About the stability of the tangent bundle of $\mathbb{P}^n$ restricted to a surface

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Abstract

Let X be a smooth projective surface over $\mathbb{C}$ and let L be a line bundle on X generated by its global sections. Let $\phi_L : X \longrightarrow \mathbb{P}^r$ be the morphism associated to L; we investigate the $\mu-$stability of $\phi_L^* T_{\mathbb{P}^r}$ with respect to L when X is either a regular surface with $p_g = 0$, a K3 surface or an abelian surface. In particular, we show that $\phi_L^* T_{\mathbb{P}^r}$ is $\mu-$stable when X is K3 and L is ample and when X is abelian and $L^2 \geq 14$.

1 Introduction

Given a line bundle L generated by its global sections on a smooth projective variety X, one can consider the kernel of the evaluation map

$$0 \longrightarrow M_L \longrightarrow H^0(X, L) \otimes \mathcal{O}_X \longrightarrow L \longrightarrow 0 \tag{1}$$

and its dual $E_L = M_L^*$.

The stability of this bundle is equivalent to that of $\phi_L^* T_{\mathbb{P}^r}$, where $\phi_L : X \longrightarrow \mathbb{P}^r$ is the morphism associated to L. It has been studied in the case of a curve by Paranjape in [9] with Ramanan and in his Ph.D. thesis [8]; in particular, the latter contains the statements on which rely all our results contained in a former paper [3] and in this one. Later Ein and Lazarsfeld showed in [4] that M_L is stable if $\deg L > 2g$ and Beauville investigated the case of degree $2g$ in [2].

The aim of this paper is to study this problem in the case of projective surfaces. Here we consider the $\mu-$stability of a sheaf with respect to a chosen linear series H, which generalises the definition given in the case of curves: a vector bundle E is said to be $\mu-$stable with respect to H if for each proper quotient sheaf F we have $\mu(F) > \mu(E)$, where $\mu(F) = \frac{c_1(F).H^{n-1}}{\operatorname{rk} F}$ is the slope of F (see [5]).

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After studying these vector bundles in Section 2, we gather some results which hold on curves in Section 3 and then in Section 4 we obtain some results about regular surfaces, including the following

Theorem 1. *Let X be a smooth projective K3 surface over $\mathbb{C}$ and let L be an ample line bundle generated by its global sections on X; then the vector bundle E_L is $\mu-$stable with respect to L.*

Finally, in Section 5 we study the case of abelian surfaces, showing the following

Theorem 2. *Let X be a smooth projective abelian surface over $\mathbb{C}$ and let L be a line bundle on X generated by its global sections such that $L^2 \geq 14$. Then the vector bundle E_L is $\mu-$stable with respect to L.*

2 Simplicity and rigidity of E_L

Let us briefly recall the geometric interpretation of E_L: since L is generated by its global sections, the morphism $\phi_L : X \longrightarrow \mathbb{P}(H^0(L)) \simeq \mathbb{P}^r$ is well-defined and we have $L = \phi_L^* \mathcal{O}_{\mathbb{P}^r}(1)$; thus, from the dual sequence of (1) and from the well-known Euler exact sequence it follows that $E_L = \phi_L^* T_{\mathbb{P}^r} \otimes L^*$ and the stability of E_L is equivalent to the stability of $\phi_L^* T_{\mathbb{P}^r}$.

In the next sections we will deal with the problem of whether or not these bundles are μ-stable, but let us first of all underline that they satisfy in almost any case a less strong property, the simplicity.

