# Normal form coordinates for the KdV equation having expansions in terms of pseudodifferential operators 

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#### Abstract

Near an arbitrary finite gap potential we construct real analytic, canonical coordinates for the KdV equation on the torus having the following two main properties: (1) up to a remainder term, which is smoothing to any given order, the coordinate transformation is a pseudodifferential operator of order 0 with principal part given by the Fourier transform and (2) the pullback of the KdV Hamiltonian is in normal form up to order three and the corresponding Hamiltonian vector field admits an expansion in terms of a paradifferential operator. Such coordinates are a key ingredient for studying the stability of finite gap solutions of the KdV equation under small, quasi-linear perturbations.


Keywords: Normal form, KdV equation, finite gap potentials, pseudodifferential operators.
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## 1 Introduction

The goal of this paper is to construct canonical coordinates for the Korteweg-de Vries (KdV) equation on the circle

$$
\begin{equation*}
\partial_{t} u=-\partial_{x}^{3} u+6 u \partial_{x} u, \quad x \in \mathbb{T}:=\mathbb{R} / \mathbb{Z} \tag{1.1}
\end{equation*}
$$

[^0]taylored for studying the stability of finite gap solutions of (1.1), also referred to as periodic multisolitons, under quasi-linear perturbations. To state our main results, we first need to make some preliminary considerations and introduce some notations. It is well known that (1.1) is well-posed on the Sobolev spaces $H^{s}=\left\{q \in H_{\mathbb{C}}^{s}: q\right.$ real valued $\}$ with $s \geq-1$ (cf [17] and references therein) where for any $s \in \mathbb{R}$,
\[

$$
\begin{equation*}
H_{\mathbb{C}}^{s} \equiv H^{s}(\mathbb{T}, \mathbb{C}):=\left\{q=\sum_{n \in \mathbb{Z}} q_{n} e^{2 \pi \mathrm{i} n x}:\|q\|_{s}<\infty\right\}, \quad\|q\|_{s}=\left(\sum_{n \in \mathbb{Z}}\langle n\rangle^{2 s}\left|q_{n}\right|^{2}\right)^{\frac{1}{2}},\langle n\rangle:=\max \{1,|n|\} \tag{1.2}
\end{equation*}
$$

\]

Note that $\int_{0}^{1} u(t, x) d x$ is a prime integral for equation (1.1). Without loss of generality, we restrict our attention to the case where $u$ has zero mean value (cf [13, Section 13]), i.e., we consider solutions $u(t, x)$ of (1.1) in $H_{0}^{s}$ with $s \geq-1$ where for any $s \in \mathbb{R}$,

$$
\begin{equation*}
H_{0}^{s}=\left\{q \in H^{s}: \int_{0}^{1} q(x) d x=0\right\} \tag{1.3}
\end{equation*}
$$

It is well known that equation (1.1) can be written as a Hamiltonian PDE, $\partial_{t} u=\partial_{x} \nabla H^{k d v}(u)$ where $\partial_{x}$ is the Gardner Poisson structure (with $\partial_{x}^{-1}$ being the corresponding symplectic structure) and $\nabla H^{k d v}$ denotes the $L^{2}$-gradient of the KdV Hamiltonian

$$
H^{k d v}(q):=\frac{1}{2} \int_{0}^{1}\left(\partial_{x} q\right)^{2} d x+\int_{0}^{1} q^{3} d x
$$

According to [13] (cf also [9]), there are canonical coordinates $x_{n}=x_{n}(q), y_{n}=y_{n}(q), n \geq 1$, defined on $L_{0}^{2} \equiv H_{0}^{0}$ so that when expressed in these coordinates, the KdV equation takes the form

$$
\dot{x}_{n}=\omega_{n}^{k d v} y_{n}, \quad \dot{y}_{n}=-\omega_{n}^{k d v} x_{n}, \quad \forall n \geq 1
$$

where $\omega_{n}^{k d v}$ denote the KdV frequencies. To be more precise, introduce for any $s \in \mathbb{R}$ the sequence space

$$
h_{0, \mathbb{C}}^{s} \equiv h^{s}(\mathbb{Z} \backslash\{0\}, \mathbb{C})=\left\{w=\left(w_{n}\right)_{n \neq 0} \subset \mathbb{C}:\|w\|_{s}<\infty\right\}, \quad\|w\|_{s}:=\left(\sum_{n \neq 0}|n|^{2 s}\left|w_{n}\right|^{2}\right)^{\frac{1}{2}}
$$

and its real subspace $h_{0}^{s}:=\left\{\left(w_{n}\right)_{n \neq 0} \in h_{0, \mathbb{C}}^{s}: w_{-n}=\bar{w}_{n} \forall n \geq 1\right\}$ and define the weighted complex coordinates $z_{ \pm n} \equiv z_{ \pm n}(q)$,

$$
\begin{equation*}
z_{n}:=\sqrt{n \pi}\left(x_{n}-\mathrm{i} y_{n}\right), \quad z_{-n}:=\sqrt{n \pi}\left(x_{n}+\mathrm{i} y_{n}\right), \quad \forall n \geq 1 \tag{1.4}
\end{equation*}
$$

where $\sqrt{\cdot} \equiv \sqrt[+]{\cdot}$ denotes the principal branch of the square root. The results in 13 imply that the transformation, referred to as Birkhoff map,

$$
\Phi^{k d v}: L_{0}^{2} \equiv H_{0}^{0} \rightarrow h_{0}^{0}, \quad q \mapsto\left(z_{n}(q)\right)_{n \neq 0}
$$

is canonical in the sense that

$$
\left\{z_{n}, z_{-n}\right\}=\int_{0}^{1} \nabla z_{n} \partial_{x} \nabla z_{-n} d x=2 \pi \mathrm{i} n, \quad \forall n \geq 1
$$

whereas the brackets between all other coordinate functions vanish, and has the property that for any $s \in \mathbb{Z}_{\geq 0}$, its restriction to $H_{0}^{s}$ is a real analytic diffeomorphism with range $h_{0}^{s}, \Phi^{k d v}: H_{0}^{s} \rightarrow h_{0}^{s}$. In terms of the coordinates $z_{n}(q), n \neq 0$, referred to as complex Birkhoff coordinates, the action variables $I_{n}(q)$ are defined by

$$
\begin{equation*}
I_{n}(q)=\frac{1}{2 \pi n} z_{n}(q) z_{-n}(q) \geq 0, \quad \forall n \geq 1 \tag{1.5}
\end{equation*}
$$

The sequences $I(q)=\left(I_{n}(q)\right)_{n \geq 1}$ fill out the whole positive quadrant $\ell_{+}^{1,1}$ of $\ell^{1,1}$ where for any $r \geq 0$, the weighted $\ell^{1}$-space $\ell^{1, r}$ is defined by

$$
\begin{equation*}
\ell^{1, r} \equiv \ell^{1, r}(\mathbb{N}, \mathbb{R}):=\left\{I=\left(I_{n}\right)_{n \geq 1} \subset \mathbb{R}: \sum_{n=1}^{\infty} n^{r}\left|I_{n}\right|<\infty\right\}, \quad \mathbb{N} \equiv \mathbb{Z}_{\geq 1} \tag{1.6}
\end{equation*}
$$

A key feature of the Birkhoff map is that the KdV Hamiltonian, expressed in the coordinates $z_{n}, n \neq 0$,

$$
H^{k d v} \circ \Psi^{k d v}: h_{0}^{1} \rightarrow \mathbb{R}, \quad \Psi^{k d v}:=\left(\Phi^{k d v}\right)^{-1},
$$

is in fact a function $\mathcal{H}^{k d v}$ of the actions $I$ alone. More precisely, $\mathcal{H}^{k d v}: \ell_{+}^{1,3} \rightarrow \mathbb{R}$ is a real analytic map. The KdV frequencies are then defined by $\omega_{n}^{k d v}:=\partial_{I_{n}} \mathcal{H}^{k d v}$. Finally, the differential $d_{0} \Phi^{k d v}: L_{0}^{2} \rightarrow h_{0}^{0}$ of $\Phi^{k d v}$ at $q=0$ is the Fourier transform $\mathcal{F}$ (cf [13, Theorem 9.8]) and hence $d_{0} \Psi^{k d v}$ the inverse Fourier transform $\mathcal{F}^{-1}$ where for any $s \in \mathbb{Z}$,

$$
\begin{equation*}
\mathcal{F}: H_{0}^{s} \rightarrow h_{0}^{s}, \quad q \mapsto\left(q_{n}\right)_{n \neq 0}, \quad q_{n}:=\int_{0}^{1} q(x) e^{-2 \pi \mathrm{i} n x} d x \tag{1.7}
\end{equation*}
$$

Let

$$
\begin{equation*}
S_{+} \subseteq \mathbb{N} \text { finite, } \quad S:=S_{+} \cup\left(-S_{+}\right) \quad \text { and } \quad S_{+}^{\perp}:=\mathbb{N} \backslash S_{+}, \quad S^{\perp}:=S_{+}^{\perp} \cup\left(-S_{+}^{\perp}\right) \tag{1.8}
\end{equation*}
$$

We denote by $M_{S} \subset L_{0}^{2}$ the manifold of $S$-gap potentials,

$$
M_{S}:=\left\{q \in L_{0}^{2}: z_{n}(q)=0 \forall n \in S^{\perp}\right\},
$$

and by $M_{S}^{o}$ the open subset of $M_{S}$, consisting of proper $S$-gap potentials,

$$
M_{S}^{o}:=\left\{q \in M_{S}: z_{n}(q) \neq 0 \forall n \in S\right\} .
$$

Note that $M_{S}$ is contained in $\cap_{s \geq 0} H_{0}^{s}$ and hence consists of $C^{\infty}$-smooth potentials and that $M_{S}^{o}$ can be parametrized by the action-angle coordinates $\theta_{S}=\left(\theta_{k}\right)_{k \in S_{+}}, I_{S}=\left(I_{k}\right)_{k \in S_{+}}$,

$$
\mathcal{M}_{S}^{o} \rightarrow M_{S}^{o},\left(\theta_{S}, I_{S}\right) \mapsto \Psi^{k d v}\left(z\left(\theta_{S}, I_{S}\right)\right), \quad \mathcal{M}_{S}^{o}:=\mathbb{T}^{S_{+}} \times \mathbb{R}_{>0}^{S_{+}}
$$

where for any $n \neq 0, z_{n}=z_{n}\left(\theta_{S}, I_{S}\right)$ is given by

$$
z_{ \pm n}:=\sqrt{2 \pi n I_{n}} e^{\mp \mathrm{i} \theta_{n}}, \quad \forall n \in S_{+}, \quad z_{n}=0, \quad \forall n \in S^{\perp}
$$

Introduce for any $s \in \mathbb{R}$

$$
h_{\perp, \mathbb{C}}^{s}:=h^{s}\left(S^{\perp}, \mathbb{C}\right), \quad h_{\perp}^{s}:=\left\{z_{\perp} \in h_{\perp, \mathbb{C}}^{s}: z_{-n}=\bar{z}_{n}, \quad \forall n \in S^{\perp}\right\}
$$

as well as the maps, related to the Fourier transform,

$$
\mathcal{F}_{\perp}: H_{0}^{s} \rightarrow h_{\perp}^{s}, q \mapsto\left(q_{n}\right)_{n \in S^{\perp}}, \quad \mathcal{F}_{\perp}^{-1}: h_{\perp}^{s} \rightarrow H_{0}^{s},\left(z_{n}\right)_{n \in S^{\perp}} \mapsto \sum_{n \in S^{\perp}} z_{n} e^{2 \pi \mathrm{inx}}
$$

By a slight abuse of notation, we view $\mathcal{M}_{S}^{o} \times h_{\perp}^{s}, s \in \mathbb{R}$, as a subset of $h_{0}^{s}$ and denote its elements by

$$
\mathfrak{x}:=\left(\theta_{S}, I_{S}, z_{\perp}\right), \quad \theta_{S}:=\left(\theta_{n}\right)_{n \in S_{+}}, \quad I_{S}:=\left(I_{n}\right)_{n \in S_{+}}, \quad z_{\perp}:=\left(z_{n}\right)_{n \in S^{\perp}}
$$

It is endowed by the standard Poisson bracket, given by

$$
\left\{I_{n}, \theta_{n}\right\}=1, \forall n \in S_{+}, \quad\left\{z_{n}, z_{-n}\right\}=2 \pi \mathrm{i} n, \quad \forall n \in S_{+}^{\perp},
$$

whereas the brackets between all other coordinate functions vanish. For any $s \in \mathbb{R}$, we denote by $\widehat{\mathfrak{x}}=$ $\left(\widehat{\theta}_{S}, \widehat{I}_{S}, \widehat{z}_{\perp}\right)$ elements in the tangent space $E_{s}$ of $\mathcal{M}_{S}^{o} \times h_{\perp}^{s}$ at any given point $\mathfrak{x} \in \mathcal{M}_{S}^{o} \times h_{\perp}^{s}$ where $E_{s}$ is given by

$$
E_{s}=\mathbb{R}^{S_{+}} \times \mathbb{R}^{S_{+}} \times h_{\perp}^{s} .
$$

Furthermore, for any $k \geq 1, \partial_{x}^{-k}: H_{\mathbb{C}}^{s} \rightarrow H_{0, \mathbb{C}}^{s+k}$ is the bounded linear operator, defined by

$$
\partial_{x}^{-k}\left[e^{2 \pi \mathrm{i} n x}\right]=\frac{1}{(2 \pi \mathrm{i} n)^{k}} e^{2 \pi \mathrm{i} n x}, \quad \forall n \neq 0, \quad \text { and } \quad \partial_{x}^{-k}[1]=0
$$

Finally, the standard inner products on $L_{0}^{2}$ and on $h_{0}^{0}$ are defined by

$$
\langle f, g\rangle \equiv\langle f, g\rangle_{L_{0}^{2}}=\int_{0}^{1} f(x) g(x) d x, \quad \forall f, g \in L_{0}^{2}, \quad\langle z, w\rangle \equiv\langle z, w\rangle_{h_{0}^{0}}=\sum_{n \neq 0} z_{n} w_{-n}, \quad \forall z, w \in h_{0}^{0}
$$

Note that $\langle\cdot, \cdot\rangle_{L_{0}^{2}}$ and $\langle\cdot, \cdot\rangle_{h_{0}^{0}}$ extend as complex valued bilinear forms to $L_{0, \mathbb{C}}^{2}$ and respectively, $h_{0, \mathbb{C}}^{0}$. In the sequel, restrictions of these inner products to various subspaces and extensions as dual pairings will be denoted in the same way and the gradient of a functional $F$ corresponding to these inner products by $\nabla F$. In more detail, for a $C^{1}$-functional $F: h_{0}^{0} \rightarrow \mathbb{C}$, one has $d F[\widehat{z}]=\sum_{n \neq 0} \widehat{z}_{n} \partial_{z_{n}} F=\langle\nabla F, \widehat{z}\rangle$ with the nth component of $\nabla F$ given by $(\nabla F)_{n}=\partial_{z_{-n}} F$. Furthermore, for any given Banach spaces $Y_{1}, Y_{2}$, we denote by $\mathcal{B}\left(Y_{1}, Y_{2}\right)$ the space of bounded linear operators from $Y_{1}$ to $Y_{2}$, endowed with the operator norm.

