Nonstandard characterizations of tensor products and monads in the theory of ultrafilters

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Abstract

We use nonstandard methods, based on iterated hyperextensions, to develop applications to Ramsey theory of the theory of monads of ultrafilters. This is performed by studying in detail arbitrary tensor products of ultrafilters, as well as by characterizing their combinatorial properties by means of their monads. This extends to arbitrary sets and properties methods previously used to study partition regular Diophantine equations on $\mathbb{N}$. Several applications are described by means of multiple examples.

1 Introduction

It is well known that ultrafilters and nonstandard analysis are closely related: on the one hand, models of nonstandard analysis are characterized, up to isomorphisms, as limit ultrapowers (see [10, Section 6.4]); on the other hand, the correspondence between elements of a nonstandard extension ${}^*X$ and ultrafilters on X was first observed (in the more general case of filters) by W.A.J. Luxemburg in [33], who introduced the concept of monad of a filter. This correspondence was then used by C. Puritz in [36, 37] and by G. Cherlin and J. Hirschfeld in [11] to produce new results about the Rudin-Keisler ordering and to characterize several classes of ultrafilters, including P-points and selective ones. Similar ideas were also pursued by S. Ng and H. Render in [35] and by A. Blass in [7].

In [28], we proved a combinatorial characterization of monads of ultrafilters in $\beta\mathbb{N}$ which made it possible to develop several applications in the study of the partition regularity of Diophantine equations[1] by means of some rather simple algebraic manipulations of hypernatural numbers. The partition regularity of Diophantine equations is a particular instance of the kind of problems that are studied in Ramsey theory, where one wants to understand which monochromatic structures can be found in some piece of arbitrary finite partitions of a given object.

The basic idea behind our nonstandard approach to Ramsey theory is that every set in a ultrafilter $\mathcal{U} \in \beta\mathbb{N}$ satisfies a prescribed property φ if and only if the monad of $\mathcal{U}$ satisfy an appropriate nonstandard version of φ. This idea has been developed in [14, 16, 17, 29, 30, 31, 32], and belongs to the family

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[1] See Theorem 45 for a definition of this notion.

of applications of nonstandard analysis in Ramsey theory, an approach that started with J. Hirschfeld in [22] and has subsequently been carried on by many authors. As R. Jin pointed out, nonstandard methods in Ramsey theory are very useful because they can be used to reduce the complexity of the mathematical objects that one needs in a proof, therefore offering a much better intuition, which allows to obtain much simpler (and shorter) proofs.

In [15], M. Di Nasso surveyed the nonstandard characterization of ultrafilters on $\mathbb{N}$, proving also several equivalent characterization of the elements of the monads of tensor products of ultrafilters. This paper can be seen as an extension of such a study, since our main aim is to characterize monads of ultrafilters and tensor products of ultrafilters on arbitrary sets, so to extend the nonstandard methods used for Diophantine equations to more general classes of problems in Ramsey theory. This requires to better understand arbitrary tensor products of ultrafilters, which are a basic important tool to develop such applications (e.g., in [5] tensor products of ultrafilters in S^n, for S semigroup, are used to obtain polynomial extensions of the Milliken-Taylor theorem). Moreover, it is helpful to characterize the Ramsey-theoretical properties of monads in terms of their combinatorial and algebraic structure for general properties, extending what we already did for Diophantine equations; such an approach could lead to unexpected applications in other related fields. It turns out that a good nonstandard framework to perform this study is given by iterated nonstandard extensions.

In Section 2 we recall the basic definitions and properties of iterated hyperextensions, providing the nonstandard framework that is used to develop the rest of the paper. In Section 3 we recall the definition of the monad of an ultrafilter. We also recall some basic properties of these monads, presenting some of their peculiar properties in iterated hyperextensions. In Section 4 we consider arbitrary tensor products of ultrafilters, we provide several equivalent characterizations of the elements in their monads and we extend the characterizations to tensor products of arbitrary (finite) length. Finally, in Section 5 we present several combinatorial properties of monads of arbitrary ultrafilters. Throughout the paper, several examples are also included to illustrate the use of such a theory in applications, as well as our main ideas.

This paper is self-contained: we only assume the reader to know the basics of ultrafilters and nonstandard analysis, in particular the notions of superstructure, transfer, ultrafilter, enlarging and saturation properties. In any case, a comprehensive reference about ultrafilters and their applications, especially in Ramsey theory, is the monograph [20]. As for nonstandard analysis, many short but rigorous presentations can be found in the literature. We suggest [2], where eight different approaches to nonstandard methods are presented, as well as the introductory book [19], which covers all the nonstandard tools that we need in this paper, except the iterated extensions that we will discuss in Section 2.

2 Iterated Hyperextensions

In this paper, we will adopt the so-called "external" approach to nonstandard analysis, based on superstructure models of nonstandard methods (see also [2, Section 3]):

Definition 1. A superstructure model of nonstandard methods is a triple

$\langle \mathbb{V}(X), \mathbb{V}(Y), * \rangle$, where

1. X is an infinite set;
2. $\mathbb{V}(X), \mathbb{V}(Y)$ are the superstructures on X, Y respectively;
3. $* : \mathbb{V}(X) \to \mathbb{V}(Y)$ is a star map, namely it satisfies the transfer principle.

We say that $\langle \mathbb{V}(X), \mathbb{V}(Y), * \rangle$ is a single superstructure model of nonstandard methods when $X = Y$.

From now on, we will use only single superstructure models of nonstandard methods. This is not restrictive: as proven in [2, Section 3] (see also [1]), every superstructure model is isomorphic to a single superstructure one.

The existence of saturated single superstructure models of nonstandard methods can be proven in different ways: we refer to [3], where single superstructure models are constructed by means of the so-called Alpha Theory, and to the nonstandard set theory $^*\mathrm{ZFC}$ introduced by M. Di Nasso in [12], where the enlarging map $*$ is defined for every set of the universe. Similar ideas have been studied, in the context of iterated ultrapowers, by K. Kunen in [27], by K. Hrbáček in [23] and by K. Hrbáček, O. Lessmann, R. O'Donovan in [24]. A clear presentation of iterated ultrapowers can also be found in [10, Section 6.5].

The main peculiarity of single superstructure models of nonstandard methods is that they allow to iterate the $*$-map. This allows to simplify certain proofs: for example, in [14] the structure $^{**}\mathbb{N}$, obtained by iterating twice the star map applied to $\mathbb{N}$, is used to give a rather short proof of Ramsey Theorem.

Iterated hyperextensions have already been studied in previous publications (e.g [14, 15, 28, 29, 30, 31, 32]). In this Section, we will recall only the main definitions and properties that will be used in the rest of the paper.

Definition 2. We define by induction the family $\langle S_n \mid n \in \mathbb{N} \rangle$ of functions $S_n : \mathbb{V}(X) \to \mathbb{V}(X)$ by setting $S_0 = id$ and, for every $n \geq 0$, $S_{n+1} = * \circ S_n$.

Let Y be a set in $\mathbb{V}(X)$. Notice that $S_2(^*Y) := {}^{**}Y$ is a nonstandard extension of both Y and $^*Y$. Intuitively, this extension resembles the extension from Y to $^*Y$. For example, if $Y = \mathbb{N}$, the fact that $^*\mathbb{N}$ is an end extension of $\mathbb{N}$, namely that

$$\forall \eta \in {}^*\mathbb{N} \setminus \mathbb{N}, \forall n \in \mathbb{N}\, \eta > n,$$

can be transferred to

$$\forall \eta \in {}^{**}\mathbb{N} \setminus {}^*\mathbb{N}, \forall n \in {}^*\mathbb{N}\, \eta > n,$$

which is the formula expressing that $^{**}\mathbb{N}$ is an end extension of $^*\mathbb{N}$.

However, not all the basic properties of the extension from Y to $^*Y$ holds also for the extension $^{**}Y$ of $^*Y$: for example, the fact that $^*A = A$ for every finite subset of $\mathbb{N}$ (as usual, we identify every number $n \in \mathbb{N}$ with $^*n$) does not hold true for $^*\mathbb{N}$. Just observe that if $\alpha \in {}^*\mathbb{N} \setminus \mathbb{N}$ then, by transfer, $^*\alpha \in {}^{**}\mathbb{N} \setminus {}^*\mathbb{N}$, hence $^*\{\alpha\} = \{^*\alpha\} \neq \{\alpha\}$.

In any case, we have the following result, which is a trivial consequence of the composition properties of elementary embeddings:

Theorem 3. *For every positive natural number n, $\langle \mathbb{V}(X), \mathbb{V}(X), S_n \rangle$ is a single superstructure model of nonstandard methods.*

In certain cases, as we will show in Section 4, it is helpful to consider the following extension of $\mathbb{N}$:

Definition 4. Let $\langle \mathbb{V}(X), \mathbb{V}(X), *\rangle$ be a superstructure model of nonstandard methods. We call ω-hyperextension of X, and denote by ${}^{\bullet}X$, the union of all hyperextensions $S_n(X)$:

$$^{\bullet}X = \bigcup_{n\in\mathbb{N}} S_n(X).$$

Since

$$\langle S_n(X) \mid n < \omega\rangle$$

is an elementary chain of extensions, we have that ${}^{\bullet}X$ is a hyperextension of X. Moreover, as ${}^{\bullet}A \supset S_n(A) \supset A$ for every $A \subseteq X$, we have the following trivial result:

Proposition 5. *Let $n \in \mathbb{N}$ and let κ be a cardinal number. Then the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) hold, where*

1. *$\langle \mathbb{V}(X), \mathbb{V}(X), *\rangle$ has the κ-enlarging property;*
2. *$\langle \mathbb{V}(X), \mathbb{V}(X), S_n\rangle$ has the κ-enlarging property;*
3. *$\langle \mathbb{V}(X), \mathbb{V}(X), \bullet\rangle$ has the κ-enlarging property.*

However, let us notice that the previous result does not hold, in general, if we substitute enlarging with saturation. In fact, ${}^{\bullet}X$ has cofinality $\aleph_0$ (which is in contrast with κ-saturation properties for $\kappa > \aleph_0$, as the cofinality is always at least as great as the cardinal saturation), since a countable right unbounded sequence in ${}^{\bullet}\mathbb{N}$ can be constructed by choosing, for every natural number n, an hypernatural number α_n in $S_{n+1}(X) \setminus S_n(X)$.

3 Monads

In the following, we will use the symbol $\star$ to denote a generic nonstandard extensions (which could be $*, S_n$ or $\bullet$), reserving to $*$ and $\bullet$ the meanings given in Section 2. We hope that this will increase the readability of the paper[2].

Monads of filters were first introduced by W.A.J. Luxembourg in [33]. In the past few years, monads of ultrafilters on $\mathbb{N}$ have been used to prove many results in combinatorial number theory, especially in the context of the partition regularity of equations (see e.g. [14, 15, 16, 17, 28, 29, 30, 31, 32]). However, it seems that to extend the range of applications of these methods, a deeper study of monads in a wider generality is needed. Our aim in this section is to start such a study. We will adopt the framework of iterated nonstandard hyperextensions, since they provide a simpler setting for the study of monads, as we are going to show.

Definition 6. Let Y be a set in $\mathbb{V}(X)$ and let $\langle \mathbb{V}(X), \mathbb{V}(X), *\rangle$ be a single superstructure model of nonstandard methods. Let $\mathcal{U}$ be an ultrafilter on Y. For every $n \in \mathbb{N}$ we let

$$\mu_n(\mathcal{U}) := \bigcap_{A\in\mathcal{U}} S_n(A)$$

[2]At least, we hope that this will not decrease the readability of the paper.

(with the agreement that $\mu(\mathcal{U}) := \mu_1(\mathcal{U})$) and

$$\mu_\infty(\mathcal{U}) := \bigcap_{A\in\mathcal{U}} {}^\bullet A.$$

Elements of $\mu_\infty(\mathcal{U})$ will be called generators of $\mathcal{U}$. Finally, when we consider a generic extension $\langle \mathbb{V}(X), \mathbb{V}(X), \star\rangle$ we will write

$$\mu_\star(\mathcal{U}) := \bigcap_{A\in\mathcal{U}} {}^\star A.$$

Notice that, equivalently, $\mu_\infty(\mathcal{U}) = \bigcup_{n\in\mathbb{N}} \mu_n(\mathcal{U})$. In general, monads can be empty if the extensions are not sufficiently enlarged. However, we have the following result:

Theorem 7. *Let Y be a set in $\mathbb{V}(X)$. Then for every $\alpha \in {}^\star Y$ the set*

$$\mathfrak{U}_\alpha := \{A \subseteq Y \mid \alpha \in {}^\star Y\}$$

is a ultrafilter on Y. Moreover, if the extension $\star : Y \to {}^\star Y$ has the $|\wp(Y)|^+$-enlarging property then $\mu_\star(\mathcal{U}) \neq \emptyset$ for every $\mathcal{U} \in \beta Y$.

Proof. That $\mathcal{U}_\alpha$ is a ultrafilter is straightforward. The second claim follows as every ultrafilter $\mathcal{U}$ on Y is a family with the finite intersection property and cardinality $|\wp(Y)|$, and the $|\wp(Y)|^+$-enlarging property hence entails that $\mu_\star(\mathcal{U}) \neq \emptyset$. □

Monads can be used to identify every ultrafilter with the trace of a principal one on a higher level: in fact, if $\alpha \in {}^\star Y$ then $\alpha \in \mu(\mathcal{U})$ if and only if $\mathcal{U} = tr_Y(P(\alpha))$, where

$$P(\alpha) := \{A \subseteq {}^\star Y \mid \alpha \in A\}$$

is the principal ultrafilter generated by α on ${}^\star Y$ and, for every ultrafilter $\mathcal{V}$ on ${}^\star Y$, we set

$$tr_Y(\mathcal{V}) := \{A \subseteq Y \mid {}^\star A \in \mathcal{V}\}.$$

Definition 8. For $\alpha, \beta \in {}^\star Y$ we will write $\alpha \sim_Y \beta$ if the ultrafilters generated on Y by α and β coincide.

A remark is in order: in previous papers, the equivalence relation $\sim_Y$ was denoted by $\sim_\mathcal{U}$ or $\underset{u}{\sim}$ (see e.g. [14, 15]). However, here we prefer to use the notation $\sim_Y$ as this equivalence relation depends on the set on which we are constructing the ultrafilters. To better explain what we mean, let $\alpha \neq \beta \in {}^*\mathbb{N}$ be such that $\alpha \sim_\mathbb{N} \beta$. Then (as we will prove in Proposition 12) ${}^*\alpha \sim_\mathbb{N} {}^*\beta$. However, ${}^*\alpha \nsim_{{}^*\mathbb{N}} {}^*\beta$! In fact, the ultrafilter generated by ${}^*\alpha$ on ${}^*\mathbb{N}$ is

$$\{A \subseteq {}^*\mathbb{N} \mid {}^*\alpha \in {}^*A\} = \{A \subseteq {}^*\mathbb{N} \mid \alpha \in A\} = P(\alpha),$$

and analogously the ultrafilter generated by ${}^*\beta$ on ${}^*\mathbb{N}$ is $P(\beta)$, and $P(\alpha) \neq P(\beta)$ since $\alpha \neq \beta$.

When we work in ω-hyperextensions, it is useful to study the relationships between sets of generators of the same ultrafilter in different extensions. To do this, let us fix a definition:

Definition 9. Let $\langle \mathbb{V}(X), \mathbb{V}(X), * \rangle$ be a single superstructure model of nonstandard methods and let Y be a set in $\mathbb{V}(X)$. We say that Y is coherent if $Y \subseteq {}^*Y$. We say that Y is completely coherent if A is coherent for every $A \subseteq Y$.

Notice that if Y is coherent then $S_n(Y) \subseteq S_m(Y)$ for every $n \leq m$.

Example 10. $\mathbb{N}$ is completely coherent, as we identify every $n \in \mathbb{N}$ with ${}^*n$. However, if $\alpha \in {}^*\mathbb{N} \setminus \mathbb{N}$, then $\{\alpha\}$ is not coherent, since ${}^*\{\alpha\} = \{{}^*\alpha\}$. Finally, $\mathbb{N} \cup \{\alpha\}$ is coherent but not completely coherent.