Proposition 1. *Let X be a smooth projective variety and L be a big line bundle generated by its global sections on X; if $\dim X \geq 2$ then E_L is simple.*

Proof. If we tensor with E_L the short exact sequence (1) in cohomology we get

$$0 \longrightarrow H^0(M_L \otimes E_L) \longrightarrow H^0(L) \otimes H^0(E_L) \xrightarrow{\alpha} H^0(L \otimes E_L) \longrightarrow$$
$$\longrightarrow H^1(M_L \otimes E_L) \longrightarrow H^0(L) \otimes H^1(E_L) \longrightarrow \cdots \tag{2}$$

Since $H^0(L^*) \cong H^1(L^*) \cong 0$ by Ramanujam-Kodaira vanishing theorem (see [7]), we also have $H^0(L)^* \cong H^0(E_L)$. Now, by tensoring the dual sequence of (1) with L we obtain in cohomology

$$0 \longrightarrow H^0(\mathcal{O}_X) \longrightarrow H^0(L) \otimes H^0(L)^* \xrightarrow{\alpha} H^0(L \otimes E_L) \longrightarrow H^1(\mathcal{O}_X) \longrightarrow \cdots \tag{3}$$

where the morphism α is the same morphism as in (2). Hence $H^0(M_L \otimes E_L) \cong H^0(\mathcal{O}_X) \cong \mathbb{C}$, i.e. E_L is simple. □

In the case of regular surfaces, under mild assumptions, which hold for example if X is a K3 surface, they are also rigid, hence providing an example of an exceptional vector bundle on such a surface.

Proposition 2. *Let X be a smooth projective regular surface and L as above; if the multiplication map $H^0(K_X)\otimes H^0(L) \to H^0(K_X\otimes L)$ is surjective, then E_L is rigid.*

Proof. The morphism α in sequence (3) is surjective because X is regular. Let us show that $H^1(E_L) \cong 0$: indeed, by tensoring (1) with K_X in cohomology we get

$$0 \longrightarrow H^0(M_L \otimes K_X) \longrightarrow H^0(L) \otimes H^0(K_X) \xrightarrow{\varphi} H^0(L \otimes K_X) \longrightarrow$$

$$\longrightarrow H^1(M_L \otimes K_X) \longrightarrow H^0(L) \otimes H^1(K_X) = 0$$

Since we assumed φ surjective, we have $H^1(E_L) \cong H^1(M_L \otimes K_X) \cong 0$ by the duality theorem. Then from the exact sequence (2) it follows that $\mathrm{Ext}^1(E_L, E_L) \cong H^1(M_L \otimes E_L) \cong 0$, i.e. E_L is rigid. □

3 Some results on vector bundles on curves

Let us briefly recall some facts about vector bundles on curves. In a former paper [3] we showed the following

Theorem 3. *Let C be a smooth projective curve of genus $g \geq 2$ over an algebraically closed field k and let L be a line bundle on C generated by its global sections such that $\deg L \geq 2g - c(C)$. Then:*

1. *E_L is semi-stable;*
2. *E_L is stable except when $\deg L = 2g$ and either C is hyperelliptic or $L \cong K(p + q)$ with $p, q \in C$.*

In the case $L = K_C$ more was already known: in [9] Paranjape and Ramanan showed the following

Theorem 4. *Let C be a smooth projective curve of genus $g \geq 2$ over $\mathbb{C}$; E_{K_C} is always semistable and it is also stable if C is not hyperelliptic.*

The proof of Theorem 3 was essentially based on the following lemma, shown by Paranjape in [8].

Lemma 1. *Let F be a vector bundle on C generated by its global sections and such that $H^0(C, F^*) = 0$; then $\deg F \geq \mathrm{rk}\, F + g - h^1(C, \det F)$. Moreover, if $h^1(C, \det F) \geq 2$ then $\deg F \geq 2\mathrm{rk}\, F + c(\det F) \geq 2\mathrm{rk}\, F + c(C)$.*

4 About regular surfaces

Before restricting to the case of regular surfaces, let us see a few statements which hold for every surface.

