ON CERTAIN ISOGENIES BETWEEN K3 SURFACES

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Abstract. We will prove that there are infinitely many families of K3 surfaces which both admit a finite symplectic automorphism and are (desingularizations of) quotients of other K3 surfaces by a symplectic automorphism. These families have an unexpectedly high dimension. We apply this result to construct "special" isogenies between K3 surfaces, which are not Galois covers between K3 surfaces, but are obtained by composing cyclic Galois covers. In the case of involutions, for any $n \in \mathbb{N}_{>0}$ we determine the transcendental lattices of the K3 surfaces which are $2^n : 1$ isogenous (by the mentioned "special" isogeny) to other K3 surfaces.

1. Introduction

K3 surfaces are complex symplectic regular surfaces; among their finite order automorphisms the ones which preserve the symplectic structure (the symplectic automorphisms) play a special role. Indeed, the quotient of a K3 surface by a finite symplectic automorphism produces a singular surface whose desingularization is again a K3 surface. This construction establishes a particular relation between different sets of K3 surfaces: the ones which admit a finite symplectic automorphism and the ones obtained as desingularization of the quotient of a K3 surface by a symplectic automorphism. In the following the latter K3 surfaces are said to be (cyclically) covered by a K3 surface and the former are said to be the cover of a K3 surface. A general K3 surface admitting a symplectic automorphism is non projective and the same holds for the desingularization of its quotient. By considering only projective K3 surfaces with the above properties one can obtain richer results and geometric properties; we denote by $\mathcal{L}_n$ the set of the projective K3 surfaces which admit an order n symplectic automorphism and by $\mathcal{M}_n$ the set of the projective K3 surfaces which are $n : 1$ cyclically covered by a K3 surface.

Thanks to several works, starting from the end of the 70's until now (see, e.g. [N2], [Mo], [vGS], [GSar1], [GSar2], [GSar3], [G2]), the sets $\mathcal{L}_n$ and $\mathcal{M}_n$ are nonempty for $n \leq 8$ and described as the union of countably many families of lattice polarized K3 surfaces. The dimension of these families is at most 11, and, recalling that the families of generic projective K3 surfaces have dimension 19, one immediately observes that the K3 surfaces which admit a finite symplectic automorphism or which are cyclically covered by a K3 surface are quite special. So, it is natural to expect that the intersection $\mathcal{L}_n \cap \mathcal{M}_n$ is small, namely smaller than the dimension of each family. Indeed, if one considers non projective K3 surfaces, this expectation is right: a generic non projective K3 surface which admits an order n symplectic

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automorphism is not $n:1$ covered by another K3 surface and vice versa, as observed in Proposition 3.1. On the other hand, there is at least one known example of a family of K3 surfaces contained in $\mathcal{L}_n \cap \mathcal{M}_n$, given by the family of the K3 surfaces which admit an elliptic fibration with an n-torsion section (see Section 3.2). This family has codimension one in some irreducible components of $\mathcal{L}_n$ and in some irreducible components of $\mathcal{M}_n$. The main result of this paper is to show that the intersection $\mathcal{L}_n \cap \mathcal{M}_n$ has components of codimension 0 in both $\mathcal{L}_n$ and $\mathcal{M}_n$.

Our purpose is to investigate more precisely the intersection between the two sets $\mathcal{L}_n$ and $\mathcal{M}_n$ and to relate it with the study of isogenies between K3 surfaces. In this paper, the term "isogeny between K3 surface" means a generically finite rational map between K3 surfaces, as in [I].

The quotient by a finite symplectic automorphism on a K3 surface X induces an isogeny between X, which admits the symplectic automorphism, and the K3 surface Y cyclically covered by X. The isogeny is birationally the quotient map, whose degree is the order of the automorphism. There are other isogenies between K3 surfaces, which are not quotient maps, see e.g. [I]. Here we discuss one such instance: given a K3 surface $Z \in \mathcal{L}_n \cap \mathcal{M}_n$, it induces an isogeny between two other K3 surfaces. Indeed, since $Z \in \mathcal{M}_n$, it is $n:1$ covered by a K3 surface X; since $Z \in \mathcal{L}_n$, it is an $n:1$ cover of a K3 surface Y. By composing these two $n:1$ maps one obtains an $n^2:1$ isogeny between X and Y. We will prove in Proposition 3.11 that generically this isogeny is not induced by a quotient map.

In Section 2 we recall some known results on the set $\mathcal{L}_n$ of K3 surfaces admitting a symplectic automorphism of order n and on the set $\mathcal{M}_n$ of the K3 surfaces $n:1$ cyclically covered by a K3 surface. In Section 3 we obtain our main results on the intersection $\mathcal{L}_n \cap \mathcal{M}_n$. In particular in Theorem 3.9 we prove:

Main theorem *For $2 \leq n \leq 8$, there are infinitely many irreducible components $\mathcal{Z}$ of $\mathcal{L}_n \cap \mathcal{M}_n$ such that $\dim(\mathcal{L}_n) = \dim(\mathcal{M}_n) = \dim \mathcal{Z}$; any such $\mathcal{Z}$ is an irreducible maximal dimensional component of both $\mathcal{L}_n$ and $\mathcal{M}_n$.*

In case $n = 2$, a more precise description of the relations among the components of $\mathcal{L}_2$ and of $\mathcal{M}_2$ is known and this allows us to obtain more specific results, contained in Section 3.6 (see Theorem 3.14 and Corollary 3.16). In particular, we describe countably many families of polarized K3 surfaces, such that there exists an isogeny between members of each family. The Néron–Severi groups and the transcendental lattices of all the surfaces involved in this isogeny are explicitly given.

In Section 3.4 we discuss an example of a K3 surface which is generic both in $\mathcal{L}_2$ and in $\mathcal{M}_2$. As a consequence of our main theorem, it admits two different degree 4 pseudo-ample divisors, and thus two different projective models. We give explicitly the relations between these two degree 4 divisors, linking two geometric constructions contained in [vGS] .

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2. Preliminary results

We recall in this section some of the definitions and results on K3 surfaces, symplectic automorphisms on K3 surfaces and quotients of K3 surfaces by their automorphisms. In the following we work with complex surfaces.

2.1. Symplectic automorphisms and cyclic covers of K3 surfaces.

Definition 2.1. *A K3 surface is a regular complex surface with trivial canonical bundle. If X is a K3 surface, we choose a generator of $H^{2,0}(X)$, (i.e. a symplectic form), we denote it by ω_X and we call it the period of the K3 surface. The second cohomology group $H^2(X,\mathbb{Z})$ of a K3 surface X equipped with the cup product is a lattice, isometric to a standard lattice which does not depend on X and is denoted by $\Lambda_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$.*

The moduli space of K3 surfaces satisfying certain prescribed geometric properties can be often described in a lattice theoretic way by using the notion of R-polarized K3 surfaces, for certain lattices R. We introduce the following notion and notation.

Definition 2.2. *Given an even lattice R which admits a primitive embedding in Λ_{K3}, we denote by $\mathcal{P}(R)$ the moduli space of isomorphism classes of R-polarized K3 surfaces, i.e. of those K3 surfaces X for which there exists a primitive embedding $R \subset \mathrm{NS}(X)$. Moreover, we will write $A < B$ in order to say that B is an overlattice of finite index of A.*

We warn the reader that this notion of lattice polarization does not imply the presence of an ample polarization as usual in algebraic geometry; nevertheless we need to work in this generality, as it is done in the seminal paper [N2] (see [N2, Section 2, §4]).

Definition 2.3. *Let X be a K3 surface, and ω_X its period. An automorphism σ of X is said to be symplectic if $\sigma^*(\omega_X) = \omega_X$.*

One of the main results on symplectic automorphisms on K3 surfaces is that the quotient of a K3 surface by a symplectic automorphism is still a K3 surface, after a birational transformation which resolves the singularities of the quotient.

Proposition 2.4. ([N2]) *Let X be a K3 surface and $\sigma \in Aut(X)$ a finite automorphism of X. Then the minimal smooth surface Y birational to X/σ is a K3 surface if and only if σ is symplectic.*

Definition 2.5. *We will say that a K3 surface Y is $n:1$ cyclically covered by a K3 surface, if there exists a pair (X,σ) such that X is a K3 surface, σ is an automorphism of order n of X and Y is birational to X/σ.*

The first mathematician who worked on symplectic automorphisms of finite order on K3 surfaces and who established the fundamental results on these automorphisms was Nikulin, in [N2]. We summarize in Theorem 2.8 and Theorem 2.10 the main results obtained in his paper, but first we recall some useful information and definitions.

If σ is a symplectic automorphism on X of order n, its linearization near the points with non trivial stabilizer is given by a 2×2 diagonal matrix with determinant 1 and thus it is of the form $\mathrm{diag}(\zeta_n^a, \zeta_n^{n-a})$ for $1 \le a \le n-1$ and ζ_n an n-th primitive root of unity. The points with non trivial stabilizer are isolated fixed points and the quotient X/σ has isolated singularities, all of type A_{m_j} where m_j+1 divides n. In particular the surface Y, which is the minimal surface resolving the singularities of X/σ, contains a configuration of smooth rational curves M_i arising from the desingularization of X/σ. This configuration depends only on n (and not on X and σ), see [N2, Section 5]. The classes of the curves M_i span the lattice $\mathbb{E}_n$ described in the following table (see [N2, Section 5]):

n	2	3	4	5	6	7	8
$\mathbb{E}_n(-1)$	$A_1^{\oplus 8}$	$A_2^{\oplus 6}$	$A_3^{\oplus 4}\oplus A_1^{\oplus 2}$	$A_4^{\oplus 4}$	$A_5^{\oplus 2}\oplus A_2^{\oplus 2}\oplus A_1^{\oplus 2}$	$A_6^{\oplus 3}$	$A_7^{\oplus 2}\oplus A_3\oplus A_1$

Since Y is the minimal model of X/σ, $\mathbb{E}_n\subset \mathrm{NS}(Y)$. In [N2, Section 6], it is proved that $\mathbb{E}_n$ is not primitive inside $\mathrm{NS}(Y)$ and that the minimal primitive sublattice of $\mathrm{NS}(Y)$ containing $\mathbb{E}_n$ depends only on n. This motivates the following definition.

