About the stability of the tangent bundle restricted to a curve

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Abstract

Let C be a smooth projective curve of genus $g \geq 2$ and let L be a line bundle on C generated by its global sections. The morphism $\phi_L : C \longrightarrow \mathbb{P}(H^0(L)) \simeq \mathbb{P}^r$ is well-defined and $\phi_L^* T_{\mathbb{P}^r}$ is the restriction to C of the tangent bundle of $\mathbb{P}^r$. Sharpening a theorem by Paranjape, we show that if $\deg L \geq 2g - c(C)$ then $\phi_L^* T_{\mathbb{P}^r}$ is semi-stable, specifying when it is also stable. We then prove the existence on many curves of a line bundle L of degree $2g - c(C) - 1$ such that $\phi_L^* T_{\mathbb{P}^r}$ is not semi-stable. Finally, we completely characterize the (semi-)stability of $\phi_L^* T_{\mathbb{P}^r}$ when C is hyperelliptic.

Résumé

Sur la stabilité du fibré tangent restreint à une courbe. Soit L un fibré en droites engendré par ses sections globales sur une courbe projective lisse C de genre $g \geq 2$. Le fibré L définit $\phi_L : C \longrightarrow \mathbb{P}(H^0(L)) \simeq \mathbb{P}^r$ et $\phi_L^* T_{\mathbb{P}^r}$ est la restriction à la courbe C du fibré tangent de $\mathbb{P}^r$. En précisant un théorème dû à Paranjape, on montre que si $\deg L \geq 2g - c(C)$ alors $\phi_L^* T_{\mathbb{P}^r}$ est semi-stable, en disant quand il est aussi stable. De plus, on montre l'existence sur plusieurs courbes d'un fibré en droites L de degré $2g - c(C) - 1$ tel que $\phi_L^* T_{\mathbb{P}^r}$ ne soit pas semi-stable. Enfin, on caractérise complètement la stabilité de $\phi_L^* T_{\mathbb{P}^r}$ si C est hyperelliptique.

1. Introduction

Let C be a smooth projective curve of genus $g \geq 2$ and let L be a line bundle on C generated by its global sections. Let M_L be the vector bundle defined by the exact sequence

$$0 \longrightarrow M_L \longrightarrow H^0(C, L) \otimes \mathcal{O}_C \xrightarrow{e_L} L \longrightarrow 0 \tag{1}$$

where e_L is the evaluation map. We denote by E_L the dual bundle of M_L: it has degree $\deg L$ and rank $h^0(C, L) - 1$. Let us briefly recall the geometric interpretation of these bundles: since L is generated by its global sections, the morphism $\phi_L : C \longrightarrow \mathbb{P}(H^0(L)) \simeq \mathbb{P}^r$ is well-defined and we have $L = \phi_L^* \mathcal{O}_{\mathbb{P}^r}(1)$; thus, from the dual sequence of (1) and from the well-known Euler exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^r} \longrightarrow H^0(C, L)^* \otimes \mathcal{O}_{\mathbb{P}^r}(1) \longrightarrow T_{\mathbb{P}^r} \longrightarrow 0 \tag{2}$$

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it follows that $E_L = \phi_L^* T_{\mathbb{P}^r} \otimes L^*$ and the stability of E_L is equivalent to the stability of $\phi_L^* T_{\mathbb{P}^r}$.

We recall the definition of the Clifford index of a curve.

Definition 1.1 *The Clifford index of a line bundle L on C is $c(L) = \deg L - 2(h^0(C, L) - 1)$.*

The Clifford index of a divisor D on C is the Clifford index of the associated line bundle $\mathcal{O}_C(D)$, i.e. $c(D) = c(\mathcal{O}_C(D)) = \deg D - 2 \dim |D|$.

The Clifford index of the curve C is $c(C) = \min\{c(L)/h^0(C, L) \geq 2,\ h^1(C, L) \geq 2\}$.