Lemma 2. *Let F be a vector bundle of rank 2 generated by its global sections on a smooth projective surface X and assume moreover that $h^0(\det F) = 2$. Then there is a short exact sequence*

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s} F \longrightarrow \det F \longrightarrow 0 \tag{4}$$

Proof. We cannot have $F = \mathcal{O}_X^2$ because $h^0(\det F) = 2$; then, since F is of rank 2 generated by its global sections, we can suppose $h^0(F) \geq 3$. Then there is a section $s \in H^0(X, F)$ which is zero only in a finite number of points and we have the following short exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s} F \longrightarrow \mathcal{I}_Z \det F \longrightarrow 0 \tag{5}$$

where Z is the zero locus of s. In cohomology we obtain

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, F) \longrightarrow H^0(X, \mathcal{I}_Z \det F) \longrightarrow \cdots$$

Since $h^0(F) \geq 3$, we get $h^0(\mathcal{I}_Z \det F) \geq 2$, but $h^0(\mathcal{I}_Z \det F) \leq h^0(\det F) = 2$. Since $\det F$ is generated by its global sections, from $h^0(\mathcal{I}_Z \det F) = h^0(\det F) = 2$ it follows that $\mathcal{I}_Z \det F = \det F$ and $Z = \varnothing$. Therefore the sequence (5) becomes (4). □

Proposition 3. *Let X be a smooth projective surface over $\mathbb{C}$ and let L be a line bundle on X generated by its global sections. Let C be a smooth irreducible curve on X such that $H^1(L \otimes \mathcal{O}_X(-C)) = 0$. Then $(E_L)_{|C} = E_{(L_{|C})} \oplus \mathcal{O}_C^r$, with $r = h^0(L \otimes \mathcal{O}_X(-C))$.*

Proof. Tensoring the exact sequence

$$0 \longrightarrow \mathcal{O}_X(-C) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_C \longrightarrow 0$$

with L, we get

$$0 \longrightarrow L \otimes \mathcal{O}_X(-C) \longrightarrow L \longrightarrow L_{|C} \longrightarrow 0$$

and hence in cohomology we have

$$0 \longrightarrow H^0(X, L \otimes \mathcal{O}_X(-C)) \longrightarrow H^0(X, L) \longrightarrow H^0(X, L_{|C}) \longrightarrow 0$$

So we have the following diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L^*_{|C} & \longrightarrow & H^0(X, L_{|C})^* \otimes \mathcal{O}_C & \longrightarrow & E_{(L_{|C})} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & L^*_{|C} & \longrightarrow & H^0(X, L)^* \otimes \mathcal{O}_C & \xrightarrow{e_L} & (E_L)_{|C} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{O}^r_C & \longrightarrow & \mathcal{O}^r_C & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array} \qquad (6)$$

By the snake lemma, the third column is exact. Moreover, the sequence splits and $(E_L)_{|C} = E_{(L_{|C})} \oplus \mathcal{O}^r_C$. □

Corollary 1. *Let X be a smooth projective regular surface over $\mathbb{C}$ such that $p_g = 0$ and let C be a smooth irreducible curve on X of genus $g \geq 2$ such that $L = \mathcal{O}_X(K_X + C)$ is generated by its global sections; then E_L is $\mu-$semistable with respect to C and it is also stable if $c(C) > 0$.*

Proof. By Proposition 3 $(E_L)_{|C} \cong E_{(L_{|C})}$, since $r = p_g = 0$; on the other hand, $L_{|C} = K_C$, so the statement follows from Theorem 4. □

When $r \neq 0$, the restriction to the curve is no longer semistable, but in the case of K3 surfaces this is enough to gain the $\mu-$stability.

Proof of Theorem 1. Let $C \in |L|$ be a smooth irreducible curve of genus $g \geq 2$. By Proposition 3 we have $(E_L)_{|C} = E_{K_C} \oplus \mathcal{O}_C$, since $L_{|C} \cong K_C$; moreover $\mu(E_L) = \frac{2g-2}{g} < 2$. Let us suppose that $g \geq 3$: if $g = 2$ then C is hyperelliptic and we will deal with the case $c(C) = 0$ later. Let F be a quotient sheaf of E_L of rank $0 < \operatorname{rk} F < g$; then $F_{|C}$ is a quotient of $(E_L)_{|C}$. There is a diagram of the form

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_C & \longrightarrow & (E_L)_{|C} & \longrightarrow & E_{K_C} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & W & \longrightarrow & F_{|C} & \longrightarrow & G \oplus \tau & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

where G is a vector bundle generated by its global sections, W is either $\mathcal{O}_C$ or 0 and τ is a torsion sheaf on C, hence $\deg W = 0$ and $\deg \tau \geq 0$. So we get $\mu(F) = \frac{\deg G + \deg \tau}{\mathrm{rk}\, F}$.