Theorem 1.1. Let $S_{+} \subseteq \mathbb{N}$ be finite and $\mathcal{K}$ be a subset of $\mathcal{M}_{S}^{o}$ of the form $\mathbb{T}^{S_{+}} \times \mathcal{K}_{1}$ where $\mathcal{K}_{1}$ is a compact subset of $\mathbb{R}_{>0}^{S_{+}}$. Then there exists an open bounded neighbourhood $\mathcal{V}$ of $\mathcal{K} \times\{0\}$ in $\mathcal{M}_{S}^{o} \times h_{\perp}^{0}$ and a canonical real analytic diffeomorphism $\Psi: \mathcal{V} \rightarrow \Psi(\mathcal{V}) \subseteq L_{0}^{2}, \mathfrak{x}=\left(\theta_{S}, I_{S}, z_{\perp}\right) \mapsto q$ with the property that $\Psi$ satisfies

$$
\Psi\left(\theta_{S}, I_{S}, 0\right)=\Psi^{k d v}\left(\theta_{S}, I_{S}, 0\right), \quad \forall\left(\theta_{S}, I_{S}, 0\right) \in \mathcal{V}
$$

and is compatible in the sense explained below with the scale of Sobolev spaces $H_{0}^{s}, s \in \mathbb{Z}_{\geq 0}$, so that the following holds:
(AE1) For any integer $N \geq 1, \Psi$ admits an asymptotic expansion on $\mathcal{V}$ of the form

$$
\Psi\left(\theta_{S}, I_{S}, z_{\perp}\right)=\Psi^{k d v}\left(\theta_{S}, I_{S}, 0\right)+\mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]+\sum_{k=1}^{N} a_{k}\left(\theta_{S}, I_{S}, z_{\perp} ; \Psi\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]+\mathcal{R}_{N}\left(\theta_{S}, I_{S}, z_{\perp} ; \Psi\right)
$$

where $\mathcal{R}_{N}\left(\theta_{S}, I_{S}, 0 ; \Psi\right)=0$ and where for any $s \in \mathbb{Z}_{\geq 0}$ and $1 \leq k \leq N$

$$
a_{k}^{\Psi}: \mathcal{V} \rightarrow H^{s}, \mathfrak{x} \mapsto a_{k}^{\Psi}(\mathfrak{x}) \equiv a_{k}(\mathfrak{x} ; \Psi), \quad \mathcal{R}_{N}^{\Psi}: \mathcal{V} \cap\left(\mathcal{M}_{S}^{o} \times h_{\perp}^{s}\right) \rightarrow H^{s+N+1}, \mathfrak{x} \mapsto \mathcal{R}_{N}^{\Psi}(\mathfrak{x}) \equiv \mathcal{R}_{N}(\mathfrak{x} ; \Psi)
$$

are real analytic maps satisfying the tame estimates of Theorem 1.2 below.
(AE2) For any $\mathfrak{x} \in \mathcal{V}$, the transpose $d \Psi(\mathfrak{x})^{t}$ (with respect to the standard inner products) of the differential $d \Psi(\mathfrak{x}): E_{1} \rightarrow H_{0}^{1}$ is a bounded operator $d \Psi(\mathfrak{x})^{t}: H_{0}^{1} \rightarrow E_{1}$. For any $\widehat{q} \in H_{0}^{1}$ and any integer $N \geq 1$, $d \Psi(\mathfrak{x})^{t}[\widehat{q}]$ admits an expansion of the form
$d \Psi(\mathfrak{x})^{t}[\widehat{q}]=\left(0,0, \mathcal{F}_{\perp}[\widehat{q}]+\mathcal{F}_{\perp} \circ \sum_{k=1}^{N} a_{k}\left(\mathfrak{x} ; d \Psi^{t}\right) \partial_{x}^{-k} \widehat{q}+\mathcal{F}_{\perp} \circ \sum_{k=1}^{N} \mathcal{A}_{k}\left(\mathfrak{x} ; d \Psi^{t}\right)[\widehat{q}] \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N}\left(\mathfrak{x} ; d \Psi^{t}\right)[\widehat{q}]$
where for any $s \in \mathbb{N}$ and $1 \leq k \leq N$,

$$
\begin{gathered}
a_{k}^{d \Psi^{t}}: \mathcal{V} \rightarrow H^{s}, \mathfrak{x} \mapsto a_{k}^{d \Psi^{t}}(\mathfrak{x}) \equiv a_{k}\left(\mathfrak{x} ; d \Psi^{t}\right) \\
\mathcal{A}_{k}^{d \Psi^{t}}: \mathcal{V} \rightarrow \mathcal{B}\left(H_{0}^{1}, H^{s}\right), \mathfrak{x} \mapsto \mathcal{A}_{k}^{d \Psi^{t}}(\mathfrak{x}) \equiv \mathcal{A}_{k}\left(\mathfrak{x} ; d \Psi^{t}\right), \\
\mathcal{R}_{N}^{d \Psi^{t}}: \mathcal{V} \cap\left(\mathcal{M}_{S}^{o} \times h_{\perp}^{s}\right) \rightarrow \mathcal{B}\left(H_{0}^{s}, E_{s+N+1}\right), \mathfrak{x} \mapsto \mathcal{R}_{N}^{d \Psi^{t}}(\mathfrak{x}) \equiv \mathcal{R}_{N}\left(\mathfrak{x} ; d \Psi^{t}\right)
\end{gathered}
$$

are real analytic maps, satisfying the tame estimates of Theorem 1.2 below. Furthermore, the coefficient $a_{1}\left(\mathfrak{x} ; d \Psi^{t}\right)$ satisfies $a_{1}\left(\mathfrak{x} ; d \Psi^{t}\right)=-a_{1}(\mathfrak{x} ; \Psi)$.
(AE3) The Hamiltonian $\mathcal{H}:=H^{k d v} \circ \Psi: \mathcal{V} \cap\left(\mathcal{M}_{S}^{o} \times h_{\perp}^{1}\right) \rightarrow \mathbb{R}$ is in normal form up to order three. More precisely,

$$
\mathcal{H}\left(\theta_{S}, I_{S}, z_{\perp}\right)=\mathcal{H}^{k d v}\left(I_{S}, 0\right)+\sum_{n \in S_{+}^{\perp}} \Omega_{n}^{k d v}\left(I_{S}, 0\right) z_{n} z_{-n}+\mathcal{P}\left(\theta_{S}, I_{S}, z_{\perp}\right)
$$

where $\mathcal{P}: \mathcal{V} \cap\left(\mathcal{M}_{S}^{o} \times h_{\perp}^{1}\right) \rightarrow \mathbb{R}$ is real analytic and satisfies $\mathcal{P}\left(\theta_{S}, I_{S}, z_{\perp}\right)=O\left(\left\|z_{\perp}\right\|_{1}\left\|z_{\perp}\right\|_{0}^{2}\right), \Omega_{n}^{k d v}:=$ $\frac{1}{2 \pi n} \omega_{n}^{k d v}, n \in S_{+}^{\perp}$, and $\omega_{n}^{k d v}$ denote the KdV frequencies introduced above. Furthermore for any integer
$N \geq 1$, there exists an integer $\sigma_{N} \geq N$ (loss of regularity) so that the gradient $\nabla \mathcal{P}\left(\theta_{S}, I_{S}, z_{\perp}\right)$ of $\mathcal{P}$ with components $\nabla_{\theta_{S}} \mathcal{P}, \nabla_{I_{S}} \mathcal{P}$, and $\nabla_{z_{\perp}} \mathcal{P}$ admits an expansion of the form

$$
\nabla \mathcal{P}\left(\theta_{S}, I_{S}, z_{\perp}\right)=\left(0,0, \mathcal{F}_{\perp} \circ \sum_{k=0}^{N} T_{a_{k}^{\nabla \mathcal{P}}} \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N}^{\nabla \mathcal{P}}\left(\theta_{S}, I_{S}, z_{\perp}\right)
$$

where for any integers $s \geq 0$ and $0 \leq k \leq N$,

$$
a_{k}^{\nabla \mathcal{P}}: \mathcal{V} \cap\left(\mathcal{M}_{S}^{o} \times h_{\perp}^{s+\sigma_{N}}\right) \rightarrow H^{s}, \quad \mathcal{R}_{N}^{\nabla \mathcal{P}}: \mathcal{V} \cap\left(\mathcal{M}_{S}^{o} \times h_{\perp}^{s \vee \sigma_{N}}\right) \rightarrow E_{s+N+1}
$$

are real analytic and satisfy the tame estimates of Theorem 1.2 below.
Here $T_{a_{k}^{\nabla \mathcal{P}}}$ denotes the operator of paramultiplication with $a_{k}^{\nabla \mathcal{P}}$ (cf Appendix E) and the diffeomorphism $\Psi: \mathcal{V} \rightarrow \Psi(\mathcal{V}) \subset L_{0}^{2}$ being compatible with the scale of Sobolev spaces $H_{0}^{s}, s \in \mathbb{Z}_{\geq 0}$, means that for any $s \in \mathbb{Z}_{\geq 0}, \Psi\left(\mathcal{V} \cap\left(\mathcal{M}_{S}^{o} \times h_{\perp}^{s}\right)\right) \subseteq H_{0}^{s}$ and $\Psi: \mathcal{V} \cap\left(\mathcal{M}_{S}^{o} \times h_{\perp}^{s}\right) \rightarrow H_{0}^{s}$ is a real analytic diffeomorphism onto its image.

In applications, it is of interest to know whether the coordinate transformation $\Psi$ preserves the reversible structure, defined by the maps $S_{\text {rev }}: L_{0}^{2} \rightarrow L_{0}^{2},\left(S_{\text {rev }} q\right)(x):=q(-x)$, and $\mathcal{S}_{\text {rev }}: \mathcal{M}_{S}^{o} \times h_{\perp}^{0} \rightarrow \mathcal{M}_{S}^{o} \times h_{\perp}^{0}$ where

$$
\begin{equation*}
\mathcal{S}_{\text {rev }}\left(\theta_{S}, I_{S}, z_{\perp}\right):=\left(\theta_{S}^{\text {rev }}, I_{S}^{\text {rev }}, z_{\perp}^{\text {rev }}\right), \quad \theta_{n}^{\text {rev }}=-\theta_{n}, \quad I_{n}^{\text {rev }}=I_{n}, \quad \forall n \in S_{+}, \quad z_{n}^{\text {rev }}=z_{-n}, \forall n \in S^{\perp} \tag{1.9}
\end{equation*}
$$

Note that for any $s \in \mathbb{Z}_{\geq 0}, S_{\text {rev }}: H_{0}^{s} \rightarrow H_{0}^{s}$ and $\mathcal{S}_{\text {rev }}: \mathcal{M}_{S}^{o} \times h_{\perp}^{s} \rightarrow \mathcal{M}_{S}^{o} \times h_{\perp}^{s}$ are linear involutions and that without loss of generality, the neighbourhood $\mathcal{V}$ of Theorem 1.1 can be chosen to be invariant under the map $\mathcal{S}_{\text {rev }}$, i.e., $\mathcal{S}_{\text {rev }}(\mathcal{V})=\mathcal{V}$.
Addendum to Theorem 1.1 The maps $\Psi: \mathcal{V} \rightarrow L_{0}^{2}, \quad \Psi^{k d v}: h_{0}^{0} \rightarrow L_{0}^{2}$, and $\mathcal{F}^{-1}: h_{0}^{0} \rightarrow L_{0}^{2}$ preserve the reversible structure, i.e.,

$$
\Psi \circ \mathcal{S}_{r e v}=S_{r e v} \circ \Psi, \quad \Psi^{k d v} \circ \mathcal{S}_{r e v}=S_{r e v} \circ \Psi^{k d v}, \quad \mathcal{F}^{-1} \circ \mathcal{S}_{r e v}=S_{r e v} \circ \mathcal{F}^{-1}
$$

and so do the maps in the asymptotic expansions (AE1) $(\mathfrak{x} \in \mathcal{V})$,

$$
a_{k}^{\Psi}\left(\mathcal{S}_{\text {rev }} \mathfrak{x}\right)=(-1)^{k} S_{\text {rev }}\left(a_{k}^{\Psi}(\mathfrak{x})\right), \quad \mathcal{R}_{N}^{\Psi}\left(\mathcal{S}_{\text {rev }} \mathfrak{x}\right)=S_{\text {rev }}\left(\mathcal{R}_{N}^{\Psi}(\mathfrak{x})\right)
$$

and the ones in the asymptotic expansions (AE2) $\left(\mathfrak{x} \in \mathcal{V} \cap\left(\mathcal{M}_{S}^{o} \times h_{\perp}^{1}\right), \widehat{q} \in H_{0}^{1}\right)$,

$$
\begin{gathered}
a_{k}\left(\mathcal{S}_{\text {rev }} \mathfrak{x} ; d \Psi^{t}\right)=(-1)^{k} S_{\text {rev }}\left(a_{k}\left(\mathfrak{x} ; d \Psi^{t}\right)\right), \quad \mathcal{A}_{k}\left(\mathcal{S}_{\text {rev }} \mathfrak{x} ; d \Psi^{t}\right)\left[S_{\text {rev }} \widehat{q}\right]=(-1)^{k} S_{\text {rev }}\left(\mathcal{A}_{k}\left(\mathfrak{x} ; d \Psi^{t}\right)[\widehat{q}]\right) \\
\mathcal{R}_{N}\left(\mathcal{S}_{\text {rev }} \mathfrak{x} ; d \Psi^{t}\right)\left[S_{\text {rev }} \widehat{q}\right]=\mathcal{S}_{\text {rev }}\left(\mathcal{R}_{N}\left(\mathfrak{x} ; d \Psi^{t}\right)[\widehat{q}]\right) .
\end{gathered}
$$

Furthermore, the Hamiltonians $H^{k d v}, \mathcal{H}=H^{k d v} \circ \Psi$, and $\mathcal{P}$ are reversible, meaning that

$$
H^{k d v} \circ S_{r e v}=H^{k d v}, \quad \mathcal{H} \circ \mathcal{S}_{\text {rev }}=\mathcal{H}, \quad \mathcal{P} \circ \mathcal{S}_{\text {rev }}=\mathcal{P}
$$

and the maps in the asymptotic expansion in (AE3) preserve the reversible structure,

$$
\begin{gathered}
a_{k}^{\nabla \mathcal{P}}\left(\mathcal{S}_{r e v} \mathfrak{x}\right)=(-1)^{k} S_{r e v}\left(a_{k}^{\nabla \mathcal{P}}(\mathfrak{x})\right), \quad \forall \mathfrak{x} \in \mathcal{V} \cap\left(\mathcal{M}_{S}^{o} \times h_{\perp}^{1+\sigma_{N}}\right), \\
\mathcal{R}_{N}^{\nabla \mathcal{P}}\left(\mathcal{S}_{r e v} \mathfrak{x}\right)=\mathcal{S}_{r e v}\left(\mathcal{R}_{N}^{\nabla \mathcal{P}}(\mathfrak{x})\right) \quad \forall \mathfrak{x} \in \mathcal{V} \cap\left(\mathcal{M}_{S}^{o} \times h_{\perp}^{1 \vee \sigma_{N}}\right)
\end{gathered}
$$

Theorem 1.2 below states tame estimates for the map $\Psi$ and the gradient $\nabla \mathcal{P}$ of the remainder term $\mathcal{P}$ in the expansion of $\mathcal{H}$. In the sequel, we denote elements in the tangent space $E_{s}:=\mathbb{R}^{S_{+}} \times \mathbb{R}^{S_{+}} \times h_{\perp}^{s}$ of $\mathcal{V} \cap\left(\mathcal{M}_{S}^{o} \times h_{\perp}^{s}\right)$ at any given point $\mathfrak{x}=\left(\theta_{S}, I_{S}, z_{\perp}\right)$ by $\widehat{\mathfrak{x}}=\left(\widehat{\theta}_{S}, \widehat{I}_{S}, \widehat{z}_{\perp}\right)$. Throughout the paper, all the stated estimates for maps hold locally uniformly with respect to their arguments.

