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Nonstandard characterisations of tensor products and monads in the theory of ultrafilters

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We use nonstandard methods, based on iterated hyperextensions, to develop applications to Ramsey theory of the theory of monads of ultrafilters. This is performed by studying in detail arbitrary tensor products of ultrafilters, as well as by characterising their combinatorial properties by means of their monads. This extends to arbitrary sets and properties methods previously used to study partition regular Diophantine equations on $\mathbb{N}$. Several applications are described by means of multiple examples.

1 Introduction

It is well known that ultrafilters and nonstandard analysis are closely related: on the one hand, models of nonstandard analysis are characterised, up to isomorphisms, as limit ultrapowers (cf. [9, § 6.4]); on the other hand, the correspondence between elements of a nonstandard extension ${}^*X$ and ultrafilters on X was first observed (in the more general case of filters) by Luxemburg in [31], who introduced the concept of monad of a filter. This correspondence was then used by Puritz, Cherlin and Hirschfeld in [10, 34, 35] to produce new results about the Rudin-Keisler ordering and to characterise several classes of ultrafilters, including P-points and selective ones. Similar ideas were also pursued by Ng and Render in [33] and by Blass in [6].

In [26], we proved a combinatorial characterisation of monads of ultrafilters in $\beta\mathbb{N}$ which made it possible to develop several applications in the study of the partition regularity of Diophantine equations[1] by means of some rather simple algebraic manipulations of hypernatural numbers. The partition regularity of Diophantine equations is a particular instance of the kind of problems that are studied in Ramsey theory, where one wants to understand which monochromatic structures can be found in some piece of arbitrary finite partitions of a given object.

The basic idea behind our nonstandard approach to Ramsey theory is that every set in a ultrafilter $\mathcal{U} \in \beta\mathbb{N}$ satisfies a prescribed property φ if and only if the monad of $\mathcal{U}$ satisfies an appropriate nonstandard version of φ. This idea has been developed in [12, 14, 15, 27–30], and belongs to the family of applications of nonstandard analysis in Ramsey theory, an approach that started with Hirschfeld in [20] and has subsequently been carried on by many authors. As Jin pointed out, nonstandard methods in Ramsey theory are very useful because they can be used to reduce the complexity of the mathematical objects that one needs in a proof, therefore offering a much better intuition, which allows to obtain much simpler (and shorter) proofs.

In [13], Di Nasso surveyed the nonstandard characterisation of ultrafilters on $\mathbb{N}$, proving also several equivalent characterisations of the elements of the monads of tensor products of ultrafilters. This paper can be seen as an extension of such a study, since our main aim is to characterise monads of ultrafilters and tensor products of ultrafilters on arbitrary sets, so to extend the nonstandard methods used for Diophantine equations to more general classes of problems in Ramsey theory. This requires to better understand arbitrary tensor products of ultrafilters, which are a basic important tool to develop such applications (e.g., in [4] tensor products of ultrafilters in S^n, for a semigroup S, are used to obtain polynomial extensions of the Milliken-Taylor theorem). Moreover, it is helpful to characterise the Ramsey-theoretical properties of monads in terms of their combinatorial and algebraic structure for general properties, extending what we already did for Diophantine equations; such an approach could lead to

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[1] Cf. Theorem 5.2 for a definition of this notion.

unexpected applications in other related fields. It turns out that a good nonstandard framework to perform this study is given by iterated nonstandard extensions.

In § 2, we recall the basic definitions and properties of iterated hyperextensions, providing the nonstandard framework that is used to develop the rest of the paper. In § 3, we recall the definition of the monad of an ultrafilter. We also recall some basic properties of these monads, presenting some of their peculiar properties in iterated hyperextensions. In § 4, we consider arbitrary tensor products of ultrafilters, we provide several equivalent characterisations of the elements in their monads and we extend the characterisations to tensor products of arbitrary (finite) length. Finally, in § 5, we present several combinatorial properties of monads of arbitrary ultrafilters. Throughout the paper, several examples are also included to illustrate the use of such a theory in applications, as well as our main ideas.

This paper is self-contained: we only assume the reader to know the basics of ultrafilters and nonstandard analysis, in particular the notions of superstructure, transfer, ultrafilter, enlarging and saturation properties. In any case, a comprehensive reference about ultrafilters and their applications, especially in Ramsey theory, is the monograph [18]. As for nonstandard analysis, many short but rigorous presentations can be found in the literature. We suggest [2], where eight different approaches to nonstandard methods are presented, as well as the introductory book [17], which covers all the nonstandard tools that we need in this paper, except the iterated extensions that we shall discuss in § 2.

2 Iterated hyperextensions

In this paper, we shall adopt the so-called "external" approach to nonstandard analysis, based on superstructure models of nonstandard methods (cf. also [2, § 3]):

Definition 2.1 A superstructure model of nonstandard methods is a triple $\langle \mathbb{V}(X), \mathbb{V}(Y), *\rangle$, where

(1) X is an infinite set, and $Y = {}^*X$;
(2) $\mathbb{N} \in \mathbb{V}(X) \cap \mathbb{V}(Y)$, and $\mathbb{N}$ is properly included in ${}^*\mathbb{N}$;
(3) $\mathbb{V}(X), \mathbb{V}(Y)$ are the superstructures on X, Y respectively;
(4) $* : \mathbb{V}(X) \to \mathbb{V}(Y)$ is a star map, viz. it satisfies the transfer principle.

We say that $\langle \mathbb{V}(X), \mathbb{V}(Y), *\rangle$ is a single superstructure model of nonstandard methods when $X = Y$.

From now on, we shall use only single superstructure models of nonstandard methods. This is not restrictive: as proven in [1], every superstructure model is isomorphic to a single superstructure one.

The existence of saturated single superstructure models of nonstandard methods can be proven in different ways: we refer to [3], where single superstructure models are constructed by means of the so-called Alpha Theory, and to the nonstandard set theory ${}^*\mathsf{ZFC}$ introduced by Di Nasso in [11], where the enlarging map $*$ is defined for every set of the universe. Similar ideas have been studied, in the context of iterated ultrapowers, by Kunen, Hrbáček, Lessmann, and O'Donovan in [21,22,25]. A clear presentation of iterated ultrapowers can also be found in [9, § 6.5].

The main peculiarity of single superstructure models of nonstandard methods is that they allow to iterate the $*$-map. This allows to simplify certain proofs: e.g., in [12] the structure ${}^{**}\mathbb{N}$, obtained by iterating twice the star map applied to $\mathbb{N}$, is used to give a rather short proof of Ramsey Theorem.

Iterated hyperextensions have already been studied in previous publications (e.g., [12, 13, 26–30]). In this Section, we shall recall only the main definitions and properties that will be used in the rest of the paper.

Definition 2.2 We define by induction the family $\langle H_n \mid n \in \mathbb{N}\rangle$ of functions $H_n : \mathbb{V}(X) \to \mathbb{V}(X)$ by setting $H_0 = \mathrm{id}$ and, for every $n \geq 0$, $H_{n+1} = * \circ H_n$.

Let Y be a set in $\mathbb{V}(X)$. Notice that $H_1({}^*Y) := {}^{**}Y$ is a nonstandard extension of both Y and ${}^*Y$. Intuitively, this extension resembles the extension from Y to ${}^*Y$. E.g., if $Y = \mathbb{N}$, the fact that ${}^*\mathbb{N}$ is an end extension of $\mathbb{N}$, viz. that

$$\forall \eta \in {}^*\mathbb{N}\setminus\mathbb{N}\, \forall n \in \mathbb{N} (\eta > n),$$

can be transferred to

$$\forall \eta \in {}^{**}\mathbb{N}\setminus{}^{*}\mathbb{N}\, \forall n \in {}^{*}\mathbb{N}(\eta > n),$$

which is the formula expressing that ${}^{**}\mathbb{N}$ is an end extension of ${}^{*}\mathbb{N}$.

However, not all the basic properties of the extension from Y to ${}^{*}Y$ holds also for the extension ${}^{**}Y$ of ${}^{*}Y$: e.g., the fact that ${}^{*}A = A$ for every finite subset of $\mathbb{N}$ (as usual, we identify every number $n \in \mathbb{N}$ with ${}^{*}n$) does not hold true for ${}^{*}\mathbb{N}$. Just observe that if $\alpha \in {}^{*}\mathbb{N}\setminus\mathbb{N}$, then, by transfer, ${}^{*}\alpha \in {}^{**}\mathbb{N}\setminus{}^{*}\mathbb{N}$, hence ${}^{*}\{\alpha\} = \{{}^{*}\alpha\} \neq \{\alpha\}$.

In any case, we have the following result, which is a trivial consequence of the composition properties of elementary embeddings:

Theorem 2.3 *For every positive natural number n, $\langle \mathbb{V}(X), \mathbb{V}(X), H_n\rangle$ is a single superstructure model of nonstandard methods.*

In certain cases, as we shall show in § 4, it is helpful to consider the following extension of X:

Let $\langle \mathbb{V}(X), \mathbb{V}(X), *\rangle$ be a superstructure model of nonstandard methods. We call the sequence $\langle H_n(X) \mid n < \omega\rangle$ an *ω-hyperextension of X*, and write ${}^{\bullet}X = \bigcup_{n\in\mathbb{N}} H_n(X)$ for the union of all hyperextensions $H_n(X)$. Since $\langle H_n(X) \mid n < \omega\rangle$ is an elementary chain of extensions, we have that ${}^{\bullet}X$ is a nonstandard extension of X.

Proposition 2.4 *Let $n \in \mathbb{N}$ and let κ be a cardinal number. Then the implications* $(1) \Rightarrow (2) \Rightarrow (3)$ *hold, where*

(1) $\langle \mathbb{V}(X), \mathbb{V}(X), *\rangle$ has the κ-enlarging property;
(2) $\langle \mathbb{V}(X), \mathbb{V}(X), H_n\rangle$ has the κ-enlarging property;
(3) $\langle \mathbb{V}(X), \mathbb{V}(X), \bullet\rangle$ has the κ-enlarging property.

Proof. $(1) \Rightarrow (2)$: By contradiction: let $n = \min\{m \in \mathbb{N} \mid \langle \mathbb{V}(X), \mathbb{V}(X), H_m\rangle$ does not have the κ-enlarging property$\}$. Let $\{A_i\}_{i<\kappa}$ be a family with the finite intersection property such that $\bigcap_{i<\kappa} H_n(A_i) = \emptyset$. Let us consider the family $\{H_{n-1}(A_i)\}_{i<\kappa}$. By transfer, this family has the finite intersection property, hence by our hypothesis we have that $\bigcap_{i<\kappa} {}^{*}H_{n-1}(A_i) \neq \emptyset$, which is absurd as ${}^{*}H_{n-1}(A_i) = H_n(A_i)$.

$(2) \Rightarrow (3)$: This is trivial, as $H_i(A) \subseteq {}^{\bullet}A$ for every set A in X. □

However, let us notice that the previous result does not hold, in general, if we substitute enlarging with saturation. In fact, ${}^{\bullet}\mathbb{N}$ has cofinality $\aleph_0$ (which is in contradiction with κ-saturation properties for $\kappa > \aleph_0$, as the cofinality is always at least as great as the cardinal saturation), since a countable right unbounded sequence in ${}^{\bullet}\mathbb{N}$ can be constructed by choosing, for every natural number n, an hypernatural number α_n in $H_{n+1}(\mathbb{N})\setminus H_n(\mathbb{N})$.

3 Monads

In the following, we shall use the symbol $\star$ to denote generic nonstandard extensions (which could be $*$, H_n, or $\bullet$), reserving to $*$ and $\bullet$ the meanings given in § 2. We hope that this will increase the readability of the paper.[2]

Monads of filters were first introduced by Luxemburg in [31]. In the past few years, monads of ultrafilters on $\mathbb{N}$ have been used to prove many results in combinatorial number theory, especially in the context of the partition regularity of equations (cf., e.g., [12–15, 26–30]). However, it seems that to extend the range of applications of these methods, a deeper study of monads in a wider generality is needed. Our aim in this section is to start such a study. We shall adopt the framework of iterated nonstandard hyperextensions, since they provide a simpler setting for the study of monads, as we are going to show.

Let Y be a set in $\mathbb{V}(X)$ and let $\langle \mathbb{V}(X), \mathbb{V}(X), *\rangle$ be a single superstructure model of nonstandard methods. Let $\mathcal{U}$ be an ultrafilter on Y. For every $n \in \mathbb{N}$ we let $\mu_n(\mathcal{U}) := \bigcap_{A\in\mathcal{U}} H_n(A)$ (with the agreement that $\mu(\mathcal{U}) := \mu_1(\mathcal{U})$) and $\mu_\infty(\mathcal{U}) := \bigcap_{A\in\mathcal{U}} {}^{\bullet}A$. Finally, when we consider a generic extension $\langle \mathbb{V}(X), \mathbb{V}(X), \star\rangle$ we shall write $\mu_\star(\mathcal{U}) := \bigcap_{A\in\mathcal{U}} {}^{\star}A$. Elements of $\mu_\star(\mathcal{U})$ will be called *generators* of $\mathcal{U}$.

In general, monads can be empty if the extensions are not sufficiently enlarged. However, we have the following result:

[2] At least, we hope that this will not decrease the readability of the paper.

Theorem 3.1 *Let Y be a set in $\mathbb{V}(X)$. Then for every $\alpha \in {}^{\star}Y$ the set $\mathfrak{U}_{\alpha}^{(Y,\star)} := \{A \subseteq Y \mid \alpha \in {}^{\star}A\}$ is a ultrafilter on Y. Moreover, if the extension $\star : Y \to {}^{\star}Y$ has the $|\wp(Y)|^{+}$-enlarging property, then $\mu_{\star}(\mathcal{U}) \neq \varnothing$ for every $\mathcal{U} \in \beta Y$.*

Proof. That $\mathfrak{U}_{\alpha}^{(Y,\star)}$ is a ultrafilter is straightforward. The second claim follows as every ultrafilter $\mathcal{U}$ on Y is a family with the finite intersection property and cardinality $|\wp(Y)|$, and the $|\wp(Y)|^{+}$-enlarging property hence entails that $\mu_{\star}(\mathcal{U}) \neq \varnothing$. □

When $\star = H_n$ we shall write $\mathfrak{U}_{\alpha}^{(Y,n)}$. Moreover, when $n = 1$ and there is no danger of confusion regarding Y (e.g., because it has already been specified that $\mathfrak{U}_{\alpha}^{(Y,n)} \in \beta Y$), we shall simply write $\mathfrak{U}_{\alpha}$.

Monads can be used to identify every ultrafilter with the trace of a principal one on a higher level: in fact, if $\alpha \in {}^{\star}Y$, then $\alpha \in \mu(\mathcal{U})$ if and only if $\mathcal{U} = \mathrm{tr}_Y(P(\alpha))$, where $P(\alpha) := \{A \subseteq {}^{\star}Y \mid \alpha \in A\}$ is the principal ultrafilter generated by α on ${}^{\star}Y$ and, for every ultrafilter $\mathcal{V}$ on ${}^{\star}Y$, we set $\mathrm{tr}_Y(\mathcal{V}) := \{A \subseteq Y \mid {}^{\star}A \in \mathcal{V}\}$.

For $\alpha, \beta \in {}^{\star}Y$ we shall write $\alpha \sim_{(Y,\star)} \beta$ if $\mathfrak{U}_{\alpha}^{(Y,\star)} = \mathfrak{U}_{\beta}^{(Y,\star)}$. When $\star = H_n$ we shall just write $\sim_{(Y,n)}$, simplified to $\sim_Y$ when $n = 1$.

A remark is in order: in previous papers, the equivalence relation $\sim_{(Y,\star)}$ was denoted by $\sim_{\mathcal{U}}$ or $\underset{u}{\sim}$ (cf., e.g., [12, 13]). However, here we decided to use the much heavier notation $\sim_{(Y,\star)}$ to highlight that this equivalence relation depends both on the set on which we are constructing the ultrafilters and, in general, on the extension that we choose. However, as we already said, we shall use much simpler notations whenever there is no danger of confusion, e.g., when $\star = *$.

To better explain what we mean, let $\alpha \neq \beta \in {}^{*}\mathbb{N}$ be such that $\alpha \sim_{(\mathbb{N},1)} \beta$. Then (as we shall prove in Proposition 3.4) ${}^{*}\alpha \sim_{(\mathbb{N},2)} {}^{*}\beta$. However, ${}^{*}\alpha \nsim_{({}^{*}\mathbb{N},1)} {}^{*}\beta$! In fact, the ultrafilter generated by ${}^{*}\alpha$ on ${}^{*}\mathbb{N}$ is $\{A \subseteq {}^{*}\mathbb{N} \mid {}^{*}\alpha \in {}^{*}A\} = \{A \subseteq {}^{*}\mathbb{N} \mid \alpha \in A\} = P(\alpha)$, and analogously the ultrafilter generated by ${}^{*}\beta$ on ${}^{*}\mathbb{N}$ is $P(\beta)$, and $P(\alpha) \neq P(\beta)$ since $\alpha \neq \beta$.

When we work in ω-hyperextensions, it is useful to study the relationships between sets of generators of the same ultrafilter in different extensions. To do this, we introduce the following concepts: Let $\langle \mathbb{V}(X), \mathbb{V}(X), * \rangle$ be a single superstructure model of nonstandard methods and let Y be a set in $\mathbb{V}(X)$. We say that Y is *coherent* if $Y \subseteq {}^{*}Y$. We say that Y is *completely coherent* if A is coherent for every $A \subseteq Y$. Notice that if Y is coherent, then $H_n(Y) \subseteq H_m(Y)$ for every $n \leq m$.

Example 3.2 The set $\mathbb{N}$ is completely coherent, as we identify every $n \in \mathbb{N}$ with ${}^{*}n$. However, if $\alpha \in {}^{*}\mathbb{N} \backslash \mathbb{N}$, then $\{\alpha\}$ is not coherent, since ${}^{*}\{\alpha\} = \{{}^{*}\alpha\}$. Finally, $\mathbb{N} \cup \{\alpha\}$ is coherent but not completely coherent.

Theorem 3.3 *Let $\langle \mathbb{V}(X), \mathbb{V}(X), * \rangle$ be a single superstructure model of nonstandard methods and let Y be a set in $\mathbb{V}(X)$. The following are equivalent:*

(1) Y is completely coherent;
(2) for all $y \in Y$, $y = {}^{*}y$.