- If $\mathrm{rk}\, G = 0$, then $\mathrm{rk}\,(F) = 1$ and we always have $\mu(F) \geq 2$. Indeed, otherwise it would be $F = \mathcal{O}_X(D)$ with $D > 0$ an effective base-point free divisor such that $D.C = 0$ or 1; we cannot have $D.C = 0$, since D is nef, hence $D^2 \geq 0$, but by the Hodge index theorem we would have $D^2 < 0$, which is a contradiction. If $D.C = 1$, by the Hodge index theorem we get $D^2 = 0$, hence $D = kE$ with $k \geq 1$ and E an elliptic curve; in fact, we have $k = 1$ because $D.C = 1$, so $h^0(D) = 2$ and $|D|$ is a pencil; then, since $C.D = 1$, C would be a section and $C^2 < 0$, impossible.

- If $\mathrm{rk}\, G > 0$, then G is generated by its global sections such that $H^0(C, G^*) = 0$; the hypothesis of Lemma 1 then hold and, since $\mu(F) \geq \frac{\deg G}{\mathrm{rk}\, F}$, we have:

  1. if $h^1(\det G) < 2$, since $g \geq 3$, then
  $$\mu(F) \geq 1 + \frac{g-2}{\mathrm{rk}\, G + 1} > 1 + \frac{g-2}{g} = \mu(E_L).$$

  2. If $h^1(\det G) \geq 2$, then
  $$\mu(F) \geq 2 + \frac{c(\det G) + \deg \tau - 2}{\mathrm{rk}\, G + 1} \geq 2 > \mu(E_L)$$
  if $c(\det G) \geq 2$, in particular if $c(C) \geq 2$, but also if $c(\det G) = 1$ and $\deg \tau > 0$.

This shows that $\mu(F) > \mu(E_L)$ in the case $c(C) \geq 2$.

We now deal with the case $c(C) = 1$. We can repeat the above proof by applying Lemma 1 and it does not work only if $h^1(\det G) \geq 2$, $\deg \tau = 0$ and $c(\det G) = 1$. If $g = 3$ then $\mu(E_L) = \frac{4}{3}$ and we always have $\mu(F) > \frac{4}{3}$.

From now on we assume $g \geq 4$; then either the curve is trigonal or a smooth plane quintic of genus $g = 6$ (see [6]).

1. If there is a $\mathfrak{g}^1_3$ on C, the only line bundles which compute the Clifford index are $\mathcal{O}_C(\mathfrak{g}^1_3)$ and $\mathcal{O}_C(K_C - \mathfrak{g}^1_3)$.

   (a) If $\det G = \mathcal{O}_C(\mathfrak{g}^1_3)$, since $h^1(\det G) \geq 2$, by Lemma 1 we have $\deg G \geq 2\mathrm{rk}\, G + 1$, hence in this case $\mathrm{rk}\, G = 1$. Then $\mathrm{rk}\, F = 2$ and $\det F_{|C} = \mathcal{O}_C(\mathfrak{g}^1_3)$; it follows that $\det F = \mathcal{O}_X(D)$ with $D.C = 3$. By the Hodge index theorem then, since $g \geq 4$, we have $D^2 \leq \frac{9}{2g-2} < 2$, so $D^2 = 0$ and $D = kE$ with $k \geq 1$ and E

an elliptic curve; since $D.C = 3$ and $C.E \geq 2$, this implies $k = 1$ and $h^0(\mathcal{O}_X(D)) = 2$; by Lemma 2, it follows from $h^1(\det F^*) = 0 = \mathrm{Ext}^1(\mathcal{O}_X, \det F)$ that $F = \mathcal{O}_X \oplus \det F$, hence $h^0(F^*) > 0$, which is impossible.

