ON CERTAIN ISOGENIES BETWEEN K3 SURFACES

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ABSTRACT. The aim of this paper is to construct "special" isogenies between K3 surfaces, which are not Galois covers between K3 surfaces, but are obtained by composing cyclic Galois covers, induced by quotients by symplectic automorphisms. We determine the families of K3 surfaces for which this construction is possible. To this purpose we will prove that there are infinitely many big families of K3 surfaces which both admit a finite symplectic automorphism and are (desingularizations of) quotients of other K3 surfaces by a symplectic automorphism.

In the case of involutions, for any $n \in \mathbb{N}_{>0}$ we determine the transcendental lattices of the K3 surfaces which are $2^n : 1$ isogenous (by a non Galois cover) to other K3 surfaces. We also study the Galois closure of the $2^2 : 1$ isogenies and we describe the explicit geometry on an example.

1. INTRODUCTION

K3 surfaces are symplectic regular surfaces and among their finite order automorphisms the ones which preserve the symplectic structure (the symplectic automorphisms) play a special role. Indeed, the quotient of a K3 surface by a finite symplectic automorphism produces a singular surface whose desingularization is again a K3 surface. This construction establishes a particular relation between different sets of K3 surfaces: the ones which admit a finite symplectic automorphism and the ones obtained as desingularization of the quotient of a K3 surfaces by a symplectic automorphism. In the following the latter K3 surfaces are said to be (cyclically) covered by a K3 surface and the former are said to be the cover of a K3 surface. We denote by $\mathcal{L}_n$ the set of the K3 surfaces which admit an order n symplectic automorphism and by $\mathcal{M}_n$ the set of the K3 surfaces which are $n : 1$ cyclically covered by a K3 surface. From now on we assume the surfaces to be projective.

Thanks to several works, starting from the end of the 70's until now (see, e.g. [N2], [Mo], [vGS], [GSar1], [GSar2] [GSar3], [G2]), the sets $\mathcal{L}_n$ and $\mathcal{M}_n$ are described as the union of countably many families of R polarized K3 surfaces, for certain known lattices R. The dimension of these families is at most 11, and, recalling that the families of generic projective K3 surfaces have dimension 19, one immediately observes that the K3 surfaces which either admit a finite symplectic automorphism or which are cyclically covered by a K3 surface are quite special. So, it is natural to expect that the intersection $\mathcal{L}_n \cap \mathcal{M}_n$ is extremely small, i.e. that a K3 surface which is both covered and cover of another K3 surface is really rare. On the other hand, there is at least one known example of a family of K3 surfaces

2010 *Mathematics Subject Classification.* Primary 14J28, 14J50; Secondary 14J10.

Key words and phrases. K3 surfaces, Quotients, Symplectic automorphisms on K3 surfaces, Galois covers between K3 surfaces, Isogenies between K3 surfaces.

contained in $\mathcal{L}_n \cap \mathcal{M}_n$, given by the family of the K3 surfaces which admit an elliptic fibration with an n-torsion section (see Section 3). This family has codimension one in the families which are components of $\mathcal{L}_n$ and of $\mathcal{M}_n$. Hence, surprisingly, the intersection $\mathcal{L}_n \cap \mathcal{M}_n$ is not so small.

The aim of this paper is to investigate more precisely the intersection between the two sets $\mathcal{L}_n$ and $\mathcal{M}_n$ and to relate it with the study of isogenies between K3 surfaces. In this paper, the term "isogeny between K3 surface" means a generically finite rational map between K3 surfaces, as in [I] and [BSV].

The quotient by a finite symplectic automorphism on a K3 surface X induces an isogeny between X, which admits the symplectic automorphism, and the K3 surface Y cyclically covered by X. The isogeny is birationally the quotient map and has of course the same order as the automorphism. There are other isogenies between K3 surfaces, which are not quotient maps, see e.g. [I] and [BSV]. Here we discuss one of these other isogenies: given a K3 surface $Z \in \mathcal{L}_n \cap \mathcal{M}_n$, it induces an $n^2 : 1$ isogeny between other two K3 surfaces. Indeed, since $Z \in \mathcal{M}_n$, it is $n : 1$ covered by a K3 surface X; since $Z \in \mathcal{L}_n$, it is an $n : 1$ cover of a K3 surface Y. By composing these two $n : 1$ maps one obtains an $n^2 : 1$ isogeny between X and Y. We will prove that generically this isogeny is not induced by a quotient map.

In Section 2 we recall some preliminary results on the set $\mathcal{L}_n$ of K3 surfaces admitting a symplectic automorphism of order n and on the set $\mathcal{M}_n$ of the K3 surfaces $n : 1$ cyclically covered by a K3 surface. In Section 3 we obtain our main results on the intersection $\mathcal{L}_n \cap \mathcal{M}_n$. In particular in Theorem 3.9 we prove:

Theorem *There are components $\mathcal{Z}$ of $\mathcal{L}_n \cap \mathcal{M}_n$ such that $\dim(\mathcal{L}_n) = \dim(\mathcal{M}_n) = \dim \mathcal{Z}$, i.e. the dimension of $\mathcal{Z}$ is the maximal possible and thus $\mathcal{Z}$ is an irreducible component of both of $\mathcal{L}_n$ and $\mathcal{M}_n$.*

As a consequence we construct $n^2 : 1$ isogenies and we prove that generically they are not quotient maps. The Section 4 contains the main results for the case $n = 2$. In addition to the results which hold for every admissible n, we also obtain the following theorem (see Theorem 4.6 and Corollary 4.8)

Theorem *For any $d, n \in \mathbb{N} > 0$, there exists a lattice $R_{d,n}$ (with $R_{d,n} \simeq R_{d',n'}$ if and only if $(d, n) = (d', n')$) and there exists a family of $R_{d,n}$-polarized K3 surfaces such that, for any $m \in \mathbb{N}_{>0}$ and any $R_{d,n}$-polarized K3 surface X there exists an $R_{d,m}$-polarized K3 surface Y isogenous to X with an isogeny of degree $2^{|n-m|}$.*

So for each $d \in \mathbb{N}_{>0}$ there are countably many families of polarized K3 surfaces, such that there exists an isogeny between members of each family.

The Néron–Severi group and the transcendental lattice of all the surfaces involved in these isogenies are explicitly given. In Section 4.3 we describe the Galois closure of the $2^2 : 1$ (non Galois) covers constructed. Moreover, in Section 4.4 we describe the geometry of a generic member X_2 of a certain maximal dimensional family of K3 surfaces which is contained in $\mathcal{L}_2 \cap \mathcal{M}_2$. The K3 surface X_2 admits two different polarizations of degree 4: one exhibits the surface X_2 as a special singular quartic in $\mathbb{P}^3$ with eight nodes, the other as smooth double cover of a quadric in $\mathbb{P}^3$. The former model is the singular quotient of another K3 surface by a symplectic involution (thus it implies that $X_2 \in \mathcal{M}_2$), the latter implies that X_2 admits a symplectic involution induced by the switching of the rulings on the quadric (thus it implies that $X_2 \in \mathcal{L}_2$). We describe both projective models of X_2 and give the explicit relation between them, providing a geometric realization of the previous lattice theoretic result which guarantees that X_2 is both covered by a K3 surface and is a cover of a K3 surface. In particular this allows us to describe a symplectic involution on the model of X_2 as singular quotient. In Section 4.5 we analyse the similar problem for two specific families of codimension 1 in $\mathcal{L}_2 \cap \mathcal{M}_2$; one of these families is totally contained in all the components of $\mathcal{L}_2$ and the other in all the components of $\mathcal{M}_2$.

Acknowledgements. Part of this project was realized while the first-named author was visiting the Max Planck Institute of Mathematics in Bonn: the first-named author is grateful to Max Planck Institute for Mathematics in Bonn for its hospitality and financial support. We are grateful to Bert van Geemen for his enlightening suggestions and to Simon Brandhorst for his precious remarks.

2. Preliminary results

We recall in this section some of the definitions and results on K3 surfaces, symplectic automorphisms on K3 surfaces and quotients of K3 surfaces by their automorphisms. In the following we work with projective surfaces.

2.1. Symplectic automorphisms and cyclic covers of K3 surfaces.

Definition 2.1. *A (projective) K3 surface is a regular projective surface with trivial canonical bundle. If X is a K3 surface, we choose a generator of $H^{2,0}(X)$, (i.e. a symplectic form), we denote it by ω_X and we call it the period of the K3 surface. The second cohomology group $H^2(X,\mathbb{Z})$ of a K3 surface X equipped with the cup product is a lattice, isometric to a standard lattice which does not depend on X and is denoted by $\Lambda_{K3} := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2}$.*

Definition 2.2. *Let X be a K3 surface, and ω_X its period. An automorphism σ of X is said to be symplectic if $\sigma^*(\omega_X) = \omega_X$.*

One of the main results on symplectic automorphisms on K3 surfaces is that the quotient of a K3 surface by a symplectic automorphism is still a K3 surface, after a birational transformation which resolves the singularities of the surface.

Proposition 2.3. ([N2]) *Let X be a K3 surface and $\sigma \in Aut(X)$ a finite automorphism of X. Then the minimal smooth surface Y birational to X/σ is a K3 surface if and only if σ is symplectic.*

Definition 2.4. *We will say that a K3 surface Y is $n:1$ cyclically covered by a K3 surface, if there exists a pair (X,σ) such that X is a K3 surface, σ is an automorphism of order n of X and Y is birational to X/σ.*

The first mathematician who worked on symplectic automorphisms of finite order on K3 surfaces and who established the fundamental results on these automorphisms was Nikulin, in [N2]. We summarize in Theorem 2.7 and Theorem 2.9 the main results obtained in his paper, but first we recall some useful information and definitions.

If σ is a symplectic automorphism on X of order n, its linearization near the points with non trivial stabilizer is given by a 2×2 diagonal matrix with determinant 1 and thus it is of the form $\mathrm{diag}(\zeta_n^a, \zeta_n^{n-a})$ for $1 \le a \le n-1$ and ζ_n an n-th primitive root of unity. So, the points with non trivial stabilizer are isolated fixed points and the quotient X/σ has isolated singularities, all of type A_{m_j} where m_j+1 divides n. In particular the surface Y, which is the minimal surface resolving the singularities of X/σ, contains smooth rational curves M_i arising from the desingularization of X/σ. The classes of these curves span a lattice isometric to $\oplus_j A_{m_j}$.

Definition 2.5. *Let Y be a K3 surface, $n:1$ cyclically covered by a K3 surface. The minimal primitive sublattice of* $\mathrm{NS}(Y)$ *containing the classes of the curves M_i, arising from the desingularization of X/σ, is denoted by $\mathbb{M}_n$.*

We observe that $\mathbb{M}_n$ is necessarily an overlattice of finite index (a priori possibly 1) of the lattice $\oplus_j A_{m_j}$ spanned by the curves M_i. The presence of a smooth cyclic cover of X/σ branched over the singular points obtained as contraction of the curves M_i suggests that there are some divisibility relations among the M_i's and thus that the index of the inclusion $\langle (M_i)_i\rangle \hookrightarrow \mathbb{M}_n$ would not be 1 (as indeed stated in Theorem 2.7).

Definition 2.6. (See [N2, Definition 4.6]) *Let σ be an order n automorphism of a K3 surface X. We will say that its action on the second cohomology group is essentially unique if there exists an isometry $g_n:\Lambda_{K3}\xrightarrow{\sim}\Lambda_{K3}$ of order n of Λ_{K3} such that for every pair (X,σ), there exists an isometry $\varphi: H^2(X,\mathbb{Z})\rightarrow\Lambda_{K3}$ such that $\sigma^*=\varphi^{-1}\circ g_n\circ\varphi$.*

Theorem 2.7. *Let X be a K3 surface and σ a finite symplectic automorphism of X of order $|\sigma|=n$. Then*

- $2\le n\le 8$ *(see* [N2, Theorem 6.3]*);*
- *the singularities of X/σ depend only on n (see* [N2, Section 5]*);*
- *the class of isometry of the lattice $\mathbb{M}_n$ depends only on n and $\mathbb{M}_n$ is an overlattice of index n of the lattice $\langle (M_i)_i\rangle$ spanned by the curves arising from the desingularization of the quotient X/σ (see* [N2, Theorem 6.3]*);*
- *the action of σ^* on $H^2(X,\mathbb{Z})$ is essentially unique (see* [N2, Theorem 4.7]*) and thus the classes of isometry of the lattices $H^2(X,\mathbb{Z})^{\sigma^*}$ and $\left(H^2(X,\mathbb{Z})^{\sigma^*}\right)^\perp$ depend only on n.*
- *The lattice $\left(H^2(X,\mathbb{Z})^{\sigma}\right)^\perp$ is primitively embedded in* $\mathrm{NS}(X)$ *(see* [N2, Lemma 4.2]*) and* $\mathrm{rank}((\Lambda_{K3}^{g_n})^\perp)=\mathrm{rank}(\mathbb{M}_n)$ *(see* [N2, Formula (8.12)]*).*

Definition 2.8. *Let X be a K3 surface with a symplectic automorphism σ of order n. Since the action of σ^* on $H^2(X,\mathbb{Z})$ is essentially unique, the lattice $\left(H^2(X,\mathbb{Z})^{\sigma^*}\right)^\perp$ is isometric to $(\Lambda_{K3}^{g_n})^\perp$ (with the notation of Definition 2.6) and we denote it by Ω_n.*

For every admissible n the lattices Ω_n were computed: in [vGS] and [Mo] if $n=2$; in [GSar2] if n is an odd prime; in [GSar3] if n is not a prime.

The lattices $\mathbb{M}_n$ were computed for every admissible n in [N2, Theorems 6.3 and 7.1].

The lattices Ω_n and $\mathbb{M}_n$ characterize the K3 surfaces admitting a symplectic automorphism of order n or a $n:1$ cyclic cover by a K3 surface respectively; indeed, the following two results hold

Theorem 2.9. (See [N2, Theorem 4.15]) *A K3 surface X admits a symplectic automorphism of order n if and only if Ω_n is primitively embedded in* $\mathrm{NS}(X)$.

Theorem 2.10. (See [GSar1, Proposition 2.3] for the case $n=2$ and [G2, Theorem 5.2] for other n) *A K3 surface Y is $n:1$ cyclically covered by a K3 surface if and only if $\mathbb{M}_n$ is primitively embedded in* $\mathrm{NS}(Y)$.

Corollary 2.11. *Let X be a projective K3 surface admitting a symplectic automorphism of order n. Then $\rho(X) \geq 1 + \mathrm{rank}(\Omega_n)$ and if $\rho(X) = 1 + \mathrm{rank}(\Omega_n)$, then $\mathrm{NS}(X)$ is an overlattice of finite index (possibly 1) of $\langle 2d\rangle \oplus \Omega_n$, for a certain $d \in \mathbb{N}_{>0}$, such that Ω_n is primitively embedded in this overlattice.*

Let Y be a projective K3 surface $n:1$ cyclically cover by a K3 surface. Then $\rho(Y) \geq 1 + \mathrm{rank}(\mathbb{M}_n)$ and if $\rho(Y) = 1 + \mathrm{rank}(\mathbb{M}_n)$, then $\mathrm{NS}(Y)$ is an overlattice of finite index (possibly 1) of $\langle 2e\rangle \oplus \mathbb{M}_n$, for a certain $e \in \mathbb{N}_{>0}$, such that $\mathbb{M}_n$ is primitively embedded in this overlattice.