Proof. For "(1)⇒(2)", let $y \in Y$. As Y is completely coherent, we have that $\{y\} \subseteq {}^{*}\{y\} = \{{}^{*}y\}$, hence $y = {}^{*}y$. For "(2)⇒(1)", let $A \subseteq Y$. Then $A \subseteq {}^{*}A$ since, for every $a \in A$, we have that $a = {}^{*}a \in {}^{*}A$. □

Proposition 3.4 *Let Y be a set in $\mathbb{V}(X)$. For every ultrafilter $\mathcal{U}$ on Y and every $n \in \mathbb{N}$, we have that ${}^{*}(\mu_n(\mathcal{U})) \subseteq \mu_{n+1}(\mathcal{U})$. Moreover, if Y is completely coherent, then the following properties hold:*

(1) $\mu_n(\mathcal{U}) \subseteq \mu_{n+1}(\mathcal{U})$;
(2) $\mu_{\infty}(\mathcal{U}) = \bigcup_{n\in\mathbb{N}} \mu_n(\mathcal{U})$;
(3) $\alpha \in \mu_{\infty}(\mathcal{U}) \Leftrightarrow {}^{*}\alpha \in \mu_{\infty}(\mathcal{U})$.

Proof. For every $A \in \mathcal{U}$ $\mu_n(\mathcal{U}) \subseteq H_n(A)$. Hence, by transfer, ${}^{*}(\mu_n(\mathcal{U})) \subseteq {}^{*}H_n(A) = H_{n+1}(A)$, and so ${}^{*}(\mu_n(\mathcal{U})) \subseteq \bigcap_{A\in\mathcal{U}} H_{n+1}(A) = \mu_{n+1}(\mathcal{U})$. Let us now assume that Y is completely coherent.

For (1), just notice that, for every $A \subseteq Y$, since A is coherent we have that $H_n(A) \subseteq H_{n+1}(A)$ for every $n \in \mathbb{N}$.

We shall now prove (2) and start with $\mu_{\infty}(\mathcal{U}) \supseteq \bigcup_{n\in\mathbb{N}} \mu_n(\mathcal{U})$: $\alpha \in \bigcup_{n\in\mathbb{N}} \mu_n(\mathcal{U})$ if and only if there exists $n \in \mathbb{N}$ such that $\alpha \in \mu_n(\mathcal{U}) = \bigcap_{A\in\mathcal{U}} H_n(A)$. As $H_n(A) \subseteq {}^{\bullet}A$ for every $A \in \mathcal{U}$, in particular we have that $\alpha \in \bigcap_{A\in\mathcal{U}} {}^{\bullet}A = \mu_{\infty}(\mathcal{U})$.

For the inclusion $\mu_\infty(\mathcal{U}) \subseteq \bigcup_{n\in\mathbb{N}} \mu_n(\mathcal{U})$, let $\alpha \in \bigcap_{A\in\mathcal{U}} {}^\bullet A$. Let $A \in \mathcal{U}$; then there exists $n \in \mathbb{N}$ such that $\alpha \in H_n(A)$. In particular, $\alpha \in H_n(Y)$. We claim that, for every other set $B \in \mathcal{U}$, $\alpha \in {}^\bullet B$ if and only if $\alpha \in H_n(B)$. In fact, assume towards a contradiction that there exists $B \in \mathcal{U}$ such that $\alpha \in {}^\bullet B \setminus H_n(B)$. In particular, we find $m > n$ such that $\alpha \in H_m(B)$. As $\alpha \notin H_n(B)$, however, we have that $\alpha \in H_n(B^c) \subseteq H_m(B^c)$ since Y is completely coherent. Therefore $\alpha \in H_m(B) \cap H_m(B^c) = \emptyset$, which is absurd. Thus this shows that $\alpha \in \bigcap_{B\in\mathcal{U}} H_n(B) = \mu_n(\mathcal{U})$.

Finally, for (3), we observe that $\alpha \in \mu_\infty(\mathcal{U})$ if and only if there is an $n \in \mathbb{N}$ such that $\alpha \in \mu_n(\mathcal{U})$. This is equivalent to the existence of an $n \in \mathbb{N}$ such that ${}^*\alpha \in \mu_{n+1}(\mathcal{U})$ which is equivalent to ${}^*\alpha \in \mu_\infty(\mathcal{U})$. □

In particular, when $Y \in \mathbb{V}(X)$ is completely coherent, for every $\alpha \in H_n(Y)$ one has that $\mathfrak{U}_\alpha^{(Y,n)} = \mathfrak{U}_\alpha^{(Y,n+1)}$, hence in such cases the notation $\mathfrak{U}_\alpha \in \beta Y$ can be used without danger of confusion.

It is well-known that functions $f : Y_1 \to Y_2$ can be lifted to $\overline{f} : \beta Y_1 \to \beta Y_2$ by defining $\overline{f}(\mathcal{U}) := \{A \subseteq Y_2 \mid f^{-1}(A) \in \mathcal{U}\}$ for every $\mathcal{U} \in \beta Y_1$. As one might expect, the monad of $\overline{f}(\mathcal{U})$ can be expressed in terms of the monad of $\mathcal{U}$, as shown in the following Theorem, that generalised similar results proven by Di Nasso in [13, Propositions 11.2.4, 11.2.10, & Theorem 11.2.7] in the context of $\mathbb{N}$, whose proof can be easily adapted to the present case:

Theorem 3.5 *Let $A, B \in \mathbb{V}(X)$ be sets, let $f : A \to B$ and let $\mathcal{U} \in \beta A$. Then the following facts hold:*

(1) If $A = B$ and $\alpha, {}^\star f(\alpha) \in \mu(\mathcal{U})$, then $\alpha = {}^\star f(\alpha)$;
(2) $\mu(\overline{f}(\mathcal{U})) = {}^\star f(\mu(\mathcal{U}))$.

4 Arbitrary tensor products and pairs

The notion of tensor product of ultrafilters is fundamental to study both several basic properties of ultrafilters (like the Rudin-Keisler order, the algebraical properties of $\beta\mathbb{N}$ and the topological properties of ultrapowers), as well as to develop many applications (e.g., to the theory of finite embeddabilities). At the best of our knowledge, most results in the literature cover the case of tensor products $\mathcal{U}_1 \otimes \mathcal{U}_2$ of ultrafilters on the same set S; in this Section, our goal is to extend these results in two directions. The first is to consider tensor products $\mathcal{U}_1 \otimes \mathcal{U}_2$ of ultrafilters on different sets, viz. where $\mathcal{U}_1 \in \beta S_1, \mathcal{U}_2 \in \beta S_2$. The second is to consider arbitrary finite tensor products of ultrafilters $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_k$.

4.1 Tensor products

A key notion in ultrafilters theory is that of tensor product of ultrafilters:

Definition 4.1 Let S_1, S_2 be sets in $\mathbb{V}(X)$ and let $\mathcal{U}_1 \in \beta S_1$, $\mathcal{U}_2 \in \beta S_2$. The tensor product of $\mathcal{U}_1$ and $\mathcal{U}_2$ is the ultrafilter on $S_1 \times S_2$ defined by $\mathcal{U}_1 \otimes \mathcal{U}_2 := \{A \subseteq S_1 \times S_2 \mid \{s_1 \in S_1 \mid \{s_2 \in S_2 \mid (s_1, s_2) \in A\} \in \mathcal{U}_2\} \in \mathcal{U}_1\}$. Moreover, we set $\beta S_1 \otimes \beta S_2 = \{\mathcal{U}_1 \otimes \mathcal{U}_2 \mid \mathcal{U}_1 \in \beta S_1, \mathcal{U}_2 \in \beta S_2\}$.

Tensor products are closely related with the notion of double limits along ultrafilters and Rudin-Keisler order (cf. [18, § 11.1] for the case $S_1 = S_2 = S$, S discrete space). However, we shall not adopt this topological point of view here. For us, tensor products are important because of the role they play in many applications, especially in combinatorial number theory.

Example 4.2 Let $(S, \cdot)$ be a semigroup. Let $f : S^2 \to S$ be $f(a, b) = a \cdot b$. Then $f(\mathcal{U}, \mathcal{V}) = \overline{f}(\mathcal{U} \otimes \mathcal{V}) = \mathcal{U} \odot \mathcal{V}$ for every $\mathcal{U}, \mathcal{V} \in \beta S$, where $\odot$ denotes the extension to βS of the operation $\cdot$ (cf., e.g., [18, § 4.1]), explicitly, $\mathcal{U} \odot \mathcal{V} := \{A \mid \{s \in S \mid \{t \in S \mid s \cdot t \in A\} \in \mathcal{V}\} \in \mathcal{U}\}$.

Example 4.3 Let $\mathcal{F} = \mathbb{N}^\mathbb{N}$ and let $H : \mathbb{N} \times \mathcal{F} \to \mathbb{N}$ be the function $H(n, f) = f(n)$. Then $H(\mathcal{U}, \mathcal{V}) = \mathcal{U} \otimes_\mathcal{F} \mathcal{V} := \{A \subseteq \mathbb{N} \mid \{n \in \mathbb{N} \mid \{f \in \mathcal{F} \mid f(n) \in A\} \in \mathcal{V}\} \in \mathcal{U}\}$.[3]

The first trivial observation about tensor products is the following:

Lemma 4.4 *If S_1 or S_2 is finite, then $\beta(S_1 \otimes S_2) = \beta S_1 \otimes \beta S_2$.*

[3] These kind of ultrafilters are important when studying combinatorial properties of $\mathbb{N}$ by means of the so-called $\mathcal{F}$-finite embeddabilities; cf. [30].

P r o o f . Let us prove the case with S_1 finite, as the other case is similar. Let $S_1 = \{a_1, \dots, a_n\}$. Let $\mathcal{U} \in \beta(S_1 \otimes S_2)$. For $i = 1, \dots, n$ let $A_i = \{(a_i, s_2) \mid s_2 \in S_2\}$. As $S_1 \times S_2 = \bigcup_{i \le n} A_i$, there exists a unique $i \le n$ such that $A_i \in \mathcal{U}$. Let[4] $\mathfrak{U}_{a_i} \in \beta S_1$ be the principal ultrafilter on a_i and let $\mathcal{U}_2 := \{A \subseteq S_2 \mid \{(a_i, s) \mid s \in A\} \in \mathcal{U}\} \in \beta S_2$. Then, by construction, we have that for every $A \subseteq S_1 \times S_2$

$$\begin{aligned} A \in \mathfrak{U}_{a_i} \otimes \mathcal{U}_2 &\Leftrightarrow \{s_1 \in S_1 \mid \{s_2 \in S_2 \mid (s_1, s_2) \in A\} \in \mathcal{U}_2\} \in \mathfrak{U}_{a_i} \\ &\Leftrightarrow \{s_2 \in S_2 \mid (a_i, s_2) \in A\} \in \mathcal{U}_2 \\ &\Leftrightarrow A \in \mathcal{U}, \end{aligned}$$

hence $\mathcal{U} = \mathfrak{U}_{a_i} \otimes \mathcal{U}_2 \in \beta S_1 \otimes \beta S_2$. □

To develop a deeper study of tensor products, our goal in this section is to give several characterisations of monads of tensor products of ultrafilters. The first question that we want to answer is: what is the relationship between $\mu_\star(\mathcal{U}) \times \mu_\star(\mathcal{V})$ and $\mu_\star(\mathcal{U} \otimes \mathcal{V})$? Let us start with a definition:

Definition 4.5 Let $\mathcal{U}_1 \in \beta S_1, \mathcal{U}_2 \in \beta S_2$. We denote by $\mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$ the filter on $S_1 \times S_2$ given by $\mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2) = \{B \in S_1 \times S_2 \mid \exists A_1 \in \mathcal{U}_1, A_2 \in \mathcal{U}_2 (A_1 \times A_2 \subseteq B)\}$.

In general, $\mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$ is just a filter and not an ultrafilter; its relationship with $\mu_\star(\mathcal{U}_1)$, $\mu_\star(\mathcal{U}_2)$ and $\mu_\star(\mathcal{U}_1 \otimes \mathcal{U}_2)$ is clarified in the following proposition.

Proposition 4.6 *Let $\mathcal{U}_1 \in \beta S_1$ and $\mathcal{U}_2 \in \beta S_2$. Then $\mu_\star(\mathcal{U}_1) \times \mu_\star(\mathcal{U}_2) = \bigcup_{\mathcal{W} \in U(\mathcal{U}_1 \times \mathcal{U}_2)} \mu_\star(\mathcal{W}) \supseteq \mu_\star(\mathcal{U}_1 \otimes \mathcal{U}_2)$, where $U(\mathcal{U}_1 \times \mathcal{U}_2) = \{\mathcal{W} \in \beta(S_1 \times S_2) \mid \mathcal{W} \supseteq \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)\}$.*

P r o o f . First of all, let us notice that $\mathcal{U}_1 \otimes \mathcal{U}_2 \in U(\mathcal{U}_1 \times \mathcal{U}_2)$, as clearly for all $A \in \mathcal{U}_1$ and $B \in \mathcal{U}_2$, we have $A \times B \in \mathcal{U}_1 \otimes \mathcal{U}_2$. Therefore we are left to prove that $\mu_\star(\mathcal{U}_1) \times \mu_\star(\mathcal{U}_2) = \bigcup_{\mathcal{W} \in U(\mathcal{U}_1 \times \mathcal{U}_2)} \mu_\star(\mathcal{W})$.

"$\subseteq$": Let $\alpha \in \mu_\star(\mathcal{U}_1)$, $\beta \in \mu_\star(\mathcal{U}_2)$. Let $\mathcal{W} = \mathfrak{U}_{(\alpha,\beta)}^{(S_1 \times S_2, \star)}$. Then $\mathcal{W} \in U(\mathcal{U}_1 \times \mathcal{U}_2)$ as, for every $A \in \mathcal{U}_1$, $B \in \mathcal{U}_2$ $(\alpha, \beta) \in {}^\star A \times {}^\star B = {}^\star(A \times B)$.

"$\supseteq$": Let $\mathcal{W} \in U(\mathcal{U}_1 \times \mathcal{U}_2)$. Let $(\alpha, \beta) \in \mu_\star(\mathcal{W})$. For every $A \in \mathcal{U}_1$, $B \in \mathcal{U}_2$ $A \times B \in \mathcal{W}$, hence $(\alpha, \beta) \in \bigcap_{A \in \mathcal{U}_1, B \in \mathcal{U}_2} {}^\star(A \times B) = \mu_\star(\mathcal{U}_1) \times \mu_\star(\mathcal{U}_2)$. □

Corollary 4.7 *For all $\alpha \in {}^\star S_1$, $\beta \in {}^\star S_2$, $\mathfrak{U}_{(\alpha,\beta)}^{(S_1 \times S_2, \star)} \supseteq \mathcal{F}(\mathfrak{U}_\gamma^{(S_1,\star)} \times \mathfrak{U}_\delta^{(S_2,\star)}) \Leftrightarrow \alpha \sim_{(S_1,\star)} \gamma, \beta \sim_{(S_2,\star)} \delta$.*

In particular, as a consequence of Proposition 4.6 we have that the map $\otimes : \beta S_1 \times \beta S_2 \to \beta(S_1 \times S_2)$ is injective but not surjective, in general. Moreover, as it is known, this entails that $\mu_\star(\mathcal{U}_1 \otimes \mathcal{U}_2) = \mu_\star(\mathcal{U}_1) \times \mu_\star(\mathcal{U}_2)$ if and only if $\mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2) = \mathcal{U}_1 \otimes \mathcal{U}_2$ (cf. also [5, Chapter 1]),

To characterise when such a situation happens, let us recall the following definitions:

Definition 4.8 Let $\mathcal{U} \in \beta S$ and let κ be a cardinal number. The norm of $\mathcal{U}$ is the cardinal $\|\mathcal{U}\| = \min_{A \in \mathcal{U}} |A|$. Moreover, $\mathcal{U}$ is κ^+-complete if for every family $\{A_i \mid i \in I\} \subseteq \mathcal{U}$ with cardinality $|I| < \kappa^+$ we have $\bigcap_{i \in I} A_i \in \mathcal{U}$.

The problem of characterising ultrafilters $\mathcal{U}_1, \mathcal{U}_2$ such that $\mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$ was already considered, and solved, by Blass in [5, § 3]. We recall (and reprove for completeness) his characterisation in the following theorem:

Theorem 4.9 *Let $\mathcal{U}_1 \in \beta S_1, \mathcal{U}_2 \in \beta S_2$. The following facts are equivalent:*

(1) $\mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$;
(2) for all $A \in \mathcal{U}_1$ and all $\{B_i \mid i \in A\} \subseteq \mathcal{U}_2$ there is a $C \in \mathcal{U}_1$ such that $\bigcap_{c \in C \cap A} B_c \in \mathcal{U}_2$.

P r o o f . "(1) $\Rightarrow$ (2)": Let $A \in \mathcal{U}_1$ and let $\{B_i \mid i \in A\} \subseteq \mathcal{U}_2$. Define $S := \bigcup_{i \in A} \{i\} \times B_i \subseteq S_1 \times S_2$. As $A \in \mathcal{U}_1$, by definition of tensor product we have that $S \in \mathcal{U}_1 \otimes \mathcal{U}_2$. But then, as $\mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$, there exist $D_1 \in \mathcal{U}_1$, $D_2 \in \mathcal{U}_2$ such that $D_1 \times D_2 \subseteq S$. Hence for every $c \in C := A \cap D_1$ we have that $D_2 \subseteq B_c$, so, in particular, $D_2 \subseteq \bigcap_{c \in C} B_c$, hence $\bigcap_{c \in C} B_c \in \mathcal{U}_2$.

[4] As S_1 is finite, it is completely coherent, hence we are allowed to use the simplified notations here.

"(2) $\Rightarrow$ (1)": Let $S \in \mathcal{U}_1 \otimes \mathcal{U}_2$. By definition, $A := \{s_1 \in S_1 \mid \{s_2 \in S_2 \mid (s_1, s_2) \in S\} \in \mathcal{U}_2\} \in \mathcal{U}_1$. For every $i \in A$ let $B_i = \{s_2 \in S_2 \mid (i, s_2) \in S\} \in \mathcal{U}_2$. Then $\{B_i \mid i \in A\} \subseteq \mathcal{U}_2$, hence by hypothesis there exists $C \in \mathcal{U}_1$ so that $B := \bigcap_{c \in C \cap A} B_c \in \mathcal{U}_2$. But then by construction $(C \cap A) \times B \subseteq S$, and so $S \in \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$. □

Example 4.10 Let $S_1 = S_2 = \mathbb{N}$. Let $\mathcal{U}_1, \mathcal{U}_2 \in \beta\mathbb{N}$. Then the following are equivalent:[5]

(1) $\mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$;
(2) there is an $i \in \{0, 1\}$ such that $\mathcal{U}_i$ is principal.