Definition 2.6. *Let Y be a K3 surface, $n:1$ cyclically covered by a K3 surface. The minimal primitive sublattice of* $\mathrm{NS}(Y)$ *containing the lattice $\mathbb{E}_n$ is denoted by $\mathbb{M}_n$.*

Definition 2.7. (See [N2, Definition 4.6]) *Let σ be an order n automorphism of a K3 surface X. We will say that its action on the second cohomology group is essentially unique if there exists an isometry $g_n:\Lambda_{K3}\xrightarrow{\sim}\Lambda_{K3}$ of order n of Λ_{K3} such that for every pair (X,σ), there exists an isometry $\varphi: H^2(X,\mathbb{Z})\rightarrow\Lambda_{K3}$ such that $\sigma^*=\varphi^{-1}\circ g_n\circ\varphi$.*

Theorem 2.8. *Let X be a K3 surface and σ a finite symplectic automorphism of X of order n. Then*

- $2\leq n\leq 8$ *(see* [N2, Theorem 6.3]*);*
- *the singularities of X/σ depend only on n (see* [N2, Section 5]*);*
- *the isometry class of the lattice $\mathbb{M}_n$ depends only on n and $\mathbb{M}_n$ is an overlattice of index n of the lattice $\mathbb{E}_n$ spanned by the curves arising from the desingularization of the quotient X/σ (see* [N2, Theorem 6.3]*);*
- *the action of σ^* on $H^2(X,\mathbb{Z})$ is essentially unique (see* [N2, Theorem 4.7]*) and thus the isometry classes of the lattices $H^2(X,\mathbb{Z})^{\sigma^*}$ and $\left(H^2(X,\mathbb{Z})^{\sigma^*}\right)^{\perp}$ depend only on n.*
- *The lattice $\left(H^2(X,\mathbb{Z})^{\sigma}\right)^{\perp}$ is primitively embedded in* $\mathrm{NS}(X)$ *(see* [N2, Lemma 4.2]*) and* $\mathrm{rank}\left(\left(H^2(X,\mathbb{Z})^{\sigma}\right)^{\perp}\right)=\mathrm{rank}(\mathbb{M}_n)$ *(see* [N2, Formula (8.12)]*).*

Definition 2.9. *Let X be a K3 surface with a symplectic automorphism σ of order n. Since the action of σ^* on $H^2(X,\mathbb{Z})$ is essentially unique, the lattice $\left(H^2(X,\mathbb{Z})^{\sigma^*}\right)^{\perp}$ is isometric to $(\Lambda_{K3}^{g_n})^{\perp}$ (with the notation of Definition 2.7) and we denote it by Ω_n.*

For every admissible n the lattices Ω_n were computed: in [Mo] if $n=2$; in [GSar1] if n is an odd prime; in [GSar3] if n is not a prime.

The lattices $\mathbb{M}_n$ were computed for every admissible n in [N2, Theorems 6.3 and 7.1].

The lattices Ω_n and $\mathbb{M}_n$ characterize the K3 surfaces admitting a symplectic automorphism of order n or a $n:1$ cyclic cover by a K3 surface respectively; indeed, the following two results hold

Theorem 2.10. (See [N2, Theorems 4.15 and 2.10]) *A K3 surface X admits a symplectic automorphism of order n if and only if Ω_n is primitively embedded in* $\mathrm{NS}(X)$.

The moduli space of the K3 surfaces (not necessarily projective) admitting a symplectic automorphism of order n is $\mathcal{P}(\Omega_n)$, which is irreducible of dimension $20 - \mathrm{rank}(\Omega_n)$.

Proof. From [N2, Theorem 4.15], Ω_n is primitively embedded in $\mathrm{NS}(X)$ if and only if X admits a symplectic automorphism of order n. The lattice Ω_n does not contain classes with self intersection equal to -2, see [N2, Theorem 4.3 b)]. It follows from [N2, Theorem 2.10], that $\mathcal{P}(\Omega_n)$ is connected. The image of $\mathcal{P}(\Omega_n)$ via the period map (see [Huy1, Chapter 6] for the definition and an extensive survey) is an irreducible set thus $\mathcal{P}(\Omega_n)$ itself is an irreducible complex space. □

Theorem 2.11. (See [N2, Section 8]; for projective cases [GSar2, Proposition 2.3] for the case $n = 2$ and [G2, Theorem 5.2] for $2 < n \leq 8$). *A K3 surface Y is $n : 1$ cyclically covered by a K3 surface if and only if $\mathbb{M}_n$ is primitively embedded in* $\mathrm{NS}(Y)$.

The moduli space of the K3 surfaces (not necessarily projective) admitting an $n : 1$ cyclic cover by a K3 surface is $\mathcal{P}(\mathbb{M}_n)$, and it has dimension $20 - \mathrm{rank}(\mathbb{M}_n)$.

Remark 2.12. The lattice $\mathbb{M}_n$ contains classes with square -2, thus by [N2, Theorem 2.10], the space $\mathcal{P}(\mathbb{M}_n)$ is not necessarily connected, and has at most a finite number of connected components.

Inside the 20-dimensional space of all complex K3 surfaces, the *Noether–Lefschetz locus* is defined as the locus of those K3 surfaces X with non-trivial $\mathrm{NS}(X)$; it is a countable union of smooth codimension one subsets. In particular, the algebraic part of the Noether–Lefschetz locus is the union of all the families $\mathcal{P}(\langle 2d\rangle)$ with $d > 0$, and it is the set of projective K3 surfaces.

In analogy, the intersection of $\mathcal{P}(\Omega_n)$, respectively of $\mathcal{P}(\mathbb{M}_n)$, with the algebraic part of the Noether–Lefschetz locus is given by the union of countably many subsets of codimension one (since Ω_n and $\mathbb{M}_n$ are negative definite lattices), giving the subset of projective K3 surfaces inside $\mathcal{P}(\Omega_n)$, respectively $\mathcal{P}(\mathbb{M}_n)$.

Corollary 2.13. *Let X be a projective K3 surface admitting a symplectic automorphism of order n. Then $\rho(X) \geq 1 + \mathrm{rank}(\Omega_n)$ and if $\rho(X) = 1 + \mathrm{rank}(\Omega_n)$, then $\mathrm{NS}(X)$ is an overlattice of finite index (possibly 1) of $\langle 2d\rangle \oplus \Omega_n$, for a certain $d \in \mathbb{N}_{>0}$, such that Ω_n is primitively embedded in this overlattice.*

Let Y be a projective K3 surface $n : 1$ cyclically covered by a K3 surface. Then $\rho(Y) \geq 1 + \mathrm{rank}(\mathbb{M}_n)$ and if $\rho(Y) = 1 + \mathrm{rank}(\mathbb{M}_n)$, then $\mathrm{NS}(Y)$ is an overlattice of finite index (possibly 1) of $\langle 2e\rangle \oplus \mathbb{M}_n$, for a certain $e \in \mathbb{N}_{>0}$, such that $\mathbb{M}_n$ is primitively embedded in this overlattice.

Proof. Since X admits a symplectic automorphism of order n, Ω_n is primitively embedded in $\mathrm{NS}(X)$. Since Ω_n is negative definite and X is projective, the orthogonal complement to Ω_n in $\mathrm{NS}(X)$ contains a class with positive self intersection, in particular it is non empty. So $\rho(X) \geq 1 + \mathrm{rank}(\Omega_n)$ and $\langle 2d\rangle \oplus \Omega_n$ is embedded in $\mathrm{NS}(X)$. Similarly one obtains the result for $\rho(Y)$ and $\mathrm{NS}(Y)$. □

Definition 2.14. *We define the following sets of K3 surfaces (which are subsets of the moduli space of K3 surfaces):*

$\mathcal{L}_n^{\mathbb{C}} := \{$*Complex K3 surfaces which admit a symplectic automorphism* σ *of order* $n\}/\cong$,

$\mathcal{L}_n := \{$*Projective K3 surfaces which admit a symplectic automorphism* σ *of order* $n\}/\cong$,

$\mathcal{M}_n^{\mathbb{C}} := \{$*Complex K3 surfaces which admit an* $n:1$ *cyclic cover by a K3 surface*$\}/\cong$,

$\mathcal{M}_n := \{$*Projective K3 surfaces which admit an* $n:1$ *cyclic cover by a K3 surface*$\}/\cong$,

where $\cong$ *denotes the equivalence relation given by isomorphism between two K3 surfaces.*

By Theorems 2.10 and 2.11, the sets $\mathcal{L}_n^{\mathbb{C}}$ and $\mathcal{M}_n^{\mathbb{C}}$ are two moduli spaces of (lattice polarized) K3 surfaces, both of dimension $20-\operatorname{rank}(\Omega_n)=20-\operatorname{rank}(\mathbb{M}_n)$. The situation is more intricate in the projective case, as shown by the following corollaries.