Proof. Since X admits a symplectic automorphism of order n, Ω_n is primitively embedded in $\mathrm{NS}(X)$. Since Ω_n is negative definite and X is projective, the orthogonal to Ω_n in $\mathrm{NS}(X)$ contains a class with a positive self intersection, in particular it is non empty. So $\rho(X) \geq 1 + \mathrm{rank}(\Omega_n)$ and $\langle 2d\rangle \oplus \Omega_n$ is embedded in $\mathrm{NS}(X)$. Similarly one obtains the result for $\rho(Y)$ and $\mathrm{NS}(Y)$. □

Definition 2.12. *We define the following sets of K3 surfaces (which are subsets of the moduli space of the K3 surfaces):*

$$\mathcal{L}_n := \{\text{K3 surfaces which admit a symplectic automorphims } \sigma \text{ of order } n\}/\cong,$$

$$\mathcal{M}_n := \{\text{K3 surfaces which admit an } n:1 \text{ cyclic cover by a K3 surface}\}/\cong,$$

where $\cong$ denotes the equivalence relation given by isomorphism between two K3 surfaces.

Given an even hyperbolic lattice R which admits a primitive embedding in Λ_{K3}, we denote by $\mathcal{P}(R)$ the moduli space of isomorphism classes of R-polarized $K3$ surfaces, i.e. of those $K3$ surfaces X for which there exists a primitive embedding $R \subset \mathrm{NS}(X)$. Moreover, we will write $A < B$ in order to say that B is an overlattice of finite index of A.

Corollary 2.13. *The set $\mathcal{L}_n$ is a union of countably many components and each of them is a family of R-polarized K3 surfaces, for an appropriate choice of the lattice R:*

$$\mathcal{L}_n = \bigcup_{d\in\mathbb{N}} \left(\bigcup_{\substack{(\langle 2d\rangle \oplus \Omega_n) < R \\ \Omega_n \subset R \text{ prim.}}} \mathcal{P}(R) \right).$$

All the components $\mathcal{P}(R)$ are equidimensional and have dimension $19 - \mathrm{rank}(\Omega_n)$.

The set $\mathcal{M}_n$ is a union of countably many components and each of them is a family of R-polarized K3 surfaces, for an appropriate choice of the lattice R:

$$\mathcal{M}_n = \bigcup_{d\in\mathbb{N}} \left(\bigcup_{\substack{(\langle 2d\rangle \oplus \mathbb{M}_n) < R \\ \mathbb{M}_n \subset R \text{ prim.}}} \mathcal{P}(R) \right).$$

All the components are equidimensional and have dimension $19 - \mathrm{rank}(\mathbb{M}_n) = 19 - \mathrm{rank}(\Omega_n)$.

Proof. Let R be an overlattice of finite index of $\langle 2d\rangle \oplus \Omega_n$ such that Ω_n is primitively embedded in it. If X is a K3 surface such that R is primitively embedded in $\mathrm{NS}(X)$, then Ω_n is primitively embedded in $\mathrm{NS}(X)$ and thus X admits a symplectic automorphism of order n, by Theorem 2.9. Vice versa, if a projective K3 surface

X admits a symplectic automorphism of order n, then there exists a $d \in \mathbb{N} > 0$ such that $\langle 2d\rangle \oplus \Omega_n$ is embedded in $\mathrm{NS}(X)$, and an overlattice R of $\langle 2d\rangle \oplus \Omega_n$ is primitively embedded in $\mathrm{NS}(X)$. So one can describe the set $\mathcal{L}_n$ as union of families $\mathcal{P}(R)$ of R-polarized K3 surfaces, where R is a proper overlattice of index r (possibly 1) of $\langle 2d\rangle \oplus \Omega_n$ for a certain $d \in \mathbb{N}$. There are countably many lattices $\langle 2d\rangle \oplus \Omega_n$ and each of them has a finite number of overlattices of finite index. So $\mathcal{L}_n$ is the union of countably many families of R-polarized K3 surfaces. The dimension of each of these families is $20 - \mathrm{rank}(R) = 20 - (1 + \mathrm{rank}(\Omega_n))$. This concludes the proof for the set $\mathcal{L}_n$.

The proof for $\mathcal{M}_n$ is similar, but one has to use the Theorem 2.10 instead of the Theorem 2.9. □

2.2. **Isogenies between K3 surfaces.** The following definition was first given by Inose in [I] in the case of K3 surfaces with Picard number 20.

Definition 2.14. *Let X and Y be two K3 surfaces. We say that X and Y are isogenous if there exists a rational map of finite degree between X and Y. This map is said to be an isogeny between X and Y and if it is generically of degree n, the map is said to be an isogeny of degree n.*

The easiest construction of an isogeny between K3 surfaces is given by the quotient by a finite symplectic automorphism, i.e. if X is a K3 surface admitting a symplectic automorphism σ of order n, then the quotient map induces an isogeny of degree n between X and Y, the minimal model of X/σ. So if $X \in \mathcal{L}_n$, then there exists $Y \in \mathcal{M}_n$ which is isogenous to X with an isogeny of degree n. Similarly if $Y \in \mathcal{M}_n$, then there exists a K3 surface $X \in \mathcal{L}_n$ which is isogenous to Y with an isogeny $X \dashrightarrow Y$ of degree n.

There exist however isogenies between K3 surfaces which are not induced by the quotient by a finite group of symplectic automorphisms: an example is given by isogenous Kummer surfaces constructed from Abelian surfaces related by an isogeny, as in [I, Proof of Thm 2], under the additional assumption that the degree is a prime $p > 7$, (see also [BSV, Example 6.5]).

Now, let us suppose that Z is a K3 surface such that $Z \in \mathcal{L}_n \cap \mathcal{M}_n$. Then, there exists a K3 surface $X \in \mathcal{L}_n$ which is isogenous to Z, with an isogeny $\rho : X \dashrightarrow Z$ of degree n, but also a K3 surface $Y \in \mathcal{M}_n$ which is isogenous to Z with an isogeny $\pi : Z \dashrightarrow Y$ of degree n. So the existence of $Z \in \mathcal{L}_n \cap \mathcal{M}_n$ allows one to construct an isogeny of degree n^2 between the two K3 surfaces X and Y, given by the composition $\pi \circ \rho : X \dashrightarrow Y$. We will show that in many cases this isogeny is not induced by a quotient by a finite group of symplectic automorphisms acting on X, see Proposition 3.11.

In the Section 3 we prove that $\mathcal{L}_n \cap \mathcal{M}_n$ is non empty if $2 \leq n \leq 8$ and then we provide examples of $n^2 : 1$ isogenies between K3 surfaces.

2.3. **Remarks on Hodge isogenies between K3 surfaces.** The Definition 2.14 is not the only notion of isogeny existing in the literature: to distinguish between the two definitions, we will talk here of *Hodge isogeny* for the notion used for example in [Bu, Huy].

Definition 2.15. *Let X and Y be two K3 surfaces. We say the X and Y are Hodge isogenous if there exists a rational Hodge isometry between $H^2(X, \mathbb{Q})$ and $H^2(Y, \mathbb{Q})$.*

Hodge isogenous K3 surfaces have been studied since foundational work of [M] and [N3], also in relation with Šafarevič's conjecture [Ša] about algebraicity of correspondences on $K3$ surfaces.

In [BSV, Prop. 3.1], the authors give a comparison between the notion of isogeny and of Hodge isogeny:

Proposition 2.16. *If $\varphi : X \dashrightarrow Y$ is an isogeny of order n, n is not a square and the rank of the transcendental lattices T_X and T_Y is odd, φ is never a Hodge isogeny.*

This follows from the fact that, under these assumptions, there cannot exist any isometry $T_X \otimes \mathbb{Q} \simeq T_Y \otimes \mathbb{Q}$. The transcendental lattice T_X of the very general K3 surface $X \in \mathcal{L}_n$ has always odd rank (see Theorem 3.9); by Proposition 2.16 if n is not a square, so if $n \neq 4$, the surface X is never Hodge isogenous to the minimal resolution of its quotient. The assumption on the degree n is in particular due to the following straightforward fact:

Lemma 2.17. *For any non degenerate lattice T and any integer $n \in \mathbb{N}$, there exists an isometry $T \otimes \mathbb{Q} \simeq T(n^2) \otimes \mathbb{Q}$.*

Proposition 2.18. *For any $n \in \mathbb{N}$, if $\varphi : X \dashrightarrow Y$ is an isogeny of degree n^2, then X and Y are Hodge isogenous.*

Proof. It is proven in [BSV, Proposition 3.2] that $T_X \otimes \mathbb{Q} \simeq T_Y \otimes \mathbb{Q}$ if and only if $T_Y \otimes \mathbb{Q} \simeq T_Y(n^2) \otimes \mathbb{Q}$, which is true by Lemma 2.17. Then Witt's theorem implies that the isometry $T_X \otimes \mathbb{Q} \simeq T_Y \otimes \mathbb{Q}$ extends to a Hodge isometry $H^2(X, \mathbb{Q}) \simeq H^2(Y, \mathbb{Q})$. □

In Proposition 3.11 we construct isogenies of degree n^2 between K3 surfaces; Proposition 2.18 implies that they are necessarily Hodge isogenies.

One of the interesting properties of Hodge isogenous K3 surfaces is that they have isomorphic rational motives, by [Huy, Theorem 0.2]. This also holds in the case described above of a K3 surface X isogenous to the minimal model Y of the quotient X/σ, as shown for example in [L, Proof of Thm 3.1] following the argument of [P], but to the knowledge of the authors it is still an open question for a general isogeny.

3. The intersection $\mathcal{L}_n \cap \mathcal{M}_n$

The main result in this section is Theorem 3.9, where we exhibit the maximal dimensional components of $\mathcal{L}_n \cap \mathcal{M}_n$. As preliminary result, we describe in §3.1 a specific family of K3 surfaces contained in $\mathcal{L}_n \cap \mathcal{M}_n$. This family is related with a special isogeny between K3 surfaces, which is induced by an isogeny between elliptic curves, see Remark 3.3.

3.1. The $(U \oplus \mathbb{M}_n)$-polarized K3 surfaces.

The $(U \oplus \mathbb{M}_n)$-polarized K3 surfaces have interesting geometric properties: this family is considered for $n = 2$ in [vGS], and for other values of n in [GSar2] and [GSar3] to find explicitly Ω_n. Here we reconsider it as example of a family of K3 surfaces contained in $\mathcal{L}_n \cap \mathcal{M}_n$.

Proposition 3.1. *Let $2 \leq n \leq 8$ and $\mathcal{U}_n := \mathcal{P}(U \oplus \mathbb{M}_n)$ be the family of the $(U \oplus \mathbb{M}_n)$-polarized K3 surfaces. Then:*

- *$\mathcal{U}_n$ is non empty and has dimension $18 - \operatorname{rank}(\mathbb{M}_n)$;*

- *if S is a K3 surface such that $S \in \mathcal{U}_n$, then S admits an elliptic fibration $\mathcal{E}_n : S \to \mathbb{P}^1$ with an n-torsion section t;*
- *$\mathcal{U}_n \subset \mathcal{L}_n \cap \mathcal{M}_n$;*
- *if $S \in \mathcal{U}_n$ and σ_t is the translation by t on $\mathcal{E}_n$, the minimal model of S/σ_t is a K3 surface in $\mathcal{U}_n$.*

Proof. The family $\mathcal{U}_n$ is non empty for each n such that $2 \leq n \leq 8$ as showed for example in [Sh, Table 2] or [G1, Table 1]. The dimension of $\mathcal{U}_n$ follows directly by the fact that the dimension of a non-empty family $\mathcal{P}(R)$ of R-polarized K3 surfaces (for a certain lattice R) is $20 - \mathrm{rank}(R)$, and in this case $R \simeq U \oplus \mathbb{M}_n$ has rank $2 + \mathrm{rank}(\mathbb{M}_n)$. The family $\mathcal{U}_n$ was considered in [G1, Proposition 4.3], where it is proved that the set of K3 surfaces admitting an elliptic fibration with a torsion section of order n coincides with the set of $(U \oplus \mathbb{M}_n)$-polarized K3 surfaces. Since $\mathbb{M}_n$ is clearly primitively embedded in $U \oplus \mathbb{M}_n$, all the K3 surfaces in $\mathcal{U}_n$ are also contained in $\mathcal{M}_n$. Moreover, let $\mathcal{E}_n : S \to \mathbb{P}^1$ be an elliptic fibration on S with an n-torsion section t. This allows to consider S as an elliptic curve over the field of functions $k(\mathbb{P}^1)$ and the presence of an n-torsion section is equivalent to the presence of an n-torsion rational point on this elliptic curve. So the translation by t is well defined and it induces an automorphism of order n on S. This is a symplectic automorphism (it is the identity on the base of the fibration and acts on the smooth fibers preserving their periods). We denote this symplectic automorphism by σ_t. Since $S \in \mathcal{U}_n$ admits a symplectic automorphism of order n, $\mathcal{U}_n \subset \mathcal{L}_n$ and thus $\mathcal{U}_n \subset \mathcal{L}_n \cap \mathcal{M}_n$. In [G1, Proposition 4.3] it is also proved that the quotient of an elliptic fibration with basis $\mathbb{P}^1$ by the translation by a torsion section is another elliptic fibration over $\mathbb{P}^1$ with an n-torsion section. Thus S/σ_t admits a smooth minimal model with an elliptic fibration with an n-torsion section and this minimal model is a K3 surface (since σ_t is a symplectic automorphism). Thus the minimal model of S/σ_t belongs to the family $\mathcal{U}_n$. □

Corollary 3.2. *For every n such that $2 \leq n \leq 8$, $\mathcal{L}_n \cap \mathcal{M}_n$ is non empty.*

Proof. The intersection $\mathcal{L}_n \cap \mathcal{M}_n$ contains at least the non empty family $\mathcal{U}_n$. □

Remark 3.3. Since both $\mathcal{L}_n$ and $\mathcal{M}_n$ are the union of $(19 - \mathrm{rank}(\mathbb{M}_n))$-dimensional families of polarized K3 surfaces, the intersection between these sets is at most $(19 - \mathrm{rank}(\mathbb{M}_n))$-dimensional. The Proposition 3.1 provides an intersection in a codimension 1 subfamily. In the Theorem 3.9 we will see that one can obtain larger intersection.

Remark 3.4. Since each $(U \oplus \mathbb{M}_n)$-polarized K3 surface has an elliptic fibration, it can be interpreted as elliptic curve over the field of rational functions in one variable. The symplectic automorphism which induces the isogeny between the two K3 surfaces in $\mathcal{U}_n$ as in Proposition 3.1 is an isogeny of the associated elliptic curve over the field of rational functions.

