \tag{6.1.12}$$

We claim the validity of

$$R^t_{ijk}f_t = W^{\varphi}_{tijk}f_t - D^{\varphi}_{ijk} - \frac{f_t}{m-1}(R^{\varphi}_{tk}\delta_{ij} - R^{\varphi}_{tj}\delta_{ik}). \tag{6.1.13}$$

We postpone its proof and we complete the proof of (6.1.11). Inserting (6.1.13) in (6.1.12) we obtain

$$\begin{aligned}0 =& C^{\varphi}_{ijk} + W^{\varphi}_{tijk}f_t - D^{\varphi}_{ijk} + \frac{1}{2(m-1)}(S^{\varphi}_k\delta_{ij} - S^{\varphi}_j\delta_{ik})\\ &- \mu(f_{ik}f_j - f_{ij}f_k) - \lambda_k\delta_{ij} + \lambda_j\delta_{ik} - \frac{f_t}{m-1}(R^{\varphi}_{tk}\delta_{ij} - R^{\varphi}_{tj}\delta_{ik}).\end{aligned}$$

Using (6.1.2) we deduce

$$\begin{aligned}\frac{1}{2(m-1)}(S^{\varphi}_k\delta_{ij} - S^{\varphi}_j\delta_{ik}) =& \frac{1}{m-1}(R^{\varphi}_{tk}f_t + \mu(f_{tk}f_t - \Delta f f_k) + (m-1)\lambda_k)\delta_{ij}\\ &- \frac{1}{m-1}(R^{\varphi}_{tj}f_t + \mu(f_{tj}f_t - \Delta f f_j) + (m-1)\lambda_j)\delta_{ik}\\ =& \frac{f_t}{m-1}(R^{\varphi}_{tk}\delta_{ij} - R^{\varphi}_{tj}\delta_{ik}) + \mu\frac{f_t}{m-1}(f_{tk}\delta_{ij} - f_{tj}\delta_{ik})\\ &+ \mu\frac{\Delta f}{m-1}(f_j\delta_{ik} - f_k\delta_{ij}) + \lambda_k\delta_{ij} - \lambda_j\delta_{ik},\end{aligned}$$

and by plugging into the above equality we infer

$$\begin{aligned}0 =& C^{\varphi}_{ijk} + W^{\varphi}_{tijk}f_t - D^{\varphi}_{ijk}\\ &- \mu\left[f_{ik}f_j - f_{ij}f_k + \frac{f_t}{m-1}(f_{tj}\delta_{ik} - f_{tk}\delta_{ij})f_t + \frac{\Delta f}{m-1}(f_k\delta_{ij} - f_j\delta_{ik})\right],\end{aligned}$$

that implies (6.1.11), using (6.1.9). It remains to prove (6.1.13). Explicitating (1.2.10) in (1.2.18) we obtain

$$R_{tijk} - W^{\varphi}_{tijk} = \frac{1}{m-2}\left[R^{\varphi}_{tj}\delta_{ik} - R^{\varphi}_{tk}\delta_{ij} + R^{\varphi}_{ik}\delta_{tj} - R^{\varphi}_{ij}\delta_{tk} - \frac{S^{\varphi}}{m-1}(\delta_{tj}\delta_{ik} - \delta_{tk}\delta_{ij})\right],$$

then, using (6.1.4), we deduce

$$\begin{aligned}R_{tijk}f_t - W^{\varphi}_{tijk}f_t =& \frac{1}{m-2}\left[f_t(R^{\varphi}_{tj}\delta_{ik} - R^{\varphi}_{tk}\delta_{ij}) + R^{\varphi}_{ik}f_j - R^{\varphi}_{ij}f_k - \frac{S^{\varphi}}{m-1}(\delta_{ik}f_j - \delta_{ij}f_k)\right]\\ =& -\frac{1}{m-2}\left[R^{\varphi}_{ij}f_k - R^{\varphi}_{ik}f_j + \frac{f_t}{m-1}(R^{\varphi}_{tk}\delta_{ij} - R^{\varphi}_{tj}\delta_{ik}) - \frac{S^{\varphi}}{m-1}(\delta_{ij}f_k - \delta_{ik}f_j)\right]\\ &- \frac{1}{m-2}\left(1 - \frac{1}{m-1}\right)f_t(R^{\varphi}_{tk}\delta_{ij} - R^{\varphi}_{tj}\delta_{ik})\\ =& -D^{\varphi}_{ijk} - \frac{f_t}{m-1}(R^{\varphi}_{tk}\delta_{ij} - R^{\varphi}_{tj}\delta_{ik})\end{aligned}$$

that is (6.1.13). □

Remark 6.1.14. Notice that, combining the first integrability condition (6.1.11) with (1.3.28),

$$[1 + (m-2)\mu]D^{\varphi}_{ijk} = e^{-\frac{3}{m-2}f}\widetilde{C}^{\varphi}_{ijk},$$

where $\widetilde{C}^{\varphi}_{ijk}$ are the components of $\widetilde{C}^{\varphi}$ in the local coframe $\{\widetilde{\theta}^i\}$ determined by the metric $\widetilde{\langle\,,\,\rangle} = e^{-\frac{2}{m-2}f}\langle\,,\,\rangle$. That implies, using (1.3.4),

$$\widetilde{C}^{\varphi} = [1+(m-2)\mu]D^{\varphi}.$$

Then D^{φ} is the description of the tensor $\widetilde{C}^{\varphi}$ in terms of the metric $\langle\,,\,\rangle$, up to constant multiplicative factor.

The second integrability condition follows by taking the divergence of (6.1.11). Indeed we have the following

Proposition 6.1.15. *In the present setting, with $m \geq 3$, in a local orthonormal coframe we have*

$$(m-2)B^{\varphi}_{ij} + \mu W^{\varphi}_{tijk} f_t f_k - \frac{m-3}{m-2} C^{\varphi}_{jik} f_k = [1+(m-2)\mu]\left(D^{\varphi}_{ijk,k} - \frac{\alpha}{m-2}\varphi^a_{kk}\varphi^a_i f_j\right). \tag{6.1.16}$$

Proof. We take the divergence of (6.1.11) and we use (6.0.1) and (1.2.45), together with (1.2.50), to obtain

$$\begin{aligned}
[1+(m-2)\mu]D^{\varphi}_{ijk,k} =& (C^{\varphi}_{ijk} + f_t W^{\varphi}_{tijk})_k \\
=& C^{\varphi}_{ijk,k} + f_{tk} W^{\varphi}_{tijk} + f_t W^{\varphi}_{tijk,k} \\
=& C^{\varphi}_{ijk,k} + (-R^{\varphi}_{tk} + \mu f_t f_k + \lambda\delta_{tk}) W^{\varphi}_{tijk} + f_k W^{\varphi}_{tjik,t} \\
=& C^{\varphi}_{ijk,k} + R^{\varphi}_{tk} W^{\varphi}_{tikj} + \mu W^{\varphi}_{tijk} f_t f_k + \lambda W^{\varphi}_{kijk} \\
&+ \left[\frac{m-3}{m-2} C^{\varphi}_{jki} + \alpha(\varphi_{ji}\varphi^a_k - \varphi^a_{jk}\varphi^a_i) + \frac{\alpha}{m-2}\varphi^a_{tt}(\varphi^a_i\delta_{jk} - \varphi^a_k\delta_{ji})\right] f_k \\
=& (m-2)B^{\varphi}_{ij} + \alpha R^{\varphi}_{kj}\varphi^a_k\varphi^a_i - \alpha\left(\varphi^a_{ij}\varphi^a_{kk} - \varphi^a_{kkj}\varphi^a_i - \frac{1}{m-2}|\tau(\varphi)|^2\delta_{ij}\right) \\
&+ \mu W^{\varphi}_{tijk} f_t f_k - \alpha\lambda\varphi^a_i\varphi^a_j \\
&+ \frac{m-3}{m-2} C^{\varphi}_{jki} f_k + \alpha(\varphi_{ji}\varphi^a_{kk} - \varphi^a_{jk} f_k \varphi^a_i) + \frac{\alpha}{m-2}\varphi^a_{tt}(\varphi^a_i f_j - \varphi^a_{kk}\delta_{ji}) \\
=& (m-2)B^{\varphi}_{ij} + \alpha\left(R^{\varphi}_{kj}\varphi^a_k + \varphi^a_{kkj} - \lambda\varphi^a_j - \varphi^a_{jk} f_k + \frac{1}{m-2}\varphi^a_{kk} f_j\right)\varphi^a_i \\
&+ \mu W^{\varphi}_{tijk} f_t f_k + \frac{m-3}{m-2} C^{\varphi}_{jki} f_k.
\end{aligned}$$

Observe that from (6.0.1) we deduce the validity of

$$R^{\varphi}_{jk}\varphi^a_k + f_{jk}\varphi^a_k = \mu\varphi^a_{kk} f_j + \lambda\varphi^a_j,$$

and by plugging it the above, together with (6.1.11), we obtain

$$\begin{aligned}
[1+(m-2)\mu]D^{\varphi}_{ijk,k} =& (m-2)B^{\varphi}_{ij} + \alpha\left(-f_{jk}\varphi^a_k + \mu\varphi^a_{kk} f_j + (\varphi^a_k f_k)_j - \varphi^a_{jk} f_k + \frac{1}{m-2}\varphi^a_{kk} f_j\right)\varphi^a_i \\
&+ \mu W^{\varphi}_{tijk} f_t f_k + \frac{m-3}{m-2} C^{\varphi}_{jki} f_k \\
=& (m-2)B^{\varphi}_{ij} + \frac{\alpha}{m-2}[1+(m-2)\mu]\varphi^a_{kk} f_j \\
&+ \mu W^{\varphi}_{tijk} f_t f_k + \frac{m-3}{m-2} C^{\varphi}_{jki} f_k,
\end{aligned}$$

and thus (6.1.16) follows. ☐

Remark 6.1.17. In case $\mu = -\frac{1}{m-2}$ from (6.1.11) and (6.1.16) we respectively obtain (2.3.3) and (2.3.4). Furthermore, when φ is constant, (6.1.11) and (6.1.16) extend, respectively, (4-5) and (4-6) of [CMMR], with $\alpha = \beta = 1$. Observe, however, that the normalization $\alpha = \beta = 1$ that we adopt here is inessential.

6.2 Vanishing of D^φ and $\tau(\varphi)$

In what follows we shall assume the following vanishing condition on φ-Bach

$$B^\varphi(\nabla f, \cdot) = 0, \tag{6.2.1}$$

the non-degeneracy condition

$$\mu \neq -\frac{1}{m-2} \tag{6.2.2}$$

and that f is proper, that is, the preimage of compact subsets is compact, and non constant. We shall comment on these assumptions in Remark 6.2.13. Our aim is now to prove the following

Proposition 6.2.3. *In the present setting, with $m \geq 3$, assume* (6.2.1) *and define the vector field $Y \in \mathfrak{X}(M)$ of components*

$$Y^k := D^\varphi_{ijk} f_i f_j. \tag{6.2.4}$$

Then, if (6.2.2) *holds, we have*

$$\frac{m-2}{2}|D^\varphi|^2 + \frac{\alpha}{m-2}|\tau(\varphi)|^2|\nabla f|^2 = div(Y). \tag{6.2.5}$$

Proof. Observe that (6.2.1) componentwise reads

$$B^\varphi_{ij} f_i = 0. \tag{6.2.6}$$

Contracting (6.1.16) against ∇f and using the symmetries of W^φ and C^φ we deduce

$$(m-2)B^\varphi_{ij} f_i = [1 + (m-2)\mu]\left(D^\varphi_{ijk,k} f_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2 f_j\right).$$

Since (6.2.2) and (6.2.6) hold, we infer from the above

$$D^\varphi_{ijk,k} f_i - \frac{\alpha}{m-2}|\tau(\varphi)|^2 f_j = 0.$$

Contracting once again against ∇f we get

$$D^\varphi_{ijk,k} f_i f_j - \frac{\alpha}{m-2}|\tau(\varphi)|^2|\nabla f|^2 = 0. \tag{6.2.7}$$

To proceed we first prove the identity

$$|D^\varphi|^2 = \frac{2}{m-2} D^\varphi_{ijk} R^\varphi_{ij} f_k. \tag{6.2.8}$$

It can be proved using the definition (6.1.4) of D^φ and its properties (6.1.6) and (6.1.7) as follows:

$$\begin{aligned}
|D^\varphi|^2 =& D^\varphi_{ijk} D^\varphi_{ijk} \\
=& \frac{1}{m-2} D^\varphi_{ijk}\left[R^\varphi_{ij} f_k - R^\varphi_{ik} f_j + \frac{1}{m-1} f_t (R^\varphi_{tk}\delta_{ij} - R^\varphi_{tj}\delta_{ik}) - \frac{S^\varphi}{m-1}(f_k\delta_{ij} - f_j\delta_{ik})\right] \\
=& \frac{1}{m-2}\left[D^\varphi_{ijk}(R^\varphi_{ij} f_k - R^\varphi_{ik} f_j) + \frac{1}{m-1} f_t (D^\varphi_{iik} R^\varphi_{tk} - D^\varphi_{iji} R^\varphi_{tj}) - \frac{S^\varphi}{m-1}(f_k D^\varphi_{iik} - f_j D^\varphi_{iji})\right] \\
=& \frac{1}{m-2} D^\varphi_{ijk} R^\varphi_{ij} f_k - \frac{1}{m-2} D^\varphi_{ikj} R^\varphi_{ij} f_k \\
=& \frac{2}{m-2} D^\varphi_{ijk} R^\varphi_{ij} f_k.
\end{aligned}$$

To obtain (6.2.5) from (6.2.7) notice that, using (6.1.6), (6.0.1) and (6.1.7)

$$\begin{aligned}
D^{\varphi}_{ijk,k}f_if_j =&(D^{\varphi}_{ijk}f_if_j)_k - D^{\varphi}_{ijk}f_{ik}f_j - D^{\varphi}_{ijk}f_if_{jk}\\
=&(D^{\varphi}_{ijk}f_if_j)_k - D^{\varphi}_{ijk}f_{ik}f_j\\
=&(D^{\varphi}_{ijk}f_if_j)_k + D^{\varphi}_{ijk}f_{ij}f_k\\
=&(D^{\varphi}_{ijk}f_if_j)_k + D^{\varphi}_{ijk}(-R^{\varphi}_{ij} + \mu f_if_j + \lambda\delta_{ij})f_k\\
=&(D^{\varphi}_{ijk}f_if_j)_k - D^{\varphi}_{ijk}R^{\varphi}_{ij}f_k + \mu D^{\varphi}_{ijk}f_if_jf_k + \lambda D^{\varphi}_{iik}f_k\\
=&(D^{\varphi}_{ijk}f_if_j)_k - D^{\varphi}_{ijk}R^{\varphi}_{ij}f_k,
\end{aligned}$$

and thus we conclude using (6.2.4) and (6.2.8). □

We are now ready to prove the first important result of this Section.