Corollary 2.15. *The set* $\mathcal{L}_n$ *is the following union of countably many components:*

$$\mathcal{L}_n=\bigcup_{d\in\mathbb{N}_{>0}}\left(\bigcup_{\substack{(\langle 2d\rangle\oplus\Omega_n)<R\\ \Omega_n\subset R \text{ prim.}}}\mathcal{P}(R)\right).$$

All the components $\mathcal{P}(R)$ *of* $\mathcal{L}_n$ *are equidimensional and have dimension* $19-\operatorname{rank}(\Omega_n)$.

The set $\mathcal{M}_n$ *is the following union of countably many components:*

$$\mathcal{M}_n=\bigcup_{e\in\mathbb{N}_{>0}}\left(\bigcup_{\substack{(\langle 2e\rangle\oplus\mathbb{M}_n)<R\\ \mathbb{M}_n\subset R \text{ prim.}}}\mathcal{P}(R)\right).$$

All the components are equidimensional and have dimension $19-\operatorname{rank}(\mathbb{M}_n)=19-\operatorname{rank}(\Omega_n)$.

Proof. Let R be an overlattice of finite index of $\langle 2d\rangle\oplus\Omega_n$ such that there exists a primitive embedding of Ω_n in R. If X is a K3 surface such that R is primitively embedded in $\operatorname{NS}(X)$, then Ω_n is primitively embedded in $\operatorname{NS}(X)$ and thus X admits a symplectic automorphism of order n, by Theorem 2.10. Vice versa, if a projective K3 surface X admits a symplectic automorphism of order n, then there exists a $d\in\mathbb{N}_{>0}$ such that $\langle 2d\rangle\oplus\Omega_n$ is embedded in $\operatorname{NS}(X)$, and an overlattice R of $\langle 2d\rangle\oplus\Omega_n$ is primitively embedded in $\operatorname{NS}(X)$. So one can describe the set $\mathcal{L}_n$ as union of families $\mathcal{P}(R)$ of R-polarized K3 surfaces, where R is a proper overlattice of index r (possibly 1) of $\langle 2d\rangle\oplus\Omega_n$ for a certain $d\in\mathbb{N}_{>0}$. There are countably many lattices $\langle 2d\rangle\oplus\Omega_n$ and each of them has a finite number of overlattices of finite index. So $\mathcal{L}_n$ is the union of countably many families of R-polarized K3 surfaces. The dimension of each of these families is $20-\operatorname{rank}(R)=20-(1+\operatorname{rank}(\Omega_n))$. This concludes the proof for the set $\mathcal{L}_n$.

The proof for $\mathcal{M}_n$ is similar, but one has to use Theorem 2.11 instead of Theorem 2.10. □

By the previous corollary one observes that the lattices $\langle 2d\rangle\oplus\Omega_n$ and $\langle 2e\rangle\oplus\mathbb{M}_n$ play a central role in the description of the families of projective K3 surfaces with symplectic automorphism and with cyclic K3 cover respectively. So we give the following definition

Definition 2.16. *For each $2\leq n\leq 8$ and each $d\in\mathbb{N}$, $d\geq 1$, we denote by $L_{d,n}$ the lattice $\langle 2d\rangle\oplus\Omega_n$. For each $2\leq n\leq 8$ and each $e\in\mathbb{N}$, $e\geq 1$, we denote by $M_{e,n}$ the lattice $\langle 2e\rangle\oplus\mathbb{M}_n$.*

2.2. **Involutions.** When $n=2$ one can give a more precise description of $\mathcal{L}_2$ and $\mathcal{M}_2$, since one is able to determine precisely all the overlattices of $L_{d,2}$ and $M_{e,2}$ which can be used to define the components of these two sets.

Proposition 2.17. [vGS, Propositions 2.2 and 2.3] (a) *The lattice Ω_2 is isometric to $E_8(-2)$ and in particular its rank is 8 so if $X\in\mathcal{L}_2$, then its Picard number is $\rho(X)\geq 9$.*

(b) *There exists an even overlattice of index two of $L_{d,2}$ in which Ω_2 is primitively embedded if and only if $d\geq 1$ is even. In this case, this lattice is unique up to isometry and denoted by $L'_{d,2}$.*

(c) *If $X\in\mathcal{L}_2$ and $\rho(X)=9$, then $\mathrm{NS}(X)$ is isometric either to $L_{d,2}$ or to $L'_{d,2}$ for a certain $d\geq 1$.*

(d)
$$\mathcal{L}_2=\bigcup_{d\in\mathbb{N}_{>0}}\mathcal{P}(L_{d,2})\cup\bigcup_{d\in 2\mathbb{N},d>0}\mathcal{P}(L'_{d,2}).$$

Proposition 2.18. (a) *The lattice $\mathbb{M}_2$ has rank 8, so if $Y\in\mathcal{M}_2$, then its Picard number is $\rho(Y)\geq 9$.*

(b) *There exists an even overlattice of index two of $M_{e,2}$ in which $\mathbb{M}_2$ is primitively embedded if and only if $e\geq 1$ is even. In this case, this lattice is unique up to isometry and denoted by $M'_{e,2}$.*

(c) *If $Y\in\mathcal{M}_2$ and $\rho(Y)=9$, then $\mathrm{NS}(Y)$ is isometric either to $M_{e,2}$ or to $M'_{e,2}$ for a certain $e\geq 1$.*

(d)
$$\mathcal{M}_2=\bigcup_{e\in\mathbb{N}_{>0}}\mathcal{P}(M_{e,2})\cup\bigcup_{e\in 2\mathbb{N},e>0}\mathcal{P}(M'_{e,2}).$$

Proof. (a) is proved by Nikulin, [N2, Proposition 7.1], (b) is proved in [GSar2, Proposition 2.2], (c) in [GSar2, Proposition 2.1], (d) follows from Corollary 2.15 and from (c). □

We observe that in the literature the lattice $\mathbb{M}_2$ is often called "Nikulin lattice" and the K3 surfaces contained in $\mathcal{M}_2$ are often called "Nikulin surfaces".

The main result which is known for involutions and is still to be proved in the more general case of symplectic automorphisms of order n is the explicit relation between the Néron–Severi group of a K3 surface which admits a symplectic involution and the Néron–Severi group of the K3 surface which is the desingularization of its quotient.

Proposition 2.19. ([GSar2, Corollary 2.2]) *Let X be a K3 surface with a symplectic involution σ and Y be the minimal resolution of X/σ. Then:*

- $\mathrm{NS}(X)\simeq L_{e,2}$ *if and only if* $\mathrm{NS}(Y)\simeq M'_{2e,2}$
- $\mathrm{NS}(X)\simeq L'_{2e,2}$ *if and only if* $\mathrm{NS}(Y)\simeq M_{e,2}$.

2.3. **Isogenies between K3 surfaces.** The following definition was first given by Inose in [I] in the case of K3 surfaces with Picard number 20.

Definition 2.20. *Let X and Y be two K3 surfaces. We say that X and Y are isogenous if there exists a rational map of finite degree between X and Y. This map is said to be an isogeny between X and Y and if it is generically of degree n, the map is said to be an isogeny of degree n.*

The easiest construction of an isogeny between K3 surfaces is given by the quotient by a finite symplectic automorphism, i.e. if X is a K3 surface admitting a symplectic automorphism σ of order n, then the quotient map induces an isogeny of degree n between X and Y, the minimal model of X/σ. So if $X \in \mathcal{L}_n$, then there exists $Y \in \mathcal{M}_n$ which is isogenous to X with an isogeny of degree n. Similarly if $Y \in \mathcal{M}_n$, then there exists a K3 surface $X \in \mathcal{L}_n$ which is isogenous to Y with an isogeny $X \dashrightarrow Y$ of degree n.

There exist however isogenies between K3 surfaces which are not induced by the quotient by a finite group of symplectic automorphisms: an example is given by isogenous Kummer surfaces constructed from Abelian surfaces related by an isogeny, as in [I, Proof of Thm 2], under the additional assumption that the degree is a prime $p > 7$, (see also [BSV, Example 6.5]).

2.4. **Remarks on Hodge isogenies between K3 surfaces.** Definition 2.20 is not the only notion of isogeny existing in the literature: to distinguish between the two definitions, we will talk here of *Hodge isogeny* for the notion used for example in [Bu, Huy2].

Definition 2.21. *Let X and Y be two K3 surfaces. We say the X and Y are Hodge isogenous if there exists a rational Hodge isometry between $H^2(X,\mathbb{Q})$ and $H^2(Y,\mathbb{Q})$.*

Hodge isogenous K3 surfaces have been studied since foundational work of [M1] and [N3], also in relation with Šafarevič's conjecture [Ša] about algebraicity of correspondences on K3 surfaces.