3.2. **Maximal dimensional components of $\mathcal{L}_n \cap \mathcal{M}_n$.** In this section we prove that there are components of $\mathcal{L}_n$ completely contained in $\mathcal{M}_n$ and vice versa. The proof is lattice theoretic: in order to obtain this result, we need some extra information on the lattices $\mathbb{M}_n$ and Ω_n. Both these lattices are primitively embedded in the Néron–Severi group of a $(U \oplus \mathbb{M}_n)$-polarized K3 surface, so we now use the K3 surfaces in the family $\mathcal{U}_n$ to compare the discriminant forms of $\mathbb{M}_n$ and Ω_n.

Proposition 3.5. *Let A_{Ω_n} (resp. $A_{\mathbb{M}_n}$) the discriminant group of Ω_n (resp. $\mathbb{M}_n$) and q_{Ω_n} (resp. $q_{\mathbb{M}_n}$) its discriminant form. Then $A_{\Omega_n} = (\mathbb{Z}/n\mathbb{Z})^{\oplus 2} \oplus A_{\mathbb{M}_n}$ and $q_{\Omega_n} = u(n) \oplus q_{\mathbb{M}_n}$, where $u(n)$ is the discriminant form of the lattice $U(n)$.*

Proof. Nikulin proved that $A_{\Omega_n} = (\mathbb{Z}/n\mathbb{Z})^{\oplus 2} \oplus A_{\mathbb{M}_n}$ in [N2, Lemma 10.2]. By the Proposition 3.1, if $\mathrm{NS}(S) \simeq U \oplus \mathbb{M}_n$, then S admits an elliptic fibration $\mathcal{E}_n : S \to \mathbb{P}^1$ with an n-torsion section t and thus a symplectic automorphism σ_t, which is the translation by t. Let us denote by F the class in $\mathrm{NS}(S)$ of the fiber of the elliptic fibration $\mathcal{E}_n$, by O the class of the zero section, by t the class of the n-torsion section, by t_i, $i = 2, \ldots, n-1$, the class of the section corresponding to the sum of t with itself i times in the Mordell–Weil group. By definition σ_t preserves the classes F and $O+t+\sum_{i=2}^{n-1} t_i$. So $U(n) \simeq \langle F, O+t+\sum_{i=2}^{n-1} t_i\rangle \subset \mathrm{NS}(S)^{\sigma_t}$ and thus $\langle F, O+t+\sum_{i=2}^{n-1} t_i\rangle^{\perp} \supset \left(\mathrm{NS}(S)^{\sigma_t}\right)^{\perp}$. Since $\mathrm{rank}(\Omega_n) = \mathrm{rank}(\mathbb{M}_n)$, $\mathrm{rank}(\Omega_n) = \rho(S) - 2$. So

$$\langle F, O+t+\sum_{i=2}^{n-1} t_i\rangle^{\perp} \supset \left(\mathrm{NS}(S)^{\sigma_t}\right)^{\perp} \simeq \Omega_n.$$

Denoted by T_S the transcendental lattice of S, it follows from $\mathrm{NS}(S) \simeq U \oplus \mathbb{M}_n$ that $q_{T_S} = -q_{\mathbb{M}_n}$. By $\left(\mathrm{NS}(S)^{\sigma_t}\right)^{\perp} \simeq \Omega_n$ one obtains that the orthogonal of Ω_n in $H^2(S,\mathbb{Z})$ is an overlattice of finite index (possibly 1) of $U(n) \oplus T_S$. Since $A_{\Omega_n} = (\mathbb{Z}/n\mathbb{Z})^2 \oplus A_{\mathbb{M}_n}$, the orthogonal of Ω_n in $H^2(S,\mathbb{Z})$ is $U(n) \oplus T_S$. So $q_{\Omega_n} \simeq -q_{U(n)\oplus T_S} = u(n) \oplus q_{\mathbb{M}_n}$. □

Lemma 3.6. *Let F be a finite abelian group with quadratic form q_F and $m \geq 2$. Let $V = \langle 2d\rangle \oplus W$ be an indefinite even non-degenerate lattice with discriminant group $A_V = (\mathbb{Z}/2d\mathbb{Z}) \oplus (\mathbb{Z}/m\mathbb{Z})^{\oplus 2} \oplus F$, with discriminant form $q_{A_V} = (\frac{1}{2d}) \oplus u(m) \oplus q_F$. If $d \equiv 0 \mod 2m$, then V admits an overlattice Z of index m with $A_Z = \mathbb{Z}/2d\mathbb{Z} \oplus F$ and $q_{A_Z} = (\frac{1}{2d}) \oplus q_F$. Moreover, Z contains primitively W.*

Proof. By assumption, there exists some integer k such that $2d = 4km$. Let h be a generator of the $\mathbb{Z}/2d\mathbb{Z}$ summand of A_V such that $h^2 = \frac{1}{2d}$, and let e_1, e_2 be a basis of the $(\mathbb{Z}/m\mathbb{Z})^{\oplus 2}$ summand in A_V such that $e_1^2 = e_2^2 = 0$ and $e_1 e_2 = -\frac{1}{m}$. We define $\epsilon := e_1 + 2ke_2$, so that $\epsilon^2 = -\frac{4k}{m}$. Then the subgroup $H := \langle (4k)h + \epsilon\rangle$ is isotropic and its orthogonal inside A_V is $H^{\perp} = \langle h - e_2, \nu\rangle \oplus F$, where $\nu := e_1 - 2ke_2$. It follows from [N1, Propostion 1.4.1 and Corollary 1.10.2] that there exists an even overlattice Z of V of index m with $A_Z \cong H^{\perp}/H = \langle h - e_2\rangle \oplus F \cong \mathbb{Z}/2d\mathbb{Z} \oplus F$, and q_{A_Z} is induced on the quotient $H^{\perp}/H$ by $(q_{A_V})_{|H^{\perp}}$, so it is exactly $(\frac{1}{2d}) \oplus q_F$. Finally, we observe that the intersection of H with A_W inside A_V is trivial, hence W is a primitive sublattice of Z. □

Corollary 3.7. *Let $2 \leq n \leq 8$, $d \in \mathbb{N}$, $d \geq 1$ and $d \equiv 0 \mod 2n$. Then $\langle 2d\rangle \oplus \Omega_n$ admits an overlattice of index n whose discriminant form is $(\frac{1}{2d}) \oplus q_{\mathbb{M}_n}$; this overlattice contains primitively Ω_n.*

Proof. It suffices to apply Lemma 3.6 to the lattice $V = \langle 2d\rangle \oplus \Omega_n$ and to recall that $q_{\Omega_n} = u(n) \oplus q_{\mathbb{M}_n}$, by Proposition 3.5. □

Definition 3.8. *For each $2 \leq n \leq 8$ and each $d \in \mathbb{N}$, $d \geq 1$, we denote by $L_{d,n}$ the lattice $\langle 2d\rangle \oplus \Omega_n$.*

For each $2 \leq n \leq 8$ and each $d \in \mathbb{N}$, $d \geq 1$, and $d \equiv 0 \mod 2n$, we denote by $L'_{d,n}$ the overlattice of index n of $L_{d,n}$ constructed in Corollary 3.7.

For each $2 \leq n \leq 8$ and each $e \in \mathbb{N}$, $e \geq 1$, we denote by $M_{e,n}$ the lattice $\langle 2e\rangle \oplus \mathbb{M}_n$.

Theorem 3.9. *Let $d \in \mathbb{N}$, $d \geq 1$ and $d \equiv 0 \mod 2n$. The lattice $L'_{d,n}$ is unique in its genus and*

$$L'_{d,n} \simeq M_{d,n}.$$

The family $\mathcal{P}(L'_{d,n})$ is $(19 - \mathrm{rank}(\Omega_n))$-dimensional and is a subset of $\mathcal{L}_n \cap \mathcal{M}_n$, i.e. each K3 surface in this family admits a symplectic automorphism of order n and is $n : 1$ cyclically covered by a K3 surface.

Proof. By the Corollary 3.7, the lattice $L'_{d,n}$ has the same discriminant group and form as the lattice $M_{d,n}$. By [N2, Proposition 7.1], the length and the rank of the lattice $\mathbb{M}_n$ are the following:

n	2	3	4	5	6	7	8
$l(\mathbb{M}_n)$	6	4	4	2	2	1	2
$\mathrm{rank}(\mathbb{M}_n)$	8	12	14	16	16	18	18

where the length $l(R)$ of a lattice R is the minimal number of generators of the discriminant group $R^\vee/R$. Since $\mathrm{rank}(M_{d,n}) = 1+\mathrm{rank}(\mathbb{M}_n)$ and, if $d \equiv 0 \mod 2n$, $l(\langle 2d\rangle \oplus \mathbb{M}_n) = 1+l(\mathbb{M}_n)$, for every admissible n and $d \equiv 0 \mod 2n$, $\mathrm{rank}(M_{d,n}) \geq 2 + l(M_{d,n})$, so by [N1, Corollary 1.13.3], there is a unique even hyperbolic lattice with the same rank, length, discriminant group and form as $M_{d,n}$. Since $L'_{d,n}$ has all the prescribed properties, we conclude that $L'_{d,n} \simeq M_{d,n}$. Moreover, by [N1, Theorem 1.14.4], if $n < 7$ the lattice $L'_{d,n} \simeq M_{d,n}$ admits a unique, up to isometry, primitive embedding in Λ_{K3}, and thus determines a $(19 - \mathrm{rank}(\Omega_n))$-dimensional family of K3 surfaces. If $n = 7, 8$, any primitive embedding of $L'_{d,n} \simeq M_{d,n}$ in the unimodular lattice Λ_{K3}, which exists by results in [GSar2, GSar3], identifies the same genus of the orthogonal complement $T_{d,n}$ of rank three and signature $(2, 1)$: we get respectively that $A_{T_{d,7}} = \mathbb{Z}/7\mathbb{Z} \oplus \mathbb{Z}/2d\mathbb{Z}$ and $A_{T_{d,8}} = \mathbb{Z}/2\mathbb{Z} \oplus Z/4\mathbb{Z} \oplus \mathbb{Z}/2d\mathbb{Z}$ with quadratic forms $q_{T_{d,7}} = \left(-\frac{4}{7}\right) \oplus \left(-\frac{1}{2d}\right)$ and $q_{T_{d,8}} = \left(\frac{1}{2}\right) \oplus \left(\frac{1}{4}\right) \oplus \left(-\frac{1}{2d}\right)$. It follows from [N1, Proposition 1.15.1] that the primitive embedding of $L'_{d,n} \simeq M_{d,n}$ in Λ_{K3} is unique, up to isometry, if and only if $T_{d,n}$ is unique in its genus and the map $O(T_{d,n}) \to O(q_{T_{d,n}})$ is surjective. By [MM, Theorem VIII.7.5], these two conditions hold in particular if the discriminant quadratic form $q_{T_{d,n}}$ is p-regular for all prime numbers $p \neq s$ and it is s-semiregular for a single prime number s. The precise (and quite technical) definition of p-regular and p-semiregular form can be found in [MM, Definition VIII.7.4]. An easy application of [MM, Lemma VIII.7.6 and VIII.7.7] implies that:

- $q_{T_{d,7}}$ is p-regular if $p \neq 7$ and it is 7-semiregular;
- $q_{T_{d,8}}$ is p-regular if $p \neq 2$ and it is 2-semiregular.

Hence, also for $n = 7, 8$, $T_{d,n}$ is unique in its genus and the primitive embedding of $L'_{d,n} \simeq M_{d,n}$ in Λ_{K3} is unique up to isometry, and thus determines a $(19-\mathrm{rank}(\Omega_n))$-dimensional family of K3 surfaces.

Each K3 surface which is $M_{d,n}$-polarized is contained in $\mathcal{L}_n \cap \mathcal{M}_n$ because there are primitive embeddings both of Ω_n and of $\mathbb{M}_n$ in its Néron–Severi group. □

Proposition 3.10. *Let $2 \leq n \leq 8$ and $d \in \mathbb{N}$, $d \geq 1$. The lattice $L_{d,n}$ is not isometric to any overlattice of finite index (possibly 1) of $M_{e,n}$, for any e. In particular if X is a K3 surface such that $\mathrm{NS}(X) \simeq L_{d,n}$, then X does not admit* *a cyclic $n:1$ cover by a K3 surface and the families of the $(L_{d,n})$-polarized K3 surfaces are not (totally) contained in $\mathcal{M}_n$.*

Proof. By Proposition 3.5, $l(\Omega_n)=2+l(\mathbb{M}_n)$. Hence $l(L_{d,n})\geq l(\Omega_n)=2+l(\mathbb{M}_n)>l(M_{e,n})$. Since any overlattice of $M_{e,n}$ has at most the length of $M_{e,n}$, the lattices $L_{d,n}$ can not be isometric to any overlattice of the lattice $M_{e,n}$. □

In conclusion we proved that there are components of $\mathcal{L}_n$ (and of $\mathcal{M}_n$) which are completely contained in $\mathcal{L}_n\cap\mathcal{M}_n$, but there are also components of $\mathcal{L}_n$, which are not contained in $\mathcal{M}_n$, and thus in $\mathcal{L}_n\cap\mathcal{M}_n$. It is also true that there are components of $\mathcal{M}_n$ which are not totally contained in $\mathcal{L}_n$ (see e.g. Theorem 4.6 for the case $n=2$.)

In the following proposition we construct an $n^2:1$ isogeny between two K3 surfaces by using a third K3 surface, which is $L'_{d,n}$-polarized, and we prove that generically this $n^2:1$ isogeny is not just the quotient by an automorphism group.

Proposition 3.11. *Let Z be a K3 surface, such that $\mathrm{NS}(Z)=L'_{d,n}$, let X be the K3 surface which is a $n:1$ cyclic cover of Z and Y be the quotient of Z by a symplectic automorphism of order n. Then there is an $n^2:1$ isogeny between X and Y but there is no finite group G of automorphisms on X such that Y is birational to X/G.*

Proof. By Theorem 3.9 the K3 surfaces Z which are $L'_{d,n}$-polarized are $n:1$ isogenous to two K3 surfaces, X and Y, respectively with the two $n:1$ isogenies $X\dashrightarrow Z$ and $Z\dashrightarrow Y$. The composition of these two isogenies is an $n^2:1$ isogeny $X\dashrightarrow Y$.