Theorem 6.2.9. *Let* $(M, \langle\, ,\, \rangle)$ *be a complete Riemannian manifold of dimension* m *with an Einstein-type structure as in* (6.0.1). *Suppose that* $m \geq 3$, *that* $\alpha > 0$, *that* (6.2.2) *and* (6.2.1) *hold and that* f *is proper and non constant. Then* $D^{\varphi} = 0$ *and* φ *is harmonic.*

Proof. Let c be a regular value of f and let Σ_c and Ω_c be its corresponding sublevel hypersurface and set, that is

$$\Omega_c := \{x \in M : f(x) \leq c\}, \quad \Sigma_c := \{x \in M : f(x) = c\} = \partial\Omega_c. \tag{6.2.10}$$

Integrating (6.2.5) on M, that holds since we are assuming the validity of (6.2.1), and applying the divergence theorem

$$\frac{m-2}{2}\int_{\Omega_c}|D^{\varphi}|^2 + \frac{\alpha}{m-2}\int_{\Omega_c}|\tau(\varphi)|^2|\nabla f|^2 = \int_{\Sigma_c}\langle Y, \nu\rangle,$$

where ν is the outward unit normal to Σ_c and Y is the vector field with components defined by (6.2.4). Since ν is in the direction of ∇f and since, using (6.1.6)

$$\langle Y, \nabla f\rangle = Y^k f_k = D^{\varphi}_{ijk}f_if_jf_k = 0,$$

we obtain

$$\frac{m-2}{2}\int_{\Omega_c}|D^{\varphi}|^2 + \frac{\alpha}{m-2}\int_{\Omega_c}|\tau(\varphi)|^2|\nabla f|^2 = 0.$$

Since c is an arbitrary regular point of f we easily conclude, letting $c \to +\infty$,

$$\frac{m-2}{2}\int_{M}|D^{\varphi}|^2 + \frac{\alpha}{m-2}\int_{M}|\tau(\varphi)|^2|\nabla f|^2 = 0.$$

and since $\alpha > 0$ and, using the second equation in (6.0.1), the vanishing of $|\tau(\varphi)|^2|\nabla f|^2$ is equivalent to the harmonicity of φ, the thesis follows at once. □

Remark 6.2.11. Note that we can give the vector field Y the following remarkable form:

$$(m-1)Y = \mathrm{Ric}^{\varphi}(\nabla f, \nabla f)\nabla f - |\nabla f|^2\mathrm{Ric}^{\varphi}(\nabla f, \cdot)^{\sharp}. \tag{6.2.12}$$

Indeed, from the definition (6.1.4) of D^{φ}

$$\begin{aligned}
D^{\varphi}_{ijk}f_i =&\frac{1}{m-2}\left[R^{\varphi}_{ij}f_if_k - R^{\varphi}_{ik}f_if_j + \frac{1}{m-1}f_t(R^{\varphi}_{tk}f_j - R^{\varphi}_{tj}f_k) - \frac{S^{\varphi}}{m-1}(f_kf_j - f_jf_k)\right]\\
=&\frac{1}{m-2}\left[\left(1 - \frac{1}{m-1}\right)R^{\varphi}_{ij}f_if_k - \left(1 - \frac{1}{m-1}\right)R^{\varphi}_{ik}f_if_j\right]\\
=&\frac{1}{m-1}f_i(R^{\varphi}_{ij}f_k - R^{\varphi}_{ik}f_j).
\end{aligned}$$

Therefore we have

$$Y^k = D^\varphi_{ijk} f_i f_j = \frac{1}{m-1}(R^\varphi_{ij} f_i f_j f_k - R^\varphi_{ik} f_i |\nabla f|^2),$$

that is (6.2.12).

Remark 6.2.13. Observe that in the degenerate case where $\mu = -\frac{1}{m-2}$, that is, when $(M, \langle\, ,\,\rangle)$ is a conformally harmonic-Einstein manifold by Theorem 2.3.5, the condition (6.2.1) is always satisfied. It follows by contracting the second integrability condition (2.3.4) against ∇f, using the skew symmetry of W^φ in the first two indexes. Observe that a sufficient condition to guarantee (6.2.1) is that $(M, \langle\, ,\,\rangle)$ is φ-Bach flat, that is, $B^\varphi = 0$. In case $m \neq 4$ this requirement is quite strong, since from Proposition 1.2.52 it implies φ is a harmonic map. On the contrary in case $m = 4$ it seems a reasonable assumption, since B^φ is traceless. Notice that if M is compact then f is always proper, then the only requirement is that the gradient Einstein-type structure is non trivial (that is, f is non constant), while, if M is noncompact and proper then f must be automatically non constant. Finally, one can prove that if $\mu = 0$ and λ is constant in the gradient Einstein-type structure then the potential function f is proper, proceeding as in Proposition 8.12 of [AMR].

6.3 The geometry of the level sets of f

Our aim is now to analize the consequences of Theorem 6.2.9, that is, the two simultaneous conditions

$$i)\ D^\varphi = 0, \quad ii)\ \tau(\varphi) = 0,$$

on the geometry of the level hypersurface $\Sigma_c = \partial\Omega_c$, defined as in (6.2.10), for a regular value of f. We fix the indexes ranges

$$1 \leq i, j, \ldots \leq m, \quad 1 \leq a, b, \ldots \leq m-1, \quad 1 \leq A, B, \ldots \leq n.$$

With respect to a local orthonormal coframe on M we have, combining $i)$ and $ii)$ above with (6.0.1),

$$\begin{cases} R^\varphi_{ij} + f_{ij} = \mu f_i f_j + \lambda\delta_{ij}, \\ \varphi^A_{ii} = 0 = \varphi^A_i f_i, \\ D^\varphi_{ijk} = 0. \end{cases} \tag{6.3.1}$$

The following Proposition provides the relation between the norm of D^φ and the curvature of the level hypersurfaces of f, it uses only the first and the last equation of (6.3.1).

Proposition 6.3.2. *Let $(M, \langle\, ,\,\rangle)$ be a Riemannian manifold of dimension $m \geq 3$ that satisfies the first equation of* (6.3.1)*. Let c be a regular value of f and let Σ_c be the corresponding level hypersurface. For $p \in \Sigma_c$ choose a local first order frame along f, that is a local orthonormal frame $\{e_i\}$ such that $e_1, \ldots, e_{m-1}$ are tangent to Σ_c and*

$$e_m = \frac{\nabla f}{|\nabla f|}.$$

Then, at p,

$$\frac{(m-2)^2}{2|\nabla f|^2}|D^\varphi|^2 = |\mathring{h}|^2|\nabla f|^2 + \frac{m-2}{m-1} R^\varphi_{am} R^\varphi_{am}, \tag{6.3.3}$$

where $\mathring{h}$ is the traceless part of h, the second fundamental form of Σ_c.

Proof. Let c be a regular value of f, $p \in \Sigma_c$ and $\{e_i\}$ a local first order frame along f, then

$$f_a = 0, \quad f_m = |\nabla f|. \tag{6.3.4}$$

Let h be the second fundamental form of Σ_c, then (see proof of Proposition 6.1 of [CMMR])

$$h = h_{ab}\theta^a \otimes \theta^b \otimes \nu,$$

where

$$\nu = \frac{\nabla f}{|\nabla f|} = e_m$$

and

$$h_{ab} = -\theta_a^m(e_b) = -\frac{f_{ab}}{|\nabla f|}. \tag{6.3.5}$$

Using the first equation of (6.3.1), that holds by hypothesis,

$$h_{ab} = \frac{1}{|\nabla f|}(R^\varphi_{ab} - \mu f_a f_b - \lambda\delta_{ab}) = \frac{1}{|\nabla f|}(R^\varphi_{ab} - \lambda\delta_{ab}). \tag{6.3.6}$$

The mean curvature h is defined as

$$\mathrm{h} := \frac{h_{aa}}{m-1}. \tag{6.3.7}$$

Tracing (6.3.6) we deduce the validity of

$$\mathrm{h} = \frac{1}{|\nabla f|}\left(\frac{S^\varphi - R^\varphi_{mm}}{m-1} - \lambda\right). \tag{6.3.8}$$

We denote by $\mathring{h}$ the traceless part of h, that is,

$$\mathring{h}_{ab} := h_{ab} - \mathrm{h}\delta_{ab}.$$

Using (6.3.6) and (6.3.8) we obtain

$$\begin{aligned}|\mathring{h}|^2 =& |h|^2 - (m-1)\mathrm{h}^2\\ =& \frac{1}{|\nabla f|^2}\left[|\mathrm{Ric}^\varphi|^2 - 2R^\varphi_{am}R^\varphi_{am} - \frac{m}{m-1}(R^\varphi_{mm})^2 - \frac{1}{m-1}(S^\varphi)^2 + \frac{2}{m-1}S^\varphi R^\varphi_{mm}\right],\end{aligned}$$

that is,

$$|\nabla f|^2|\mathring{h}|^2 = |\mathrm{Ric}^\varphi|^2 - 2R^\varphi_{am}R^\varphi_{am} - \frac{m}{m-1}(R^\varphi_{mm})^2 - \frac{1}{m-1}(S^\varphi)^2 + \frac{2}{m-1}S^\varphi R^\varphi_{mm}. \tag{6.3.9}$$

Then we compute $|D^\varphi|^2$ on M. A long and tedius computation yields the validity, where $\nabla f \neq 0$, of the following

$$\frac{(m-2)^2}{2|\nabla f|^2}|D^\varphi|^2 = |\mathrm{Ric}^\varphi|^2 - \frac{m}{m-1}R^\varphi_{ma}R^\varphi_{ma} - \frac{m}{m-1}(R^\varphi_{mm})^2 - \frac{1}{m-1}(S^\varphi)^2 + \frac{2}{m-1}S^\varphi R^\varphi_{mm}. \tag{6.3.10}$$

Indeed, using the definition (6.1.4) of D^φ we obtain, simplifying and rearranging the terms,

$$\frac{(m-2)^2}{2}|D^\varphi|^2 = |\mathrm{Ric}^\varphi||\nabla f|^2 - \frac{m}{m-1}(R^\varphi)^2_{ij}f_if_j - \frac{1}{m-1}(S^\varphi)^2|\nabla f|^2 + \frac{2}{m-1}S^\varphi R^\varphi_{ij}f_if_j,$$

where

$$(R^\varphi)^2_{ij} = R^\varphi_{ik}R^\varphi_{jk}.$$

Using (6.3.4) we deduce immediately

$$(R^\varphi)^2_{ij}f_if_j = [R^\varphi_{am}R^\varphi_{am} + (R^\varphi_{mm})^2]|\nabla f|^2, \quad R^\varphi_{ij}f_if_j = R^\varphi_{mm}|\nabla f|^2,$$

so that, from the above, we conclude the validity of (6.3.10).

By plugging (6.3.9) in (6.3.10) we deduce the validity of (6.3.3). □

Remark 6.3.11. In the assumptions of Proposition 6.3.2, using also the third equation of (6.3.1), that is, $D^\varphi = 0$ then Σ_c is totally umbilical, that is

$$\mathring{h} = 0,$$

or equivalently

$$h_{ab} = \frac{1}{|\nabla f|}\left(\frac{S^\varphi - R^\varphi_{mm}}{m-1} - \lambda\right)\delta_{ab}, \tag{6.3.12}$$

and for every $a = 1, \dots, m-1$

$$R^\varphi_{am} = 0. \tag{6.3.13}$$

Then, by plugging (6.3.12) in (6.3.6) we obtain

$$\frac{1}{|\nabla f|}(R^\varphi_{ab} - \lambda\delta_{ab}) = \frac{1}{|\nabla f|}\left(\frac{S^\varphi - R^\varphi_{mm}}{m-1} - \lambda\right)\delta_{ab},$$

that is,

$$R^\varphi_{ab} = \frac{S^\varphi - R^\varphi_{mm}}{m-1}\delta_{ab}. \tag{6.3.14}$$

In the following Proposition also the second equation of (6.3.1) comes into play.

Proposition 6.3.15. *In the assumptions and the notations above, that is all the equations of* (6.3.1) *are satisfied and c is a regular value of f, the quantities $|\nabla f|$, S^φ and λ and the mean curvature h are constants on Σ_c. In particular Σ_c is totally umbilical hypersurface of $(M, \langle\, ,\, \rangle)$ with constant mean curvature. Moreover ${}^{\Sigma_c}S^\psi$ is constant on Σ_c, where*

$${}^{\Sigma_c}S^\psi = {}^{\Sigma_c}S - \alpha|d\psi|^2_{\Sigma_c}$$

is the ψ-scalar curvature of the Riemannian manifold $(\Sigma_c, \langle\, ,\, \rangle_{\Sigma_c})$, where $\langle\, ,\, \rangle_{\Sigma_c}$ is the metric induced on Σ_c and $\psi := \varphi|_{\Sigma_c}$.

Proof. We use the notations of Proposition 6.3.2. Clearly, using (6.3.4),

$$\frac{|\nabla f|^2_a}{2} = f_{ia}f_i = f_{ma}|\nabla f|,$$

hence from the first equation of (6.3.1) we obtain

$$\frac{|\nabla f|^2_a}{2} = (-R^\varphi_{ma} + \mu f_m f_a + \lambda\delta_{ma})|\nabla f| = (-R^\varphi_{ma} + \mu f_m f_a)|\nabla f|.$$

Using in it once again (6.3.4), together with (6.3.13) we deduce

$$|\nabla f|^2_a = 0. \tag{6.3.16}$$

Hence $|\nabla f|$ is a positive constant on Σ_c (it is positive because c is a regular value for f).

The fact that $d\varphi(\nabla f) = 0$ implies the validity

$$\varphi^A_m = 0, \tag{6.3.17}$$

indeed, using (6.3.4),

$$\varphi^A_i f_i = \varphi^A_m|\nabla f|,$$

and since $|\nabla f|$ is positive we conclude that (6.3.17) holds.

The fact that the immersion is totally umbilical gives

$$(m-2)\mathrm{h}_b = R_{mb}. \tag{6.3.18}$$

Indeed, by Codazzi equation (see, for instance, (1.145) of [AMR])

$$h_{ab,c} = h_{ac,b} - R^m_{abc}, \tag{6.3.19}$$

hence summing on $a = c$ we get
$$h_{ab,a} = h_{aa,b} - R^m_{aba},$$
that is, using (6.3.7),
$$h_{ab,a} = (m-1)\mathrm{h}_b - R_{mb}$$
Since the immersion is totally umbilical
$$h_{ab,a} = \mathrm{h}_b,$$
hence the above relation reads
$$\mathrm{h}_b = (m-1)\mathrm{h}_b - R_{mb},$$
that is (6.3.18).

Using the definition (1.2.2), (6.3.13) and (6.3.17),
$$R_{mb} = R^\varphi_{mb} + \alpha\varphi^A_m\varphi^A_b = 0. \tag{6.3.20}$$
hence from (6.3.18) we get that Σ_c has constant mean curvature h.

Using (6.1.2) with $i = b$ and (6.3.4) we get
$$\frac{1}{2}S^\varphi_b = R^\varphi_{mb}|\nabla f| + \mu f_{mk}|\nabla f| + (m-1)\lambda_b,$$
so that, using (6.3.13) and the first equation of (6.3.1) we conclude
$$\frac{1}{2}S^\varphi_b = \mu(-R^\varphi_{mb} + \mu f_b f_m + \lambda\delta_{mb})|\nabla f| + (m-1)\lambda_b,$$
that is, using once again (6.3.13) and (6.3.4),
$$\frac{1}{2}S^\varphi_b = (m-1)\lambda_b.$$
It follows that
$$\frac{1}{2}S^\varphi - (m-1)\lambda \tag{6.3.21}$$
is constant on Σ_c. In particular, if we show that S^φ is constant on Σ_c we can conclude also that λ is constant on Σ_c. To show that S^φ is constant on Σ_c we first observe that (6.3.8) can be rewritten as
$$|\nabla f|\mathrm{h} = \frac{S^\varphi - R^\varphi_{mm}}{m-1} - \lambda, \tag{6.3.22}$$
or equivalently
$$(m-1)|\nabla f|\mathrm{h} = S^\varphi - R^\varphi_{mm} - (m-1)\lambda = \left(\frac{1}{2}S^\varphi - (m-1)\lambda\right) + \frac{1}{2}S^\varphi - R^\varphi_{mm},$$
and since both $|\nabla f|\mathrm{h}$ and (6.3.21) are constants on Σ_c we can conclude that also
$$\frac{1}{2}S^\varphi - R^\varphi_{mm}$$
is constant on Σ_c. Then it is sufficient to show that R^φ_{mm} is constant to obtain that S^φ is constant and then conclude. For this purpose, observe that using the first equation of (6.3.1), (6.3.4), (6.3.14) and (6.3.22)
$$f_{aa} = -R^\varphi_{aa} + \mu f_a f_a + \lambda\delta_{aa} = -\left(\frac{S^\varphi - R^\varphi_{mm}}{m-1}\right)(m-1) + (m-1)\lambda = -(m-1)|\nabla f|\mathrm{h},$$
that is,
$$f_{aa} = -(m-1)|\nabla f|\mathrm{h}, \tag{6.3.23}$$

hence f_{aa} is constant on Σ_c. Using (6.1.2) with $i = m$ and (6.3.4) we can conclude

$$\begin{aligned}\frac{1}{2}S^{\varphi}_m =& R^{\varphi}_{km} f_k + \mu(f_{km}f_k - \Delta f f_m) + (m-1)\lambda_m\\ =& R^{\varphi}_{am} f_a + R^{\varphi}_{mm}|\nabla f| + \mu(f_{am}f_a + f_{mm}|\nabla f| - \Delta f|\nabla f|) + (m-1)\lambda_m\\ =& R^{\varphi}_{mm}|\nabla f| + \mu(f_{mm} - \Delta f)|\nabla f| + (m-1)\lambda_m\\ =& R^{\varphi}_{mm}|\nabla f| - \mu f_{aa}|\nabla f| + (m-1)\lambda_m,\end{aligned}$$

hence

$$\frac{1}{2}S^{\varphi}_m - R^{\varphi}_{mm}|\nabla f| - (m-1)\lambda_m$$

is constant on Σ_c. Then, using once again that $|\nabla f|$ and (6.3.21) are constants on Σ_c,

$$\begin{aligned}0 =& \left(\frac{1}{2}S^{\varphi}_m - R^{\varphi}_{mm}|\nabla f| - (m-1)\lambda_m\right)_a\\ =& \left(\frac{1}{2}S^{\varphi} - (m-1)\lambda\right)_{am} - R^{\varphi}_{mm,a}|\nabla f|\\ =& - R^{\varphi}_{mm,a}|\nabla f|,\end{aligned}$$

so that R^{φ}_{mm} is constant on Σ_c and the proof of the constancy of S^{φ} on Σ_c is concluded.