Clifford's theorem states that $c(C) \geq 0$, with equality if and only if C is hyperelliptic; moreover, for any divisor D on C, $c(D) = c(K - D)$.

Remark 1 By the Riemann-Roch theorem, $c(L) = 2g - \deg L - 2h^1(C, L)$ for any line bundle L.

In [3], by using the properties of this invariant, Paranjape proves the following

Proposition 1.2 *Let C be a smooth projective curve of genus $g \geq 2$ and let L be a line bundle on C generated by its global sections. If $c(C) \geq c(L)$ then E_L is semi-stable. If $h^1(C, L) = 1$ and $c(C) > 0$ or $c(C) > c(L)$ then E_L is also stable.*

By completing his proof we show the following

Theorem 1.3 *Let C be a smooth projective curve of genus $g \geq 2$ and let L be a line bundle on C generated by its global sections such that $\deg L \geq 2g - c(C)$. Then:*

(i) E_L is semi-stable;

(ii) E_L is stable except when $\deg L = 2g$ and either C is hyperelliptic or $L \cong K(p + q)$ with $p, q \in C$.

If C is a smooth projective $d-$gonal curve of genus $g \geq 2$ with Clifford index $c(C) = d - 2 < \frac{g-2}{2}$, we then prove the existence of a line bundle L of degree $2g - c(C) - 1$ such that E_L is not semi-stable. Moreover, a theorem by Schneider (see [4]) states that on a general smooth curve E_L is always semi-stable: our proof also shows that one cannot replace semi-stable by stable in this statement.

Finally, we completely characterize the (semi-)stability of E_L when C is hyperelliptic.

2. Proof of Theorem 1.3

We first need a lemma, shown by Paranjape in [3].

Lemma 2.1 *Let F be a vector bundle on C generated by its global sections and such that $H^0(C, F^*) = 0$; then $\deg F \geq \operatorname{rk} F + g - h^1(C, \det F)$ and equality holds if and only if $F = E_L$, where $L = \det F$. Moreover, if $h^1(C, \det F) \geq 2$ then $\deg F \geq 2\operatorname{rk} F + c(C)$ and if equality holds then $F = E_L$.*

The canonical bundle K is generated by its global sections and there is an exact sequence

$$0 \longrightarrow K^* \longrightarrow H^0(C, K)^* \otimes \mathcal{O}_C \longrightarrow E_K \longrightarrow 0$$

thus in cohomology we have

$$0 \longrightarrow H^0(K^*) \longrightarrow H^0(K)^* \otimes H^0(\mathcal{O}_C) \longrightarrow H^0(E_K) \longrightarrow H^1(K^*) \xrightarrow{\varphi} H^0(K)^* \otimes H^1(\mathcal{O}_C) \longrightarrow \cdots \tag{3}$$

The map φ is the dual map of $m : H^0(K) \otimes H^0(K) \to H^0(K^2)$, so it is injective by Noether's theorem (see [1], Chap.III); moreover, $H^0(C, K^*) = 0$. As a consequence $H^0(C, E_K) \simeq H^0(C, K)^* = H^1(C, \mathcal{O}_C)$ and $h^0(C, E_K) = g$.

Now we have all the tools necessary to prove Theorem 1.3.

Proof of Theorem 1.3. By Remark 1, if $\deg L \geq 2g - c(C)$ a fortiori $c(C) \geq c(L)$. By definition, $\deg E_L = c(L) + 2\operatorname{rk} E_L$ and $h^0(C, L) = \operatorname{rk} E_L + 1$, hence it follows by the Riemann-Roch theorem that $\deg E_L = \operatorname{rk} E_L + g - h^1(C, L)$.

Let F be a quotient bundle of E_L; then F satisfies the hypothesis of Lemma 2.1, because it is spanned by its global sections since E_L is and $H^0(C, F^*) \subset H^0(C, E_L^*) = 0$.