If there exists a group of automorphism G as required, it has to be a group of symplectic automorphisms (otherwise the quotient X/G would not be birational to a K3 surface). So X should admit a group of symplectic automorphisms of order n^2. If X admits a group G of symplectic automorphisms, then $\left(\mathrm{NS}(X)^G\right)^\perp$ is a lattice (analogous to Ω_n) which is unique in most of the cases. Its rank depends only on G and it is known for every admissible G, see [H]. In particular, if $2\leq n\leq 8$, for every group G_{n^2} of order n^2 acting symplectically on a K3 surface, the rank of the lattice is $\mathrm{rank}\left(\left(\mathrm{NS}(X)^{G_{n^2}}\right)^\perp\right)>\mathrm{rank}(\Omega_n)$. Hence, if a K3 surface X admits G_{n^2} as group of symplectic automorphisms, $\rho(X)>1+\Omega_n=\rho(Z)$. But X and Z are isogenous, hence $\rho(X)=\rho(Z)$ and thus X can not admit a group of symplectic automorphisms of order n^2. □

Remark 3.12. In Proposition 3.11 we proved that the $n^2:1$ isogeny $X\dashrightarrow Y$ is not induced by a quotient map, so that the rational map $X\dashrightarrow Y$ is not a Galois cover. Let us denote by V its Galois closure, hence V is a surface such that both X and Y are birational to Galois quotients of V. Denoted by G the Galois group of the cover $V\dashrightarrow Y$ and by H the subgroup of G which is the Galois group of $V\dashrightarrow X$, H is not a normal subgroup of G, otherwise the rational map $X\dashrightarrow Y$ would be a Galois cover with Galois group G/H. The Kodaira dimension of the surface V is non negative (since V covers K3 surfaces), but moreover V can not be a K3 surface. This can be proved applying the same argument of the proof of Proposition 3.11: if V were K3 surface, it would admit G as group of symplectic automorphism, but its Picard number is not big enough. We give more details on the construction of V and G in the case $n=2$, see Section 4.3

4. Involutions

In this section we restrict our attention to the case of the symplectic involutions (i.e. $n = 2$). In this case several more precise and deep results are known about the relations between K3 surfaces admitting a symplectic involutions and K3 surfaces which are their quotients, hence we can improve the general results of the previous section and we can describe explicit examples. In particular we obtain: a complete description of the maximal dimensional components of the intersection $\mathcal{L}_2 \cap \mathcal{M}_2$, in Theorem 4.6; infinite families of K3 surfaces such that for each K3 surface in a family there is another one in another family which is isogenous to it, in Corollary 4.8; geometric examples in Sections 4.4 and 4.5.

4.1. Preliminary results on symplectic involutions and Nikulin surfaces.

For historical reasons, we refer to the K3 surfaces in $\mathcal{M}_2$ (i.e. the K3 surfaces which are cyclically 2 : 1 covered by a K3 surface) as the Nikulin surfaces and to the lattice $\mathbb{M}_2$ as the Nikulin lattice, denoted by $N (:= \mathbb{M}_2)$. So we have

Definition 4.1. *A Nikulin surface Y is a K3 surface which is the minimal resolution of the quotient of a K3 surface X by a symplectic involution σ. The minimal primitive sublattice of* $\mathrm{NS}(Y)$ *containing the curves arising from the desingularization of X/σ is denoted by N and it is called Nikulin lattice.*

If σ is a symplectic involution on a K3 surface X, then the fixed locus of σ on X consists of 8 isolated points. The quotient surface X/σ has 8 singularities of type A_1, so the minimal resolution X contains 8 disjoint rational curves, which are the exceptional divisors over the singular points of X/σ. Called N_i, $i = 1, \ldots, 8$ their classes in the Néron–Severi group, the class $\left(\sum_{i=1}^{8} N_i\right)/2$ is contained in the Néron–Severi group. Indeed the union of the eight disjoint rational curves is the branch locus of the double cover $\widetilde{X} \to Y$, where $\widetilde{X}$ is the blow up of X in the 8 points fixed by σ. We will say that a set of rational curves is an even set if the sum of their classes divided by 2 is contained in the Néron–Severi group and that a set of nodes is an even set if the curves resolving these nodes form an even set.

Proposition 4.2. ([N2, Section 6]) *The Nikulin lattice is an even negative definite lattice of rank 8 and its discriminant form is the same as the one of $U(2)^3$. It contains 16 classes with self-intersection -2, i.e. $\pm N_i$, $i = 1, \ldots, 8$. A $\mathbb{Z}$-basis of N is given by $\left(\sum_{i=1}^{8} N_i\right)/2$, N_i, $i = 1, \ldots 7$.*

As in the previous section, we will denote by $M_{e,2}$ the lattice $\langle 2e \rangle \oplus N$.

Proposition 4.3. (a) *A K3 surface Y is a Nikulin surface if and only if the lattice N is primitively embedded in* $\mathrm{NS}(Y)$.

(b) *The minimal Picard number of a Nikulin surface is 9.*

(c) *There exists an even overlattice of index two of $M_{e,2}$ in which N is primitively embedded if and only if e is even. In this case, this lattice is unique up to isometry and denoted by $M'_{e,2}$.*

(d) *If Y is a Nikulin surface with Picard number 9, then* $\mathrm{NS}(Y)$ *is isometric either to $M_{e,n}$ or to $M'_{e,n}$ for a certain e.*

Proof. The point (a) is Theorem 2.10, the point (b) is proved in Corollary 2.11 in the case $n = 2$. The point (c) is proved in [GSar1, Proposition 2.2] and the point (d) in [GSar1, Proposition 2.1]. □

In the case $n = 2$, the lattice Ω_2 is known to be isometric to $E_8(-2)$. As in the previous section, we will denote by $L_{d,2}$ the lattice $\langle 2d\rangle \oplus E_8(-2)$ and by $L'_{d,2}$ the overlattice of index two of $L_{d,2}$ such that $L'_{d,2}$ is even and $E_8(-2)$ is primitively embedded in $L'_{d,2}$.

Proposition 4.4. (a) *A K3 surface X admits a symplectic involution σ if and only if the lattice $E_8(-2)$ is primitively embedded in* NS(X).

(b) *If X admits a symplectic involution, $\rho(X) \geq 9$.*

(c) *There exists an even overlattice of index two of $L_{d,2}$ in which $E_8(-2)$ is primitively embedded if and only if d is even. In this case, this lattice is unique up to isometry and is $L'_{d,2}$.*

(d) *If X is a K3 surface admitting a symplectic involution and with Picard number 9, then* NS(X) *is isometric either to $L_{d,2}$ or to $L'_{d,2}$ for a certain d.*

Proof. The Proposition follows directly by [vGS, Propositions 2.2 and 2.3] (and the points (a), (b) and (c) were already proved in the more general setting of automorphisms of order n in the previous Section). □

The main result which is known for involutions and is not yet stated in the more general case of symplectic automorphisms of order n, is the explicit relation between the Néron–Severi group of a K3 surface which admits a symplectic involution and the Néron–Severi group of the K3 surface which is its quotient.

Proposition 4.5. ([GSar1, Corollary 2.2]) *Let X be a K3 surface with a symplectic involution σ and Y be the minimal resolution of X/σ. Then:*

- NS$(X) \simeq L_{e,2}$ *if and only if* NS$(Y) \simeq M'_{2e,2}$
- NS$(X) \simeq L'_{2e,2}$ *if and only if* NS$(Y) \simeq M_{e,2}$.

4.2. **The intersection $\mathcal{L}_2\cap\mathcal{M}_2$ and infinite towers of isogenous K3 surfaces.** Since we know the structure of all the possible Néron–Severi groups of Nikulin surfaces of minimal Picard number (by Proposition 4.3) and all the possible Néron–Severi groups of K3 surfaces of minimal Picard number admitting a symplectic involution (by Proposition 4.4), we are able to give the following refinement of the Theorem 3.9 and of the Proposition 3.10.

Theorem 4.6. *A Nikulin surface Y such that $\rho(Y) = 9$ admits a symplectic involution if and only if* NS$(Y) \simeq M_{2d,2}(\simeq L'_{2d,2})$.

A K3 surface X admitting a symplectic involution such that $\rho(X) = 9$ is a Nikulin surface if and only if NS$(X) \simeq L'_{2d,2}(\simeq M_{2d,2})$.

So

$$\mathcal{L}_2 \cap \mathcal{M}_2 \supset \bigcup_{d\in\mathbb{N}_{>0},\ d\equiv 0(2)} \mathcal{P}(M_{2d,2}).$$

Proof. By Theorem 3.9 $M_{2d,2} \simeq L'_{2d,2}$ and thus if NS$(Y) \simeq M_{2d,2}$, then Y admits a symplectic involution. Similarly if NS$(X) \simeq L'_{2d,2}$, X is a Nikulin surface. It remains to prove that if a K3 surface is in $\mathcal{L}_2 \cap \mathcal{M}_2$, and its Picard number is 9, then its Néron–Severi can not be isometric to $M'_{e,2}$, to $M_{e,2}$ for an odd e or to $L_{f,2}$ with $f \in \mathbb{N}_{>0}$. The argument is similar to the one of Proposition 3.10.

By Proposition 4.4, if Y is a Nikulin surface, its Néron–Severi group is either isometric to $M_{e,2}$ or to $M'_{2e,2}$. By Proposition 4.3 if X is a K3 surface admitting a symplectic involution, its Néron–Severi group is either isometric to $L_{d,2}$ or to $L'_{2d,2}$. So if a K3 surface has both properties (i.e. it is in $\mathcal{L}_2 \cap \mathcal{M}_2$ and has Picard number 9), its Néron–Severi group is isometric both to a lattice in $\{M_{e,2}, M'_{2e,2}\}$ and to a lattice in $\{L_{d,2}, L'_{2d,2}\}$. Hence we are looking for pairs of lattices, one in $\{M_{e,2}, M'_{2e,2}\}$ and one in $\{L_{d,2}, L'_{2d,2}\}$, which are isometric. If two lattices are isometric, they have the same length. We observe that $l(M_{e,2}) = 1 + l(N) = 7$, $l(M'_{2e,2}) = 1+l(N)-2 = 5$, $l(L_{d,2}) = 1+l(\Omega_2) = 9$, $l(L'_{2d,2}) = 1+l(\Omega_2)-2 = 7$. In particular, the unique possible pair of lattices as required is given by $(M_{e,2}, L'_{2d,2})$. Since if two lattices are isometric they have the same discriminant, one obtains that $e = 2d$. □

Corollary 4.7. *Two Nikulin surfaces Y and $\hat{Y}$ with Picard number 9 are isogenous by a chain of quotients by involutions if and only if one of the following equivalent conditions hold:*
(i) $\mathrm{NS}(Y) \simeq M_{d,2}$, $\mathrm{NS}(\hat{Y}) = M_{e,2}$, *and there exists $m \in \mathbb{N}_{>0}$ such that either $d = 2^m e$ or $e = 2^m d$;*
(ii) $T_Y \simeq U \oplus U \oplus N \oplus \langle -2d\rangle$, $T_{\hat{Y}} \simeq U \oplus U \oplus N \oplus \langle -2e\rangle$ *and there exists $m \in \mathbb{N}_{>0}$ such that either $d = 2^m e$ or $e = 2^m d$.*

Proof. We can assume that $\hat{Y}$ is obtained by iterated quotients from Y. Then Y admits a symplectic involution σ and, by Theorem 4.6, there exists an even d such that $\mathrm{NS}(Y) \simeq M_{d,2} \simeq L'_{d,2}$. So Y is the cover of a K3 surface Z with Néron–Severi group $M_{d/2,2}$ (by Proposition 4.5). If d is odd, then the process stops and $\hat{Y}$ is necessarily Z; otherwise, $\mathrm{NS}(Z) \simeq M_{d,2} \simeq L'_{d,2}$ and Z is the cover of a K3 surface Z with Néron–Severi group $M_{d/4,2}$. Iterating, if possible, this process m times, one obtains Nikulin surfaces with Néron–Severi group isometric to $M_{d/2^m,2}$. In particular, one never obtains lattices isometric to $M'_{e,2}$ (for a certain e) as Néron–Severi groups of a Nikulin surface obtained by iterated quotients from Y.

Vice versa, if $\mathrm{NS}(\hat{Y}) \simeq M_{e,2}$ for a certain e, $\hat{Y}$ is covered by a K3 surface W with $\mathrm{NS}(W) \simeq L'_{2e,e} \simeq M_{2e,2}$ (by Proposition 4.5). So W is a Nikulin surface, 2 : 1 covered by a K3 surface with Néron–Severi group isometric to $L'_{4e,e} \simeq M_{4e,2}$. Reiterating this process m times one obtains that $\hat{Y}$ is isogenous to a Nikulin surface whose Néron–Severi lattice is isometric to $M_{h,e}$ with $h = 2^m d$.

The equivalent statement for the transcendental lattice follows by the fact that if the Néron–Severi group of a K3 surface is isometric to $M_{d,2}$, then its transcendental lattice is isometric to $U \oplus U \oplus N \oplus \langle -2d\rangle$ (since the discriminant form of the latter is minus the discriminant form of $M_{d,2}$, and in this case the transcendental lattice is uniquely determined by its genus). □

We determined an infinite number of infinite series of Nikulin surfaces of Picard number 9 related by iterated quotients by symplectic involutions. More precisely we proved the following.

Corollary 4.8. *For every $d \in \mathbb{N}$, if $\mathrm{NS}(Y) \simeq M_{d,2}$ there exists an infinite number of K3 surfaces Y_m isogenous to Y. In particular for each m there exists at least one K3 surface Y_m with an isogeny of degree 2^m to Y whose Néron–Severi group is isometric to $\mathrm{NS}(Y_m) = M_{2^m d,2}$. The transcendental lattice of Y is $T_Y \simeq U \oplus U \oplus N \oplus \langle -2d\rangle$ and for each m the one of Y_m is $T_{Y_m} \simeq U \oplus U \oplus N \oplus \langle -2^{m+1}d\rangle$.*

Remark 4.9. The $M_{2^{m+2}d,2}$-polarized K3 surfaces can be interpreted as moduli spaces of twisted sheaves on $M_{2^m d,2}$-polarized K3 surfaces. Let e_1, e_2 be a standard basis of the first copy of U inside the K3 lattice Λ_{K3} and choose a primitive embedding of $M_{2^{m+2}d,2}$ in Λ_{K3} so that a generator of $\langle 2^{m+1}d\rangle$ is $e_1+2^m de_2$ and N is embedded in $U^\perp$. Given $S\in\mathcal{P}(M_{2^m d,2})$ generic, the transcendental lattice is $T_S=U^{\oplus 2}\oplus N\oplus\langle -2^{m+1}d\rangle$, and with our previous choice it is easy to see that $\langle -2^{m+1}d\rangle$ is generated by $t:=e_1-2^m de_2$. The B-field $B=\frac{e_2}{2}\in H^2(S,\mathbb{Q})$ is a lift for the Brauer class $\beta: T_S\to\mathbb{Z}/2\mathbb{Z}$ given by $v\mapsto (v,2B)$. It is easy to see that $T(S,B)\cong\ker\beta=U^{\oplus 2}\oplus N\oplus\langle -2^{m+3}d\rangle\subset H^*(S,\mathbb{Z})$, where the last summand is spanned by $(0,2t,1)$. Moreover, the orthogonal of $T(S,B)$ inside the Mukai lattice $H^*(S,\mathbb{Z})$ is the generalized Picard group $\mathrm{Pic}(S,B)$, which is the sublattice spanned by $f_1:=(0,0,1)$, $f_2:=(2,e_2,0)$, $f_3:=(0,e_1+2^m de_2,0)$ and $(0,b_i,0)$ with $b_1,\cdots,b_8\in M_{2^{m+2}d,2}$ a basis of the lattice N, and has quadratic form

$$\begin{pmatrix} 0 & 2 & 0\\ 2 & 0 & 1\\ 0 & 1 & 2^{m+1}d\end{pmatrix}\oplus N.$$

The isotropic element $v:=2^m df_2-f_3$ now satisfies $((\mathbb{Z}v)^\perp\cap\mathrm{Pic}(S,B))/\mathbb{Z}v\simeq M_{2^{m+2}d,2}$. Hence, the moduli space of stable twisted sheaves $M_v(S,\beta)$ is a smooth $M_{2^{m+2}d,2}$-polarized K3 surface. It is an interesting open question to see whether the isogeny of degree 4 which we constructed here coincides with the one induced by a twisted universal family on $S\times M_v(S,\beta)$ or not (for further details see [Huy, Theorem 0.1]).