We denote by ψ the restriction of φ on Σ_c, that is,

$$\psi = \varphi \circ \imath, \tag{6.3.24}$$

where $\imath : \Sigma_c \to M$ is the inclusion. By definition

$$d\psi = \psi^A_a \theta^a \otimes E_A,$$

where

$$\psi^*\omega^A = \psi^A_a \theta^a.$$

Observe that

$$\psi^A_a = \varphi^A_a, \tag{6.3.25}$$

indeed, using (6.3.24) and $\varphi^*\omega^A = \varphi^A_i \theta^i$, we get

$$\psi^*\omega^A = \imath^*\varphi^*\omega^A = \imath^*(\varphi^A_i\theta^i) = \varphi^A_i \imath^*\theta^i = \varphi^A_a\theta^a,$$

where we used the standard identification

$$\imath^*\theta^a = \theta^a$$

and the fact that

$$\imath^*\theta^m = 0.$$

Notice that, from (6.3.25) and (6.3.17),

$$|d\psi|^2_{\Sigma_c} = \psi^A_a\psi^A_a = \varphi^a_i\varphi^a_i = |d\varphi|^2.$$

As a consequence, the ψ-scalar curvature of Σ_c, endowed with the metric $\imath^*\langle\, ,\,\rangle := \langle\, ,\,\rangle_{\Sigma_c}$, is given by

$${}^{\Sigma_c}S^{\psi} = {}^{\Sigma_c}S - \alpha|d\varphi|^2, \tag{6.3.26}$$

where ${}^{\Sigma_c}S$ is the scalar curvature of $(\Sigma_c, \langle\, ,\,\rangle_{\Sigma_c})$.

The Gauss equations (see, for instance, (1.139) of [AMR]) are given by

$${}^{\Sigma_c}R_{abcd} = R_{abcd} + h_{ac}h_{bd} - h_{ad}h_{bc},$$

and since the immersion is totally umbilical, the above reads

$$^{\Sigma_c}R_{abcd} = R_{abcd} + \mathrm{h}^2(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}).$$

Then, summing on $a = c$

$$^{\Sigma_c}R_{bd} = R_{bd} - R_{mbmd} + (m-2)\mathrm{h}^2\delta_{bd}. \tag{6.3.27}$$

Summing the above on $b = d$

$$^{\Sigma_c}S = S - 2R_{mm} + (m-2)(m-1)\mathrm{h}^2. \tag{6.3.28}$$

Using (6.3.26), (6.3.28) and the definition of φ-scalar curvature of $(M, \langle\, , \,\rangle)$ we obtain

$$^{\Sigma_c}S^{\psi} = S^{\varphi} - 2R_{mm} + (m-2)(m-1)\mathrm{h}^2.$$

Moreover, using the definition (1.2.2) and (6.3.17),

$$R^{\varphi}_{mm} = R_{mm} - \alpha\varphi^A_m\varphi^A_m = R_{mm},$$

and since we already showed that R^{φ}_{mm}, S^{φ} and h are constants on Σ_c we get from the above that $^{\Sigma_c}S^{\psi}$ is constant too, concluding the proof. □

Our aim now it to show that $(\Sigma_c, \langle\, , \,\rangle_{\Sigma_c})$ is harmonic-Einstein with respect to α and ψ, for a regular value c of f. In order to prove it we need the following result.

Proposition 6.3.29. *In the assumptions above $C^{\varphi} = 0$ on $\{x \in M : \nabla f(x) \neq 0\}$.*

Remark 6.3.30. In the assumptions of Proposition 6.3.29, if the potential function f is real analytic in harmonic coordinates, then $(M, \langle\, , \,\rangle)$ is φ-Cotton flat. Indeed, since $C^{\varphi} = 0$ on $\{x \in M : \nabla f(x) \neq 0\}$, if $\{x \in M : \nabla f(x) \neq 0\}$ is dense on M we get $C^{\varphi} = 0$ on M, by continuity. Assume by contradiction $\{x \in M : \nabla f(x) \neq 0\}$ is not dense on M, then there exists an open subset of M such that f is constant on it. Then, since f is real analytic in harmonic coordinates, f is constant on all M. Contradiction.

Remark 6.3.31. Following the argument in Proposition 2.4 of [HPW] (inspired by Theorem 5.26 of [B], that relies on the work of De Turk and Kazdan), it is easy to get that, for a general structure (6.0.1), g, f and $d\varphi$ are real analytic in harmonic coordinates, at least when λ is constant. Indeed, when $\mu \neq 0$, rewriting (6.0.1) in terms of $u := e^{\mu f}$ we get

$$\begin{cases} \mu u \mathrm{Ric}^{\varphi} - \alpha\mu\varphi^*\langle\, , \,\rangle_N - \mathrm{Hess}(u) - \mu\lambda u\langle\, , \,\rangle = 0 \\ u\tau(\varphi) + \dfrac{1}{\mu}d\varphi(\nabla u). \end{cases} \tag{6.3.32}$$

Since λ is constant, the Hamilton equation (7.1.8) holds, that in terms of u is given by

$$u\Delta u + \frac{1-\mu}{\mu}|\nabla u|^2 + \lambda u^2 - \Lambda = 0, \tag{6.3.33}$$

where Λ is a real constant. Coupling (6.3.32) and (6.3.33) we obtain the following system, in harmonic coordinates $(x^1, \ldots, x^m)$,

$$\begin{cases} \dfrac{\mu u}{2}g^{kt}\dfrac{\partial^2 g_{ij}}{\partial x^k \partial x^t} + \dfrac{\partial^2 u}{\partial x^i \partial x^j} + \ldots = 0 \\ ug^{kt}\dfrac{\partial^2 u}{\partial x^k \partial x^t} + \ldots = 0 \\ ug^{kt}\dfrac{\partial^2 \varphi^a_i}{\partial x^k \partial x^t} + \ldots = 0, \end{cases}$$

where the dots denote lower order terms and, on the domain of $(x^1, \ldots, x^m)$,

$$g = g_{ij}dx^i \otimes dx^j, \quad d\varphi = \varphi^a_i dx^i \otimes \frac{\partial}{\partial y^a},$$

where $(y^1, \dots, y^n)$ are local coordinates on M. Proceeding as in the proof of Proposition 6.3.29 we conclude that g_{ij}, u and φ^a_i are real analytic on the domain of $(x^1, \dots, x^m)$. When $\mu = 0$ we couple (6.0.1) with (7.1.9), we set $u = -f$ and we proceed as above.

Proof (of Proposition 6.3.29). We fix $x \in M$ such that $\nabla f(x) \neq 0$ and we set $c := f(x)$. Since c is a regular value we may take a local first order frame $\{e_i\}$ along f. By the first integrability condition (6.1.11), since we are assuming the validity of the third equation of (6.3.1) we deduce

$$C^{\varphi}_{ijk} = -f_t W^{\varphi}_{tijk}. \tag{6.3.34}$$

Hence, contracting the above against ∇f, by the symmetries of W^{φ} and using (6.3.4)

$$0 = -f_i f_t W^{\varphi}_{tijk} = f_i C^{\varphi}_{ijk} = C^{\varphi}_{mjk}|\nabla f|.$$

Then $C^{\varphi}_{mjk} = 0$. Since Σ_c is totally umbilical with constant mean curvature h is parallel, that is $h_{ab,c} = 0$. Then, from Codazzi's equation (6.3.19) we get

$$R_{mabc} = 0, \tag{6.3.35}$$

But then, explicitating the decomposition (1.2.18)

$$R_{mabc} = W^{\varphi}_{mabc} + \frac{1}{m-2}\left[R^{\varphi}_{mb}\delta_{ac} - R^{\varphi}_{mc}\delta_{ab} + R^{\varphi}_{ac}\delta_{mb} - R^{\varphi}_{ab}\delta_{mc} - \frac{S^{\varphi}}{m-1}(\delta_{mb}\delta_{ac} - \delta_{mc}\delta_{ab})\right],$$

and since (6.3.13) holds we conclude from the above equality and (6.3.35) that

$$W^{\varphi}_{mabc} = 0. \tag{6.3.36}$$

Therefore, from (6.3.34), using (6.3.4) and (6.3.36) we obtain

$$C^{\varphi}_{abc} = -f_t W^{\varphi}_{tabc} = -f_d W^{\varphi}_{dabc} - |\nabla f| W^{\varphi}_{mabc} = 0.$$

By the symmetries of C^{φ} it remains only to prove

$$C^{\varphi}_{amb} = 0 \tag{6.3.37}$$

First of all observe that

$$R^{\varphi}_{am,k}\theta^k = \frac{S^{\varphi} - mR^{\varphi}_{mm}}{m-1}\theta^m_a, \tag{6.3.38}$$

in fact from the definition of covariant derivative, since (6.3.13) holds,

$$\begin{aligned}
0 =& dR^{\varphi}_{am} \\
=& R^{\varphi}_{km}\theta^k_a + R^{\varphi}_{ak}\theta^k_m + R^{\varphi}_{am,k}\theta^k \\
=& R^{\varphi}_{bm}\theta^b_a + R^{\varphi}_{mm}\theta^m_a + R^{\varphi}_{ab}\theta^b_m + R^{\varphi}_{am}\theta^m_m + R^{\varphi}_{am,k}\theta^k \\
=& R^{\varphi}_{mm}\theta^m_a + R^{\varphi}_{ab}\theta^b_m + R^{\varphi}_{am,k}\theta^k
\end{aligned}$$

and thus, using also (6.3.14) from the above equality we obtain

$$\begin{aligned}
R^{\varphi}_{am,k}\theta^k =& -R^{\varphi}_{mm}\theta^m_a - R^{\varphi}_{ab}\theta^b_m \\
=& -R^{\varphi}_{mm}\theta^m_a - \frac{S^{\varphi} - R^{\varphi}_{mm}}{m-1}\delta_{ab}\theta^b_m \\
=& -R^{\varphi}_{mm}\theta^m_a - \frac{S^{\varphi} - R^{\varphi}_{mm}}{m-1}\theta^a_m \\
=& \frac{S^{\varphi} - mR^{\varphi}_{mm}}{m-1}\theta^m_a,
\end{aligned}$$

that is (6.3.38). Now we are going to prove

$$R^{\varphi}_{am,m} = 0. \tag{6.3.39}$$

Observe that, by taking $i = a$ and $j = m$ in the first equation of (6.3.1) we obtain

$$R^{\varphi}_{am} + f_{am} = \mu f_a f_m,$$

and thus, using (6.3.13) and (6.3.4), we deduce

$$f_{am} = 0. \tag{6.3.40}$$

Moreover, taking the covariant derivative of the first equation of (6.3.1) we infer

$$R^{\varphi}_{ij,k} + f_{ijk} = \mu f_{ik} f_j + \mu f_i f_{jk} + \lambda_k \delta_{ij}$$

that for $i = m = k$ and $j = a$ reads as

$$R^{\varphi}_{am,m} + f_{mam} = \mu f_{mm} f_a + \mu f_m f_{am}.$$

Using (6.3.40) and (6.3.4) into the above we immediately get (6.3.39).

From (6.3.39) and (6.3.38) we infer

$$R^{\varphi}_{am,b}\theta^b = R^{\varphi}_{am,k}\theta^k = \frac{S^{\varphi} - mR^{\varphi}_{mm}}{m-1}\theta^m_a.$$

Using (6.3.5) in the above equality we deduce the validity of

$$R^{\varphi}_{am,b} = \frac{S^{\varphi} - mR^{\varphi}_{mm}}{m-1}\theta^m_a(e_b) = \frac{1}{|\nabla f|}\frac{mR^{\varphi}_{mm} - S^{\varphi}}{m-1} f_{ab}. \tag{6.3.41}$$

Then we finally obtain, using (1.2.42), (6.3.14) and (6.3.41)

$$\begin{aligned}
C^{\varphi}_{abm} =& R^{\varphi}_{ab,m} - R^{\varphi}_{am,b} - \frac{1}{2(m-1)} S^{\varphi}_m \delta_{ab} \\
=& \left(\frac{S^{\varphi} - R^{\varphi}_{mm}}{m-1}\delta_{ab}\right)_m + \frac{1}{|\nabla f|}\frac{S^{\varphi} - mR^{\varphi}_{mm}}{m-1} f_{ab} - \frac{1}{2(m-1)} S^{\varphi}_m \delta_{ab} \\
=& \frac{S^{\varphi}_m - R^{\varphi}_{mm,m}}{m-1}\delta_{ab} - \frac{1}{2(m-1)} S^{\varphi}_m \delta_{ab} + \frac{1}{|\nabla f|}\frac{S^{\varphi} - mR^{\varphi}_{mm}}{m-1} f_{ab} \\
=& \frac{1}{2(m-1)} S^{\varphi}_m \delta_{ab} - \frac{1}{m-1} R^{\varphi}_{mm,m}\delta_{ab} + \frac{1}{|\nabla f|}\frac{S^{\varphi} - mR^{\varphi}_{mm}}{m-1} f_{ab}.
\end{aligned}$$

Moreover, since φ is harmonic, from (1.2.26),

$$\frac{1}{2} S^{\varphi}_m = R^{\varphi}_{im,i} = R^{\varphi}_{cm,c} + R^{\varphi}_{mm,m}.$$

By inserting it in the above equality we deduce the validity of

$$C^{\varphi}_{abm} = \frac{1}{m-1} R^{\varphi}_{cm,c}\delta_{ab} + \frac{1}{|\nabla f|}\frac{S^{\varphi} - mR^{\varphi}_{mm}}{m-1} f_{ab}. \tag{6.3.42}$$

Taking the trace of (6.3.41) and using (6.3.23) we have

$$R^{\varphi}_{cm,c} = -(mR^{\varphi}_{mm} - S^{\varphi})\mathrm{h}. \tag{6.3.43}$$

On the other hand, using (6.3.5) and the fact that the immersion is totally umbilical we obtain

$$\frac{1}{|\nabla f|}\frac{S^{\varphi} - mR^{\varphi}_{mm}}{m-1} f_{ab} = -\frac{S^{\varphi} - mR^{\varphi}_{mm}}{m-1} h_{ab} = \frac{mR^{\varphi}_{mm} - S^{\varphi}}{m-1}\mathrm{h}\delta_{ab}. \tag{6.3.44}$$

Using (6.3.43) and (6.3.44) in (6.3.42) we conclude the validity of (6.3.37), then the proof is completed. □

We are now able to prove the following Proposition, as announced before.