Theorem 1.2. Let $N, l \in \mathbb{N}$. Then under the same assumptions as in Theorem 1.1, the following estimates hold:
(Est1) For any $\mathfrak{x}=\left(\theta_{S}, I_{S}, z_{\perp}\right) \in \mathcal{V}, 1 \leq k \leq N, \widehat{\mathfrak{x}}_{1}, \ldots, \widehat{\mathfrak{x}}_{l} \in E_{0}, s \in \mathbb{Z}_{\geq 0}$,

$$
\left\|a_{k}^{\Psi}(\mathfrak{x})\right\|_{s} \lesssim_{s, k} 1, \quad\left\|d^{l} a_{k}^{\Psi}(\mathfrak{x})\left[\widehat{\mathfrak{x}}_{1}, \ldots, \widehat{\mathfrak{x}}_{l}\right]\right\|_{s} \lesssim_{s, k, l} \prod_{j=1}^{l}\left\|\widehat{\mathfrak{x}}_{j}\right\|_{0} .
$$

Simlarly, for any $\mathfrak{x} \in \mathcal{V} \cap\left(\mathcal{M}_{S}^{o} \times h_{\perp}^{s}\right), \widehat{\mathfrak{x}}_{1}, \ldots, \widehat{\mathfrak{x}}_{l} \in E_{s}, s \in \mathbb{Z}_{\geq 0}$,

$$
\left\|\mathcal{R}_{N}^{\Psi}(\mathfrak{x})\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}, \quad\left\|d^{l} \mathcal{R}_{N}^{\Psi}(\mathfrak{x})\left[\widehat{\mathfrak{x}}_{1}, \ldots, \widehat{\mathfrak{x}}_{l}\right]\right\|_{s+N+1} \lesssim_{s, N, l} \sum_{j=1}^{l}\left\|\widehat{\mathfrak{x}}_{j}\right\|_{s} \prod_{i \neq j}\left\|\widehat{\mathfrak{x}}_{i}\right\|_{0}+\left\|z_{\perp}\right\|_{s} \prod_{j=1}^{l}\left\|\widehat{\mathfrak{x}}_{j}\right\|_{0}
$$

(Est2) For any $\mathfrak{x}=\left(\theta_{S}, I_{S}, z_{\perp}\right) \in \mathcal{V} \cap\left(\mathcal{M}_{S}^{o} \times h_{\perp}^{1}\right), 1 \leq k \leq N, \quad \widehat{\mathfrak{x}}_{1}, \ldots, \widehat{\mathfrak{x}}_{l} \in E_{1}, s \in \mathbb{N}$,

$$
\left\|a_{k}^{d \Psi^{t}}(\mathfrak{x})\right\|_{s} \lesssim_{s, k} 1+\left\|z_{\perp}\right\|_{1}, \quad\left\|d^{l} a_{k}^{d \Psi^{t}}(\mathfrak{x})\left[\widehat{\mathfrak{x}}_{1}, \ldots, \widehat{\mathfrak{x}}_{l}\right]\right\|_{s} \lesssim_{s, k, l} \prod_{j=1}^{l}\left\|\widehat{\mathfrak{x}}_{j}\right\|_{1}
$$

and

$$
\left\|\mathcal{A}_{k}^{d \Psi^{t}}(\mathfrak{x})\right\|_{s} \lesssim_{s, k} 1+\left\|z_{\perp}\right\|_{1}, \quad\left\|d^{l} \mathcal{A}_{k}^{d \Psi^{t}}(\mathfrak{x})\left[\widehat{\mathfrak{x}}_{1}, \ldots, \widehat{\mathfrak{x}}_{l}\right]\right\|_{s} \lesssim_{s, k, l} \prod_{j=1}^{l}\left\|\widehat{\mathfrak{x}}_{j}\right\|_{1}
$$

Similarly, for any $\mathfrak{x} \in \mathcal{V} \cap\left(\mathcal{M}_{S}^{o} \times h_{\perp}^{s}\right), \widehat{\mathfrak{x}}_{1}, \ldots, \widehat{\mathfrak{x}}_{l} \in E_{s}, \quad \widehat{q} \in H_{0}^{s}, s \in \mathbb{N}$,

$$
\| \mathcal{R}_{N}^{d \Psi^{t}}(\mathfrak{x})\left[\widehat{q}\left\|_{s+N+1} \lesssim_{s, N}\right\| \widehat{q}\left\|_{s}+\right\| z_{\perp}\left\|_{s}\right\| \widehat{q} \|_{1}\right.
$$

$$
\left\|d^{l}\left(\mathcal{R}_{N}^{d \Psi^{t}}(\mathfrak{x})[\widehat{q}]\right)\left[\widehat{\mathfrak{x}}_{1}, \ldots, \widehat{\mathfrak{x}}_{l}\right]\right\|_{s+N+1} \lesssim_{s, N, l}\|\widehat{q}\|_{s} \prod_{j=1}^{l}\left\|\widehat{\mathfrak{x}}_{j}\right\|_{1}+\|\widehat{q}\|_{1} \sum_{j=1}^{l}\left\|\widehat{\mathfrak{x}}_{j}\right\|_{s} \prod_{i \neq j}\left\|\widehat{\mathfrak{x}}_{i}\right\|_{1}+\|\widehat{q}\|_{1}\left\|z_{\perp}\right\|_{s} \prod_{j=1}^{l}\left\|\widehat{\mathfrak{x}}_{j}\right\|_{1}
$$

(Est3) For any $s \in \mathbb{Z}_{\geq 0}, \mathfrak{x}=\left(\theta_{S}, I_{S}, z_{\perp}\right) \in \mathcal{V} \cap\left(\mathcal{M}_{S}^{o} \times h_{\perp}^{s+\sigma_{N}}\right),\left\|z_{\perp}\right\|_{\sigma_{N}} \leq 1,1 \leq k \leq N, \widehat{\mathfrak{x}}_{1}, \ldots, \widehat{\mathfrak{x}}_{l} \in E_{s+\sigma_{N}}$,

$$
\left\|a_{k}^{\nabla \mathcal{P}}(\mathfrak{x})\right\|_{s} \lesssim_{s, k}\left\|z_{\perp}\right\|_{s+\sigma_{N}}, \quad\left\|d^{l} a_{k}^{\nabla \mathcal{P}}(\mathfrak{x})\left[\widehat{\mathfrak{x}}_{1}, \ldots, \widehat{\mathfrak{x}}_{l}\right]\right\|_{s} \lesssim_{s, k, l} \sum_{j=1}^{l}\left\|\widehat{\mathfrak{x}}_{j}\right\|_{s+\sigma_{N}} \prod_{n \neq j}\left\|\widehat{\mathfrak{x}}_{n}\right\|_{\sigma_{N}}+\left\|z_{\perp}\right\|_{s+\sigma_{N}} \prod_{j=1}^{l}\left\|\widehat{\mathfrak{x}}_{j}\right\|_{\sigma_{N}}
$$

For any $s \in \mathbb{Z}_{\geq 0}, \mathfrak{x} \in \mathcal{V} \cap\left(\mathcal{M}_{S}^{o} \times h_{\perp}^{s \vee \sigma_{N}}\right)$ with $\left\|z_{\perp}\right\|_{\sigma_{N}} \leq 1, \widehat{\mathfrak{x}} \in E_{s \vee \sigma_{N}}$,
$\left\|\mathcal{R}_{N}^{\nabla \mathcal{P}}(\mathfrak{x})\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{\sigma_{N}}\left\|z_{\perp}\right\|_{s \vee \sigma_{N}}, \quad\left\|d \mathcal{R}_{N}^{\nabla \mathcal{P}}(\mathfrak{x})[\mathfrak{x}]\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{\sigma_{N}}\|\widehat{\mathfrak{x}}\|_{s \vee \sigma_{N}}+\left\|z_{\perp}\right\|\left\|_{s \vee \sigma_{N}}\right\| \hat{\mathfrak{x}} \|_{\sigma_{N}}$, and if in addition $\widehat{\mathfrak{x}}_{1}, \ldots, \widehat{\mathfrak{x}}_{l} \in E_{s \vee \sigma_{N}}, l \geq 2$,

$$
\left\|d^{l} \mathcal{R}_{N}^{\nabla \mathcal{P}}(\mathfrak{x})\left[\widehat{\mathfrak{x}}_{1}, \ldots, \widehat{\mathfrak{x}}_{l}\right]\right\|_{s+N+1} \lesssim_{s, N, l} \sum_{j=1}^{l}\left\|\widehat{\mathfrak{x}}_{j}\right\|_{s \vee \sigma_{N}} \prod_{n \neq j}\left\|\widehat{\mathfrak{x}}_{n}\right\|_{\sigma_{N}}+\left\|z_{\perp}\right\|_{s \vee \sigma_{N}} \prod_{j=1}^{l}\left\|\widehat{\mathfrak{x}}_{j}\right\|_{\sigma_{N}}
$$

Applications: The Birkhoff coordinates are well suited to study the initial value problem of (1.1) (cf e.g. [17], [11] and references therein) and semilinear perturbations of (1.1) (cf e.g. [13], 19]). However, when equation (1.1) is expressed in Birkhoff coordinates, various features of the KdV equation and its perturbations such as being partial differential equations, get lost. On the other hand, due to the expansions (AE1) - (AE3), the coordinates of Theorem 1.1 allow to preserve the essence of such features and in the form stated turn out to be well suited to study quasi-linear perturbations of the KdV equation.
Outline of the construction: In his pioneering work [19], Kuksin presents a general scheme for proving KAM-type theorems for integrable PDEs in one space dimension such as the KdV or the sine-Gordon (sG) equations, which possess a Lax pair formulation and admit finite dimensional integrable subsystems foliated by invariant tori. Expanding on work of Krichever [18], Kuksin considers bounded integrable subsystems of
such a PDE which admit action-angle coordinates. They are complemented by infinitely many coordinates whose construction is based on a set of time periodic solutions, referred to as Floquet solutions of the PDE, obtained by linearizing the PDE under consideration along a solution evolving in the integrable subsystem. It turns out that the resulting coordinate transformation is typically not symplectic. Extending arguments of Moser and Weinstein to the given infinite dimensional setup (see [19], Lemma 1.4 and Section 1.7), he constructs a second coordinate transformation so that the composition of the two transformations become symplectic. We follow Kuksin's scheme of the proof by constructing $\Psi$ as the composition of $\Psi_{L} \circ \Psi_{C}$ of two transformations. The $\Psi_{L}$ is given by the Taylor expansion of $\Psi^{k d v}$ of order one in the normal direction $z_{\perp}$ around $\left(\theta_{S}, I_{S}, 0\right)$,

$$
\Psi^{k d v}\left(\theta_{S}, I_{S}, 0\right)+d \Psi^{k d v}\left(\theta_{S}, I_{S}, 0\right)\left[\left(0,0, z_{\perp}\right)\right] .
$$

The neighbourhood $\mathcal{V}$ of $\mathcal{K} \times\{0\}$ is chosen sufficiently small so that by the inverse function theorem, $\Psi_{L}$ is a real analytic diffeomorphism onto its image. Using that $\Psi_{L}$ is given in terms of the Birkhoff map $\Psi^{k d v}$, we prove in a first step that $\Psi_{L}$ admits an asymptotic expansion and tame estimates corresponding to the ones of Theorems 1.1, 1.2. In a second step we establish the corresponding results for the symplectic corrector $\Psi_{C}$. The methods developed in this paper also apply to the defocusing NLS equation and can be used to provide corresponding asymptotic expansions and estimates, thus complementing our previous work [12] on this equation.
Comments: In view of the definition of $\Psi_{L}$, the map $\Psi=\Psi_{L} \circ \Psi_{C}$ can be considered as a symplectic version of the Taylor expansion of $\Psi^{k d v}$ of order 1 in normal directions at points of $\mathcal{M}_{S}^{o} \times\{0\}$ and hence as a locally defined symplectic approximation of $\Psi^{k d v}$. Theorem 1.1 in particular says that $\Psi\left(\theta_{S}, I_{S}, z_{\perp}\right)-\mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]$ maps $\mathcal{V} \cap\left(\mathcal{M}_{S}^{o} \times h_{\perp}^{s}\right)$ into $H_{0}^{s+1}$ for any $s \geq 0$, i.e., that it is one-smoothing. Such a property has previously been established for the Birkhoff map $\Psi^{k d v}$ near 0 by Kuksin-Perelman [20], Theorem 0.2, and proved to hold on the entire phase space by Kappeler-Schaad-Topalov [16. Theorem 1.1 says that for the map $\Psi$, a much stronger property holds: up to a remainder term which is $(N+1)$-smoothing, $\Psi$ is a (nonlinear) pseudodifferential operator acting on $\mathcal{F}_{\perp}^{-1}\left(h_{\perp}^{0}\right)$.

Organization: The maps $\Psi_{L}$ and $\Psi_{C}$ are studied in Section 2 and respectively, Section 3. The expansion of the KdV Hamiltonian in the new coordinates is treated in Section 4 and a summary of the proofs of Theorem 1.1 and Theorem 1.2 is given in Section 5. In Appendix $\boxed{A}$ - Appendix D, we present results needed for the analysis of the map $\Psi_{L}$ in Section 2 and in Appendix Ewe review material from the pseudodifferential and paradifferential calculus.

## 2 The map $\Psi_{L}$

In this section we define and study the map $\Psi_{L}$ described in Section 1 First let us introduce some more notation. For $S \subset \mathbb{Z}$ finite as in (1.8), denote by $h_{S}^{0} \subset \mathbb{C}^{S}$ the subspace given by

$$
\begin{equation*}
h_{S}^{0}:=\left\{z_{S}=\left(z_{n}\right)_{n \in S} \in \mathbb{C}^{S}: z_{-n}=\bar{z}_{n} \forall n \in S_{+}\right\} \tag{2.1}
\end{equation*}
$$

By a slight abuse of terminology, for any $s \in \mathbb{Z}$, we identify $h_{0}^{s}$ with $h_{S}^{0} \times h_{\perp}^{s}$ and write $\left(z_{S}, z_{\perp}\right)$ for $z \in h_{0}^{s}$. According to the Addendum to Theorem A. 1 at the end of Appendix A the space $M_{S}$ of $S$-gap potentials is viewed as a (real analytic) submanifold of the weighted Sobolev space $H_{0}^{w_{*}}$ and the restriction of $\Phi^{k d v}$ to $M_{S}$ yields a real analytic diffeomorphism, $\Phi^{k d v}{ }_{\mid M_{S}}: M_{S} \rightarrow h_{S}^{0}$. We endow $h_{S}^{0}$ with the pull back of the standard Poisson structure on $h_{0}^{0}$ by the natural embedding $h_{S}^{0} \hookrightarrow h_{0}^{0}$, where the standard Poisson structure is the one for which $\left\{z_{n}, z_{-n}\right\}=2 \pi$ in for any $n \geq 1$ whereas the Poisson brackets among all the other coordinates vanish.