In [BSV, Prop. 3.1], the authors give a comparison between the notion of isogeny and of Hodge isogeny:

Proposition 2.22. *If $\varphi : X \dashrightarrow Y$ is an isogeny of degree n, n is not a square and the rank of the transcendental lattices T_X and T_Y is odd, φ is never a Hodge isogeny.*

This follows from the fact that, under these assumptions, there cannot exist any isometry $T_X \otimes \mathbb{Q} \simeq T_Y \otimes \mathbb{Q}$. The transcendental lattice T_X of the very general K3 surface $X \in \mathcal{L}_n$ has always odd rank (see Theorem 3.9); by Proposition 2.22 if n is not a square, so if $n \neq 4$, the surface X is never Hodge isogenous to the minimal resolution of its quotient. The assumption on the degree n is in particular due to the following straightforward fact:

Lemma 2.23. *For any lattice T and any integer $n \in \mathbb{N}$, there exists an isometry $T \otimes \mathbb{Q} \simeq T(n^2) \otimes \mathbb{Q}$.*

Proposition 2.24. *For any $n \in \mathbb{N}$, if $\varphi : X \dashrightarrow Y$ is an isogeny of degree n^2, then X and Y are Hodge isogenous.*

Proof. It is proven in [BSV, Proposition 3.2] that $T_X \otimes \mathbb{Q} \simeq T_Y \otimes \mathbb{Q}$ if and only if $T_Y \otimes \mathbb{Q} \simeq T_Y(n^2) \otimes \mathbb{Q}$, which is true by Lemma 2.23. Then Witt's theorem implies that the isometry $T_X \otimes \mathbb{Q} \simeq T_Y \otimes \mathbb{Q}$ extends to a Hodge isometry $H^2(X, \mathbb{Q}) \simeq H^2(Y, \mathbb{Q})$. □

One of the interesting properties of Hodge isogenous K3 surfaces is that they have isomorphic rational motives, by [Huy2, Theorem 0.2]. This also holds in the case described above of a K3 surface X isogenous to the minimal model Y of the quotient X/σ, as shown for example in [L, Proof of Thm 3.1] following the argument of [P], but to the knowledge of the authors it is still an open question for a general isogeny.

3. The intersection $\mathcal{L}_n \cap \mathcal{M}_n$

The main result in this section is Theorem 3.9, where we exhibit the maximal dimensional components of $\mathcal{L}_n \cap \mathcal{M}_n$. As preliminary results, in Section 3.1 we discuss the non projective case (where the analogue of Theorem 3.9 does not hold) and we describe in Section 3.3 a specific family of K3 surfaces contained in $\mathcal{L}_n \cap \mathcal{M}_n$. This family is related with a special isogeny between K3 surfaces, which is induced by an isogeny between elliptic curves. In Section 3.6 we consider the case $n = 2$: under this assumption Theorem 3.9 can be improved to Theorem 3.14, and this gives as biproduct point-wise isogenies between infinitely many families of K3 surfaces (see Corollary 3.16). In Section 3.4 a geometric example of the lattice theoretic result of Theorem 3.9 is given.

3.1. Non-projective K3 surfaces.

A general complex K3 surface which admits a finite order symplectic automorphism (resp. a cyclic cover by a K3 surface) is non projective, by Theorem 2.10 (resp. Theorem 2.11). Here we prove that a general K3 surface admitting an order n symplectic automorphism is not $m : 1$ cyclically covered by a K3 surface. In particular the set $\mathcal{L}_n^{\mathbb{C}}$ does not coincide with the set $\mathcal{M}_n^{\mathbb{C}}$ and if these sets intersect, the irreducible components of their intersection have positive codimension in both of them.

Proposition 3.1. (*a*) *Let X be a K3 surface in $\mathcal{L}_n^{\mathbb{C}}$ with the minimum possible Picard number, so* $\mathrm{NS}(X) \simeq \Omega_n$. *Then X does not admit an $m : 1$ cyclic cover by a K3 surface for any possible m, i.e. $X \notin \mathcal{M}_m^{\mathbb{C}}$ for any m.*

(*b*) *Let Y be a K3 surface in $\mathcal{M}_n^{\mathbb{C}}$ with the minimum possible Picard number, so* $\mathrm{NS}(Y) \simeq \mathbb{M}_n$. *Then Y does not admit an order m symplectic automorphism for any $m \geq n$, , i.e. $Y \notin \mathcal{L}_m^{\mathbb{C}}$ for any $m \geq n$.*

Proof. If X admits an order n symplectic automorphism and its Picard number is minimal, then $\mathrm{NS}(X) \simeq \Omega_n$ (by Theorem 2.10). By [N2, Lemma 4.2], the lattice Ω_n does not contain elements of square -2. In particular it cannot contain $\mathbb{M}_m$, which is an overlattice of finite index of a root lattice. So, by Theorem 2.11, X does not admit a finite cyclic cover by a K3 surface.

If Y admits an $n : 1$ cyclic cover by a K3 surface and its Picard number is minimal, then $\mathrm{NS}(Y) \simeq \mathbb{M}_n$ (by Theorem 2.11). By comparison of the lattices $\mathbb{M}_n$ and Ω_n one can directly check that they never have the same discriminant group (see also Proposition 3.5), so $\mathbb{M}_n$ is not isometric to Ω_n, which proves that Y cannot admit a symplectic automorphism of order n. Moreover, if $m \geq n$, then

$\mathrm{rank}(\Omega_m) \geq \mathrm{rank}(\mathbb{M}_n)$, and Ω_m is never isometric to $\mathbb{M}_n$. So Y does not admit a symplectic automorphism of order $m \geq n$. □

In Proposition 3.1 case (b), one cannot erase the condition $m \geq n$, as shown by the following proposition.

Proposition 3.2. *If Y is a K3 surface such that $\mathrm{NS}(Y) = \mathbb{M}_8$, then $Y \in \mathcal{L}_2^{\mathbb{C}} \cap \mathcal{M}_8^{\mathbb{C}}$.*

Proof. If $\mathrm{NS}(Y) = \mathbb{M}_8$, the transcendental lattice of Y is a lattice with signature $(3,1)$ and discriminant form $\mathbb{Z}/2\mathbb{Z}\left(\frac{1}{2}\right) \times \mathbb{Z}/4\mathbb{Z}\left(\frac{1}{4}\right)$ (see [GSar3]). By [N1, Theorem 1.13.2], the transcendental lattice is uniquely determined by these data and so it is $T_Y \simeq U \oplus \langle 2\rangle \oplus \langle 4\rangle$. This lattice admits a primitive embedding in $U\oplus U\oplus U$, and thus in the lattice $T := U \oplus U \oplus U \oplus E_8(-2)$. The lattice T is the transcendental lattice of a K3 surface, whose Néron–Severi group N is an even lattice with signature $(0, 8)$ and discriminant form $-q_T = u(2)^{\oplus 4}$. There is a unique lattice with these properties up to isometries, which is $E_8(-2)$. Indeed, since $\mathrm{rank}(N) = l(N)$ there is a basis $\{b_1, \ldots, b_8\}$ of N such that $\{b_i/2\}_i$ generates A_N. The discriminant form $u(2)$ takes values in $\mathbb{Z}/2\mathbb{Z}$, thus

$$\frac{b_i}{2}\frac{b_j}{2} \in \begin{cases} \mathbb{Z} & \text{if } i = j \\ \frac{1}{2}\mathbb{Z} & \text{if } i \neq j \end{cases} \quad \text{so } b_i b_j \in \begin{cases} 4\mathbb{Z} & \text{if } i = j \\ 2\mathbb{Z} & \text{if } i \neq j \end{cases}$$

Hence there exists an even lattice R such that $N = R(2)$, and R is necessarily unimodular. The only possibility is $R = E_8(-1)$ and $N = E_8(-2)$.

By $T_Y \hookrightarrow T$, their orthogonal complements in Λ_{K3} satisfy

$$\mathrm{NS}(Y) \simeq \mathbb{M}_8 \simeq T_Y^{\perp} \hookleftarrow T^{\perp} \simeq E_8(-2) \simeq \Omega_2.$$

By Theorem 2.10, $Y \in \mathcal{L}_2^{\mathbb{C}}$. □

3.2. **The $(U \oplus \mathbb{M}_n)$-polarized K3 surfaces.** In view of Section 3.1, one could ask if there exist K3 surfaces in $\mathcal{L}_n^{\mathbb{C}} \cap \mathcal{M}_n^{\mathbb{C}}$. In this section we consider a family of projective K3 surfaces which answers positively this question; it has codimension 2 in $\mathcal{L}_n^{\mathbb{C}} \cap \mathcal{M}_n^{\mathbb{C}}$ and codimension 1 in $\mathcal{L}_n$ and $\mathcal{M}_n$.

The $(U \oplus \mathbb{M}_n)$-polarized K3 surfaces have interesting geometric properties: this family is considered for $n = 2$ in [vGS], and for other values of n in [GSar1] and [GSar3], to find explicitly Ω_n. Here we reconsider it as example of a family of K3 surfaces contained in $\mathcal{L}_n \cap \mathcal{M}_n$.