Theorem 11. *Let $\langle \mathbb{V}(X), \mathbb{V}(X), * \rangle$ be a single superstructure model of nonstandard methods and let Y be a set in $\mathbb{V}(X)$. The following are equivalent:*

1. *Y is completely coherent;*
2. *for all $y \in Y$ $y = {}^*y$.*

Proof. (1) $\Rightarrow$ (2) Let $y \in Y$. As Y is completely coherent, we have that $\{y\} \subseteq {}^*\{y\} = \{{}^*y\}$, hence $y = {}^*y$.

(2) $\Rightarrow$ (1) Let $A \subseteq Y$. Then $A \subseteq {}^*A$ since, for every $a \in A$, we have that $a = {}^*a \in {}^*A$. □

Proposition 12. *Let Y be a set in $\mathbb{V}(X)$. For every ultrafilter $\mathcal{U}$ on Y, for every $n \in \mathbb{N}$ we have that ${}^*\mu_n(\mathcal{U}) \subseteq \mu_{n+1}(\mathcal{U})$. Moreover, if Y is completely coherent then the following properties hold:*

1. *$\mu_n(\mathcal{U}) \subseteq \mu_{n+1}(\mathcal{U})$;*
2. *$\mu_\infty(\mathcal{U}) = \bigcup_{n \in \mathbb{N}} \mu_n(\mathcal{U})$;*
3. *$\alpha \in \mu_\infty(\mathcal{U}) \Leftrightarrow {}^*\alpha \in \mu_\infty(\mathcal{U})$.*

Proof. For every $A \in \mathcal{U}$ $\mu_n(\mathcal{U}) \subseteq S_n(A)$. Hence, by transfer, ${}^*\mu_n(\mathcal{U}) \subseteq {}^*S_n(A) = S_{n+1}(A)$, and so ${}^*\mu_n(\mathcal{U}) \subseteq \bigcap_{A \in \mathcal{U}} S_{n+1}(A) = \mu_{n+1}(\mathcal{U})$.

Let us now assume that Y is completely coherent.

(1) Just notice that, for every $A \subseteq Y$, since A is coherent we have that $S_n(A) \subseteq S_{n+1}(A)$ for every $n \in \mathbb{N}$.

(2) $\mu_\infty(\mathcal{U}) \supseteq \bigcup_{n \in \mathbb{N}} \mu_n(\mathcal{U})$: $\alpha \in \bigcup_{n \in \mathbb{N}} \mu_n(\mathcal{U})$ if and only if there exists $n \in \mathbb{N}$ such that $\alpha \in \mu_n(\mathcal{U}) = \bigcap_{A \in \mathcal{U}} S_n(A)$. As $S_n(A) \subseteq {}^\bullet A$ for every $A \in \mathcal{U}$, in particular we have that $\alpha \in \bigcap_{A \in \mathcal{U}} {}^\bullet A = \mu_\infty(\mathcal{U})$.

$\mu_\infty(\mathcal{U}) \subseteq \bigcup_{n \in \mathbb{N}} \mu_n(\mathcal{U})$: Let $\alpha \in \bigcap_{A \in \mathcal{U}} {}^\bullet A$. Let $A \in \mathcal{U}$; then there exists $n \in \mathbb{N}$ such that $\alpha \in S_n(A)$. In particular, $\alpha \in S_n(Y)$. We claim that, for every other set $B \in \mathcal{U}$, $\alpha \in {}^\bullet B$ if and only if $\alpha \in S_n(B)$. In fact, assume by contrast that there exists $B \in \mathcal{U}$ such that $\alpha \in {}^\bullet B \setminus S_n(B)$. In particular, we find $m > n$ such that $\alpha \in S_m(B)$. As $\alpha \notin S_n(B)$, however, we have that $\alpha \in S_n(B^c) \subseteq S_m(B^c)$ since Y is completely coherent. Therefore $\alpha \in S_m(B) \cap S_m(B^c) = \emptyset$, which is absurd. Thus this shows that $\alpha \in \bigcap_{B \in \mathcal{U}} S_n(B) = \mu_n(\mathcal{U})$.

(3) $\alpha \in \mu_\infty(\mathcal{U}) \Leftrightarrow \exists n \in \mathbb{N}\, \alpha \in \mu_n(\mathcal{U}) \Leftrightarrow \exists n \in \mathbb{N}\, {}^*\alpha \in \mu_{n+1}(\mathcal{U}) \Leftrightarrow {}^*\alpha \in \mu_\infty(\mathcal{U})$. □

It is well-known that functions $f : Y_1 \to Y_2$ can be lifted to $\overline{f} : \beta Y_1 \to \beta Y_2$ by setting for every $\mathcal{U} \in \beta Y_1$

$$\overline{f}(\mathcal{U}) := \left\{ A \subseteq Y_2 \mid f^{-1}(A) \in \mathcal{U} \right\}.$$

As one might expect, the monad of $\overline{f}(\mathcal{U})$ can be expressed in terms of the monad of $\mathcal{U}$, as shown in the following Theorem, that generalized similar results proven by M. Di Nasso[3] in [15, Propositions 11.2.4,11.2.10 and Theorem 11.2.7] in the context of $\mathbb{N}$:

Theorem 13. *Let $A, B \in \mathbb{V}(X)$ be sets, let $f : A \to B$ and let $\mathcal{U} \in \beta A$. Then the following facts hold:*

1. *If $A = B$ and $\alpha, {}^{\star}f(\alpha) \in \mu(\mathcal{U})$ then $\alpha = {}^{\star}f(\alpha)$;*
2. *$\mu(\overline{f}(\mathcal{U})) = {}^{\star}f(\mu(\mathcal{U}))$.*

Proof. The proof in the general case is identical to that given for $A = B = \mathbb{N}$ in ${}^{*}\mathbb{N}$, see [15, Propositions 11.2.4,11.2.10 and Theorem 11.2.7]. $\square$

4 Arbitrary tensor products and pairs

The notion of tensor product of ultrafilters is fundamental to study both several basic properties of ultrafilters (like the Rudin-Keisler order, the algebraical properties of $\beta\mathbb{N}$ and the topological properties of ultrapowers), as well as to develop many applications (for example, to the theory of finite embeddabilities). At the best of our knowledge, most results in the literature cover the case of tensor products $\mathcal{U}_1 \otimes \mathcal{U}_2$ of ultrafilters on the same set S; in this Section, our goal is to extend these results in two directions. The first is to consider tensor products $\mathcal{U}_1 \otimes \mathcal{U}_2$ of ultrafilters on different sets, namely where $\mathcal{U}_1 \in \beta S_1, \mathcal{U}_2 \in \beta S_2$. The second is to consider arbitrary finite tensor products of ultrafilters $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_k$.

4.1 Tensor products

A key notion in ultrafilters theory is that of tensor product of ultrafilters:

Definition 14. Let S_1, S_2 be sets in $\mathbb{V}(X)$ and let $\mathcal{U}_1 \in \beta S_1, \mathcal{U}_2 \in \beta S_2$. The tensor product of $\mathcal{U}_1$ and $\mathcal{U}_2$ is the unique ultrafilter on $S_1 \times S_2$ such that for every $A \subseteq S_1 \times S_2$

$$A \in \mathcal{U}_1 \otimes \mathcal{U}_2 \Leftrightarrow \{s_1 \in S_1 \mid \{s_2 \in S_2 \mid (s_1, s_2) \in A\} \in \mathcal{U}_2\} \in \mathcal{U}_1.$$

Moreover, we set

$$\beta S_1 \otimes \beta S_2 = \{\mathcal{U}_1 \otimes \mathcal{U}_2 \mid \mathcal{U}_1 \in \beta S_1, \mathcal{U}_2 \in \beta S_2\}.$$

Tensor products are closely related with the notion of double limits along ultrafilters and Rudin-Keisler order (see [20, Section 11.1] for the case $S_1 = S_2 = S$, S discrete space). However, we will not adopt this topological point of view here. For us, tensor products are important because of the role they play in many applications, especially in combinatorial number theory.

Example 15. Let $(S, \cdot)$ be a semigroup. Let $f : S^2 \to S$ be $f(a, b) = a \cdot b$. Then $f(\mathcal{U}, \mathcal{V}) = \overline{f}(\mathcal{U} \otimes \mathcal{V}) = \mathcal{U} \odot \mathcal{V}$ for every $\mathcal{U}, \mathcal{V} \in \beta S$.

[3] Notice that in [15] these properties are proven in the case $X = Y = \mathbb{N}$; however, these arguments used in these proofs work also in the present case.

Example 16. Let $\mathcal{F}=\mathbb{N}^{\mathbb{N}}$ and let $H:\mathbb{N}\times\mathcal{F}\to\mathbb{N}$ be the function $H(n,f)=f(n)$. Then[4]

$$H(\mathcal{U},\mathcal{V})=\mathcal{U}\otimes_{\mathcal{F}}\mathcal{V}:=\{A\subseteq\mathbb{N}\mid\{n\in\mathbb{N}\mid\{f\in\mathcal{F}\mid f(n)\in A\}\in\mathcal{V}\}\in\mathcal{U}\}.$$

The first trivial observation about tensor products is the following:

Lemma 17. *If S_1 or S_2 is finite then $\beta(S_1\otimes S_2)=\beta S_1\otimes\beta S_2$.*

Proof. Let us prove the case with S_1 finite, as the other case is similar. Let $S_1=\{a_1,\dots,a_n\}$. Let $\mathcal{U}\in\beta(S_1\otimes S_2)$. For $i=1,\dots,n$ let $A_i=\{(a_i,s_2)\mid s_2\in S_2\}$. As $S_1\times S_2=\bigcup_{i\leq n}A_i$, there exists a unique $i\leq n$ such that $A_i\in\mathcal{U}$. Let $\mathfrak{U}_{a_i}\in\beta S_1$ be the principal ultrafilter on a_i and let

$$\mathcal{U}_2=\overline{\pi_2}(\mathcal{U}):=\{A\subseteq S_2\mid\{(a_i,s)\mid s\in A\}\in\mathcal{U}\}\in\beta S_2.$$

Then, by construction, we have that for every $A\subseteq S_1\times S_2$

$$A\in\mathfrak{U}_{a_i}\otimes\mathcal{U}_2\Leftrightarrow\{s_1\in S_1\mid\{s_2\in S_2\mid(s_1,s_2)\in A\}\in\mathcal{U}_2\}\in\mathfrak{U}_{a_i}\Leftrightarrow$$

$$\{s_2\in S_2\mid(a_i,s_2)\in A\}\in\mathcal{U}_2\Leftrightarrow A\in\mathcal{U},$$

hence $\mathcal{U}=\mathfrak{U}_{a_i}\otimes\mathcal{U}_2\in\beta S_1\otimes\beta S_2$. □

To develop a deeper study of tensor products, our goal in this section is to give several characterizations of monads of tensor products of ultrafilters. The first question that we want to answer is: what is the relationship between $\mu(\mathcal{U})\times\mu(\mathcal{V})$ and $\mu(\mathcal{U}\otimes\mathcal{V})$? Let us start with a definition:

Definition 18. Let $\mathcal{U}_1\in\beta S_1,\mathcal{U}_2\in\beta S_2$. We denote by $\mathcal{F}(\mathcal{U}_1,\mathcal{U}_2)$ the filter on $S_1\times S_2$ given by

$$\mathcal{F}(\mathcal{U}_1\times\mathcal{U}_2)=\{B\in S_1\times S_2\mid\exists A_1\in\mathcal{U}_1,A_2\in\mathcal{U}_2\text{ s.t. }A_1\times A_2\subseteq B\}.$$

In general, $\mathcal{F}(\mathcal{U}_1\times\mathcal{U}_2)$ is just a filter and not an ultrafilter; its relationship with $\mu(\mathcal{U}_1)$, $\mu(\mathcal{U}_2)$ and $\mu(\mathcal{U}_1\otimes\mathcal{U}_2)$ is clarified in the following Proposition.

Proposition 19. *Let $\mathcal{U}_1\in\beta S_1$ and $\mathcal{U}_2\in\beta S_2$. Then*

$$\mu(\mathcal{U}_1)\times\mu(\mathcal{U}_2)=\bigcup_{\mathcal{W}\in U(\mathcal{U}_1\times\mathcal{U}_2)}\mu(\mathcal{W})\supseteq\mu(\mathcal{U}_1\otimes\mathcal{U}_2),\tag{1}$$

where $U(\mathcal{U}_1\times\mathcal{U}_2)=\{\mathcal{W}\in\beta(S_1\times S_2)\mid\mathcal{W}\supseteq\mathcal{F}(\mathcal{U}_1\times\mathcal{U}_2)\}$.

Proof. First of all, let us notice that $\mathcal{U}_1\otimes\mathcal{U}_2\in U(\mathcal{U}_1\times\mathcal{U}_2)$, as clearly $A\times B\in\mathcal{U}_1\otimes\mathcal{U}_2\,\forall A\in\mathcal{U}_1,B\in\mathcal{U}_2$. Therefore we are left to prove that $\mu(\mathcal{U}_1)\times\mu(\mathcal{U}_2)=\bigcup_{\mathcal{W}\in U(\mathcal{U}_1\times\mathcal{U}_2)}\mu(\mathcal{W})$.

$\subseteq$: Let $\alpha\in\mu(\mathcal{U}_1),\beta\in\mu(\mathcal{U}_2)$. Let $\mathcal{W}=\mathfrak{U}_{(\alpha,\beta)}$. Then $\mathcal{W}\in U(\mathcal{U}_1\times\mathcal{U}_2)$ as, for every $A\in\mathcal{U}_1,B\in\mathcal{U}_2$ $(\alpha,\beta)\in{}^\star A\times{}^\star B={}^\star(A\times B)$.

$\supseteq$: Let $\mathcal{W}\in U(\mathcal{U}_1\times\mathcal{U}_2)$. Let $(\alpha,\beta)\in\mu(\mathcal{W})$. For every $A\in\mathcal{U}_1,B\in\mathcal{U}_2$ $A\times B\in\mathcal{W}$, hence

$$(\alpha,\beta)\in\bigcap_{A\in\mathcal{U}_1,B\in\mathcal{U}_2}{}^\star(A\times B)=\mu(\mathcal{U}_1)\times\mu(\mathcal{U}_2).$$ □

[4] These kind of ultrafilters are important when studying combinatorial properties of $\mathbb{N}$ by means of the so-called $\mathcal{F}$-finite embeddabilities, see [32].

Corollary 20. *For every $\alpha \in {}^{\star}S_1, \beta \in {}^{\star}S_2$ $\mathfrak{U}_{(\alpha,\beta)} \supseteq \mathcal{F}(\mathfrak{U}_\alpha \times \mathfrak{U}_\beta) \Leftrightarrow \alpha \sim_{S_1} \gamma, \beta \sim_{S_2} \delta$.*

In particular, as a consequence of Proposition 19 we have that the map $\otimes : \beta S_1 \times \beta S_2 \to \beta(S_1 \times S_2)$ is injective but not surjective, in general. Moreover, as it is known, this entails that $\mu(\mathcal{U}_1 \otimes \mathcal{U}_2) = \mu(\mathcal{U}_1) \times \mu(\mathcal{U}_2)$ if and only if $\mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2) = \mathcal{U}_1 \otimes \mathcal{U}_2$ (see also [6, Chapter 1]),

To characterize when such a situation happens, let us recall the following definitions:

Definition 21. Let $\mathcal{U} \in \beta S$ and let κ be a cardinal number. The norm of $\mathcal{U}$ is the cardinal

$$\|\mathcal{U}\| = \min_{A \in \mathcal{U}} |A|.$$

Moreover, $\mathcal{U}$ is κ^+-complete if for every family $\{A_i \mid i \in I\} \subseteq \mathcal{U}$ with cardinality $|I| < \kappa^+$ we have $\bigcap_{i \in I} A_i \in \mathcal{U}$.

The problem of characterizing ultrafilters $\mathcal{U}_1, \mathcal{U}_2$ such that $\mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$ was already considered, and solved, by A. Blass in [6, Section 4]. We recall (and reprove for completeness) his characterization in the following Theorem:

Theorem 22. *Let $\mathcal{U}_1 \in \beta S_1, \mathcal{U}_2 \in \beta S_2$. The following facts are equivalent:*

1. $\mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$;
2. $\forall A \in \mathcal{U}_1 \, \forall \{B_i \mid i \in A\} \subseteq \mathcal{U}_2 \, \exists C \in \mathcal{U}_1 \, \bigcap_{c \in C \cap A} B_c \in \mathcal{U}_2$.