Consider the partially linearized inverse Birkhoff map

$$
\begin{equation*}
\Psi_{L}: h_{S}^{0} \times h_{\perp}^{0} \rightarrow L_{0}^{2}, \quad\left(z_{S}, z_{\perp}\right) \mapsto \Psi^{k d v}\left(z_{S}, 0\right)+d_{\perp} \Psi^{k d v}\left(z_{S}, 0\right)\left[z_{\perp}\right] \tag{2.2}
\end{equation*}
$$

where $d_{\perp} \Psi^{k d v}\left(z_{S}, 0\right)$ denotes the Fréchet derivative of the map $z_{\perp} \mapsto \Psi^{k d v}\left(z_{S}, z_{\perp}\right)$, evaluated at the point $\left(z_{S}, 0\right)$. By Theorem A.1, $\Psi_{L}$ is a real analytic map.

Proposition 2.1. The map $\Psi_{L}$ has the following properties: (i) For any $z_{S} \in h_{S}^{0}$,

$$
\Psi_{L}\left(z_{S}, 0\right)=\Psi^{k d v}\left(z_{S}, 0\right) \quad \text { and } \quad d \Psi_{L}\left(z_{S}, 0\right)=d \Psi^{k d v}\left(z_{S}, 0\right)
$$

(ii) For any compact subset $\mathcal{K} \subseteq h_{S}^{0}$ there exists an open neighbourhood $\mathcal{V}$ of $\mathcal{K} \times\{0\}$ in $h_{0}^{0} \equiv h_{S}^{0} \times h_{\perp}^{0}$
 diffeomorphism onto its image. The neighborhood $\mathcal{V}$ is chosen of the form $\mathcal{V}_{S} \times \mathcal{V}_{\perp}$ where $\mathcal{V}_{S}$ is an open, bounded neighborhood of $\mathcal{K}$ in $h_{S}^{0}$ and $\mathcal{V}_{\perp}$ is an open ball in $h_{\perp}^{0}$ of sufficiently small radius, centered at zero. (iii) For any $z=\left(z_{S}, z_{\perp}\right) \in \mathcal{V}$ and $\widehat{z}=\left(\hat{z}_{S}, \hat{z}_{\perp}\right) \in h_{S}^{0} \times h_{\perp}^{0}$,

$$
\begin{equation*}
d \Psi_{L}(z)[\widehat{z}]=d \Psi_{L}\left(z_{S}, 0\right)[\widehat{z}]+d_{S}\left(d_{\perp} \Psi^{k d v}\left(z_{S}, 0\right)\left[z_{\perp}\right]\right)\left[\widehat{z}_{S}\right] \tag{2.3}
\end{equation*}
$$

where the linear map $d \Psi_{L}\left(z_{S}, 0\right)=d \Psi^{k d v}\left(z_{S}, 0\right)$ is canonical and $d_{S}\left(d_{\perp} \Psi^{k d v}\left(z_{S}, 0\right)\left[z_{\perp}\right]\right)$ denotes the Fréchet derivative of the map

$$
\mathcal{V}_{S} \rightarrow L_{0}^{2}, z_{S} \mapsto d_{\perp} \Psi^{k d v}\left(z_{S}, 0\right)\left[z_{\perp}\right] .
$$

Proof. (i) The stated formulas follow from the definition of $\Psi_{L}$ in a straightforward way. (ii) In view of Theorem A. 1 the claimed statements can be proved by using the same arguments as in the proof of the corresponding results for the defocusing NLS equation in [12, Proposition 3.1]. Item (iii) is proved in a straightforward way.

In a next step we want to analyze $d_{\perp} \Psi^{k d v}\left(z_{S}, 0\right)$ further. Consider the Hamiltonian vector fields $\partial_{x} \nabla_{q} z_{ \pm n}$, $n \geq 1$, corresponding to the Hamiltonians given by the complex Birkhoff coordinates $z_{ \pm n}$. Since $\Phi^{k d v}$ is canonical in the sense that $\left\{z_{n}, z_{-n}\right\}=2 \pi$ in for any $n \neq 0$ whereas the brackets among all the other coordinates vanish, it follows that for any $q \in L_{0}^{2}$ and $n \geq 1$,

$$
d_{q} \Phi^{k d v}\left[\partial_{x} \nabla_{q} z_{ \pm n}\right]=X_{z_{ \pm n}}
$$

where $X_{z_{ \pm n}}$ are the constant vector fields on $h_{0, \mathbb{C}}^{0}$ given by

$$
X_{z_{n}}=-2 \pi \mathrm{i} n e^{(-n)}, \quad X_{z_{-n}}=2 \pi \mathrm{i} n e^{(n)}
$$

and $e^{(n)}, e^{(-n)}$ are the standard basis elements in the sequence space $h_{0, \mathbb{C}}^{0}$,

$$
e^{(n)}=\left(\delta_{n, k}\right)_{k \neq 0}, \quad e^{(-n)}=\left(\delta_{-n, k}\right)_{k \neq 0}
$$

(Here we extended $d_{q} \Phi^{k d v}: L_{0}^{2} \rightarrow h_{0}^{0}$ as a $\mathbb{C}$-linear map $L_{0, \mathbb{C}}^{2} \rightarrow h_{0, \mathbb{C}}^{0}$ ). Hence for any $n \geq 1$,

$$
\begin{equation*}
\left(d_{q} \Phi^{k d v}\right)^{-1}\left[e^{(n)}\right]=\frac{1}{2 \pi \mathrm{i} n} \partial_{x} \nabla_{q} z_{-n}, \quad\left(d_{q} \Phi^{k d v}\right)^{-1}\left[e^{(-n)}\right]=-\frac{1}{2 \pi \mathrm{i} n} \partial_{x} \nabla_{q} z_{n} \tag{2.4}
\end{equation*}
$$

It then follows from [13, Theorem 9.5, 9.7] and the formulas (B.5), (B.6) for the functions $H_{n}, G_{n}$ together with their properties stated in Proposition B. 1 that for any $q \in M_{S}$ and $n \in S_{+}^{\perp}$

$$
\begin{equation*}
\partial_{x} \nabla_{q} z_{n}=\sqrt{n \pi} \partial_{x} \nabla_{q}\left(x_{n}-\mathrm{i} y_{n}\right)=\sqrt{n \pi} \frac{\xi_{n}}{2} e^{-\mathrm{i} \beta_{n}} \partial_{x}\left(H_{n}-\mathrm{i} G_{n}\right)^{2} \tag{2.5}
\end{equation*}
$$

and similarly

$$
\begin{equation*}
\partial_{x} \nabla_{q} z_{-n}=\sqrt{n \pi} \partial_{x} \nabla_{q}\left(x_{n}+\mathrm{i} y_{n}\right)=\sqrt{n \pi} \frac{\xi_{n}}{2} e^{\mathrm{i} \beta_{n}} \partial_{x}\left(H_{n}+\mathrm{i} G_{n}\right)^{2} \tag{2.6}
\end{equation*}
$$

where $\beta_{n}=\sum_{\ell \in S_{+}} \beta_{\ell}^{n}\left(c f\left[13\right.\right.$, Theorem 8.5]) and $\xi_{n}=\sqrt{8 I_{n} / \gamma_{n}}$ (cf [13, Theorem 7.3]). Since $q \in M_{S}$ and $n \in S_{+}^{\perp}$ one has $\gamma_{n}(q)=0$ and the factor $\xi_{n}(q)$ is obtained by a limiting argument. By a slight abuse of terminology, we denote this limit also by $\sqrt{8 I_{n}(q) / \gamma_{n}^{2}(q)}$. The formulas (2.5) - (2.6) allow to express $\partial_{x} \nabla_{q} z_{ \pm n}$ in terms of the Floquet solutions $f_{n}(x) \equiv f_{n}(x, q)$ as follows (cf Appendix B for notations).

Proposition 2.2. For any $q \in M_{S}$ and $n \in S_{+}^{\perp}$,

$$
\partial_{x} \nabla_{q} z_{n}=\sqrt{n \pi} \frac{\xi_{n}}{2} e^{-\mathrm{i} \beta_{n}}\left(-\frac{2 \dot{m}_{2}\left(\tau_{n}\right)}{\ddot{\Delta}\left(\tau_{n}\right)}\right) \partial_{x} f_{-n}^{2}, \quad \partial_{x} \nabla_{q} z_{-n}=\sqrt{n \pi} \frac{\xi_{n}}{2} e^{\mathrm{i} \beta_{n}}\left(-\frac{2 \dot{m}_{2}\left(\tau_{n}\right)}{\ddot{\Delta}\left(\tau_{n}\right)}\right) \partial_{x} f_{n}^{2} .
$$

Hence by (2.4)

$$
\left(d_{q} \Phi^{k d v}\right)^{-1}\left[e^{(n)}\right]=\sqrt{n \pi} \xi_{n} e^{\mathrm{i} \beta_{n}} \frac{\dot{m}_{2}\left(\tau_{n}\right)}{\ddot{\Delta}\left(\tau_{n}\right)} \frac{-1}{2 \pi \mathrm{i} n} \partial_{x} f_{n}^{2}, \quad\left(d_{q} \Phi^{k d v}\right)^{-1}\left[e^{(-n)}\right]=\sqrt{n \pi} \xi_{n} e^{-\mathrm{i} \beta_{n}} \frac{\dot{m}_{2}\left(\tau_{n}\right)}{\ddot{\Delta}\left(\tau_{n}\right)} \frac{1}{2 \pi \mathrm{i} n} \partial_{x} f_{-n}^{2}
$$

Remark 2.1. At $q=0, \sqrt{n \pi} \xi_{n}=1, \beta_{n}=0, \frac{\dot{m}_{2}\left(\tau_{n}\right)}{\dot{\Delta}\left(\tau_{n}\right)}=-1$ and $f_{ \pm n}(x)=e^{ \pm \pi \mathrm{i} n x}$ for any $n \geq 1$, confirming that $\left(d_{0} \Phi^{k d v}\right)^{-1}\left[e^{( \pm n)}\right]=e^{ \pm 2 \pi \mathrm{i} n x} \quad(c f[13])$.

For any given $q \in M_{S}$ and $n \in S_{+}^{\perp}$, introduce the functions $W_{ \pm n}(x) \equiv W_{ \pm n}(x, q)$ given by

$$
\begin{equation*}
W_{n}(x):=\sqrt{n \pi} \xi_{n} \frac{\dot{m}_{2}\left(\tau_{n}\right)}{\ddot{\Delta}\left(\tau_{n}\right)} e^{\mathrm{i} \beta_{n}} \frac{-1}{2 \pi \mathrm{i} n} \partial_{x} f_{n}^{2}(x), \quad W_{-n}(x):=\sqrt{n \pi} \xi_{n} \frac{\dot{m}_{2}\left(\tau_{n}\right)}{\ddot{\Delta}\left(\tau_{n}\right)} e^{-\mathrm{i} \beta_{n}} \frac{1}{2 \pi \mathrm{i} n} \partial_{x} f_{-n}^{2}(x) \tag{2.7}
\end{equation*}
$$

We record that for any $n \in S_{+}^{\perp}, W_{-n}=\overline{W_{n}}$ since $f_{-n}=\overline{f_{n}}$. Combining Proposition 2.1] and Proposition 2.2 one obtains the following formula for the $\operatorname{map} \Psi_{L}$ :

Corollary 2.1. For any $z=\left(z_{S}, z_{\perp}\right) \in \mathcal{V}$, one has $d_{\perp} \Psi^{k d v}\left(z_{S}, 0\right)\left[z_{\perp}\right]=\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]$ where

$$
\begin{equation*}
\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right](x):=\sum_{n \in S^{\perp}} z_{n} W_{n}(x, q), \quad q=\Psi^{k d v}\left(z_{S}, 0\right) \tag{2.8}
\end{equation*}
$$

Note that $\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]=\Psi_{L}(z)-q$ is linear in $z_{\perp}$. Since $q \in M_{S}$ is a finite gap potential, it is $C^{\infty}$-smooth and so is $W_{n}(x, q)$. Next we want to show that $\Psi_{L}(z)$ admits an expansion of the type stated in Theorem 1.1 , Recall from the Addendum to Theorem A. 1 at the end of Appendix A that for any $q \in M_{S}, V_{q, S}^{*}$ denotes a neighborhood of $q$, consisting of complex valued $S$-gap potentials in the weighted Sobolev space $H_{0, \mathbb{C}}^{w_{*}}$ so that the restriction of $\Phi^{k d v}$ to $V_{q, S}^{*}$ is a real analytic diffeomorphism onto its image $\Phi^{k d v}\left(V_{q, S}^{*}\right) \subset h_{S, \mathbb{C}}^{0}$. Combining Theorem C. 1 and Lemma C.4-Lemma C. 6 of Appendix C and using that

$$
\sum_{n \in S^{\perp}} z_{n} \frac{1}{(2 \pi \mathrm{i} n)^{k}} e^{2 \pi \mathrm{i} n x}=\partial_{x}^{-k}\left(\sum_{n \in S^{\perp}} z_{n} e^{2 \pi \mathrm{i} n x}\right)=\partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]
$$

one obtains the following
Theorem 2.1. (i) Let $q \in M_{S}$ and $N \in \mathbb{N}$. Then for any $p \in V_{q, S}^{*}, W_{n}(x) \equiv W_{n}(x, p), n \in S^{\perp}$, has an expansion as $|n| \rightarrow \infty$ of the form

$$
\begin{equation*}
W_{n}(x, p)=e^{2 \pi \mathrm{in} x}\left(1+\sum_{k=1}^{N} \frac{W_{k}^{a e}(x, p)}{(2 \pi \mathrm{i} n)^{k}}+\frac{\mathcal{R}_{N}^{W_{n}}(x, p)}{(2 \pi \mathrm{i} n)^{N+1}}\right) \tag{2.9}
\end{equation*}
$$

where for any $s \in \mathbb{Z}_{\geq 0}$, $W_{k}^{a e}: V_{q, S}^{*} \rightarrow H_{\mathbb{C}}^{s}, p \mapsto W_{k}^{a e}(\cdot, p), k \geq 1$, are real analytic and $\mathcal{R}_{N}^{W_{n}}: V_{q, S}^{*} \rightarrow$ $H_{\mathbb{C}}^{s}, p \mapsto \mathcal{R}_{N}^{W_{n}}(\cdot, p), n \in S^{\perp}$, are analytic and satisfy for any $j \geq 0$,

$$
\sup _{0 \leq x \leq 1}\left|\partial_{x \in S^{\perp}}^{j} \mathcal{R}_{N}^{W_{n}}(x, p)\right| \leq C_{N, j}
$$