In fact, the direction "(2) $\Rightarrow$ (1)" is straightforward. On the other hand, let us assume (1), and assume that $\mathcal{U}_1$ and $\mathcal{U}_2$ are not principal. Let A be any set in $\mathcal{U}_1$ and, for every $a \in A$, let $B_a = \{n \in \mathbb{N} \mid n > a\} \in \mathcal{U}_2$. By Theorem 4.9 there exists $C \in \mathcal{U}_1$ such that $\bigcap_{a \in A \cap C} B_a \in \mathcal{U}_2$. And this cannot be, as $A \cap C$ is infinite (since $\mathcal{U}_1$ is not principal) and hence $\bigcap_{a \in A \cap C} B_a = \varnothing$.

We want to generalise the previous example and solve the following two problems:

(1) For which $\mathcal{U}_1 \in \beta S_1$ do we have that $\mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$ for all $\mathcal{U}_2 \in \beta S_2$?
(2) For which $\mathcal{U}_2 \in \beta S_2$ do we have that $\mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$ for all $\mathcal{U}_1 \in \beta S_1$?

Although it is not evident from Theorem 4.9, the property $\mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$ is symmetic in $\mathcal{U}_1, \mathcal{U}_2$, in the sense that $\mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$ if and only if $\mathcal{U}_2 \otimes \mathcal{U}_1 = \mathcal{F}(\mathcal{U}_2 \times \mathcal{U}_1)$ (this basic observation was pointed out to us by Blass, cf. also [5, Corollary 9, § 3]). Therefore problems (1) and (2) are equivalent: a solution of the first entails directly a solution of the second. And the second problem is rather simple to solve:

Theorem 4.11 *Let $\mathcal{U}_2 \in \beta S_2$ and let $\kappa = |S_1|$. The following are equivalent:*

(1) $\mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$ for all $\mathcal{U}_1 \in \beta S_1$;
(2) $\mathcal{U}_2$ is κ^+-complete.

Proof. "(1) $\Rightarrow$ (2)": Without loss of generality, we can assume that $S_1 = \kappa$. Towards a contradiction, let us suppose that $\mathcal{U}_2$ is not κ^+-complete. Let $\lambda := \min\{\mu \mid \exists F \subseteq \mathcal{U}(|F| = \mu \wedge \bigcap_{B \in F} B \notin \mathcal{U}_2)\}$. As $\mathcal{U}_2$ is not κ^+-complete, $\lambda \le \kappa$. Let $\{B_i \mid i < \lambda\} \subseteq \mathcal{U}_2$ be such that $\bigcap_{i<\lambda} B_i \notin \mathcal{U}_2$. For every $i < \lambda$ we set $D_i = \bigcap_{j \le i} B_j$. By the definition of λ we have that every $D_i \in \mathcal{U}_2$, as it is an intersection of fewer than λ elements of $\mathcal{U}_2$, and clearly $D_i \supseteq D_j$ for every $i \le j$. Now let $\mathcal{U}_1 \in \beta\lambda \subseteq \beta\kappa$ be an ultrafilter that extends the filter of co-initial sets on λ, so that every set $C \in \mathcal{U}_1$ is cofinal in λ. By hypothesis, $\mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$. If we set $A = \lambda$, by Theorem 4.9 we deduce that there is a $C \in \mathcal{U}_1$ such that $\bigcap_{c \in C} D_c \in \mathcal{U}_2$. But C is cofinal in λ and $\{D_i\}_{i<\lambda}$ is a decreasing sequence, so $\bigcap_{c \in C} D_c = \bigcap_{i<\lambda} D_i = \bigcap_{i<\lambda} B_i \notin \mathcal{U}_2$, which is absurd.

"(2) $\Rightarrow$ (1)": Let $A \in \mathcal{U}_1$ and let $\{B_i \mid i \in A\} \subseteq \mathcal{U}_2$. Then, as $|A| \le \kappa$, by κ^+-completeness $\bigcap_{i \in A} B_i \in \mathcal{U}_2$, hence the condition of Theorem 4.9 is fulfilled by setting $C = A$. □

Let us call a ultrafilter $\mathcal{U}_2 \in \beta S_2$ a *factorising ultrafilter* if $\mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$ for all $\mathcal{U}_1 \in \beta S_1$. If $\lambda = |S_2|$, from the previous theorem we deduce that

(i) if $\lambda \le \kappa$, then the only factorising ultrafilters are the principal ones;
(ii) if $\lambda > \kappa$, then nonprincipal factorising ultrafilter $\mathcal{U}_2 \in \beta S_2$ might or might not exist: e.g., if $\lambda = \kappa^+$, then such a nonprincipal factorising ultrafilter exists if and only if κ^+ is measurable (the existence of such ultrafilters is consistent with ZF but not with ZFC).

4.2 Tensor pairs

As discussed above, tensor products are very important to develop several applications of ultraproducts theory. Therefore, if one wants to follow a nonstandard perspective, it becomes fundamental to characterise tensor products in terms of their monads. To do so, we introduce the following definition:

[5] This is a well-known fact: cf., e.g., [13, Remark 11.5.5].

Definition 4.12 Let $(\alpha, \beta) \in {}^{\star}(S_1 \times S_2)$. We say that (α, β) is a $\star$-tensor pair if $\mathfrak{U}_{(\alpha,\beta)}^{(S_1\times S_2,\star)} = \mathfrak{U}_{\alpha}^{(S_1,\star)} \otimes \mathfrak{U}_{\beta}^{(S_2,\star)}$.

Following the usual convention for the case $\star = *$, we call $*$-tensor pairs just *tensor pairs*.

As, in general, $\beta S_1 \otimes \beta S_2 \subsetneq \beta(S_1 \times S_2)$, not all pairs $(\alpha, \beta) \in {}^{\star}(S_1 \times S_2)$ are $\star$-tensor pairs. When $S_1 = S_2 = \mathbb{N}$, $\star = *$ many properties of tensor pairs have been proven (in the context of non-iterated hyperextensions) by Di Nasso in [13] (cf. also [26]). We plan to show that most of these characterisations can be extended (sometimes in an even more general form) to arbitrary tensor pairs, with some simplifications given by the possibility of iterating the star map.

The main advantage when working in iterated hyperextensions is that they allow to write down easily generators of tensor products:

Theorem 4.13 *Let $n, m \in \mathbb{N}$, let $S_1, S_2 \in \mathbb{V}(X)$ be sets with S_1 completely coherent, let $\mathcal{U} \in \beta S_1$, $\mathcal{V} \in \beta S_2$ and let $\alpha \in \mu_n(\mathcal{U})$, $\beta \in \mu_m(\mathcal{V})$. Then $(\alpha, H_n(\beta)) \in \mu_{n+m}(\mathcal{U} \otimes \mathcal{V})$.*

Proof. Let $A \subseteq S_1 \times S_2$. Then

$$A \in \mathfrak{U}_{\alpha}^{(S_1,n)} \otimes \mathfrak{U}_{\beta}^{(S_2,m)} \Leftrightarrow \{s_1 \in S_1 \mid \{s_2 \in S_2 \mid (s_1, s_2) \in A\} \in \mathfrak{U}_{\beta}^{(S_2,m)}\} \in \mathfrak{U}_{\alpha}^{(S_1,n)}.$$

Now, by definition, $\{s_2 \in S_2 \mid (s_1, s_2) \in A\} \in \mathfrak{U}_{\beta}^{(S_2,m)}$ if and only if $\beta \in H_m(\{s_2 \in S_2 \mid (s_1, s_2) \in A\}) = \{s_2 \in H_m(S_2) \mid (s_1, s_2) \in H_m(A)\}$, as $H_m(s_1) = s_1$ for every $s_1 \in S_1$, since S_1 is completely coherent. Then $\{s_1 \in S_1 \mid \{s_2 \in S_2 \mid (s_1, s_2) \in A\} \in \mathfrak{U}_{\beta}^{(S_2,m)}\} \in \mathfrak{U}_{\alpha}^{(S_1,n)}$ if and only if $\{s_1 \in S_1 \mid (s_1, \beta) \in H_m(A)\} \in \mathfrak{U}_{\alpha}^{(S_1,n)}$ if and only if $(\alpha, H_n(\beta)) \in H_{n+m}(A)$. □

Corollary 4.14 *Let $(S, \cdot) \in \mathbb{V}(X)$ be a semigroup. Assume that S is completely coherent. Let $\mathcal{U}, \mathcal{V} \in \beta S$, let $\alpha \in \mu_n(\mathcal{U})$, $\beta \in \mu_m(\mathcal{V})$. Then $\alpha \cdot H_n(\beta) \in \mu_{n+m}(\mathcal{U} \odot \mathcal{V})$.*

Proof. $\mathcal{U} \odot \mathcal{V} = \overline{f}(\mathcal{U} \otimes \mathcal{V})$, where $f : S^2 \to S$ is the function that maps every pair $(a, b) \in S^2$ in $a \cdot b$. Hence we can conclude by applying Theorem 3.5.(2) as, by Theorem 4.13, $(\alpha, H_n(\beta)) \in \mu_{n+m}(\mathcal{U} \otimes \mathcal{V})$. □

In Theorem 3.1, we showed that ${}^{\bullet}Y/\sim_Y \cong \beta Y$, provided that the extension $\bullet$ is sufficiently enlarged. Theorem 4.13 allows to refine this result when $Y = S$ is a semigroup, provided that $\bullet$ is sufficiently enlarged and S is completely coherent: if we let $\widetilde{\odot} : {}^{\bullet}S^2 \to {}^{\bullet}S$ be the map such that, $\alpha \,\widetilde{\odot}\, \beta = \alpha \cdot H_{h(\alpha)}(\beta)$ for every $\alpha, \beta \in {}^{\bullet}S$, where $h(\alpha) = \min\{n \in \mathbb{N} \mid \alpha \in H_n(S)\}$, we get that $(\beta S, \widetilde{\odot})$ and $({}^{\bullet}S, \odot)/\sim_Y$ are isomorphic as semigroups.[6]

To simplify the notation, from now on we shall assume that $(\alpha, \beta) \in {}^{*}(S_1 \times S_2)$, as the characterisation for the general cases where $\alpha \in H_n(S_1)$, $\beta \in H_m(S_2)$ can be analogously deduced from Theorem 4.13. Therefore we shall simply talk about tensor pairs (although in Theorem 4.15 we shall still use the heavier notations to avoid as much as possible the danger of confusion).

In the case of non-iterated hyperextensions, tensor pairs have been studied mostly for the product $\mathbb{N} \times \mathbb{N}$, in order to characterise certain properties of ultraproducts. In this case, a characterisation was given by Puritz in [35], where he proved that (α, β) is a tensor pair if and only if $\alpha < \mathrm{er}(\beta)$, where $\mathrm{er}(\beta) = \{{}^{*}f(\beta) \mid f : \mathbb{N} \to \mathbb{N}, {}^{*}f(\beta) \in {}^{*}\mathbb{N}\backslash\mathbb{N}\}$.

In ${}^{**}\mathbb{N}$, it is very simple to see that Puritz's characterisation is not symmetric, in the sense that the condition $\beta > \mathrm{er}(\alpha)$ does not entail that (α, β) is a tensor pair. In fact, let α be a prime number in ${}^{*}\mathbb{N}\backslash\mathbb{N}$ and let $\beta = ({}^{*}\alpha)^{\alpha}$. Then $\beta > \mathrm{er}(\alpha)$, as $({}^{**}f)(\alpha) = ({}^{*}f)(\alpha) \in {}^{*}\mathbb{N}$ for every $f \in \mathbb{N}^{\mathbb{N}}$, whilst $\beta \in {}^{**}\mathbb{N}\backslash{}^{*}\mathbb{N}$. However, if f is the function such that, if $n = p_1^{a_1} \cdot \dots \cdot p_h^{a_h} \in \mathbb{N}$ is the factorisation of n as product of distinct prime numbers, then $f(n) = \max_{j=1,\dots,h} a_j$, we have that $({}^{**}f)(\beta) = \alpha$, hence by Puritz's characterisation (α, β) is not a tensor pair.

The main problem in extending Puritz's characterisation to arbitrary sets is that it uses the order relation on $\mathbb{N}$, whilst arbitrary products of sets might not be ordered. In [13], several equivalent characterisation of Puritz's condition for tensor pairs in $\mathbb{N} \times \mathbb{N}$ were given by Di Nasso. In the following theorem, we adapt these characterisations to arbitrary tensor pairs on $S_1 \times S_2$, and we also introduce two new characterisations in terms of preservations of tensor pairs via standard functions that will be useful to find several characterisations in our examples. Although the characterisations are given for $\star = *$, the general formulation can be easily obtained by minor modifications in the proofs.

[6] This result could be improved to a topological isomorphism by introducing the star topology on ${}^{\bullet}S$, but we shall not consider this topological approach here.

Theorem 4.15 *Let $S_1, S_2 \in \mathbb{V}(X)$ be sets, with S_1 completely coherent, and let $(\alpha_1, \alpha_2) \in {}^*(S_1 \times S_2)$. The following are equivalent:*

(1) (α_1, α_2) *is a tensor pair;*
(2) $\mathfrak{U}^{(S_1\times S_2,1)}_{(\alpha_1,\alpha_2)} = \mathfrak{U}^{(S_1\times S_2,2)}_{(\alpha_1,{}^*\alpha_2)}$;
(3) $\mathfrak{U}^{(S_1\times S_2,1)}_{(\alpha_1,\alpha_2)} \subseteq \mathfrak{U}^{(S_1\times S_2,2)}_{(\alpha_1,{}^*\alpha_2)}$;
(4) $\mathfrak{U}^{(S_1\times S_2,2)}_{(\alpha_1,{}^*\alpha_2)} \subseteq \mathfrak{U}^{(S_1\times S_2,1)}_{(\alpha_1,\alpha_2)}$;
(5) *for all* $F : S_1 \to \wp(S_2)$, *if* $\alpha_2 \in {}^*F(\alpha_1)$, *then* ${}^*F(\alpha_1) \in {}^*\mathfrak{U}^{(S_2,1)}_{\alpha_2}$;
(6) *for all* $F : S_1 \to \wp(S_2)$, *if* ${}^*F(\alpha_1) \in {}^*\mathfrak{U}^{(S_2,1)}_{\alpha_2}$, *then* $\alpha_2 \in {}^*F(\alpha_1)$;
(7) *for all sets* S_3, S_4 *with* S_3 *completely coherent and all functions* $f : S_1 \to S_3,\ g : S_2 \to S_4$ $({}^*f(\alpha_1), {}^*g(\alpha_2))$ *is a tensor pair;*
(8) *there exist sets* S_3, S_4 *with* S_3 *completely coherent, bijections* $f : S_1 \to S_3,\ g : S_2 \to S_4$ *such that* $({}^*f(\alpha_1), {}^*g(\alpha_2))$ *is a tensor pair;*
(9) *for all* $A \subseteq S_1 \times S_2$, *if* $(s_1, \alpha_2) \in {}^*A$ *for all* $s_1 \in S_1$, *then* $(\alpha_1, \alpha_2) \in {}^*A$;
(10) *for all* $A \subseteq S_1 \times S_2$, *if* $(\alpha_1, \alpha_2) \in {}^*A$, *then there exists* $s_1 \in S_1$ *such that* $(s_1, \alpha_2) \in {}^*A$.

Proof. "(1) ⇔ (2)": This is an immediate consequence of Theorem 4.13.

"(2) ⇔ (3) ⇔ (4)": These equivalences are trivial, as inclusion between ultrafilters on the same set means equality.

"(1) ⇒ (5)": Let $F : S_1 \to \wp(S_2)$ be given. Let $A = \{(s_1, s_2) \mid s_2 \in F(s_1)\}$. Assume that $\alpha_2 \in {}^*F(\alpha_1)$. In particular, this means that $(\alpha_1, \alpha_2) \in {}^*A$. By the hypothesis, it follows that $A \in \mathfrak{U}^{(S_1,1)}_{\alpha_1} \otimes \mathfrak{U}^{(S_2,1)}_{\alpha_2}$, viz. $\{s_1 \in S_1 \mid \{s_2 \in S_2 \mid (s_1, s_2) \in A\} \in \mathfrak{U}^{(S_2,1)}_{\alpha_2}\} \in \mathfrak{U}^{(S_1,1)}_{\alpha_1}$, which means that

$$\begin{aligned}\alpha_1 &\in {}^*\{s_1 \in S_1 \mid \{s_2 \in S_2 \mid (s_1, s_2) \in A\} \in \mathfrak{U}^{(S_2,1)}_{\alpha_2}\}\\ &= \{\sigma_1 \in {}^*S_1 \mid \{\sigma_2 \in {}^*S_2 \mid (\sigma_1, \sigma_2) \in {}^*A\} \in {}^*\mathfrak{U}^{(S_2,1)}_{\alpha_2}\}.\end{aligned}$$

In particular, then $\{\sigma_2 \in {}^*S_2 \mid (\alpha_1, \sigma_2) \in {}^*A\} \in {}^*\mathfrak{U}^{(S_2,1)}_{\alpha_2}$, and we conclude as $\{\sigma_2 \in {}^*S_2 \mid (\alpha_1, \sigma_2) \in {}^*A\} = {}^*F(\alpha_1)$.

"(5) ⇒ (1)": Let $A \in \mathfrak{U}^{(S_1\times S_2,1)}_{(\alpha_1,\alpha_2)}$, i.e., $(\alpha_1, \alpha_2) \in {}^*A$. Let $F : S_1 \to \wp(S_2)$ be given by $F(s_1) := \{s_2 \in S_2 \mid (s_1, s_2) \in A\}$ for all $s_1 \in S_1$. Then $\alpha_2 \in {}^*F(\alpha_1)$ as $(\alpha_1, \alpha_2) \in {}^*A$. By the hypothesis, ${}^*F(\alpha_1) \in {}^*\mathfrak{U}^{(S_2,1)}_{\alpha_2}$. But ${}^*F(\alpha_1) = \{\sigma_2 \in {}^*S_2 \mid (\alpha_1, \sigma_2) \in {}^*A\}$, therefore $\alpha_1 \in {}^*\{s_1 \in S_1 \mid \{s_2 \in S_2 \mid (s_1, s_2) \in A\} \in \mathfrak{U}^{(S_2,1)}_{\alpha_2}\}$, which means that $A \in \mathfrak{U}^{(S_1,1)}_{\alpha_1} \otimes \mathfrak{U}^{(S_2,1)}_{\alpha_2}$.

"(1) ⇔ (6)" can be proven similarly to the equivalence "(1) ⇔ (5)".