(b) If $\det G = \mathcal{O}_C(K_C - \mathfrak{g}^1_3)$ we have $\deg G = 2g-5$ and $\mathrm{rk}\, G \leq g-3$ by Lemma 1, hence

$$\mu(F) \geq \frac{2g-5}{\mathrm{rk}\, G + 1} \geq \frac{2g-5}{g-2} = 2 - \frac{1}{g-2} > \mu(E_L)$$

if $g > 4$. If $g = 4$ we have $\deg G = 3$ and we fall in the former case.

2. If there is a $\mathfrak{g}^2_5$ on C, the genus is $g = 6$ and the only line bundle which computes the Clifford index is $\mathcal{O}_C(\mathfrak{g}^2_5) \cong \mathcal{O}_C(K_C - \mathfrak{g}^2_5)$.

   If $\det G = \mathcal{O}_C(\mathfrak{g}^2_5)$, since $h^1(\det G) \geq 2$, by Lemma 1 $\deg G \geq 2\mathrm{rk}\, G+1$, hence $\mathrm{rk}\, G \leq 2$ and $\mathrm{rk}\, F \leq 3$. Therefore we get

$$\mu(F) = \frac{5}{\mathrm{rk}\, G + 1} \geq \frac{5}{3} = \mu(E_L)$$

   Let us investigate whether equality can hold or not; suppose that $\mathrm{rk}\, F = 3$. Since F is of rank > 2 generated by its global sections, there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow F \longrightarrow V \longrightarrow 0 \tag{7}$$

   with V of rank 2 generated by its global sections such that $\det V = \det F = \mathcal{O}_X(D)$ with $D.C = 5$. By the Hodge index theorem then $D^2 \leq 2$; however the case $D^2 = 2$ cannot occur, since otherwise $(C - 2D)^2 = -2$ and by Riemann-Roch theorem at least one between $C-2D$ and $2D-C$ would be effective, contradicting $(C-2D).C = 0$ and the ampleness of C. If $D^2 = 0$, then $D = kE$ with $k \geq 1$ and E an elliptic curve; since $D.C = 5$ and $C.E \geq 2$, this implies $k = 1$ and $h^0(\mathcal{O}_X(D)) = 2$, so by Lemma 2 there is a short exact sequence

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\ s\ } V \longrightarrow \det V \longrightarrow 0$$

   and in cohomology we obtain $h^1(V^*) = h^1(V) = 0$. As a consequence we have $\mathrm{Ext}^1(\mathcal{O}_X, V) = 0$ and $F = \mathcal{O}_X \oplus V$, impossible since it would imply $h^0(F^*) > 0$.

Then $\mu(F) > \mu(E_L)$ also if $c(C) = 1$.

Suppose now that C is a hyperelliptic curve; in this case (see [1], pag.129), the morphism $\phi_L : X \longrightarrow \mathbb{P}^g$ induces a double covering $\pi : X \longrightarrow F$ where

$F \subset \mathbb{P}^g$ is a rational surface of degree $g-1$ which is either smooth or a cone over a rational normal curve. If $g=2$ then $F=\mathbb{P}^2$ (see [1], pag.129) and it is well-known that its tangent bundle is $\mu-$stable (see [5] Section 1.4) with respect to $\mathcal{O}_{\mathbb{P}^2}(1)$. If $g \geq 3$, let $i : F \hookrightarrow \mathbb{P}^g$ be the embedding and $H = i^*\mathcal{O}_{\mathbb{P}^g}(1)$ the ample hyperplane section of F such that $\pi^* H = L$; we have $H^2 = g-1$.