The constants $C_{N, j}$ can be chosen locally uniformly for $p \in V_{q, S}^{*}$. By a slight abuse of terminology, in the sequel, we will view $W_{k}^{a e}(\cdot, q)$ and $\mathcal{R}_{N}^{W_{n}}(\cdot, q)$ as functions of $z_{S}$,

$$
W_{k}^{a e}\left(\cdot, z_{S}\right) \equiv W_{k}^{a e}\left(\cdot, \Psi^{k d v}\left(z_{S}, 0\right)\right), \quad \mathcal{R}_{N}^{W_{n}}\left(\cdot, z_{S}\right) \equiv \mathcal{R}_{N}^{W_{n}}\left(\cdot, \Psi^{k d v}\left(z_{S}, 0\right)\right)
$$

(ii) For any $z_{S} \in h_{S}^{0}$, the linear operator $\Psi_{1}\left(z_{S}\right)$, given by

$$
\Psi_{1}\left(z_{S}\right): h_{\perp}^{0} \rightarrow L_{0}^{2}, \widehat{z}_{\perp} \mapsto \Psi_{1}\left(z_{S}\right)\left[\widehat{z}_{\perp}\right]=\sum_{n \in S^{\perp}} \widehat{z}_{n} W_{n}(\cdot, q), \quad q=\Psi^{k d v}\left(z_{S}, 0\right)
$$

has the property that for any $s \in \mathbb{Z}_{\geq 0}$, its restriction to $h_{\perp}^{s}$ is a bounded linear operator $h_{\perp}^{s} \rightarrow H_{0}^{s}$. Furthermore, up to a remainder, the operator $\Psi_{1}\left(z_{S}\right): h_{\perp}^{0} \rightarrow L_{0}^{2}$ is a pseudodifferential operator of order 0 . More precisely, $\Psi_{1}\left(z_{S}\right)$ has an expansion to any order $N \geq 1$ of the form

$$
\Psi_{1}\left(z_{S}\right)=\left(I d+\sum_{k=1}^{N} a_{k}\left(z_{S} ; \Psi_{1}\right) \partial_{x}^{-k}\right) \circ \mathcal{F}_{\perp}^{-1}+\mathcal{R}_{N}\left(z_{S} ; \Psi_{1}\right)
$$

where

$$
a_{k}\left(z_{S} ; \Psi_{1}\right):=W_{k}^{a e}\left(\cdot, z_{S}\right), k \geq 1, \quad \mathcal{R}_{N}\left(z_{S} ; \Psi_{1}\right)\left[\widehat{z}_{\perp}\right](x):=\sum_{n \in S^{\perp}} \widehat{z}_{n} \frac{\mathcal{R}_{N}^{W_{n}}\left(x, z_{S}\right)}{(2 \pi \mathrm{i} n)^{N+1}} e^{2 \pi \mathrm{i} n x}
$$

For any $s \geq 0$, the restriction of $\mathcal{R}_{N}\left(z_{S} ; \Psi_{1}\right)$ to $h_{\perp}^{s}$ defines a bounded linear operator $h_{\perp}^{s} \rightarrow H^{s+N+1}$ and the map

$$
h_{S}^{0} \rightarrow \mathcal{B}\left(h_{\perp}^{s}, H^{s+N+1}\right), z_{S} \mapsto \mathcal{R}_{N}\left(z_{S} ; \Psi_{1}\right)
$$

is real analytic. Corresponding properties hold for the map $\Psi_{L}: \mathcal{V} \rightarrow L_{0}^{2}$, defined in Proposition [2.1,

$$
\begin{equation*}
\Psi_{L}\left(z_{S}, z_{\perp}\right)=q+\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]=q+\mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]+\sum_{k=1}^{N} a_{k}\left(z_{S} ; \Psi_{L}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]+\mathcal{R}_{N}\left(z_{S} ; \Psi_{L}\right)\left[z_{\perp}\right] \tag{2.10}
\end{equation*}
$$

where

$$
a_{k}\left(z_{S} ; \Psi_{L}\right):=a_{k}\left(z_{S} ; \Psi_{1}\right), \quad k \geq 1 \quad \mathcal{R}_{N}\left(z_{S} ; \Psi_{L}\right):=\mathcal{R}_{N}\left(z_{S} ; \Psi_{1}\right)
$$

Remark 2.2. (i) Note that the pseudodifferential operator $\left(I d+\sum_{k=1}^{N} a_{k}\left(z_{S} ; \Psi_{1}\right) \partial_{x}^{-k}\right) \circ \mathcal{F}_{\perp}^{-1}$ defines a bounded linear operator $h_{\perp}^{s} \rightarrow H^{s}$ for any $s \in \mathbb{R}$ whereas the remainder $\mathcal{R}_{N}\left(z_{S} ; \Psi_{1}\right)$ defines a bounded linear operator $h_{\perp}^{s} \rightarrow H^{s+N+1}$ for any $s \geq-N-1$.
(ii) Whenever possible, we will use similar notation for the coefficients of the expansion of the various quantities such as $\Psi_{1}\left(z_{S}\right)$. If the coefficients are operators, we use the upper case letter $A$ and write $\mathcal{A}_{k}$ for the kth coefficient, whereas when they are functions (or operators, defined as the multiplication by a function), we use the lower case letter $a$ and write $a_{k}$ for the $k$ 'th coefficient. The quantity, which is expanded, is indicated as an argument of $\mathcal{A}_{k}$ and $a_{k}$.
(iii) The fact that up to a remainder term, $\Psi_{L}\left(z_{S}, \cdot\right)$ is given by the pseudodifferential operator of order 0 , $\left(I d+\sum_{k=1}^{N} a_{k}\left(z_{S} ; \Psi_{1}\right) \partial_{x}^{-k}\right) \circ \mathcal{F}_{\perp}^{-1}$, acting on the scale of Hilbert spaces $h_{\perp}^{s}, s \in \mathbb{Z}_{\geq 0}$, is at the heart of this paper. The result shows that the differential of the Birkhoff map $z \mapsto \Psi^{k d v}(z)$ at a finite gap potential, has distinctive features.

A straightforward application of Theorem 2.1(ii) yields an expansion of the transpose operator $\Psi_{1}\left(z_{S}\right)^{t}$ of $\Psi_{1}\left(z_{S}\right)$. Since $\mathcal{F}_{\perp}^{-1}$ is the restriction of the inverse of the Fourier transform to $h_{\perp}^{0}$, the transpose $\mathcal{F}_{\perp}^{-t}:=$ $\left(\mathcal{F}_{\perp}^{-1}\right)^{t}$ of $\mathcal{F}_{\perp}^{-1}$ with respect to the standard inner products in $L_{0}^{2}$ and $h_{\perp}^{0}$ is given by the Fourier transform, i.e., for any $\widehat{\widehat{q}} \in L_{0}^{2}$,

$$
\left\langle\mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right], \widehat{q}\right\rangle=\int_{0}^{1} \sum_{n \in S^{\perp}} z_{n} e^{2 \pi \mathrm{i} n x} \widehat{q}(x) d x=\sum_{n \in S^{\perp}} z_{n} \int_{0}^{1} \widehat{q}(x) e^{2 \pi \mathrm{i} n x} d x=\sum_{n \in S^{\perp}} z_{n} \widehat{q}_{-n}=\left\langle z_{\perp}, \mathcal{F}_{\perp} \widehat{q}\right\rangle .
$$

Corollary 2.2. For any $z_{S} \in h_{S}^{0}, q=\Psi^{k d v}\left(z_{S}, 0\right)$, and $N \in \mathbb{N}, \Psi_{1}\left(z_{S}\right)^{t}: L^{2} \rightarrow h_{\perp}^{0}, \widehat{q} \mapsto\left(\left\langle W_{-n}(\cdot, q), \widehat{q}\right\rangle\right)_{n \in S^{\perp}}$ has an expansion of the form

$$
\begin{equation*}
\Psi_{1}\left(z_{S}\right)^{t}=\mathcal{F}_{\perp} \circ\left(I d+\sum_{k=1}^{N} a_{k}\left(z_{S} ; \Psi_{1}^{t}\right) \partial_{x}^{-k}\right)+\mathcal{R}_{N}\left(z_{S} ; \Psi_{1}^{t}\right) \tag{2.11}
\end{equation*}
$$

where for any $s \geq 0$, the coefficients $h_{S}^{0} \rightarrow H^{s}, z_{S} \mapsto a_{k}\left(z_{S} ; \Psi_{1}^{t}\right), k \geq 1$, and the remainder $h_{S}^{0} \rightarrow$ $\mathcal{B}\left(H^{s}, h_{\perp}^{s+N+1}\right)$, $z_{S} \mapsto \mathcal{R}_{N}\left(z_{S} ; \Psi_{1}^{t}\right)$, are real analytic. Furthermore, $a_{k}\left(z_{S} ; \Psi_{1}^{t}\right)=-a_{k}\left(z_{S} ; \Psi_{1}\right)$. Corresponding properties hold for the map

$$
\mathcal{V} \rightarrow \mathcal{B}\left(L^{2}, h_{0}^{0}\right), z \mapsto d \Psi_{L}(z)^{t} .
$$

For any $z \in \mathcal{V}, N \in \mathbb{N}, d \Psi_{L}(z)^{t}$ has an expansion of the form

$$
\begin{equation*}
d \Psi_{L}(z)^{t}=\left(0, \mathcal{F}_{\perp} \circ\left(I d+\sum_{k=1}^{N} a_{k}\left(z ; d \Psi_{L}^{t}\right) \partial_{x}^{-k}\right)\right)+\mathcal{R}_{N}\left(z ; d \Psi_{L}^{t}\right) \tag{2.12}
\end{equation*}
$$

where $a_{k}\left(z ; d \Psi_{L}^{t}\right)=a_{k}\left(z_{S} ; \Psi_{1}^{t}\right)$ and where for any integer $s \geq 0, \mathcal{V} \cap h_{0}^{s} \rightarrow B\left(L^{2}, h_{0}^{s+N+1}\right), z \mapsto \mathcal{R}_{N}\left(z ; d \Psi_{L}^{t}\right)$ is real analytic.

Remark 2.3. Again we record that the pseudodifferential operator $\mathcal{F}_{\perp} \circ\left(\operatorname{Id}+\sum_{k=1}^{N} a_{k}\left(z_{S} ; \Psi_{1}^{t}\right) \partial_{x}^{-k}\right)$ defines a bounded linear operator $H^{s} \rightarrow h_{\perp}^{s}$ for any $s \in \mathbb{R}$ whereas the remainder $\mathcal{R}_{N}\left(z_{S} ; \Psi_{1}^{t}\right)$ defines a bounded linear operator $H^{s} \rightarrow h_{\perp}^{s+N+1}$ for any $s \geq-N-1$.

Proof. By Theorem[2.1(ii), $\Psi_{1}\left(z_{S}\right)=\mathcal{F}_{\perp}^{-1}+\sum_{k=1}^{N} a_{k}\left(z_{S} ; \Psi_{1}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}+\mathcal{R}_{N}\left(z_{S} ; \Psi_{1}\right)$ where for any $z_{\perp} \in h_{\perp}^{0}$,

$$
\mathcal{R}_{N}\left(z_{S} ; \Psi_{1}\right)\left[z_{\perp}\right](x):=\sum_{n \in S^{\perp}} z_{n} \frac{\mathcal{R}_{N}^{W_{n}}\left(x, z_{S}\right)}{(2 \pi \mathrm{i} n)^{N+1}} e^{2 \pi \mathrm{i} n x}
$$

Note that the functions $a_{k}\left(z_{S} ; \Psi_{1}\right)(x), k \geq 1$, are real valued. Taking into account that $\mathcal{F}_{\perp}^{-t}=\mathcal{F}_{\perp}$ and $\left(\partial_{x}^{-k}\right)^{t}=(-1)^{k} \partial_{x}^{-k}$, the expansion of the transpose $\Psi_{1}\left(z_{S}\right)^{t}$ of $\Psi_{1}\left(z_{S}\right)$ then reads

$$
\Psi_{1}\left(z_{S}\right)^{t}=\mathcal{F}_{\perp}+\mathcal{F}_{\perp} \circ \sum_{k=1}^{N}(-1)^{k} \partial_{x}^{-k} \circ a_{k}\left(z_{S} ; \Psi_{1}\right)+\left(\mathcal{R}_{N}\left(z_{S} ; \Psi_{1}\right)\right)^{t}
$$

By Theorem 2.1(i), for any $\widehat{q} \in H^{s}$,

$$
\left(\mathcal{R}_{N}\left(z_{S} ; \Psi_{1}\right)\right)^{t}[\widehat{q}]=\left(\frac{1}{(2 \pi \mathrm{i} n)^{N+1}} \int_{0}^{1} \widehat{q}(x) \mathcal{R}_{N}^{W_{n}}\left(x, z_{S}\right) e^{2 \pi \mathrm{i} n x} d x\right)_{n \in S^{\perp}} \in h_{\perp}^{s+N+1}
$$

$\left(\mathcal{R}_{N}\left(z_{S} ; \Psi_{1}\right)\right)^{t}: H^{s} \rightarrow h_{\perp}^{s+N+1}$ is bounded, and the map $h_{S}^{0} \rightarrow \mathcal{B}\left(H^{s}, h_{\perp}^{s+N+1}\right), z_{S} \mapsto \mathcal{R}_{N}\left(z_{S} ; \Psi_{1}^{t}\right)$, is real analytic. Since by Lemma E. 2 and the notation introduced there,

$$
\partial_{x}^{-k} \circ a_{k}\left(z_{S} ; \Psi_{1}\right)=a_{k}\left(z_{S} ; \Psi_{1}\right) \partial_{x}^{-k}+\sum_{j=1}^{N-k} C_{j}(k)\left(\partial_{x}^{j} a_{k}\left(z_{S} ; \Psi_{1}\right)\right) \partial_{x}^{-k-j}+\mathcal{R}_{N, k, 0}^{\psi d o}\left(a_{k}\left(z_{S} ; \Psi_{1}\right)\right)
$$

one sees that $\Psi_{1}\left(z_{S}\right)^{t}$ admits an expansion of the form,

$$
\Psi_{1}\left(z_{S}\right)^{t}=\mathcal{F}_{\perp}+\mathcal{F}_{\perp} \circ \sum_{k=1}^{N} a_{k}\left(z_{S} ; \Psi_{1}^{t}\right) \partial_{x}^{-k}+\mathcal{R}_{N}\left(z_{S} ; \Psi_{1}^{t}\right)
$$

where $a_{k}\left(z_{S} ; \Psi_{1}^{t}\right), k \geq 1$, and $\mathcal{R}_{N}\left(z_{S} ; \Psi_{1}^{t}\right)$ satisfy the claimed properties. Since

$$
d \Psi_{L}(z)[\widehat{z}]=\left(d_{S} \Psi^{k d v}\left(z_{S}, 0\right)+d_{S}\left(\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)\right)\left[\widehat{z}_{S}\right]+\Psi_{1}\left(z_{S}\right)\left[\widehat{z}_{\perp}\right]
$$

the claimed properties of $d \Psi_{L}(z)^{t}$ follow from the ones of $\Psi_{1}\left(z_{S}\right)^{t}$.
Using results of Appendix D and Appendix C one obtains the following properties of the functions $W_{n}$, $n \in S^{\perp}$, and the map $\Psi_{L}$ with regard to the reversible structure, introduced in Section 1 .