4.3. **The Galois closure of $2^2:1$ covers.** Let X_d be a K3 surface such that $\mathrm{NS}(X_d)=M_{d,2}$. Then $X_{2d}\in\mathcal{L}_2\cap\mathcal{M}_2$ and there are the two Galois covers $X_{4d}\dashrightarrow X_{2d}$ and $X_{2d}\dashrightarrow X_d$. The composition of these two maps is a $2^2:1$ isogeny, not induced by a Galois cover, by Proposition 3.11. As observed in Remark 3.12 there exist a surface V, a group $G\subset\mathrm{Aut}(V)$ and a subgroup H of G such that V/G is birational to X_d, and V/H is birational to X_{4d}. Here we construct the surface V and the group G, proving the following

Proposition 4.10. *The group G is the dihedral group of order 8 and V is a $(\mathbb{Z}/2\mathbb{Z})^2$ Galois cover of X_{2d}, whose branch locus B is the union of 16 smooth rational curves. If B is normal crossing, then V is a positive Kodaira dimension smooth surface such that $h^{1,0}(V)=0$ and $h^{2,0}(V)\geq 35$.*

To prove the Proposition one constructs a $(\mathbb{Z}/2\mathbb{Z})^2$ Galois cover of X_{2d} (see Section 4.3.1) by a surface denoted by V. Then one constructs a $(\mathbb{Z}/2\mathbb{Z})^2$ Galois cover of X_d (see Section 4.3.2), and eventually one proves that these two covers can be pasted to obtain a unique Galois cover by the dihedral group of order 8 (see Section 4.3.3). In order to obtain the $(\mathbb{Z}/2\mathbb{Z})^2$ covers one compares the branch loci of the $2:1$ maps $X_{4d}\dashrightarrow X_{2d}$ and $X_{2d}\dashrightarrow X_d$.

Here we do not consider the Galois closure of $2^n:1$ isogenies given in Corollary 4.8, but for any fixed n a priori one can iterate the previous process.

In the following we will call the $(\mathbb{Z}/2\mathbb{Z})^2$ Galois covers bidouble covers as in [C], where all the basic definitions and properties of these covers can be found.

4.3.1. *A bidouble cover of the surface X_{2d}.* Let us denote by $N_1,\dots,N_8\subset X_{2d}$ the rational curves which are the branch locus of the double cover $X_{4d}\dashrightarrow X_{2d}$. Let us denote by σ_{2d} the symplectic involution on X_{2d} such that X_d is birational to X_{2d}/σ_{2d}. The curves N_i, $i=1,\dots,8$ are not preserved by σ_{2d}, since $\langle H\rangle:=\mathrm{NS}(X_{2d})^{\sigma_{2d}^*}$ is positive definite and more precisely $H^2=4d$. Set $N_i':=\sigma_{2d}(N_i)$.

Hence we found two even sets of eight rational curves on X_{2d}: $\{N_1, \ldots, N_8\}$ and $\{N_1', \ldots, N_8'\}$. Let $D_1 := \sum_{i=1}^{8} N_i$, $D_2 := \sum_{i=1}^{8} N_i'$, $D_3 := 0$ and $2L_i := D_j + D_k$, $i, j, k \in \{1, 2, 3\}$. The six divisors D_j, L_i, $j, i = 1, 2, 3$ in $\mathrm{NS}(X_{2d})$ satisfy the conditions which define a bidouble cover, so there exists a surface V such that $(\mathbb{Z}/2\mathbb{Z})^2 \in \mathrm{Aut}(V)$ and $V/(\mathbb{Z}/2\mathbb{Z})^2$ is (birational to) X_{2d} (see [C, Section 2]). Moreover, there are three surfaces which are double covers of X_{2d} branched respectively along the curves supported on $2L_1$, $2L_2$ and $2L_3$; all of them are $2:1$ covered by V. Since $L_2 = \sum_{i=1}^{8} N_i/2$, the double cover of X_{2d} branched on $2L_2$ is a non-minimal model of X_{4d}. We denote the cover branched on the curves in the support of $2L_2$ by $\widetilde{X_{4d}}$. Similarly the double cover of X_{2d} branched on $\cup_i N_i'$ is the blow up of a K3 surface, X_{4d}', in 8 points and it will be denoted by $\widetilde{X_{4d}'}$. The Néron–Severi group of the K3 surface X_{4d}' is determined by the one of X_{2d}, by Proposition 4.5, and thus it is isometric to M_{4d}. We obtain the following diagram:

$$\begin{array}{ccccc} & & V & & \\ & \swarrow & \downarrow & \searrow & \\ W & & \widetilde{X_{4d}} & & \widetilde{X_{4d}'} \\ & \searrow & \downarrow{\scriptstyle \widetilde{\pi_{4d}}} & \swarrow{\scriptstyle \widetilde{\pi_{4d}'}} & \\ & & X_{2d} & & \end{array} \tag{4.1}$$

The surfaces W and V have non negative Kodaira dimension, because they are covers of K3 surfaces.

Let us now suppose that the intersections $N_i \cap N_j'$ are transversal, and thus both the branch divisors of $W \to X_{2d}$ and of $V \to X_{2d}$ are normal crossing. Under this assumption, V is smooth and the birational invariants of W and V depend only on L_j^2, for $j = 1, 2, 3$. The surface W is the double cover of X_{2d} branched on the reducible curve which is the support of $2L_3$, i.e. on the curve $\bigcup_{i=1}^{8}(N_i \cup N_i')$. Since $N_i + N_i'$ is an effective (σ_{2d}^*)-invariant divisor and H is the ample generator of $\mathrm{NS}(X_{2d})^{\sigma_{2d}^*}$, there exists a positive integer k_i such that $N_i + N_i' = k_i H$. Then $L_3^2 = \left(\sum_{i=1}^{8} k_i H\right)^2 \Big/ 4 = d\left(\sum_{i=1}^{8} k_i\right)^2$ and

$$\chi(W) = 4 + \frac{d}{2}\left(\sum_{i=1}^{8} k_i\right)^2, \ h^{2,0}(W) = 3 + \frac{d}{2}\left(\sum_{i=1}^{8} k_i\right)^2 \text{ and } h^{1,0}(W) = 0.$$

The singularities of W are in the inverse image of the singular points of $\bigcup_{i=1}^{8}(N_i \cup N_i')$ and V is a double cover of W branched on its singular points. The invariants of V can be computed by [C, Section 2], from which one obtains

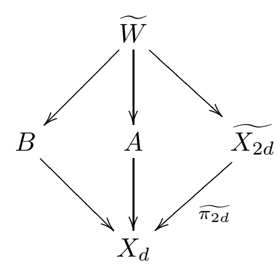
$$h^{2,0}(V) = h^{2,0}(W) = 3 + \frac{d}{2}\left(\sum_{i=1}^{8} k_i\right)^2 \geq 3 + 32d \geq 35, \quad h^{1,0}(V) = h^{1,0}(W) = 0.$$

Hence V is a surface with non negative Kodaira dimension, $h^{2,0}(V) \geq 35$ and $h^{1,0}(V) = 0$, so its Kodaira dimension is necessarily positive.

4.3.2. *A bidouble cover of the surface X_d.* The surface X_d is the desingularization of the quotient of X_{2d}/σ_{2d} and we will denote by $R_1,\ldots,R_8$ the eight disjoint rational curves resolving the singularities of X_{2d}/σ_{2d}. Equivalently, the double cover of X_d branched on $\bigcup_i R_i$ is birational to X_{2d}. Denoted by $\pi_{2d}: X_{2d}\rightarrow X_{2d}/\sigma_{2d}$ the quotient map, one observes that $\pi_{2d}(N_i)=\pi_{2d}(N_i')$, $i=1,\ldots,8$, and $\pi_{2d}(N_i)$ is a rational curve singular in the points $\pi_{2d}(N_i\cap N_i')$. We denote by $\overline{N_i}$ the strict transform on X_d of the curve $\pi_{2d}(N_i)$. The curves $\overline{N_i}$ could be singular and the set $\{\overline{N_1},\ldots,\overline{N_8}\}$ is a divisible set. Moreover, since $N_i+N_i'=k_iH\subset \mathrm{NS}(X_{2d})^{\sigma_{2d}^*}$, one has $(\pi_{2d})_*(N_i+N_i')\subset \mathrm{NS}(X_{2d}/\sigma_{2d})$. Hence $\overline{N_i}\subset (N^{\perp_{\mathrm{NS}(X_d)}})$.

The sets $\{R_1,\ldots,R_8\}$ and $\{\overline{N_1},\ldots\overline{N_8}\}$ are two 2-divisible sets of curves, which allow us to construct a bidouble cover of X_d, whose data are $\Delta_1:=\sum_{i=1}^8 R_i$, $\Delta_2:=\sum_{i=1}^8\overline{N_i}$, $\Delta_3:=\sum_{i=1}^8\left(R_i+\overline{N}_i\right)$, $2\Gamma_i:=\Delta_j+\Delta_k$, with $\{i,j,k\}=\{1,2,3\}$. The double cover $\widetilde{X_{2d}}\rightarrow X_d$ is branched over $\cup_i R_i$, i.e. the curve in the support of $2\Gamma_2$. It induces a double cover of $\widetilde{X_{2d}}$ branched over $\cup_i\left(\widetilde{N_i}+\widetilde{N_i'}\right)$, where $\widetilde{N_i}$ (resp. $\widetilde{N_i'}$) is the strict transform on $\widetilde{X_{2d}}$ of the curve N_i (resp. N_i'). Let us denote by $\widetilde{W}$ the surface double cover of $\widetilde{X_{2d}}$ branched on $\cup_i\left(\widetilde{N_i}+\widetilde{N_i'}\right)$. So we have the following diagram:

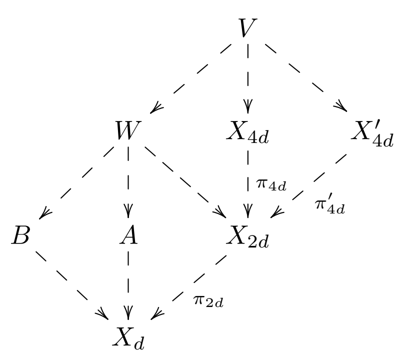
(4.2)



where $\widetilde{\pi_{2d}}:\widetilde{X_{2d}}\rightarrow X_d$ is induced by π_{2d}.

4.3.3. *The $\mathcal{D}_4$ cover of X_d.* Both the diagrams (4.2) and (4.1) induce a $2:1$ rational map $W \dashrightarrow X_{2d}$, which is (birationally) the double of cover of X_{2d} branched on $\bigcup_{i=1}^8(N_i\cup N_i')$. Hence these diagrams can be pasted to obtain the following, where all the arrows are rational maps of generically degree 2

(4.3)

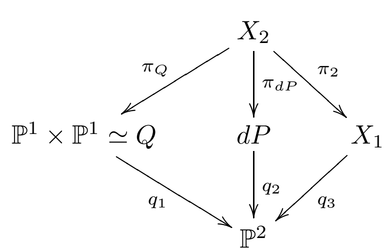


We already proved that the $4:1$ covers $X_{4d}\dashrightarrow X_d$ and $X_{4d}'\dashrightarrow X_d$ are not Galois covers in Proposition 3.11. On the other hand, the cover $4:1$ $W\rightarrow X_d$ is a Galois cover (indeed a bidouble cover), by construction. Since $V \dashrightarrow X_{2d}$ is constructed as bidouble cover, the cover involution of the cover $W \dashrightarrow X_{2d}$, lifts to an involution of V (which is the cover involution of $V \dashrightarrow X_{4d}$). Hence one obtains that the cover $V \dashrightarrow X_d$ is a Galois $8:1$ cover. The cover group G is an order 8 group, which admits non normal subgroups of order 2 (otherwise $X_{4d} \dashrightarrow X_d$ should be a Galois cover). Hence $G \simeq \mathcal{D}_4$, the dihedral group of order 8. We recall that $\mathcal{D}_4 :=< s, r | s^2 = 1, r^4 = 1, rs = sr^{-1} >$. The center H of G is $H :=< r^2 >$ and the quotient of V by H is birational to W. So we conclude that the Galois cover is given by the surface V on which acts the group $G = \mathcal{D}_4$.

4.4. **The K3 surface $X_2 \in \mathcal{L}_2 \cap \mathcal{M}_2$.** By Proposition 4.6, for every even d, if a K3 surface X_d is such that $\mathrm{NS}(X_d) \simeq L'_{d,2}$, then X_d admits a symplectic involution and it is $2:1$ cyclically covered by a K3 surface. Here we describe geometrically these properties for the minimum possible value of d, i.e. for $d = 2$: let X_2 be a K3 surface with $\mathrm{NS}(X_2) \simeq L'_{2,2}$. It admits an involution σ and by Proposition 4.5 the K3 surface Y_1 which is the desingularization of X_2/σ is such that $\mathrm{NS}(Y_1) = M_{1,2} \simeq \langle 2 \rangle \oplus N$. Since $\mathrm{NS}(X_2) \simeq M_{2,2} \simeq \langle 4 \rangle \oplus N$ (by Proposition 4.6), the surface X_2 is $2:1$ covered by a K3 surface X_4, whose Néron–Severi group is $\mathrm{NS}(X_4) \simeq L'_{4,2}$ (by Proposition 4.5). Since X_2 is a Nikulin surface, there are 8 disjoint rational curves, which resolve the singularities of the quotient of X_4 by a symplectic involution. Thus the surface X_2 admits two different descriptions according to the interpretation of it as K3 surface with a symplectic involution or as Nikulin surface. These descriptions are associated to different projective models, induced by different (pseudo)ample divisors. Here we recall these descriptions and we explain how to pass from one to the other.