Proof. (1) $\Rightarrow$ (2) Let $A \in \mathcal{U}_1$ and let $\{B_i \mid i \in A\} \subseteq \mathcal{U}_2$. Let $S \subseteq S_1 \times S_2$ be the set

$$S = \bigcup_{i \in A} \{i\} \times B_i.$$

As $A \in \mathcal{U}_1$, by definition of tensor product we have that $S \in \mathcal{U}_1 \otimes \mathcal{U}_2$. But then, as $\mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$, there exist $D_1 \in \mathcal{U}_1, D_2 \in \mathcal{U}_2$ such that $D_1 \times D_2 \subseteq S$. Hence for every $c \in C := A \cap D_1$ we have that $D_2 \subseteq B_c$, so in particular

$$D_2 \subseteq \bigcap_{c \in C} B_c,$$

hence $\bigcap_{c \in C} B_c \in \mathcal{U}_2$.

(2) $\Rightarrow$ (1) Let $S \in \mathcal{U}_1 \otimes \mathcal{U}_2$. By definition,

$$A := \{s_1 \in S_1 \mid \{s_2 \in S_2 \mid (s_1, s_2) \in S\} \in \mathcal{U}_2\} \in \mathcal{U}_1.$$

For every $i \in A$ let $B_i = \{s_2 \in S_2 \mid (i, s_2) \in S\} \in \mathcal{U}_2$. Then $\{B_i \mid i \in A\} \subseteq \mathcal{U}_2$, hence by hypothesis there exists $C \in \mathcal{U}_1$ so that $B := \bigcap_{c \in C \cap A} B_c \in \mathcal{U}_2$. But then by construction $(C \cap A) \times B \subseteq S$, and so $S \in \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$. $\square$

Example 23. Let $S_1 = S_2 = \mathbb{N}$. Let $\mathcal{U}_1, \mathcal{U}_2 \in \beta\mathbb{N}$. Then the following are equivalent[5]:

1. $\mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$;
2. $\exists i \in \{0, 1\}$ such that $\mathcal{U}_i$ is principal.

[5] This is a well-known fact: see e.g. [15, Remark 11.5.5].

In fact, that (2) $\Rightarrow$ (1) is straightforward. On the other hand, let us assume (1), and assume that $\mathcal{U}_1$ and $\mathcal{U}_2$ are not principal. Let A be any set in $\mathcal{U}$ and, for every $a \in A$, let $B_a = \{n \in \mathbb{N} \mid n > a\} \in \mathcal{U}_2$. By Theorem 22 there exists $C \in \mathcal{U}_1$ such that $\bigcap_{a \in A \cap C} B_a \in \mathcal{U}_2$. And this cannot be, as $A \cap C$ is infinite (since $\mathcal{U}_1$ is not principal) and hence $\bigcap_{a \in A \cap C} B_a = \emptyset$.

We want to generalize the previous example and solve the following two problems:

1. For which $\mathcal{U}_1 \in \beta S_1$ we have that $\forall \mathcal{U}_2 \in \beta S_2\ \mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$?
2. For which $\mathcal{U}_2 \in \beta S_2$ we have that $\forall \mathcal{U}_1 \in \beta S_1\ \mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$?

Although it is not evident from Theorem 22, the property $\mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$ is symmetic in $\mathcal{U}_1$,$\mathcal{U}_2$, in the sense that

$$\mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2) \Leftrightarrow \mathcal{U}_2 \otimes \mathcal{U}_1 = \mathcal{F}(\mathcal{U}_2 \times \mathcal{U}_1)$$

(this basic observations was pointed out to us by A. Blass, see also [6, Corollary 9, Section 4]). Therefore problems 1 and 2 are equivalent: a solution of the first entails directly a solution of the second. And the second problem is rather simple to solve:

Theorem 24. *Let $\mathcal{U}_2 \in \beta S_2$ and let $\kappa = |S_1|$. The following are equivalent:*

1. $\forall \mathcal{U}_1 \in \beta S_1 \mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$;

2. $\mathcal{U}_2$ *is* κ^+*-complete.*

Proof. (1) $\Rightarrow$ (2) Without loss of generality, we can assume that $S_1 = \kappa$. By contrast, let us suppose that $\mathcal{U}_2$ is not κ^+-complete. Let

$$\lambda = \min \left\{ \mu \mid \exists F \subseteq \mathcal{U} \text{ with } |F| = \mu \text{ and } \bigcap_{B \in F} B \notin \mathcal{U}_2 \right\}.$$

As $\mathcal{U}_2$ is not κ^+-complete, $\lambda \leq \kappa$. Let $\{B_i \mid i < \lambda\} \subseteq \mathcal{U}_2$ be such that $\bigcap_{i<\lambda} B_i \notin \mathcal{U}_2$. For every $i < \lambda$ we set $D_i = \bigcap_{j \leq i} B_j$. By the definition of λ we have that every $D_i \in \mathcal{U}_2$, as it is an intersection of fewer than λ elements of $\mathcal{U}_2$, and clearly $D_i \supseteq D_j$ for every $i \leq j$. Now let $\mathcal{U}_1 \in \beta\lambda \subseteq \beta\kappa$ be an ultrafilter that extends the filter of co-initial sets on λ, so that every set $C \in \mathcal{U}_1$ is cofinal in λ. By hypothesis, $\mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$. If we set $A = \lambda$, by Theorem 22 we deduce that $\exists C \in \mathcal{U}_1 \bigcap_{c \in C} D_c \in \mathcal{U}_2$. But C is cofinal in λ and $\{D_i\}_{i<\lambda}$ is a decreasing sequence, so $\bigcap_{c \in C} D_c = \bigcap_{i<\lambda} D_i = \bigcap_{i<\lambda} B_i \notin \mathcal{U}_2$, which is absurd.

(2) $\Rightarrow$ (1) Let $A \in \mathcal{U}_1$ and let $\{B_i \mid i \in A\} \subseteq \mathcal{U}_2$. Then, as $|A| \leq \kappa$, by κ^+-completeness $\bigcap_{i \in A} B_i \in \mathcal{U}_2$, hence the condition of Theorem 22 is fulfilled by setting $C = A$. □

Let us call a ultrafilter $\mathcal{U}_2 \in \beta S_2$ such that $\forall \mathcal{U}_1 \in \beta S_1 \mathcal{U}_1 \otimes \mathcal{U}_2 = \mathcal{F}(\mathcal{U}_1 \times \mathcal{U}_2)$ a factorizing ultrafilter. If $\lambda = |S_2|$, from the previous Theorem we deduce that:

- if $\lambda \leq \kappa$ then the only factorizing ultrafilters are the principal ones;
- if $\lambda > \kappa$ then nonprincipal factorizing ultrafilter $\mathcal{U}_2 \in \beta S_2$ might or might not exist: for example, if $\lambda = \kappa^+$then such a nonprincipal factorizing ultrafilter exists if and only if κ^+ is measurable (the existence of such ultrafilters is consistent with ZF but not with ZFC).

4.2 Tensor pairs

As discussed above, tensor products are very important to develop several applications of ultrafunctions theory. Therefore, if one wants to follow a non-standard perspective, it becomes fundamental to characterize tensor products in terms of their monads. To do so, we introduce the following definition:

Definition 25. Let $(\alpha,\beta)\in {}^{\star}(S_1\times S_2)$. We say that (α,β) is a tensor pair (notation: (α,β) TT) if $\mathfrak{U}_{(\alpha,\beta)}=\mathfrak{U}_\alpha\otimes\mathfrak{U}_\beta$.

As, in general, $\beta S_1\otimes\beta S_2\subsetneq\beta\left(S_1\times S_2\right)$, not all pairs $(\alpha,\beta)\in{}^{\star}(S_1\times S_2)$ are tensor pairs. When $S_1=S_2=\mathbb{N}$, many properties of tensor pairs have been proven (in the context of non-iterated hyperextensions) by M. Di Nasso in [15] (see also [28]). We plan to show that most of these characterizations can be extended (sometimes in an even more general form) to arbitrary tensor pairs, with some simplifications given by the possibility of iterating the star map.

The main advantage when working in iterated hyperextensions is that they allow to write down easily generators of tensor products:

Theorem 26. *Let $n,m\in\mathbb{N}$, let $S_1,S_2\in\mathbb{V}(X)$ be sets with S_1 completely coherent, let $\mathcal{U}\in\beta S_1$, $\mathcal{V}\in\beta S_2$ and let $\alpha\in\mu_n(\mathcal{U})$, $\beta\in\mu_m(\mathcal{V})$. Then $(\alpha,S_n(\beta))\in\mu_{n+m}\left(\mathcal{U}\otimes\mathcal{V}\right)$.*

Proof. Let $A\subseteq S_1\times S_2$. Then

$$A\in\mathfrak{U}_\alpha\otimes\mathfrak{U}_\beta\Leftrightarrow\{s_1\in S_1\mid\{s_2\in S_2\mid(s_1,s_2)\in A\}\in\mathfrak{U}_\beta\}\in\mathfrak{U}_\alpha.$$

Now, by definition, $\{s_2\in S_2\mid(s_1,s_2)\in A\}\in\mathfrak{U}_\beta$ if and only if

$$\beta\in S_m\left(\{s_2\in S_2\mid(s_1,s_2)\in A\}\right)=\{s_2\in S_m\left(S_2\right)\mid(s_1,s_2)\in S_m\left(A\right)\},$$

as $S_m\left(s_1\right)=s_1$ for every $s_1\in S_1$, since S_1 is completely coherent. Then

$$\{s_1\in S_1\mid\{s_2\in S_2\mid(s_1,s_2)\in A\}\in\mathfrak{U}_\beta\}\in\mathfrak{U}_\alpha\Leftrightarrow$$

$$\{s_1\in S_1\mid(s_1,\beta)\in S_m(A)\}\in\mathfrak{U}_\alpha\Leftrightarrow(\alpha,S_n(\beta))\in S_{n+m}(A).\qquad\square$$

Corollary 27. *Let $(S,\cdot)\in\mathbb{V}(X)$ be a semigroup. Assume that S is completely coherent. Let $\mathcal{U},\mathcal{V}\in\beta S$, let $\alpha\in\mu_n(\mathcal{U})$, $\beta\in\mu_m(\mathcal{V})$. Then $\alpha\cdot S_n\left(\beta\right)\in\mu_{n+m}\left(\mathcal{U}\odot\mathcal{V}\right)$.*

Proof. $\mathcal{U}\odot\mathcal{V}=\overline{f}\left(\mathcal{U}\otimes\mathcal{V}\right)$, where $f:S^2\to S$ is the function that maps every pair $(a,b)\in S^2$ in $a\cdot b$. Hence we can conclude by applying Theorem 13.(2) as, by Theorem 26, $(\alpha,S_n(\beta))\in\mu_{n+m}\left(\mathcal{U}\otimes\mathcal{V}\right)$. $\square$

Remark 28. In Theorem 7 we showed that ${}^{\bullet}Y/_{\sim_Y}\cong\beta Y$. Theorem 26 allows to refine this result when $Y=S$ is a semigroup: if we let $\odot:{}^{\bullet}S^2\to{}^{\bullet}S$ be the map such that, for every $\alpha,\beta\in{}^{\bullet}S$

$$\alpha\odot\beta=\alpha\cdot S_{h(a)}(\beta),$$

where $h(\alpha)=\min\{n\in\mathbb{N}\mid\alpha\in S_n(S)\}$, we get that $(\beta S,\odot)$ and $\left({}^{\bullet}S,\odot\right)/_{\sim_Y}$ are isomorphic as semigroups[6].

[6]This result could be improved to a topological isomorphism by introducing the star topology on ${}^{\bullet}S$, but we will not consider this topological approach here.

To simplify the notations, from now on we will assume that $(\alpha,\beta) \in {}^*(S_1 \times S_2)$, as the characterization for the general cases where $\alpha \in S_n(S_1)$, $\beta \in S_m(S_2)$ can be analogously deduced from Theorem 26.

In the case of non-iterated hyperextensions, tensor pairs have been studied mostly for the product $\mathbb{N} \times \mathbb{N}$, in order to characterize certain properties of ultraproducts. In this case, a characterization was given by C. Puritz in [37], where he proved that

$$(\alpha,\beta)\ \mathrm{TT} \Leftrightarrow \alpha < er(\beta),$$

where

$$er(\beta) = \left\{ {}^*f(\beta) \mid f : \mathbb{N} \to \mathbb{N}, {}^*f(\beta) \in {}^*\mathbb{N} \setminus \mathbb{N}. \right\}$$

Remark 29. In ${}^{**}\mathbb{N}$ it is very simple to see that Puritz's characterization is not symmetric, in the sense that the condition $\beta > er(\alpha)$ does not entail that (α,β) is TT. In fact, let α be a prime number in ${}^*\mathbb{N}$ and let $\beta = ({}^*\alpha)^{\alpha}$. Then $\beta > er(\alpha)$, as $({}^{**}f)(\alpha) = ({}^*f)(\alpha) \in {}^*\mathbb{N}$ for every $f \in \mathbb{N}^{\mathbb{N}}$, whilst $\beta \in {}^{**}\mathbb{N} \setminus {}^*\mathbb{N}$. However, if f is the function such that, if $n = p_1^{a_1} \cdot \dots \cdot p_h^{a_h} \in \mathbb{N}$ is the factorization of n as product of distinct prime numbers, then

$$f(n) = \max_{j=1,\dots,h} a_j,$$

we have that $({}^{**}f)(\beta) = \alpha$, hence by Puritz's characterization (α,β) is not TT.

The main problem in extending Puritz's characterization to arbitrary sets is that it uses the order relation on $\mathbb{N}$, whilst arbitrary products of sets might not be ordered. In [15], several equivalent characterization of Puritz's condition for tensor pairs in $\mathbb{N} \times \mathbb{N}$ were given by M. Di Nasso in [15]. In the following theorem, we adapt these characterizations to arbitrary tensor pairs on $S_1 \times S_2$, and we also introduce two new characterizations in terms of preservations of tensor pairs via standard functions.

Theorem 30. *Let $S_1, S_2 \in \mathbb{V}(X)$ be sets and let $(\alpha_1,\alpha_2) \in {}^*(S_1 \times S_2)$. The following are equivalent:*

1. *(α_1,α_2) is a tensor pair;*
2. *$(\alpha_1,\alpha_2) \sim_{S_1 \times S_2} (\alpha_1, {}^*\alpha_2)$;*
3. *$\forall A \subseteq S_1 \times S_2$ if $(\alpha_1,\alpha_2) \in {}^*A$ then $\exists s \in S_1$ such that $(s,\alpha_2) \in {}^*A$;*
4. *$\forall A \subseteq S_1 \times S_2$ if $(s,\alpha_2) \in {}^*A$ $\forall s \in S_1$ then $(\alpha_1,\alpha_2) \in {}^*A$;*
5. *for every sets S_3, S_4, for every functions $f : S_1 \to S_3$, $g : S_2 \to S_4$ $\left({}^*f(\alpha_1), {}^*g(\alpha_2)\right)$ is a tensor pair;*
6. *there exist sets S_3, S_4, a bijection $f : S_1 \to S_3$ and an injective function $g : S_2 \to S_4$ such that $\left({}^*f(\alpha_1), {}^*g(\alpha_2)\right)$ is a tensor pair.*

Proof. (1)⇒(2) This is an immediate consequence of Theorem 26.

(2)⇒(3) Let $A \subseteq S_1 \times S_2$ be such that $(\alpha_1,\alpha_2) \in {}^*A$. As $(\alpha_1,\alpha_2) \sim_{S_1 \times S_2} (\alpha_1, {}^*\alpha_2)$, we have that $(\alpha_1, {}^*\alpha_2) \in {}^{**}A$, namely

$$\alpha_1 \in {}^*\{s \in S_1 \mid (s,\alpha_2) \in {}^*A\},$$

hence $\{s \in S_1 \mid (s, \alpha_2) \in {}^*A\} \neq \emptyset$.

(3)⇒(4) This is straightforward, as (4) is the contrapositive of (3) applied to A^c.