Proposition 3.3. *Let $2 \leq n \leq 8$ and $\mathcal{U}_n := \mathcal{P}(U \oplus \mathbb{M}_n)$ be the family of the $(U \oplus \mathbb{M}_n)$-polarized K3 surfaces. Then:*

- *$\mathcal{U}_n$ is non empty and has dimension $18 - \mathrm{rank}(\mathbb{M}_n)$;*
- *if S is a K3 surface such that $S \in \mathcal{U}_n$, then S admits an elliptic fibration $\mathcal{E}_n : S \to \mathbb{P}^1$ with an n-torsion section t;*
- *if $S \in \mathcal{U}_n$ and σ_t is the translation by t on $\mathcal{E}_n$, the minimal model of S/σ_t is a K3 surface in $\mathcal{U}_n$;*
- *$\mathcal{U}_n \subset \mathcal{L}_n \cap \mathcal{M}_n$.*

Proof. In [G1, Proposition 4.3] it is proved that the set of K3 surfaces admitting an elliptic fibration with a torsion section of order n coincides with $\mathcal{P}(U \oplus \mathbb{M}_n)$, of dimension $20 - (\mathrm{rank}(U \oplus \mathbb{M}_n)) = 18 - \mathrm{rank}(\mathbb{M}_n)$. Since $\mathbb{M}_n$ is clearly primitively embedded in $U\oplus\mathbb{M}_n$, all the K3 surfaces in $\mathcal{U}_n$ are also contained in $\mathcal{M}_n$. Moreover, let $\mathcal{E}_n : S \to \mathbb{P}^1$ be an elliptic fibration on S with an n-torsion section t. This allows to consider S as an elliptic curve over the field of functions $k(\mathbb{P}^1)$ and the presence of an n-torsion section is equivalent to the presence of an n-torsion rational point on this elliptic curve. The translation by t is well defined and it induces an automorphism σ_t of order n on S, which is symplectic (it is the identity on the base of the fibration and acts on the smooth fibers preserving their periods). Since any $S \in \mathcal{U}_n$ admits a symplectic automorphism of order n, $\mathcal{U}_n \subset \mathcal{L}_n$ and thus $\mathcal{U}_n \subset \mathcal{L}_n \cap \mathcal{M}_n$. In [G1, Proposition 4.3] it is also proved that the quotient of $\mathcal{E}_n$ by σ_t is another elliptic fibration, denoted by $\mathcal{E}_n/\sigma_t$, with an n-torsion section. Thus the minimal model of S/σ_t is a K3 surface belonging to the family $\mathcal{U}_n$. By considering $\mathcal{E}_n$ and $\mathcal{E}_n/\sigma_t$ as elliptic curves over $k\left(\mathbb{P}^1\right)$, the isogeny between S and the minimal model of S/σ_t is the isogeny between these elliptic curves (induced by the translation). □

Corollary 3.4. *For every n such that $2 \leq n \leq 8$, $\mathcal{L}_n \cap \mathcal{M}_n$ is non empty.*

Proof. The intersection $\mathcal{L}_n \cap \mathcal{M}_n$ contains at least the non empty family $\mathcal{U}_n$. □

3.3. **Maximal dimensional components of $\mathcal{L}_n \cap \mathcal{M}_n$.** In this section we prove that there are components of $\mathcal{L}_n$ completely contained in $\mathcal{M}_n$ and vice versa. The proof is lattice theoretic: in order to obtain this result, we need some extra information on the lattices $\mathbb{M}_n$ and Ω_n. Both these lattices are primitively embedded in the Néron–Severi group of a $(U \oplus \mathbb{M}_n)$-polarized K3 surface, so we now use the K3 surfaces in the family $\mathcal{U}_n$ to compare the discriminant forms of $\mathbb{M}_n$ and Ω_n.

Proposition 3.5. *Let A_{Ω_n} (resp. $A_{\mathbb{M}_n}$) the discriminant group of Ω_n (resp. $\mathbb{M}_n$) and q_{Ω_n} (resp. $q_{\mathbb{M}_n}$) its discriminant form. Then $A_{\Omega_n} = (\mathbb{Z}/n\mathbb{Z})^{\oplus 2} \oplus A_{\mathbb{M}_n}$ and $q_{\Omega_n} = u(n) \oplus q_{\mathbb{M}_n}$, where $u(n)$ is the discriminant form of the lattice $U(n)$.*

Proof. Nikulin proved that $A_{\Omega_n} = (\mathbb{Z}/n\mathbb{Z})^{\oplus 2} \oplus A_{\mathbb{M}_n}$ in [N2, Lemma 10.2]. By the Proposition 3.3, if $\mathrm{NS}(S) \simeq U \oplus \mathbb{M}_n$, then S admits an elliptic fibration $\mathcal{E}_n : S \to \mathbb{P}^1$ with an n-torsion section t and thus a symplectic automorphism σ_t, which is the translation by t. Let us denote by F the class in $\mathrm{NS}(S)$ of the fiber of the elliptic fibration $\mathcal{E}_n$, by O the class of the zero section, by t the class of the n-torsion section, by t_i, $i = 2, \ldots, n-1$, the class of the section corresponding to the sum of t with itself i times in the Mordell–Weil group. By definition σ_t preserves the classes F and $O+t+\sum_{i=2}^{n-1} t_i$. Since all the sections are disjoint (because they are torsion sections), $U(n) \simeq \langle F, O+t+\sum_{i=2}^{n-1} t_i\rangle \subset \mathrm{NS}(S)^{\sigma_t}$. We consider the orthogonal complement in the Néron–Severi group and we obtain $\langle F, O+t+\sum_{i=2}^{n-1} t_i\rangle^{\perp} \supset (\mathrm{NS}(S)^{\sigma_t})^{\perp} \simeq \Omega_n$. Since $\mathrm{rank}(\Omega_n) = \mathrm{rank}(\mathbb{M}_n) = \rho(S) - 2$, we have an inclusion between two primitive sublattices of $\mathrm{NS}(S)$, which have the same rank. It follows that

$$\langle F, O + t + \sum_{i=2}^{n-1} t_i\rangle^{\perp} \simeq \Omega_n.$$

Denoted by T_S the transcendental lattice of S, $q_{T_S} = -q_{\mathrm{NS}(S)} = -q_{\mathbb{M}_n}$. By $(\mathrm{NS}(S)^{\sigma_t})^{\perp} \simeq \Omega_n$ one obtains that the orthogonal complement of Ω_n in $H^2(S,\mathbb{Z})$ is an overlattice of finite index (possibly 1) of $U(n) \oplus T_S$. Since $A_{\Omega_n} = (\mathbb{Z}/n\mathbb{Z})^2 \oplus A_{\mathbb{M}_n}$, the orthogonal complement of Ω_n in $H^2(S,\mathbb{Z})$ is $U(n) \oplus T_S$. So $q_{\Omega_n} = -q_{U(n)\oplus T_S} = u(n) \oplus q_{\mathbb{M}_n}$. □

Lemma 3.6. *Let F be a finite abelian group with quadratic form q_F and $m \geq 2$. Let W be an even non-degenerate lattice with discriminant group $A_W = (\mathbb{Z}/m\mathbb{Z})^{\oplus 2} \oplus F$,* *discriminant form* $q_{A_W} = u(m) \oplus q_F$*; denote* $V = \langle 2d\rangle \oplus W$*. If* $d \equiv 0 \mod m$*, then* V *admits an overlattice* Z *of index* m *with* $A_Z = \mathbb{Z}/2d\mathbb{Z} \oplus F$ *and* $q_{A_Z} = (\frac{1}{2d}) \oplus q_F$*. Moreover,* Z *contains* W *as a primitive sublattice.*

Proof. By assumption, on $A_V = (\mathbb{Z}/2d\mathbb{Z}) \oplus A_W$ the discriminant form is $q_{A_V} = (\frac{1}{2d}) \oplus u(m) \oplus q_F$; moreover, there exists some integer k such that $2d = 2km$. Let h be a generator of the $\mathbb{Z}/2d\mathbb{Z}$ summand of A_V such that $h^2 = \frac{1}{2d}$, and let e_1, e_2 be a basis of the $(\mathbb{Z}/m\mathbb{Z})^{\oplus 2}$ summand in A_V such that $e_1^2 = e_2^2 = 0$ and $e_1e_2 = -\frac{1}{m}$. We define $\epsilon := e_1 + ke_2$, so that $\epsilon^2 = -\frac{2k}{m}$. Then the subgroup $H := \langle (2k)h + \epsilon\rangle$ is isotropic and its orthogonal complement inside A_V is $H^\perp = \langle h + e_2, e_1 - ke_2\rangle \oplus F$. It follows from [N1, Propostion 1.4.1] that there exists an even overlattice Z of V of index m with $A_Z \cong H^\perp/H = \langle h + e_2\rangle \oplus F \cong \mathbb{Z}/2d\mathbb{Z} \oplus F$, and q_{A_Z} is induced on the quotient $H^\perp/H$ by $(q_{A_V})_{|H^\perp}$, so it is exactly $(\frac{1}{2d}) \oplus q_F$. Finally, we observe that the intersection of H with A_W inside A_V is trivial, hence W is a primitive sublattice of Z. □

Corollary 3.7. *Let* $2 \le n \le 8$*,* $d \in \mathbb{N}$*,* $d \ge 1$ *and* $d \equiv 0 \mod n$*. Then* $L_{d,n}$ *admits an overlattice of index* n *whose discriminant form is* $(\frac{1}{2d}) \oplus q_{\mathbb{M}_n}$*; this overlattice contains* Ω_n *as a primitive sublattice.*

Proof. It suffices to apply Lemma 3.6 to the lattice $L_{d,n} = \langle 2d\rangle \oplus \Omega_n$ and to recall that $q_{\Omega_n} = u(n) \oplus q_{\mathbb{M}_n}$, by Proposition 3.5. □

Definition 3.8. *For each* $2 \le n \le 8$ *and each* $d \in \mathbb{N}$*,* $d \ge 1$*, and* $d \equiv 0 \mod n$*, we denote by* $L'_{d,n}$ *the overlattice of index* n *of* $L_{d,n}$ *constructed in Corollary 3.7.*

We observe that the construction of $L'_{d,n}$ given in Corollary 3.7 is coherent with the definition of $L'_{d,2}$ given in Proposition 2.17, since $L'_{d,2}$ is the unique overlattice of $L_{d,2}$ which contains Ω_2 as a primitive sublattice.