Proposition 6.3.45. *In the assumptions above, for every regular value c of f, Σ_c is harmonic Einstein with respect the induced metric $\langle\, ,\,\rangle_{\Sigma_c}$, α and the restriction ψ of φ on Σ_c, that is,*

$$\begin{cases} {}^{\Sigma_c}Ric - \alpha\psi^*\langle\, ,\,\rangle_N = \dfrac{{}^{\Sigma_c}S^\psi}{m-1}\langle\, ,\,\rangle_{\Sigma_c} \\ {}^{\Sigma_c}\tau(\psi) = 0. \end{cases} \tag{6.3.46}$$

Proof. First of all we prove that $\psi : (\Sigma_c, \langle\, ,\,\rangle_{\Sigma_c}) \to (N, \langle\, ,\,\rangle_N)$ is harmonic. Taking the covariant derivative of $d\varphi(\nabla f) = 0$ we get

$$\varphi^A_{ij} f_i + \varphi^A_i f_{ij} = 0.$$

Using (6.3.4) and (6.3.17) into the above we get

$$\varphi^A_{mj}|\nabla f| + \varphi^A_a f_{aj} = 0.$$

By choosing $j = m$, since $f_{am} = 0$ (see (6.3.40)), we get

$$\varphi^A_{mm}|\nabla f| = 0,$$

that implies

$$\varphi^A_{mm} = 0. \tag{6.3.47}$$

Now, by definition of covariant derivative,

$$\psi^A_{ab}\theta^b = d\psi^A_a - \psi^A_b\theta^b_a + \psi^B_a\omega^A_B \tag{6.3.48}$$

and

$$\varphi^A_{ai}\theta^i = d\varphi^A_a - \varphi^A_i\theta^i_a + \varphi^B_a\omega^A_B.$$

Using (6.3.17) into the above and we get

$$\varphi^A_{ab}\theta^b + \varphi^A_{am}\theta^m = d\varphi^A_a - \varphi^A_b\theta^b_a + \varphi^B_a\omega^A_B,$$

restricting it to Σ_c, recalling (6.3.25), yields

$$\varphi^A_{ab}\theta^b = d\psi^A_a - \psi^A_b\theta^b_a + \psi^B_a\omega^A_B. \tag{6.3.49}$$

Comparing (6.3.48) with (6.3.49) we deduce

$$\psi^A_{ab} = \varphi^A_{ab}.$$

Using the above, that φ is harmonic and (6.3.47) we conclude that also ψ is harmonic, indeed

$$\psi^A_{aa} = \varphi^A_{aa} = \varphi^A_{ii} - \varphi^A_{mm} = 0.$$

Equivalently one can prove that ψ is harmonic using formula (1.180) of [AMR], that is,

$${}^{\Sigma_c}\tau(\psi) = \nabla d\varphi(d\imath(e_a), d\imath(e_a)) + d\varphi(\tau(\imath)).$$

Indeed, since φ is harmonic and $\nabla d\varphi(\nabla f, \nabla f) = 0$ (that is (6.3.47)),

$$\nabla d\varphi(d\imath(e_a), d\imath(e_a)) = \nabla d\varphi(e_a, e_a) = \tau(\varphi) - \frac{1}{|\nabla f|^2}\nabla d\varphi(\nabla f, \nabla f) = 0,$$

and since $\imath$ is an isometric immersion

$$\tau(\imath) = h_{aa}e_m = \frac{(m-1)\mathrm{h}}{|\nabla f|}\nabla f,$$

hence, using $d\varphi(\nabla f) = 0$ and that $|\nabla f|$ and h are constants on Σ_c,

$$d\varphi(\tau(\imath)) = \frac{(m-1)\mathrm{h}}{|\nabla f|} d\varphi(\nabla f) = 0.$$

It only remains to prove that the traceless part of ψ-Ricci vanishes. Recall that $C^\varphi = 0$ on Σ_c, from Proposition 6.3.29. Hence using also the third equation (6.3.1) the first integrability condition (6.1.11) implies

$$0 = C^\varphi_{ijk} + f_t W^\varphi_{tijk} = |\nabla f| W^\varphi_{mijk},$$

thus

$$W^\varphi_{mijk} = 0. \tag{6.3.50}$$

From the decomposition (1.2.18), using (6.3.50) we obtain

$$R_{mamb} = \frac{1}{m-2}\left(R^\varphi_{ab} + R^\varphi_{mm}\delta_{ab} - \frac{S^\varphi}{m-1}\delta_{ab}\right), \tag{6.3.51}$$

indeed

$$\begin{aligned} R_{mamb} =& W^\varphi_{mamb} + \frac{1}{m-2}(A^\varphi_{mm}\delta_{ab} - A^\varphi_{mb}\delta_{am} + A^\varphi_{ab} - A^\varphi_{ma}\delta_{bm}) \\ =& \frac{1}{m-2}\left(R^\varphi_{mm}\delta_{ab} - \frac{S^\varphi}{2(m-1)}\delta_{ab} + R^\varphi_{ab} - \frac{S^\varphi}{2(m-1)}\delta_{ab}\right) \\ =& \frac{1}{m-2}\left(R^\varphi_{ab} + R^\varphi_{mm}\delta_{ab} - \frac{S^\varphi}{m-1}\delta_{ab}\right). \end{aligned}$$

By plugging (6.3.14) into (6.3.51) we obtain

$$R_{mamb} = \frac{1}{m-2}\left(\frac{S^\varphi - R^\varphi_{mm}}{m-1}\delta_{ab} + R^\varphi_{mm}\delta_{ab} - \frac{S^\varphi}{m-1}\delta_{ab}\right) = \frac{R^\varphi_{mm}}{m-1}\delta_{ab}.$$

By Gauss formula (6.3.27), using the above and (1.2.2),

$${}^{\Sigma_c}R_{bd} = R^\varphi_{bd} + \alpha\varphi^A_b\varphi^A_d - \frac{R^\varphi_{mm}}{m-1}\delta_{bd} + (m-2)\mathrm{h}^2\delta_{bd}.$$

By plugging (6.3.14) into the above and using (6.3.25),

$${}^{\Sigma_c}R_{bd} - \alpha\psi^A_b\psi^A_d = \left(\frac{S^\varphi - 2R^\varphi_{mm}}{m-1} + (m-2)\mathrm{h}^2\right)\delta_{bd}.$$

that implies the validity of the first equation of (6.3.46). □

6.4 Main results

We are now ready to prove the most important results of this section.

Theorem 6.4.1. *Let $(M, \langle\,,\,\rangle)$ be a complete Riemannian manifold of dimension $m \geq 3$ endowed with a gradient Einstein-type structure as* (6.0.1) *for some $\alpha > 0$, $\varphi : M \to (N, \langle\,,\,\rangle_N)$ smooth, $\mu \in \mathbb{R}$ with* (6.2.2) *and $\lambda, f \in \mathcal{C}^\infty(M)$ with f proper and non-constant. Assume that* (6.2.1) *holds.*

For every regular level set Σ of f there exists a connected neighbourghood $\mathcal{U}$ of Σ, an open interval $I \ni 0$ and an isometry $\phi : I \times_{e^{-2H}} \Sigma \to \mathcal{U}$. Moreover $\varphi \circ \phi = \psi \circ \pi_\Sigma$ on $\mathcal{U}$, where $\psi := \varphi|_\Sigma$ and $\pi_\Sigma : I \times \Sigma \to \Sigma$ is the canonical projection. Furthermore, $\mathcal{U}$ is foliated by totally umbilical hypersurfaces Σ_r with constant mean curvature $H'(r)$ with leaves that are harmonic-Einstein with respect to the induced metric, α and $\psi_r := \varphi|_{\Sigma_r}$, where $r \in I$ and $\Sigma_0 = \Sigma$.

Proof. Our assumptions permits to apply Theorem 6.2.9 to deduce that φ must be harmonic and D^{φ} must vanish on M. Let Σ be a regular level set of f, that is $|\nabla f| \neq 0$ on Σ (it exists by Sard's theorem, since f is non-constant). In a connected neighborhood $\mathcal{U}$ of Σ which does not contain any critical point of f the potential function f only depends on the signed distance r to the hypersurface Σ. Hence, by a suitable change of variable, we can express the metric tensor $\langle \, , \, \rangle$ as

$$dr \otimes dr + g_{ab}\theta^a \otimes \theta^b,$$

where $g_{ab} = g_{ab}(r, x)$ and $r \in I := (r_*, r^*)$ for some maximal $r_* \in [-\infty, 0)$ and $r^* \in (0, +\infty]$, where $\theta^1, \dots, \theta^{m-1}$ is the local coframe on Σ induced by a local first order frame along f.

The change of variable is performed in the following way: the integral curves of vector field $Y := \frac{\nabla f}{|\nabla f|}$ are unit speed geodesic orthogonal to Σ and the flow of Y gives rise to a smooth map $\phi : I \times \Sigma \to \mathcal{U}$, that coincide with the normal exponential map of Σ. The map ϕ is an isometry when we endow $I \times \Sigma$ with the metric

$$\phi^*\langle \, , \, \rangle = dr \otimes dr + (\phi_r)^*\langle \, , \, \rangle_\Sigma,$$

where $\phi_r : \Sigma \to M$ is defined by $\phi_r(x) = \phi(r, x)$.

In particular, since ϕ_0 is the inclusion of Σ in M,

$$g_{ab}(0, \cdot)\theta^a \otimes \theta^b = (\phi_0)^*\langle \, , \, \rangle_\Sigma = \langle \, , \, \rangle_\Sigma,$$

that is,

$$g_{ab}(0, \cdot) = \delta_{ab}.$$

By definition of Lie derivative, using (6.3.5) and that every $\Sigma_r := \phi(\{r\} \times \Sigma)$ is totally umbilical with constant mean curvature $\mathrm{h}(r)$, since, as proved in Proposition 6.3.15, it is a level set with respect to a regular value of f, we obtain

$$\frac{\partial g_{ab}}{\partial r} = (\mathcal{L}_{\nabla r}\langle \, , \, \rangle)(e_a, e_a) = 2\mathrm{Hess}(r)(e_a, e_a) = 2\frac{f_{ab}}{|\nabla f|} = -2h_{ab} = -2\mathrm{h}g_{ab},$$

that is,

$$\frac{\partial g_{ab}}{\partial r}(r, \theta) = -2\mathrm{h}(r)g_{ab}(r, \theta).$$

Thus we deduce the validity of

$$g_{ab}(r, \cdot) = e^{-2H(r)}g_{ab}(0, \cdot), \qquad \text{where} \qquad H(r) = \int_0^r \mathrm{h}.$$

This proves that on $\mathcal{U}$ the metric $\langle \, , \, \rangle$ takes the form of a warped product metric

$$dr \otimes dr + e^{-2H}\langle \, , \, \rangle_\Sigma,$$

where H is a function on (r_*, r^*).

Moreover, since $d\varphi(\nabla f) = 0$ then $\varphi \circ \phi$ does not depend on r (using that $\mathcal{U}$ is connected). Hence $\varphi \circ \phi$ is the lifting to $I \times \Sigma$ of the map $\psi = \varphi|_\Sigma$, that is, $\varphi = \psi \circ \pi_\Sigma$.

Notice, finally, that $(\Sigma_r, \langle \, , \, \rangle_{\Sigma_r})$ is harmonic-Einstein with respect to ψ_r by Proposition 6.3.45 for every $r \in I$. □

Remark 6.4.2. Notice that the vanishing of D^{φ} implies that the φ-Schouten tensor is a Codazzi tensor in the conformal metric

$$\widetilde{\langle \, , \, \rangle} = e^{-\frac{2}{m-2}f}\langle \, , \, \rangle.$$

Moreover, it is easy to see that in this conformal metric the φ-Schouten tensor has at most two eigenvalues, one of multiplicity 1 and the other of multiplicity $m - 1$. Indeed, proceeding as in the proof of Theorem 2.3.5,

$$\widetilde{\mathrm{Ric}}^{\varphi} = \left(\mu + \frac{1}{m-2}\right) df \otimes df + e^{\frac{2}{m-2}f}\left(\frac{\Delta_f f}{m-2} + \lambda\right)\widetilde{\langle \, , \, \rangle},$$

hence

$$\widetilde{A}^\varphi = \left(\mu + \frac{1}{m-2}\right) df \otimes df + \eta\widetilde{\langle\, ,\,\rangle},$$

where η is an appropriate smooth function on M.

The presence of a Codazzi tensor with two eigenvalues of multiplicity 1 and $m-1$ with constant trace forces the manifold to be locally a warped product of an $m-1$-dimensional Riemannian manifolds on an interval of $\mathbb{R}$, this is a classical result by A. Derdzinski. In our situation we did not rely on this general result because the trace of the φ-Schouten tensor is not constant, in general (but one can rely on the generalization obtained in [Me] by G. Merton to obtain the local structure of a warped product).

When λ is constant we have

Theorem 6.4.3. *Let $(M, \langle\, ,\,\rangle)$ be a complete Riemannian manifold of dimension $m \geq 3$ endowed with a gradient Einstein-type structure as* (6.0.1) *for some $\alpha > 0$, $\varphi : M \to (N, \langle\, ,\,\rangle_N)$ smooth, $\mu, \lambda \in \mathbb{R}$ with* (6.2.2) *and $f \in \mathcal{C}^\infty(M)$ with f non-constant and, in case $\mu \neq 0$, proper. Assume that* (6.2.1) *holds. Then*

$$C^\varphi = 0, \quad \tau(\varphi) = 0, \quad B^\varphi = 0$$

and

$$\mathcal{W}^\varphi(T^\varphi) = 0, \tag{6.4.4}$$

where $\mathcal{W}^\varphi$ is the endomorphism of $S^2_0(M)$ defined by (2.3.16)*. In particular, if $\mathcal{W}^\varphi$ is injective, then the gradient Einstein-type structure* (6.0.1) *reduces to a harmonic-Einstein structure.*

Proof. From Theorem 6.2.9 we have $D^\varphi = 0$ and $\tau(\varphi) = 0$. Notice that, when $\mu = 0$, f is proper from Remark 6.2.13. Then, from Proposition 6.3.29, combined with Remark 6.3.30 and Remark 6.3.31, we have $C^\varphi = 0$ on M. Then the first integrability (6.1.11) yields

$$W^\varphi_{tijk} f_t = 0. \tag{6.4.5}$$

Moreover, from (6.1.16) we get that $(M, \langle\, ,\,\rangle)$ is φ-Bach flat. Then, from the definition of φ-Bach (1.2.50) we get, using that φ is harmonic, $C^\varphi = 0$,

$$0 = (m-2)B^\varphi_{ij} = W^\varphi_{tikj} R^\varphi_{tk} - \alpha R^\varphi_{kj} \varphi^a_k \varphi^a_i.$$

Using the symmetries of B^φ, Ric^φ and W^φ we also have

$$0 = (m-2)B^\varphi_{ji} = W^\varphi_{tjki} R^\varphi_{tk} - \alpha R^\varphi_{ki} \varphi^a_k \varphi^a_j = W^\varphi_{tikj} R^\varphi_{tk} - \alpha R^\varphi_{ki} \varphi^a_k \varphi^a_j,$$

so that, summing with the above

$$2W^\varphi_{tikj} R^\varphi_{tk} - \alpha \varphi^a_k (R^\varphi_{kj} \varphi^a_i + R^\varphi_{ki} \varphi^a_j) = 0. \tag{6.4.6}$$

Then we conclude the validity of (6.4.4). Indeed, from (2.3.16) and (6.4.6), using the definition of T^φ and (1.2.22),

$$\begin{aligned}
\mathcal{W}^\varphi(T^\varphi)_{ij} =& W^\varphi_{tikj} T^\varphi_{tk} - \frac{\alpha}{2} \varphi^A_t (\varphi^A_i \delta_{kj} + \varphi^A_j \delta_{ki}) T^\varphi_{tk} \\
=& W^\varphi_{tikj} R^\varphi_{tk} - \frac{S^\varphi}{m} W^\varphi_{kikj} - \frac{\alpha}{2} \varphi^A_t (\varphi^A_i T^\varphi_{tj} + \varphi^A_j T^\varphi_{ti}) \\
=& \frac{1}{2} \left(2W^\varphi_{tikj} R^\varphi_{tk} - \alpha \varphi^A_t (\varphi^A_i R^\varphi_{tj} + \varphi^A_j R^\varphi_{ti}) \right) + \frac{\alpha}{2} \frac{S^\varphi}{m} \varphi^A_t (\varphi^a_i \delta_{tj} + \varphi^A_j \delta_{ti}) - \frac{S^\varphi}{m} W^\varphi_{kikj} \\
=& \frac{\alpha}{m} S^\varphi \varphi^A_i \varphi^A_j - \frac{\alpha}{m} S^\varphi \varphi^A_i \varphi^A_j = 0.
\end{aligned}$$

If $\mathcal{W}^\varphi$ is injective then $T^\varphi = 0$, that is, since φ is harmonic, $(M, \langle\, ,\,\rangle)$ is harmonic-Einstein. ☐

Remark 6.4.7. Notice that in the assumption of the Proposition above we only required that $\mathcal{W}^\varphi$ is injective. Assuming that φ is non-constant one may ask, equivalently, that the pair $(\langle\, , \rangle, \varphi)$ is generic in the sense of Definition 2.3.17. This is due to the fact that, since φ is harmonic, by unique continuation it cannot be constant on an open set unless it is constant on the whole M. Hence the zeros of $d\varphi$ can be located only at isolated points (and they are not of infinite order).