On the surface F we have the short exact sequence

$$0 \longrightarrow H^* \longrightarrow H^0(F,H)^* \otimes \mathcal{O}_F \longrightarrow E_H \longrightarrow 0 \qquad (8)$$

We know that the curve H is rational, so $p_a(H)=0$; we consider a smooth curve $\Gamma \in |2H|$. By the adjunction formula we have $0 = p_a(H) = 1+\frac{1}{2}(H^2 + H.K_F)$, so we get $H.K_F = -H^2-2 = -g-1$; using the adjunction formula once more we then obtain

$$p_a(\Gamma) = 1 + \frac{1}{2}(\Gamma^2 + \Gamma.K_F) = 1 + 2H^2 + H.K_F = g-2$$

Since $g \geq 3$ we have $p_a(\Gamma) \geq 1$. Since H is ample, we deduce $H^0(F, \mathcal{O}_F(-H)) = H^1(F, \mathcal{O}_F(-H)) = 0$ (see [7]). Then from the short exact sequence

$$0 \longrightarrow \mathcal{O}_F(H-\Gamma) \longrightarrow \mathcal{O}_F(H) \longrightarrow \mathcal{O}_\Gamma(H) \longrightarrow 0$$

and from the associated cohomology sequence it follows that $H^0(F,\mathcal{O}_F(H)) \cong H^0(F, \mathcal{O}_\Gamma(H))$, hence $(E_H)_{|\Gamma} = E_{\mathcal{O}_\Gamma(H)}$.

Moreover, $\deg \mathcal{O}_\Gamma(H) = H.\Gamma = 2g-2 > 2p_a(\Gamma) = 2g-4$. Since $\mathcal{O}_\Gamma(H)$ is a line bundle on a smooth projective curve Γ of genus ≥ 1 of degree $> 2p_a(\Gamma)$, $(E_H)_{|\Gamma}$ is stable (see [4]).

Since E_H is $\mu-$stable with respect to $2H$, it is also $\mu-$stable with respect to H and this yields the $\mu-$stability of E_L with respect to L, because π is a double covering (see [5], Lemma 3.2.2). □

Remark. Throughout the proof the ampleness of L is used only when C is a smooth plane quintic of genus $g=6$ to show that we cannot have equality between slopes. Indeed, if we only assume that L is generated by its global sections and $L^2 \geq 2$ then E_L is still $\mu-$semistable with respect to L and also $\mu-$stable unless C is a smooth plane quintic of genus $g=6$.

5 About abelian surfaces

In this section we study the same problem when X is an abelian surface over $\mathbb{C}$ and we give the proof of Theorem 2.

Proposition 4. *Let X be an abelian surface over $\mathbb{C}$; then there is no irreducible hyperelliptic curve of genus $g \geq 6$ and no irreducible trigonal curve of genus $g \geq 8$ on X.*

Proof. Take $d = 2$ or 3 and suppose that there is a $d-$gonal irreducible curve C of genus $g \geq 2d + 2$ on X. Then there is an exact sequence of sheaves on X

$$0 \longrightarrow F^* \longrightarrow H^0(g^1_d) \otimes \mathcal{O}_X \longrightarrow \mathcal{O}_C(g^1_d) \longrightarrow 0$$

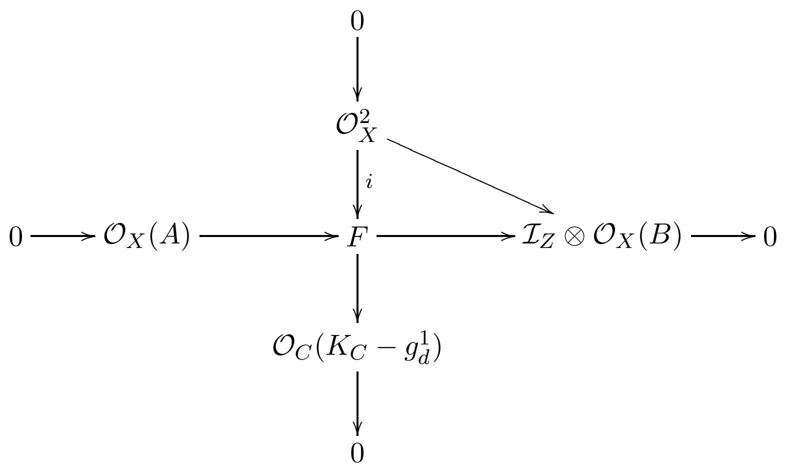
where F is a vector bundle of rank 2 such that $c_1(F) = C$ and $c_2(F) = d$. Dualising the above exact sequence we get

$$0 \longrightarrow \mathcal{O}_X^2 \longrightarrow F \longrightarrow \mathcal{O}_C(K_C - g^1_d) \longrightarrow 0$$