Theorem 3.9. *Let* $2 \le n \le 8$*,* $d \in \mathbb{N}$*,* $d \ge 1$ *and* $d \equiv 0 \mod n$*. The lattice* $L'_{d,n}$ *is unique in its genus and*

$$L'_{d,n} \simeq M_{d,n}.$$

The family $\mathcal{P}(L'_{d,n})$ *is* $(19 - \mathrm{rank}(\Omega_n))$*-dimensional and is an irreducible component of* $\mathcal{L}_n \cap \mathcal{M}_n$*, i.e. each K3 surface in this family admits a symplectic automorphism of order* n *and is* $n:1$ *cyclically covered by a K3 surface.*

Proof. By Corollary 3.7, the lattice $L'_{d,n}$ has the same discriminant group and form of the lattice $M_{d,n}$. By [N2, Proposition 7.1], the length and the rank of the lattice $\mathbb{M}_n$ are the following:

n	2	3	4	5	6	7	8
$l(\mathbb{M}_n)$	6	4	4	2	2	1	2
$\mathrm{rank}(\mathbb{M}_n)$	8	12	14	16	16	18	18

where the length $l(R)$ of a lattice R is the minimal number of generators of the discriminant group $R^\vee/R$. Since $\mathrm{rank}(M_{d,n}) = 1 + \mathrm{rank}(\mathbb{M}_n)$ and, if $d \equiv 0 \mod n$, $l(M_{d,n}) = 1 + l(\mathbb{M}_n)$, for every admissible n and $d \equiv 0 \mod n$, $\mathrm{rank}(M_{d,n}) \ge 2 + l(M_{d,n})$, so by [N1, Corollary 1.13.3], there is a unique even hyperbolic lattice with the same rank, length, discriminant group and form as $M_{d,n}$. Since $L'_{d,n}$ has all the prescribed properties, we conclude that $L'_{d,n} \simeq M_{d,n}$. Moreover, by [N1, Theorem 1.14.4], if $n < 7$ the lattice $L'_{d,n} \simeq M_{d,n}$ admits a unique, up to isometry, primitive embedding in Λ_{K3}, and thus determines a $(19-\mathrm{rank}(\Omega_n))$-dimensional family of K3 surfaces. If $n=7,8$, any primitive embedding of $L'_{d,n}\simeq M_{d,n}$ in the unimodular lattice Λ_{K3}, which exists by results in [GSar1, GSar3], determines the same genus of the orthogonal complement $T_{d,n}$ of rank three and signature $(2,1)$: we get respectively that $A_{T_{d,7}}=\mathbb{Z}/7\mathbb{Z}\oplus\mathbb{Z}/2d\mathbb{Z}$ and $A_{T_{d,8}}=\mathbb{Z}/2\mathbb{Z}\oplus\mathbb{Z}/4\mathbb{Z}\oplus\mathbb{Z}/2d\mathbb{Z}$ with quadratic forms $q_{T_{d,7}}=\left(-\frac{4}{7}\right)\oplus\left(-\frac{1}{2d}\right)$ and $q_{T_{d,8}}=\left(\frac{1}{2}\right)\oplus\left(\frac{1}{4}\right)\oplus\left(-\frac{1}{2d}\right)$. It follows from [N1, Proposition 1.15.1] that the primitive embedding of $L'_{d,n}\simeq M_{d,n}$ in Λ_{K3} is unique, up to isometry, if and only if $T_{d,n}$ is unique in its genus and the map $O(T_{d,n})\to O(q_{T_{d,n}})$ is surjective. By [MM, Theorem VIII.7.5], these two conditions hold in particular if the discriminant quadratic form $q_{T_{d,n}}$ is p-regular for all prime numbers $p\neq s$ and it is s-semiregular for a single prime number s. The precise (and quite technical) definition of p-regular and p-semiregular form can be found in [MM, Definition VIII.7.4]. An easy application of [MM, Lemma VIII.7.6 and VIII.7.7] implies that:

- $q_{T_{d,7}}$ is p-regular if $p\neq 7$ and it is 7-semiregular;
- $q_{T_{d,8}}$ is p-regular if $p\neq 2$ and it is 2-semiregular.

Hence, also for $n=7,8$, $T_{d,n}$ is unique in its genus and the primitive embedding of $L'_{d,n}\simeq M_{d,n}$ in Λ_{K3} is unique up to isometry, and thus determines a $(19-\mathrm{rank}(\Omega_n))$-dimensional family of K3 surfaces.

Each K3 surface which is $M_{d,n}$-polarized is contained in $\mathcal{L}_n\cap\mathcal{M}_n$ because there are primitive embeddings both of Ω_n and of $\mathbb{M}_n$ in its Néron–Severi group. □

Proposition 3.10. *Let $2\leq n\leq 8$ and $d\in\mathbb{N}$, $d\geq 1$. The lattice $L_{d,n}$ is not isometric to any overlattice of finite index (possibly 1) of $M_{e,n}$, for any e. In particular if X is a K3 surface such that $\mathrm{NS}(X)\simeq L_{d,n}$, then X does not admit a cyclic $n:1$ cover by a K3 surface and the families $\mathcal{P}(L_{d,n})$ are not contained in $\mathcal{M}_n$.*

Proof. By Proposition 3.5, $l(\Omega_n)=2+l(\mathbb{M}_n)$. Hence $l(L_{d,n})\geq l(\Omega_n)=2+l(\mathbb{M}_n)>l(M_{e,n})$. Since any finite index overlattice of $M_{e,n}$ has at most the length of $M_{e,n}$, the lattices $L_{d,n}$ cannot be isometric to any overlattice of $M_{e,n}$. □

In conclusion we proved that there are components of $\mathcal{L}_n$ (and of $\mathcal{M}_n$) which are contained in $\mathcal{L}_n\cap\mathcal{M}_n$, but there are also components of $\mathcal{L}_n$ which are not contained in $\mathcal{M}_n$, and thus in $\mathcal{L}_n\cap\mathcal{M}_n$. It is also true that there are components of $\mathcal{M}_n$ which are not contained in $\mathcal{L}_n$ (see e.g. Theorem 3.14 for the case $n=2$.)

In the following proposition we construct an $n^2:1$ isogeny between two K3 surfaces by using a third K3 surface, which is $L'_{d,n}$-polarized, and we prove that generically this $n^2:1$ isogeny is not just the quotient by an automorphism group.

Proposition 3.11. *Let Z be a K3 surface such that $\mathrm{NS}(Z)=L'_{d,n}$, let X be a K3 surface which is an $n:1$ cyclic cover of Z and let Y be the quotient of Z by a symplectic automorphism of order n. Then there is an $n^2:1$ isogeny between X and Y but there is no finite group G of order n^2 of automorphisms on X such that Y is birational to X/G.*

Proof. By Theorem 3.9 the K3 surfaces Z which are $L'_{d,n}$-polarized are $n:1$ isogenous to two K3 surfaces, X and Y, respectively with the two $n:1$ isogenies $X\dashrightarrow Z$ and $Z\dashrightarrow Y$. The composition of these two isogenies is an $n^2:1$ isogeny $X\dashrightarrow Y$.

If G is the Galois group of a generically $n^2:1$ map $X\dashrightarrow Y$, then G is a group of symplectic automorphisms on X (otherwise the quotient X/G would not be birational to a K3 surface). So X should admit a group G of symplectic automorphisms of order n^2. Such groups are classified in [M2], and the isometry classes of the lattices $\left(\mathrm{NS}(X)^G\right)^{\perp}$ are known, see [H]. In particular, if $2 \leq n \leq 8$, for every group G_{n^2} of order n^2 acting symplectically on a K3 surface, the rank of the lattice is $\mathrm{rank}\left(\left(\mathrm{NS}(X)^{G_{n^2}}\right)^{\perp}\right) > \mathrm{rank}(\Omega_n)$. Hence, if a K3 surface X admits G_{n^2} as group of symplectic automorphisms, $\rho(X) > 1+\mathrm{rank}(\Omega_n) = \rho(Z)$. But X and Z are isogenous, hence $\rho(X) = \rho(Z)$ and thus X cannot admit a group of symplectic automorphisms of order n^2. □

Remark 3.12. The isogenies given in Proposition 3.11 are of degree n^2 between K3 surfaces, hence Proposition 2.24 implies that they are necessarily Hodge isogenies.

3.4. **An example.** Let us consider a K3 surface X_4 whose Néron–Severi group is isometric to $\langle 4\rangle \oplus \mathbb{M}_2$. By Proposition 2.19, X_4 is the quotient of a K3 surface X_8 whose Néron–Severi group is $\mathrm{NS}(X_8) = (\langle 8\rangle \oplus \Omega_2)'$. In [vGS, Section 3.7] the K3 surface X_8 and its quotient by a symplectic involution are described, and this gives a geometric description of X_4. It is a quartic surface in $\mathbb{P}^3$ with eight nodes, which are the complete intersection of the quartic surface with two specific quadric surfaces. A basis of $\mathrm{NS}(X_4)$ is $\mathcal{H} := \{H, N_1, \ldots, N_7, \sum_{i=1}^8 N_i/2\}$, where H is the class of the hyperplane section of the quartic, and N_i are the classes of the smooth rational curves on X_4 contracted to nodes by the map $\varphi_{|H|}$, given by the pseudoample divisor H.

