About the stability of the tangent bundle restricted to a curve

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Abstract

Let $C$ be a smooth projective curve of genus $g \geq 2$ and let $L$ be a line bundle on $C$ generated by its global sections. The morphism $φ_L : C \to \mathbb{P}(H^0(L)) \simeq \mathbb{P}^r$ is well-defined and $φ_L^*T_{\mathbb{P}^r}$ is the restriction to $C$ of the tangent bundle of $\mathbb{P}^r$. Sharpening a theorem by Paranjape, we show that if $\deg L \geq 2g - c(C)$ then $φ_L^*T_{\mathbb{P}^r}$ is semi-stable, specifying when it is also stable. We then prove the existence on many curves of a line bundle $L$ of degree $2g - c(C) - 1$ such that $φ_L^*T_{\mathbb{P}^r}$ is not semi-stable. Finally, we completely characterize the (semi-)stability of $φ_L^*T_{\mathbb{P}^r}$ when $C$ is hyperelliptic.

1. Introduction

Let $C$ be a smooth projective curve of genus $g \geq 2$ and let $L$ be a line bundle on $C$ generated by its global sections. Let $M_L$ be the vector bundle defined by the exact sequence

$$0 \to M_L \to H^0(C, L) \otimes \mathcal{O}_C \xrightarrow{\epsilon_L} L \to 0$$

where $\epsilon_L$ is the evaluation map. We denote by $E_L$ the dual bundle of $M_L$: it has degree $\deg L$ and rank $h^0(C, L) - 1$. Let us briefly recall the geometric interpretation of these bundles: since $L$ is generated by its global sections, the morphism $φ_L : C \to \mathbb{P}(H^0(L)) \simeq \mathbb{P}^r$ is well-defined and we have $L = φ_L^*\mathcal{O}_{\mathbb{P}^r}(1)$; thus, from the dual sequence of (1) and from the well-known Euler exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^r} \to H^0(C, L)^* \otimes \mathcal{O}_{\mathbb{P}^r}(1) \to T_{\mathbb{P}^r} \to 0$$

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it follows that $E_L = \phi_L^* T_C \otimes L^*$ and the stability of $E_L$ is equivalent to the stability of $\phi_L^* T_C$.

We recall the definition of the Clifford index of a curve.

**Definition 1.1** The Clifford index of a line bundle $L$ on $C$ is $c(L) = \deg L - 2(h^0(C, L) - 1)$.

The Clifford index of a divisor $D$ on $C$ is the Clifford index of the associated line bundle $\mathcal{O}_C(D)$, i.e. $c(D) = c(\mathcal{O}_C(D)) = \deg D - 2 \dim |D|$.

The Clifford index of the curve $C$ is $c(C) = \min \{c(L)/h^0(C, L) \geq 2, h^1(C, L) \geq 2\}$.

Clifford’s theorem states that $c(C) \geq 0$, with equality if and only if $C$ is hyperelliptic; moreover, for any divisor $D$ on $C$, $c(D) = c(K - D)$.

**Remark 1** By the Riemann-Roch theorem, $c(L) = 2g - \deg L - 2h^1(C, L)$ for any line bundle $L$.

In [3], by using the properties of this invariant, Paranjape proves the following

**Proposition 1.2** Let $C$ be a smooth projective curve of genus $g \geq 2$ and let $L$ be a line bundle on $C$ generated by its global sections. If $c(C) \geq c(L)$ then $E_L$ is semi-stable. If $h^1(C, L) = 1$ and $c(C) > 0$ or $c(C) > c(L)$ then $E_L$ is also stable.

By completing his proof we show the following

**Theorem 1.3** Let $C$ be a smooth projective curve of genus $g \geq 2$ and let $L$ be a line bundle on $C$ generated by its global sections such that $\deg L \geq 2g - c(C)$. Then:

(i) $E_L$ is semi-stable;

(ii) $E_L$ is stable except when $\deg L = 2g$ and either $C$ is hyperelliptic or $L \cong K(p + q)$ with $p, q \in C$.

If $C$ is a smooth projective $d$-gonal curve of genus $g \geq 2$ with Clifford index $c(C) = d - 2 < \frac{g + 2}{2}$, we then prove the existence of a line bundle $L$ of degree $2g - c(C) - 1$ such that $E_L$ is not semi-stable. Moreover, a theorem by Schneider (see [4]) states that on a general smooth curve $E_L$ is always semi-stable; our proof also shows that one cannot replace semi-stable by stable in this statement.

Finally, we completely characterize the (semi-)stability of $E_L$ when $C$ is hyperelliptic.