By [vGS, Section 3.5], any K3 surface X_2 such that $\mathrm{NS}(X_2) \simeq L'_{2,2}$ is described as bidouble cover of $\mathbb{P}^2$ as follows: one considers two smooth plane curves B and C_0 of degree respectively 4 and 2 in $\mathbb{P}^2$. The double cover of $\mathbb{P}^2$ branched on $B \cup C_0$ is a surface singular in eight points, the inverse image of $B \cap C_0$. The resolution of this surface is the K3 surface X_1 such that $\mathrm{NS}(X_1) \simeq M_{1,2}$ and the eight rational curves arising from this resolution will be denoted by R_i, $i = 1, \ldots, 8$. The curves $R_1, \ldots, R_8$ form an even set of rational curves on X_1 and the double cover of X_1 branched on $\cup_i R_i$ is, by construction, a K3 surface X_2 such that $\mathrm{NS}(X_2) \simeq L'_{2,2}$. The choice of the curves $B \cup C_0$ totally determines the surfaces X_1 and X_2. To construct the bidouble cover one considers also the double cover of $\mathbb{P}^2$ branched on C_0 and the double cover of $\mathbb{P}^2$ branched on B. The first surface is a quadric $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$, the latter a del Pezzo surface of degree 2, denoted in the following by dP. Hence one has the following diagram, where all the arrows are rational maps of degree 2:

(4.4)



The Néron–Severi group of X_2 is isometric to $L'_{2,2}$, hence it is an overlattice of index two of $\langle 4\rangle\oplus E_8(-2)$. The linear system of the ample divisor L, orthogonal to $E_8(-2)$ in $L'_{2,2}$ exhibits X_2 as double cover of a quadric $\mathbb{P}^1\times\mathbb{P}^1$ in $\mathbb{P}^3$. One can assume that the class generating $L'_{2,2}/L_{2,2}$ is $E_1 := (L+e_1)/2$, where e_i is a standard basis of $E_8(-2)$ (i.e. $e_ie_{i+1}=2$ if $i=1,\ldots 6$, $e_3e_8=2$, $(e_i)^2=-4$ and the other intersections are 0). Then the divisor E_1 is a nef divisor and the map associated to its linear system $\varphi_{|E_1|}: X_2\to\mathbb{P}^1$ is a genus 1 fibration. The action of σ^* on $\mathrm{NS}(X_2)$ is the identity on the subspace $\langle L\rangle$ and minus the identity on the subspace $L^\perp\simeq E_8(-2)$. So the image of E_1 by σ^* is the nef divisor $E_2 := (L-e_1)/2 = E_1-e_1$ (see [vGS, Section 3.5]). The two maps $\varphi_{|E_i|}$, $i=1,2$ are the maps on the rulings of the quadric $Q\subset\mathbb{P}^3$ image of the map $\varphi_{|L|}=\varphi_{|E_1+E_2|}$. In particular the set of divisors $\{E_1,e_1,\ldots e_8\}$ is a basis of $\mathrm{NS}(X_2)$.

By (4.4), it follows that X_2 admits three commuting involutions, the covering involutions of the three double covers π_Q, π_{dP}, π_2. The latter involution is the symplectic involution σ, the others will be denoted by ι_Q and ι_{dP} respectively.

Proposition 4.11. *The involutions ι_Q and ι_{dP} are non-symplectic involutions and their composition is the symplectic involution σ. The group $\langle\iota_Q,\iota_{dP}\rangle$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and it is the Galois group of the $2^2:1$ cover $\pi: X_2 \dashrightarrow \mathbb{P}^2$.*

The induced three involutions on $\mathrm{NS}(X_2)$ act as follows on the basis $\{E_1,e_1,\ldots e_8\}$:

$$\begin{array}{llll}
\sigma^*(E_1)=E_1-e_1, & \sigma^*(e_i)=-e_i, & i=1,\ldots,8 & \\
\iota_Q^*(E_1)=E_1, & \iota_Q^*(e_1)=e_1, & \iota_Q^*(e_2)=-e_1-e_2, & \iota_Q^*(e_j)=-e_j,\\
\iota_{dP}^*(E_1)=E_1-e_1, & \iota_{dP}^*(e_1)=-e_1, & \iota_{dP}^*(e_2)=e_1+e_2 & \iota_{dP}^*(e_j)=e_j,
\end{array}$$

where $j=3,\ldots,8$.

Proof. The action of σ is minus the identity on $(\mathrm{NS}(X_2)^\sigma)^\perp\simeq E_8(-2)\subset\mathrm{NS}(X_2)$ and we chose the basis of $\mathrm{NS}(X_2)$ in such a way that the divisors e_i, $i=1,\ldots,8$ span exactly $(\mathrm{NS}(X_2)^\sigma)^\perp\simeq E_8(-2)$. Moreover we chose L to be the orthogonal to $\langle e_i\rangle_{i=1,\ldots 8}$ and thus $\sigma^*(L)=L$. By the definition of $E_1(=(L+e_1)/2)$ one obtains $\sigma^*(E_1)=(L-e_1)/2=E_1-e_1$.

The automorphism ι_Q is such that X_2/ι_Q is a rational surface and thus ι_Q is non symplectic and X_2/ι_Q is smooth. Since X_2/ι_Q is $\mathbb{P}^1\times\mathbb{P}^1$, $\mathrm{rank}(\mathrm{NS}(X_2)^{\iota_Q})=2$ and $\mathrm{NS}(X_2)^{\iota_Q}$ is generated by the divisors which induce the maps $X_2\to\mathbb{P}^1$ given by the composition of the quotient map $\pi_Q: X_2\to\mathbb{P}^1\times\mathbb{P}^1$ with the projection on the first, respectively second, factor. These maps are $\varphi_{|E_1|}: X_2\to\mathbb{P}^1$ and $\varphi_{|E_2|}: X_2\to\mathbb{P}^1$. So $\mathrm{NS}(X_2)^{\iota_Q}=\langle E_1,E_2\rangle$ and ι_Q acts as minus the identity on $(\mathrm{NS}(X_2)^{\iota_Q})^\perp$. So $\iota_Q^*(e_j)=-e_j$ if $j=3,\ldots,8$, $\iota_Q^*(E_1)=E_1$, and $\iota_Q^*(E_2)=\iota_Q^*(E_1-e_1)=E_1-\iota_Q^*(e_1)=E_1-e_1=E_2$. It follows $\iota_Q^*(e_1)=e_1$. In order to find the image of e_2 it suffices to recall that ι_Q^* is an involution and that $\left(\iota_Q^*(e_2)\right)\cdot\left(\iota_Q^*(D)\right)=e_2D$ for any divisor $D\in\mathrm{NS}(X_2)$. The group $\langle\iota_Q,\sigma\rangle$ is by construction the Galois group of the cover $\pi: X_2\to\mathbb{P}^2$, so it is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$ and contains three different involutions, each of them is the composition of the other two. In particular $\iota_{dP}=\iota_Q\circ\sigma$ and so $\iota_{dP}^*=\sigma^*\circ\iota_Q^*$ and ι_{dP} is non-symplectic. □

We already observed that the classes $E_1 := (L+e_1)/2$ and $E_2 := (L-e_1)/2$ induce two elliptic fibrations $\varphi_{|E_i|}: X_2\to\mathbb{P}^1$. By the properties of these elliptic fibrations we will be able to identify the classes of irreducible rational curves on X_2 and in particular 8 classes which span the Nikulin lattice. The following proposition gives the explicit isometry between $L'_{2,2}$ and $M_{2,2}$ and shows directly that the surface X_2 admits a 2:1 rational double cover by another K3 surface, thus it provides an explicit geometric interpretation of Theorem 3.9 in the case $n = 2$.

Proposition 4.12. *Both the genus 1 fibrations $\varphi_{|E_1|} : X_2 \to \mathbb{P}^1$ and $\varphi_{|E_2|} : X_2 \to \mathbb{P}^1$ have no reducible fibers and 8 disjoint sections which can be chosen to generate the Mordell–Weil group (which is isomorphic to $\mathbb{Z}^7$). One can choose these sections, for each fibration, in such a way that 7 sections are in common, the eighth section of $\varphi_{|E_1|}$ is a 5-section for $\varphi_{|E_2|}$ and vice versa the eighth section of $\varphi_{|E_2|}$ is a 5-section for $\varphi_{|E_1|}$. The eight sections of $\varphi_{|E_1|}$ (resp. $\varphi_{|E_2|}$) chosen as above form an even set of eight disjoint rational curves, so X_2 is a Nikulin surface and* $\mathrm{NS}(X_2) \simeq M_{2,2}$.

Proof. Since one has a basis of $\mathrm{NS}(X_2)$, one can compute explicitly the sublattice $E_1^{\perp} := \{D \in \mathrm{NS}(X_2) \simeq L'_{2,2} | DE_1 = 0\}$ and one observes that it is $P(2)$ for a certain degenerate even lattice P. In particular there are no (-2)-classes orthogonal to E_1 in $\mathrm{NS}(X_2)$ and thus the fibration $\varphi_{|E_1|}$ does not admit reducible fibers. The fibration $\varphi_{|E_2|} : X_2 \to \mathbb{P}^1$ is the image of $\varphi_{|E_1|}$ for the automorphism σ, so also $\varphi_{|E_2|}$ has no reducible fibers too.

To conclude the proof it suffices to exhibit the classes of the irreducible rational curves with the required properties. Let us assume that N_i is a class such that $N_i^2 = -2$, $N_iL > 0$, $N_iE_1 = 1$. Then N_i is the class of an effective divisor (by $N_iL > 0$), supported on a (possibly reducible) curve. If N_i is irreducible, then it is a section of $\varphi_{|E_1|}$. Otherwise it should be the sum of a section and some irreducible components of reducible fibers, but there are no reducible fibers in the genus 1 fibration $\varphi_{|E_1|}$. So N_i is a section of $\varphi_{|E_1|}$. All the classes listed below satisfy the conditions $N_i^2 = -2$, $N_iL > 0$, $N_iE_1 = 1$, so they are supported on irreducible rational curves, all sections of E_1:

$N_1 = E_1 + e_2$; $N_2 = E_1 + e_2 + e_3$;
$N_3 = E_1 + e_2 + e_3 + e_4$; $N_4 = E_1 + e_2 + e_3 + e_4 + e_5$;
$N_5 = E_1 + e_2 + e_3 + e_4 + e_5 + e_6$; $N_6 = E_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7$;
$N_7 = E_1 - 2e_1 - 3e_2 - 5e_3 - 4e_4 - 3e_5 - 2e_6 - e_7 - 3e_8$; $N_8 = 3E_1 + e_2 - e_8$.

Since $N_iN_j = 0$ for every $i, j = 1, \dots, 8$, $i \neq j$, the curves N_i are disjoint. Moreover $(\sum_{i=1}^{8} N_i)/2 \in \mathrm{NS}(X_2)$, so $\{N_1, \dots, N_8\}$ is an even set of disjoint rational curves and thus there is a $2:1$ cover branched on these rational curves, i.e. $X_2 \in \mathcal{M}_2$.

The curves N_i $i = 1, \dots 7$ intersect both E_1 and E_2 in 1 point. So the fibrations $\varphi_{|E_1|}$ and $\varphi_{|E_2|}$ share 7 sections.

The divisor $H := 6E_1 - e_1 + 2e_2 - 2e_8$ is a pseudoample divisor of self intersection 4 which is orthogonal to all the N_i's. So $N(X_2) \simeq M_{2,2}$.

The class $N_8'' := 3E_1 - 2e_1 + e_2 - e_8$ is a section of $\varphi_{|E_2|}$ and a 5 section of $\varphi_{|E_1|}$. The class $\left(\sum_{i=1}^{7} N_i + N_8''\right)/2 \in \mathrm{NS}(X_2)$, so $\{N_1, N_2, N_3, N_4, N_5, N_6, N_7, N_8''\}$ is an even set of disjoint rational curves. □

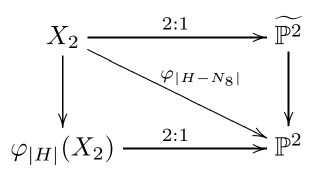
The explicit knowledge of the change of bases from $\{E_1, e_1, \dots e_8\}$ to $\{H, N_1, \dots, N_7, \sum_{i=1}^{8} N_i/2\}$ given in the proof of Proposition 4.12 allows one to obtain some interesting geometric characterizations of the K3 surfaces with $\mathrm{NS}(X_2) \simeq L'_{2,2}$. Indeed, let S be a K3 surface of Picard number 9 and which satisfies one of the following conditions:

- S admits an elliptic fibration $\mathcal{E} : S \to \mathbb{P}^1$ without reducible fibers and admitting 8 disjoint sections, $P_1, \dots, P_8$, such that $(\sum_i P_i)/2 \in \mathrm{NS}(S)$.

- S admits an elliptic fibration $\mathcal{E} : S \to \mathbb{P}^1$ without reducible fibers with zero section O. The Mordell–Weil group of $\mathcal{E}$ is generated by 7 sections, $P_1, \ldots, P_7$, such that $\{O, P_1, \ldots, P_6\}$ are mutually disjoint and P_7 intersects the zero section in 12 points and the other sections P_i, $i = 1, \ldots, 6$ in 6 points.
- S admits two elliptic fibrations $\mathcal{E}$ and $\mathcal{F}$ with class of the fiber E and F respectively such that $EF = 2$. Let us assume that there are 7 orthogonal rational curves such that 6 are sections of both the fibrations, the seventh is section of one fibration and a 5-section for the others.

Then S satisfies also the other conditions, it admits a symplectic involution switching $\mathcal{E}$ and $\mathcal{F}$ and S is a Nikulin surface. In particular $\mathrm{NS}(X_2) \simeq L'_{2,2} \simeq M_{2,2}$. Indeed, any of the above set of data of fibrations and sections is enough to exhibit the lattice $L'_{2,2}$ as the Néron–Severi group of X_2, as it follows by the proof of Proposition 4.12.

The map $\varphi_{|H|} : X_2 \to \mathbb{P}^3$ exhibits X_2 as a singular quartic in $\mathbb{P}^3$ and its eight nodes are $\varphi_{|H|}(N_i)$ for $i = 1, \ldots, 8$. It is well known that the projection of a nodal quartic from a node gives a model of the same K3 surface as a double cover of $\mathbb{P}^2$ branched on a sextic. In particular, let us consider the projection by the node $\varphi_{|H|}(N_8)$, induced by the linear system $|H - N_8|$. We thus have a $2:1$ map $\varphi_{|H-N_8|} : X_2 \to \mathbb{P}^2$, which contracts the 7 curves N_i to seven nodes of the branch sextic. Hence we obtain the following diagram, where the vertical arrows are contractions of 7 curves and the horizontal arrows are $2:1$ maps:



In particular $\widetilde{\mathbb{P}^2}$ is the blow up of $\mathbb{P}^2$ in the seven points $\varphi_{|H-N_8|}(N_i)$ (which are the singular points of the branch locus of the map $X_2 \to \mathbb{P}^2$) and thus $\widetilde{\mathbb{P}^2}$ is a del Pezzo surface of degree 2. The cover involution of the double cover $X_2 \to \widetilde{\mathbb{P}^2}$ is an involution i, such that $i^*(H - N_8) = H - N_8$, $i^*(N_i) = N_i$, $i = 1, \ldots 7$ and $\iota^*(N_8) = 2H - 3N_8$. One is now able to rewrite the action of i^* on the basis $\{L, e_1 \ldots, e_8\}$ and one finds that $i^* = \iota^*_{dP}$ (where ι^*_{dP} is as in Proposition 4.11). Thus, we the notation of (4.4), one obtains $\widetilde{\mathbb{P}^2} = dP$, $\iota_{dP} = i$ and the map π_{dP} is induced by the projection of $\varphi_{|H|}(X_2)$ from the node $\varphi_{|H|}(N_8)$. The even set $\{N_1, \ldots, N_7, N''_8\}$ is nothing but the image of the even set $\{N_1, \ldots, N_8\}$ for the action of ι_{dP}.