As an application of Theorem 6.4.3 and the results of Section 4.2 we obtain

Corollary 6.4.8. *Let $(M, \langle\, , \rangle)$ be a Riemannian manifold of dimension $m \geq 3$ endowed with a non trivial gradient Einstein-type structure as* (6.0.1) *for some $\alpha > 0$, $\varphi : M \to (N, \langle\, , \rangle_N)$ smooth, $\mu, \lambda \in \mathbb{R}$ with* (6.2.2) *and $f \in \mathcal{C}^\infty(M)$. Assume that* (6.2.1) *holds and that $\mathcal{W}^\varphi$ is injective. Then M is non-compact.*
Assume moreover that $(M, \langle\, , \rangle)$ is complete and that f has exactly one critical point.

i) If $\mu = 0$ then $\lambda \neq 0$, φ is constant and $(M, \langle\, , \rangle)$ is isometric to the Euclidean space $\mathbb{R}^m$. Moreover f is given by (4.2.21) *for some $b \in \mathbb{R}^m$ and $c \in \mathbb{R}$.*

ii) If $\mu \neq 0$ and that f is proper then φ is constant, $(M, \langle\, , \rangle)$ is isometric to the hyperbolic space of constant sectional curvature $\kappa = \frac{S^\varphi}{m}$ and of dimension m and λ is given by

$$\lambda = \kappa \frac{(m-1)\mu + 1}{\mu}.$$

Proof. Assume by contradiction that M is compact. From Theorem 6.4.3 we get that $(M, \langle\, , \rangle)$ is harmonic-Einstein. Then, using Corollary 4.2.32 if $\mu = 0$ and Corollary 4.2.33 if $\mu \neq 0$, we conclude, among the other things, that λ is non-constant. Contradiction. Now assume that $(M, \langle\, , \rangle)$ is complete and, in case $\mu \neq 0$, that f is proper. Then, once again using Theorem 6.4.3, $(M, \langle\, , \rangle)$ is harmonic-Einstein. If $\mu = 0$ we get the validity of *iii*) *a*) of Theorem 4.2.19. Indeed, since λ is non-constant, *i*) and *ii*) cannot hold and, since f has exactly one critical point, *iii*) *b*) cannot hold too. If $\mu \neq 0$, since λ is constant, we get the validity of *ii*) of Theorem 4.2.25. □

Chapter 7

Einstein-type structures with λ constant

In the following we consider a complete Riemannian manifold $(M, \langle\, ,\, \rangle)$ with a gradient Einstein-type structure of the form

$$\begin{cases} \mathrm{Ric}^{\varphi} + \mathrm{Hess}(f) - \mu df \otimes df = \lambda \langle\, ,\, \rangle \\ \tau(\varphi) = d\varphi(\nabla f), \end{cases} \tag{7.0.1}$$

where $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ is a smooth map, $\alpha \in \mathbb{R} \setminus \{0\}$, $\mu, \lambda \in \mathbb{R}$ and $f \in \mathcal{C}^{\infty}(M)$.

We begin with Section 7.1 in which we generalize the classical Hamilton-type identities, a very important tool in the study of Ricci solitons, to Einstein-type structures.

In Section 7.2 we deduce from the presence of a complete Einstein-type structure, that in turns imply a lower bound on the generalized Bakry-Émery Ricci tensor, some restrictions on the volume growth of geodesic balls. As a consequence, assuming eventually that the density energy of φ and $|\nabla f|^2$ are bounded, we obtain the validity of the maximum principle at infinity for the operator Δ_f and also compactness or f-parabolicity, under some additional assumptions on the parameters involved.

In Section 7.3, by applying the weak maximum principle, we provide estimates on the infimum of the φ-scalar curvature in the complete case, when $\alpha > 0$. We are to obtain those estimates only for $0 \leq \mu \leq 1$ and, if $\mu \neq 0$, eventually assuming that the potential (or the smallest eigenvalue of its hessian) is bounded from above. In case $\mu = 0$ we are able also to study also the generic case.

In the final Section, Section 7.4, we obtain a Bochner-type formula, dealing once again with the generic case. Using this formula we prove that, under some assumptions on the parameters involved, the φ-scalar curvature and eventually some integrability conditions, the Einstein-type structure reduces to a harmonic-Einstein structure.

Some of the results of this Chapter are part of a joint work with Marco Rigoli.

7.1 Hamilton-type identities

In case $\mu = 0$ and φ is constant (7.0.1) yields the Ricci soliton system

$$\mathrm{Ric} + \mathrm{Hess}(f) = \lambda \langle\, ,\, \rangle. \tag{7.1.1}$$

In this situation we have the well known identity due to Hamilton,

$$\nabla S = 2\mathrm{Ric}(\nabla f, \cdot)^{\sharp}. \tag{7.1.2}$$

The latter, in turns, gives rise to the celebrated Hamilton identity

$$S + |\nabla f|^2 - 2\lambda f = \Lambda, \tag{7.1.3}$$

for some constant $\Lambda \in \mathbb{R}$. Note that in case $\lambda \neq 0$ one can add a constant to f to obtain $\Lambda = 0$. We shall generalize (7.1.2) and (7.1.3) to the Einstein-type structure (7.0.1). The equation corresponding to (7.1.2), with λ non-constant, is given in a local orthonormal coframe by (3.1.16), which we report here for the sake of convenience

$$\frac{1}{2}S^{\varphi}_j = R^{\varphi}_{kj}f_k - \mu\Delta f f_j + \mu f_k f_{kj} + (m-1)\lambda_j. \tag{7.1.4}$$

Observe that for $\mu = 0$ and for λ and φ constants (7.1.4) reduces to (7.1.2). Next we extend (7.1.3) in the following

Proposition 7.1.5. *Let $(M, \langle\, ,\, \rangle)$ be a Riemannian manifold with an Einstein-type structure as in* (7.0.1) *for $\alpha \in \mathbb{R} \setminus \{0\}$, $\mu, \lambda \in \mathbb{R}$, $f \in \mathcal{C}^{\infty}(M)$ and $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ smooth. Then there exists $\Lambda \in \mathbb{R}$ such that, if $\mu \neq 0$:*

$$S^{\varphi} - (\mu - 1)|\nabla f|^2 + \left(\frac{1}{\mu} - m\right)\lambda = \frac{\Lambda}{\mu}e^{2\mu f}, \tag{7.1.6}$$

and if $\mu = 0$:

$$S^{\varphi} + |\nabla f|^2 - 2\lambda f = m\lambda - \Lambda. \tag{7.1.7}$$

As a consequence we have the validity of the following equations, if $\mu \neq 0$:

$$\Delta_f f = \frac{\lambda}{\mu} - \frac{\Lambda}{\mu}e^{2\mu f}, \tag{7.1.8}$$

and if $\mu = 0$:

$$\Delta_f f = \Lambda - 2\lambda f. \tag{7.1.9}$$

Remark 7.1.10. Observe that in (7.1.8) and (7.1.9) the map $\varphi : M \to (N, \langle\, ,\, \rangle_N)$ and the constant α do not appear. This observation enables us to extends some results on quasi-Einstein manifolds that relies on the validity of generalized Hamilton-type identities to gradient Einstein-type structures.

Proof. In the assumptions above, but with $\lambda \in \mathcal{C}^{\infty}(M)$, we claim the validity of the following equation

$$(\Delta_f f + (m-2)\lambda)_j - 2f_j(\mu\Delta_f f - \lambda) = 0. \tag{7.1.11}$$

Towards this aim we trace the first equation of (7.0.1) to obtain

$$m\lambda = S^{\varphi} + \Delta f - \mu|\nabla f|^2. \tag{7.1.12}$$

Taking the covariant derivative and inserting into (7.1.4) we deduce

$$\frac{1}{2}S^{\varphi}_j = R^{\varphi}_{kj}f_k - \mu\Delta f f_j + \mu f_k f_{kj} - \lambda_j + (S^{\varphi} + \Delta f - \mu|\nabla f|^2)_j,$$

that is,

$$\frac{1}{2}S^{\varphi}_j + (\Delta f)_j + R^{\varphi}_{ij}f_i = \mu\Delta f f_j + \mu f_{ij}f_i + \lambda_j.$$

Contracting the first equation of (7.0.1) against ∇f we infer

$$R^{\varphi}_{ij}f_i + f_{ij}f_i = \mu|\nabla f|^2 f_j + \lambda f_j,$$

and replacing into the above yields

$$\frac{1}{2}S^{\varphi}_j + (\Delta f)_j - f_{ij}f_i + \mu|\nabla f|^2 f_j + \lambda f_j = \mu\Delta f f_j + \mu f_{ij}f_i + \lambda_j,$$

that is,

$$S^{\varphi}_j = -2(\Delta f)_j + 2(1+\mu)f_{ij}f_i - 2\mu|\nabla f|^2 f_j - 2\lambda f_j + 2\mu\Delta f f_j + 2\lambda_j \tag{7.1.13}$$

The covariant derivative of (7.1.12) yields

$$S^{\varphi}_j + (\Delta f)_j = 2\mu f_i f_{ij} + m\lambda_j, \tag{7.1.14}$$

and by plugging (7.1.13) in (7.1.14) we obtain

$$-2(\Delta f)_j + 2(1+\mu) f_{ij} f_i - 2\mu|\nabla f|^2 f_j - 2\lambda f_j + 2\mu\Delta f f_j + 2\lambda_j + (\Delta f)_j = 2\mu f_i f_{ij} + m\lambda_j,$$

that implies (7.1.11).

Now, assuming λ constant (7.1.11) can be rewritten as

$$(\Delta_f f)_j - 2f_j(\mu\Delta_f f - \lambda) = 0. \tag{7.1.15}$$

- If $\mu \neq 0$ from (7.1.15) we deduce

  $$\left(\Delta_f f - \frac{\lambda}{\mu}\right)_j - 2\mu f_j\left(\Delta_f f - \frac{\lambda}{\mu}\right) = 0.$$

  It follows that the function

  $$v := \left(\Delta_f f - \frac{\lambda}{\mu}\right) e^{-2\mu f}$$

  is a constant, say $-\frac{\Lambda}{\mu}$, where $\Lambda \in \mathbb{R}$, on M. Indeed, using the above

  $$v_j = \left[\left(\Delta_f f - \frac{\lambda}{\mu}\right)_j - 2\mu f_j\left(\Delta_f f - \frac{\lambda}{\mu}\right)\right] e^{-2\mu f} = 0.$$

  Observe that since $v = -\frac{\Lambda}{\mu}$ we have the validity of (7.1.8). To deduce the validity of (7.1.6) it is sufficient to plug (7.1.12), that is,

  $$\Delta_f f = -S^{\varphi} + (\mu - 1)|\nabla f|^2 + m\lambda, \tag{7.1.16}$$

  into (7.1.8).

- If $\mu = 0$ then (7.1.15) becomes

  $$(\Delta_f f)_j + 2\lambda f_j = 0,$$

  and thus, since λ is constant,

  $$(\Delta_f f + 2\lambda f)_j = 0.$$

  Then the function

  $$v := \Delta_f f + 2\lambda f$$

  is constant on M, say it is equal to Λ for some $\Lambda \in \mathbb{R}$. Hence (7.1.9) holds. By plugging (7.1.16) into (7.1.9) we get the validity of (7.1.7). □

Remark 7.1.17. It is worth to observe that when $m \geq 3$ and

$$\mu = -\frac{1}{m-2}, \tag{7.1.18}$$

or equivalently when $(M, \langle\, ,\, \rangle)$ is conformally harmonic-Einstein, equations (7.1.6) and (7.1.8) holds for $\lambda \in \mathcal{C}^{\infty}(M)$, see Theorem 2.3.5. This can also be seen directly, in fact from the proof of the Proposition above, in case (7.1.18) holds, equation (7.1.11) becomes

$$(\Delta_f f + (m-2)\lambda)_j - 2f_j\left(-\frac{1}{m-2}\Delta_f f - \lambda\right) = 0,$$

that is,

$$(\Delta_f f + (m-2)\lambda)_j + \frac{2}{m-2} f_j (\Delta_f f + (m-2)\lambda) = 0.$$

Then, setting

$$v := (\Delta_f f + (m-2)\lambda) e^{\frac{2}{m-2} f},$$

it is easy to see that v is constant on M.

Remark 7.1.19. During the proof of the Proposition above, in case $\mu \neq 0$ the choice of the constant Λ may seems unpleasant. The motivation the choice we made is contained in Theorem 2.5.26. Indeed, in the assumptions and notations of Theorem 2.5.26, the constant Λ is the Einstein constant of the space $(F, \langle \, , \, \rangle_F)$. Compare (7.1.8) with (2.5.30), where $\mu = \frac{1}{d}$.

7.2 Weighted volume growth for gradient Einstein-type structures

The validity on a complete Riemannian manifold $(M, \langle \, , \, \rangle)$ of dimension m of a system of the type

$$\mathrm{Ric} + \mathrm{Hess}(v) - \frac{1}{\gamma} dv \otimes dv \geq -(\gamma + m - 1) G \circ r \langle \, , \, \rangle, \tag{7.2.1}$$

where $r(x) := \mathrm{dist}_M(x, o)$ is the geodesic distance of $x \in M$ to a fixed origin $o \in M$, for some $v \in \mathcal{C}^\infty(M)$, $\gamma \in \mathbb{R}^+$ and some continuous function $G : \mathbb{R}^+_0 \to \mathbb{R}^+_0$, implies some restriction on the weighted volume growth of geodesic balls. The same applies to the system

$$\mathrm{Ric} + \mathrm{Hess}(v) \geq -(\gamma + m - 1) G \circ r \langle \, , \, \rangle,$$

that, for the sake of brevity, we shall indicate as the case $\gamma = +\infty$.

Indeed, in case $\gamma > 0$, the left hand side of (7.2.1) is the generalized Bakry-Emery Ricci tensor Ric^γ_v of $(M, \langle \, , \, \rangle)$ introduced by Z. Qian in [Q], so that we can write (7.2.1) in the form

$$\mathrm{Ric}^\gamma_v \geq -(\gamma + m - 1) G \circ r \langle \, , \, \rangle. \tag{7.2.2}$$

Inequality (7.2.2) enables us to estimate from above the weighted volume of geodesic balls

$$\mathrm{vol}_v(B_r) := \int_{B_r} e^{-v},$$

via Theorem 2.4 of [MRS] whenever G has an appropriate behaviour at infinity. Of course in the estimate will enter the parameter γ. Indeed, let g be a positive solution (if any) of

$$\begin{cases} g'' - Gg \geq 0 \text{ on } \mathbb{R}^+_0 \\ g(0) = 0, \quad g'(0) = 1. \end{cases} \tag{7.2.3}$$

Then (7.2.2), together with completeness of $(M, \langle \, , \, \rangle)$, implies via Theorem 2.4 of [MRS], for r large enough,

$$\mathrm{vol}_v(B_r) \leq C \int_0^r g^{\gamma + m - 1}, \tag{7.2.4}$$

for some constant $C > 0$. Note that, and this is important, the upper bound in (7.2.4) only depends on G via g but not on v.