2. Proof of Theorem 1.3

We first need a lemma, shown by Paranjape in [3].

**Lemma 2.1** Let $F$ be a vector bundle on $C$ generated by its global sections and such that $H^0(C, F^*) = 0$; then $\deg F \geq \rk F + g - h^1(C, \det F)$ and equality holds if and only if $F = E_L$, where $L = \det F$. Moreover, if $h^1(C, \det F) \geq 2$ then $\deg F \geq 2\rk F + c(C)$ and if equality holds then $F = E_L$.

The canonical bundle $K$ is generated by its global sections and there is an exact sequence

$$0 \to K^* \to H^0(C, K)^* \otimes \mathcal{O}_C \to E_K \to 0$$

thus in cohomology we have

$$0 \to H^0(K^*) \to H^0(K)^* \otimes H^0(\mathcal{O}_C) \to H^0(E_K) \to H^1(K^*) \to \cdots$$

(3)

The map $\varphi$ is the dual map of $m : H^0(K) \otimes H^0(K) \to H^0(K^2)$, so it is injective by Noether’s theorem (see [1], Chap.III); moreover, $H^0(C, K^*) = 0$. As a consequence $H^0(C, E_K) \simeq H^0(C, K)^* \simeq H^1(C, \mathcal{O}_C)$ and $h^0(C, E_K) = g$.

Now we have all the tools necessary to prove Theorem 1.3.

**Proof of Theorem 1.3.** By Remark 1, if $\deg L \geq 2g - c(C)$ a fortiori $c(C) \geq c(L)$. By definition, $\deg E_L = c(L) + 2\rk E_L$ and $h^0(C, L) = \rk E_L + 1$, hence it follows by the Riemann-Roch theorem that $\deg E_L = \rk E_L + g - h^1(C, L)$.

Let $F$ be a quotient bundle of $E_L$; then $F$ satisfies the hypothesis of Lemma 2.1, because it is spanned by its global sections since $E_L$ is and $H^0(C, F^*) \subset H^0(C, E_L^*) = 0$. 2
Therefore, if $h^1(C, \text{det } F) \geq 2$ we have $\deg F \geq 2 \text{rk } F + c(C)$; then

$$
\frac{\mu(F) - \mu(E_L)}{\text{rk } F} \geq \frac{c(C)}{\text{rk } F} - \frac{c(L)}{\text{rk } E_L} = \frac{\text{rk } E_L \cdot c(C) - \text{rk } F \cdot c(L)}{\text{rk } F \cdot \text{rk } E_L} = \frac{(\text{rk } E_L - \text{rk } F) \cdot c(C) + \text{rk } F \cdot (c(C) - c(L))}{\text{rk } F \cdot \text{rk } E_L} \geq 0
$$

since $\text{rk } E_L > \text{rk } F > 0$ and $c(C) \geq c(L)$. Moreover, the inequality is strict if $c(C) > 0$ or if $C$ is hyperelliptic and $\deg L \geq 2g + 1$, because $L$ is non-special and $c(L) < 0$.

If $h^1(C, \text{det } F) < 2$ we still have $\deg F \geq \text{rk } F + g - h^1(C, \text{det } F)$, hence

$$
\frac{\mu(F) - \mu(E_L)}{\text{rk } F} \geq \frac{g - h^1(\text{det } F)}{\text{rk } E_L} = \frac{(g - h^1(\text{det } F)) \cdot (\text{rk } E_L - \text{rk } F) + \text{rk } F \cdot [h^1(L) - h^1(\text{det } F)]}{\text{rk } F \cdot \text{rk } E_L} > 0
$$

provided that $h^1(C, L) > h^1(C, \text{det } F)$, since $g - h^1(C, \text{det } F) > 0$ follows from the hypothesis that $h^1(C, \text{det } F) < 2$ and $g \geq 2$.

The only case remaining is $0 = h^1(C, L) < h^1(C, \text{det } F) = 1$. We have $\deg F = \deg(\text{det } F) \leq 2g - 2$, otherwise we should have $h^1(C, \text{det } F) = 0$; then, a fortiori, we have $\text{rk } F \leq g - 1$. It then follows from the previous inequalities that

$$
\mu(F) - \mu(E_L) \geq \frac{(g - 1)(\text{rk } E_L - \text{rk } F) - \text{rk } F}{\text{rk } F \cdot \text{rk } E_L} \geq \frac{(g - 1)(\text{rk } E_L - \text{rk } F - 1)}{\text{rk } F \cdot \text{rk } E_L} \geq 0
$$

Thus we have shown that we always have $\mu(F) - \mu(E_L) \geq 0$, i.e. $E_L$ is semi-stable. In order to gain the stability of $E_L$, we still need to prove that $\mu(F) - \mu(E_L) > 0$ when $0 = h^1(C, L) < h^1(C, \text{det } F) = 1$.