By Theorem 3.9 the lattice $\langle 4\rangle \oplus \mathbb{M}_2$ is isometric to $(\langle 4\rangle \oplus \Omega_2)'$, so $\mathrm{NS}(X_4) \simeq (\langle 4\rangle \oplus \Omega_2)'$. This gives a different description of the geometry of X_4 and a different basis for the Néron–Severi group. Indeed in [vGS, Section 3.5] the K3 surface X_4 such that $\mathrm{NS}(X_4) \simeq (\langle 4\rangle \oplus \Omega_2)'$ is described as double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ and it is shown that it is endowed with a symplectic involution which switches the two fibrations induced by the projections on the factors of $\mathbb{P}^1 \times \mathbb{P}^1$. A $\mathbb{Z}$-basis of $(\langle 4\rangle \oplus \Omega_2)'$ is given by $\mathcal{S} := \{(L+e_1)/2, e_1, \ldots, e_8\}$, where L is the ample divisor of degree 4 such that $\varphi_{|L|}$ exhibits X_4 as double cover of the quadric $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$, and $\{e_1, \ldots, e_8\}$ spans $\Omega_2 \simeq E_8(-2)$ (more precisely $e_i^2 = -4$, $e_i e_{i+1} = 2$ for $i = 1, \ldots, 6$, $e_3 e_8 = 2$ and the other intersections are zeros). The classes $E_1 := (L + e_1)/2$ and $E_2 := (L - e_1)/2$ induce the fibrations $X_4 \to \mathbb{P}^1$, given by the projections onto the two factors of $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 3.9 implies that there exists a change of basis between the two bases $\mathcal{H}$ and $\mathcal{S}$ of $\mathrm{NS}(X_4)$, but this change of basis cannot be given in the general setting of the theorem, indeed it depends on the degree of the polarizations which appear in the Néron–Severi group. Here we give the explicit change of basis in our example, i.e. for the surface X_4. The basis $\mathcal{H}$ expressed in terms of the basis $\mathcal{S}$ is

$$\begin{array}{l} H = 6E_1 - e_1 + 2e_2 - 2e_8; \quad N_1 = E_1 + e_2; \quad N_2 = E_1 + e_2 + e_3; \\ N_3 = E_1 + e_2 + e_3 + e_4; \quad N_4 = E_1 + e_2 + e_3 + e_4 + e_5; \\ N_5 = E_1 + e_2 + e_3 + e_4 + e_5 + e_6; \quad N_6 = E_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7; \\ N_7 = E_1 - 2e_1 - 3e_2 - 5e_3 - 4e_4 - 3e_5 - 2e_6 - e_7 - 3e_8; \quad N_8 = 3E_1 + e_2 - e_8. \end{array}$$

The polarization L, expressed in terms of the basis $\mathcal{H}$ is $L := 3H - \sum_{i=1}^7 N_i - 3N_8$ and the classes E_1 and E_2 are $E_1 := H - (\sum_{i=1}^8 N_i)/2$ and $E_2 := 2H - (\sum_{i=1}^8 N_i)/2 - 2N_8$.

In [vGS] these two models are described as models of different K3 surfaces, since the isometry between the lattices $(\langle 4\rangle \oplus \Omega_2)'$ and $\langle 4\rangle \oplus \mathbb{M}_2$ was not known. The possibility to write explicitly the change of basis between $\mathcal{S}$ and $\mathcal{H}$ allows one to compare the two different models described for X_4. For example one observes that the polarization L has degree 3 with respect to the model $\varphi_{|H|}(X_4)$ as singular quartic and that E_1 is given by a pencil of genus 1 curves passing through all the nodes of the quartic $\varphi_{|H|}(X_4)$.

3.5. **Remarks on $\mathcal{L}_n^{\mathbb{C}} \cap \mathcal{M}_n^{\mathbb{C}}$.** By Proposition 3.1, the dimension of $\mathcal{L}_n^{\mathbb{C}} \cap \mathcal{M}_n^{\mathbb{C}}$ is strictly less than the dimension of $\mathcal{L}_n^{\mathbb{C}}$. Since the components of $\mathcal{L}_n$ (resp. $\mathcal{M}_n$) have codimension 1 in $\mathcal{L}_n^{\mathbb{C}}$ (resp. $\mathcal{M}_n^{\mathbb{C}}$), by Theorem 3.14, $\mathcal{L}_n^{\mathbb{C}}$ and $\mathcal{M}_n^{\mathbb{C}}$ intersect in codimension 1 at least in the maximal dimensional components of $\mathcal{L}_n \cap \mathcal{M}_n$. It is natural to ask if there are K3 surfaces in the intersection $\mathcal{L}_n^{\mathbb{C}} \cap \mathcal{M}_n^{\mathbb{C}}$ which are non projective.

We do not have a complete answer, but we can observe that the set $\mathcal{L}_n^{\mathbb{C}} \cap \mathcal{M}_n^{\mathbb{C}}$ surely contains non projective K3 surfaces and an example is provided by Proposition 3.2. Indeed in Proposition 3.2 it is proved that a non-projective K3 surface X admitting an $8:1$ cyclic cover by a K3 surface, also admits an order 2 symplectic automorphism. If X is $8:1$ covered by a K3 surface, it is also $2:1$ covered by a K3 surface (because, by construction, there exists a primitive embedding between the lattices $\mathbb{M}_2 \hookrightarrow \mathbb{M}_8$). So X is a non projective K3 surface which admits an order 2 automorphism and is $2:1$ covered by a K3 surface, i.e. a non-projective K3 surface in $\mathcal{L}_2^{\mathbb{C}} \cap \mathcal{M}_2^{\mathbb{C}}$.

Moreover, we can observe that the "negative analogues" of the lattices $L'_{n,d}$ and $M_{n,d}$ have not so good properties as the lattices $L'_{n,d}$ and $M_{n,d}$, in the following sense: let us consider the lattices $\langle 2d\rangle \oplus \Omega_n$ and $\langle 2d\rangle \oplus \mathbb{M}_n$ with $d \leq 0$ (the case $d > 0$ is the projective case considered in Theorem 3.9). If $d \equiv 0 \mod n$, there exists an overlattice of index n of $\langle 2d\rangle \oplus \Omega_n$ (the proof is as in Lemma 3.6) and we denote this overlattice by $(\langle 2d\rangle \oplus \Omega_n)'$. The discriminant groups and forms of the negative definite lattices $(\langle 2d\rangle \oplus \Omega_n)'$ and $\langle 2d\rangle \oplus \mathbb{M}_n$ are the same. Nevertheless, one cannot conclude that $(\langle 2d\rangle \oplus \Omega_n)'$ and $\langle 2d\rangle \oplus \mathbb{M}_n$ are isometric. Indeed, there is at least one case in which they are not: if $n = 2$ and $d = -2$, as shown in the following lemma.

Lemma 3.13. *Let $\langle -4\rangle \oplus \Omega_2$ be the lattice generated by $h, e_1, \ldots e_8$ with the following intersection properties $h^2 = e_i^2 = -4$, $e_ie_{i+1} = 2$ if $1 \leq i \leq 6$, $e_3e_8 = 2$ and the other intersections are zeros. Let $(\langle -4\rangle \oplus \Omega_2)'$ be the overlattice of index 2 of $\langle -4\rangle \oplus \Omega_2$ obtained by adding the class $v := (h + e_1)/2$ to $\langle -4\rangle \oplus \Omega_2$. The lattice $(\langle -4\rangle \oplus \Omega_2)'$ is not isometric to $\langle -4\rangle \oplus \mathbb{M}_2$.*

Proof. A $\mathbb{Z}$-basis for the lattice $(\langle -4\rangle \oplus \Omega_2)'$ is $\{v, e_1, \ldots e_8\}$ and the unique vectors with length -2 are $\pm v$, $\pm(v - e_1)$ (one can check with a machine computation). In particular there are 4 vectors with length -2. The lattice $\mathbb{M}_2$ contains exactly 16 vectors with length -2 (they are 8 up to sign), see [N2, Section 6] for the complete description of $\mathbb{M}_2$. Thus the lattice $\langle -4\rangle \oplus \mathbb{M}_2$ contains more than 4 vectors with length -2 and cannot be isometric to $(\langle -4\rangle \oplus \Omega_2)'$. □

3.6. **Involutions.** In this section we restrict our attention to the case of symplectic involutions (i.e. $n = 2$) improving the general results of Section 3.3; indeed we obtain a complete description of the maximal dimensional components of the intersection $\mathcal{L}_2 \cap \mathcal{M}_2$, in Theorem 3.14. Moreover we are able to construct infinite families of K3 surfaces related by point-wise isogenies, see Corollary 3.16.

Theorem 3.14. *A K3 surface Y such that $Y \in \mathcal{M}_2$ and $\rho(Y) = 9$ admits a symplectic involution if and only if* $\mathrm{NS}(Y) \simeq M_{2d,2}(\simeq L'_{2d,2})$.

A K3 surface X such that $X \in \mathcal{L}_2$ and $\rho(X) = 9$ admits a $2:1$ cover by a K3 surface if and only if $\mathrm{NS}(X) \simeq L'_{2d,2}(\simeq M_{2d,2})$.

So the only maximal dimensional components in $\mathcal{L}_2 \cap \mathcal{M}_2$ are $\mathcal{P}(M_{2d,2})$ for any $d \in \mathbb{N}_{>0}$.