"(5) ⇒ (7)": We prove this by contradiction: assume that there exist sets S_3, S_4, with S_3 completely coherent, and functions $f : S_1 \to S_3,\ g : S_2 \to S_4$ such that $({}^*f(\alpha_1), {}^*g(\alpha_2))$ is not a tensor pair. By hypothesis, there exists $G : S_3 \to \wp(S_4)$ such that ${}^*g(\alpha_2) \in {}^*G({}^*f(\alpha_1))$ but ${}^*G({}^*f(\alpha_1)) \notin {}^*\mathfrak{U}^{(S_4,1)}_{{}^*g(\alpha_2)}$. Let $F : S_1 \to \wp(S_2)$ be defined by $F(s_1) := g^{-1}(G(f(s_1)))$ for all $s_1 \in S_1$. As ${}^*g(\alpha_2) \in {}^*G({}^*f(\alpha_1))$, we have that $\alpha_2 \in {}^*F(\alpha_1)$. Then, by hypothesis ${}^*F(\alpha_1) \in {}^*\mathfrak{U}^{(S_2,1)}_{\alpha_2}$, viz. $\alpha_1 \in {}^*\{s_1 \in S_1 \mid F(s_1) \in \mathfrak{U}^{(S_2,1)}_{\alpha_2}\} = {}^*\{s_1 \in S_1 \mid g^{-1}(G(f(s_1))) \in \mathfrak{U}^{(S_2,1)}_{\alpha_2}\} = {}^*\{s_1 \in S_1 \mid G(f(s_1)) \in \mathfrak{U}^{(S_4,1)}_{{}^*g(\alpha_2)}\} = \{\sigma_1 \in {}^*S_1 \mid {}^*G({}^*f(\sigma_1)) \in {}^*\mathfrak{U}^{(S_4,1)}_{{}^*g(\alpha_2)}\}$, therefore ${}^*G({}^*f(\alpha_1)) \in {}^*\mathfrak{U}^{(S_4,1)}_{{}^*g(\alpha_2)}$, which is absurd.

"(7) ⇒ (8)": Just set $S_1 = S_3$, $S_2 = S_4$, $f = \mathrm{id}_{S_1}$, $g = \mathrm{id}_{S_2}$.

"(8) ⇒ (5)": We prove this by contradiction: assume that there exists $F : S_1 \to \wp(S_2)$ such that $\alpha_2 \in {}^*F(\alpha_1)$ but ${}^*F(\alpha_1) \notin {}^*\mathfrak{U}^{(S_2,1)}_{\alpha_2}$. Let $G : S_3 \to \wp(S_4)$ be defined as follows: $G(s_3) = g(F(f^{-1}(s_3)))$. Notice that this definition makes sense as f is bijective, hence invertible. Then the bijectivity of f implies that ${}^*G({}^*f(\alpha_1)) = {}^*g({}^*F({}^*f^{-1}({}^*f(\alpha_1)))) = {}^*g({}^*F(\alpha_1))$, hence ${}^*g(\alpha_2) \in {}^*G({}^*f(\alpha_1))$ as $\alpha_2 \in {}^*F(\alpha_1)$. By our hypothesis, it follows that ${}^*G({}^*f(\alpha_1)) \in {}^*\mathfrak{U}^{(S_4,1)}_{{}^*g(\alpha_2)}$, viz. ${}^*f(\alpha_1) \in \{\sigma_3 \in {}^*S_3 \mid {}^*G(\sigma_3) \in {}^*\mathfrak{U}^{(S_4,1)}_{{}^*g(\alpha_2)}\} = {}^*\{s_3 \in S_3 \mid G(s_3) \in \mathfrak{U}^{(S_4,1)}_{{}^*g(\alpha_2)}\} = {}^*\{s_3 \in S_3 \mid G(s_3) \in g(\mathfrak{U}^{(S_2,1)}_{\alpha_2})\} = {}^*\{s_3 \in S_3 \mid g(F(f^{-1}(s_3))) \in g(\mathfrak{U}^{(S_2,1)}_{\alpha_2})\} = {}^*\{s_3 \in S_3 \mid F(f^{-1}(s_3)) \in \mathfrak{U}^{(S_2,1)}_{\alpha_2}\}$, as g is bijective. Hence ${}^*F(\alpha_1) \in {}^*\mathfrak{U}^{(S_2,1)}_{\alpha_2}$, which is absurd.

"(9) ⇔ (10)" is trivial.

"(1) $\Rightarrow$ (9)": Let $A \subseteq S_1 \times S_2$. If $(s_1, \alpha_2) \in {}^*A$ for every $s_1 \in S_1$, then $\{s_1 \in S_1 \mid \{s_2 \in S_2 \mid (s_1, s_2) \in A\} \in \mathfrak{U}_{\alpha_2}^{(S_2,1)}\} = S_1 \in \mathfrak{U}_{\alpha_1}^{(S_1,1)}$, hence $(\alpha_1, \alpha_2) \in {}^*A$ by hypothesis.

"(9) $\Rightarrow$ (1)": Let $A \subseteq S_1 \times S_2$, with $A \in \mathfrak{U}_{(\alpha_1,\alpha_2)}^{(S_1 \times S_2,1)}$. For all $s_1 \in S_1$ let $A_{s_1} := \{s_2 \in S_2 \mid (s_1, s_2) \in A\}$, and let $F : S_1 \to \wp(S_2)$ be defined as follows for every $s_1 \in S_1$:

$$F(s_1) = \begin{cases} A_{s_1}, & \text{if } A_{s_1} \in \mathfrak{U}_{\alpha_2}^{(S_2,1)}; \\ A_{s_1}^{\mathrm{c}}, & \text{otherwise.} \end{cases}$$

By construction, for all $s_1 \in S_1$, we have that $F(s_1) \in \mathfrak{U}_{\alpha_2}^{(S_2,1)}$. In particular, if $X_F = \{(s_1, s_2) \mid s_2 \in F(s_1)\}$ then for all $s_1 \in S_1$, we have that $(s_1, \alpha_2) \in {}^*X_F$. By our hypothesis, it follows that $(\alpha_1, \alpha_2) \in {}^*X_F$, i.e., $\alpha_2 \in {}^*F(\alpha_1)$. Therefore there are two cases:

Case 1: ${}^*F(\alpha_1) = \{\sigma_2 \in {}^*S_2 \mid (\alpha_1, \sigma_2) \in {}^*A\}$. In this case, ${}^*F(\alpha_1) \in {}^*\mathfrak{U}_{\alpha_2}^{(S_2,1)}$, hence $\alpha_1 \in {}^*\{s_1 \in S_1 \mid \{s_2 \in S_2 \mid (s_1, s_2) \in A\} \in \mathfrak{U}_{\alpha_2}^{(S_2,1)}\}$, and so $A \in \mathfrak{U}_{\alpha_1}^{(S_1,1)} \otimes \mathfrak{U}_{\alpha_2}^{(S_2,1)}$.

Case 2: ${}^*F(\alpha_1) = \{\sigma_2 \in {}^*S_2 \mid (\alpha_1, \sigma_2) \in {}^*A^{\mathrm{c}}\}$. But then $(\alpha_1, \alpha_2) \notin {}^*A$, which is absurd. In particular, this shows that the only case that can happen is *Case 1*, therefore $\mathfrak{U}_{(\alpha_1,\alpha_2)}^{(S_1 \times S_2,1)} = \mathfrak{U}_{\alpha_1}^{(S_1,1)} \otimes \mathfrak{U}_{\alpha_2}^{(S_2,1)}$. □

To prove that Theorem 4.15 entails Puritz result and allows for a simple characterisation of tensor pairs in many cases,[7] let us introduce the following definition:

Definition 4.16 Let S_1, S_2 be given sets. Let $Y \subseteq \beta(S_1 \times S_2)$. We say that $(\alpha, \beta) \in {}^*(S_1 \times S_2)$ is a Y-tensor pair if it is a tensor pair and $\mathfrak{U}_{(\alpha,\beta)} \in Y$.

The basic observation is the following: if (α, β) is a Y-tensor pair then $(\alpha, \beta) \in {}^*A$ for every $A \in \bigcap_{\mathcal{U} \in Y} \mathcal{U}$ (i.e., (α, β) generates the filter $\bigcap_{\mathcal{U} \in Y} \mathcal{U}$).

Example 4.17 If $S_1 = S_2 = \mathbb{N}$ and $Y = \{\mathcal{U} \otimes \mathcal{V} \mid \mathcal{U}, \mathcal{V} \in \beta\mathbb{N} \backslash \mathbb{N}\}$, then Y-tensor pairs are tensor pairs with both entries infinite and, as $\Delta = \{(n, m) \mid n < m\} \in \mathcal{W}$ for every $\mathcal{W} \in Y$, this shows that $(\alpha, \beta) \in {}^*\Delta$ for every Y-tensor pair (α, β). But then, by applying Theorem 4.15.(7) with $S_3 = S_4 = \mathbb{N}$, we deduce that for every $f, g : \mathbb{N} \to \mathbb{N}$ with ${}^*f(\alpha_1), {}^*g(\alpha_2)$ infinite we have $({}^*f(\alpha_1), {}^*g(\alpha_2)) \in {}^*\Delta$, viz. $f(\alpha_1) < \mathrm{er}(\alpha_2)$ for all $f : \mathbb{N} \to \mathbb{N}$, which is one implication in Puritz's characterisation.

Example 4.18 Let $S_1 = S_2 = \mathbb{Z}$. Let $\alpha, \beta \in {}^*\mathbb{Z} \backslash \mathbb{Z}$. Let $A_1 = \{z \in \mathbb{Z} \mid z \geq 0, z \equiv 0 \mod 2\}$, $A_2 = \{z \in \mathbb{Z} \mid z > 0, z \equiv 1 \mod 2\}$, $A_3 = \{z \in \mathbb{Z} \mid z < 0, z \equiv 0 \mod 2\}$, and $A_4 = \{z \in \mathbb{Z} \mid z < 0, z \equiv 1 \mod 2\}$. Let i, j be such that $\alpha \in {}^*A_i$, $\beta \in {}^*A_j$ and let $f, g : \mathbb{Z} \to \mathbb{N}$ be bijections such that f coincides with the absolute value on A_i and g coincides with the absolute value on A_j. Then from conditions (7) and (8) in Theorem 4.15 we deduce that (α, β) is a tensor pair if and only if $(|\alpha|, |\beta|)$ is a tensor pair, viz. (α, β) is a tensor pair if and only if $|\alpha| < {}^*h(|\beta|)$ for all $h : \mathbb{N} \to \mathbb{N}$ such that ${}^*h(|\beta|) \notin \mathbb{N}$, and it is hence straightforward to see that (α, β) is a tensor pair if and only if $|\alpha| < |{}^*h(\beta)|$ for all $h : \mathbb{Z} \to \mathbb{Z}$ such that ${}^*h(\beta) \notin \mathbb{Z}$.

Example 4.19 Let $S_1 = S_2 = \mathbb{Q}$. In $\beta\mathbb{Q}$ there are three kinds of ultrafilters:

(i) principal ones, viz. ultrafilters $\mathcal{U} \in \beta\mathbb{Q}$ such that $\mu(\mathcal{U}) = \{q\}$ for some $q \in \mathbb{Q}$;
(ii) quasi-principal, viz. ultrafilters $\mathcal{U} \in \beta\mathbb{Q}$ such that $\mu(\mathcal{U})$ consists of finite nonstandard elements, in which case it is very simple to show that there exists $q \in \mathbb{Q}$ such that $\mu(\mathcal{U}) \subset \mathrm{mon}(q) = \{\xi \in {}^*\mathbb{Q} \mid \xi - q \text{ is infinitesimal}\}$;
(iii) infinite ultrafilters, viz. ultrafilters $\mathcal{U} \in \beta\mathbb{Q}$ such that $\mu(\mathcal{U})$ consists of infinite elements.

Now let $(\alpha, \beta) \in {}^*(\mathbb{Q} \times \mathbb{Q})$. When is it that (α, β) is a tensor pair? As always, this is the case if $\{\alpha, \beta\} \cap \mathbb{Q} \neq \varnothing$. If $\{\alpha, \beta\} \cap \mathbb{Q} = \varnothing$, we distinguish three cases:

(1) both α and β are infinite;
(2) both α and β are finite;
(3) one is infinite, one is finite.

[7] To the best of our knowledge, the characterisations given in Examples 4.18, 4.19, 4.20 are new.

Notice that, as $(\varepsilon, {}^*\varepsilon)$ is a tensor pair for every infinitesimal $\varepsilon \in {}^*\mathbb{Q}$, Puritz's characterisation does not hold (directly) in our present case (as ${}^*\varepsilon < \varepsilon$ for every positive infinitesimal ε).

As, by Theorem 4.15.(7), we know that (α, β) is a tensor type if and only if $(-\alpha, \beta)$ and $(\alpha, -\beta)$ are, we reduce to consider the case $\alpha > 0$, $\beta > 0$. Moreover, let us observe that we can reduce to case (1). In fact, if η is any finite element in ${}^*\mathbb{Q}_{>0}\backslash\mathbb{Q}$, let $f_\eta : \mathbb{Q}\backslash\{\mathrm{st}(\eta)\} \to \mathbb{Q}_{>0}$ be the function such that for all $q \in \mathbb{Q}\backslash \mathrm{st}(\eta)$, we have that $f_\eta(q) = \frac{1}{q-\mathrm{st}(\eta)}$. Then $f_\eta(\eta)$ is infinite and, as this function is bijective, by points (7) and (8) of Theorem 4.15 we get that

(i) if α is finite then (α, β) is a tensor pair if and only if $(f_\alpha(\alpha), \beta)$ is a tensor pair;
(ii) if β is finite then (α, β) is a tensor pair if and only if $(\alpha, f_\beta(\beta))$ is a tensor pair.

So we are left to study case (1). As a simple necessary criterion, from Theorem 4.15.(7) we get that if (α, β) is a tensor pair then also the pair of hypernatural parts $(\lfloor\alpha\rfloor, \lfloor\beta\rfloor)$ is a tensor pair. This fact can be refined: as $\Delta_{\mathbb{Q}} = \{(q_1, q_2) \in \mathbb{Q}^2 \mid q_2 > q_1\} \in \mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta$ whenever α, β are positive and infinite, we get from Theorem 4.15.(7) that it must be $\alpha < \mathrm{er}_{\mathbb{Q}_{>0}}(\beta)$, where $\mathrm{er}_{\mathbb{Q}_{>0}}(\beta) = \{{}^*f(\beta) \mid f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}, {}^*f(\beta) \text{ is infinite}\}$. Let us show that the converse holds as well. Let $\alpha < \mathrm{er}_{\mathbb{Q}_{>0}}(\beta)$. We prove this by contradiction: assume that (α, β) is not a tensor pair. Then by Theorem 4.15.(9) there exists $A \subseteq \mathbb{Q}^2$ such that $(q, \beta) \in {}^*A$ for every $q \in \mathbb{Q}$ but $(\alpha, \beta) \notin {}^*A$. Let $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ be defined by $f(q) := \min\{n \in \mathbb{N} \mid \exists s \in \mathbb{Q}_{>0}\, (s < n + 1 \wedge (s, q) \notin A)\}$ for all $q \in \mathbb{Q}_{>0}$. As $(q, \beta) \in {}^*A$ for every $q \in \mathbb{Q}$ we have that ${}^*f(\beta)$ is infinite. And, as $(\alpha, \beta) \notin {}^*A$, we have that ${}^*f(\beta) \leq \alpha$, which is absurd.

Example 4.20 A similar proof can be used to show that, for every infinite $\alpha, \beta \in {}^*\mathbb{R}_{>0}$, (α, β) is a tensor pair if and only if $\alpha < \mathrm{er}_{\mathbb{R}_{>0}}(\beta)$, where $\mathrm{er}_{\mathbb{R}_{>0}}(\beta) = \{{}^*f(\beta) \mid f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}, {}^*f(\beta) \text{ is infinite}\}$, and following ideas similar to those of Example 4.19 this can be used to characterise tensor pairs in $\mathbb{R}^2$. This can be used also to characterise certain ultrafilters in $\beta\mathbb{C}$: as $\mathbb{C} \cong \mathbb{R}^2$, e.g., we have that ultrafilters in $\beta\mathbb{C}$ of the form $\mathcal{U} \oplus i\mathcal{V}$, with $\mathcal{U}, \mathcal{V} \in \beta\mathbb{R}$, are generated by hypercomplex numbers of the form $\alpha + i\beta$ where (α, β) is a tensor pair in $\mathbb{R}^2$.

Moreover, as $\mathcal{F} := \mathbb{N}^{\mathbb{N}}$ is in bijection with $\mathbb{R}$, from Theorem 4.15.(7) we get a characterisation of tensor pairs in $\mathcal{F}^2$ and, since $\mathbb{N}$ can be embedded in $\mathcal{F}$ just mapping any natural number n to the constant function with value n, this gives a characterisation of tensor pairs in $\mathbb{N} \times \mathcal{F}$ and $\mathcal{F} \times \mathbb{N}$. This characterisation is quite implicit; however, Theorem 4.15 can be used to give explicit necessary and sufficient conditions even in this case: in fact, for $\alpha \in {}^*\mathbb{N}$ and $\varphi \in {}^*\mathcal{F}$ we have the following.

"Necessary": if (α, φ) is a tensor pair, then $({}^*f(\alpha), {}^*J(\varphi))$ is a tensor pair for every $f \in \mathcal{F}$, $J : \mathcal{F} \to \mathbb{N}$. In particular, by letting for every $n \in \mathbb{N}$ J_n be the evaluation in $\mathbb{N}$, we get that if (α, φ) is a tensor pair, then $(\alpha, \varphi(n))$ is a tensor pair in ${}^*\mathbb{N}^2$ for every $n \in \mathbb{N}$.

"Sufficient": $(\alpha, {}^*\varphi)$ is a tensor pair. In particular, if we let $\mathcal{V} := \mathfrak{U}_\alpha^{(\mathbb{N},1)} \otimes_{\mathcal{F}} \mathfrak{U}_\varphi^{(\mathcal{F},1)} \in \beta\mathbb{N}$ be the ultrafilter such that for all $A \subseteq \mathbb{N}$, we have $A \in \mathcal{V}$ if and only if $\{n \in \mathbb{N} \mid \{f \in \mathcal{F} \mid f(n) \in A\} \in \mathfrak{U}_\varphi\} \in \mathfrak{U}_\alpha$, we get that $({}^*\varphi)(\alpha) \in \mu_2(\mathcal{V})$.

4.3 Tensor k-tuples

If we consider products of k sets, the natural generalisation of tensor pairs are tensor k-tuples.