In Section 4.3 we proved that the construction of the $\mathcal{D}_4$ Galois cover of X_d is totally determined by two sets of rational curves in X_d, i.e. the sets $\{R_1, \ldots, R_8\}$ and $\{\overline{N}_1, \ldots, \overline{N}_8\}$. In particular in the case we are now considering, i.e. if $d = 1$, the curves R_i, $i = 1, \ldots, 8$ were already considered in the diagram (4.4) and are mapped by $q_3 : X_1 \to \mathbb{P}^2$ to the eight singular points of the branch sextic $B \cap C_0$. We now describe the curves $\overline{N}_i$, by giving their image as plane curves $q_3(\overline{N_i}) \subset \mathbb{P}^2$.

By construction, $q_3(\overline{N_i}) = \pi(N_i)$, where $\pi : X_2 \dashrightarrow \mathbb{P}^2$ is the rational $2^2 : 1$ map given in (4.4).

Proposition 4.13. *For each $i = 1, \ldots, 7$, the curve $\pi(N_i) \subset \mathbb{P}^2$ is a bitangent line to the quartic B. The curve $\pi(N_8)$ is a rational irreducible sextic $D \subset \mathbb{P}^2$ which is tangent to $B \cup C_0$ in all their intersection points. The curves $\pi(N_i)$, $i = 1, \ldots, 8$ split in X_1, the orbit of N_i with respect to $\langle \iota_Q, \iota_{dP} \rangle$ consists of two rational curves if $i = 1, \ldots, 7$ and of the four curves if $i = 8$.*

Proof. The surface dP is a degree 2 del Pezzo surface, and then it is naturally endowed with an involution i, which is the cover involution of the $2 : 1$ map $dP \to \mathbb{P}^2$ (see [DO, Chaptes VII, Section 4] for details on del Pezzo surfaces of degree 2 and its involution). The double cover $q_2 : dP \to \mathbb{P}^2$ is branched on $B \subset \mathbb{P}^2$. Since dP is a del Pezzo surface of degree 2, there is a set of 7 disjoint (-1)-curves on dP, denoted by p_i, $i = 1, \ldots, 7$ and such that $\beta_{dP} : dP \to \mathbb{P}^2$ is a contraction of these (-1)-curves. The plane curves $q_2(p_i) \subset \mathbb{P}^2$ are 7 lines which are bitangent to B, and each of them splits in the double cover into two rational curves p_i, $i(p_i)$, $i = 1, \ldots, 7$. So we have the following commutative diagram:

$$\begin{array}{ccccc} X_2 & \xrightarrow{2:1} & dP & \xrightarrow[q_2]{2:1} & \mathbb{P}^2 \supset B \cup C_0 \\ \downarrow{\scriptstyle \beta_{X_2}} & \searrow{\scriptstyle \varphi_{|H-N_8|}} & \downarrow{\scriptstyle \beta_{dP}} & & \\ \varphi_{|H|}(X_2) & \xrightarrow[2:1]{} & \mathbb{P}^2 & & \end{array} \qquad (4.5)$$

where β_{X_2} contracts the curves N_i, $i = 1, \ldots, 8$. For $i = 1, \ldots, 7$, one has $\varphi_{|H-N_8|}(N_i) = \beta_{dP}(p_i)$. Each of the 7 lines $q_2(p_i) \subset \mathbb{P}^2$ is bitangent to B, and intersects C_0 transversally. So $q_2(p_i)$ does not split in the double cover $q_1 : Q \to \mathbb{P}^2$, which is branched on C_0. In particular, $q_1^{-1}(q_2(p_i)) = q_1^{-1}(\pi(N_i))$ is an irreducible smooth rational curve for $i = 1, \ldots, 7$. So, for each $i = 1, \ldots, 7$, $\pi^{-1}(\pi_{dP}(p_i)) \subset X_2$ consists of a pair of rational curves, switched by ι_Q, preserved by ι_{dP} and then switched by $\sigma = \iota_Q \circ \iota_{dP}$. This can also be checked directly on the classes of the curves N_i by using Propositions 4.11 and 4.12, indeed $\iota_Q^*(N_i) = N_i$ and $\iota_{dP}(N_i) = \sigma(N_i) \neq N_i$ for $i = 1, \ldots, 7$. Since for each $i \in \{1, \ldots, 7\}$ the curve N_i is a section of both the elliptic fibrations $|E_1|$ and $|E_2|$ on X_2, the curve $q_1(N_i) \subset Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$ is a curve of bidegree $(1, 1)$ if $i = 1, \ldots 7$.

It remains to describe the curve $N_8 \subset X_2$. The orbit of N_8 is given by the four classes $N_8 = 3E_1 + e_2 - e_8$, $N_8' := \sigma(N_8) = 3E_1 - 3e_1 - e_2 + e_8$; $N_8'' = \iota_{dP}(N_8) = 3E_1 - 2e_1 + e_2 - e_8$, $N_8''' := \sigma(\iota_{dP}(N_8)) = 3E_1 - e_1 - e_2 + e_8$. Since $N_8 + N_8' + N_8'' + N_8''' \simeq 12E_1 - 6e_1 \simeq 6L$ the curve $\pi(N_8)$ is a sextic C_8 in $\mathbb{P}^2$, which splits in all the double covers Q, dP and X_2 of $\mathbb{P}^2$. The sextic C_8 is a rational curve (since it is the image of rational curves) and thus has 10 nodes. Moreover, since C_8 splits in the double covers, $C_8 \cap C_0$ consists of 6 points with multiplicity two and $C_8 \cap B$ consists of 12 points with multiplicity two. The inverse image of C_8 in Q consists of two rational curves, one of bidegree $(1, 5)$ which is the common image of N_8 and N_8''' and one of bidegree $(5, 1)$ which is the common image of N_8' and N_8''. The bidegrees of these curves are obtained by the fact that N_8 is a section of E_1 and a 5-section of E_2, and N_8' is a section of E_1 and a 5-section of E_2. The inverse image $q_3^{-1}(C_8)$ in X_1 consists of two rational curves, one is $\pi_2(N_8) = \pi_2(N_8')$ and it is the curve denoted by $\overline{N_8}$ in Section 4.3.2, the other is $\pi_2(N_8'') = \pi_2(N_8''')$. □

Remark 4.14. As in the proof of Proposition 4.13 one is able to determine the image of the curves in the linear systems $|E_i|$ under the map π. The orbit of E_1 for $\langle \iota_Q^*, \iota_{dP}^* \rangle$ is $\{E_1, E_2\}$, thus we have two elliptic fibrations which are switched by σ and by ι_{dP} but each of them is preserved by ι_Q. Since $E_1 + E_2 = L$, a curve $F_1 \in |E_1|$ is mapped to a line f_1 in $\mathbb{P}^2$. Moreover, for a general F_1, $q_1^{-1}(f_1)$ is the union of the two curves $\pi_Q(F_1)$ and $\pi_Q(\sigma(F_1))$. Hence the line f_1 is tangent to the conic C_0 (which is the branch locus of $q_1 : Q \to \mathbb{P}^2$). The line f_1 does not splits for the covers q_2 and q_3 and in particular the class $(q_3)_*(E_1)$ induces an elliptic fibration on X_1. So $q_3^{-1}(f_1)$ is a genus 1 curve. This implies that $q \cap B$ consists of 4 disjoint points (which are the branch points of the $2:1$ cover $q_3^{-1}(f_1) \to f_1$). Hence the 1-dimensional linear system of genus 1 curve in $|E_1|$ is mapped by π to the 1-dimensional linear system of lines tangent to the conic C_0. The same holds true for the 1-dimensional linear system $|E_2|$, since $\sigma^*(E_1) = E_2$.

By definition the $2:1$ map $q_2 : dP \to \mathbb{P}^2$ is the anticanonical map, and then, denoted by h the class of a line in $\mathbb{P}^2$, $q_2^*(h) = -K_{dP}$. So $q_2^{-1}(f_1)$ is a genus 1 curve in the anticanonical system, with the special property that it intersects the curve $q_2^{-1}(C_0)$ with even multiplicity in each of their intersection points. Hence the curve $q_2^{-1}(f_1) \subset dP$ splits in the double cover $\pi_{dP} : X_2 \to dP$. With the notation of (4.5), the curve $\beta_{dP}(q_2^{-1}(f_1))$ is a plane cubic tangent to $\beta_{dP}(q_2^{-1}(C_0))$.

Until now our point of view was to consider X_2 as a surface with a symplectic automorphism σ and to determine its structure as Nikulin surface, but one can consider the reverse problem: given a Nikulin surface with Néron–Severi group $M_{2,2}$, it has a very natural model as quartic in $\mathbb{P}^3$ with 8 nodes. To reconstruct the structure of this surface as double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ admitting a symplectic involution one has to identify the two elliptic fibrations $\varphi_{|E_1|} : X_2 \to \mathbb{P}^1$ and $\varphi_{|E_2|} : X_2 \to \mathbb{P}^1$. We gave a change of basis from $\{E_1, e_1, \ldots, e_8\}$ to $\{H, N_1, \ldots, N_7, \sum_{i=1}^8 N_1/2\}$ in proof of Proposition 4.12. Its inverse allows us to find the class of E_1 in terms of the classes H and N_i, $i = 1, \ldots, 8$, in particular $E_1 = H - (\sum_{i=1}^8 N_i)/2$. The curves in this linear system are mapped to cubics by the linear system $|H - N_8|$. So, given a curve $F_1 \in |E_1|$, $c := \varphi_{|H-N_8|}(F_1)$ is a cubic curve in $\mathbb{P}^2$ and $\varphi^{-1}_{|H-N_8|}(c)$ consists of two curves, whose linear systems are $E_1 = H - (\sum_{i=1}^8 N_i)/2$ and $E_2 = 2H - (\sum_{i=1}^8 N_i)/2 - 2N_8$ respectively. Their sum exhibits S as double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ admitting the required symplectic involution.

In [vGS, Section 3.7] an equation of X_2 is given, starting from a description of a K3 surface X_4 such that $\mathrm{NS}(X_4) = L'_{4,2}$. The surface X_4 is given as complete intersections of three quadrics in $\mathbb{P}^5_{(y_0:y_1:x_0:\ldots:x_3)}$ of the form

$$\left\{ \begin{array}{l} y_0^2 = Q_1(x_0 : x_1 : x_2 : x_3) \\ y_1^2 = Q_2(x_0 : x_1 : x_2 : x_3) \\ y_0 y_1 = Q_3(x_0 : x_1 : x_2 : x_3) \end{array} \right. \tag{4.6}$$

Each complete intersection with equation (4.6) admits a symplectic involution induced by the projective transformation

$$(y_0 : y_1 : x_0 : x_1 : x_2 : x_3) \mapsto (-y_0 : -y_1 : x_0 : x_1 : x_2 : x_3).$$

As shown in [vGS, Section 3.7], a singular model of the quotient surface is given by

$$Q_1(x_0 : x_1 : x_2 : x_3) Q_2(x_0 : x_1 : x_2 : x_3) = Q_3^2(x_0 : x_1 : x_2 : x_3) \subset \mathbb{P}^3_{(x_0:x_1:x_2:x_3)}. \tag{4.7}$$

By Proposition 4.5, the smooth model of the quotient surface (4.7) has Néron–Severi group isometric to $M_{2,2}$, i.e. it is the surface $X_2 \simeq S$ and the map to $\mathbb{P}^3_{(x_0:x_1:x_2:x_3)}$ is given by the linear system of the pseudo ample polarization H (with the notation of Proposition 4.12).

Let us consider the pencil of quadrics $\mathcal{P}_t := \{Q_1 = tQ_3\} \subset \mathbb{P}^3$. It cuts on X_2 a pencil of curves, whose class is $2H - \sum_{i=1}^{8} N_i$, since all the quadrics in $\mathcal{P}_t$ pass through the 8 points in $Q_1 \cap Q_2 \cap Q_3$, which are the singular points of the surfaces in (4.7). For almost every t, $\mathcal{P}_t$ cuts two genus 1-curves on the surfaces in (4.7): one is the complete intersection $Q_1 \cap Q_2$ (and does not depend on t) the other is $(Q_1 - tQ_3) \cap (Q_3 - tQ_2)$. So, the first curve is a fixed component of the linear system $2H - \sum_{i=1}^{8} N_i$, the latter is a movable curve. The curves $Q_1 \cap Q_2$ and $(Q_1 - tQ_3) \cap (Q_3 - tQ_2)$ intersect transversally in the singular points of the quartic (4.7). So, they have no intersection points in the blow up X_2 of the quartic (4.7) in its singular points. Hence the curves $Q_1 \cap Q_2$ and $(Q_1 - tQ_3) \cap (Q_3 - tQ_2)$ are two fibers of the same fibration $X_2 \to \mathbb{P}^1_t$ and are represented by the same divisor in $\mathrm{NS}(X_2)$. It is necessarily $\left(2H - \sum_{i=1}^{8} N_i\right)/2 = H - \left(\sum_{i=1}^{8} N_i\right)/2$. This is the divisor E_1 considered above so we conclude that if the surface $S \simeq X_2$ is embedded in $\mathbb{P}^3$ as a quartic of the form $Q_1Q_2 = Q_3^2$, then the elliptic fibration E_1 is cut out by $\mathcal{P}_t := \{Q_1 = tQ_3\}$. The elliptic fibration E_2 is the image of E_1 under ι_{dP}, which is the involution induced by the projection from the node $\varphi_{|H|}(N_8)$ of $\varphi_{|H|}(X_2)$.

4.5. **Special 10-dimensional subfamilies of $\mathcal{L}_2$ and $\mathcal{M}_2$.** In Proposition 3.1 we discussed the family $\mathcal{U}_n$ of $(U \oplus \mathbb{M}_n)$-polarized K3 surfaces, proving that it is contained in $\mathcal{L}_n \cap \mathcal{M}_n$ and it has codimension 1 in this space. This holds for every admissible n, so in particular for $n = 2$. Here we reconsider this family, since it also has interesting properties with respect to the components of $\mathcal{M}_2$: it is contained in the common intersection of all the irreducible components of $\mathcal{M}_2$. We discuss the analogous property for the components of $\mathcal{L}_2$, identifying another interesting family of K3 surfaces, which has codimension 1 in each component of $\mathcal{L}_2$. More precisely the aim of this section is to prove the following:

- There exists an irreducible connected 10-dimensional subvariety of the moduli space of K3 surfaces (it is $\mathcal{U}_2$) which is properly contained in all the families of Nikulin surfaces. Moreover all the K3 surfaces in this subvariety also admit a symplectic involution.
- There exists an irreducible connected 10-dimensional subvariety of the moduli space of K3 surfaces which is properly contained in all the families of K3 surfaces admitting a symplectic involution. All the K3 surfaces in this subvariety are also Nikulin surfaces.