Therefore, if $h^1(C, \det F) \geq 2$ we have $\deg F \geq 2\mathrm{rk}\, F + c(C)$; then

$$\mu(F) - \mu(E_L) \geq \frac{c(C)}{\mathrm{rk}\, F} - \frac{c(L)}{\mathrm{rk}\, E_L} = \frac{\mathrm{rk}\, E_L \cdot c(C) - \mathrm{rk}\, F \cdot c(L)}{\mathrm{rk}\, F \cdot \mathrm{rk}\, E_L} = \frac{(\mathrm{rk}\, E_L - \mathrm{rk}\, F)\cdot c(C) + \mathrm{rk}\, F \cdot (c(C) - c(L))}{\mathrm{rk}\, F \cdot \mathrm{rk}\, E_L} \geq 0$$

since $\mathrm{rk}\, E_L > \mathrm{rk}\, F > 0$ and $c(C) \geq c(L)$. Moreover, the inequality is strict if $c(C) > 0$ or if C is hyperelliptic and $\deg L \geq 2g + 1$, because L is non-special and $c(L) < 0$.

If $h^1(C, \det F) < 2$ we still have $\deg F \geq \mathrm{rk}\, F + g - h^1(C, \det F)$, hence

$$\mu(F) - \mu(E_L) \geq \frac{g - h^1(\det F)}{\mathrm{rk}\, F} - \frac{g - h^1(L)}{\mathrm{rk}\, E_L} = \frac{\left[g - h^1(\det F)\right]\cdot(\mathrm{rk}\, E_L - \mathrm{rk}\, F) + \mathrm{rk}\, F \cdot \left[h^1(L) - h^1(\det F)\right]}{\mathrm{rk}\, F \cdot \mathrm{rk}\, E_L} > 0$$

provided that $h^1(C, L) \geq h^1(C, \det F)$, since $g - h^1(C, \det F) > 0$ follows from the hypothesis that $h^1(C, \det F) < 2$ and $g \geq 2$.

The only case remaining is $0 = h^1(C, L) < h^1(C, \det F) = 1$. We have $\deg F = \deg(\det F) \leq 2g - 2$, otherwise we should have $h^1(C, \det F) = 0$; then, a fortiori, we have $\mathrm{rk}\, F \leq g - 1$. It then follows from the previous inequalities that

$$\mu(F) - \mu(E_L) \geq \frac{(g-1)(\mathrm{rk}\, E_L - \mathrm{rk}\, F) - \mathrm{rk}\, F}{\mathrm{rk}\, F \cdot \mathrm{rk}\, E_L} \geq \frac{(g-1)\cdot(\mathrm{rk}\, E_L - \mathrm{rk}\, F - 1)}{\mathrm{rk}\, F \cdot \mathrm{rk}\, E_L} \geq 0 \tag{4}$$

Thus we have shown that we always have $\mu(F) - \mu(E_L) \geq 0$, i.e. E_L is semi-stable. In order to gain the stability of E_L, we still need to prove that $\mu(F) - \mu(E_L) > 0$ when $0 = h^1(C, L) < h^1(C, \det F) = 1$.

Suppose that $\mu(E_L) = \mu(F)$; by (4), we then have $(g-1)\cdot\mathrm{rk}\, E_L - g\cdot\mathrm{rk}\, F = 0$. Since $g \geq 2$, it follows that $(g-1)|\mathrm{rk}\, F \leq g - 1$, i.e. $\mathrm{rk}\, F = g - 1$, and $\mathrm{rk}\, E_L = g$; hence $\deg E_L = g + \mathrm{rk}\, E_L = 2g$ and $\mu(E_L) = 2$. Therefore, if $\deg L \neq 2g$ we cannot have $\mu(E_L) = \mu(F)$ and E_L is stable.

If $\deg L = 2g$ then E_L is stable provided that $c(C) > 0$ and $L \not\cong K(p + q)$ with $p, q \in C$.