(4)⇒(5) By contrast: assume that there exists sets S_3, S_4 and functions $f : S_1, \to S_3$, $g : S_2 \to S_4$ such that $\left({}^*f(\alpha_1), {}^*g(\alpha_2)\right)$ is not a tensor pair. Let $A \subseteq S_3 \times S_4$ be such that $\left(s_3, {}^*g(\alpha_2)\right) \in {}^*A$ for every $s_3 \in S_3$ but $\left({}^*f(\alpha_1), {}^*g(\alpha_2)\right) \notin {}^*A$. Let

$$X_A = \{(s_1, s_2) \in S_1 \times S_2 \mid (f(s_1), g(s_2)) \in A\}.$$

By construction, $(s_1, \alpha_2) \in {}^*X_A$ for every $s_1 \in S_1$, hence by (4) $\alpha_1 \in {}^*X_A$, namely $\left({}^*f(\alpha_1), {}^*g(\alpha_2)\right) \in {}^*A$, which is absurd.

(5)⇒(6) This is straightforward.

(5)⇒(1) Just set $S_1 = S_3$, $S_2 = S_4$, $f = id_{S_1}$, $g = id_{S_2}$.

(6)⇒(1) By contrast, assume that (α_1, α_2) is not a tensor pair. By the equivalence (5)⇔(1), there exists $A \subseteq S_1 \times S_3$ such that $(s, \alpha_2) \in {}^*A\ \forall s \in S_1$ but $(\alpha_1, \alpha_2) \notin {}^*A$. Let

$$X_A = (f, g)(A) = \{(s_3, s_4) \mid \exists (s_1, s_2) \in A\, f(s_1) = s_3, g(s_2) = s_4\}.$$

As f is surjective and $(s, \alpha_2) \in {}^*A\ \forall s \in S_1$, we have that $\left(s_3, {}^*g(\alpha_2)\right) \in {}^*X_A$ for every $s_3 \in S_3$. Hence $\left({}^*f(\alpha_1), {}^*g(\alpha_2)\right) \in {}^*X_A$, as $\left({}^*f(\alpha_1), {}^*g(\alpha_2)\right)$ is a tensor pair. But as $X = (f, g)(A)$ and f, g are 1-1, we deduce that $(\alpha_1, \alpha_2) \in {}^*A$, which is absurd. $\square$

To prove that Theorem 30 entails Puritz result and allows for a simple characterization of tensor pairs in many cases[7], let us introduce the following definition:

Definition 31. Let S_1, S_2 be given sets. Let $Y \subseteq \beta(S_1 \times S_2)$. We say that $(\alpha, \beta) \in {}^*(S_1 \times S_2)$ is a Y-tensor pair if it is a tensor pair and $\mathfrak{U}_{(\alpha,\beta)} \in Y$.

The basic observation is the following:

Remark 32. If (α, β) is a Y-tensor pair then $(\alpha, \beta) \in {}^*A$ for every $A \in \bigcap_{\mathcal{U} \in Y} \mathcal{U}$ (i.e., (α, β) generates the filter $\bigcap_{\mathcal{U} \in Y} \mathcal{U}$).

Example 33. If $S_1 = S_2 = \mathbb{N}$ and $Y = \{\mathcal{U} \otimes \mathcal{V} \mid \mathcal{U}, \mathcal{V} \in \beta\mathbb{N} \setminus \mathbb{N}\}$ then Y-tensor pairs are tensor pairs with both entries infinite and, as

$$\Delta = \{(n, m) \mid n < m\} \in \mathcal{W}$$

for every $\mathcal{W} \in Y$, this shows that $(\alpha, \beta) \in {}^*\Delta$ for every tensor pair (α, β). But then, by applying Theorem 30.(5) with $S_3 = S_4 = \mathbb{N}$, we deduce that for every $f, g : \mathbb{N} \to \mathbb{N}$ with ${}^*f(\alpha_1), {}^*g(\alpha_2)$ infinite we have $\left({}^*f(\alpha_1), {}^*g(\alpha_2)\right) \in {}^*\Delta$, namely

$$f(\alpha_1) < er(\alpha_2)\ \forall f : \mathbb{N} \to \mathbb{N},$$

which is one implication in Puritz's characterization.

Example 34. Let $S_1 = S_2 = \mathbb{Z}$. Let $\alpha, \beta \in {}^*\mathbb{Z} \setminus \mathbb{Z}$. Let

$$A_1 = \{z \in \mathbb{Z} \mid z \geq 0, z \equiv 0 \mod 2\},\ A_2 = \{z \in \mathbb{Z} \mid z > 0, z \equiv 1 \mod 2\},$$
$$A_3 = \{z \in \mathbb{Z} \mid z < 0, z \equiv 0 \mod 2\},\ A_4 = \{z \in \mathbb{Z} \mid z < 0, z \equiv 1 \mod 2\}.$$

[7] To the best of our knowledge, the characterizations given in Examples 34, 35, 36 are new.

Let i, j be such that $\alpha \in {}^*A_i, \beta \in {}^*A_j$ and let $f, g : \mathbb{Z} \to \mathbb{N}$ be bijections such that f coincides with the absolute value on A_i and g coincides with the absolute value on A_j. Then from conditions (5) and (6) in Theorem 30 we deduce that (α, β) is a tensor pair iff $(|\alpha|, |\beta|)$ is a tensor pair, namely

$$(\alpha, \beta)\,\mathrm{TT} \Leftrightarrow |\alpha| < {}^*h\,(|\beta|)\ \forall h : \mathbb{N} \to \mathbb{N}\,\text{s.t. } {}^*h\,(|\beta|) \notin \mathbb{N},$$

and it is hence straightforward to see that

$$(\alpha, \beta)\ \mathrm{TT} \Leftrightarrow |\alpha| < |{}^*h(\beta)|\,\forall h : \mathbb{Z} \to \mathbb{Z}\,\text{s.t. } {}^*h\,(\beta) \notin \mathbb{Z}.$$

Example 35. Let $S_1 = S_2 = \mathbb{Q}$. In $\beta\mathbb{Q}$ there are three kinds of ultrafilters:

- principal ones, namely ultrafilters $\mathcal{U} \in \beta\mathbb{Q}$ such that $\mu(\mathcal{U}) = \{q\}$ for some $q \in \mathbb{Q}$;
- quasi-principal, namely ultrafilters $\mathcal{U} \in \beta\mathbb{Q}$ such that $\mu(\mathcal{U})$ consists of finite nonstandard elements, in which case it is very simple to show that there exists $q \in \mathbb{Q}$ such that

$$\mu(\mathcal{U}) \subset mon(q) = \{\xi \in {}^*\mathbb{Q} \mid \xi - q\,\text{is infinitesimal}\}\,;$$

- infinite ultrafilters, namely ultrafilters $\mathcal{U} \in \beta\mathbb{Q}$ such that $\mu(\mathcal{U})$ consists of infinite elements.

Now let $(\alpha, \beta) \in {}^*(\mathbb{Q} \times \mathbb{Q})$. When is it that (α, β) is a tensor pair? As always, this is the case if $\{\alpha, \beta\} \cap \mathbb{Q} \neq \emptyset$. If $\{\alpha, \beta\} \cap \mathbb{Q} = \emptyset$, we distinguish three cases:

1. both α and β are infinite;
2. both α and β are finite;
3. one is infinite, one is finite.

Notice that, as $(\varepsilon, {}^*\varepsilon)$ is a tensor pair for every infinitesimal $\varepsilon \in {}^*\mathbb{Q}$, Puritz's characterization does not hold (directly) in our present case (as ${}^*\varepsilon < \varepsilon$ for every positive infinitesimal ε).

As, by Theorem 30.(5), we know that (α, β) is a tensor type iff $(-\alpha, \beta)$ and $(\alpha, -\beta)$ are, we reduce to consider the case $\alpha > 0, \beta > 0$. Moreover, let us observe that we can reduce to case (1). In fact, if η is any finite element in ${}^*\mathbb{Q}_{>0} \setminus \mathbb{Q}$, let $f_\eta : \mathbb{Q} \setminus \{st(\eta)\} \to \mathbb{Q}_{>0}$ be the function such that

$$\forall q \in \mathbb{Q}\, f_\eta(q) = \frac{1}{q - st(\eta)}.$$

Then $f_\eta(\eta)$ is infinite and, as this function is bijective, by points (5) and (6) of Theorem 30 we get that

- if α is finite then (α, β) is TT iff $(f_\alpha(\alpha), \beta)$ is TT;
- if β is finite then (α, β) is TT iff $(\alpha, f_\beta(\beta))$ is TT.

So we are left to study case (1). As a simple necessary criterion, from Theorem 30.(5) we get that if (α, β) is TT then also the pair of hypernatural parts $(\lfloor\alpha\rfloor, \lfloor\beta\rfloor)$ is TT. This fact can be refined: as

$$\Delta_{\mathbb{Q}} = \left\{(q_1, q_2) \in \mathbb{Q}^2 \mid q_2 > q_1\right\} \in \mathfrak{U}_\alpha \otimes \mathfrak{U}_\beta$$

whenever α, β are positive and infinite, we get from Theorem 30.(5) that it must be $\alpha < er_{\mathbb{Q}_{>0}}(\beta)$, where

$$er_{\mathbb{Q}_{>0}}(\beta) = \{{}^*f(\beta) \mid f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}, {}^*f(\beta) \text{ is infinite}\}.$$

Let us show that the converse holds as well. Let $\alpha < er_{\mathbb{Q}_{>0}}(\beta)$. By contrast, assume that (α, β) is not TT. Then by Theorem 30.(4) there exists $A \subseteq \mathbb{Q}^2$ such that $(q, \beta) \in {}^*A$ for every $q \in \mathbb{Q}$ but $(\alpha, \beta) \notin {}^*A$. Let $f : \mathbb{Q}_{>0} \to \mathbb{Q}_{>0}$ be such that

$$\forall q \in \mathbb{Q}\, f(q) := \min\{n \in \mathbb{N} \mid \exists s \in \mathbb{Q}_{>0}\ (s < n+1) \text{ and } (s, q) \notin A\}.$$

As $(q, \beta) \in {}^*A$ for every $q \in \mathbb{Q}$ we have that ${}^*f(\beta)$ is infinite. And, as $(\alpha, \beta) \notin {}^*A$, we have that ${}^*f(\beta) \leq \alpha$, which is absurd.

Example 36. A similar proof can be used to show that, for every infinite $\alpha, \beta \in {}^*\mathbb{R}_{>0}$, (α, β) is TT iff $\alpha < er_{\mathbb{R}_{>0}}(\beta)$, where

$$er_{\mathbb{R}_{>0}}(\beta) = \{{}^*f(\beta) \mid f : \mathbb{R}_{>0} \to \mathbb{R}_{>0}, {}^*f(\beta) \text{ is infinite}\},$$

and following ideas similar to those of Example 35 this can be used to characterize tensor pairs in $\mathbb{R}^2$. This can be used also to characterize certain ultrafilters in $\beta\mathbb{C}$: as $\mathbb{C} \cong \mathbb{R}^2$, for example we have that ultrafilters in $\beta\mathbb{C}$ of the form $\mathcal{U} \oplus i\mathcal{V}$, with $\mathcal{U}, \mathcal{V} \in \beta\mathbb{R}$, are generated by hypercomplex numbers of the form $\alpha + i\beta$ where (α, β) is TT in $\mathbb{R}^2$.

Moreover, as $\mathcal{F} := \mathbb{N}^{\mathbb{N}}$ is in bijection with $\mathbb{R}$, from Theorem 30.(6) we get a characterization of tensor pairs in $\mathcal{F}^2$ and, since $\mathbb{N}$ can be embedded in $\mathcal{F}$ just mapping any natural number n to the constant function with value n, this gives a characterization of tensor pairs in $\mathbb{N} \times \mathcal{F}$ and $\mathcal{F} \times \mathbb{N}$. This characterization is quite implicit; however, Theorem 30 can be used to give explicit necessary and sufficient conditions even in this case: in fact, for $\alpha \in {}^*\mathbb{N}$ and $\varphi \in {}^*\mathcal{F}$ we have that

- Necessary: if (α, φ) is TT then $({}^*f(\alpha), {}^*H(\varphi))$ is TT for every $f \in \mathcal{F}$, $H : \mathcal{F} \to \mathbb{N}$. In particular, by letting for every $n \in \mathbb{N}$ H_n be the evaluation in $\mathbb{N}$, we get that if (α, φ) is TT then $(\alpha, \varphi(n))$ is TT in ${}^*\left(\mathbb{N}^2\right)$ for every $n \in \mathbb{N}$.

- Sufficient: $(\alpha, {}^*\varphi)$ is TT. In particular, if we let $\mathcal{V} := \mathfrak{U}_\alpha \otimes_{\mathcal{F}} \mathfrak{U}_\varphi \in \beta\mathbb{N}$ be the ultrafilter such that $\forall A \subseteq \mathbb{N}$

$$A \in \mathcal{V} \Leftrightarrow \{n \in \mathbb{N} \mid \{f \in \mathcal{F} \mid f(n) \in A\} \in \mathfrak{U}_\varphi\} \in \mathfrak{U}_\alpha,$$

  we get that $({}^*\varphi)(\alpha) \in \mu(\mathcal{V})$ (these ultrafilters are studied in [32], where they are used to study several Ramsey-theoretical combinatorial properties of $\mathbb{N}$).

4.3 Tensor k-ples

If we consider products of k sets, the natural generalization of tensor pairs are tensor k-ples.

Definition 37. Let $S_1, \ldots, S_k$ be sets and, for every $i \leq k$, let $\mathcal{U}_i \in \beta S_i$. The tensor product $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_k$ is the unique ultrafilter on $S_1 \times \cdots \times S_k$ such that, for every $A \subseteq S_1 \times \cdots \times S_k$ we have that $A \in \mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_k$ if and only if

$$\{s_1 \in S_1 \mid \{s_2 \in S_2 \mid \ldots \{s_k \in S_k \mid (s_1, \ldots, s_k) \in A\} \in \mathcal{U}_k\} \ldots\} \in \mathcal{U}_1.$$

We say that $(\alpha_1, \ldots, \alpha_k) \in {}^*(S_1 \times \cdots \times S_k)$ is a tensor k-ple if $\mathfrak{U}_{(\alpha_1,\ldots,\alpha_k)} = \mathfrak{U}_{\alpha_1} \otimes \cdots \otimes \mathfrak{U}_{\alpha_k}$.

It is immediate to prove that $\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_k$ is an ultrafilter and that the operation $\otimes$ is associative (modulo the usual identification of products $(S_1 \times S_2) \times S_3 = S_1 \times (S_2 \times S_3) = S_1 \times S_2 \times S_3$), see e.g. [5, Appendix]. This allows to characterize tensor k-ples in terms of pairs:

Theorem 38. *Let $k \geq 1$, let $S_1, \ldots, S_k, S_{k+1}$ be given sets and let $(\alpha_1, \ldots, \alpha_{k+1}) \in {}^*(S_1 \times \cdots \times S_{k+1})$. The following facts are equivalent:*

1. *$(\alpha_1, \ldots, \alpha_{k+1})$ is a tensor $(k+1)$-ple;*
2. *$((\alpha_1, \ldots, \alpha_k), \alpha_{k+1})$ is a tensor pair and $(\alpha_1, \ldots, \alpha_k)$ is a tensor k-ple;*
3. *(α_k, α_{k+1}) is a tensor pair and $(\alpha_1, \ldots, \alpha_k)$ is a tensor k-ple;*
4. *$\forall i \leq k$ (α_i, α_{i+1}) is a tensor pair.*

Proof. (1)⇒(2) By hypothesis, $\mathfrak{U}_{(\alpha_1,\ldots,\alpha_{k+1})} = \mathfrak{U}_{\alpha_1} \otimes \cdots \otimes \mathfrak{U}_{\alpha_{k+1}}$. By the associativity of tensor products, $\mathfrak{U}_{\alpha_1} \otimes \cdots \otimes \mathfrak{U}_{\alpha_{k+1}} = (\mathfrak{U}_{\alpha_1} \otimes \cdots \otimes \mathfrak{U}_{\alpha_k}) \otimes \mathfrak{U}_{\alpha_{k+1}}$. Let $\mathcal{V} = \mathfrak{U}_{\alpha_1} \otimes \cdots \otimes \mathfrak{U}_{\alpha_k}$. Then $((\alpha_1, \ldots, \alpha_k), \alpha_{k+1}) \in \mu(\mathcal{V} \otimes \mathcal{U})$, namely $((\alpha_1, \ldots, \alpha_k), \alpha_{k+1})$ is a tensor pair, and $(\alpha_1, \ldots, \alpha_k) \in \mu(\mathfrak{U}_{\alpha_1} \otimes \cdots \otimes \mathfrak{U}_{\alpha_k})$, namely $(\alpha_1, \ldots, \alpha_k)$ is a tensor k-ple.