In case $\gamma = +\infty$, that is,

$$\mathrm{Ric}_v \geq -(m-1) G \circ r \langle \, , \, \rangle,$$

the estimates corresponding to (7.2.4) are given in Proposition 8.11 of [AMR], that is,

$$\mathrm{vol}_v(\partial B_r) \leq e^{C(r-\varepsilon) + \int_\varepsilon^r \left(\int_\varepsilon^t (m-1) G \right) dt}$$

for some constants $\varepsilon, C > 0$ and $r \geq \varepsilon$ and as a consequence

$$\mathrm{vol}_v(B_r) \leq D + \int_0^r e^{Cs + \int_\varepsilon^s \left(\int_\varepsilon^t (m-1)G\right)dt} ds \tag{7.2.5}$$

with C, ε as above, $D > 0$ a constant and $r \in \mathbb{R}_0^+$.

In particular, when $G \equiv \Sigma$ for some $\Sigma \in \mathbb{R}$, $\gamma = +\infty$ and $G \equiv \Sigma$ for some $\Sigma \in \mathbb{R}$, that is,

$$\mathrm{Ric}_v \geq -(m-1)\Sigma\langle\, , \,\rangle \tag{7.2.6}$$

from (7.2.5) we obtain the estimate

$$\mathrm{vol}_v(B_r) \leq D + B\int_0^r e^{\frac{(m-1)\Sigma}{2}t^2 + Ct} dt \qquad \text{for } r >> 1 \tag{7.2.7}$$

and some constants $B, C, D > 0$.

We point out that for if $\gamma > 0$ and $\Sigma < 0$, Qian, Theorem 5 in [Q], shows that the complete manifold $(M, \langle\, , \,\rangle)$ satisfying

$$\mathrm{Ric}_v^\gamma \geq -(\gamma + m - 1)\Sigma\langle\, , \,\rangle$$

has to be compact. For $\gamma = +\infty$ and $\Sigma < 0$ a complete Riemannian manifold $(M, \langle\, , \,\rangle)$ satisfying (7.2.7) can be non-compact (to see this it is sufficient to consider the Gaussian shrinker gradient Ricci soliton structure on the Euclidean space). Nevertheless, the following Proposition holds.

Proposition 7.2.8. *Let $(M, \langle\, , \,\rangle)$ be a complete Riemannian manifold such that* (7.2.6) *holds for some $v \in \mathcal{C}^\infty(M)$ and for some constant $\Sigma < 0$. Then $(M, \langle\, , \,\rangle)$ is v-parabolic.*

Recall that $(M, \langle\, , \,\rangle)$ is said to be *v-parabolic* if every bounded above v-subharmonic function (that is, subharmonic with respect to Δ_v) on M is constant. To prove the above Proposition we observe that Theorem A of [RS] can be easily adapted in the weighted setting, obtaining

Theorem 7.2.9. *Let $(M, \langle\, , \,\rangle)$ be a complete Riemannian manifold, let $v \in \mathcal{C}^\infty(M)$ and assume that*

$$vol_v(\partial B_r)^{-1} \notin L^1(+\infty). \tag{7.2.10}$$

Then M is parabolic with respect to Δ_v.

Proof (of Proposition 7.2.8). Our assumptions implies the validity of (7.2.7). Notice that (7.2.10) is implied by

$$\frac{r}{\mathrm{vol}_v(B_r)} \notin L^1(+\infty).$$

To prove the above observe that, using (7.2.7), for $r >> 1$,

$$\frac{r}{\mathrm{vol}_v(B_r)} \geq \frac{r}{D + B\int_0^r e^{\frac{(m-1)\Sigma}{2}t^2 + Ct} dt}$$

and that, using L'Hôpital's rule and $\Sigma < 0$,

$$\lim_{r \to +\infty} \frac{r}{D + B\int_0^r e^{\frac{(m-1)\Sigma}{2}t^2 + Ct} dt} = \lim_{r \to +\infty} Be^{-\frac{(m-1)\Sigma}{2}r^2 - Cr} = +\infty,$$

concluding the proof. □

For $\Sigma \geq 0$ we have

Proposition 7.2.11. *Let $(M, \langle\, , \,\rangle)$ be a complete Riemannian manifold and $v \in \mathcal{C}^\infty(M)$. Assume* (7.2.6) *holds for some $\Sigma \in \mathbb{R}$. Then the weak maximum principle at infinity for Δ_v holds. As a consequence, the L^1-Liouville property for v-subharmonic functions holds.*

Recall that the L^1-*Liouville property for v-subharmonic functions* holds if every $u \in Lip_{loc}(M)$ solution of $\Delta_v u \leq 0$ on M and satisfying $0 \leq u \in L^1(M, e^{-v})$ is constant.

Proof. From Theorem 3.11 of [PRS] (see Theorem 9 of [PRiS] and the discussion above), the validity of the weak maximum principle at infinity for Δ_v is guaranteed in case

$$\frac{r}{\log \operatorname{vol}_v(B_r)} \notin L^1(+\infty). \tag{7.2.12}$$

As remarked above (7.2.6) yields (7.2.7) for some constants $B, C, D > 0$, so that, by a computation we obtain that (7.2.12) holds. Now the validity of the L^1-Liouville property for v-subharmonic functions follows immediately from the validity of the weak maximum principle for Δ_v, see for instance Theorem 24 of [PRiS]. □

Remark 7.2.13. The Proposition above appears also in [W], see Lemma 3.8.

In the presence of a gradient-Einstein type structure on a complete Riemannian manifold we naturally have the validity of a system of the type (7.2.1), as we now show.

Proposition 7.2.14. *Let $(M, \langle\, , \,\rangle)$ be a complete Riemannian manifold with a gradient Einstein-type structure as in* (7.0.1) *for some $f \in \mathcal{C}^\infty(M)$, $\varphi : M \to (N, \langle\, , \,\rangle_N)$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\mu, \lambda \in \mathbb{R}$. Let $o \in M$ be a fixed origin and $r(x) := dist_M(x, o)$ the geodesic distance of $x \in M$ from o. Let $K, F : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ be such that, if $\alpha < 0$*

$$|d\varphi|^2 \leq K \circ r \tag{7.2.15}$$

and if $\mu < 0$

$$|\nabla f|^2 \leq F \circ r. \tag{7.2.16}$$

Then, denoting for every $t \in \mathbb{R}$

$$t_+ := \begin{cases} t \text{ if } t \geq 0 \\ 0 \text{ if } t < 0 \end{cases} \qquad t_- := \begin{cases} 0 \text{ if } t \geq 0 \\ t \text{ if } t < 0 \end{cases}$$

we have

$$Ric_f \geq -(m-1) G \circ r \langle\, , \,\rangle, \tag{7.2.17}$$

where

$$G = -\frac{\lambda + \mu_- F + \alpha_- K}{+m-1}. \tag{7.2.18}$$

Proof. The following inequalities, in the sense of quadratic forms, hold:

$$0 \leq \varphi^* \langle\, , \,\rangle_N \leq |d\varphi|^2 \langle\, , \,\rangle.$$

Hence using the first equation of (7.0.1) we obtain, in case $\alpha > 0$

$$\mathrm{Ric} + \mathrm{Hess}(f) - \mu df \otimes df \geq \lambda \langle\, , \,\rangle$$

while in case $\alpha < 0$, using (7.2.15),

$$\mathrm{Ric} + \mathrm{Hess}(f) - \mu df \otimes df \geq (\lambda + \alpha K \circ r) \langle\, , \,\rangle.$$

In conclusion, for every $\alpha \in \mathbb{R} \setminus \{0\}$,

$$\mathrm{Ric} + \mathrm{Hess}(f) - \mu df \otimes df \geq (\lambda + \alpha_- K \circ r) \langle\, , \,\rangle. \tag{7.2.19}$$

In case $\mu \geq 0$, (7.2.19) gives

$$\mathrm{Ric}_f \geq (\lambda + \alpha_- K \circ r) \langle\, , \,\rangle.$$

Notice that

$$df \otimes df \leq |\nabla f|^2 \langle\, , \,\rangle.$$

Hence, in case $\mu < 0$ we get, from (7.2.19), using (7.2.16),

$$\mathrm{Ric}_f \geq (\lambda + \alpha_- K \circ r + \mu F \circ r)\langle\, , \,\rangle.$$

We then conclude the validity of (7.2.17). □

As a consequence of Proposition 7.2.14 and the Propositions above we have

Proposition 7.2.20. *Let $(M, \langle\, , \,\rangle)$ be a complete Riemannian manifold with a gradient Einstein-type structure as in (7.0.1) for some $f \in \mathcal{C}^\infty(M)$, $\varphi : M \to (N, \langle\, , \,\rangle_N)$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\mu, \lambda \in \mathbb{R}$. In case $\alpha < 0$ assume*

$$(|d\varphi|^2)^* := \sup_M |d\varphi|^2 < +\infty$$

and in case $\mu < 0$ assume

$$(|\nabla f|^2)^* := \sup_M |\nabla f|^2 < +\infty.$$

Then

i) The weak maximum principle at infinity for Δ_f and the L^1-Liouville property for f-subharmonic functions hold;

ii) M is compact in case $\mu > 0$ and either $\alpha, \lambda > 0$ or $\alpha < 0$ and $\lambda > |\alpha|(|d\varphi|^2)^*$;

iii) $(M, \langle\, , \,\rangle)$ is f-parabolic in case $\mu = 0$ and either $\alpha, \lambda > 0$ or $\alpha < 0$ and $\lambda > |\alpha|(|d\varphi|^2)^*$;

iv) $(M, \langle\, , \,\rangle)$ is f-parabolic in case $\mu < 0$ and either $\alpha > 0$ and $\lambda > |\mu|(|\nabla f|^2)^*$ or $\alpha < 0$ and

$$\lambda > |\mu|(|\nabla f|^2)^* + |\alpha|(|d\varphi|^2)^*.$$

Proof. From Proposition 7.2.14, by choosing $K \equiv (|d\varphi|^2)^*$ in case $\alpha < 0$ and $K \equiv (|d\varphi|^2)^*$ in case $\mu < 0$, we have

$$\mathrm{Ric}_f \geq -(m-1)\Sigma\langle\, , \,\rangle \tag{7.2.21}$$

with

$$\Sigma := -\frac{\lambda + (\alpha(|d\varphi|^2)^*)_- + (\mu(|\nabla f|^2)^*)_-}{m-1} \in \mathbb{R} \tag{7.2.22}$$

(where we are using the convention $(+\infty)_- = 0$). Then *i*) follows from Proposition 7.2.11, *ii*) from Theorem 5 of [Q] and finally *iii*) and *iv*) follows from Proposition 7.2.8. □

7.3 φ-scalar curvature estimates

Assume $(M, \langle\, , \,\rangle)$ is a Riemannian manifold of dimension $m \geq 2$, $\varphi : M \to (N, \langle\, , \,\rangle_N)$ a smooth map, $X \in \mathfrak{X}(M)$, $\alpha \in \mathbb{R} \setminus \{0\}$, $\lambda \in \mathbb{R}$ such that

$$\begin{cases} \mathrm{Ric}^\varphi + \dfrac{1}{2}\mathcal{L}_X\langle\, , \,\rangle = \lambda\langle\, , \,\rangle \\ \tau(\varphi) = d\varphi(X). \end{cases} \tag{7.3.1}$$

Recall that the Einstein-type structure (7.3.1) is trivial if $X = 0$. For for the sake of the reader we report here (3.1.14), that is,

$$\frac{1}{2}\Delta_X S^\varphi + |T^\varphi|^2 + \alpha|\tau(\varphi)|^2 + \frac{S^\varphi}{m}(S^\varphi - m\lambda) = 0, \tag{7.3.2}$$

where T^φ denotes the traceless part of the φ-Ricci tensor

$$T^\varphi := \mathrm{Ric}^\varphi - \frac{S^\varphi}{m}\langle\, ,\,\rangle.$$

Taking the trace of the first equation of (7.3.1) we get

$$S^\varphi + \mathrm{div}(X) = m\lambda. \tag{7.3.3}$$

In the next Proposition we obtain the φ-scalar curvature estimates in the complete case, when $\alpha > 0$.

Proposition 7.3.4. *Let $(M, \langle\, ,\,\rangle)$ be a complete Riemannian manifold of dimension $m \geq 2$ with an Einstein-type structure as in (7.3.1) with $\varphi : M \to (N, \langle\, ,\,\rangle_N)$ a smooth map, $X \in \mathfrak{X}(M)$, $\alpha > 0$ and $\lambda \in \mathbb{R}$. Denoting $S^\varphi_* := \inf_M S^\varphi$ we have $S^\varphi_* > -\infty$. Moreover*

i) If $\lambda < 0$ then

$$m\lambda \leq S^\varphi_* \leq 0.$$

If there exists $x_0 \in M$ such that $S^\varphi(x_0) = m\lambda$ then the structure (7.3.1) reduces to a harmonic-Einstein structure and X is a vertical Killing vector field. Furthermore, if M is compact then (7.3.1) is trivial.

If $S^\varphi_* = 0$ then either $S^\varphi > 0$ on M or $(M, \langle\, ,\,\rangle)$ is flat and φ is a constant map;

ii) If $\lambda = 0$ then

$$S^\varphi_* = 0.$$

Then either $S^\varphi > 0$ on M or $(M, \langle\, ,\,\rangle)$ is φ-Ricci flat and X is a vertical Killing vector field. Furthermore, if M is compact then X is parallel;

iii) If $\lambda > 0$ then

$$0 \leq S^\varphi_* \leq m\lambda.$$

If there exists $x_0 \in M$ such that $S^\varphi(x_0) = 0$ then $(M, \langle\, ,\,\rangle)$ is flat and φ is a constant map.

If $S^\varphi_* = m\lambda$ either $S^\varphi > m\lambda$ or the structure (7.3.1) reduces to a harmonic-Einstein structure, X is a vertical Killing vector field and M is compact.

Proof. The proof of this Theorem follows closely the proof of Theorem 8.2 of [AMR]. Since $\alpha > 0$ we have

$$\mathrm{Ric}_X \geq \mathrm{Ric}^\varphi + \frac{1}{2}\mathcal{L}_X\langle\, ,\,\rangle = \lambda\langle\, ,\,\rangle,$$

hence from the results of Section 8.2 of [AMR] we deduce the validity of the Omori-Yau maximum principle for the operator Δ_X on M. Moreover, since $\alpha > 0$, from (7.3.2) we deduce the validity of

$$\frac{1}{2}\Delta_X S^\varphi \leq -\frac{S^\varphi}{m}(S^\varphi - m\lambda). \tag{7.3.5}$$

We set $u := -S^\varphi$ so that (7.3.5) gives

$$\frac{1}{2}\Delta_X u \geq \lambda u + \frac{u^2}{m}.$$

We are in position to apply Theorem 3.6 of [AMR] with the choices

$$F(t) = t^2, \quad \varphi(u, |\nabla u|) = \lambda u + \frac{u^2}{m}.$$

As consequences $u^* < \infty$ and

$$\lambda u^* + \frac{(u^*)^2}{m} \leq 0,$$

that is, u^* is included between 0 and $-m\lambda$ and thus S^φ_* is included between 0 and $m\lambda$.

- Assume $\lambda < 0$. Then $m\lambda \leq S^{\varphi}_{*} \leq 0$.

  Assume that for some $x_0 \in M$ we have $S^{\varphi}(x_0) = m\lambda$. Then $S^{\varphi} \geq S^{\varphi}_{*} \geq m\lambda$ so that the function $v := S^{\varphi} - m\lambda$ is non-negative on M. Using (7.3.5) we obtain

$$\frac{1}{2}\Delta_X v = \frac{1}{2}\Delta_X S^{\varphi} \leq -\frac{S^{\varphi}}{m}(S^{\varphi} - m\lambda) = -\frac{v + m\lambda}{m}v = -\frac{v^2}{m} - \lambda v \leq -\lambda v,$$

  that is,

$$\frac{1}{2}\Delta_X v + \lambda v \leq 0,$$

  An application of the minimum principle in [GT], exactly as in Theorem 8.2 of [AMR], gives that $v \equiv 0$ on M, that is, $S^{\varphi} \equiv m\lambda$ on M. Then from (7.3.2) we infer the validity of

$$|T^{\varphi}|^2 + \alpha|\tau(\varphi)|^2 = 0, \tag{7.3.6}$$

  and, since $\alpha \in \mathbb{R}^+$, $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein. Moreover, since $S^{\varphi} = m\lambda$ we deduce from the first equation of (7.3.1) that X is a Killing vector field and from the second that is vertical, that is, $d\varphi(X) = 0$. From Proposition 4.1.21, since S^{φ} is negative, we get that $X = 0$ if M is compact.