Proof. By Theorem 3.9 $M_{2d,2} \simeq L'_{2d,2}$ and thus if $\mathrm{NS}(Y) \simeq M_{2d,2}$, then Y admits a symplectic involution. Similarly if $\mathrm{NS}(X) \simeq L'_{2d,2}$, X is contained in $\mathcal{M}_2$. It remains to prove that if a K3 surface is in $\mathcal{L}_2 \cap \mathcal{M}_2$, and its Picard number is 9, then its Néron–Severi cannot be isometric to $M'_{e,2}$, to $M_{e,2}$ for an odd e or to $L_{f,2}$ with $f \in \mathbb{N}_{>0}$. The argument is similar to that of Proposition 3.10.

By Proposition 2.18, if $Y \in \mathcal{M}_2$ its Néron–Severi group is either isometric to $M_{e,2}$ or to $M'_{2e,2}$. By Proposition 2.17 if $X \in \mathcal{L}_2$, its Néron–Severi group is either isometric to $L_{d,2}$ or to $L'_{2d,2}$. So if a K3 surface has both properties (i.e. it is in $\mathcal{L}_2 \cap \mathcal{M}_2$ and has Picard number 9), its Néron–Severi group is isometric both to a lattice in $\{M_{e,2}, M'_{2e,2}\}$ and to a lattice in $\{L_{d,2}, L'_{2d,2}\}$. Hence we are looking for pairs of lattices, one in $\{M_{e,2}, M'_{2e,2}\}$ and one in $\{L_{d,2}, L'_{2d,2}\}$, which are isometric. If two lattices are isometric, they have the same length. We observe that $l(M_{e,2}) = 1 + l(\mathbb{M}_2) = 7$, $l(M'_{2e,2}) = 1 + l(\mathbb{M}_2) - 2 = 5$, $l(L_{d,2}) = 1 + l(\Omega_2) = 9$, $l(L'_{2d,2}) = 1 + l(\Omega_2) - 2 = 7$. In particular, the unique possible pair of lattices as required is given by $\{M_{e,2}, L'_{2d,2}\}$. Since the discriminant of two isometric lattices is the same, one obtains that $e = 2d$. □

By Theorem 3.9 we know that there are K3 surfaces $X \in \mathcal{L}_n \cap \mathcal{M}_n$ with minimal possible Picard number. Thus we know that there exists a surface $Z \in \mathcal{L}_n$ which is a $n:1$ cyclic cover of X (and also a surface Y which is $n:1$ cyclically covered by X). But for $n \neq 2$, we do not know if the K3 surface Z is also contained in $\mathcal{M}_n$, because we do not know its Néron–Severi group. If $n = 2$, the situation is different, thanks to Proposition 2.19. Indeed by Proposition 2.19, one obtains the following corollary, which says that if Z is a $2:1$ cover of a K3 surface X in $\mathcal{L}_2 \cap \mathcal{M}_2$, then Z itself is contained in $\mathcal{L}_2 \cap \mathcal{M}_2$.

Corollary 3.15. *Let Y and $\hat{Y}$ be two K3 surfaces in $\mathcal{M}_2$ with Picard number 9. They are isogenous by a chain of quotients by involutions if and only if one of the following equivalent conditions holds:*
(i) $\mathrm{NS}(Y) \simeq M_{d,2}$, $\mathrm{NS}(\hat{Y}) = M_{e,2}$, *and there exists $m \in \mathbb{N}_{>0}$ such that either $d = 2^m e$ or $e = 2^m d$;*
(ii) $T_Y \simeq U \oplus U \oplus \mathbb{M}_2 \oplus \langle -2d \rangle$, $T_{\hat{Y}} \simeq U \oplus U \oplus \mathbb{M}_2 \oplus \langle -2e \rangle$ *and there exists $m \in \mathbb{N}_{>0}$ such that either $d = 2^m e$ or $e = 2^m d$.*

Proof. We can assume that $\hat{Y}$ is obtained by iterated quotients from Y. Then Y admits a symplectic involution σ and, by Theorem 3.14, there exists an even d such that $\mathrm{NS}(Y) \simeq M_{d,2} \simeq L'_{d,2}$. So Y is the cover of a K3 surface Z with Néron–Severi group $M_{d/2,2}$ (by Proposition 2.19). If $d/2$ is odd, then the process stops and $\hat{Y}$ is necessarily Z; otherwise, $\mathrm{NS}(Z) \simeq M_{d/2,2} \simeq L'_{d/2,2}$ and Z is the cover of a K3 surface Z with Néron–Severi group $M_{d/4,2}$. Iterating, if possible, this process m times, one obtains K3 surfaces with Néron–Severi group isometric to $M_{d/2^m,2}$. In particular, one never obtains lattices isometric to $M'_{e,2}$ (for any e) as Néron–Severi groups of a K3 surface obtained by iterated quotients from Y.

Vice versa, if $\mathrm{NS}(\hat{Y}) \simeq M_{e,2}$ for a certain e, $\hat{Y}$ is covered by a K3 surface W with $\mathrm{NS}(W) \simeq L'_{2e,e} \simeq M_{2e,2}$ (by Proposition 2.19). So W is a K3 surface, $2:1$ covered by a K3 surface with Néron–Severi group isometric to $L'_{4e,e} \simeq M_{4e,2}$. Reiterating this process m times one obtains that $\hat{Y}$ is isogenous to a K3 surface whose Néron–Severi lattice is isometric to $M_{2^m e,2}$.

The equivalent statement for the transcendental lattice follows by the fact that if the Néron–Severi group of a K3 surface is isometric to $M_{d,2}$, then its transcendental lattice is isometric to $U \oplus U \oplus \mathbb{M}_2 \oplus \langle -2d \rangle$. Indeed, the discriminant form of the latter is minus the discriminant form of $M_{d,2}$, because the discriminant group of $\mathbb{M}_2$ is $(\mathbb{Z}/2\mathbb{Z})^{\oplus 6}$ and so $q_{\mathbb{M}_2} = -q_{\mathbb{M}_2}$. Moreover in this case the transcendental lattice is uniquely determined by its genus. □

We determined an infinite number of infinite series of K3 surfaces of Picard number 9 related by iterated quotients by symplectic involutions. More precisely we prove the following.

Corollary 3.16. *For every $d \in \mathbb{N}_{>0}$ and $m \in \mathbb{N}$, if $Y_0 \in \mathcal{P}(M_{d,2})$ there exists a K3 surface $Y_m \in \mathcal{P}(M_{2^m d,2})$ with an isogeny of degree 2^m to Y_0. If Y_0 is very general, i.e.* $\mathrm{NS}(Y_0) \simeq M_{d,2}$*, the transcendental lattice of Y_m is $T_{Y_m} \simeq U \oplus U \oplus \mathbb{M}_2 \oplus \langle -2^{m+1} d \rangle$.*

Proof. If $Y_0 \in \mathcal{P}(M_{d,2})$, then it is $2:1$ covered by a K3 surface Y_1, by Theorem 2.11. By [GSar2, Remark after Corollary 2.2], $Y_1 \in \mathcal{P}(L'_{2d,2})$ and by Theorem 3.14 this is equivalent to $Y_1 \in \mathcal{P}(M_{2d,2})$. By iterating this process m times, we obtain a K3 surface $Y_m \in \mathcal{P}(M_{2^m d,2})$ which is a $2^m : 1$ cover of Y_0. The statement for very general Y_0 follows directly from Corollary 3.15. □

Remark 3.17. Corollary 3.16 says that there are infinite towers of families of K3 surfaces, one for each odd $d > 0$:

$$\cdots \longrightarrow \mathcal{P}(M_{16d,2}) \longrightarrow \mathcal{P}(M_{8d,2}) \longrightarrow \mathcal{P}(M_{4d,2}) \longrightarrow \mathcal{P}(M_{2d,2}) \longrightarrow \mathcal{P}(M_{d,2})$$

where arrows refer to point-wise defined isogenies of degree 2. So, denoted by Y_k a very general K3 surface in $\mathcal{P}(M_{2^k d,2})$, Y_k is isogenous to a K3 surface $Y_h \in \mathcal{P}(M_{2^h d,2})$ and the rational motives of Y_k and Y_h are isomorphic, see [Huy2], [L], [P].

In particular, Y_k and Y_h are isogenous but not Hodge isogenous if $|k-h| \equiv 1 \mod 2$ and they are isogenous and Hodge isogenous if $|k-h| \equiv 0 \mod 2$.

The transcendental lattice of Y_{m+2} is

$$T_{2^{m+2}d} := U^{\oplus 2} \oplus \mathbb{M}_2 \oplus \langle -2^{m+2} d \rangle.$$

The lattice $T_{2^m d}$ is an overlattice of finite index of $T_{2^{m+2}d}$ such that $T_{2^m d}/T_{2^{m+2}d} \simeq \mathbb{Z}/2\mathbb{Z}$. As a consequence, Y_{m+2} can be interpreted as moduli spaces of twisted sheaves on Y_m. On the contrary, it is not possible to realize Y_{m+2} as moduli space of twisted sheaves on Y_{m+1}, since, by [M1, Corollary 6.5], this would imply the existence of a Hodge isometry between $T_{2^{m+2}d} \otimes \mathbb{Q}$ and $T_{2^{m+1}d} \otimes \mathbb{Q}$, in contradiction with Proposition 2.22.

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