Proposition 4.15. *There exists an overlattice of index 2 of $U(2) \oplus N$, denoted by $(U(2) \oplus N)'$, which is isometric to $U \oplus N$ and such that for any $d \in \mathbb{N}_{\geq 1}$, both the lattice $M_{d,2}$ and the lattice $M'_{2d,2}$ are primitively embedded in $(U(2) \oplus N)'$. Hence all the irreducible components of the 11-dimensional families of Nikulin surfaces properly contain the 10-dimensional family $\mathcal{U}_2 = \mathcal{P}(U \oplus N)$.*

Proof. Let us consider the lattice $U(2) \oplus N$. Let u_1 and u_2 be the basis of $U(2)$ such that $u_j^2 = 0$, $j = 1, 2$ and $u_1u_2 = 2$, and let $w_{i,j}$, $i = 1, 2$, $j = 1, 2, 3$, be a set of vectors in N such that $w_{i,h}/2$ are contained in the discriminant group of $N \subset U(2) \oplus N$. Moreover we assume that the discriminant form on $w_{i,j}/2$,

$i=1,2$, $j=1,2,3$ is $u(2)^3$. The vector $v:=(u_1+u_2+w_{1,1}+w_{2,1})/2$ is isotropic in $A_{U(2)\oplus N}$, and the lattice obtained by adding the vector v to $U(2)\oplus N$ is an even overlattice of index 2 of $U(2)\oplus N$. Let us call it $(U(2)\oplus N)'$. The discriminant group of $(U(2)\oplus N)'$ is generated by $(u_1+w_{1,1})/2$, $(u_1+w_{1,2})/2$, $w_{i,j}$, $i=1,2$, $j=2,3$ and its discriminant form is $u(2)^3$. There is a unique, up to isometry, even hyperbolic lattice with rank 10, length 6 and prescribed discriminant form $u(2)^3$. Hence $(U(2)\oplus N)'\simeq U\oplus N$.

To give a primitive embedding of $M_{d,2}\simeq\langle 2d\rangle\oplus N$ in $U\oplus N$ it suffices to give a primitive embedding of $\langle 2d\rangle$ in U, for example the embedding $\begin{pmatrix}1\\d\end{pmatrix}\hookrightarrow U$ is primitive.

To give a primitive embedding of $M'_{2d,2}$ in $(U(2)\oplus N)'$ we consider a primitive embedding of $M_{2d,2}$ in $U(2)\oplus N$, which extends primitively to their overlattices. As above, a primitive embedding of $M_{2d,2}\simeq\langle 4d\rangle\oplus N$ in $U(2)\oplus N$ is induced by a primitive embedding of $\langle 4d\rangle$ in $U(2)$. We fix this embedding to be $\langle u_1+du_2\rangle\hookrightarrow U(2)$. This induces a primitive embedding of $M'_{2d,2}$ in $(U(2)\oplus N)'\simeq U\oplus N$. □

Let S be a K3 surface with $\mathrm{NS}(S)\simeq U\oplus N$. Then S admits an elliptic fibration with 8 reducible fibers of type I_2 and a 2-torsion section. We denote by F the class of the fiber of this fibration, by O the class of the zero section, by t the 2-torsion section and by C_i^j, $i=0,1$, $j=1,\dots,8$ the i-th component of the j-th fiber (with the usual assumption that the 0-component meets the zero section). A basis of $U\oplus N$ is then given by F, $F+O$, C_1^j, $j=1,\dots 7$, $\left(\sum_{j=1}^8 C_1^j\right)/2=2F+O-t$. The translation by a 2-torsion section is a symplectic involution, denoted by σ and classically called van Geemen–Sarti involution. Its action is $F\leftrightarrow F$, $O\leftrightarrow t$, $C_1^j\leftrightarrow C_0^j$. The sublattice of $\mathrm{NS}(S)$ invariant for σ is $\mathrm{NS}(S)^\sigma\simeq\langle F,s+t\rangle\simeq U(2)$. This exhibits the Néron–Severi group $\mathrm{NS}(S)\simeq U\oplus N$ as an overlattice (necessarily of index 2^2) of $U(2)\oplus E_8(-2)$, since $(\mathrm{NS}(S)^\sigma)^\perp\simeq E_8(-2)$. Chosen a positive integer e, the divisor $v:=F-e(s+t)$ has the following properties: $v^2=-4e$, v is invariant and $v^{\perp_{\mathrm{NS}(S)}}\simeq L'_{2e,2}$. In particular the van Geemen–Sarti involution on S induces the symplectic involution whose action on $\mathrm{NS}(S)$ is -1 on $E_8(-2)\hookrightarrow L'_{2e,2}\simeq v^\perp$ and $+1$ on v. Hence the isometry σ^* on $\mathrm{NS}(S)$ extends the isometry associated to the symplectic involution on $L'_{2e,2}$-polarized K3 surfaces, once an embedding $L'_{2e,2}\hookrightarrow(U\oplus N)$ as in the proof of Proposition 4.15 is fixed.

Now we consider the analogous problem on the irreducible components of $\mathcal{L}_n$.

Proposition 4.16. *The 10-dimensional family $\mathcal{P}(U\oplus E_8(-2))$ is properly contained in all the families $\mathcal{P}(L_{e,2})$ and $\mathcal{P}(L'_{2e,2})$.*

The lattice $U\oplus E_8(-2)$ is isometric to the lattice $U(2)\oplus N$, hence all the members of the family $\mathcal{P}(U\oplus E_8(-2))$ are Nikulin surfaces.

Proof. The primitive embedding of $L_{e,2}\simeq\langle 2e\rangle\oplus E_8(-2)$ in $U\oplus E_8(-2)$ is induced by the primitive embedding of $\langle 2e\rangle\simeq\left\langle\begin{pmatrix}1\\e\end{pmatrix}\right\rangle$ in U, as in the proof of Proposition 4.15. We observe that $U\oplus E_8(-2)$ is an overlattice of index 2 of $U(2)\oplus E_8(-2)$. Indeed, similarly to what we did in proof of Proposition 4.15, we consider the basis u_1 and u_2 of $U(2)\subset U(2)\oplus E_8(-2)$ and the vectors $w_{i,j}/2$ $i=1,2$, $j=1,2,3,4$ in $A_{U(2)\oplus E_8(-2)}$ such that the discriminant form on $u_1/2$, $u_2/2$ and $w_{i,j}/2$, $i=1,2$, $j=1,\dots,4$ is $u(2)^5$. By adding $v=(u_1+u_2+w_{1,1}+w_{2,1})/2$ to $U(2)\oplus E_8(-2)$

one obtains an even overlattice $(U(2)\oplus E_8(-2))'$ of index 2 of $U(2)\oplus E_8(-2)$, which is isometric to $U\oplus E_8(-2)$. Hence the primitive embedding of $L'_{2e,2}$ in $(U(2)\oplus E_8(-2))'\simeq U\oplus E_8(-2)$ is induced by a primitive embedding of $\langle 4e\rangle$ in $U(2)$, which is given by $\langle 4e\rangle\simeq\left\langle\begin{pmatrix}1\\e\end{pmatrix}\right\rangle$ in $U(2)$.

Since $L_{e,2}$ and $L'_{2e,2}$ are primitively embedded in $U\oplus E_8(-2)$ and they determine uniquely their orthogonal complement in Λ_{K3}, the families $\mathcal{P}(L_{e,2})$ and $\mathcal{P}(L'_{2e,2})$ properly contain the family $\mathcal{P}(U\oplus E_8(-2))$.

The isometry between the lattices $U\oplus E_8(-2)$ and $U(2)\oplus N$ follows by observing that they are lattices with rank 10, length 8 and the same discriminant form. □

Let $\mathcal{E}_R: R\rightarrow\mathbb{P}^1$ be a rational elliptic surface (i.e. R is a rational surface endowed with an elliptic fibration $\mathcal{E}_R$). It is known that a base change of order 2 on this elliptic fibration branched on two smooth fibers induces an elliptic fibration $\mathcal{E}_S: S\rightarrow\mathbb{P}^1$ on a K3 surface S. If the fibration $\mathcal{E}_R$ has no reducible fibers, then $\mathrm{NS}(S)\simeq U\oplus E_8(-2)$, see e.g. [GSal, Proposition 4.6]. More in general the family of the K3 surfaces obtained by a base change of order 2 on a rational elliptic surface, is the family $\mathcal{P}\left(U\oplus E_8(-2)\right)$, see e.g. [GSal].

Proposition 4.17. *The 10-dimensional family $\mathcal{P}(U\oplus E_8(-2))$ is the family $\mathcal{R}$ of the K3 surfaces obtained by a base change of order two on a rational elliptic fibration $\mathcal{E}_R: R\rightarrow\mathbb{P}^1$. Let S be a general member of $\mathcal{R}$ and let $\mathcal{E}_S$ be the elliptic fibration induced by $\mathcal{E}_R$: $\mathcal{E}_S$ has no reducible fibers and its Mordell–Weil rank is equal to 8. The symplectic involution σ on S preserves $\mathcal{E}_S$. Denoted by $\widetilde{S/\sigma}$ the desingularization of S/σ, $\mathrm{NS}(\widetilde{S/\sigma})\simeq U\oplus D_4\oplus D_4$.*

Proof. We already observed that S is obtained by a base change of order 2 by $\mathcal{E}_R: R\rightarrow\mathbb{P}^1$. Then $\mathcal{E}_S$ admits an involution ι which acts only on the basis of the fibration, and which is the deck involution of the generically 2:1 cover $R\rightarrow S$. The involution ι maps fibers of $\mathcal{E}_S$ to other fibers and in particular preserves the class of the fiber and of the sections, i.e. it acts trivially on the Néron–Severi group. Thus it preserves the elliptic fibration (cf. [GSal, Proposition 4.6]). Moreover, ι preserves exactly two fibers of $\mathcal{E}_S$ (the ramification fibers of the cover $R\rightarrow S$). The elliptic fibration $\mathcal{E}_S$ is preserved also by the elliptic involution ϵ, which preserves the classes of the fiber and of the zero section (i.e. a set of generators of U in $\mathrm{NS}(S)\simeq U\oplus E_8(-2)$). The composition $\sigma:=\iota\circ\epsilon$ is a symplectic involution which acts trivially on $U\hookrightarrow U\oplus E_8(-2)\simeq\mathrm{NS}(S)$ and as $-\mathrm{id}$ on $E_8(-2)\hookrightarrow U\oplus E_8(-2)\simeq\mathrm{NS}(S)$. Thus σ is a symplectic involution whose fixed locus consists of 4 points on each of the two fibers preserved by ι. Hence the elliptic fibration $\mathcal{E}_S: S\rightarrow\mathbb{P}^1$ induces an elliptic fibration on $\widetilde{S/\sigma}$ whose generic fiber is a copy of the two fibers of $\mathcal{E}_S$ switched by σ. The images of the two fibers preserved by ι are two fibers of type I_0^*. The Picard number of $\widetilde{S/\sigma}$ is 10, which is also the rank of the trivial lattice of an elliptic fibration with two fibers of type I_0^*. We conclude that there are no sections of infinite order for the elliptic fibration induced by $\mathcal{E}_S$ on $\widetilde{S/\sigma}$ and that $\mathrm{NS}(\widetilde{S/\sigma})\simeq U\oplus D_4\oplus D_4$. □

We observe that $U\oplus D_4\oplus D_4\not\simeq U\oplus E_8(-2)$ since their discriminant groups are different, so $\mathrm{NS}(S)\not\simeq\mathrm{NS}(\widetilde{S/\sigma})$.

By Proposition 4.16, if S is a K3 surface such that $\mathrm{NS}(S) \simeq U \oplus E_8(-2)$, then it admits a symplectic involution (described in the proof of Proposition 4.17) and it is also $2:1$ cyclically covered by a K3 surface. So it admits a 2-divisible set of rational curves, which we describe here. As observed S is obtained by a base change of order 2 on R. Since R is a rational elliptic surface, it is the blow up of $\mathbb{P}^2$ in nine points which are the base points of a pencil of generically smooth cubics. So S, which is a 2:1 double cover of R branched on two smooth fibers, is a generically $2:1$ cover of $\mathbb{P}^2$ branched in the union of two smooth cubics C_1 and C_2 (see e.g. [GSal, Section 2.2]). The branch locus is singular in the nine points $C_1 \cap C_2$. We denote by H the genus 2 divisor on S such that $\varphi_{|H|}: S \to \mathbb{P}^2$ is this $2:1$ cover of $\mathbb{P}^2$ and by D_i, $i = 0, \ldots 8$, the classes of the rational curves contracted by $\varphi_{|H|}$ to the nine singular points of the branch locus. By construction the fiber of the fibration $\mathcal{E}_S$ is the class of C_1 (and of C_2), i.e. $\left(3H - \sum_{i=0}^{8} D_i\right)/2$. The curves in the linear system $|H - D_0|$ (and in $|H - D_1|$ respectively) on S are mapped to lines of a pencil in $\mathbb{P}^2$, with base point $\varphi_{|H|}(D_0)$ (with base point $\varphi_{|H|}(D_1)$ respectively). Each line in this pencil meets the branch in 4 points (with the exception of $\varphi_{|H|}(D_0)$), and so its inverse image in S is a $2:1$ cover of a rational curve branched in 4 points. So the curves in $|H - D_0|$ (resp. $|H - D_1|$) are genus 1 curves and $|H - D_0|$ and $|H - D_1|$ induce two genus 1 fibrations on S (see [GSal, Proposition 3.8]). Since $(H - D_0)(H - D_1) = 2$, the map $\varphi_{|2H - D_0 - D_1|}$ is a generically $2:1$ map to $\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$ (see [SD]). It contracts the 8 mutually disjoint rational curves D_i, $i = 2, \ldots, 8$, and $H - D_0 - D_1$. The last contracted curve is the pullback of the line through the two points $\varphi_{|H|}(D_0)$ and $\varphi_{|H|}(D_1)$. The $2:1$ map $\varphi_{|H-D_0|} \times \varphi_{|H-D_1|}: S \to \mathbb{P}^1 \times \mathbb{P}^1$ is induced by the $2:1$ map $\varphi_{|H|}: S \to \mathbb{P}^2$ via the birational transformation $\beta: \mathbb{P}^2 \to \mathbb{P}^1 \times \mathbb{P}^1$, which is the blow up of $\mathbb{P}^2$ in the two points $\varphi_{|H|}(D_0)$ and $\varphi_{|H|}(D_1)$ followed by the contraction of the line through these points. So the branch locus of the $2:1$ cover $S \to \mathbb{P}^1 \times \mathbb{P}^1$ splits in the union of two genus 1 curves of bidegree $(2,2)$, which are the images of the two cubics C_1 and C_2 under the birational transformation β. The curves $\beta(C_1)$ and $\beta(C_2)$ intersect in 8 points in $\mathbb{P}^1 \times \mathbb{P}^1$, which are the images of the curves D_i, $i = 2, \ldots, 8$, and $H - D_0 - D_1$. The classes of the pullback of $\beta(C_1)$ and $\beta(C_2)$ on S coincide and each of them is represented by the class $\left(2(H - D_0) + 2(H - D_1) - \sum_{i=2}^{8} D_i - (H - D_0 - D_1)\right)/2$. So the set of curves $\{D_2, \ldots D_8, H - D_0 - D_1\}$ is an even set.

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