Definition 4.21 Let $S_1, \dots, S_k$ be sets and, for every $i \leq k$, let $\mathcal{U}_i \in \beta S_i$. The tensor product $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_k$ is the unique ultrafilter on $S_1 \times \cdots \times S_k$ such that, for every $A \subseteq S_1 \times \cdots \times S_k$ we have that $A \in \mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_k$ if and only if $\{s_1 \in S_1 \mid \{s_2 \in S_2 \mid \dots \{s_k \in S_k \mid (s_1, \dots, s_k) \in A\} \in \mathcal{U}_k\} \dots\} \in \mathcal{U}_1$. We say that $(\alpha_1, \dots, \alpha_k) \in {}^*(S_1 \times \cdots \times S_k)$ is a *tensor k-tuple* if $\mathfrak{U}_{(\alpha_1,\dots,\alpha_k)}^{(S_1\times\cdots\times S_k,1)} = \mathfrak{U}_{\alpha_1}^{(S_1,1)} \otimes \cdots \otimes \mathfrak{U}_{\alpha_k}^{(S_k,1)}$.

It is easy to prove that $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_k$ is an ultrafilter and that the operation $\otimes$ is associative (modulo the usual identification of products $(S_1 \times S_2) \times S_3 = S_1 \times (S_2 \times S_3) = S_1 \times S_2 \times S_3$), cf., e.g., [4, Appendix]. This allows to characterise tensor k-tuples in terms of pairs:

Theorem 4.22 *Let $k \geq 1$, let $S_1, \dots, S_k, S_{k+1}$ be given sets and let $(\alpha_1, \dots, \alpha_{k+1}) \in {}^*(S_1 \times \cdots \times S_{k+1})$. The following facts are equivalent:*

(1) $(\alpha_1, \dots, \alpha_{k+1})$ *is a tensor* $(k+1)$*-tuple;*
(2) $((\alpha_1, \dots, \alpha_k), \alpha_{k+1})$ *is a tensor pair and* $(\alpha_1, \dots, \alpha_k)$ *is a tensor* k*-tuple;*
(3) (α_k, α_{k+1}) *is a tensor pair and* $(\alpha_1, \dots, \alpha_k)$ *is a tensor* k*-tuple;*
(4) for all $i \le k$, (α_i, α_{i+1}) *is a tensor pair.*

Proof. As there is no danger of confusion, we shall use the simplified notations $\mathfrak{U}_{(\alpha_1,\dots,\alpha_{k+1})}$ instead of $\mathfrak{U}^{(S_1\times\cdots\times S_{k+1},1)}_{(\alpha_1,\dots,\alpha_{k+1})}$, and similarly for $\mathfrak{U}_{\alpha_i}$ for $i \le k+1$.

"(1) $\Rightarrow$ (2)": By hypothesis, $\mathfrak{U}_{(\alpha_1,\dots,\alpha_{k+1})} = \mathfrak{U}_{\alpha_1} \otimes \cdots \otimes \mathfrak{U}_{\alpha_{k+1}}$. By the associativity of tensor products, $\mathfrak{U}_{\alpha_1} \otimes \cdots \otimes \mathfrak{U}_{\alpha_{k+1}} = (\mathfrak{U}_{\alpha_1} \otimes \cdots \otimes \mathfrak{U}_{\alpha_k}) \otimes \mathfrak{U}_{\alpha_{k+1}}$. Let $\mathcal{V} = \mathfrak{U}_{\alpha_1} \otimes \cdots \otimes \mathfrak{U}_{\alpha_k}$. Then $((\alpha_1, \dots, \alpha_k), \alpha_{k+1}) \in \mu(\mathcal{V} \otimes \mathcal{U})$, viz. $((\alpha_1, \dots, \alpha_k), \alpha_{k+1})$ is a tensor pair, and $(\alpha_1, \dots, \alpha_k) \in \mu(\mathfrak{U}_{\alpha_1} \otimes \cdots \otimes \mathfrak{U}_{\alpha_k})$, viz. $(\alpha_1, \dots, \alpha_k)$ is a tensor k-tuple.

"(2) $\Rightarrow$ (1)": $\mathfrak{U}_{(\alpha_1,\dots,\alpha_k,\alpha_{k+1})} = \mathfrak{U}_{(\alpha_1,\dots,\alpha_k)} \otimes \mathfrak{U}_{\alpha_{k+1}}$ as $((\alpha_1, \dots, \alpha_k), \alpha_{k+1})$ is a tensor pair, and $\mathfrak{U}_{(\alpha_1,\dots,\alpha_k)} = \mathfrak{U}_{\alpha_1} \otimes \cdots \otimes \mathfrak{U}_{\alpha_k}$ as $(\alpha_1, \dots, \alpha_k)$ is a tensor k-tuple.

"(2) $\Rightarrow$ (3)": We prove this by contradiction: assume that (α_k, α_{k+1}) is not a tensor pair. Let $A \subseteq S_k \times S_{k+1}$ be such that for all $s_k \in S_k$, we have $(s_k, \alpha_{k+1}) \in {}^*A$ but $(\alpha_k, \alpha_{k+1}) \notin {}^*A$. Let $B = \{(s_1, \dots, s_{k+1}) \in S_1 \times \cdots \times S_{k+1} \mid (s_k, s_{k+1}) \in A\}$. By construction, $((s_1, \dots, s_k), \alpha_{k+1}) \in {}^*B$ for all $(s_1, \dots, s_k) \in S_1 \times \cdots \times S_k$. As $((\alpha_1, \dots, \alpha_k), \alpha_{k+1})$ is a tensor pair, this entails that $((\alpha_1, \dots, \alpha_k), \alpha_{k+1}) \in {}^*B$, hence $(\alpha_k, \alpha_{k+1}) \in {}^*A$, which is absurd.

"(3) $\Rightarrow$ (2)": We prove this by contradiction: assume that $((\alpha_1, \dots, \alpha_k), \alpha_{k+1})$ is not a tensor pair. Let $T = S_1 \times \cdots \times S_k$ and let $A \subseteq T \times S_{k+1}$ be such that for all $t \in T$, we have that $(t, \alpha_{k+1}) \in {}^*A$, but $((\alpha_1, \dots, \alpha_k), \alpha_{k+1}) \notin {}^*A$. Define $B = \{(s_k, s_{k+1}) \in S_k \times S_{k+1} \mid \forall (s_1, \dots, s_{k-1}) \in S_1 \times \cdots \times S_{k-1} ((s_1, \dots, s_k), s_{k+1}) \in A\}$. By construction, $(s_k, \alpha_{k+1}) \in {}^*B$ for all $s_k \in S_k$, hence, as (α_k, α_{k+1}) is a tensor pair, we have that $(\alpha_k, \alpha_{k+1}) \in {}^*B = \{(\eta_k, \eta_{k+1}) \in {}^*(S_k \times S_{k+1}) \mid \forall (\sigma_1, \dots, \sigma_{k-1}) \in {}^*(S_1 \times \cdots \times S_{k-1})((\sigma_1, \dots, \sigma_{k-1}, \eta_k), \eta_{k+1}) \in {}^*A\}$, hence $((\alpha_1, \dots, \alpha_k), \alpha_{k+1}) \in {}^*A$, which is absurd.

"(3) $\Rightarrow$ (4)": By induction on k. If $k = 1$ there is nothing to prove. Now let us assume the claim to hold for $k \ge 1$ and let us prove it for $k+1$. By inductive hypothesis, as (3) $\Leftrightarrow$ (1), $(\alpha_1, \dots, \alpha_k)$ is a tensor k-tuple if and only if for all $i \le k-1$, (α_i, α_{i+1}) is a tensor pair, so the claim is proven.

"(4) $\Rightarrow$ (3)" is immediate by induction. □

Example 4.23 If $S_i = \mathbb{N}$ for every $i = 1, \dots, k$, we get the following equivalence: if for all $i \le k+1$ $\alpha_i \in {}^*\mathbb{N} \setminus \mathbb{N}$, then $(\alpha_1, \dots, \alpha_{k+1})$ is a tensor $(k+1)$-tuple if and only if $\alpha_i < \mathrm{er}(\alpha_{i+1})$ for every $i \le k$.

Notice that, as a trivial corollary of Theorem 4.22, we obtain that the relation of "being a tensor pair" is transitive:

Corollary 4.24 *For every* $(\alpha_1, \alpha_2, \alpha_3) \in {}^*(S_1 \times S_2 \times S_3)$, *if* (α_1, α_2) *and* (α_2, α_3) *are tensor pairs, then* (α_1, α_3) *is a tensor pair.*

Proof. As (α_1, α_2) and (α_2, α_3) are tensor pairs, from Theorem 4.22 we deduce that $((\alpha_1, \alpha_2), \alpha_3)$ is a tensor pair. Let us now assume that (α_1, α_3) is not a tensor pair. Let $A \subseteq S_1 \times S_3$ be such that $(s_1, \alpha_3) \in {}^*A$ for every $s_1 \in S_1$ but $(\alpha_1, \alpha_3) \notin {}^*A$. Let $B \subseteq S_1 \times S_2 \times S_3$ be defined as follows: $(s_1, s_2, s_3) \in B$ if and only if $(s_1, s_3) \in A$. Then $(s_1, s_2, \alpha_3) \in {}^*B$ for every $(s_1, s_2) \in S_1 \times S_2$, and so (as $((\alpha_1, \alpha_2), \alpha_3)$ is a tensor pair) we have that $((\alpha_1, \alpha_2), \alpha_3) \in {}^*B$, therefore $(\alpha_1, \alpha_3) \in {}^*A$, which is absurd. □

Using this fact, it is possible to add the following equivalent characterisation to Theorem 4.22:

Theorem 4.25 *Let* $k \ge 1$, *let* $S_1, \dots, S_k, S_{k+1}$ *be given sets and let* $(\alpha_1, \dots, \alpha_{k+1}) \in {}^*(S_1 \times \cdots \times S_{k+1})$. *The following facts are equivalent:*

(1) $(\alpha_1, \dots, \alpha_{k+1})$ *is a tensor* $(k+1)$*-tuple;*
(2) for all $F = \{i_1 < \cdots < i_\ell\} \subseteq \{1, \dots, k+1\}$, $(\alpha_{i_1}, \dots, \alpha_{i_\ell})$ *is a tensor* ℓ*-tuple.*

Proof. The implication (2) $\Rightarrow$ (1) is trivial (just set $F = \{1, \dots, k+1\}$).

To prove the other implication, by the transitivity of the relation of being a tensor pair we have (using the characterisation (4) of Theorem 4.22) that for every $j \le \ell - 1$ (α_j, α_{j+1}) is a tensor pair, so all pairs $(\alpha_{i_j}, \alpha_{i_{j+1}})$ are tensor pairs. Hence from the equivalence (1) $\Leftrightarrow$ (4) in Theorem 4.22 we deduce that $(\alpha_{i_1}, \dots, \alpha_{i_\ell})$ is a tensor ℓ-tuple. □

Finally, by iterating inductively the proof of Theorem 4.13, we obtain the following result:

Theorem 4.26 *Let $S_1, \dots, S_{k+1} \in \mathbb{V}(X)$ be sets, with $S_1, \dots, S_k$ completely coherent. For every $i \leq k+1$ let $\mathcal{U}_i \in \beta S_i$ and let $\alpha_i \in \mu(\mathcal{U}_i)$. Then $(\alpha_1, {}^*\alpha_2, \dots, H_k(\alpha_{k+1})) \in \mu_{k+1}(\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_{k+1})$.*

As a straightforward corollary we get the following characterisation of tensor k-tuples in iterated hyperextensions:

Corollary 4.27 *Let $S_1, \dots, S_{k+1} \in \mathbb{V}(X)$ be sets, with $S_1, \dots, S_k$ completely coherent. For every $i \leq k+1$ let $\alpha_i \in {}^*S_i$. The following facts are equivalent:*

(1) $(\alpha_1, \dots, \alpha_{k+1})$ *is a tensor pair;*
(2) $(\alpha_1, \dots, \alpha_{k+1}) \sim_{(S_1 \times \cdots \times S_{k+1}, k+1)} (\alpha_1, {}^*\alpha_2, \dots, H_k(\alpha_{k+1}))$.

5 Combinatorial properties of monads

In recent years, several open problems in Ramsey theory regarding the partition regularity (and the partial partition regularity) of formulas have been solved (cf., e.g., [14, 16, 23, 32]). Moreover, in many cases nonstandard approaches based on the algebraical properties of the monads of ultrafilters have been used to extend several known results in new directions (cf., e.g., [8, 12–15, 27, 30]). In this section our goal is to give a unified formulation of all these nonstandard approaches (which will be obtained in Theorem 5.4), as well as to extend these methods to new directions: the study of partial partition regularity and the partition regularity of formulas with internal parameters.

5.1 Partition regularity of existential formulas

In all this section we let $Y \in \mathbb{V}(X)$ be a set, and we shall work with the extension $*$ (except in some examples). The formulation of the results in the general case of an extension $\star$ can be obtained analogously. We shall be concerned with the notion of "partition regularity":[8]

Definition 5.1 A family $\mathfrak{F} \subseteq \wp(Y)$ is partition regular if for every $k \in \mathbb{N}$, for every partition $Y = A_1 \cup \cdots \cup A_k$ there exists $i \leq k$ such that $A_i \in \mathfrak{F}$.

The relationship between partition regular families and ultrafilters is a well known fact in combinatorial number theory;[9] in [26], this characterisation was expressed by means of properties of monads in the case of families of witnesses of the partition regularity of Diophantine equations, a field rich in very interesting open problems.

Theorem 5.2 *Let $P(x_1, \dots, x_n) \in \mathbb{Z}[x_1, \dots, x_n]$. The following facts are equivalent:*

(1) the equation $P(x_1, \dots, x_n) = 0$ is partition regular on $\mathbb{N}$, viz. the family $\mathfrak{F}_P = \{A \subseteq \mathbb{N} \mid \exists a_1, \dots, a_n \in A\ P(a_1, \dots, a_n) = 0\}$ is partition regular;
(2) there exists an ultrafilter $\mathcal{U} \in \beta\mathbb{N}$ and generators $\alpha_1, \dots, \alpha_n \in \mu_\infty(\mathcal{U})$ such that $P(\alpha_1, \dots, \alpha_n) = 0$.

This characterisation has been subsequently used in a series of paper [12–15, 27, 29] to study the partition regularity of several classes of nonlinear polynomials. In this section we want to show how this characterisation can be extended to study the partition regularity of several families of subsets of arbitrary sets.[10]

Let us start with some preliminaries. In all this section, when we talk about "formulas" we mean first order formulas with bounded quantifiers[11] in the language of the superstructure $\mathbb{V}(X)$ (cf. [17, Chapter 13]), and when

[8] In this paper, we call "partition regularity" what is sometimes called "weak partition regularity". We shall not consider the notion of "strong partition regularity", except briefly in Example 5.25. Cf. also [18] for a discussion of the two notions.

[9] A family of subsets of Y is partition regular if and only if it contains an ultrafilter on Y. These notions are also closely related to co-ideals; for a thorough treatment of co-ideals in Ramsey Theory, we refer to [37].

[10] Some results of this section already appeared, in a much weaker form, in [26].

[11] We adopt a slight abuse of language here: the kind of formulas we work with are those introduced in Definition 5.3, which contain some unbounded quantifiers. However, the notion we are interested in is that of a set $A \subseteq Y$ witnessing these formulas, and when we adopt this notion there are no more unbounded quantifiers to be handled, as every unbounded quantifier $Q_i x_i$ ($Q_i \in \{\forall, \exists\}$) is substituted with $Q_i x_i \in A$. For this reason, we believe that this slight abuse of language should not create too much confusion.

we write a formula as $\varphi(x_1, \dots, x_n)$ we mean that $x_1, \dots, x_n$ are all and only variables appearing in φ. We say that a formula $\varphi(x_1, \dots, x_n)$ is totally open if all its variables are free.

Definition 5.3 Let $\varphi(x_1, \dots, x_n, y_1, \dots, y_m)$ be a totally open formula, let $S_1, \dots, S_m \in \mathbb{V}(X)$ be sets and, for $i = 1, \dots, m$, let $Q_i \in \{\exists, \forall\}$. The existential closure of $\varphi(x_1, \dots, x_n, y_1, \dots, y_m)$ with constraints $\{Q_i y_i \in S_i \mid i \leq m\}$ is the formula $E_{\overrightarrow{Qy} \in \overrightarrow{S}}(\varphi(x_1, \dots, x_n, y_1, \dots, y_m))$:

$$\exists x_1 \dots \exists x_n \, Q_1 y_1 \in S_1 \dots Q_m y_m \in S_m \, \varphi(x_1, \dots, x_n, y_1, \dots, y_m).$$

When $m = 0$ we shall use the notation $E(\varphi(x_1, \dots, x_n))$, and $E(\varphi(x_1, \dots, x_n))$ will be called the existential closure of $\varphi(x_1, \dots, x_n)$. Similarly, the universal closure of $\varphi(x_1, \dots, x_n, y_1, \dots, y_m)$ with constraints $\{Q_i y_i \in S_i \mid i \leq m\}$ is the formula $U_{\overrightarrow{Qy} \in \overrightarrow{S}}(\varphi(x_1, \dots, x_n, y_1, \dots, y_m))$:

$$\forall x_1 \dots \forall x_n \, Q_1 y_1 \in S_1 \dots Q_m y_m \in S_m \, \varphi(x_1, \dots, x_n, y_1, \dots, y_m).$$

When $m = 0$ we shall use the notation $U(\varphi(x_1, \dots, x_n))$, and $U(\varphi(x_1, \dots, x_n))$ will be called the universal closure of $\varphi(x_1, \dots, x_n)$. Given a totally open formula $\varphi(x_1, \dots, x_n, y_1, \dots, y_m)$, a set of constraints $\{Q_i y_i \in S_i \mid i \leq m\}$ and a set $A \subseteq Y$, $E_{\overrightarrow{Qy} \in \overrightarrow{S}}(\varphi(x_1, \dots, x_n, y_1, \dots, y_m))$ is said to be modeled by A, denoted by

$$A \models E_{\overrightarrow{Qy} \in \overrightarrow{S}}(\varphi(x_1, \dots, x_n, y_1, \dots, y_m)),$$

if the formula

$$\exists x_1 \in A \dots \exists x_n \in A \, Q_1 y_1 \in S_1 \dots Q_m y_m \in S_m \, \varphi(x_1, \dots, x_n, y_1, \dots, y_m)$$

holds true. Similarly, we say that A models $U_{\overrightarrow{Qy} \in \overrightarrow{S}}(\varphi(x_1, \dots, x_n, y_1, \dots, y_m))$, denoted by

$$A \models U_{\overrightarrow{Qy} \in \overrightarrow{S}}(\varphi(x_1, \dots, x_n, y_1, \dots, y_m)),$$

if the formula

$$\forall x_1 \in A \dots \forall x_n \in A \, Q_1 y_1 \in S_1 \dots Q_m y_m \in S_m \, \varphi(x_1, \dots, x_n)$$

holds true.