It follows from the assumption on the genus that $c_1(F)^2 - 4c_2(F) = 2g - 2 - 4d > 0$, so F is Bogomolov unstable (see [10]). Therefore, there exists a line bundle $\mathcal{O}_X(A)$ on X such that $\mu(\mathcal{O}_X(A)) > \mu(F)$, i.e. $2A.C > C^2$, and we have an exact sequence

$$0 \longrightarrow \mathcal{O}_X(A) \longrightarrow F \longrightarrow \mathcal{I}_Z \otimes \mathcal{O}_X(B) \longrightarrow 0$$

with $A + B = C$, $A.B + \deg \mathcal{I}_Z = d$ and $(A - B)^2 > 0$ (see [10]). Hence we can construct the following diagram



Since i is an isomorphism outside C, $h^0(\mathcal{I}_Z \otimes \mathcal{O}_X(B)) > 0$ and B is effective. By the Hodge index theorem $A^2B^2 \leq (A.B)^2 \leq d^2$. Since $K_X = 0$, A^2 and B^2 are even numbers and $A^2 > B^2$ because $2A.C > C^2$, hence we must have $B^2 \leq 2$.

If $B^2 = 2$, then $d = 3$ and $A^2 = 4$ and we would have $6 - 2A.B > 0$, so $A.B \leq 2$ in contradiction with $A^2B^2 = 8$. Therefore $B^2 = 0$, which means that $B = kE$ where E is an elliptic curve and $k \geq 1$; on the other hand we know that $0 \leq A.B \leq d$. In fact $A.B > 0$, otherwise by the Hodge index theorem it would follow $B = 0$ against the fact that $h^0(\mathcal{I}_Z \otimes \mathcal{O}_X(B)) > 0$; hence $1 \leq kA.E \leq d$. Since $A.E = 1$ would imply that A itself is elliptic, the only possibility is $k = 1$ and $A.B > 1$. In this case we have $h^0(B) = 1$, hence by the snake lemma we have the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & \mathcal{O}_X^2 & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle s} & & \downarrow & & \downarrow{\scriptstyle \sigma} & & \\
0 & \longrightarrow & \mathcal{O}_X(A) & \longrightarrow & F & \longrightarrow & \mathcal{I}_Z \otimes \mathcal{O}_X(B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tau & \longrightarrow & \mathcal{O}_C(K_C - g^1_d) & \longrightarrow & \tau' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where τ and τ' are two torsion sheaves with support respectively on the zero-locus of s and σ. Hence the exactness of the third line implies that C is reducible, against our assumptions. □

Proof of Theorem 2. Since L is generated by its global sections such that $L^2 \geq 14$, the general member of $|L|$ is a smooth irreducible curve of genus $g \geq 8$. Hence, given a non-zero $\alpha \in \mathrm{Pic}^0(X)$, we can find $C \in |L \otimes \alpha^{-1}|$ smooth irreducible of genus $g \geq 8$. The $\mu-$stability of E_L with respect to L is equivalent to the $\mu-$stability of E_L with respect to C. Since we have $H^0(\alpha) = H^1(\alpha) = 0$, it follows from Proposition 3 that $(E_L)_{|C} \cong E_{(L_{|C})}$. Moreover, $L_{|C} \cong K_C \otimes \alpha_{|C}$, so by Theorem 3 E_L is $\mu-$stable with respect to C if $c(C) \geq 2$. By the hypothesis on the genus of C and by Proposition 4 the cases $c(C) = 0, 1$ cannot occur, so there is nothing more to prove. □

Remark. In the case $g(C) \leq 7$ the same proof shows the $\mu-$stability of E_L if $c(C) \geq 2$. Moreover, it is possible to show that E_L is $\mu-$stable with respect to L also if either C is a smooth plane quintic of genus $g = 6$ or if C is a trigonal curve of genus $g = 4$.

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