(2)⇒(1) $\mathfrak{U}_{(\alpha_1,\ldots,\alpha_k,\alpha_{k+1})} = \mathfrak{U}_{(\alpha_1,\ldots,\alpha_k)} \otimes \mathfrak{U}_{\alpha_{k+1}}$ as $((\alpha_1, \ldots, \alpha_k), \alpha_{k+1})$ is a tensor pair, and $\mathfrak{U}_{(\alpha_1,\ldots,\alpha_k)} = \mathfrak{U}_{\alpha_1} \otimes \cdots \otimes \mathfrak{U}_{\alpha_k}$ as $(\alpha_1, \ldots, \alpha_k)$ is a tensor k-ple.

(2)⇒(3) By contrast, assume that (α_k, α_{k+1}) is not a tensor pair. Let $A \subseteq S_k \times S_{k+1}$ be such that $\forall s_k \in S_k$ $(s_k, \alpha_{k+1}) \in {}^*A$ but $(\alpha_k, \alpha_{k+1}) \notin {}^*A$. Let

$$B = \{(s_1, \ldots, s_{k+1}) \in S_1 \times \cdots \times S_{k+1} \mid (s_k, s_{k+1}) \in A\}.$$

By construction, $\forall (s_1, \ldots, s_k) \in S_1 \times \cdots \times S_k$ $((s_1, \ldots, s_k), \alpha_{k+1}) \in {}^*B$. As $((\alpha_1, \ldots, \alpha_k), \alpha_{k+1})$ is a tensor pair, this entails that $((\alpha_1, \ldots, \alpha_k), \alpha_{k+1}) \in {}^*B$, hence $(\alpha_k, \alpha_{k+1}) \in {}^*A$, which is absurd.

(3)⇒(2) By contrast, assume that $((\alpha_1, \ldots, \alpha_k), \alpha_{k+1})$ is not a tensor pair. Let $T = S_1 \times \cdots \times S_k$ and let $A \subseteq T \times S_{k+1}$ be such that $\forall t \in T$ $(t, \alpha_{k+1}) \in {}^*A$ but $((\alpha_1, \ldots, \alpha_k), \alpha_{k+1}) \notin {}^*A$. Let $B \subseteq S_k \times S_{k+1}$ be the set

$$B = \{(s_k, s_{k+1}) \in S_k \times S_{k+1} \mid \forall (s_1, \ldots, s_{k-1}) \in S_1 \times \cdots \times S_{k-1} \\ ((s_1, \ldots, s_k), s_{k+1}) \in A\}.$$

By construction, $\forall s_k \in S_k$ $(s_k, s_{k+1}) \in {}^*B$ hence, as (α_k, α_{k+1}) is TT, we have that $(\alpha_k, \alpha_{k+1}) \in {}^*B$, where

$$ {}^*B = \{(\eta_k, \eta_{k+1}) \in {}^*S_k \times S_{k+1} \mid \forall (\sigma_1, \dots, \sigma_{k-1}) \in {}^*S_1 \times \cdots \times S_{k-1} \\ ((\sigma_1, \dots, \sigma_{k-1}, \eta_k), \eta_{k+1}) \in {}^*A\}, $$

hence $((\alpha_1, \dots, \alpha_k), \alpha_{k+1}) \in {}^*A$, which is absurd.

(3)⇒(4) By induction on k. If $k = 1$ there is nothing to prove. Now let us assume the claim to hold for $k \geq 1$ and let us prove it for $k + 1$. By inductive hypothesis, as (3)⇔(1), $(\alpha_1, \dots, \alpha_k)$ is a tensor k-ple if and only if $\forall i \leq k - 1$ (α_i, α_{i+1}) is a tensor pair, so the claim is proven.

(4)⇒(3) This is immediate by induction. □

Example 39. If $S_i = \mathbb{N}$ for every $i = 1, \dots, k$, we get the following equivalence: if $\forall i \leq k\, \alpha_i \in {}^*\mathbb{N} \setminus \mathbb{N}$ then $(\alpha_1, \dots, \alpha_{k+1})$ is a tensor $(k + 1)$-ple if and only if $\alpha < er\,(\alpha_{i+1})$ for every $i \leq k$.

Notice that, as a trivial corollary of Theorem 38, we obtain that the relation of "being a tensor pair" is transitive:

Corollary 40. *For every $(\alpha_1, \alpha_2, \alpha_3) \in {}^*(S_1 \times S_2 \times S_3)$, if (α_1, α_2) and (α_2, α_3) are TT then (α_1, α_3) is TT.*

Proof. As (α_1, α_2) and (α_2, α_3) are TT, from Theorem 38 we deduce that $((\alpha_1, \alpha_2), \alpha_3)$ is TT. Let us now assume that (α_1, α_3) is not TT. Let $A \subseteq S_1 \times S_3$ be such that $(s_1, \alpha_3) \in {}^*A$ for every $s_1 \in S_1$ but $(\alpha_1, \alpha_3) \notin {}^*A$. Let $B \subseteq S_1 \times S_2 \times S_3$ be defined as follows: $(s_1, s_2, s_3) \in B$ if and only if $(s_1, s_3) \in A$. Then $(s_1, s_2, \alpha_3) \in {}^*B$ for every $(s_1, s_2) \in S_1 \times S_2$, and so (as $((\alpha_1, \alpha_2), \alpha_3)$ is TT) we have that $((\alpha_1, \alpha_2), \alpha_3) \in {}^*B$, therefore $(\alpha_1, \alpha_3) \in {}^*A$, which is absurd. □

Using this fact, it is possible to add the following equivalent characterization to Theorem 38:

Theorem 41. *Let $k \geq 1$, let $S_1, \dots, S_k, S_{k+1}$ be given sets and let $(\alpha_1, \dots, \alpha_{k+1}) \in {}^*(S_1 \times \cdots \times S_{k+1})$. The following facts are equivalent:*

1. *$(\alpha_1, \dots, \alpha_{k+1})$ is a tensor $(k + 1)$-ple;*
2. *$\forall F = \{i_1 < \cdots < i_l\} \subseteq \{1, \dots, k + 1\}$ $(\alpha_{i_1}, \dots, \alpha_{i_l})$ is a tensor l-ple.*

Proof. The implication (2) ⇒ (1) is trivial (just set $F = \{1, \dots, k + 1\}$).

To prove the other implication, by the transitivity of the relation of being TT we have (using the characterization (4) of Theorem 38) that for every $i \leq l - 1$ (α_i, α_{i+1}) is TT. Hence from the equivalence (1)⇔(4) in Theorem 38 we deduce that $(\alpha_{i_1}, \dots, \alpha_{i_l})$ is a tensor l-ple. □

Finally, by iterating inductively the proof of Theorem 26, we obtain the following result:

Theorem 42. *Let $S_1, \dots, S_{k+1} \in \mathbb{V}(X)$ be sets, with $S_1, \dots, S_k$ completely coherent. For every $i \leq k + 1$ let $\mathcal{U}_i \in \beta S_i$ and let $\alpha_i \in \mu(\mathcal{U}_i)$. Then $(\alpha_1, {}^*\alpha_2, \dots, S_k(\alpha_{k+1})) \in \mu(\mathcal{U}_1 \otimes \cdots \otimes \mathcal{U}_{k+1})$.*

As a straightforward corollary we get the following characterization of tensor k-ples in iterated hyperextensions:

Corollary 43. *Let $S_1, \ldots, S_{k+1} \in \mathbb{V}(X)$ be sets, with $S_1, \ldots, S_k$ completely coherent. For every $i \leq k+1$ let $\alpha_i \in {}^*S_i$. The following facts are equivalent:*

1. $(\alpha_1, \ldots, \alpha_{k+1})$ *is TT;*

2. $(\alpha_1, \ldots, \alpha_{k+1}) \sim_{S_1 \times \cdots \times S_{k+1}} (\alpha_1, {}^*\alpha_2, \ldots, S_k(\alpha_{k+1}))$.

5 Combinatorial properties of monads

In recent years, several open problems in Ramsey theory regarding the partition regularity (and the partial partition regularity) of formulas have been solved (see e.g. [16, 18, 25, 34]). Moreover, in many cases nonstandard approaches based on the algebraical properties of the monads of ultrafilters have been used to extend several known results in new directions (see e.g. [9, 14, 15, 16, 17, 29, 32]). In this Section our goal is to give a unified formulation of all these nonstandard approaches (which will be obtained in Theorem 47), as well as to extend these methods to new directions: the study of partial partition regularity and the partition regularity of formulas with internal parameters.

5.1 Partition regularity of existential formulas

In all this section we let $Y \in \mathbb{V}(X)$ be a set. We will be concerned with the notion of "partition regularity[8]":

Definition 44. A family $\mathfrak{F} \subseteq \wp(Y)$ is partition regular if for every $k \in \mathbb{N}$, for every partition $X = A_1 \cup \cdots \cup A_k$ there exists $i \leq k$ such that $A_i \in \mathfrak{F}$.

The relationship between partition regular families and ultrafilters is a well known fact in combinatorial number theory[9]; in [28], this characterization was expressed by means of properties of monads in the case of families of witnesses of the partition regularity of Diophantine equations, a field rich in very interesting open problems.

Theorem 45. *Let $P(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$. Then following facts are equivalent:*

1. *the equation $P(x_1, \ldots, x_n) = 0$ is partition regular on $\mathbb{N}$, namely the family*

$$\mathfrak{F}_P = \{A \subseteq \mathbb{N} \mid \exists a_1, \ldots, a_n \in A \, P(a_1, \ldots, a_n) = 0\}$$

   is partition regular;

2. *there exists an ultrafilter $\mathcal{U} \in \beta\mathbb{N}$ and generators $\alpha_1, \ldots, \alpha_n \in \mu(\mathcal{U})$ such that $P(\alpha_1, \ldots, \alpha_n) = 0$.*

[8] In this paper we will always say "partition regularity" meaning what is sometimes called "weak partition regularity". We will not consider the notion of "strong partition regularity", see also [20] for a discussion of the two notions.

[9] A family of subsets of Y is partition regular if and only if it contains an ultrafilter on Y. These notions are also closely related to co-ideals; for a thorough treatment of co-ideals in Ramsey Theory, we refer to [40].

This characterization has been subsequently used in a series of paper [14, 15, 16, 17, 29, 31] to study the partition regularity of several classes of nonlinear polynomials. In this section we want to show how this characterization can be extended to study the partition regularity of several families of subsets of arbitrary sets[10].

Let us start with some preliminaries. In all this section, when we talk about "formulas" we mean first order formulas with bounded quantifiers[11] in the language of the superstructure $\mathbb{V}(X)$ (see [19, Chapter 13]), and when we write a formula as $\phi(x_1, ..., x_n)$ we mean that $x_1, \dots, x_n$ are all and only variables appearing in ϕ. We say that a formula $\phi(x_1, ..., x_n)$ is totally open if all its variables are free.

Definition 46. Let $\phi(x_1, \dots, x_n, y_1, \dots, y_m)$ be a totally open formula, let $S_1, \dots, S_m \in \mathbb{V}(X)$ be sets and, for $i = 1, \dots, m$, let $Q_i \in \{\exists, \forall\}$. The existential closure of $\phi(x_1, \dots, x_n, y_1, \dots, y_m)$ with constraints $\{Q_i y_i \in S_i \mid i \leq m\}$ is the formula

$$E_{\overrightarrow{Qy} \in \overrightarrow{S}}(\phi(x_1, \dots, x_n, y_1, \dots, y_m)) : \\ \exists x_1 \dots \exists x_n \, Q_1 y_1 \in S_1 \dots Q_m y_m \in S_m \, \phi(x_1, \dots, x_n, y_1, \dots, y_m).$$

When $m = 0$ we will use the notation $E(\phi(x_1, \dots, x_n))$, and $E(\phi(x_1, \dots, x_n))$ will be called the existential closure of $\phi(x_1, \dots, x_n)$.

Similarly, the universal closure of $\phi(x_1, \dots, x_n, y_1, \dots, y_m)$ with constraints $\{Q_i y_i \in S_i \mid i \leq m\}$ is the sentence

$$U_{\overrightarrow{Qy} \in \overrightarrow{S}}(\phi(x_1, \dots, x_n, y_1, \dots, y_m)) : \\ \forall x_1 \dots \forall x_n \, Q_1 y_1 \in S_1 \dots Q_m y_m \in S_m \, \phi(x_1, \dots, x_n, y_1, \dots, y_m).$$

When $m = 0$ we will use the notation $U(\phi(x_1, \dots, x_n))$, and $U(\phi(x_1, \dots, x_n))$ will be called the universal closure of $\phi(x_1, \dots, x_n)$.

Given a totally open formula $\phi(x_1, \dots, x_n, y_1, \dots, y_m)$, a set of constraints $\{Q_i y_i \in S_i \mid i \leq m\}$ and a set $A \subseteq Y$, $E_{\overrightarrow{Qy} \in \overrightarrow{S}}(\phi(x_1, \dots, x_n, y_1, \dots, y_m))$ is said to be modeled by A (notation: $A \models E_{\overrightarrow{Qy} \in \overrightarrow{S}}(\phi(x_1, \dots, x_n, y_1, \dots, y_m))$) if the formula

$$\exists x_1 \in A \dots \exists x_n \in A \, Q_1 y_1 \in S_1 \dots Q_m y_m \in S_m \, \phi(x_1, \dots, x_n, y_1, \dots, y_m)$$

holds true. Similarly, we say that A models $U_{\overrightarrow{Qy} \in \overrightarrow{S}}(\phi(x_1, \dots, x_n, y_1, \dots, y_m))$ (notation: $A \models U_{\overrightarrow{Qy} \in \overrightarrow{S}}(\phi(x_1, \dots, x_n, y_1, \dots, y_m))$) if the formula

$$\forall x_1 \in A \dots \forall x_n \in A \, Q_1 y_1 \in S_1 \dots Q_m y_m \in S_m \, \phi(x_1, \dots, x_n)$$

holds true.

[10] Some results of this section already appeared, in a much weaker form, in [28].

[11] We adopt a slight abuse of language here: the kind of formulas we work with are those introduced in Definition 46, which contain some unbounded quantifiers. However, the notion we are interested in is that of a set $A \subseteq Y$ witnessing these formulas, and when we adopt this notion there are no more unbounded quantifiers to be handled, as every unbounded quantifier $Q_i x_i$ ($Q_i \in \{\forall, \exists\}$) is substituted with $Q_i x_i \in A$. For this reason, we believe that this slight abuse of language should not create too much confusion.

$E_{\overrightarrow{Qy}\in\overrightarrow{S}}\left(\phi\left(x_1,\ldots,x_n,y_1,\ldots,y_m\right)\right)$ (resp. $U_{\overrightarrow{Qy}\in\overrightarrow{S}}\left(\phi\left(x_1,\ldots,x_n,y_1,\ldots,y_m\right)\right)$) is said to be partition regular on Y if for every $k\in\mathbb{N}$, for every partition $Y=A_1\cup\cdots\cup A_k$ there exists $i\leq k$ such that $A_i\models E_{\overrightarrow{Qy}\in\overrightarrow{S}}\left(\phi\left(x_1,\ldots,x_n,y_1,\ldots,y_m\right)\right)$ (resp. $A_i\models U_{\overrightarrow{Qy}\in\overrightarrow{S}}\left(\phi\left(x_1,\ldots,x_n,y_1,\ldots,y_m\right)\right)$).