The formula $E_{\overrightarrow{Qy} \in \overrightarrow{S}}(\varphi(x_1, \dots, x_n, y_1, \dots, y_m))$ or the formula $U_{\overrightarrow{Qy} \in \overrightarrow{S}}(\varphi(x_1, \dots, x_n, y_1, \dots, y_m))$ is said to be partition regular on Y if for every $k \in \mathbb{N}$, for every partition $Y = A_1 \cup \dots \cup A_k$ there exists $i \leq k$ such that $A_i \models E_{\overrightarrow{Qy} \in \overrightarrow{S}}(\varphi(x_1, \dots, x_n, y_1, \dots, y_m))$ (or $A_i \models U_{\overrightarrow{Qy} \in \overrightarrow{S}}(\varphi(x_1, \dots, x_n, y_1, \dots, y_m))$).

Our main result in this section is the following theorem, which generalises Theorem 5.2 to arbitrary existential formulas and sets with constraints:[12]

Theorem 5.4 *Let $\varphi(x_1, \dots, x_n, y_1, \dots, y_m)$ be a totally open formula and, for $i = 1, \dots, m$, let $S_i \in \mathbb{V}(X)$ and $Q_i \in \{\exists, \forall\}$. Let $Y \in \mathbb{V}(X)$, and assume that the hyperextension $*$ is $|\wp(Y)|^+$-enlarging. The following are equivalent:*

(1) $E_{\overrightarrow{Qy} \in \overrightarrow{S}}(\varphi(x_1, \dots, x_n, y_1, \dots, y_m))$ is partition regular on Y;

(2) there are $\alpha_1 \sim_Y \dots \sim_Y \alpha_n \in {}^*Y$ such that the sentence

$$Q_1 y_1 \in {}^*S_1, \dots, Q_m y_m \in {}^*S_m {}^*\varphi(\alpha_1, \dots, \alpha_n, y_1, \dots, y_m)$$

holds true;

(3) there exists a ultrafilter $\mathcal{U} \in \beta Y$ such that for every set $A \in \mathcal{U}$ we have that

$$A \models E_{\overrightarrow{Qy} \in \overrightarrow{S}}(\varphi(x_1, \dots, x_n, y_1, \dots, y_m)).$$

Proof. "(1) $\Rightarrow$ (2)". Let $\mathrm{Par}(Y)$ be the set of all possible finite partitions of Y. Given partitions $P_1(Y) = A_{1,1} \cup \dots \cup A_{1,k_1}, \dots, P_m(Y) = A_{m,1} \cup \dots \cup A_{m,k_m}$, we let $P(P_1, \dots, P_m)$ be the partition generated

[12] We have included in this theorem also the known characterisation of partition regular families in terms of ultrafilters, providing a new rather simple nonstandard proof that uses monads. Notice that the analogous formulation of this Theorem for $H_n, \bullet$ can be easily obtained following an analogous proof.

by $P_1, \ldots, P_m$, viz. the partition $Y = \bigcup_{(i_1,\ldots,i_m)\in K} \bigcap_{1\le \ell\le m} A_{\ell,i_\ell}$, where $K = [1, k_1] \times \ldots \times [1, k_m]$. Now, for every partition $P(Y) = A_1 \cup \ldots \cup A_m$ let $I_{P(Y)}$ be the set of all partitions refining $P(Y)$, viz. $I_{P(Y)} = \{f : Y \to [1, k] \mid k \in \mathbb{N}, \forall i \le k\, \exists! j \le m(f^{-1}(i) \subseteq A_j)\}$.[13] The family $\{I_{P(Y)}\}_{P\in \mathrm{Par}(Y)}$ has the finite intersection property, since $I_{P_1(Y)} \cap \ldots \cap I_{P_m(Y)} \supseteq I_{P(P_1\ldots,P_m)}$. By enlarging, there exists a hyperfinite partition ${}^*Y = A_1 \cup \ldots \cup A_\lambda$ that refines all finite partitions of Y. As $E_{\overrightarrow{Qy}\in \overrightarrow{S}}(\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m))$ is partition regular on Y, by transfer ${}^*E_{\overrightarrow{Qy}\in \overrightarrow{S}}(\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m))$ is partition regular on ${}^*Y$, hence there exists $i \le \lambda$, $\alpha_1, \ldots, \alpha_n \in A_i$, such that $Q_1\beta_1 \in {}^*S_1, \ldots, Q_m\beta_m \in {}^*S_m\, {}^*\varphi(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$ holds true. To conclude the proof, we show that $\alpha_1 \sim_Y \ldots \sim_Y \alpha_n$. In fact, as $A_1 \cup \ldots \cup A_\lambda$ refines all finite partitions on Y, for every $i \le \lambda$ it is straightforward to show that the set $U_i = \{A \subseteq Y \mid A_i \subseteq {}^*A\}$ is an ultrafilter, and so $A_i \subseteq \bigcap_{A\in U_i} {}^*A = \mu(U_i)$, hence $\alpha_1, \ldots, \alpha_n \in \mu(U_i)$ are all $\sim_Y$-equivalent.

"(2) $\Rightarrow$ (3)": Let $\mathcal{U}$ be the ultrafilter generated by $\alpha_1, \ldots, \alpha_n$. Let $A \in \mathcal{U}$. By hypothesis, $\mu(\mathcal{U}) \models {}^*E_{\overrightarrow{Qy}\in \overrightarrow{S}}(\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m))$ and, since the formula $E_{\overrightarrow{Qy}\in \overrightarrow{S}}(\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m))$ is existential in $\overrightarrow{x} = (x_1, \ldots, x_n)$ and $\mu(\mathcal{U}) \subseteq {}^*A$, this entails that ${}^*A \models {}^*E_{\overrightarrow{Qy}\in \overrightarrow{S}}(\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m))$, so we can conclude by transfer.

"(3) $\Rightarrow$ (1)": This is straightforward from the definitions, as for every finite partition $Y = A_1 \cup \cdots \cup A_k$ there exists $i \le k$ such that $A_i \in \mathcal{U}$. □

The characterisation of partition regular Diophantine equations is a particular case of the previous theorem, where we let $m = 0$, $Y = \mathbb{N}$ and, given a polynomial $P(x_1, \ldots, x_n)$, $\varphi(x_1, \ldots, x_n)$ is the formula $P(x_1, \ldots, x_n) = 0$.

Definition 5.5 If φ is a partition regular formula, we call an ultrafilter $\mathcal{U}$ a *φ-ultrafilter* and say that $\mathcal{U}$ *witnesses* φ (in symbols: $\mathcal{U} \models \varphi$) if for all $A \in \mathcal{U}$, we have that $A \models \varphi$.

In particular, the proof of Theorem 5.4 shows that, for any ultrafilter $\mathcal{U} \in \beta Y$,

$$\mathcal{U} \models E_{\overrightarrow{Qy}\in \overrightarrow{S}}(\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m))$$

if and only if there are $\alpha_1, \ldots, \alpha_n \in \mu(\mathcal{U})$ such that

$$Q_1\beta_1 \in {}^*S_1, \ldots, Q_m\beta_m \in {}^*S_m {}^*\varphi(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m)$$

holds true.

Example 5.6 (This example appears, in the weaker form $m = 0$, also in [30, Theorem 4.2].) Let S be a semigroup, and let $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be an homogeneous totally open formula with constraints $Q_1r_1 \in R_1, \ldots, Q_mr_m \in R_m$, in the sense that for all $s_1, \ldots, s_n, t \in S$ if $Q_1r_1 \in R_1 \ldots Q_mr_m \in R_m\ \varphi(s_1, \ldots, s_n, r_1, \ldots, r_m)$ holds true, then $Q_1r_1 \in R_1 \ldots Q_mr_m \in R_m\ \varphi(t \cdot s_1, \ldots, t \cdot s_n, r_1, \ldots, r_m)$ and $Q_1r_1 \in R_1 \ldots Q_mr_m \in R_m\ \varphi(s_1 \cdot t, \ldots, s_n \cdot t, r_1, \ldots, r_m)$ hold true. Let us also assume that

(1) S is completely coherent;
(2) for all $\alpha_1, \ldots, \alpha_n \in {}^*S$, if $Q_1r_1 \in {}^*R_1 \ldots Q_mr_m \in {}^*R_m\, {}^*\varphi(\alpha_1, \ldots, \alpha_n, r_1, \ldots, r_m)$ holds true, then also $Q_1r_1 \in {}^{**}R_1 \ldots Q_mr_m \in {}^{**}R_m\, {}^{**}\varphi(\alpha_1, \ldots, \alpha_n, r_1, \ldots, r_m)$ holds true.[14]

Then $I_\varphi = \{\mathcal{U} \in \beta S \mid \mathcal{U} \models E_{\overrightarrow{Qy}\in \overrightarrow{R}}(\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m))\}$ is a closed bilateral ideal in βS. Closure is trivial; now let $\mathcal{U} \in I_\varphi$ and $\mathcal{V} \in \beta S$. Let $\alpha_1 \sim_S \cdots \sim_S \alpha_n \in \mu(\mathcal{U})$ be such that $Q_1y_1 \in {}^*R_1 \ldots Q_my_m \in {}^*R_m\, {}^*\varphi(\alpha_1, \ldots, \alpha_n, y_1, \ldots, y_m)$ holds, and let $\beta \in \mu(\mathcal{V})$. Then we have:

(i) $\mathcal{U} \odot \mathcal{V} \in I_\varphi$ as, by Corollary 4.14, $\alpha_i \cdot {}^*\beta \in \mu_2(\mathcal{U} \odot \mathcal{V})$ for every $i \le n$, and $Q_1y_1 \in {}^{**}R_1 \ldots Q_my_m \in {}^{**}R_m\, {}^{**}\varphi(\alpha_1 \cdot {}^*\beta, \ldots, \alpha_n \cdot {}^*\beta, y_1, \ldots, y_m)$ holds as φ is homogeneous, $\alpha_1, \ldots, \alpha_n \in {}^{**}S$ as S is completely coherent and $Q_1y_1 \in {}^{**}R_1 \ldots Q_my_m \in {}^{**}R_m\, {}^{**}\varphi(\alpha_1, \ldots, \alpha_n, y_1, \ldots, y_m)$ holds by our assumption (2);

[13] As usual, we are identifying partitions and functions with finite image.
[14] This is the case, e.g., if $S = \mathbb{N}$ and φ is the formula stating that a certain given polynomial $P(x_1, \ldots, x_n) = 0$.

(ii) similarly, $\mathcal{V} \odot \mathcal{U} \in I_\varphi$ as, by Corollary 4.14, $\beta \cdot {}^*\alpha_i \in \mu_2(\mathcal{V} \odot \mathcal{U})$ for every $i \leq n$, and $Q_1 y_1 \in {}^{**}R_1 \ldots Q_m y_m \in {}^{**}R_m\, {}^{**}\varphi(\beta \cdot {}^*\alpha_1, \ldots, \beta \cdot {}^*\alpha_n, y_1, \ldots, y_m)$ holds as

$$Q_1 y_1 \in {}^{**}R_1 \ldots Q_m y_m \in {}^{**}R_m\, {}^{**}\varphi({}^*\alpha_1, \ldots, {}^*\alpha_n, y_1, \ldots, y_m)$$

holds by transfer, φ is homogeneous and $\beta \in {}^{**}S$ since S is completely coherent.[15]

Example 5.7 In [23], Khalfalah and Szemer'edi proved that, for every polynomial $P(y)$ such that $2 \mid P(y)$ for some $y \in \mathbb{Z}$, the formula $\exists x_1, x_2, \exists y \in \mathbb{Z}\, x_1 + x_2 = P(y)$ is partition regular[16] on $\mathbb{Z}$. By Theorem 5.4, there exist $\alpha_1 \sim_{(\mathbb{Z},1)} \alpha_2$ and $\beta \in {}^*\mathbb{Z}$ such that $\alpha_1 + \alpha_2 = P(\beta)$. Similarly, in [16], Frantzikinakis and Host proved the partition regularity of the formulas $\exists x_1, x_2 \exists y_1 \in \mathbb{Z}\, 16x_1^2 + 9x_2^2 = y_1^2$ and $\exists x_1, x_2 \exists y_1 \in \mathbb{Z}\, x_1^2 - x_1x_2 + x_2^2 = y_1^2$. Once again, by Theorem 5.4, there exist $\alpha_1 \sim_{(\mathbb{Z},1)} \alpha_2$ and $\beta \in {}^*\mathbb{Z}$ such that $16\alpha_1^2 + 9\alpha_2^2 = \beta_1^2$ and there exists $\eta_1 \sim_{(\mathbb{Z},1)} \eta_2$ and $\mu_1 \in {}^*\mathbb{Z}$ such that $\eta_1^2 - \eta_1\eta_2 + \eta_2^2 = \mu_1^2$. Notice that both these formulas are homogeneous, hence by Example 5.6 we get that the sets of ultrafilters witnessing them are closed bilateral ideals in $(\beta\mathbb{N}, \odot)$ (hence, in particular, any ultrafilter in the minimal closed bilateral ideal $\overline{K(\beta\mathbb{N}, \odot)}$ witnesses both of them).

Example 5.8 In [32], Moreira solved a long standing open problem, proving the partition regularity on $\mathbb{N}$ of the formula[17] $\exists x_1, x_2, x_3 \exists y \in \mathbb{N}\, (x_1 + y = x_2) \wedge (x_1 \cdot y = x_3)$. By Theorem 5.4, this entails the existence of an ultrafilter $\mathcal{U} \in \beta\mathbb{N}$ and $\alpha_1, \alpha_2, \alpha_3 \in \mu(\mathcal{U})$, $\beta \in {}^*\mathbb{N}$ such that $\alpha_1 + \beta = \alpha_2$ and $\alpha_1 \cdot \beta = \alpha_3$.

In most cases, however, one is interested in full partition regularity, viz. in the case of Definition 5.3 where $m = 0$.

Example 5.9 A very well-known fact in combinatorial number theory is that every additively idempotent ultrafilter in $\beta\mathbb{N}$ is a Schur ultrafilter, viz. it witnesses the partition regularity on $\mathbb{N}$ of the formula $\exists x, y, z\, x + y = z$ (cf. [36] for the original combinatorial proof of this result, and [18] for the ultrafilters version). This fact can be seen directly also as a consequence of Theorem 5.4. In fact, let $\mathcal{U}$ be idempotent and let $\alpha \in \mu(\mathcal{U})$. Then ${}^*\alpha \in \mu_2(\mathcal{U})$ and $\alpha + {}^*\alpha \in \mu_2(\mathcal{U} \oplus \mathcal{U}) = \mu_2(\mathcal{U})$ by Corollary 4.14, hence letting $\alpha_1 = \alpha, \alpha_2 = {}^*\alpha$ and $\alpha_3 = \alpha + {}^*\alpha$ we get the thesis, as $\alpha_1 \sim_{(\mathbb{N},2)} \alpha_2 \sim_{(\mathbb{N},2)} \alpha_3$. This allows us to conclude, as the inclusion $\mu(\mathcal{U}) \subseteq \mu_2(\mathcal{U})$ follows from Proposition 3.4.(1).

The characterisation of partition regularity by means of ultrafilters allows to use a iterative process to produce new partition regular formulas. The following is a generalisation of [14, Lemma 2.1], where this result was framed and proven restricting to the context of partition regular equations:

Theorem 5.10 *Let $\varphi(x, y_1, \ldots, y_n)$ be a totally open formula, let $S_1, \ldots, S_m \in \mathbb{V}(X)$ be sets and, for $i = 1, \ldots, m$, let $Q_i \in \{\exists, \forall\}$. Assume that*

$$\exists x\, Q_1 y_1 \in S_1 \ldots Q_n y_n \in S_n\, \varphi(x, y_1, \ldots, y_n)$$

is a partition regular formula, and that $\mathcal{U} \in \beta Y$ is one of its witnesses. Then for every set $A \in \mathcal{U}$ the set $I_A(\varphi) := \{a \in A \mid Q_1 y_1 \in S_1 \ldots Q_n y_n \in S_n \varphi(a, y_1, \ldots, y_n)$ holds true$\} \in \mathcal{U}$. Moreover, let $\psi(x, z_1 \ldots, z_m)$ be another totally open formula, let $R_1, \ldots, R_m \in \mathbb{V}(X)$ be sets and, for i $= 1, \ldots, m$, let $\widetilde{Q_i} \in \{\exists, \forall\}$. Assume that $\mathcal{U}$ witnesses also the partition regularity of $\exists x\, \widetilde{Q}_1 z_1 \in R_1 \ldots \widetilde{Q}_m z_m \in R_m\, \psi(x, z_1, \ldots, z_m)$. Then $\mathcal{U}$ witnesses the formula

$$\exists x\, Q_1 y_1 \in S_1 \ldots Q_n y_n \in S_n \widetilde{Q_1} z_1 \in R_1 \widetilde{Q_m} z_m \in R_m \varphi(x, y_1, \ldots, y_n) \wedge \psi(x, z_1, \ldots, z_m),$$

which is then partition regular.

Proof. Towards a contradiction, assume that there exists $A \in \mathcal{U}$ such that $I_A(\varphi) \notin \mathcal{U}$. Then $A \setminus I_A \in \mathcal{U}$, but $A \setminus I_A(\varphi) \models \neg(\exists x\, Q_1 y_1 \in S_1 \ldots Q_n y_n \in S_n\, \varphi(x, y_1, \ldots, y_n))$, hence $\mathcal{U}$ is not a witness of the partition regularity of $\exists x\, Q_1 y_1 \in S_1 \ldots Q_n y_n \in S_n\, \varphi(x, y_1, \ldots, y_n)$, which is absurd.

[15] Notice that to prove this inclusion we never used our assumption (2) on φ.

[16] However, as a consequence of [14, Theorem 3.10], if we drop the constraint $y \in \mathbb{Z}$, the formula $\exists x_1, x_2, y\;\; x_1 + x_2 = P(y)$ is not partition regular on $\mathbb{Z}$.

[17] If we drop the constraint $y \in \mathbb{N}$, the problem of the partition regularity of the formula $\exists x_1, x_2, x_3, x_4\, (x_1 + x_4 = x_2) \wedge (x_1 \cdot x_4 = x_3)$ is still open.