  Now assume that $S^{\varphi}_{*} = 0$. Then or $S^{\varphi} > 0$ on M or otherwise there exists $x_0 \in M$ such that $S^{\varphi}(x_0) = S^{\varphi}_{*} = 0$. In the latter case $S^{\varphi} \geq S^{\varphi}_{*} = 0 \geq m\lambda$ and, from (7.3.5),

$$\frac{1}{2}\Delta_X S^{\varphi} \leq -\frac{S^{\varphi}}{m}(S^{\varphi} - m\lambda) \leq 0.$$

  Applying the strong minimum principle $S^{\varphi} \equiv 0$ on M. Then, as before, the structure $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein. In particular

$$\frac{1}{2}\mathcal{L}_X\langle\, ,\, \rangle = \lambda\langle\, ,\, \rangle.$$

  By a result known to Tashiro, Theorem 4.1 of [T], $(M, \langle\, ,\, \rangle)$ is flat. Then $S \equiv 0$ on M and, since also $S^{\varphi} \equiv 0$, we obtain

$$0 = S^{\varphi} = S - \alpha|d\varphi|^2 = -\alpha|d\varphi|^2.$$

  Since $\alpha \in \mathbb{R}^+$, φ must be a constant map.

- Assume $\lambda = 0$, then $S^{\varphi}_{*} = 0$ and thus or $S^{\varphi} > 0$ on M or otherwise $S^{\varphi}(x_0) = 0$ for some $x_0 \in M$. In the latter case, proceeding as above we get $S^{\varphi} \equiv 0$ on M, hence $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein with vanishing φ-scalar curvature, or equivalently, φ-Ricci flat. Since $\lambda = 0$, X is a vertical Killing vector field. From Proposition 4.1.21 we get that X is parallel if M is compact.

- Assume $\lambda > 0$, then $0 \leq S^{\varphi} \leq m\lambda$.

  Assume there exists $x_0 \in M$ such that $S^{\varphi}(x_0) = S^{\varphi}_{*} = 0$. Then, using (7.3.5) we obtain

$$\frac{1}{2}\Delta_X S^{\varphi} \leq -\frac{S^{\varphi}}{m}(S^{\varphi} - m\lambda) = -\frac{(S^{\varphi})^2}{m} + \lambda S^{\varphi} \leq \lambda S^{\varphi},$$

  that is,

$$\frac{1}{2}\Delta_X S^{\varphi} - \lambda S^{\varphi} \leq 0,$$

  hence from the minimum principle of [GT] we obtain $S^{\varphi} \equiv 0$ on M. Then, as above, $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein and X is a vertical homothetic vector field. Moreover, once again by the result known to Tashiro, we obtain that $(M, \langle\, ,\, \rangle)$ is flat and φ is constant.

  Now assume there exists $x_0 \in M$ such that $S^{\varphi}(x_0) = S^{\varphi}_{*} = m\lambda$. Then $S^{\varphi} \geq S^{\varphi}_{*} = m\lambda \geq 0$ so that the function $v := S^{\varphi} - m\lambda$ is non-negative on M. Using (7.3.5) we obtain

$$\frac{1}{2}\Delta_X v = \frac{1}{2}\Delta_X S^{\varphi} \leq -\frac{S^{\varphi}}{m}(S^{\varphi} - m\lambda) \leq 0.$$

The strong minimum principle gives that $v \equiv 0$ on M, that is, $S^\varphi \equiv m\lambda$ on M. Then, as before, we get that $(M, \langle\,,\,\rangle)$ is harmonic-Einstein and X is a vertical Killing vector field. Finally, since $\alpha > 0$ we have $\mathrm{Ric} \geq \mathrm{Ric}^\varphi$ and since $(M, \langle\,,\,\rangle)$ is harmonic-Einstein with $S^\varphi > 0$, by Myer's theorem, we get that M is compact. □

In the gradient case we can be more specific, combining the Theorem above with Theorem 4.2.19.

Theorem 7.3.7. *Let $(M, \langle\,,\,\rangle)$ be a complete Riemannian manifold of dimension $m \geq 2$, $\varphi : M \to (N, \langle\,,\,\rangle_N)$ a smooth map, $f \in \mathcal{C}^\infty(M)$, $\alpha > 0$, $\lambda \in \mathbb{R}$ such that*

$$\begin{cases} Ric^\varphi + Hess(f) = \lambda\langle\,,\,\rangle \\ \tau(\varphi) = d\varphi(\nabla f). \end{cases} \tag{7.3.8}$$

Denoting $S_*^\varphi := \inf_M S^\varphi$ we have $S_*^\varphi > -\infty$. Moreover

i) If $\lambda < 0$ then

$$m\lambda \leq S_*^\varphi \leq 0.$$

If there exists $x_0 \in M$ such that $S^\varphi(x_0) = m\lambda$ then f is constant.

If $S_*^\varphi = 0$ then either $S^\varphi > 0$ on M or $(M, \langle\,,\,\rangle)$ is isometric to the euclidean space $\mathbb{R}^m$ and φ is a constant map. Moreover, the potential f can be expressed on $\mathbb{R}^m$ as $f(x) = \frac{\lambda}{2}|x|^2 + \langle b, x\rangle + c$ for some $b \in \mathbb{R}^m$ and $c \in \mathbb{R}$, for every $x \in \mathbb{R}^m$.

ii) If $\lambda = 0$ then

$$S_*^\varphi = 0.$$

Then either $S^\varphi > 0$ on M or, if f is non constant, $(M, \langle\,,\,\rangle)$ splits as the Riemannian product of $\mathbb{R}$ with a totally geodesic ψ-Ricci flat hypersurface Σ, where $\psi := \varphi|_\Sigma$. Moreover $\varphi = \psi \circ \pi_\Sigma$ on $\mathbb{R} \times \Sigma$, where $\pi_\Sigma : \mathbb{R} \times \Sigma \to \Sigma$ is the canonical projection and the function f can be expressed on $\mathbb{R} \times \Sigma$ as

$$f(t, x) = at + b \quad \text{for every } t \in \mathbb{R} \text{ and } x \in \Sigma, \tag{7.3.9}$$

for some $a > 0$ and $b \in \mathbb{R}$ such that $\Sigma = f^{-1}(\{b\})$.

iii) If $\lambda > 0$ then

$$0 \leq S_*^\varphi \leq m\lambda.$$

If there exists $x_0 \in M$ such that $S^\varphi(x_0) = 0$ then $(M, \langle\,,\,\rangle)$ is isometric to the euclidean space $\mathbb{R}^m$ and φ is a constant map. Moreover, the potential f can be expressed on $\mathbb{R}^m$ as $f(x) = \frac{\lambda}{2}|x|^2 + \langle b, x\rangle + c$ for some $b \in \mathbb{R}^m$ and $c \in \mathbb{R}$, for every $x \in \mathbb{R}^m$.

If $S_*^\varphi = m\lambda$ either $S^\varphi > m\lambda$ or M is compact and f is constant.

Proof. We can apply Proposition 7.3.4 with $X = \nabla f$. The only thing we need to observe to prove the Theorem is that, from Theorem 4.2.19, the only possibility when λ is constant and $S^\varphi \neq 0$ to have that (7.3.8) reduces to a harmonic-Einstein manifold is that f is constant. The other cases follows from *a*) and *b*) of *iii*) in Theorem 4.2.19. □

Now we deal with the case $\mu \neq 0$. As a consequence of (3.1.17) we provide an estimate on $S_*^\varphi := \inf_M S^\varphi$, assuming $\alpha > 0$ and $0 < \mu \leq 1$. Precisely we prove

Theorem 7.3.10. *Let $(M, \langle\,,\,\rangle)$ be a complete Riemannian manifold of dimension m with a gradient Einstein-type structure as in (7.0.1) with $\alpha > 0$, $0 < \mu \leq 1$, $\lambda \in \mathbb{R}$, $f \in \mathcal{C}^\infty(M)$ and $\varphi : M \to (N, \langle\,,\,\rangle_N)$ a smooth map. If $\lambda \leq 0$ assume that $f_* > -\infty$, where f_* is the infimum of f on M, or that the smallest eigenvalue of $Hess(f)$ is bounded from below. Denoting $S_*^\varphi := \inf_M S^\varphi$ we have $S_*^\varphi > +\infty$. Moreover* *i) If $\lambda > 0$ then M is compact and*

$$\frac{(m-1)\mu}{1+(m-1)\mu}m\lambda \leq S^{\varphi}_{*} \leq m\lambda.$$

If $\mu \neq 1$, then

$$\frac{(m-1)\mu}{1+(m-1)\mu}m\lambda < S^{\varphi}_{*} \leq m\lambda$$

and $S^{\varphi}_{*} = m\lambda$, that is, $S^{\varphi}(x_0) = S^{\varphi}_{*}$ for some $x_0 \in M$, if and only f is constant.

ii) If $\lambda = 0$ then

$$S^{\varphi}_{*} = 0.$$

Moreover, if $\mu \neq 1$, or $S^{\varphi} > 0$ on M or otherwise f is constant.

iii) If $\lambda < 0$ then

$$m\lambda \leq S^{\varphi}_{*} \leq \frac{(m-1)\mu}{1+(m-1)\mu}m\lambda.$$

If $\mu \neq 1$, then $S^{\varphi}(x_0) = m\lambda$ for some $x_0 \in M$ if and only f is constant.

Proof. The proof of this theorem follows closely the proof of Theorem 3 of [R]. Equation (3.1.17) can be written, since λ is constant and $\mu > 0$, as

$$\frac{1}{2}\Delta_{(1+2\mu)f}S^{\varphi} =(\mu-1)(\alpha|\tau(\varphi)|^2 + |T^{\varphi}|^2) - \frac{(m-1)\mu+1}{m}(S^{\varphi}-m\lambda)\left(S^{\varphi} - \frac{(m-1)\mu m}{1+(m-1)\mu}\lambda\right). \tag{7.3.11}$$

We set $u := -S^{\varphi}$ so that (7.3.11) takes the form

$$\frac{1}{2}\Delta_{(1+2\mu)f}u =(1-\mu)(\alpha|\tau(\varphi)|^2 + |T^{\varphi}|^2) + \frac{(m-1)\mu+1}{m}(u+m\lambda)\left(u + \frac{(m-1)\mu}{1+(m-1)\mu}m\lambda\right). \tag{7.3.12}$$

Since $\mu \leq 1$ we deduce

$$\frac{1}{2}\Delta_{(1+2\mu)f}u \geq \frac{(m-1)\mu+1}{m}(u+m\lambda)\left(u + \frac{(m-1)\mu}{1+(m-1)\mu}m\lambda\right)$$

on M. We now set

$$g := (1+2\mu)f$$

so that

$$\frac{1}{2}\Delta_{g}u \geq \frac{(m-1)\mu+1}{m}(u+m\lambda)\left(u + \frac{(m-1)\mu}{1+(m-1)\mu}m\lambda\right), \tag{7.3.13}$$

or equivalently, in terms of S^{φ},

$$\frac{1}{2}\Delta_{g}S^{\varphi} \leq -\frac{(m-1)\mu+1}{m}(S^{\varphi}-m\lambda)\left(S^{\varphi} - \frac{(m-1)\mu}{1+(m-1)\mu}m\lambda\right). \tag{7.3.14}$$

i) If $\lambda > 0$ then, from $\mu > 0$ and Theorem 5 of [Q], M is compact and since $S^{\varphi}_{*} = S^{\varphi}(x_0)$ for some $x_0 \in M$ we deduce, from (7.3.14), that

$$\frac{(m-1)\mu}{1+(m-1)}m\lambda \leq S^{\varphi}_{*} \leq m\lambda.$$

We now show that the left inequality above is strict if $\mu \neq 1$. Indeed, assume by contradiction $S^{\varphi}_{*} = \frac{(m-1)\mu}{1+(m-1)\mu}m\lambda$. Because of (7.3.14) the non-negative function

$$v := S^{\varphi} - \frac{(m-1)\mu}{1+(m-1)\mu}m\lambda$$

satisfies

$$\frac{1}{2}\Delta_g v \leq -\frac{(m-1)\mu+1}{m}\left(v - \frac{1}{1+(m-1)\mu}m\lambda\right)v = -\frac{(m-1)\mu+1}{m}v^2 + \lambda v \leq \lambda v.$$

Since M is compact v attains its minimum, that is zero, and from the minimum principle, see page 35 of [GT], we deduce that v vanishes identically. Hence

$$S^{\varphi} \equiv \frac{(m-1)\mu}{1+(m-1)\mu}m\lambda. \tag{7.3.15}$$

From (7.3.11) we then have

$$(1-\mu)(\alpha|\tau(\varphi)|^2 + |T^{\varphi}|^2) = 0,$$

so that, since $\mu < 1$ and $\alpha > 0$, $(M, \langle\, ,\, \rangle)$ is a harmonic-Einstein manifold. To obtain the contradiction, since λ is constant, we use Corollary 4.2.33 combined with the fact that f cannot be constant. Indeed, if f is constant then $S^{\varphi} = m\lambda$, that is impossible since (7.3.15) holds.

Suppose now that $S^{\varphi}* = m\lambda$; then

$$S^{\varphi} \geq S^{\varphi}_{*} = m\lambda \geq \frac{(m-1)\mu}{1+(m-1)\mu}m\lambda,$$

hence from (7.3.14) we deduce

$$\frac{1}{2}\Delta_g S^{\varphi} \leq 0.$$

Since M is compact we infer that $S^{\varphi} \equiv S^{\varphi}_{*}$. Once again from (7.3.11) we obtain that $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein. Then f is constant because, if by contradiction f is non-constant from Corollary 4.2.33 we deduce that also λ is non-constant, a contradiction.

If $\lambda \leq 0$ we show that the weak maximum principle hold for Δ_g if either $f_* > -\infty$ or the smallest eigenvalue of $\mathrm{Hess}(f)$ is bounded from below.

- Assume at first $f_* > -\infty$. We want to prove that

$$\frac{r}{\log \mathrm{vol}_g B_r} \notin L^1(+\infty),$$

and then conclude as in the proof of Proposition 7.2.11. From the definition of g we immediately obtain $\mathrm{vol}_g(B_r) \leq e^{-2\mu f_*}\mathrm{vol}_f(B_r)$ and since $(M, \langle\, ,\, \rangle)$ we know, from the proof of Proposition 7.2.11, that

$$\frac{r}{\log \mathrm{vol}_f B_r} \notin L^1(+\infty),$$

we are able to conclude the validity of the weak maximum principle at infinity for Δ_g.

- Now suppose that the smallest eigenvalue of $\mathrm{Hess}(f)$ is bounded from below. The first equation of (7.0.1) can be written in terms of g as

$$\mathrm{Ric} + \mathrm{Hess}(g) - \frac{\mu}{(1+2\mu)^2}dg \otimes dg = \lambda\langle\, ,\, \rangle + 2\mu\mathrm{Hess}(f) + \alpha\varphi^*\langle\, ,\, \rangle_N,$$

so that, using that $\alpha, \mu > 0$ and that the smallest eigenvalue of $\mathrm{Hess}(f)$ is bounded from below,

$$\mathrm{Ric}^{\gamma}_g \geq \lambda\langle\, ,\, \rangle, \quad \gamma := \frac{(1+2\mu)^2}{\mu} > 0.$$

Then, from Proposition 7.2.20, the weak maximum principle for Δ_g also holds in this case.