Our main result in this Section is the following Theorem, which generalizes Theorem 45 to arbitrary existential formulas and sets with constraints[12]:

Theorem 47. *Let $\phi\left(x_1,\ldots,x_n,y_1,\ldots,y_m\right)$ be a totally open formula and, for $i=1,\ldots,m$, let $S_i\in\mathbb{V}(X)$ and $Q_i\in\{\exists,\forall\}$. Let $Y\in\mathbb{V}(X)$. The following are equivalent:*

1. *$E_{\overrightarrow{Qy}\in\overrightarrow{S}}\left(\phi\left(x_1,\ldots,x_n,y_1,\ldots,y_m\right)\right)$ is partition regular on Y;*
2. *$\exists\alpha_1\sim_Y\cdots\sim_Y\alpha_n\in{}^*Y$ such that the sentence $Q_1y_1\in{}^*S_1,\ldots,Q_my_m\in{}^*S_m\ {}^*\phi\left(\alpha_1,\ldots,\alpha_n,y_1,\ldots,y_m\right)$ holds true;*
3. *there exists a ultrafilter $\mathcal{U}\in\beta X$ such that for every set $A\in\mathcal{U}$ we have that*
$$A\models E_{\overrightarrow{Qy}\in\overrightarrow{S}}\left(\phi\left(x_1,\ldots,x_n,y_1,\ldots,y_m\right)\right).$$

Proof. (1)⇒(2) First, let us fix a notation. Let $Par(Y)$ be the set of all possible finite partitions of Y. Given partitions $P_1(Y)=A_{1,1}\cup\cdots\cup A_{1,k_1},\ldots,P_m(Y)=A_{m,1}\cup\cdots\cup A_{m,k_m}$, we let $P\left(P_1,\ldots,P_m\right)$ be the partition generated by $P_1,\ldots,P_m$, namely the partition

$$Y=\bigcup_{(i_1,\ldots,i_m)\in K}\ \bigcap_{1\leq l\leq m}A_{l,i_l},$$

where $K=[1,k_1]\times\cdots\times[1,k_m]$. Now, for every partition $P(Y)=A_1\cup\cdots\cup A_m$ let $I_{P(Y)}$ be the set of all partitions refining $P(Y)$, namely[13]

$$I_{P(Y)}=\left\{f:Y\to[1,k]\mid k\in\mathbb{N},\forall i\leq k\ \exists!j\leq m\,\text{such that}\,f^{-1}(j)\subseteq A_i\right\}.$$

The family $\left\{I_{P(Y)}\right\}_{P\in Par(Y)}$ has the FIP, since $I_{P_1(Y)}\cap\cdots\cap I_{P_m(Y)}\supseteq I_{P(P_1\ldots,P_m)}$. By enlarging, there exists a hyperfinite partition ${}^*Y=A_1\cup\cdots\cup A_\lambda$ that refines all finite partitions of Y. As $E_{\overrightarrow{Qy}\in\overrightarrow{S}}\left(\phi\left(x_1,\ldots,x_n,y_1,\ldots,y_m\right)\right)$ is partition regular on X, by transfer ${}^*E_{\overrightarrow{Qy}\in\overrightarrow{S}}\left(\phi\left(x_1,\ldots,x_n,y_1,\ldots,y_m\right)\right)$ is partition regular on ${}^*Y$, hence there exists $i\leq\lambda$, $\alpha_1,\ldots,\alpha_n\in A_i$, such that $Q_1\beta_1\in{}^*S_1,\ldots,Q_m\beta_m\in{}^*S_m\ {}^*\phi\left(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_m\right)$ holds true. To conclude the proof, we show that $\alpha_1\sim_Y\cdots\sim_Y\alpha_n$. In fact, as $A_1\cup\cdots\cup A_\lambda$ refines all finite partitions on Y, for every $i\leq\lambda$ it is straightforward to show that the set

$$U_i=\{A\subseteq Y\mid A_i\subseteq{}^*A\}$$

is an ultrafilter, and so $A_i\subseteq\bigcap_{A\in F_i}{}^*A=\mu\left(U_i\right)$, hence $\alpha_1,\ldots,\alpha_n\in\mu\left(U_i\right)$ are all $\sim_Y$-equivalent.

[12] We have included in this Theorem also the known characterization of partition regular families in terms of ultrafilters, providing a new rather simple nonstandard proof that uses monads.

[13] As usual, we are identifying partitions and functions with finite image.

(2)⇒(3) Let $\mathcal{U}$ be the ultrafilter generated by $\alpha_1, \ldots, \alpha_n$. Let $A \in \mathcal{U}$. By hypothesis, $\mu(\mathcal{U}) \models {}^*E_{\overrightarrow{Qy} \in \overrightarrow{S}}\left(\phi\left(x_1, \ldots, x_n, y_1, \ldots, y_m\right)\right)$ and, since the formula $E_{\overrightarrow{Qy} \in \overrightarrow{S}}\left(\phi\left(x_1, \ldots, x_n, y_1, \ldots, y_m\right)\right)$ is existential in $\overrightarrow{x} = (x_1, \ldots, x_n)$ and $\mu(\mathcal{U}) \subseteq {}^*A$, this entails that ${}^*A \models {}^*E_{\overrightarrow{Qy} \in \overrightarrow{S}}\left(\phi\left(x_1, \ldots, x_n, y_1, \ldots, y_m\right)\right)$, so we can conclude by transfer.

(3)⇒(1) This is straightforward from the definitions, as for every finite partition $Y = A_1 \cup \cdots \cup A_k$ there exists $i \leq k$ such that $A_i \in \mathcal{U}$. □

Remark 48. The characterization of partition regular Diophantine equations is a particular case of the previous Theorem, where we let $m = 0$, $Y = \mathbb{N}$ and, given a polynomial $P\left(x_1, \ldots, x_n\right)$, $\phi\left(x_1, \ldots, x_n\right)$ is the formula $P\left(x_1, \ldots, x_n\right) = 0$.

Definition 49. If ϕ is a partition regular formula, every ultrafilter $\mathcal{U}$ such that $\forall A \in \mathcal{U}\, A \models \phi$ will be called a ϕ-ultrafilter (in this case, we will also say that $\mathcal{U}$ witnesses ϕ). We will write $\mathcal{U} \models \phi$ to mean that $\mathcal{U}$ is a ϕ ultrafilter.

In particular, the proof of Theorem 47 shows that, for any ultrafilter $\mathcal{U} \in \beta Y$, $\mathcal{U} \models E_{\overrightarrow{Qy} \in \overrightarrow{S}}\left(\phi\left(x_1, \ldots, x_n, y_1, \ldots, y_m\right)\right)$ if and only $\exists \alpha_1, \ldots, \alpha_n \in \mu(\mathcal{U})$ such that $Q_1\beta_1 \in {}^*S_1, \ldots, Q_m\beta_m \in {}^*S_m \; {}^*\phi\left(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m\right)$ holds true.

Example 50. (This example appears, in the weaker form $m = 0$, also in [32, Theorem 4.2].) Let S be a semigroup, and let $\phi\left(x_1, \ldots, x_n, y_1, \ldots, y_m\right)$ be an homogeneous totally open formula with constraints $Q_1R_1, \ldots, Q_mR_m$, in the sense that $\forall s_1, \ldots, s_n, t \in S$ if $Q_1r_1 \in R_1 \ldots Q_mr_m \in R_m\, \phi\left(s_1, \ldots, s_n, r_1, \ldots, r_m\right)$ holds true then $Q_1\widetilde{r_1} \in R_1 \ldots Q_m\widetilde{r_m} \in R_m \phi\left(t \cdot s_1, \ldots, t \cdot s_n, r_1, \ldots, r_m\right)$ holds true. Then

$$I_\phi = \left\{\mathcal{U} \in \beta S \mid \mathcal{U} \models E_{\overrightarrow{Qy} \in \overrightarrow{R}}\left(\phi\left(x_1, \ldots, x_n, y_1, \ldots, y_m\right)\right)\right\}$$

is a closed bilateral ideal in βS. Closure is trivial; now let $\mathcal{U} \in I_\phi$ and $\mathcal{V} \in \beta S$. Let $\alpha_1 \sim_S \cdots \sim_S \alpha_n \in \mu(\mathcal{U})$ be such that $Q_1y_1 \in {}^*R_1 \ldots Q_my_m \in {}^*R_m \; {}^*\phi\left(\alpha_1, \ldots, \alpha_n, y_1, \ldots, y_m\right)$ holds, and let $\beta \in \mu(\mathcal{V})$. Then:

- $\mathcal{U} \odot \mathcal{V} \in I_\phi$ as, by Corollary 27, $\alpha_i \cdot {}^*\beta \in \mu(\mathcal{U} \odot \mathcal{V})$ for every $i \leq n$, and $Q_1y_1 \in {}^*R_1 \ldots Q_my_m \in {}^*R_m \; {}^*\phi\left(\alpha_1, \ldots, \alpha_n, y_1, \ldots, y_m\right)$ holds as ${}^*\phi\left(\alpha_1, \ldots, \alpha_n, y_1, \ldots, y_m\right)$ holds and ϕ is homogeneous;
- similarly, $\mathcal{V} \odot \mathcal{U} \in I_\phi$ as, by Corollary 27, $\beta \cdot {}^*\alpha \in \mu(\mathcal{V} \odot \mathcal{U})$ for every $i \leq n$, and $Q_1y_1 \in {}^*R_1 \ldots Q_my_m \in {}^*R_m \; {}^*\phi\left(\beta \cdot {}^*\alpha_1, \ldots, \beta \cdot {}^*\alpha_n, y_1, \ldots, y_m\right)$ holds as ${}^*\phi\left({}^*\alpha_1, \ldots, {}^*\alpha_n, y_1, \ldots, y_m\right)$ holds and ϕ is homogeneous.

Example 51. In [25], A. Khalfalah and E. Szemerèdi proved that, for every polynomial $P(y)$ such that $2 \mid P(y)$ for some $y \in \mathbb{Z}$, the formula $\exists x_1, x_2, \exists y \in \mathbb{Z}\, x_1 + x_2 = P(y)$ is partition regular[14] on $\mathbb{N}$. By Theorem 47, there exists $\alpha_1 \sim_{\mathbb{Z}} \alpha_2$ and $\beta \in {}^*\mathbb{Z}$ such that $\alpha_1 + \alpha_2 = P(\beta)$. Similarly, in [18] N. Frantzikinakis and B. Host proved the partition regularity of the formulas $\exists x_1, x_2 \exists y_1 \in \mathbb{Z}\, 16x_1^2 + 9x_2^2 = y_1^2$ and $\exists x_1, x_2 \exists y_1 \in \mathbb{Z}\, x_1^2 - x_1x_2 + x_2^2 = y_1^2$. Once again, by Theorem 47, there exists $\alpha_1 \sim_{\mathbb{Z}} \alpha_2$ and $\beta \in {}^*\mathbb{Z}$ such that $16\alpha_1^2 + 9\alpha_2^2 = \beta_1^2$ and there exists $\eta_1 \sim_{\mathbb{Z}} \eta_2$ and $\mu_1 \in {}^*\mathbb{Z}$ such that $\eta_1^2 - \eta_1\eta_2 + \eta_2^2 = \mu_1^2$. Notice that

[14] However, as a consequence of [16, Theorem 3.10], if we drop the constraint $y \in \mathbb{Z}$, the formula $\exists x_1, x_2, y \;\; x_1 + x_2 = P(y)$ is not partition regular on $\mathbb{Z}$.

both these formulas are homogeneous, hence by Example 50 we get that the sets of ultrafilters witnessing them are closed bilateral ideals in $\overline{(\beta\mathbb{N},\odot)}$ (hence, in particular, any ultrafilter in the minimal bilateral ideal $\overline{K(\beta\mathbb{N},\odot)}$ witnesses both of them).

Example 52. In [34], J. Moreira solved a long standing open problem, proving the partition regularity on $\mathbb{N}$ of the formula[15] $\exists x_1, x_2, x_3 \exists y \in \mathbb{N}\ (x_1 + y = x_2) \wedge (x_1 \cdot y = x_3)$. By Theorem 47, this entails the existence of an ultrafilter $\mathcal{U} \in \beta\mathbb{N}$ and $\alpha_1, \alpha_2, \alpha_3 \in \mu(\mathcal{U}), \beta \in {}^*\mathbb{N}$ such that $\alpha_1 + \beta = \alpha_2$ and $\alpha_1 \cdot \beta = \alpha_3$.

In most cases, however, one is interested in full partition regularity, namely in the case of Definition 46 where $m = 0$.

Example 53. A very well-know fact in combinatorial number theory is that every idempotent ultrafilter is a Schur ultrafilter, namely it witnesses the partition regularity of the formula $\exists x, y, z\, x+y = z$ (see [39] for the original combinatorial proof of this result, and [20] for the ultrafilters version). This fact can be seen directly also as a consequence of Theorem 47. In fact, let $\mathcal{U}$ be idempotent and let $\alpha \in \mu(\mathcal{U})$. Then ${}^*\alpha \in \mu_2(\mathcal{U})$ and $\alpha + {}^*\alpha \in \mu_3(\mathcal{U} \oplus \mathcal{U}) = \mu_3(\mathcal{U})$ by Corollary 27, hence letting $\alpha_1 = \alpha, \alpha_2 = {}^*\alpha$ and $\alpha_3 = \alpha + {}^*\alpha$ we get the thesis.

The characterization of partition regularity by means of ultrafilters allows to use a iterative process to produce new partition regular formulas. The following is a generalization of [16, Lemma 2.1], where this result was framed and proven restricting to the context of partition regular equations:

Theorem 54. *Let $\phi(x, y_1, \ldots, y_n)$ be a totally open formula, let $S_1, \ldots, S_m \in \mathbb{V}(X)$ be sets and, for $i = 1, \ldots, m$, let $Q_i \in \{\exists, \forall\}$. Assume that*

$$\exists x\, Q_1 y_1 \in S_1 \ldots Q_n y_n \in S_n\, \phi(x, y_1, \ldots, y_n)$$

is a partition regular formula, and that $\mathcal{U} \in \beta Y$ is one of its witnesses. Then for every set $A \in \mathcal{U}$ the set

$$I_A(\phi) := \{a \in A \mid Q_1 y_1 \in S_1 \ldots Q_n y_n \in S_n \phi(a, y_1, \ldots, y_n) \text{ holds true}\} \in \mathcal{U}.$$

Moreover, let $\psi(x, z_1 \ldots, z_m)$ be another totally open formula, let $R_1, \ldots, R_m \in \mathbb{V}(X)$ be sets and, for $i = 1, \ldots, m$, let $\widetilde{Q_i} \in \{\exists, \forall\}$. Assume that $\mathcal{U}$ witnesses also the partition regularity of $\exists x \widetilde{Q}_1 z_1 \in R_1 \ldots \widetilde{Q}_m z_m \in R_m\, \psi(x, z_1, \ldots, z_n)$. Then $\mathcal{U}$ witnesses the formula

$$\exists x\, Q_1 y_1 \in S_1 \ldots Q_n y_n \in S_n \widetilde{Q_1} z_1 \in R_1 \widetilde{Q_m} z_m \in R_m \\ \phi(x, y_1, \ldots, y_n) \wedge \psi(x, z_1, \ldots, z_m),$$

which is then partition regular.

Proof. By contrast, assume that there exists $A \in \mathcal{U}$ such that $I_A(\phi) \notin \mathcal{U}$. Then $A \setminus I_A \in \mathcal{U}$, but

$$A \setminus I_A(\phi) \models \neg(\exists x\, Q_1 y_1 \in S_1 \ldots Q_n y_n \in S_n\, \phi(x, y_1, \ldots, y_n)),$$

[15] If we drop the constraint $y \in \mathbb{N}$, the problem of the partition regularity of the formula $\exists x_1, x_2, x_3, x_4\ (x_1 + x_4 = x_2) \wedge (x_1 \cdot x_4 = x_3)$ is still open.

hence $\mathcal{U}$ is not a witness of the partition regularity of $\exists x\, Q_1 y_1 \in S_1 \ldots Q_n y_n \in S_n\, \phi\left(x, y_1, \ldots, y_n\right)$, which is absurd.