Addendum to Theorem 2.1 (i) For any $z_{S} \in h_{S}^{0}, q=\Psi^{k d v}\left(z_{S}, 0\right)$ satisfies $S_{r e v} q=\Psi^{k d v}\left(\mathcal{S}_{r e v}\left(z_{S}, 0\right)\right)$ and for any $n \in S^{\perp}, x \in \mathbb{R}, W_{n}\left(x, S_{r e v} q\right)=W_{-n}(-x, q)$ as well as $(k \geq 1, N \geq 1)$

$$
\begin{equation*}
W_{k}^{a e}\left(x, \mathcal{S}_{\text {rev }} z_{S}\right)=(-1)^{k} W_{k}^{a e}\left(-x, z_{S}\right), \quad \mathcal{R}_{N}^{W_{n}}\left(x, \mathcal{S}_{r e v} z_{S}\right)=(-1)^{N+1} \mathcal{R}_{N}^{W_{-n}}\left(-x, z_{S}\right) \tag{2.13}
\end{equation*}
$$

(ii) For any $z=\left(z_{S}, z_{\perp}\right) \in h_{S}^{0} \times h_{\perp}^{0}$ and $x \in \mathbb{R}$,

$$
\left(\Psi_{1}\left(\mathcal{S}_{\text {rev }} z_{S}\right)\left[\mathcal{S}_{\text {rev }} z_{\perp}\right]\right)(x)=\left(\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)(-x)
$$

As a consequence, for any $z \in \mathcal{V}, x \in \mathbb{R}$

$$
\begin{equation*}
\left(\Psi_{L}\left(\mathcal{S}_{\text {rev }} z\right)\right)(x)=\left(\Psi_{L}(z)\right)(-x), \quad\left(\mathcal{R}_{N}\left(\mathcal{S}_{\text {rev }} z_{S} ; \Psi_{1}\right)\left[\mathcal{S}_{\text {rev }} z_{\perp}\right]\right)(x)=\left(\mathcal{R}_{N}\left(z_{S} ; \Psi_{1}\right)\left[z_{\perp}\right]\right)(-x) \tag{2.14}
\end{equation*}
$$

(iii) For any $z_{S} \in h_{S}^{0}$ and $\widehat{q} \in L_{0}^{2}$, one has $\Psi_{1}\left(\mathcal{S}_{\text {rev }} z_{S}\right)^{t}\left[S_{\text {rev }} \widehat{q}\right]=\mathcal{S}_{\text {rev }}\left(\Psi_{1}\left(z_{S}\right)^{t}[\widehat{q}]\right)$. As a consequence, for any $k \geq 1$ and $N \geq 1$,

$$
a_{k}\left(\mathcal{S}_{\text {rev }} z_{S} ; \Psi_{1}^{t}\right)(x)=(-1)^{k} a_{k}\left(z_{S} ; \Psi_{1}^{t}\right)(-x), \quad \mathcal{R}_{N}\left(\mathcal{S}_{\text {rev }} z_{S} ; \Psi_{1}^{t}\right)\left[S_{\text {rev }} \widehat{q}\right]=\mathcal{S}_{\text {rev }}\left(\mathcal{R}_{N}\left(z_{S} ; \Psi_{1}^{t}\right)[\widehat{q}]\right)
$$

Proof of Addendum to Theorem 2.1 (i) By the Addendum to Theorem C. 1 we know that for any $q \in M_{S}$ and $n \in S_{+}^{\perp}, f_{ \pm n}\left(x, S_{r e v} q\right)=f_{\mp n}(-x, q)$. Furthermore, one has $\xi_{n}\left(S_{r e v} q\right)=\xi_{n}(q), \Delta\left(\lambda, S_{r e v} q\right)=$ $\Delta(\lambda, q), m_{2}\left(\lambda, S_{r e v} q\right)=m_{2}(\lambda, q)\left(\right.$ Lemma D.1) and $\beta_{n}\left(S_{r e v} q\right)=-\beta_{n}(q)$ (Corollary D.1). In view of the definition (2.7) of $W_{ \pm n}$ it then follows that $W_{ \pm n}\left(x, S_{r e v} q\right)=W_{\mp n}(-x, q)$ and in turn, comparing the expansion (2.9) of $W_{ \pm n}\left(x, S_{r e v} q\right)$ with the one of $W_{\mp n}(-x, q)$, one obtains the identities (2.13). (ii) By (i) one has for any $z_{S} \in h_{S}^{0}$ and $q=\Psi^{k d v}\left(z_{S}, 0\right)$,

$$
\left(\Psi_{1}\left(\mathcal{S}_{r e v} z_{S}\right)\left[\mathcal{S}_{r e v} z_{\perp}\right]\right)(x)=\sum_{n \in S^{\perp}} z_{-n} W_{n}\left(x, S_{r e v} q\right)=\sum_{n \in S^{\perp}} z_{-n} W_{-n}(-x, q)=\left(\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)(-x)
$$

as well as $W_{k}^{a e}\left(x, \mathcal{S}_{r e v} z_{S}\right)=(-1)^{k} W_{k}^{a e}\left(-x, z_{S}\right)$ and $\left(\mathcal{R}_{N}\left(\mathcal{S}_{r e v} z_{S} ; \Psi_{1}\right)\left[\mathcal{S}_{r e v} z_{\perp}\right]\right)(x)=\left(\mathcal{R}_{N}\left(z_{S} ; \Psi_{1}\right)\left[z_{\perp}\right]\right)(-x)$. By (2.10), the claimed identities (2.14) then follow. (iii) Recall that for any $z_{S} \in h_{S}^{0}, \widehat{q} \in L_{0}^{2}$, one has $\Psi_{1}\left(z_{S}\right)^{t}[\widehat{q}]=\left(\left\langle W_{-n}(\cdot, q), \widehat{q}\right\rangle\right)_{n \in S^{\perp}}$. It then follows from item (i) that

$$
\Psi_{1}\left(\mathcal{S}_{\text {rev }} z_{S}\right)^{t}\left[S_{\text {rev }} \widehat{q}\right]=\mathcal{S}_{\text {rev }}\left(\Psi_{1}\left(z_{S}\right)^{t}[\widehat{q}]\right)
$$

Comparing the expansion (2.11) for $\mathcal{S}_{\text {rev }}\left(\Psi_{1}\left(\mathcal{S}_{\text {rev }} z_{S}\right)^{t}[\widehat{q}]\right)$ with the one for $\Psi_{1}\left(z_{S}\right)^{t}\left[S_{\text {rev }} \widehat{q}\right]$ and taking into account that $\mathcal{S}_{\text {rev }} \circ \mathcal{F}_{\perp}=\mathcal{F}_{\perp} \circ S_{\text {rev }}$ and $\partial_{x} \circ S_{\text {rev }}=-S_{\text {rev }} \circ \partial_{x}$ one sees that for any $k \geq 1$,

$$
a_{k}\left(\mathcal{S}_{\text {rev }} z_{S} ; \Psi_{1}^{t}\right)(x)=(-1)^{k} a_{k}\left(z_{S} ; \Psi_{1}^{t}\right)(-x), \quad \mathcal{R}_{N}\left(\mathcal{S}_{\text {rev }} z_{S} ; \Psi_{1}^{t}\right)\left[S_{\text {rev }} \widehat{q}\right]=\mathcal{S}_{\text {rev }}\left(\mathcal{R}_{N}\left(z_{S} ; \Psi_{1}^{t}\right)[\widehat{q}]\right)
$$

In the remaining part of this section we describe the pull back $\Psi_{L}^{*} \Lambda_{G}$ of the symplectic form $\Lambda_{G}$ by the map $\Psi_{L}$, defined in Proposition 2.1 where $\Lambda_{G}$, defined by the Gardner Poisson structure, is given by

$$
\Lambda_{G}[\widehat{u}, \widehat{v}]=\left\langle\partial_{x}^{-1} \widehat{u}, \widehat{v}\right\rangle=\int_{0}^{1}\left(\partial_{x}^{-1} \widehat{u}\right)(x) \widehat{v}(x) d x, \quad \forall \widehat{u}, \widehat{v} \in L_{0}^{2}
$$

Note that $\Lambda_{G}=d \lambda_{G}$ where the one form $\lambda_{G}$, defined on $L_{0}^{2}$, is given by

$$
\begin{equation*}
\lambda_{G}(u)[\widehat{v}]=\left\langle\partial_{x}^{-1} u, \widehat{v}\right\rangle=\int_{0}^{1}\left(\partial_{x}^{-1} u\right)(x) \widehat{v}(x) d x, \quad \forall u, \widehat{v} \in L_{0}^{2} \tag{2.15}
\end{equation*}
$$

To compute the pull back of $\Lambda_{G}$ by $\Psi_{L}$, note that for any $z=\left(z_{S}, z_{\perp}\right) \in \mathcal{V}=\mathcal{V}_{S} \times \mathcal{V}_{\perp}$, the derivative $d \Psi_{L}(z)$, when written in $1 \times 2$ matrix form, is given by (cf (2.8))

$$
\begin{equation*}
d \Psi_{L}(z)=d \Psi_{L}\left(z_{S}, 0\right)+\left(d_{S}\left(\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right) \quad 0\right)=\left(d_{S} \Psi^{k d v}\left(z_{S}, 0\right) \quad \Psi_{1}\left(z_{S}\right)\right)+\left(d_{S}\left(\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right) \quad 0\right) \tag{2.16}
\end{equation*}
$$

For any $\widehat{z}=\left(\widehat{z}_{S}, \widehat{z}_{\perp}\right), \widehat{w}=\left(\widehat{w}_{S}, \widehat{w}_{\perp}\right) \in h_{0}^{0}$ one has

$$
\begin{aligned}
& \left(\Psi_{L}^{*} \Lambda_{G}\right)(z)[\widehat{z}, \widehat{w}]=\Lambda_{G}\left[d \Psi_{L}(z)[\widehat{z}], d \Psi_{L}(z)[\widehat{w}]\right]=\left\langle\partial_{x}^{-1} d \Psi_{L}(z)[\widehat{z}], d \Psi_{L}(z)[\widehat{w}]\right\rangle \\
& =\left\langle\partial_{x}^{-1}\left(d \Psi_{L}\left(z_{S}, 0\right)[\widehat{z}]\right)+\partial_{x}^{-1}\left(d_{S}\left(\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)\left[\widehat{z}_{S}\right]\right), d \Psi_{L}\left(z_{S}, 0\right)[\widehat{w}]+d_{S}\left(\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)\left[\widehat{w}_{S}\right]\right\rangle .
\end{aligned}
$$

Since by construction, $d \Psi_{L}\left(z_{S}, 0\right): h_{0}^{0} \rightarrow L_{0}^{2}$ is symplectic, one has

$$
\left(\Psi_{L}^{*} \Lambda_{G}\right)\left(z_{S}, 0\right)=\Lambda
$$

where $\Lambda$ is the symplectic form on $h_{0}^{0}$,

$$
\begin{equation*}
\Lambda[\widehat{z}, \widehat{w}]:=\left\langle J^{-1} \widehat{z}, \widehat{w}\right\rangle=\sum_{n \neq 0} \frac{1}{2 \pi \mathrm{i} n} \widehat{z}_{n} \widehat{w}_{-n}, \quad \forall \widehat{z}, \widehat{w} \in h_{0}^{0} \tag{2.17}
\end{equation*}
$$

and $J^{-1}$ denotes the inverse of the diagonal operator, acting on the scale of Hilbert spaces $h_{0}^{s}, s \in \mathbb{R}$,

$$
\begin{equation*}
J: h_{0}^{s+1} \rightarrow h_{0}^{s}:\left(z_{n}\right)_{n \neq 0} \mapsto\left(2 \pi \mathrm{i} n z_{n}\right)_{n \neq 0} \tag{2.18}
\end{equation*}
$$

Note that $\Lambda=d \lambda$ where $\lambda$ is the one form on $h_{0}^{0}$,

$$
\begin{equation*}
\lambda(z)[\widehat{w}]:=\left\langle J^{-1} z, \widehat{w}\right\rangle=\sum_{n \neq 0} \frac{1}{2 \pi \mathrm{i} n} z_{n} \widehat{w}_{-n}, \quad \forall z, \widehat{w} \in h_{0}^{0} \tag{2.19}
\end{equation*}
$$

Altogether we have that

$$
\begin{equation*}
\left(\Psi_{L}^{*} \Lambda_{G}\right)(z)[\widehat{z}, \widehat{w}]=\Lambda[\widehat{z}, \widehat{w}]+\Lambda_{L}(z)[\widehat{z}, \widehat{w}], \quad \Lambda_{L}(z)[\widehat{z}, \widehat{w}]=\langle\mathcal{L}(z)[\widehat{\widehat{z}}], \widehat{w}\rangle \tag{2.20}
\end{equation*}
$$

where the operator $\mathcal{L}(z): h_{S}^{0} \times h_{\perp}^{0} \rightarrow h_{S}^{0} \times h_{\perp}^{0}$ has the form

$$
\mathcal{L}(z)=\left(\begin{array}{cc}
\mathcal{L}_{S}^{S}(z) & \mathcal{L}_{S}^{\perp}(z)  \tag{2.21}\\
\mathcal{L}_{\perp}^{S}(z) & 0
\end{array}\right)
$$

with $\mathcal{L}_{S}^{S}(z): h_{S}^{0} \rightarrow h_{S}^{0}, \mathcal{L}_{S}^{\perp}(z): h_{\perp}^{0} \rightarrow h_{S}^{0}$, and $\mathcal{L}_{\perp}^{S}: h_{S}^{0} \rightarrow h_{\perp}^{0}$ given by

$$
\begin{align*}
\mathcal{L}_{S}^{S}(z) & :=\left(d_{S} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)^{t} \partial_{x}^{-1}\left(d_{S} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)+\left(d_{S} \Psi^{k d v}\left(z_{S}, 0\right)\right)^{t} \partial_{x}^{-1}\left(d_{S} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right) \\
& +\left(d_{S} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)^{t} \partial_{x}^{-1} d_{S} \Psi^{k d v}\left(z_{S}, 0\right)  \tag{2.22}\\
\mathcal{L}_{S}^{\perp}(z) & :=\left(d_{S} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)^{t} \partial_{x}^{-1} \Psi_{1}\left(z_{S}\right), \quad \mathcal{L}_{\perp}^{S}(z):=\Psi_{1}\left(z_{S}\right)^{t} \partial_{x}^{-1}\left(d_{S} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)
\end{align*}
$$