Suppose that $\mu(E_L) = \mu(F)$; by (4), we then have $(g - 1)\text{rk } E_L - g \cdot \text{rk } F = 0$. Since $g \geq 2$, it follows that $(g - 1)\text{rk } F \leq g - 1$, i.e. $\text{rk } F = g - 1$, and $\text{rk } E_L = g$; hence $\deg E_L = g + \text{rk } E_L = 2g$ and $\mu(E_L) = 2$.

Therefore, if $\deg L \neq 2g$ we cannot have $\mu(E_L) = \mu(F)$ and $E_L$ is stable.

If $\deg L = 2g$ then $E_L$ is stable provided that $c(C) > 0$ and $L \not\cong K(p + q)$ with $p, q \in C$.

Indeed, since $\deg F = \text{rk } F \cdot \mu(F) = 2g - 2$ and $h^1(C, \text{det } F) = 1$, we have $\deg F \cong K$. As a consequence we have $\text{rk } F + g - h^1(C, \text{det } F) = 2g - 2 = \deg F$, so $F = E_K$ by Lemma 2.1. On the other hand, $F$ is a quotient of $E_L$, so there is an exact sequence

$$
0 \to W \to E_L \to F \to 0
$$

(5)

where $W$ is a sub-bundle of $E_L$ of degree 2 and rank 1. The associated exact sequence of cohomology then is

$$
0 \to H^0(C, W) \to H^0(C, E_L) \to H^0(C, E_K) \to H^1(C, W) \to \cdots
$$

From the exact sequence of cohomology associated to the dual sequence of (1) we see that $h^0(C, E_L) \geq g + 1$ and $h^0(C, E_K) = g$ since $c(C) > 0$; hence $\varphi$ cannot be injective, i.e. $H^0(C, W) \neq 0$. Thus $W \cong O_C(p + q)$ with $p, q \in C$. Furthermore, it follows from (5) that

$$
L = \det E_L = \det W \otimes \det F = W \otimes K = K(p + q),
$$

which concludes the proof of Theorem 1.3 since this is not possible under our hypothesis. □

3. Some line bundles of degree $2g - c(C) - 1$ with non semi-stable $E_L$

Theorem 1.3 is the best possible result that one can obtain if looking for properties of all curves.

**Proposition 3.1** Let $C$ be a smooth projective $\ell$-gonal curve of genus $g \geq 2$ such that the Clifford index is $c(C) = d - 2 < \frac{g + 2}{2}$; there exists a line bundle $L$ of degree $\deg L = 2g - c(C) - 1$ on $C$ generated by its global sections and non-special such that $E_L$ is not semi-stable.
Proof. By the hypothesis, $g^1_3$ computes the Clifford index. We put $N = \mathcal{O}_C(K - g^1_3)$; it is a line bundle of degree $2g - c(C) - 4$ and by the Riemann-Roch theorem $h^0(N) = g - c(C) - 1$. Moreover $N$ is spanned by its global sections: assume that there exists $q \in C$ such that $h^0(N(-q)) = h^0(N)$, or equivalently $h^1(N(-q)) = h^1(N) + 1$; then, by Serre’s duality, we have $h^0(g^1_3 + q) = h^0(g^1_3) + 1 = 3$, i.e. $g^1_3 + q = g^3_{d+1}$, and this is not possible because we would have $c(g^3_{d+1}) = d - 3 < c(C)$.

Let $E$ be an effective divisor of degree 3 on $C$; we can choose $E$ in such a way that $L = N \otimes \mathcal{O}_C(E)$ is a line bundle of degree $\deg L = 2g - c(C) - 1$, non-special and spanned by its global sections. Indeed, we have $h^1(L) = 0$ because $h^1(L) = h^0(g^1_3 - E) = 0$ for a general effective divisor $E$; moreover $L$ is generated by its global sections if and only if $h^1(L(-p)) = h^1(L) = 0$ for any $p \in C$ and if $E$ is a general effective divisor of degree 3 we have $h^1(L(-p)) = h^0(g^1_3 - E + p) = 0$.