As for the second claim, let $A \in \mathcal{U}$. Then $I_A(\phi)$ and $I_A(\psi)$ belong to $\mathcal{U}$, hence $I_A(\phi) \cap I_A(\psi) \in \mathcal{U}$, and

$$\begin{aligned} I_A(\phi) \cap I_A(\psi) \models \exists x\, Q_1 y_1 \in S_1 \ldots Q_n y_n \in S_n \widetilde{Q_1} z_1 \in R_1 \widetilde{Q_m} z_m \in R_m \\ \phi\left(x, y_1, \ldots, y_n\right) \wedge \psi\left(x, z_1, \ldots, z_m\right). \end{aligned}$$

Since this formula is existential, this entails that

$$\begin{aligned} A \models \exists x\, Q_1 y_1 \in S_1 \ldots Q_n y_n \in S_n \widetilde{Q_1} z_1 \in R_1 \widetilde{Q_m} z_m \in R_m \\ \phi\left(x, y_1, \ldots, y_n\right) \wedge \psi\left(x, z_1, \ldots, z_m\right), \end{aligned}$$

hence our claim is proven. □

Example 55. Let $X = \mathbb{N}$. For every $n \in \mathbb{N}$ let ϕ_n be the formula

$$\phi_n\left(x_1, \ldots, x_n, y_1, \ldots, y_n\right) := \bigwedge_{i \leq n} \left(\sum_{j \leq i} x_j = y_j \right)$$

and let $E\left(\phi_n\right)$ be the existential closure of ϕ_n. Hence, for every $A \in \mathbb{N}$ we have that $A \models E\left(\phi_n\right)$ if and only if A contains a subset $\{a_1, \ldots, a_n\}$ of n elements such that all ordered sums $a_1 + a_2, a_1 + a_2 + a_3$ and so on lie in A. By Schur's Theorem (see [39]) we know that $E\left(\phi_2\right)$ is partition regular. Let $\mathcal{U}$ be a $E\left(\phi_2\right)$ ultrafilter (which, from now on, we will call a Schur ultrafilter). We claim that $\forall n \in \mathbb{N}\, \mathcal{U} \models E\left(\phi_n\right)$. We prove this by induction on n.

If $n = 2$, the claim coincides with our hypothesis.

Now let $n > 2$, let us suppose the claim true for $n-1$, and let us prove it for n. By hypothesis and by inductive hypothesis, we have that $\mathcal{U}$ is a Schur and a $E\left(\phi_n\right)$-ultrafilter. In particular, $\mathcal{U}$ witnesses the formulas[16]

$$\exists z \left(\exists x_1 \exists x_2\, x_1 + x_2 = z\right)$$

and

$$\exists z \left(\exists x_3 \ldots \exists x_n \exists y_2 \ldots \exists y_n\ \left(z = y_2\right) \wedge \bigwedge_{i=3}^{n} \left(z + \sum_{j=3}^{i} x_j = y_i \right) \right),$$

hence by Theorem 54 $\mathcal{U}$ witnesses the formula

$$\begin{aligned} \exists z \left(\exists x_1 \exists x_2\, x_1 + x_2 = z\right) \wedge \\ \left(\exists x_3 \ldots \exists x_n \exists y_2 \ldots \exists y_n\ \left(z = y_2\right) \wedge \bigwedge_{i=3}^{n} \left(z + \sum_{j=3}^{i} x_j = y_j \right) \right) \end{aligned}$$

therefore (by renaming the variables and by letting $y_1 = x_1$) $\mathcal{U}$ witnesses the partition regularity of the formula

$$\exists x_1 \ldots \exists x_n \exists y_1 \ldots \exists y_n \bigwedge_{i \leq n} \left(\sum_{j \leq i} x_j = y_j \right),$$

as desired.

[16]The apparently strange naming of the variables is chosen to make more transparent the argument, at least in our hopes.

5.2 Partition regularity of arbitrary formulas

Even if, in most cases, applications regard existential closures of totally open formulas, characterizations similar to that of Theorem 47 hold also in other cases.

Corollary 56. *Let $\phi(x_1,\ldots,x_n,y_1,\ldots,y_m)$ be a totally open formula and, for $i=1,\ldots,m$, let $S_i\in\mathbb{V}(X)$ and $Q_i\in\{\exists,\forall\}$. Let $Y\in\mathbb{V}(X)$ and $\mathcal{U}\in\beta Y$. Then the following conditions are equivalent:*

1. *there is a set A in $\mathcal{U}$ that satisfies $U_{\overrightarrow{Qy}\in\overrightarrow{S}}(\phi(x_1,\ldots,x_n,y_1,\ldots,y_m))$;*
2. *for every $\alpha_1,\ldots,\alpha_n$ in $\mu(\mathcal{U})$ the sentence $Q_1y_1\in{}^*S_1,\ldots,Q_my_m\in{}^*S_m$ ${}^*\phi(\alpha_1,\ldots,\alpha_n,y_1,\ldots,y_m)$ holds true.*

Proof. This is just Theorem 47 applied to the existential closure of $\neg\phi$. □

A useful consequence of Corollary 56 is that, in some cases, the existence of a generator with some property implies that this property is shared by all other generators:

Corollary 57. *Let $\phi(x,y_1,\ldots,y_n)$ be a totally open formula and, for $i=1,\ldots,m$, let $S_i\in\mathbb{V}(X)$ and $Q_i\in\{\exists,\forall\}$. Let $Y\in\mathbb{V}(X)$, and let $\mathcal{U}$ be an ultrafilter in βY that witnesses $E_{\overrightarrow{Qy}\in\overrightarrow{S}}(\phi(x,y_1,\ldots,y_m))$. Then the formula*

$$\forall\alpha\in\mu(\mathcal{U})\ {}^*\phi(\alpha,y_1,\ldots,y_n)$$

holds true.

Proof. By Theorem 54 the set $I_Y(\phi)=\{a\in Y\mid\phi(a,y_1,\ldots,y_n)\text{ holds true}\}\in\mathcal{U}$, namely there is a set Y in $\mathcal{U}$ such that $\forall y\in Y\,\phi(y,y_1,\ldots,y_n)$ holds true. The conclusion hence follows straightforwardly from Corollary 56. □

Example 58. Let $\mathcal{U}\models\exists x,y_1,y_2\,y_1+y_2=x$. In particular, for every set $A\in\mathcal{U}$ we have that $\mathcal{U}$ witnesses $\exists x\,\exists y_1,y_2\in A\ (y_1+y_2=x)$. Hence from Corollary 57 we deduce that $\forall\alpha\in\mu(\mathcal{U})\,\exists\beta_1,\beta_2\in{}^*A$ such that $\alpha=\beta_1+\beta_2$. By saturation, this entails that $\forall\alpha\in\mu(\mathcal{U})\,\exists\beta_1,\beta_2\in\mu(\mathcal{U})$ such that $\alpha=\beta_1+\beta_2$.

Example 59. Let $Y=\mathbb{N}$. The formulas

$$\phi(d,x,y,z):\ \exists x,y,z,d\,y-x=z-y=d$$

and

$$\psi(d,u,v):\ \exists d,u,v\,u+v=d$$

are both partition regular and homogeneous. Hence from Example 50 we deduce that every ultrafilter $\mathcal{U}\in\overline{K(\beta\mathbb{N},\odot)}$ (the minimal closed bilateral ideal in the semigroup $(\beta\mathbb{N},\odot)$) witnesses both $\phi(d,x,y,z)$ and $\psi(d,u,v)$. Therefore, by Corollary 57 we get that for every set $A\in\mathcal{U}$ there exists an arithmetic progression in A of length 3 with a common difference in A that can be written as a sum of elements of A and, analogously, that every set $A\in\mathcal{U}$ contains elements x,y,z that are increments in arithmetic progressions of length 3 and such that $x+y=z$. Moreover, if $\mathcal{U}\odot\mathcal{U}=\mathcal{U}\in\overline{K(\beta\mathbb{N},\odot)}$ then $\mathcal{U}$ witnesses also the formula $\varphi(d,u,v):\ \exists d,u,v\,u\cdot v=d$, hence, again by Corollary 57, we get that every set $A\in\mathcal{U}$ contains an arithmetic progression in A of length 3 with a common difference in A that can be written as a product of elements of A.

Example 60. Selective ultrafilters admit several equivalent characterizations (see e.g. [8]). One of them says that $\mathcal{U}$ is a selective ultrafilter on Y if and only if every function $f : Y \to Y$ is $\mathcal{U}$-equivalent to either an injective or a constant function, namely there exists $A \in Y$ such that $f|_A$ is injective or constant. By Corollary 56, this is equivalent to say that for every $f : Y \to Y$ the function ${}^*f$ is injective or constant on $\mu(\mathcal{U})$.

Let us consider the case $Y = \mathbb{N}$. In this case, it is simple to see that "injective" can be substituted with "strictly increasing". Let $P(x) \in \mathbb{Z}[x]$. Let $\{a_n \mid n \in \mathbb{N}\}$ be the sequence inductively defined as follows: $a_0 = 0$ and, for every $n \geq 0$,

$$a_{n+1} = \min\{m \in \mathbb{N} \mid m > |P(j)| \; \forall j \leq a_n\}.$$

Let $f_P : \mathbb{N} \to \mathbb{N}$ be the function such that

$$\forall m \in \mathbb{N}\, f(m) = \max\{a_n \mid a_n \leq m\}.$$

As $f^{-1}(m)$ is finite for every $m \in \mathbb{N}$, there exists $A \in \mathcal{U}$ such that $f_P|_A$ is increasing. Hence we have that

$$\forall P(x) \in \mathbb{Z}[x]\, \forall \alpha, \beta \in \mu(\mathcal{U})\, (\alpha < \beta) \Rightarrow (|P(\alpha)| < \beta). \tag{2}$$

if $\alpha < \beta$ then $|P(\alpha)| < \beta$. As a consequence, we have that no selective ultrafilter is Schur: in fact, if $\mathcal{U}$ is a selective Schur ultrafilter, by Theorem 47 there are $\alpha, \beta, \gamma \in \mu(\mathcal{U})$ such that $\alpha + \beta = \gamma$ and, if $\alpha \geq \beta$, this means that $\alpha < \gamma \leq 2\alpha$, which is in contrast with the characterization (2).

Example 61. The result of Example 60 can be generalized. First of all, from characterization (2) we deduce immediately the following strengthening:

$$\forall n \in \mathbb{N}\, \forall P(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]\; \forall \alpha_1, \ldots, \alpha_n, \beta \in \mu(\mathcal{U})$$
$$(\alpha_1, \ldots, \alpha_n < \beta) \Rightarrow (|P(\alpha_1, \ldots, \alpha_n)| < \beta); \tag{3}$$

in fact, if $P(x_1, \ldots, x_n) = \sum_{i=1}^{k} c_i x_1^{e_{1,i}} \cdot \cdots \cdot x_n^{e_{n,i}}$ then

$$|P(x_1, \ldots, x_n)| = \left|\sum_{i=1}^{k} c_i x_1^{e_{1,i}} \cdot \cdots \cdot x_n^{e_{n,j}}\right| \leq \sum_{i=1}^{k} \left|c_i x_1^{e_{1,i}} \cdot \cdots \cdot x_n^{e_{n,i}}\right|,$$

hence if $\alpha = \max\{\alpha_i \mid i \leq n\}$ then $|P(\alpha_1, \ldots, \alpha_n)| \leq \sum_{i=1}^{k} |c_i|\, \alpha^{\sum_{j=1}^{n} e_{n,j}}$, so we conclude by characterization (2).

Now we use fact (3) to prove that for every polynomial $P(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ and for every selective ultrafilter $\mathcal{U}$, $\mathcal{U}$ is not a witness of the partition regularity of the formula

$$\exists x_1, \ldots, x_n \left(\bigwedge_{1 \leq i < j \leq n} x_i \neq x_j\right) \wedge P(x_1, \ldots, x_n) = 0. \tag{4}$$

We proceed by induction. If $n = 2$, the claim is trivial, as in this case by Rado's Theorem[17] (see [38]) the only partition regular polynomial in two variables is $x - y$.

[17] The fundamental Rado's Theorem for linear equations states as follows.

Theorem (Rado). *A linear polynomial $\sum_{i=1}^{n} c_i x_i$ is partition regular if and only if there exists a non empty subset $I \subseteq \{1, \ldots, n\}$ such that $\sum_{i \in I} c_i = 0$.*

Now let $n > 2$ and let us assume the claim to be true for $n-1$. Assume, by contrast, that there exists a polynomial $P(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$ and a selective ultrafilter $\mathcal{U}$ that witnesses the partition regularity of the formula (4). Then, by Theorem 47 we can find mutually distinct elements $\alpha_1, \ldots, \alpha_n \in \mu(\mathcal{U})$ such that $P(\alpha_1, \ldots, \alpha_n) = 0$. By rearranging the indexes, if necessary, we can assume that $\alpha_n = \max\{\alpha_i \mid i \leq n\}$.

Let $P(x_1, \ldots, x_n) = \sum_{i=1}^{k} c_i x_1^{e_{1,i}} \cdot \cdots \cdot x_n^{e_{n,i}}$, let $J = \{i \in [1,k] \mid e_{n,i} > 0\}$, let $Q(x_1, \ldots, x_n) = \sum_{i \in J} c_i x_1^{e_{1,i}} \cdot \cdots \cdot x_n^{e_{n,i}}$ and $R(x_1, \ldots, x_{n-1}) = \sum_{i \notin J} c_i x_1^{e_{1,i}} \cdot \cdots \cdot x_{n-1}^{e_{n-1,i}}$. As $P(\alpha_1, \ldots, \alpha_n) = 0$, we have that

$$|Q(\alpha_1, \ldots, \alpha_n)| = |R(\alpha_1, \ldots, \alpha_{n-1})|.$$

From characterization (3) we have that $|R(\alpha_1, \ldots, \alpha_{n-1})| < \alpha_n$. We consider two cases:

- If $Q(\alpha_1, \ldots, \alpha_n) \neq 0$ then $|Q(\alpha_1, \ldots, \alpha_n)| \geq \alpha_n$, as $Q(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$, $x_n \mid Q(x_1, \ldots, x_n)$ and $Q(\alpha_1, \ldots, \alpha_n) \neq 0$, hence it cannot be $|Q(\alpha_1, \ldots, \alpha_n)| = |R(\alpha_1, \ldots, \alpha_{n-1})|$ and we have reached an absurd;
- If $Q(\alpha_1, \ldots, \alpha_n) = 0$ then $R(x_1, \ldots, x_{n-1}) = 0$, and we can conclude by using the inductive hypothesis.

5.3 Combinatorial properties with internal parameters

As shown in our examples, Theorem 47 can be used to prove several properties of monads. This result can be strengthened, in saturated extensions, taking into account also internal parameters:

Theorem 62. *Let $\overrightarrow{p} := (p_1, \ldots, p_k)$, where $p_1, \ldots, p_k$ are internal objects in $\mathbb{V}(X)$. Let $S_1, \ldots, S_m$ be internal sets in $\mathbb{V}(X)$ and, for every $i = 1, \ldots, m$ let $Q_i \in \{\exists, \forall\}$. Let $\mathcal{U} \in \beta Y$ and let $\phi(x_1, \ldots, x_n, y_1, \ldots, y_m, z_1, \ldots, z_k)$ be a totally open formula. The following facts are equivalent:*

1. *$\forall A \in \mathcal{U}\, \exists \alpha_1 \ldots \alpha_n \in {}^*A\, Q_1\beta_1 \in S_1 \ldots Q_m\beta_m \in S_m\ {}^*\phi\left(\overrightarrow{\alpha}, \overrightarrow{\beta}, \overrightarrow{p}\right)$ holds true, where $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\overrightarrow{\beta} = (\beta_1, \ldots, \beta_m)$;*
2. *$\exists \alpha_1 \ldots \alpha_n \in \mu(\mathcal{U})\, Q_1\beta_1 \in S_1 \ldots Q_1\beta_m \in S_m\ {}^*\phi\left(\overrightarrow{\alpha}, \overrightarrow{\beta}, \overrightarrow{p}\right)$ holds true, where $\overrightarrow{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\overrightarrow{\beta} = (\beta_1, \ldots, \beta_m)$.*

Proof. (1)⇒(2) For every $A \in \mathcal{U}$ let

$$I_A = \{(\alpha_1, \ldots, \alpha_m) \in {}^*A^m \mid Q_1\beta_1 \in S_1 \ldots Q_m\beta_m \in S_m\ {}^*\phi(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m, p_1, \ldots, p_k) \text{ holds true}\}.$$

The family $\{I_A\}_{A \in \mathcal{U}}$ has the finite intersection property as $I_{A_1} \cap I_{A_2} = I_{A_1 \cap A_2}$, and every set I_A is internal by the internal definition principle. Hence, by saturation the formula

$$\exists \alpha_1 \ldots \alpha_n \in \mu(\mathcal{U})\, Q_1\beta_1 \in S_1 \ldots Q_m\beta_m \in S_m\ {}^*\phi(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m, p_1, \ldots, p_k)$$

holds true.