For any $z=\left(z_{S}, z_{\perp}\right) \in \mathcal{V}$, the operators $\mathcal{L}(z), \mathcal{L}_{S}^{S}(z), \mathcal{L}_{S}^{\perp}(z)$, and $\mathcal{L}_{\perp}^{S}(z)$ are bounded. In the sequel, we will often write the operators (2.22) in the following way

$$
\begin{align*}
\mathcal{L}_{S}^{S}(z)\left[\widehat{z}_{S}\right] & =\left(\left\langle\partial_{x}^{-1} d_{S}\left(\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)\left[\widehat{z}_{S}\right], \partial_{z_{n}} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right\rangle\right)_{n \in S}+\left(\left\langle\partial_{x}^{-1} d_{S}\left(\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)\left[\widehat{z}_{S}\right], \partial_{z_{n}} \Psi^{k d v}\left(z_{S}, 0\right)\right\rangle\right)_{n \in S} \\
& +\left(\left\langle\partial_{x}^{-1} d_{S}\left(\Psi^{k d v}\left(z_{S}, 0\right)\right)\left[\widehat{z}_{S}\right], \partial_{z_{n}} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right\rangle\right)_{n \in S} \\
\mathcal{L}_{S}^{\perp}(z)\left[\widehat{z}_{\perp}\right] & =\left(\left\langle\partial_{x}^{-1} \Psi_{1}\left(z_{S}\right)\left[\widehat{z}_{\perp}\right], \partial_{z_{n}} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right\rangle\right)_{n \in S} \\
\mathcal{L}_{\perp}^{S}(z)\left[\widehat{z}_{S}\right] & =\Psi_{1}\left(z_{S}\right)^{t} \partial_{x}^{-1} d_{S}\left(\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)\left[\widehat{z}_{S}\right]=\left(\left\langle W_{-n}(\cdot, q), \partial_{x}^{-1} d_{S}\left(\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)\left[\widehat{z}_{S}\right]\right\rangle\right)_{n \in S^{\perp}} \tag{2.23}
\end{align*}
$$

where $q=\Psi^{k d v}\left(z_{S}, 0\right)$. The operators $\mathcal{L}_{S}^{S}(z), \mathcal{L}_{S}^{\perp}(z)$, and $\mathcal{L}_{\perp}^{S}(z)$ satisfy the following properties.
Lemma 2.1. (i) The maps

$$
\mathcal{V} \rightarrow \mathcal{B}\left(h_{S}^{0}, h_{S}^{0}\right), z \mapsto \mathcal{L}_{S}^{S}(z), \quad \mathcal{V} \rightarrow \mathcal{B}\left(h_{\perp}^{0}, h_{S}^{0}\right), z \mapsto \mathcal{L}_{S}^{\perp}(z)
$$

are real analytic. Furthermore, the following estimates hold: for any $z=\left(z_{S}, z_{\perp}\right) \in \mathcal{V}, \widehat{z}_{S} \in h_{S}^{0}$, and $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{0}, l \geq 1$,

$$
\left\|\mathcal{L}_{S}^{S}(z)\left[\widehat{z}_{S}\right]\right\| \lesssim\left\|\widehat{z}_{S}\right\|\left\|z_{\perp}\right\|_{0}, \quad\left\|d^{l}\left(\mathcal{L}_{S}^{S}(z)\left[\widehat{z}_{S}\right]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\| \lesssim_{l}\left\|\widehat{z}_{S}\right\| \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
$$

and if in addition, $\widehat{z}_{\perp} \in h_{\perp}^{0}$,

$$
\left\|\mathcal{L}_{S}^{\perp}(z)\left[\widehat{z}_{\perp}\right]\right\| \lesssim\left\|z_{\perp}\right\|_{0}\left\|\widehat{z}_{\perp}\right\|_{0}, \quad\left\|d^{l}\left(\mathcal{L}_{S}^{\perp}(z)\left[\widehat{z}_{\perp}\right]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\| \lesssim_{l}\left\|\widehat{z}_{\perp}\right\|_{0} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
$$

(ii) For any $z=\left(z_{S}, z_{\perp}\right) \in \mathcal{V}, \mathcal{L}_{\perp}^{S}(z)$ has an expansion of arbitrary order $N \geq 2$,

$$
\begin{equation*}
\mathcal{L}_{\perp}^{S}(z)=\mathcal{F}_{\perp} \circ \sum_{k=2}^{N} \mathcal{A}_{k}\left(z_{S} ; \mathcal{L}_{\perp}^{S}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]+\mathcal{R}_{N}\left(z ; \mathcal{L}_{\perp}^{S}\right) \tag{2.24}
\end{equation*}
$$

where for any $s \geq 0, k \geq 1$, the maps

$$
\mathcal{V}_{S} \mapsto \mathcal{B}\left(h_{S}^{0}, H^{s}\right), z_{S} \mapsto \mathcal{A}_{k}\left(z_{S} ; \mathcal{L}_{\perp}^{S}\right), \quad \mathcal{V} \cap h_{0}^{s} \rightarrow \mathcal{B}\left(h_{S}^{0}, h_{\perp}^{s+N+1}\right), z \mapsto \mathcal{R}_{N}\left(z ; \mathcal{L}_{\perp}^{S}\right)
$$

are real analytic. In particular, the operator $\mathcal{L}_{\perp}^{S}(z)$ is two smooothing. More precisely, for any $s \geq 0$,

$$
\mathcal{V} \cap h_{0}^{s} \rightarrow \mathcal{B}\left(h_{S}^{0}, h_{\perp}^{s+2}\right), z \mapsto \mathcal{L}_{\perp}^{S}(z)
$$

is real analytic. The coefficients $\mathcal{A}_{k}\left(z_{S} ; \mathcal{L}_{\perp}^{S}\right)$ are independent of $z_{\perp}$ and satisfy for any $s \geq 0, z_{S} \in \mathcal{V}_{S}$, $\widehat{z}_{S} \in h_{S}^{0}, \widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{0}, l \geq 1$, the following estimates

$$
\left\|\mathcal{A}_{k}\left(z_{S} ; \mathcal{L}_{\perp}^{S}\right)\left[\widehat{z}_{S}\right]\right\|_{s} \lesssim_{s, k}\left\|\widehat{z}_{S}\right\|, \quad\left\|d^{l}\left(\mathcal{A}_{k}\left(z_{S} ; \mathcal{L}_{\perp}^{S}\right)\left[\widehat{z}_{S}\right]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s} \lesssim_{s, k, l}\left\|\widehat{z}_{S}\right\| \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
$$

Furthermore, for any $s \in \mathbb{Z}_{\geq 0}, z=\left(z_{S}, z_{\perp}\right) \in \mathcal{V} \cap h_{0}^{s}$, $\widehat{z}_{S} \in h_{S}^{0}$, and $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s}, l \geq 1, \mathcal{R}_{N}\left(z ; \mathcal{L}_{\perp}^{S}\right)\left[\widehat{z}_{S}\right]$ satisfies $\left\|\mathcal{R}_{N}\left(z ; \mathcal{L}_{\perp}^{S}\right)\left[\widehat{z}_{S}\right]\right\|_{s+N+1} \lesssim_{s, N}\left\|\widehat{z}_{S}\right\|\left\|z_{\perp}\right\|_{s}$ and

$$
\left\|d^{l}\left(\mathcal{R}_{N}\left(z ; \mathcal{L}_{\perp}^{S}\right)\left[\widehat{z}_{S}\right]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+N+1} \lesssim_{s, N, l}\left\|\widehat{z}_{S}\right\|\left(\sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{0}+\left\|z_{\perp}\right\|_{s} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}\right)
$$

(iii) As a consequence, for any integer $s \geq 0$, the map $\mathcal{V} \cap h_{0}^{s} \rightarrow \mathcal{B}\left(h_{0}^{0}, h_{0}^{s+2}\right), z \mapsto \mathcal{L}(z)$ is real analytic. Furthermore, for any $z=\left(z_{S}, z_{\perp}\right) \in \mathcal{V} \cap h_{0}^{s}$ and $\widehat{z} \in h_{0}^{0}$, it satisfies the estimates

$$
\begin{equation*}
\|\mathcal{L}(z)[\widehat{z}]\|_{s+2} \leq C(s ; \mathcal{L})\|\widehat{z}\|_{o}\left\|z_{\perp}\right\|_{s} \tag{2.25}
\end{equation*}
$$

and if in addition $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s}, l \geq 1$, one has

$$
\begin{equation*}
\left\|d^{l}(\mathcal{L}(z)[\widehat{z}])\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+2} \leq C(s, l ; \mathcal{L})\|\widehat{z}\|_{0}\left(\sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{0}+\left\|z_{\perp}\right\|_{s} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}\right) \tag{2.26}
\end{equation*}
$$

for some constants $C(s ; \mathcal{L}) \geq 1, C(s, l ; \mathcal{L}) \geq 1$.
Remark 2.4. Recall that by Remark 2.2( $(i), \partial_{x}^{-1} \Psi_{1}\left(z_{S}\right): h_{\perp}^{-1} \rightarrow H_{0}^{0}$ is a bounded linear operator for any $z \in \mathcal{V}$. Since $\partial_{z_{n}} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right] \in H_{0}^{0}$ for any $n \in S$, it then follows that $\mathcal{L}_{S}^{\perp}(z): h_{\perp}^{-1} \rightarrow h_{S}^{0}$ and in turn $\mathcal{L}(z): h_{0}^{-1} \rightarrow h_{0}^{0}$ are bounded linear operators. Estimates, corresponding to the ones for $\mathcal{L}_{S}^{\perp}(z)$ and $\mathcal{L}(z)$ of Lemma 2.1, continue to hold, when these operators are extended to $h_{\perp}^{-1}$ and, respectively, $h_{0}^{-1}$.

Proof. The lemma follows in a straightforward way by using the properties of the maps $\Psi_{1}\left(z_{S}\right)$ and $\Psi_{1}\left(z_{S}\right)^{t}$ (cf Lemma [2.2) and the expansion of the composition $\partial_{x}^{-n} \circ a \partial_{x}^{-k}$ (cf Lemma E.2 in Appendix E).

Finally, we discuss the properties of the symplectic forms $\Lambda_{G}, \Lambda$, and $\Psi_{L}^{*} \Lambda_{G}$ with respect to the reversible structures introduced in Section [1] First note that for any $\widehat{u}, \widehat{v} \in L_{0}^{2}$,

$$
\left(S_{r e v}^{*} \Lambda_{G}\right)[\widehat{u}, \widehat{v}]=\Lambda_{G}\left[S_{r e v} \widehat{u}, S_{r e v} \widehat{v}\right]=\int_{0}^{1} \partial_{x}^{-1}(\widehat{u}(-x)) \widehat{v}(-x) d x=-\Lambda_{G}[\widehat{u}, \widehat{v}]
$$

and similarly, for any $\widehat{z}, \widehat{w} \in h_{0}^{0}$,

$$
\left(\mathcal{S}_{r e v}^{*} \Lambda\right)[\widehat{z}, \widehat{w}]=\Lambda\left[\mathcal{S}_{r e v} \widehat{z}, \mathcal{S}_{r e v} \widehat{w}\right]=\sum_{n \neq 0} \frac{1}{2 \pi i n} \widehat{z}_{-n} \widehat{w}_{n}=-\Lambda[\widehat{z}, \widehat{w}]
$$

By the Addendum to Theorem 2.1, the pullback $\mathcal{S}_{\text {rev }}^{*} \Psi_{L}^{*} \Lambda_{G}$ can then be computed as

$$
\left(\mathcal{S}_{r e v}^{*} \Psi_{L}^{*}\right) \Lambda_{G}=\Psi_{L}^{*}\left(S_{r e v}^{*} \Lambda_{G}\right)=-\Psi_{L}^{*} \Lambda_{G}
$$

implying together with (2.20) that

$$
\mathcal{S}_{r e v}^{*} \Lambda_{L}=-\Lambda_{L} .
$$

It then follows that the operators $\mathcal{L}_{S}^{S}(z), \mathcal{L}_{S}^{\perp}(z)$, and $\mathcal{L}_{\perp}^{S}(z)$ have the following symmetry properties.
Addendum to Lemma 2.1 For any $z=\left(z_{S}, z_{\perp}\right) \in h_{S}^{0} \times h_{\perp}^{0}$ and any $\widehat{z}_{S} \in h_{S}^{0}, \widehat{z}_{\perp} \in h_{\perp}^{0}$,

$$
\begin{gathered}
\mathcal{L}_{S}^{S}\left(\mathcal{S}_{r e v} z\right)\left[\mathcal{S}_{\text {rev }} \widehat{z}_{S}\right]=-\mathcal{S}_{\text {rev }}\left(\mathcal{L}_{S}^{S}(z)\left[\widehat{z}_{S}\right]\right) \\
\left.\mathcal{L}_{S}^{\perp}\left(\mathcal{S}_{r e v} z\right)\left[\mathcal{S}_{\text {rev }} \widehat{z}_{\perp}\right]=-\mathcal{S}_{\text {rev }}\left(\mathcal{L}_{S}^{\perp}(z)\left[\widehat{z}_{\perp}\right]\right), \quad \mathcal{L}_{\perp}^{S}\left(\mathcal{S}_{\text {rev }} z\right)\left[\mathcal{S}_{r e v} \widehat{z}_{S}\right]\right)=-\mathcal{S}_{\text {rev }}\left(\mathcal{L}_{\perp}^{S}(z)\left[\widehat{z}_{S}\right]\right) .
\end{gathered}
$$

By (2.24) it then follows that

$$
\begin{gathered}
\mathcal{A}_{k}\left(\mathcal{S}_{\text {rev }} z_{S} ; \mathcal{L}_{\perp}^{S}\right)\left[\mathcal{S}_{\text {rev }} \widehat{z}_{S}\right](x)=-(-1)^{k} \mathcal{A}_{k}\left(z_{S} ; \mathcal{L}_{\perp}^{S}\right)\left[\widehat{z}_{S}\right](-x), \\
\mathcal{R}_{N}\left(\mathcal{S}_{\text {rev }} z ; \mathcal{L}_{\perp}^{S}\right)\left[\mathcal{S}_{\text {rev }} \widehat{z}_{S}\right]=-\mathcal{S}_{\text {rev }}\left(\mathcal{R}_{N}\left(z ; \mathcal{L}_{\perp}^{S}\right)\left[\widehat{z}_{S}\right]\right) .
\end{gathered}
$$