Using Theorem 3.6 of [AMR] we conclude that $u^* := \sup_M u < +\infty$ and that

$$-\frac{(m-1)\mu}{1+(m-1)\mu} m\lambda \leq u^* \leq -m\lambda,$$

and as a consequence we immediately get the bounds on S^φ_*.

ii) Let $\lambda = 0$, the bounds on S^φ_* gives $S^\varphi_* = 0$. In this case (7.3.14) gives $\Delta_g S^\varphi \leq 0$ so that either $S^\varphi > 0$ on M or $S^\varphi \equiv 0$. In the latter case, if $\mu \neq 1$, from (7.3.11), we obtain that the Einstein-type structure (7.0.1) reduces to a harmonic-Einstein structure. From *iii*) of Theorem 4.2.25, since λ is constant, we conclude that f is constant.

iii) Let $\lambda < 0$. If $S^\varphi(x_0) = m\lambda$ for some $x_0 \in M$ then, from (7.3.14), the function $v := S^\varphi - m\lambda$ is non-negative and satisfies,

$$\frac{1}{2}\Delta_g v \leq -\frac{(m-1)\mu+1}{m} v\left(v + \frac{1}{1+(m-1)\mu} m\lambda\right) = -\frac{(m-1)\mu+1}{m} v^2 - \lambda v \leq -\lambda v,$$

that is,

$$\Delta_g v + 2\lambda v \leq 0,$$

so that, since v attains its minimum, from the minimum principle $v \equiv 0$. Then $S^\varphi \equiv m\lambda$ and, from (7.3.11), the Einstein-type structure (7.0.1) reduces to a harmonic-Einstein structure in case $\mu \neq 1$. Assume, by contradiction, f is non-constant. Since λ is constant, from *ii*) of Theorem 4.2.25 we obtain

$$\lambda = \frac{S^\varphi}{m} \frac{\mu(m-1)+1}{\mu(m-1)},$$

that contradicts $S^\varphi \equiv m\lambda$. □

7.4 Some triviality results

Formula (7.4.3), contained in the Proposition below, is a Bochner-type formula for Einstein-type structures.

Proposition 7.4.1. *Let $(M, \langle\, ,\,\rangle)$ be an m-dimensional Riemannian manifold with an Einstein-type structure as*

$$\begin{cases} Ric + \frac{1}{2}\mathcal{L}_X\langle\, ,\,\rangle - \mu X^\flat \otimes X^\flat - \alpha\varphi^*\langle\, ,\,\rangle_N = \lambda\langle\, ,\,\rangle \\ \tau(\varphi) = d\varphi(X), \end{cases} \tag{7.4.2}$$

with $\lambda \in \mathcal{C}^\infty(M)$, $X \in \mathfrak{X}(M)$, $\mu \in \mathbb{R}$, $\alpha \in \mathbb{R} \setminus \{0\}$ and $\varphi : M \to (N, \langle\, ,\,\rangle_N)$ smooth. Then

$$\frac{1}{2}\Delta_X |X|^2 = |\nabla X|^2 + \alpha|\tau(\varphi)|^2 + [(2\mu m - 1)\lambda - 2\mu S^\varphi]|X|^2 + \mu(2\mu - 1)|X|^4 - (m-2)\langle\nabla\lambda, X\rangle, \tag{7.4.3}$$

Proof. The generalized Bochner formula is given by, see Lemma 8.1 of [AMR],

$$\frac{1}{2}\Delta|X|^2 = |\nabla X|^2 + \operatorname{div}(\mathcal{L}_X\langle\, ,\,\rangle)(X) - \langle\nabla \operatorname{div}(X), X\rangle - \operatorname{Ric}(X, X). \tag{7.4.4}$$

Taking the trace of the first equation of (7.4.2) we get

$$\operatorname{div}(X) = -S^\varphi + \mu|X|^2 + m\lambda. \tag{7.4.5}$$

Then

$$\langle\nabla \operatorname{div}(X), X\rangle = -\langle\nabla S^\varphi, X\rangle + \mu\langle\nabla|X|^2, X\rangle + m\langle\nabla\lambda, X\rangle. \tag{7.4.6}$$

By definition of the φ-Ricci tensor we have

$$\mathrm{Ric}(X,X) = \mathrm{Ric}^{\varphi}(X,X) + \alpha|d\varphi(X)|^2, \tag{7.4.7}$$

so that, using (7.4.2) and

$$\mathcal{L}_X\langle\, ,\,\rangle(X,X) = \langle\nabla|X|^2, X\rangle$$

we infer

$$\mathrm{Ric}(X,X) = -\frac{1}{2}\langle\nabla|X|^2, X\rangle + \mu|X|^4 + \lambda|X|^2 + \alpha|\tau(\varphi)|. \tag{7.4.8}$$

Finally, using the first equation of (7.4.2) we obtain

$$\mathrm{div}(\mathcal{L}_X\langle\, ,\,\rangle)(X) = -2\mathrm{div}(\mathrm{Ric}^{\varphi})(X) + 2\mu\mathrm{div}(X^{\flat}\otimes X^{\flat})(X) + 2\langle\nabla\lambda, X\rangle.$$

From the generalized Schur's identity (1.2.26) and the second equation of (7.4.2) we deduce

$$\mathrm{div}(\mathrm{Ric}^{\varphi})(X) = \frac{1}{2}\langle\nabla S^{\varphi}, X\rangle - \alpha|\tau(\varphi)|^2,$$

so that from the above relation and

$$\mathrm{div}(X^{\flat}\otimes X^{\flat})(X) = \mathrm{div}(X)|X|^2 + \frac{1}{2}\langle\nabla|X|^2, X\rangle$$

we get

$$\mathrm{div}(\mathcal{L}_X\langle\, ,\,\rangle)(X) = -\langle\nabla S^{\varphi}, X\rangle + 2\alpha|\tau(\varphi)|^2 + \mu\langle\nabla|X|^2, X\rangle + 2\mu\mathrm{div}(X)|X|^2 + 2\langle\nabla\lambda, X\rangle. \tag{7.4.9}$$

By plugging (7.4.6), (7.4.8) and (7.4.9) in (7.4.4) we have

$$\frac{1}{2}\Delta|X|^2 = |\nabla X|^2 + \alpha|\tau(\varphi)|^2 + 2\mu\mathrm{div}(X)|X|^2 - (m-2)\langle\nabla\lambda, X\rangle + \frac{1}{2}\langle\nabla|X|^2, X\rangle - \mu|X|^4 - \lambda|X|^2,$$

that is (7.4.3), using once again (7.4.5) and the definition of Δ_X. □

Proposition 7.4.10. *Let $(M,\langle\, ,\,\rangle)$ be a complete Riemannian manifold of dimension $m \geq 2$ with an Einstein-type structure as in* (7.4.2) *with $X \in \mathfrak{X}(M)$, $\lambda \in \mathbb{R}$, $\mu > \frac{1}{2}$, $\alpha > 0$ and $\varphi : M \to (N,\langle\, ,\,\rangle_N)$ smooth. If*

$$(S^{\varphi})^* := \sup_M S^{\varphi} < +\infty, \tag{7.4.11}$$

then

$$(|X|^2)^* := \sup_M |X|^2 \leq 2\frac{(S^{\varphi})^* - \left(m - \frac{1}{2\mu}\right)\lambda}{2\mu - 1}. \tag{7.4.12}$$

As a consequence, if

$$(S^{\varphi})^* \leq \left(m - \frac{1}{2\mu}\right)\lambda, \tag{7.4.13}$$

then (3.0.2) *reduces to a harmonic-Einstein structure.*

Proof. First of all observe that the Omori-Yau maximum principle holds for Δ_X. Indeed, since $\mu, \alpha > 0$, from the first equation of (3.0.2) we deduce

$$\mathrm{Ric}_X \geq \lambda\langle\, ,\,\rangle. \tag{7.4.14}$$

As shown in Proposition 8.7, and the discussion above, of [AMR], (7.4.14) is sufficient to obtain the validity of the Omori-Yau maximum principle for Δ_X. Since λ is constant and $\alpha > 0$, (7.4.3) gives

$$\frac{1}{2}\Delta_X|X|^2 \geq [(2\mu m - 1)\lambda - 2\mu S^{\varphi}]|X|^2 + \mu(2\mu - 1)|X|^4.$$

Moreover, using (7.4.11) and $\mu > 0$ in the above and setting $u := |X|^2$ we get

$$\frac{1}{2}\Delta_X u \geq [(2\mu m - 1)\lambda - 2\mu(S^\varphi)^*]u + \mu(2\mu - 1)u^2.$$

Since $\mu > \frac{1}{2}$ the constant $\mu(2\mu - 1)$ is positive, hence from Theorem 3.6 of [AMR], with the choices

$$F(t) = t^2 \text{ for every } t \in \mathbb{R}, \quad \varphi(u, |\nabla u|) := [(2\mu m - 1)\lambda - 2\mu(S^\varphi)^*]u + \mu(2\mu - 1)u^2,$$

we conclude $u^* < +\infty$ and

$$[(2\mu m - 1)\lambda - 2\mu(S^\varphi)^* + \mu(2\mu - 1)u^*]u^* \leq 0.$$

Hence, from the above,

$$u^* \leq 2\frac{(S^\varphi)^* - \left(m - \frac{1}{2\mu}\right)\lambda}{2\mu - 1},$$

that is, (7.4.12). Clearly, if (7.4.13) holds, from (7.4.12) we immediately get that $X = 0$. $\square$

Remark 7.4.15. Assume that $\langle\, , \rangle$ is a complete φ-static metric on M, in the sense of Definition 2.5.54, that is,

$$\begin{cases} \mathrm{Ric}^\varphi + \mathrm{Hess}(f) - df \otimes df = \lambda\langle\, , \rangle \\ \tau(\varphi) = d\varphi(\nabla f) \end{cases} \tag{7.4.16}$$

with $f \in \mathcal{C}^\infty(M)$, $\lambda \in \mathbb{R}$, $\alpha > 0$ and $\varphi : M \to (N, \langle\, , \rangle_N)$ smooth and

$$\Delta_f f = \lambda. \tag{7.4.17}$$

From Remark 2.5.57 we know that if f is non constant then M is non-compact. Taking the trace of the first equation of (7.4.16),

$$S^\varphi + \Delta_f f = m\lambda,$$

so that, using (7.4.17),

$$S^\varphi = (m - 1)\lambda.$$

As a consequence of the above the condition (7.4.13), with $X = \nabla f$, is satisfied if and only if $\lambda \geq 0$. Then, if (7.4.16) is non-trivial, that is, f is non constant, then $(M, \langle\, , \rangle)$ is non-compact and $\lambda < 0$.

Now we provide some triviality results for gradient Einstein-type structure with potential function f satisfying the integrability condition (7.4.22) below for some $1 < p < +\infty$. To prove the next Proposition we shall use a modification of Theorem 1.1 of [PRS05], that we report here for the sake of the reader,

Theorem 7.4.18. *Let $(M, \langle\, ,\rangle)$ be a complete Riemannian manifold and let $f \in \mathcal{C}^\infty(M)$. Assume that $u \in Lip_{loc}(M)$ satisfy*

$$u\Delta_f u e^{-f} \geq 0 \quad \textit{weakly on } M. \tag{7.4.19}$$

If, for some $p \in (1, +\infty)$,

$$\left(\int_{\partial B_r} |u|^p e^{-f}\right)^{-1} \notin L^1(+\infty), \tag{7.4.20}$$

then u is constant.

Proposition 7.4.21. *Let $(M, \langle\, , \rangle)$ be a complete Riemannian manifold of dimension m with a gradient Einstein-type structure as in (7.0.1) with $\alpha \in \mathbb{R} \setminus \{0\}$, $\mu, \lambda \in \mathbb{R}$, $f \in \mathcal{C}^\infty(M)$ and $\varphi : M \to (N, \langle\, , \rangle_N)$ a smooth map. We denote by $(S^\varphi)^*$ and S^φ_*, respectively, the supremum and the infimum of S^φ on M. Suppose*

$$|\nabla(e^{-\frac{f}{p}})| \in L^p(M), \tag{7.4.22}$$

for some $p \in (1, +\infty)$, $\alpha > 0$ and that one the following conditions hold *i)* $\mu = \frac{1}{2}$, $(S^{\varphi})^{*} < (m-1)\lambda$;

ii) $\mu = 0$, $\lambda < 0$;

iii) $\mu < 0$, $S^{\varphi}_{*} \geq \left(m - \frac{1}{2\mu}\right)\lambda$.

Then f is constant and $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein.

Proof. Since $\lambda \in \mathbb{R}$ and $X = \nabla f$ equation (7.4.3) becomes

$$\frac{1}{2}\Delta_f |\nabla f|^2 = |\mathrm{Hess}(f)|^2 + \alpha|\tau(\varphi)|^2 + (2\mu\lambda m - \lambda - 2\mu S^{\varphi})|\nabla f|^2 + \mu(2\mu - 1)|\nabla f|^4. \tag{7.4.23}$$

Recall that, from Kato's inequality,

$$|\nabla|\nabla f||^2 \leq |\mathrm{Hess}(f)|^2 \quad \text{weakly on } M.$$

Then we infer

$$\frac{1}{2}\Delta_f |\nabla f|^2 = |\nabla f|\Delta_f|\nabla f| + |\nabla|\nabla f||^2 \leq |\nabla f|\Delta_f|\nabla f| + |\mathrm{Hess}(f)|^2 \quad \text{weakly on } M.$$

Combining the above with (7.4.23) and using $\alpha > 0$, we obtain

$$|\nabla f|\Delta_f|\nabla f| \geq (2\mu m\lambda - 2\mu S^{\varphi} - \lambda)|\nabla f|^2 + \mu(2\mu - 1)|\nabla f|^4 \quad \text{weakly on } M.$$

If anyone of *i*), *ii*) or5 *iii*) hold it is easy to prove the validity of, for some positive constant c and $q \in \{2, 4\}$:

$$|\nabla f|\Delta_f|\nabla f| \geq c|\nabla f|^q \geq 0 \quad \text{weakly on } M. \tag{7.4.24}$$

Then we are in position to apply Theorem 7.4.18 with the choice of $u = |\nabla f|$. Indeed, (7.4.22) is equivalent to $|\nabla f| \in L^p(M, e^{-f})$ and the latter guarantees the validity of (7.4.20). Moreover also (7.4.19) holds. As a consequence of Theorem 7.4.18 we have that $|\nabla f|$ is constant and, since (7.4.24) holds, the only possibility is that f is constant. Then $(M, \langle\, ,\, \rangle)$ is harmonic-Einstein. ☐

Remark 7.4.25. To prove Proposition 7.4.21 we may consider, instead of one of the assumptions *i*), *ii*) or *iii*) the assumption

iv) $\mu > 0$, $\lambda < 0$, $\Lambda < 0$ and
$$f_{*} \geq \frac{1}{2\mu}\log\left(\frac{\lambda}{2\Lambda}\right),$$
where $f_{*} := \inf_M f$ and Λ is the constant appearing in (7.1.8).

Indeed, (7.4.23) can be rewritten, using the trace of the first equation in (7.0.1), as

$$\frac{1}{2}\Delta_f |\nabla f|^2 = |\mathrm{Hess}(f)|^2 + \alpha|\tau(\varphi)|^2 + |\nabla f|^2(2\mu\Delta f - \lambda - \mu|\nabla f|^2),$$

or equivalently,

$$\frac{1}{2}\Delta_f |\nabla f|^2 = |\mathrm{Hess}(f)|^2 + \alpha|\tau(\varphi)|^2 + |\nabla f|^2(2\mu\Delta_f f + \mu|\nabla f|^2 - \lambda).$$

Proceeding as in the proof of the Proposition we get

$$|\nabla f|\Delta_f|\nabla f| \geq |\nabla f|^2(2\mu\Delta_f f + \mu|\nabla f|^2 - \lambda) \quad \text{weakly on } M. \tag{7.4.26}$$

We use (7.1.8) to obtain, from (7.4.26),

$$|\nabla f|\Delta_f|\nabla f| \geq (\lambda - 2\Lambda e^{2\mu f} + \mu|\nabla f|^2)|\nabla f|^2 \quad \text{weakly on } M.$$

The hypothesis $iv)$ guarantees the validity of

$$\lambda - 2\Lambda e^{2\mu f} \geq 0,$$

hence from the above we get

$$|\nabla f|\Delta_f|\nabla f| \geq \mu|\nabla f|^4 \quad \text{weakly on } M,$$

and thus we can conclude, as in the proof of the Proposition.

Notice that if f is constant, from (7.1.8) we have

$$\Lambda e^{2\mu f} = \lambda.$$

Hence $\lambda = 0$ if and only if $\Lambda = 0$ and, if this is the case, f can be an arbitrary constant. On the other hand, if λ is different from zero then $\lambda\Lambda > 0$ and

$$f = \frac{1}{2\mu}\log\left(\frac{\lambda}{\Lambda}\right).$$

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