(1)⇒(2) Just notice that $\mu(\mathcal{U}) \subseteq {}^*A$ for every $A \in \mathcal{U}$ by definition. □

Corollary 63. *Let $\overrightarrow{p} := (p_1, \dots, p_k)$ where $p_1, \dots, p_k$ are internal objects in $\mathbb{V}(X)$. Let $S_1, \dots, S_m$ be internal sets in $\mathbb{V}(X)$ and, for every $i = 1, \dots, m$ let $Q_i \in \{\exists, \forall\}$. Let $\mathcal{U} \in \beta Y$ and let $\phi(x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_k)$ be a totally open formula. The following facts are equivalent:*

1. *$\exists A \in \mathcal{U} \forall \alpha_1 \dots \alpha_n \in {}^*A\, Q_1\beta_1 \in S_1 \dots Q_m\beta_m \in S_m\ {}^*\phi\left(\overrightarrow{\alpha}, \overrightarrow{\beta}, \overrightarrow{p}\right)$ holds true, where $\overrightarrow{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\overrightarrow{\beta} = (\beta_1, \dots, \beta_m)$;*
2. *$\forall \alpha_1 \dots \alpha_n \in \mu(\mathcal{U})\, Q_1\beta_1 \in S_1 \dots Q_m\beta_m \in S_m\ {}^*\phi\left(\overrightarrow{\alpha}, \overrightarrow{\beta}, \overrightarrow{p}\right)$ holds true, where $\overrightarrow{\alpha} = (\alpha_1, \dots, \alpha_n)$ and $\overrightarrow{\beta} = (\beta_1, \dots, \beta_m)$.*

Proof. Just apply Theorem 62 to $\neg\phi(x_1, \dots, x_n, y_1, \dots, y_m, z_1, \dots, z_k)$. $\square$

Example 64. Let $X = \mathbb{Q}$. Let $\mathcal{U}$ be a positive infinite ultrafilter in $\beta\mathbb{Q}$ (in the sense of Example 35). We claim that $\mu(\mathcal{U})$ is right and left unbounded in the set $Inf({}^*\mathbb{Q})$ of positive infinite elements of ${}^*\mathbb{Q}$. By contrast, assume that there are $\beta_1, \beta_2 \in Inf({}^*\mathbb{Q})$ such that $\beta_1 < \alpha < \beta_2$ for every $\alpha \in \mu(\mathcal{U})$. Then by Corollary 63 we have that there exists $A \in \mathcal{U}$ such that $\beta_1 < \alpha < \beta_2$ for every $\alpha \in {}^*A$. However:

- β_1 cannot exist, as $A \subseteq {}^*A$ and $q < \alpha_1$ for every $q \in Inf({}^*\mathbb{Q})$;
- α_2 cannot exist, as every set $B \in \mathcal{U}$ is right unbounded (and so is ${}^*B$ by transfer).

Example 65. Let $X = \mathbb{N}^{\mathbb{N}}$. Let $\mathcal{U}$ be an ultrafilter in βX and let $\alpha_1, \alpha_2 \in {}^*\mathbb{N}$. Then every generator φ of $\mathcal{U}$ maps α_1 into α_2 if and only if there is a set $B \in \mathcal{U}$ such that every function in B maps α_1 into α_2. For example, if $\alpha_1 \in \mathbb{N}$ and $\alpha_2 \in {}^*\mathbb{N} \setminus \mathbb{N}$ this means that no ultrafilter has this property, as if a function $f \in B$ then ${}^*f(\alpha_1) \in \mathbb{N}$.

We conclude by considering another version of the partition regularity of properties where multiple ultrafilters are considered at once[18]. Such notions applies to partial partition regular properties in Ramsey theory, which includes several fundamental results proven recently in the area (see e.g. [18, 34]).

Example 66. In [9, 28, 30] it has been introduced and studied the notion of finite embeddability between subsets of $\mathbb{N}$. In [32], this notion has been extended to arbitrary families of functions and semigroups. In particular, if $(S, \cdot)$ is a commutative[19] semigroup, a set $A \subseteq S$ is finitely embeddable in a set $B \subseteq S$ (notation: $A \leq_{fe} B$) iff for every finite subset $F \subseteq S$ there exists $t \in S$ such that $t \cdot F \subseteq B$. If we fix the cardinality n of the finite set F, we can rewrite this property as

$$\forall a_1, \dots, a_n \in A\, \exists b_1, \dots, b_n \in B\, \exists t \in S \bigwedge_{i \leq n} (a_i \cdot t = b_i).$$

This notion has been extended to ultrafilters in [30]: a ultrafilter $\mathcal{U} \in \beta S$ is finitely embeddable in $\mathcal{V} \in \beta S$ (notation: $\mathcal{U} \leq_{fe} \mathcal{V}$) if and only if for every

[18] Similar ideas, but in a rather different context, appeared in [7].

[19] In a very similar way, we can work with non-commutative semigroups; however, this means considering the different notions of right and left finite embeddability, and we prefer to avoid such complications here.

set $B \in \mathcal{V}$ there exists $A \in \mathcal{U}$ such that $A \leq_{fe} B$. Once again, if in $\leq_{fe}$ we fix the cardinality of the finite sets to be embedded, we can rewrite the finite embeddability between ultrafilters as follows:

$$\forall A \in \mathcal{V}\, \exists B \in \mathcal{U}\, \forall a_1, \dots, a_n \in A\, \exists b_1, \dots, b_n \in B\, \exists t \in S \bigwedge_{i \leq n} (a_i + t = b_i)\,.$$

We want to give a nonstandard characterization of properties like that expressed in Example 66. For the sake of simplicity, we give it for an alternation $\forall - \exists$ of two ultrafilters; similar characterizations for arbitrary finite amounts of ultrafilters and different alternations of quantifiers can be analogously deduced.

Theorem 67. *Let $\overrightarrow{p} := (p_1, \dots, p_k)$, where $p_1, \dots, p_k$ are internal objects in $\mathbb{V}(X)$. Let $S_1, \dots, S_h$ be internal sets in $\mathbb{V}(X)$. Let $\mathcal{U}, \mathcal{V} \in \beta Y$ and let $\phi(x_1, \dots, x_n, y_1, \dots, y_m, t_1, \dots, t_h, z_1, \dots, z_k)$ be a totally open formula. Assume that the extension ${}^*Y$ is $|Y|^+$-saturated. The following facts are equivalent:*

1. *$\forall A \in \mathcal{U} \exists B \in \mathcal{V}\, \forall \beta_1, \dots, \beta_m \in {}^*B\, \exists \alpha_1, \dots, \alpha_n \in {}^*A\, \exists s_1 \in S_1 \dots \exists s_h \in S_h\, {}^*\phi\left(\overrightarrow{\alpha}, \overrightarrow{\beta}, \overrightarrow{s}, \overrightarrow{p}\right)$, where $\overrightarrow{\alpha} = (\alpha_1, \dots, \alpha_n)\,, \overrightarrow{\beta} = (\beta_1, \dots, \beta_m)\,, \overrightarrow{s} = (s_1, \dots, s_h)\,, \overrightarrow{p} = (p_1, \dots, p_k)$;*

2. *$\forall \beta_1 \dots \beta_n \in \mu(\mathcal{U})\, \exists \alpha_1 \dots \alpha_m \in \mu(\mathcal{V})\, \exists s_1 \in S_1 \dots \exists s_h \in S_h\, {}^*\phi\left(\overrightarrow{\alpha}, \overrightarrow{\beta}, \overrightarrow{s}, \overrightarrow{p}\right)$ holds true, where $\overrightarrow{\alpha} = (\alpha_1, \dots, \alpha_n)\,, \overrightarrow{\beta} = (\beta_1, \dots, \beta_m)\,, \overrightarrow{s} = (s_1, \dots, s_h)\,, \overrightarrow{p} = (p_1, \dots, p_k)$.*

Proof. We will use the notations $\overrightarrow{\alpha} = (\alpha_1, \dots, \alpha_n)\,, \overrightarrow{\beta} = (\beta_1, \dots, \beta_m)\,, \overrightarrow{s} = (s_1, \dots, s_h)\,, \overrightarrow{p} = (p_1, \dots, p_k)$ throughout the proof.

(1)⇒(2) Let $\overrightarrow{\beta} \in \mu(\mathcal{U})^n$. As $\mu(\mathcal{V}) \subseteq {}^*B$ for every $B \in \mathcal{V}$, we have that for every $A \in \mathcal{U}$ the set

$$I_A := \left\{\overrightarrow{\alpha} \in {}^*A^n \mid \exists s_1 \in S_1 \dots \exists s_h \in S_h\, {}^*\phi\left(\overrightarrow{\alpha}, \overrightarrow{\beta}, \overrightarrow{s}, \overrightarrow{p}\right) \text{ holds true}\right\} \neq \emptyset.$$

As I_A is internal and $\{I_A\}_{A \in \mathcal{U}}$ has the FIP, by saturation $|Y|^+$-saturation we have that $\bigcap_{A \in \mathcal{U}} I_A \neq \emptyset$, and we conclude as if $\bigcap_{A \in \mathcal{U}} I_A \subseteq \mu(\mathcal{U})^n$.

(2) ⇒ (1) Let $A \in \mathcal{U}$. By using ${}^*A$ as a parameter, we see that the thesis is a straightforward consequence of Corollary 63. □

Example 68. Let us consider the finite embeddability. Let $(S, \cdot)$ be a commutative semigroup with identity and let $\mathcal{U}, \mathcal{V} \in \beta S$. From Theorem 67 we deduce that, for every $n \in \mathbb{N}$, the following two conditions are equivalent:

- $\mathcal{U} \leq_{fe} \mathcal{V}$;
- $\forall \beta_1 \dots \beta_n \in \mu(\mathcal{U}) \exists \sigma \in {}^*S$ such that $\sigma \cdot \beta_1, \dots, \sigma \cdot \beta_n \in \mu(\mathcal{V})$.

In particular, as $\mathcal{V} \in \beta S$ is such that $\forall \mathcal{U} \in \beta S \mathcal{U} \leq_{fe} \mathcal{V}$ if and only if $\mathcal{V} \in \overline{K(\beta S, \odot)}$ (this result has been proven in [32, Theorem 4.13]), we obtain the equivalence between the following two properties:

- $\mathcal{V} \in \overline{K(\beta S, \odot)}$;
- $\forall n \in \mathbb{N}, \forall \beta_1 \sim_S \dots \sim_S \beta_n \in {}^*S\, \exists \sigma \in {}^*S$ such that $\sigma \cdot \beta_1, \dots, \sigma \cdot \beta_n \in \mu(\mathcal{V})$.

Finally, as $\mathcal{V} \in \overline{K(\beta S, \odot)}$ if and only if every set $A \in \mathcal{V}$ is piecewise syndetic in $(S, \cdot)$ (see e.g. [20, Theorem 4.40]), from Theorem 62 we obtain the following characterization[20] of piecewise syndetic subsets of S: $A \subseteq S$ is piecewise syndetic if and only if

$$\forall n \in \mathbb{N}, \forall \beta_1 \sim_S \cdots \sim_S \beta_n \in {}^*S \, \exists \sigma \in {}^*S \, \text{such that} \, \sigma \cdot \beta_1, \ldots, \sigma \cdot \beta_n \in {}^*S.$$

Example 69. Finite embeddabilities can be generalized to arbitrary families of functions $\mathcal{F} : S^n \to S$ (see [32]). In particular, let $S = \mathbb{N}$ and $\mathcal{F} : \mathbb{N} \to \mathbb{N}$ be the family of affinities

$$\mathcal{F} := \{f_{a,b} : \mathbb{N} \to \mathbb{N} \mid \forall n \in \mathbb{N} \, f_{a,b}(n) = an + b\}.$$

We say that a set $A \subseteq \mathbb{N}$ is $\mathcal{F}$-finitely embeddable in $B \subseteq \mathbb{N}$ (notation: $A \leq_{\mathcal{F}} B$) if for every finite set $F \subseteq A$ there exists $f \in \mathcal{F}$ such that $f(A) \subseteq B$. Of course, this notion is related to that of AP-rich set (namely, of a set that contains arbitrarily long arithmetic progressions): in fact, it is straightforward to see that $B \subseteq \mathbb{N}$ is AP-rich if and only if $A \leq_{\mathcal{F}} B$ for every $A \subseteq \mathbb{N}$. $\mathcal{F}$-finite embeddability can be extended to ultrafilters as follows: we say that an ultrafilter $\mathcal{U} \in \beta\mathbb{N}$ is $\mathcal{F}$-finitely embeddable in $\mathcal{V} \in \beta\mathbb{N}$ if for every set $B \in \mathcal{V}$ there exists $A \in \mathcal{U}$ such that $A \leq_{\mathcal{F}} B$. Again, from Theorem 67 we deduce that, for every $n \in \mathbb{N}$, the following two conditions are equivalent:

- $\mathcal{U} \leq_{\mathcal{F}} \mathcal{V}$;
- $\forall \beta_1 \ldots \beta_n \in \mu(\mathcal{U}) \exists \sigma, \rho \in {}^*\mathbb{N}$ such that $\sigma \cdot \beta_1 + \rho, \ldots, \sigma \cdot \beta_n + \rho \in \mu(\mathcal{V})$.

In [32] we proved that $\mathcal{V} \in \beta\mathbb{N}$ is such that $\forall \mathcal{U} \in \beta\mathbb{N} \mathcal{U} \leq_{fe} \mathcal{V}$ if and only if every set $A \in \mathcal{V}$ is AP-rich. In particular, Theorem 67 entails the equivalence between the following two properties:

- every set $A \in \mathcal{V}$ is AP-rich;
- $\forall n \in \mathbb{N}, \forall \beta_1 \sim_S \cdots \sim_S \beta_n \in {}^*\mathbb{N} \, \exists \sigma, \rho \in {}^*\mathbb{N} \, \sigma \cdot \beta_1 + \rho, \ldots, \sigma \cdot \beta_n + \rho \in \mu(\mathcal{V})$.

In particular, as the family of AP-rich sets is strongly partition regular, from Theorem 62 we obtain the following characterization of AP-rich sets: $A \subseteq \mathbb{N}$ is AP-rich if and only if

$$\forall n \in \mathbb{N}, \forall \beta_1 \sim_S \cdots \sim_S \beta_n \in {}^*\mathbb{N} \, \exists \sigma, \rho \in {}^*\mathbb{N} \, \text{such that} \, \sigma \cdot \beta_1 + \rho, \ldots, \sigma \cdot \beta_n + \rho \in {}^*\mathbb{N}.$$

Example 70. Of course, similar ideas to that introduced in Theorem 67 can be used to study other properties involving multiple ultrafilters and partition regularity. In [26], the authors proved that it is consistent with ZFC that for every finite coloring of $\mathbb{R}$ there is an infinite set $X \subseteq \mathbb{R}$ such that $X + X$ is monochromatic (whilst in [21] it was proven that also the negation of this statement is consistent with ZFC). Of course, without loss of generality we can assume that X is also monochromatic (not necessarily of the same colour of $X + X$). In terms of ultrafilters, this means that there are ultrafilters $\mathcal{U}, \mathcal{V} \in \beta\mathbb{R}$ such that

$$\forall A \in \mathcal{U} \exists B \in \mathcal{V} \text{ such that } B + B \subseteq A. \tag{5}$$

[20] Notice that this characterization resembles that of thick subsets of S: a set $A \subseteq S$ is thick if and only if for every $s_1, \ldots, s_n \in S$ there exists $t \in S$ such that $t \cdot s_1, \ldots t \cdot s_n \in A$, i.e. (by transfer) if for every $\beta_1, \ldots, \beta_n \in {}^*S$ there exists $\sigma \in {}^*S$ such that $\sigma \cdot \beta_1, \ldots \sigma \cdot \beta_n \in {}^*A$.

With arguments similar to those used in the proof of Theorem 67, it is simple to show that this property is equivalent to the following nonstandard fact:

$$\exists \Gamma \in {}^*\mathcal{V} \text{ such that } \Gamma + \Gamma \subseteq \mu(\mathcal{U}). \tag{6}$$

Finally, by noticing that (5) can be rewritten as

$$\forall A \in \mathcal{U} \forall B \in \mathcal{V} \exists C \in \wp(B) \cap \mathcal{V} \text{ such that } C + C \subseteq A,$$

we can strenghten (6) to

$$\exists \Gamma \in {}^*\mathcal{V} \cap \wp(\mu(\mathcal{V})) \text{ such that } \Gamma + \Gamma \subseteq \mu(\mathcal{U}).$$

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