Indeed, since $\deg F = \mathrm{rk}\, F \cdot \mu(F) = 2g - 2$ and $h^1(C, \det F) = 1$, we have $\det F \cong K$. As a consequence we have $\mathrm{rk}\, F + g - h^1(C, \det F) = 2g - 2 = \deg F$, so $F = E_K$ by Lemma 2.1. On the other hand, F is a quotient of E_L, so there is an exact sequence

$$0 \longrightarrow W \longrightarrow E_L \longrightarrow F \longrightarrow 0 \tag{5}$$

where W is a sub-bundle of E_L of degree 2 and rank 1. The associated exact sequence of cohomology then is

$$0 \longrightarrow H^0(C, W) \longrightarrow H^0(C, E_L) \xrightarrow{\varphi} H^0(C, E_K) \longrightarrow H^1(C, W) \longrightarrow \cdots$$

From the exact sequence of cohomology associated to the dual sequence of (1) we see that $h^0(C, E_L) \geq g+1$ and $h^0(C, E_K) = g$ since $c(C) > 0$; hence φ cannot be injective, i.e. $H^0(C, W) \neq 0$. Thus $W \cong \mathcal{O}_C(p + q)$ with $p, q \in C$. Furthermore, it follows from (5) that

$$L = \det E_L = \det W \otimes \det F = W \otimes K = K(p + q),$$

which concludes the proof of Theorem 1.3 since this is not possible under our hypothesis. $\square$

3. Some line bundles of degree $2g - c(C) - 1$ with non semi-stable E_L

Theorem 1.3 is the best possible result that one can obtain if looking for properties of all curves.

Proposition 3.1 *Let C be a smooth projective d-gonal curve of genus $g \geq 2$ such that the Clifford index is $c(C) = d - 2 < \frac{g-2}{2}$; there exists a line bundle L of degree $\deg L = 2g - c(C) - 1$ on C generated by its global sections and non-special such that E_L is not semi-stable.*

Proof. By the hypothesis, $\mathfrak{g}^1_d$ computes the Clifford index. We put $N = \mathcal{O}_C(K - \mathfrak{g}^1_d)$: it is a line bundle of degree $2g - c(C) - 4$ and by the Riemann-Roch theorem $h^0(N) = g - c(C) - 1$. Moreover N is spanned by its global sections: assume that there exists $q \in C$ such that $h^0(N(-q)) = h^0(N)$, or equivalently $h^1(N(-q)) = h^1(N) + 1$; then, by Serre's duality, we have $h^0(\mathfrak{g}^1_d + q) = h^0(\mathfrak{g}^1_d) + 1 = 3$, i.e. $\mathfrak{g}^1_d + q = \mathfrak{g}^2_{d+1}$, and this is not possible because we would have $c(\mathfrak{g}^2_{d+1}) = d - 3 < c(C)$.

Let E be an effective divisor of degree 3 on C; we can choose E in such a way that $L = N \otimes \mathcal{O}_C(E)$ is a line bundle of degree $\deg L = 2g - c(C) - 1$, non-special and spanned by its global sections. Indeed, we have $h^1(L) = 0$ because $h^1(L) = h^0(\mathfrak{g}^1_d - E) = 0$ for a general effective divisor E; moreover L is generated by its global sections if and only if $h^1(L(-p)) = h^1(L) = 0$ for any $p \in C$ and if E is a general effective divisor of degree 3 we have $h^1(L(-p)) = h^0(\mathfrak{g}^1_d - E + p) = 0$.

Since we have supposed that E is effective, $H^0(L \otimes N^*)) \neq 0$, so we have an inclusion $N \hookrightarrow L$. Hence M_N is a sub-bundle of M_L, or equivalently E_N is a quotient bundle of E_L. Since $\operatorname{rk} E_L = g - c(C) - 1$ and $\operatorname{rk} E_N = h^0(N) - 1 = g - c(C) - 2$, we have

$$\mu(E_N) = 2 + \frac{c(C)}{g - c(C) - 2} < \mu(E_L) = 2 + \frac{c(C) + 1}{g - c(C) - 1} \tag{6}$$

whenever $c(C) < \frac{g-2}{2}$. It then follows that E_L is not semi-stable. $\square$

Remark 2 If C is a curve of genus $g \geq 2$ with Clifford index c, in most cases C is $(c+2)-$gonal: see [2] for further details.