As for the second claim, let $A \in \mathcal{U}$. Then $I_A(\varphi)$ and $I_A(\psi)$ belong to $\mathcal{U}$, hence $I_A(\varphi) \cap I_A(\psi) \in \mathcal{U}$, and

$$I_A(\varphi) \cap I_A(\psi) \models \exists x\, Q_1 y_1 \in S_1 \ldots Q_n y_n \in S_n \widetilde{Q_1} z_1 \in R_1 \widetilde{Q_m} z_m \in R_m \\ \varphi(x, y_1, \ldots, y_n) \wedge \psi(x, z_1, \ldots, z_m).$$

Since this formula is existential, this entails that

$$A \models \exists x\, Q_1 y_1 \in S_1 \ldots Q_n y_n \in S_n \widetilde{Q_1} z_1 \in R_1 \widetilde{Q_m} z_m \in R_m \varphi(x, y_1, \ldots, y_n) \wedge \psi(x, z_1, \ldots, z_m),$$

hence our claim is proven. □

Example 5.11 Let $X = \mathbb{N}$. For every $n \in \mathbb{N}$ let φ_n be the formula

$$\varphi_n(x_1, \ldots, x_n, y_1, \ldots, y_n) := \bigwedge_{i \leq n} \Big(\sum_{j \leq i} x_j = y_i\Big)$$

and let $E(\varphi_n)$ be the existential closure of φ_n. Hence, for every $A \in \mathbb{N}$ we have that $A \models E(\varphi_n)$ if and only if A contains a subset $\{a_1, \ldots, a_n\}$ of n elements such that all ordered sums $a_1 + a_2, a_1 + a_2 + a_3$ and so on lie in A. By Schur's Theorem (cf. [36]) we know that $E(\varphi_2)$ is partition regular. Let $\mathcal{U}$ be a $E(\varphi_2)$ ultrafilter (which, from now on, we shall call a Schur ultrafilter). We claim that for all $n \in \mathbb{N}$, $\mathcal{U} \models E(\varphi_n)$. We prove this by induction on n.

If $n = 2$, the claim coincides with our hypothesis.

Now let $n > 2$, let us suppose the claim true for $n - 1$, and let us prove it for n. By hypothesis and by inductive hypothesis, we have that $\mathcal{U}$ is a Schur and a $E(\varphi_{n-1})$-ultrafilter. In particular, $\mathcal{U}$ witnesses the formulas[18]

$$\exists z(\exists x_1 \exists x_2\, x_1 + x_2 = z)$$

and

$$\exists z(\exists x_3 \ldots \exists x_n \exists y_2 \ldots \exists y_n\, (z = y_2) \wedge \bigwedge_{i=3}^{n} \Big(z + \sum_{j=3}^{i} x_j = y_i\Big)),$$

hence by Theorem 5.10, $\mathcal{U}$ witnesses the formula

$$\exists z(\exists x_1 \exists x_2\, x_1 + x_2 = z) \wedge (\exists x_3 \ldots \exists x_n \exists y_2 \ldots \exists y_n\, (z = y_2) \wedge \bigwedge_{i=3}^{n} \Big(z + \sum_{j=3}^{i} x_j = y_j\Big))$$

therefore (by renaming the variables and by letting $y_1 = x_1$), $\mathcal{U}$ witnesses the partition regularity of the formula

$$\exists x_1 \ldots \exists x_n \exists y_1 \ldots \exists y_n \bigwedge_{i \leq n} \Big(\sum_{j \leq i} x_j = y_j\Big),$$

as desired.

5.2 Partition regularity of arbitrary formulas

Even if, in most cases, applications regard existential closures of totally open formulas, characterisations similar to that of Theorem 5.4 hold also in other cases.

Corollary 5.12 *Let $\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m)$ be a totally open formula and, for $i = 1, \ldots, m$, let $S_i \in \mathbb{V}(X)$ and $Q_i \in \{\exists, \forall\}$. Let $Y \in \mathbb{V}(X)$, $\mathcal{U} \in \beta Y$ and assume that the hyperextension $*$ is $|\wp(Y)|^+$-enlarging. Then the following conditions are equivalent:*

(1) there is a set A in $\mathcal{U}$ that satisfies $U_{\overrightarrow{Qy} \in \overrightarrow{S}}(\varphi(x_1, \ldots, x_n, y_1, \ldots, y_m))$;
(2) for every $\alpha_1, \ldots, \alpha_n$ in $\mu(\mathcal{U})$ the sentence $Q_1 y_1 \in {}^*S_1, \ldots, Q_m y_m \in {}^*S_m {}^*\varphi(\alpha_1, \ldots, \alpha_n, y_1, \ldots, y_m)$ holds true.

[18] We hope that the apparently strange naming of the variables makes the argument more transparent.

P r o o f . This is just Theorem 5.4 applied to the existential closure of $\neg\varphi$. □

A useful consequence of Corollary 5.12 is that, in some cases, the existence of a generator with some property implies that this property is shared by all other generators:

Corollary 5.13 *Let $\varphi(x, y_1, \dots, y_n)$ be a totally open formula and, for $i = 1, \dots, m$, let $S_i \in \mathbb{V}(X)$ and $Q_i \in \{\exists, \forall\}$. Let $Y \in \mathbb{V}(X)$ and assume that the hyperextension $*$ is $|\wp(Y)|^+$-enlarging. Let $\mathcal{U}$ be an ultrafilter in βY that witnesses $E_{\overrightarrow{Qy}\in\overrightarrow{S}}(\varphi(x, y_1, \dots, y_m))$. Then the formula*

$$\forall\alpha \in \mu(\mathcal{U})\, Q_1 y_1 \in {}^*S_1 \dots {}^*Q_n y_n \in {}^*S_n\, {}^*\varphi(\alpha, y_1, \dots, y_n)$$

holds true.

P r o o f . By Theorem 5.10, we have that $I_Y(\varphi) = \{a \in Y \mid Q_1 y_1 \in S_1 \dots Q_n y_n \in S_n\, \varphi(a, y_1, \dots, y_n)$ holds true$\} \in \mathcal{U}$, viz. there is a set A in $\mathcal{U}$ such that $\forall y \in A\, Q_1 y_1 \in S_1 \dots Q_n y_n \in S_n\, \varphi(y, y_1, \dots, y_n)$ holds true. The conclusion hence follows straightforwardly from Corollary 5.12. □

In all the following examples, we assume the extensions to be properly saturated.

Example 5.14 Let $\mathcal{U} \models \exists x, y_1, y_2\; y_1 + y_2 = x$. In particular, for every set $A \in \mathcal{U}$ we have that $\mathcal{U}$ witnesses $\exists x\, \exists y_1, y_2 \in A\, (y_1 + y_2 = x)$. Hence from Corollary 5.13 we deduce that for all $\alpha \in \mu(\mathcal{U})$ there are $\beta_1, \beta_2 \in {}^*A$ such that $\alpha = \beta_1 + \beta_2$. By saturation, this entails that for all $\alpha \in \mu(\mathcal{U})$ there are $\beta_1, \beta_2 \in \mu(\mathcal{U})$ such that $\alpha = \beta_1 + \beta_2$.

Example 5.15 Let $Y = \mathbb{N}$. The formulas $\varphi(d, x, y, z) : \exists x, y, z, d\; y - x = z - y = d$ and $\psi(d, u, v) : \exists d, u, v\; u + v = d$ are both partition regular and homogeneous. Hence from Example 5.6 we deduce that every ultrafilter $\mathcal{U} \in \overline{K(\beta\mathbb{N}, \odot)}$ (the minimal closed bilateral ideal in the semigroup $(\beta\mathbb{N}, \odot)$) witnesses both $\varphi(d, x, y, z)$ and $\psi(d, u, v)$. Therefore, by Corollary 5.13 we get that for every set $A \in \mathcal{U}$ there exists an arithmetic progression in A of length 3 with a common difference in A that can be written as a sum of elements of A: in fact, let $\alpha_1, \alpha_2, \alpha_3, \gamma \in \mu(\mathcal{U})$ be such that $\alpha_2 - \alpha_1 = \alpha_3 - \alpha_2 = \gamma$. Then, by Corollary 5.13 we can write $\gamma = \beta_1 + \beta_2$ for some $\beta_1, \beta_2 \in \mu(\mathcal{U})$, therefore there are $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2 \in \mu(\mathcal{U})$ such that $\alpha_2 - \alpha_1 = \alpha_3 - \alpha_2 = \beta_1 + \beta_2$, and we conclude by Theorem 5.4. Analogously, we can prove that every set $A \in \mathcal{U}$ contains elements x, y, z that are increments in arithmetic progressions of length 3 and such that $x + y = z$. Moreover, if $\mathcal{U} \odot \mathcal{U} = \mathcal{U} \in \overline{K(\beta\mathbb{N}, \odot)}$, then $\mathcal{U}$ witnesses also the formula $\varphi(d, u, v) : \exists d, u, v\; u \cdot v = d$, hence, again by Corollary 5.13, we get that every set $A \in \mathcal{U}$ contains an arithmetic progression in A of length 3 with a common difference in A that can be written as a product of elements of A.

Example 5.16 Selective ultrafilters admit several equivalent characterisations (cf., e.g., [7]). One of them says that $\mathcal{U}$ is a selective ultrafilter on Y if and only if every function $f : Y \to Y$ is $\mathcal{U}$-equivalent to either an injective or a constant function, viz. there exists $A \in \mathcal{U}$ such that $f|_A$ is injective or constant. By Corollary 5.12, this is equivalent to say that for every $f : Y \to Y$ the function ${}^*f$ is injective or constant on $\mu(\mathcal{U})$.

Let us consider the case $Y = \mathbb{N}$. In this case, it is simple to see that "injective" can be substituted with "strictly increasing". Let $P(x) \in \mathbb{Z}[x]$. Let $\{a_n \mid n \in \mathbb{N}\}$ be the sequence inductively defined as follows: $a_0 = 0$ and, for every $n \geq 0$, $a_{n+1} = \min\{m \in \mathbb{N} \mid m > |P(j)|\, \forall j \leq a_n\}$.

Let $f_P : \mathbb{N} \to \mathbb{N}$ be the function such that for all $m \in \mathbb{N}$, $f_P(m) = \max\{a_n \mid a_n \leq m\}$. As $f_P^{-1}(m)$ is finite for every $m \in \mathbb{N}$, there exists $A \in \mathcal{U}$ such that $f_P|_A$ is increasing. Hence we have that

$$\forall P(x) \in \mathbb{Z}[x] \forall\alpha, \beta \in \mu(\mathcal{U})(\alpha < \beta) \Rightarrow (|P(\alpha)| < \beta). \tag{1}$$

As a consequence, we have that no selective ultrafilter is Schur: in fact, if $\mathcal{U}$ is a selective Schur ultrafilter, by Theorem 5.4 there are $\alpha, \beta, \gamma \in \mu(\mathcal{U})$ such that $\alpha + \beta = \gamma$ and, if $\alpha \geq \beta$, this means that $\alpha < \gamma \leq 2\alpha$, which is in contradiction with the characterisation (1).

Example 5.17 The result of Example 5.16 can be generalised. First of all, from characterisation (1) we deduce immediately the following strengthening:

$$\forall n \in \mathbb{N}\, \forall P(x_1, \dots, x_n) \in \mathbb{Z}[x_1, \dots, x_n]\, \forall\alpha_1, \dots, \alpha_n, \beta \in \mu(\mathcal{U})$$
$$(\alpha_1, \dots, \alpha_n < \beta) \Rightarrow (|P(\alpha_1, \dots, \alpha_n)| < \beta); \tag{2}$$

in fact, if $P(x_1,\dots,x_n)=\sum_{i=1}^{k} c_i x_1^{e_{1,i}}\cdot\dots\cdot x_n^{e_{n,i}}$, then

$$|P(x_1,\dots,x_n)| = |\sum_{i=1}^{k} c_i x_1^{e_{1,i}}\cdot\dots\cdot x_n^{e_{n,j}}| \leq \sum_{i=1}^{k} |c_i x_1^{e_{1,i}}\cdot\dots\cdot x_n^{e_{n,i}}|,$$

hence, if $\alpha=\max\{\alpha_i \mid i\leq n\}$, then $|P(\alpha_1,\dots,\alpha_n)|\leq\sum_{i=1}^{k}|c_i|\alpha^{\sum_{j=1}^{n} e_{n,j}}$, so we conclude by characterisation (1).

Now we use fact (2) to prove that for every polynomial $P(x_1,\dots,x_n)\in\mathbb{Z}[x_1,\dots,x_n]$ and for every selective ultrafilter $\mathcal{U}$, $\mathcal{U}$ is not a witness of the partition regularity of the formula

$$\exists x_1,\dots,x_n\,(\bigwedge_{1\leq i<j\leq n} x_i\neq x_j)\wedge P(x_1,\dots,x_n)=0. \tag{3}$$

We proceed by induction.

If $n=1$, the claim follows as for every infinite hypernatural number α, for every polynomial $P(x)\in\mathbb{Z}[x]$, if $P\neq 0$, then $P(\alpha)\neq 0$: in fact, if $P\neq 0$, then the equation $P(x)=0$ has only a finite amount of solutions, hence the claim is a trivial consequence of transfer.

Now let $n>1$ and let us assume the claim to be true for $n-1$. Assume, towards a contradiction, that there exists a polynomial $P(x_1,\dots,x_n)\in\mathbb{Z}[x_1,\dots,x_n]$ and a selective ultrafilter $\mathcal{U}$ that witnesses the partition regularity of the formula (3). Then, by Theorem 5.4 we can find mutually distinct elements $\alpha_1,\dots,\alpha_n\in\mu(\mathcal{U})$ such that $P(\alpha_1,\dots,\alpha_n)=0$. By rearranging the indexes, if necessary, we can assume that $\alpha_n=\max\{\alpha_i\mid i\leq n\}$.

Let $P(x_1,\dots,x_n)=\sum_{i=1}^{k} c_i x_1^{e_{1,i}}\cdot\dots\cdot x_n^{e_{n,i}}$, let $J=\{i\in[1,k]\mid e_{n,i}>0\}$, let $Q(x_1,\dots,x_n)=\sum_{i\in J} c_i x_1^{e_{1,i}}\cdot\dots\cdot x_n^{e_{n,i}}$ and $R(x_1,\dots,x_{n-1})=\sum_{i\notin J} c_i x_1^{e_{1,i}}\cdot\dots\cdot x_{n-1}^{e_{n-1,i}}$. As $P(\alpha_1,\dots,\alpha_n)=0$, we have that $|Q(\alpha_1,\dots,\alpha_n)|=|R(\alpha_1,\dots,\alpha_{n-1})|$. From characterisation (2) we have that $|R(\alpha_1,\dots,\alpha_{n-1})|<\alpha_n$. We consider two cases:

(i) If $Q(\alpha_1,\dots,\alpha_n)\neq 0$, then $|Q(\alpha_1,\dots,\alpha_n)|\geq\alpha_n$, as $Q(x_1,\dots,x_n)\in\mathbb{Z}[x_1,\dots,x_n]$, $x_n\mid Q(x_1,\dots,x_n)$ and $Q(\alpha_1,\dots,\alpha_n)\neq 0$, hence it cannot be $|Q(\alpha_1,\dots,\alpha_n)|=|R(\alpha_1,\dots,\alpha_{n-1})|$ and we have reached an absurd;

(ii) If $Q(\alpha_1,\dots,\alpha_n)=0$, then $R(x_1,\dots,x_{n-1})=0$, and we can conclude by using the inductive hypothesis.

5.3 Combinatorial properties with internal parameters

As shown in our examples, Theorem 5.4 can be used to prove several properties of monads. This result can be strengthened, in saturated extensions, taking into account also internal parameters:

Theorem 5.18 *Let $Y\in\mathbb{V}(X)$, and assume that the hyperextension $*$ is $|\wp(Y)|^+$-saturated. Let $\overrightarrow{p}:=(p_1,\dots,p_k)$, where $p_1,\dots,p_k$ are internal objects in $\mathbb{V}(X)$. Let $S_1,\dots,S_m$ be internal sets in $\mathbb{V}(X)$ and, for every $i=1,\dots,m$ let $Q_i\in\{\exists,\forall\}$. Let $\mathcal{U}\in\beta Y$ and let $\varphi(x_1,\dots,x_n,y_1,\dots,y_m,z_1,\dots,z_k)$ be a totally open formula. The following facts are equivalent:*

(1) for all $A\in\mathcal{U}$ the formula $\exists\alpha_1\dots\alpha_n\in{}^*A\; Q_1\beta_1\in S_1\dots Q_m\beta_m\in S_m\;{}^*\varphi(\overrightarrow{\alpha},\overrightarrow{\beta},\overrightarrow{p})$ holds true, where $\overrightarrow{\alpha}=(\alpha_1,\dots,\alpha_n)$ and $\overrightarrow{\beta}=(\beta_1,\dots,\beta_m)$;

(2) $\exists\alpha_1\dots\alpha_n\in\mu(\mathcal{U})\; Q_1\beta_1\in S_1\dots Q_m\beta_m\in S_m\;{}^*\varphi(\overrightarrow{\alpha},\overrightarrow{\beta},\overrightarrow{p})$ holds true, where $\overrightarrow{\alpha}=(\alpha_1,\dots,\alpha_n)$ and $\overrightarrow{\beta}=(\beta_1,\dots,\beta_m)$.

Proof. "(1) $\Rightarrow$ (2)": For every $A\in\mathcal{U}$ let $I_A=\{(\alpha_1,\dots,\alpha_m)\in{}^*A^m\mid Q_1\beta_1\in S_1\dots Q_m\beta_m\in S_m{}^*\varphi(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_m,p_1,\dots,p_k)$ holds true$\}$. The family $\{I_A\}_{A\in\mathcal{U}}$ has the finite intersection property as $I_{A_1}\cap I_{A_2}=I_{A_1\cap A_2}$, and every set I_A is internal by the internal definition principle. Hence, by saturation the formula

$$\exists\alpha_1\dots\alpha_n\in\mu(\mathcal{U})\;Q_1\beta_1\in S_1\dots Q_m\beta_m\in S_m\;{}^*\varphi(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_m,p_1,\dots,p_k)$$

holds true.