Since we have supposed that $E$ is effective, $H^0(L \otimes N^*) \neq 0$, so we have an inclusion $N \hookrightarrow L$. Hence $M_N$ is a sub-bundle of $M_L$, or equivalently $E_N$ is a quotient bundle of $E_L$. Since $\text{rk} E_L = g - c(C) - 1$ and $\text{rk} E_N = h^0(N) - 1 = g - c(C) - 2$, we have

$$\mu(E_N) = 2 + \frac{c(C)}{g - c(C) - 2} < \mu(E_L) = 2 + \frac{c(C) + 1}{g - c(C) - 1}$$

whenever $c(C) < \frac{g - 2}{2}$. It then follows that $E_L$ is not semi-stable. \(\square\)

Remark 2 If $C$ is a curve of genus $g \geq 2$ with Clifford index $c$, in most cases $C$ is $(c+2)$-gonal; see [2] for further details.

Remark 3 The hypothesis that $c(C) < \frac{g - 2}{2}$ leaves out only the case $c(C) = \left[\frac{g - 2}{2}\right]$, i.e. the general one; however, in [4] Schneider shows the following

Proposition 3.2 Let $C$ be a general smooth curve of genus $g \geq 3$. If $L$ is a line bundle on $C$ generated by its global sections, then $E_L$ is semi-stable.

It is worth underlining that one cannot replace semi-stable by stable: if $C$ is a general curve of even genus $g = 2n$ we know that

$$c(C) = \left[\frac{g - 1}{2}\right] = n - 1 = \frac{g - 2}{2},$$

so the proof of Proposition 3.1 shows that $E_L$ is not stable, since one obtains $\mu(E_N) = \mu(E_L)$.

4. The case of hyperelliptic curves

In the case of hyperelliptic curves we completely characterize the stability of $E_L$.

Proposition 4.1 Let $C$ be a smooth projective hyperelliptic curve of genus $g \geq 2$, let $L$ be a line bundle on $C$ generated by its global sections and such that $h^0(C, L) \geq 3$ and let $H = \mathcal{O}_C(g^1_3)$. Then:

(i) $E_L$ is stable if and only if $\deg L \geq 2g + 1$;

(ii) $E_L$ is semi-stable if and only if $\deg L \geq 2g$ or there exists an integer $k > 0$ such that $L = H^\otimes k$.

Proof. By Theorem 1.3, if $\deg L \geq 2g$ then $E_L$ is semi-stable and if $\deg L \geq 2g + 1$ then $E_L$ is stable.

On the other hand $E_L$ is not stable if $\deg L = 2g$, in which case $\mu(E_L) = 2$. Indeed, we show that $H$ is a quotient bundle of $E_L$ of same slope. We know that there is a surjection $E_L \twoheadrightarrow M_L$; if and only if $H^0(C, M_L \otimes H) \neq 0$. From the exact sequence (1) we get an exact sequence

$$0 \rightarrow H^0(C, M_L \otimes H) \rightarrow H^0(C, L) \otimes H^0(C, H) \rightarrow H^0(C, L \otimes H) \rightarrow \cdots$$

We then have $\dim H^0(C, L) \otimes H^0(C, H) = 2g + 2 > g + 3 = h^0(C, L \otimes H)$, so $H^0(C, M_L \otimes H) \neq 0$.

If $0 < \deg L \leq 2g - 1$ we always have $c(L) \geq 0$. If $c(L) = 0$ then $E_L$ is semi-stable, as it follows from the proof of Theorem 1.3: if $F$ is a quotient bundle of $E_L$, the inequality $\mu(F) - \mu(E_L) \geq 0$ still holds in each case.
Using again the exact sequence (8), since \( h^0(C, L) \geq 3 \), we have
\[
\dim H^0(C, L) \otimes H^0(C, H) = 2h^0(C, L) > h^0(C, L) + 2 \geq h^0(C, L \otimes H).
\]
Therefore, \( H^0(C, M_L \otimes H) \neq 0 \) and there is a surjection \( E_L \twoheadrightarrow H \); furthermore,
\[
\mu(E_L) = 2 + \frac{c(L)}{h^0(C, L) - 1}
\]
and \( \mu(H) = 2 \). Thus if \( c(L) > 0 \) then \( \mu(E_L) > \mu(H) \) and \( E_L \) is not semi-stable; else, if \( c(L) = 0 \), 
\( \mu(E_L) = \mu(H) \) and \( E_L \) is not stable.

The proposition then follows by Clifford’s theorem: since \( C \) is hyperelliptic and \( \deg L > 0 \), \( c(L) = 0 \) if and only if there exists an integer \( k > 0 \) such that \( L = H^\otimes k \).  

\[\blacksquare\]

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References