Remark 3 The hypothesis that $c(C) < \frac{g-2}{2}$ leaves out only the case $c(C) = \left[\frac{g-1}{2}\right]$, i.e. the general one; however, in [4] Schneider shows the following

Proposition 3.2 *Let C be a general smooth curve of genus $g \geq 3$. If L is a line bundle on C generated by its global sections, then E_L is semi-stable.*

It is worth underlining that one cannot replace semi-stable by stable: if C is a general curve of even genus $g = 2n$ we know that

$$c(C) = \left[\frac{g-1}{2}\right] = n - 1 = \frac{g-2}{2}, \tag{7}$$

so the proof of Proposition 3.1 shows that E_L is not stable, since one obtains $\mu(E_N) = \mu(E_L)$.

4. The case of hyperelliptic curves

In the case of hyperelliptic curves we completely characterize the stability of E_L.

Proposition 4.1 *Let C be a smooth projective hyperelliptic curve of genus $g \geq 2$, let L be a line bundle on C generated by its global sections and such that $h^0(C, L) \geq 3$ and let H be $\mathcal{O}_C(\mathfrak{g}^1_2)$. Then:*

(i) E_L is stable if and only if $\deg L \geq 2g + 1$;

(ii) E_L is semi-stable if and only if $\deg L \geq 2g$ or there exists an integer $k > 0$ such that $L = H^{\otimes k}$.

Proof. By Theorem 1.3, if $\deg L \geq 2g$ then E_L is semi-stable and if $\deg L \geq 2g + 1$ then E_L is stable.

On the other hand E_L is not stable if $\deg L = 2g$, in which case $\mu(E_L) = 2$. Indeed, we show that H is a quotient bundle of E_L of same slope. We know that there is a surjection $E_L \twoheadrightarrow H$ if and only if there is an inclusion $H^* \rightarrowtail M_L$, if and only if $H^0(C, M_L \otimes H) \neq 0$. From the exact sequence (1) we get an exact sequence

$$0 \longrightarrow H^0(C, M_L \otimes H) \longrightarrow H^0(C, L) \otimes H^0(C, H) \longrightarrow H^0(C, L \otimes H) \longrightarrow \cdots \tag{8}$$

We then have $\dim H^0(C, L) \otimes H^0(C, H) = 2g + 2 > g + 3 = h^0(C, L \otimes H)$, so $H^0(C, M_L \otimes H) \neq 0$.

If $0 < \deg L \leq 2g - 1$ we always have $c(L) \geq 0$. If $c(L) = 0$ then E_L is semi-stable, as it follows from the proof of Theorem 1.3: if F is a quotient bundle of E_L, the inequality $\mu(F) - \mu(E_L) \geq 0$ still holds in each case.

Using again the exact sequence (8), since $h^0(C,L) \geq 3$, we have

$$\dim H^0(C,L) \otimes H^0(C,H) = 2h^0(C,L) > h^0(C,L) + 2 \geq h^0(C, L \otimes H).$$

Therefore, $H^0(C, M_L \otimes H) \neq 0$ and there is a surjection $E_L \twoheadrightarrow H$; furthermore,

$$\mu(E_L) = 2 + \frac{c(L)}{h^0(C,L) - 1}$$

and $\mu(H) = 2$. Thus if $c(L) > 0$ then $\mu(E_L) > \mu(H)$ and E_L is not semi-stable; else, if $c(L) = 0$, $\mu(E_L) = \mu(H)$ and E_L is not stable.

The proposition then follows by Clifford's theorem: since C is hyperelliptic and $\deg L > 0$, $c(L) = 0$ if and only if there exists an integer $k > 0$ such that $L = H^{\otimes k}$. $\square$

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