"(1) $\Rightarrow$ (2)": Just notice that $\mu(\mathcal{U})\subseteq{}^*A$ for every $A\in\mathcal{U}$ by definition. □

Corollary 5.19 *Let $Y \in \mathbb{V}(X)$, and assume that the hyperextension $*$ is $|\wp(Y)|^+$-saturated. Let $\overrightarrow{p} := (p_1, \dots, p_k)$ where $p_1, \dots, p_k$ are internal objects in $\mathbb{V}(X)$. Let $S_1, \dots, S_m$ be internal sets in $\mathbb{V}(X)$ and, for every $i = 1, \dots, m$ let $Q_i \in \{\exists, \forall\}$. Let $\mathcal{U} \in \beta Y$ and let $\varphi(x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_k)$ be a totally open formula. The following facts are equivalent:*

(1) there is an $A \in \mathcal{U}$ such that $\forall \alpha_1 \dots \alpha_n \in {}^*A\ Q_1\beta_1 \in S_1 \dots Q_m\beta_m \in S_m\ {}^*\varphi(\overrightarrow{\alpha}, \overrightarrow{\beta}, \overrightarrow{p})$ holds true, where $\overrightarrow{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\overrightarrow{\beta} = (\beta_1, \dots, \beta_m)$;

(2) $\forall \alpha_1 \dots \alpha_n \in \mu(\mathcal{U})\ Q_1\beta_1 \in S_1 \dots Q_m\beta_m \in S_m\ {}^*\varphi(\overrightarrow{\alpha}, \overrightarrow{\beta}, \overrightarrow{p})$ holds true, where $\overrightarrow{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\overrightarrow{\beta} = (\beta_1, \dots, \beta_m)$.

P r o o f . Just apply Theorem 5.18 to $\neg\varphi(x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_k)$. □

Example 5.20 Let $X = \mathbb{Q}$. Let $\mathcal{U}$ be a positive infinite ultrafilter in $\beta\mathbb{Q}$ (in the sense of Example 4.19). We claim that $\mu(\mathcal{U})$ is right and left unbounded in the set $\mathrm{Inf}({}^*\mathbb{Q})$ of positive infinite elements of ${}^*\mathbb{Q}$. Towards a contradiction, assume that there are $\beta_1, \beta_2 \in \mathrm{Inf}({}^*\mathbb{Q})$ such that $\beta_1 < \alpha < \beta_2$ for every $\alpha \in \mu(\mathcal{U})$. Then by Corollary 5.19 we have that there exists $A \in \mathcal{U}$ such that $\beta_1 < \alpha < \beta_2$ for every $\alpha \in {}^*A$. However:

(i) β_1 cannot exist, as $A \subseteq {}^*A$ and $q < \beta_1$ for every $q \in A$;
(ii) β_2 cannot exist, as every set $B \in \mathcal{U}$ is right unbounded (and so is ${}^*B$ by transfer).

Example 5.21 Let $X = \mathbb{N}^\mathbb{N}$. Let $\mathcal{U}$ be an ultrafilter in βX and let $\alpha_1, \alpha_2 \in {}^*\mathbb{N}$. Then every generator φ of $\mathcal{U}$ maps α_1 into α_2 if and only if there is a set $B \in \mathcal{U}$ such that every function in B maps α_1 into α_2. E.g., if $\alpha_1 \in \mathbb{N}$ and $\alpha_2 \in {}^*\mathbb{N}\backslash\mathbb{N}$ this means that no ultrafilter has this property, as if a function $f \in B$, then ${}^*f(\alpha_1) \in \mathbb{N}$.

We conclude by considering another version of the partition regularity of properties where multiple ultrafilters are considered at once.[19] Such notions apply to partial partition regular properties in Ramsey theory, which includes several fundamental results proven recently in the area (cf., e.g., [16, 32]).

Example 5.22 In [8, 26, 28] it has been introduced and studied the notion of finite embeddability between subsets of $\mathbb{N}$. In [30], this notion has been extended to arbitrary families of functions and semigroups. In particular, if $(S, \cdot)$ is a commutative[20] semigroup, a set $A \subseteq S$ is finitely embeddable in a set $B \subseteq S$ (notation: $A \leq_{\mathrm{fe}} B$) if and only if for every finite subset $F \subseteq A$ there exists $t \in S$ such that $t \cdot F \subseteq B$, viz. if and only if

$$\forall n \in \mathbb{N}\, \forall a_1, \dots, a_n \in A\, \exists b_1, \dots, b_n \in B\, \exists t \in S\, \bigwedge_{i \leq n} (a_i \cdot t = b_i).$$

This notion has been extended to ultrafilters in [28]: a ultrafilter $\mathcal{U} \in \beta S$ is *finitely embeddable* in $\mathcal{V} \in \beta S$ (notation: $\mathcal{U} \leq_{\mathrm{fe}} \mathcal{V}$) if and only if for every set $B \in \mathcal{V}$ there exists $A \in \mathcal{U}$ such that $A \leq_{\mathrm{fe}} B$, viz. if and only if

$$\forall B \in \mathcal{V}\, \exists A \in \mathcal{U}\, \forall n \in \mathbb{N}\, \forall a_1, \dots, a_n \in A\, \exists b_1, \dots, b_n \in B\, \exists t \in S\, \bigwedge_{i \leq n} (a_i + t = b_i).$$

We want to give a nonstandard characterisation of monads that allows to study certain properties like that expressed in Example 5.22. For the sake of simplicity, we give it for an alternation $\forall/\exists$ of two ultrafilters; similar characterisations for arbitrary finite amounts of ultrafilters and different alternations of quantifiers can be analogously deduced.

Theorem 5.23 *Let $Y \in \mathbb{V}(X)$. Let $\overrightarrow{p} := (p_1, \dots, p_k)$, where $p_1, \dots, p_k$ are internal objects in $\mathbb{V}(X)$. Let $S_1, \dots, S_h$ be internal sets in $\mathbb{V}(X)$. Let $\mathcal{U}, \mathcal{V} \in \beta Y$ and let $\varphi(x_1, \dots, x_n, y_1, \dots, y_m, t_1, \dots, t_h, z_1, \dots, z_k)$ be a totally open formula. Assume that the extension ${}^*Y$ is $|Y|^+$-saturated. The following facts are equivalent:*

(1) for all $A \in \mathcal{U}$ there is a $B \in \mathcal{V}$ such that

$$\forall \beta_1, \dots, \beta_m \in {}^*B\, \exists \alpha_1, \dots, \alpha_n \in {}^*A\, \exists s_1 \in S_1 \dots \exists s_h \in S_h\, {}^*\varphi(\overrightarrow{\alpha}, \overrightarrow{\beta}, \overrightarrow{s}, \overrightarrow{p}),$$

[19] Similar ideas, but in a rather different context, appeared in [6].

[20] In a very similar way, we can work with non-commutative semigroups; however, this means considering the different notions of right and left finite embeddability, and we prefer to avoid such complications here.

holds true, where $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\overrightarrow{\beta} = (\beta_1, \ldots, \beta_m)$, $\overrightarrow{s} = (s_1, \ldots, s_h)$*;*

(2) $\forall \beta_1 \ldots \beta_n \in \mu(\mathcal{V})\, \exists \alpha_1 \ldots \alpha_m \in \mu(\mathcal{U})\, \exists s_1 \in S_1 \ldots \exists s_h \in S_h\, {}^*\varphi(\overrightarrow{\alpha}, \overrightarrow{\beta}, \overrightarrow{s}, \overrightarrow{p})$ *holds true, where* $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_n)$, $\overrightarrow{\beta} = (\beta_1, \ldots, \beta_m)$, $\overrightarrow{s} = (s_1, \ldots, s_h)$.

P r o o f. "(1) $\Rightarrow$ (2)": Let $\overrightarrow{\beta} \in \mu(\mathcal{V})^n$. As $\mu(\mathcal{V}) \subseteq {}^*B$ for every $B \in \mathcal{V}$, we have that for every $A \in \mathcal{U}$ the set $I_A := \{\overrightarrow{\alpha} \in {}^*A^n \mid \exists s_1 \in S_1 \ldots \exists s_h \in S_h\, {}^*\varphi(\overrightarrow{\alpha}, \overrightarrow{\beta}, \overrightarrow{s}, \overrightarrow{p}) \text{ holds true}\} \neq \varnothing$. As I_A is internal and $\{I_A\}_{A \in \mathcal{U}}$ has the finite intersection property, by $|\wp(Y)|^+$-saturation we have that $\bigcap_{A \in \mathcal{U}} I_A \neq \varnothing$, and we conclude as $\bigcap_{A \in \mathcal{U}} I_A \subseteq \mu(\mathcal{U})^n$.

"(2) $\Rightarrow$ (1)": Let $A \in \mathcal{U}$. By using ${}^*A$ as a parameter, we see that the thesis is a straightforward consequence of Corollary 5.19. □

Example 5.24 Let us consider the finite embeddability. Let $(S, \cdot)$ be a commutative semigroup with identity and let $\mathcal{U}, \mathcal{V} \in \beta S$. As a consequence of Theorem 5.23, the following two properties are equivalent:[21]

(i) for all $n \in \mathbb{N}$ and all $A \in \mathcal{V}$ there is a $B \in \mathcal{U}$ such that $\forall b_1, \ldots, b_n \in B\, \exists s \in S\; s \cdot b_1, \ldots, s \cdot b_n \in A$;
(ii) for all $n \in \mathbb{N}$ and all $\beta_1, \ldots, \beta_n \in \mu(\mathcal{U})$ there is a $\sigma \in {}^*S$ such that $\sigma \cdot \beta_1, \ldots, \sigma \cdot \beta_n \in \mu(\mathcal{V})$.

Notice that if $\mathcal{U}$ is finitely embeddable in $\mathcal{V}$, then the first property holds trivially. In particular, as $\mathcal{V} \in \beta S$ is such that $\forall \mathcal{U} \in \beta S\, \mathcal{U} \leq_{\mathrm{fe}} \mathcal{V}$ if and only if $\mathcal{V} \in \overline{K(\beta S, \odot)}$ (this result has been proven in [30, Theorem 4.13]), we obtain the implication (1) $\Rightarrow$ (2), where

(1) $\mathcal{V} \in \overline{K(\beta S, \odot)}$;
(2) for all $n \in \mathbb{N}$ and all $\beta_1 \sim_S \cdots \sim_S \beta_n \in {}^*S$ there is a $\sigma \in {}^*S$ such that $\sigma \cdot \beta_1, \ldots, \sigma \cdot \beta_n \in \mu(\mathcal{V})$.

Finally, as $\mathcal{V} \in \overline{K(\beta S, \odot)}$ if and only if every set $A \in \mathcal{V}$ is piecewise syndetic in $(S, \cdot)$ (cf., e.g., [18, Corollary 4.41]), from Theorem 5.18 we obtain the following property[22] of piecewise syndetic subsets of S: if $A \subseteq S$ is piecewise syndetic, then for all $n \in \mathbb{N}$ and all $\beta_1 \sim_S \cdots \sim_S \beta_n \in {}^*S$ there is a $\sigma \in {}^*S$ such that $\sigma \cdot \beta_1, \ldots, \sigma \cdot \beta_n \in {}^*A$.

Example 5.25 Finite embeddabilities can be generalised to arbitrary families of functions $\mathcal{F} : S^n \to S$ (cf. [30]). In particular, let $S = \mathbb{N}$ and $\mathcal{F} \subseteq \mathbb{N}^{\mathbb{N}}$ be the family of affinities $\mathcal{F} := \{f_{a,b} : \mathbb{N} \to \mathbb{N} \mid \forall n \in \mathbb{N}\; f_{a,b}(n) = an + b\}$. We say that a set $A \subseteq \mathbb{N}$ is $\mathcal{F}$-finitely embeddable in $B \subseteq \mathbb{N}$ (notation: $A \leq_{\mathcal{F}} B$) if for every finite set $F \subseteq A$ there exists $f \in \mathcal{F}$ such that $f(A) \subseteq B$. Of course, this notion is related to that of AP-rich set (viz. of a set that contains arbitrarily long arithmetic progressions): in fact, it is straightforward to see that $B \subseteq \mathbb{N}$ is AP-rich if and only if $A \leq_{\mathcal{F}} B$ for every $A \subseteq \mathbb{N}$. $\mathcal{F}$-finite embeddability can be extended to ultrafilters as follows: we say that an ultrafilter $\mathcal{U} \in \beta\mathbb{N}$ is $\mathcal{F}$-finitely embeddable in $\mathcal{V} \in \beta\mathbb{N}$ if for every set $B \in \mathcal{V}$ there exists $A \in \mathcal{U}$ such that $A \leq_{\mathcal{F}} B$. An argument similar to that of Example 5.24 allows to show the implication (1) $\Rightarrow$ (2), where

(1) $\mathcal{U} \leq_{\mathcal{F}} \mathcal{V}$;
(2) for all $n \in \mathbb{N}$ and all $\beta_1 \ldots \beta_n \in \mu(\mathcal{U})$, there are $\sigma, \varrho \in {}^*\mathbb{N}$ such that $\sigma \cdot \beta_1 + \varrho, \ldots, \sigma \cdot \beta_n + \varrho \in \mu(\mathcal{V})$.

In [30] we proved that $\mathcal{V} \in \beta\mathbb{N}$ is such that $\forall \mathcal{U} \in \beta\mathbb{N}\, \mathcal{U} \leq_{\mathcal{F}} \mathcal{V}$ if and only if every set $A \in \mathcal{V}$ is AP-rich. By the implication discussed above, we know that such an ultrafilter $\mathcal{V}$ has the following property: for all $n \in \mathbb{N}$ and all $\beta_1 \sim_{\mathbb{N}} \cdots \sim_{\mathbb{N}} \beta_n \in {}^*\mathbb{N}$ there are $\sigma, \varrho \in {}^*\mathbb{N}$ such that $\sigma \cdot \beta_1 + \varrho, \ldots, \sigma \cdot \beta_n + \varrho \in \mu(\mathcal{V})$. In particular, as the family of AP-rich sets is strongly partition regular[23] (cf., e.g., [30, Theorem 6.3]), from Theorem 5.18

[21] More precisely, for all $n \in \mathbb{N}$ the properties:

(i) $\forall A \in \mathcal{V}\, \exists B \in \mathcal{U}\, \forall b_1, \ldots, b_n \in B\, \exists s \in S\; s \cdot b_1, \ldots, s \cdot b_n \in A$;
(ii) $\forall \beta_1, \ldots, \beta_n \in \mu(\mathcal{U})\, \exists \sigma \in {}^*S$ such that $\sigma \cdot \beta_1, \ldots, \sigma \cdot \beta_n \in \mu(\mathcal{V})$,

are equivalent; our claim is a trivial consequence of this fact.

[22] Notice that this property resembles a well-known characterisation of thick subsets of S: a set $A \subseteq S$ is thick if and only if for every $s_1, \ldots, s_n \in S$ there exists $t \in S$ such that $t \cdot s_1, \ldots t \cdot s_n \in A$, i.e., (by transfer) if for every $\beta_1, \ldots, \beta_n \in {}^*S$ there exists $\sigma \in {}^*S$ such that $\sigma \cdot \beta_1, \ldots \sigma \cdot \beta_n \in {}^*A$.

[23] If S is a set, a family $\mathcal{G} \subseteq \wp(S)$ is strongly partition regular if and only if for every $A \in \mathcal{G}$, for every finite partition $A = A_1 \cup \cdots \cup A_n$ there exists $i \leq n$ such that $A_i \in \mathcal{G}$. In terms of ultrafilters, this property can be reformulated as follows: $\mathcal{G}$ is strongly partition regular if and

we obtain the following characterisation of AP-rich sets: $A \subseteq \mathbb{N}$ is AP-rich if and only if for all $n \in \mathbb{N}$ and all $\beta_1 \sim_{\mathbb{N}} \cdots \sim_{\mathbb{N}} \beta_n \in {}^*\mathbb{N}$, there are $\sigma, \varrho \in {}^*\mathbb{N}$ such that $\sigma \cdot \beta_1 + \varrho, \ldots, \sigma \cdot \beta_n + \varrho \in {}^*A$. In fact, if A is AP-rich, then there exists $\mathcal{U} \in \beta\mathbb{N}$ such that $A \in \mathcal{U}$ and every set $B \in \mathcal{U}$ is AP-rich, so for all $n \in \mathbb{N}$ and all $\beta_1 \sim_{\mathbb{N}} \cdots \sim_{\mathbb{N}} \beta_n \in {}^*\mathbb{N}$, there are $\sigma, \varrho \in {}^*\mathbb{N}$ such that $\sigma \cdot \beta_1 + \varrho, \ldots, \sigma \cdot \beta_n + \varrho \in \mu(\mathcal{U})$ and we conclude as $\mu(\mathcal{V}) \subseteq {}^*A$.

For the reverse implication, let A satisfy our hypothesis, and let $\beta_1 \sim_{\mathbb{N}} \cdots \sim_{\mathbb{N}} \beta_n$ form an arithmetic progression of length[24] n. Then $\sigma \cdot \beta_1 + \varrho, \ldots, \sigma \cdot \beta_n + \varrho$ is a length n arithmetic progression in ${}^*A$. As this property holds $\forall n \in \mathbb{N}$, we deduce that A is AP-rich.

Example 5.26 Of course, similar ideas to that introduced in Theorem 5.23 can be used to study other properties involving multiple ultrafilters and partition regularity. In [24], the authors proved that it is consistent with ZFC that for every finite coloring of $\mathbb{R}$ there is an infinite set $X \subseteq \mathbb{R}$ such that $X + X$ is monochromatic (whilst in [19] it was proven that also the negation of this statement is consistent with ZFC). Of course, without loss of generality we can assume that X is also monochromatic (not necessarily of the same colour of $X + X$). In terms of ultrafilters, this means that there are ultrafilters $\mathcal{U}, \mathcal{V} \in \beta\mathbb{R}$ such that

$$\text{for all } A \in \mathcal{U} \text{ there is a } B \in \mathcal{V} \text{ such that } B + B \subseteq A. \tag{4}$$

With arguments similar to those used in the proof of Theorem 5.23, it is simple to show that this property is equivalent to the following nonstandard fact:

$$\text{there is a } \Gamma \in {}^*\mathcal{V} \text{ such that } \Gamma + \Gamma \subseteq \mu(\mathcal{U}). \tag{5}$$

Finally, by noticing that (4) can be rewritten as

$$\text{for all } A \in \mathcal{U} \text{ and all } B \in \mathcal{V} \text{ there is a } C \in \wp(B) \cap \mathcal{V} \text{ such that } C + C \subseteq A,$$

we can strenghten (5) to obtain that it is consistent with ZFC to assume that there is $\Gamma \in {}^*\mathcal{V} \cap \wp(\mu(\mathcal{V}))$ such that $\Gamma + \Gamma \subseteq \mu(\mathcal{U})$.

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only if $\forall A \in \mathcal{G}\, \exists \mathcal{U} \in \beta S\, \mathcal{U} \subseteq \mathcal{G}$. Cf. [18, Theorem 3.1] for a proof of this equivalence; notice that in the reference strongly partition regular families are called partition regular.

[24] Such a configuration exists thanks to Theorem 5.4, as the property of containing an arithmetic progression of length n is partition regular.

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