## 3 The map $\Psi_{C}$

In this section we construct the symplectic corrector $\Psi_{C}$. Our approach is based on a well known method of Moser and Weinstein, implemented for an infinite dimensional setup in [19] (cf also [12]). We begin by briefly outlining the construction. At the end of Section 2, we introduce the symplectic forms $\Lambda$ and $\Psi_{L}^{*} \Lambda_{G}$. They are defined on $\mathcal{V}=\mathcal{V}_{S} \times \mathcal{V}_{\perp}$ and are related as follows $\left(z \in \mathcal{V}, \widehat{z}, \widehat{w} \in h_{0}^{0}\right)$

$$
\Psi_{L}^{*} \Lambda_{G}(z)[\widehat{z}, \widehat{w}]=\Lambda[\widehat{z}, \widehat{w}]+\Lambda_{L}(z)[\widehat{z}, \widehat{w}], \quad \Lambda_{L}(z)[\widehat{z}, \widehat{w}]=\langle\mathcal{L}(z)[\widehat{z}], \widehat{w}\rangle
$$

where $\mathcal{L}(z)$ is the operator defined by (2.21). Our candidate for $\Psi_{C}$ is $\Psi_{X}^{0,1}$ where $X \equiv X(\tau, z)$ is a nonautonomous vector field, defined for $z \in \mathcal{V}$ and $0 \leq \tau \leq 1$, so that $\left(\Psi_{X}^{0,1}\right)^{*}\left(\Psi_{L}^{*} \Lambda_{G}\right)=\Lambda$. The flow $\Psi_{X}^{\tau_{0}, \tau}$, corresponding to the vector field $X$, is required to be well defined on a neighborhood $\mathcal{V}^{\prime}$ (cf Lemma 3.4) for $0 \leq \tau_{0}, \tau \leq 1$ and to satisfy the standard normalization conditions $\Psi_{X}^{\tau_{0}, \tau_{0}}(z)=z$ for any $z \in \mathcal{V}^{\prime}$ and $0 \leq \tau_{0} \leq 1$. To find $X$ with the desired properties, introduce the one parameter family of two forms,

$$
\Lambda_{\tau}(z)=\Lambda+\tau \Lambda_{L}(z), \quad 0 \leq \tau \leq 1
$$

Note that $\Lambda_{0}=\Lambda, \Lambda_{1}=\Psi_{L}^{*} \Lambda_{G}$, and $\left(\Psi_{X}^{0,0}\right)^{*} \Lambda_{0}=\Lambda_{0}$. The desired identity $\left(\Psi_{X}^{0,1}\right)^{*} \Lambda_{1}=\Lambda_{0}$ then follows if one can show that $\left(\Psi_{X}^{0, \tau}\right)^{*} \Lambda_{\tau}$ is independent of $\tau$, that is, $\partial_{\tau}\left(\left(\Psi_{X}^{0, \tau}\right)^{*} \Lambda_{\tau}\right)=0$. By Cartan's identity,

$$
\partial_{\tau}\left(\left(\Psi_{X}^{0, \tau}\right)^{*} \Lambda_{\tau}\right)=\left(\Psi_{X}^{0, \tau}\right)^{*}\left(\partial_{\tau} \Lambda_{\tau}+d\left(\Lambda_{\tau}[X(\tau, \cdot), \cdot]\right)\right)=\left(\Psi_{X}^{0, \tau}\right)^{*}\left(\Lambda_{L}+d\left(\Lambda_{\tau}[X(\tau, \cdot), \cdot]\right)\right)
$$

Hence we need to choose the vector field $X(\tau, z)$ in such a way that

$$
\begin{equation*}
\Lambda_{L}(z)+d\left(\Lambda_{\tau}(z)[X(\tau, z), \cdot]\right)=0, \quad \Lambda_{\tau}(z)[X(\tau, z), \cdot]=\left\langle J^{-1} \mathcal{L}_{\tau}(z)[X(\tau, z)], \cdot\right\rangle \tag{3.1}
\end{equation*}
$$

where for any $0 \leq \tau \leq 1$ and $z \in \mathcal{V}$, the operator $\mathcal{L}_{\tau}(z): h_{0}^{0} \rightarrow h_{0}^{0}$ is defined by

$$
\begin{equation*}
\mathcal{L}_{\tau}(z):=\operatorname{Id}+\tau J \mathcal{L}(z) \tag{3.2}
\end{equation*}
$$

and where $J^{-1}$ is the inverse of the diagonal operator $J$, defined by (2.18). In a next step we want to rewrite $\Lambda_{L}(z)$ as the differential of a properly chosen one form. First note that since $\Lambda_{G}=d \lambda_{G}$ (cf (2.15)) and $\Lambda=d \lambda(c f(2.19))$, the two form $\Lambda_{L}$ is closed, $\Lambda_{L}=d\left(\lambda_{1}-\lambda_{0}\right)$ where $\lambda_{1}:=\Psi_{L}^{*} \lambda_{G}$ and $\lambda_{0}:=\lambda$. Furthermore, by Lemma [2.1] $\mathcal{L}\left(z_{S}, 0\right)=0$ and hence $\Lambda_{L}\left(z_{S}, 0\right)=0$ for any $z_{S} \in \mathcal{V}_{S}$. It then follows by the Poincaré Lemma (cf e.g. [12, Appendix 1]) that $d \lambda_{L}=\Lambda_{L}$ where

$$
\lambda_{L}(z)[\widehat{z}]:=\int_{0}^{1}\left\langle\mathcal{L}\left(z_{S}, t z_{\perp}\right)\left[0, z_{\perp}\right],\left(\widehat{z}_{S}, t \widehat{z}_{\perp}\right)\right\rangle d t \stackrel{\sqrt{2.21}}{=} \int_{0}^{1}\left\langle\mathcal{L}_{S}^{\perp}\left(z_{S}, t z_{\perp}\right)\left[z_{\perp}\right], \widehat{z}_{S}\right\rangle d t
$$

Since $\mathcal{L}_{S}^{\perp}\left(z_{S}, t z_{\perp}\right)=t \mathcal{L}_{S}^{\perp}\left(z_{S}, z_{\perp}\right)(\operatorname{cf}(2.23))$ one is then led to

$$
\begin{equation*}
\lambda_{L}(z)[\widehat{z}]=\langle\mathcal{E}(z), \widehat{z}\rangle, \quad \mathcal{E}(z):=\left(\mathcal{E}_{S}(z), 0\right) \in h_{S}^{0} \times h_{\perp}^{0}, \quad z \in \mathcal{V}, \widehat{z} \in h_{0}^{0} \tag{3.3}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{E}_{S}: \mathcal{V} \rightarrow h_{S}^{0}, z \mapsto \mathcal{E}_{S}(z):=\frac{1}{2} \mathcal{L}_{S}^{\perp}(z)\left[z_{\perp}\right] . \tag{3.4}
\end{equation*}
$$

In view of (3.1), we will choose $X$ so that

$$
\begin{equation*}
\mathcal{E}(z)+J^{-1} \mathcal{L}_{\tau}(z)[X(\tau, z)]=0, \quad \forall z \in \mathcal{V}, 0 \leq \tau \leq 1 \tag{3.5}
\end{equation*}
$$

Arguing as in the proof of [12, Lemma 4.1] one can show that after shrinking the ball $\mathcal{V}_{\perp}$, if needed, $\mathcal{L}_{\tau}(z)$ is invertible for any $0 \leq \tau \leq 1$ and $z \in \mathcal{V}$. In view of Lemma 2.1, the following version of [12, Lemma 4.1] holds:

Lemma 3.1. After shrinking the ball $\mathcal{V}_{\perp} \subset h_{\perp}^{0}$ in $\mathcal{V}=\mathcal{V}_{S} \times \mathcal{V}_{\perp}$, if needed, for any $s \geq 0, z \in \mathcal{V} \cap h_{0}^{s}$, and $\tau \in[0,1]$, the operator $\mathcal{L}_{\tau}(z): h_{0}^{s} \rightarrow h_{0}^{s}$ is invertible with inverse $\mathcal{L}_{\tau}(z)^{-1}: h_{0}^{s} \rightarrow h_{0}^{s}$ given by the Neumann series,

$$
\begin{equation*}
\mathcal{L}_{\tau}(z)^{-1}=I d+\sum_{n \geq 1}(-1)^{n}(\tau J \mathcal{L}(z))^{n} \tag{3.6}
\end{equation*}
$$

Furthermore, for any $s \geq 0$, the map

$$
[0,1] \times\left(\mathcal{V} \cap h_{0}^{s}\right) \rightarrow \mathcal{B}\left(h_{0}^{s}, h_{0}^{s+1}\right), \quad(\tau, z) \mapsto \mathcal{L}_{\tau}(z)^{-1}-I d=-\tau J \mathcal{L}(z) \mathcal{L}_{\tau}(z)^{-1}
$$

is real analytic and the following estimates hold: for any $z \in \mathcal{V} \cap h_{0}^{s}, 0 \leq \tau \leq 1, \widehat{z}, \widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s}, l \geq 1$,

$$
\begin{gathered}
\left\|\left(\mathcal{L}_{\tau}(z)^{-1}-I d\right)[\widehat{z}]\right\|_{s+1} \lesssim_{s}\left\|z_{\perp}\right\|_{s}\|\widehat{z}\|_{0} \\
\left\|d^{l}\left(\left(\mathcal{L}_{\tau}(z)^{-1}-I d\right)[\widehat{z}]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+1} \lesssim_{s, l}\|\widehat{z}\|_{0} \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{0}+\|\widehat{z}\|_{0}\left\|z_{\perp}\right\|_{s} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
\end{gathered}
$$

Note that by (3.4) and (2.23), $\mathcal{E}(z)$ and hence $\lambda_{L}(z)$ are quadratic expressions in $z_{\perp}$. Applying Lemma 2.1 to $\mathcal{E}(z)$, one obtains the following estimates:

Lemma 3.2. The map $\mathcal{V} \rightarrow h_{S}^{0} \times h_{\perp}^{0}, z \mapsto \mathcal{E}(z)=\left(\mathcal{E}_{S}(z), 0\right)$ is real analytic. Furthermore, for any $z \in \mathcal{V}$, $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{0}, l \geq 1$, one has

$$
\left\|\mathcal{E}_{S}(z)\right\| \lesssim\left\|z_{\perp}\right\|_{0}^{2}, \quad\left\|d \mathcal{E}_{S}(z)\left[\widehat{z}_{1}\right]\right\| \lesssim\left\|z_{\perp}\right\|_{0}\left\|\widehat{z}_{1}\right\|_{0}, \quad \quad\left\|d^{l} \mathcal{E}_{S}(z)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\| \lesssim l \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}, \quad l \geq 2
$$

Since $\mathcal{L}_{\tau}(z)$ is invertible (cf Lemma 3.1), equation (3.5) can be solved for $X(\tau, z)$,

$$
\begin{equation*}
X(\tau, z):=-\mathcal{L}_{\tau}(z)^{-1}[J \mathcal{E}(z)], \quad \forall z \in \mathcal{V}, \tau \in[0,1] \tag{3.7}
\end{equation*}
$$

Note that by Lemma 3.2, $J \mathcal{E}(z)$ is $C^{\infty}$-smooth. Hence it follows from Lemma3.1 that for any integer $s \geq 0$, $z \in \mathcal{V} \cap h_{0}^{s}$

$$
X(\tau, z)=-J \mathcal{E}(z)-\left(\mathcal{L}_{\tau}(z)^{-1}-\mathrm{Id}\right)[J \mathcal{E}(z)]=-J \mathcal{E}(z)+\tau J \mathcal{L}(z)[X(\tau, z)]
$$

Lemma 3.1 and Lemma 3.2 then lead to the following results (cf Lemma 12, 4.3]).
Lemma 3.3. For any $s \geq 0$, the non-autonomous vector field

$$
X:[0,1] \times\left(\mathcal{V} \cap h_{0}^{s}\right) \rightarrow h_{0}^{s+1}
$$

is real analytic and the following estimates hold: for any $z \in \mathcal{V} \cap h_{0}^{s}, 0 \leq \tau \leq 1, \widehat{z} \in h_{0}^{s}$,

$$
\left.\|X(\tau, z)\|_{s+1} \lesssim_{s}\left\|z_{\perp}\right\|_{s}\|z\|_{0}, \quad \| d X(\tau, z)\right)\left[\widehat{z}_{1}\right]\left\|_{s+1} \lesssim_{s}\right\| z_{\perp}\left\|_{s}\right\| \widehat{z}\left\|_{0}+\right\| z_{\perp}\left\|_{0}\right\| \widehat{z} \|_{s}
$$

and for any $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s}, l \geq 2$,

$$
\left\|d^{l} X(\tau, z)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+1} \lesssim_{s, l} \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s} \prod_{i \neq j}^{l}\left\|\widehat{z}_{i}\right\|_{0}+\left\|z_{\perp}\right\|_{s} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
$$

By a standard contraction argument, there exists an open neighborhood $\mathcal{V}_{S}^{\prime} \subset \mathcal{V}_{S}$ of $\mathcal{K} \subset h_{S}^{0}$ and a ball $\mathcal{V}_{\perp}^{\prime} \subset \mathcal{V}_{\perp}$, centered at 0 , so that for any $\tau, \tau_{0} \in[0,1]$, the flow map $\Psi_{X}^{\tau_{0}, \tau}$ of the non-autonomous differential equation $\partial_{\tau} z=X(\tau, z)$ is well defined on $\mathcal{V}^{\prime}:=\mathcal{V}_{S}^{\prime} \times \mathcal{V}_{\perp}^{\prime}$ and

$$
\begin{equation*}
\Psi_{X}^{\tau_{0}, \tau}: \mathcal{V}^{\prime} \rightarrow \mathcal{V} \tag{3.8}
\end{equation*}
$$

is real analytic. Arguing as in the proof of [12, Lemma 4.4] one shows that $\Psi_{X}^{\tau_{0}, \tau}-i d$ is one smoothing. More precisely, the following holds.

Lemma 3.4. Shrinking the ball $\mathcal{V}_{\perp}^{\prime} \subset h_{\perp}^{0}$ in $\mathcal{V}^{\prime}=\mathcal{V}_{S}^{\prime} \times \mathcal{V}_{\perp}^{\prime}$, if needed, it follows that for any $s \geq 0$, $\tau_{0}, \tau \in[0,1]$, the map $\Psi_{X}^{\tau_{0}, \tau}-i d: \mathcal{V}^{\prime} \cap h_{0}^{s} \rightarrow h_{0}^{s+1}$ is real analytic and for any $z \in \mathcal{V}^{\prime} \cap h_{0}^{s}, 0 \leq \tau_{0}, \tau \leq 1$, $\widehat{z} \in h_{0}^{s}$,

$$
\left\|\Psi_{X}^{\tau_{0}, \tau}(z)-z\right\|_{s+1} \lesssim_{s}\left\|z_{\perp}\right\|_{s}\left\|z_{\perp}\right\|_{0}, \quad\left\|\left(d \Psi_{X}^{\tau_{0}, \tau}(z)-I d\right)[\widehat{z}]\right\|_{s+1} \lesssim_{s}\left\|z_{\perp}\right\|_{s}\|\widehat{z}\|_{0}+\left\|z_{\perp}\right\|_{0}\|\widehat{z}\|_{s}
$$

and for any $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s}, l \geq 2$,

$$
\left\|d^{l} \Psi_{X}^{\tau_{0}, \tau}(z)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+1} \lesssim s \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s} \prod_{i \neq j}\left\|\widehat{\widetilde{z}}_{i}\right\|_{0}+\left\|z_{\perp}\right\|_{s} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
$$

Our aim is to derive expansions for the flow maps $\Psi_{X}^{\tau_{0}, \tau}(z)$. To this end we derive such expansions for $\mathcal{L}_{\tau}(z)^{-1}$ and in turn for the vector field $X(\tau, z)$. Recall that $\mathcal{L}_{\tau}(z)^{-1}$ is given by the Neumann series (3.6) and hence we first derive an expansion for the operators $(J \mathcal{L}(z))^{n}$. It is convenient to introduce the projections

$$
\begin{equation*}
\Pi_{S}: h_{S}^{0} \times h_{\perp}^{0} \rightarrow h_{S}^{0} \times h_{\perp}^{0},\left(\widehat{z}_{S}, \widehat{z}_{\perp}\right) \mapsto\left(\widehat{z}_{S}, 0\right), \quad \Pi_{\perp}: h_{S}^{0} \times h_{\perp}^{0} \rightarrow h_{S}^{0} \times h_{\perp}^{0},\left(\widehat{z}_{S}, \widehat{z}_{\perp}\right) \mapsto\left(0, \widehat{z}_{\perp}\right) \tag{3.9}
\end{equation*}
$$

and the maps

$$
\begin{equation*}
\pi_{S}: h_{S}^{0} \times h_{\perp}^{0} \rightarrow h_{S}^{0}, \quad z=\left(z_{S}, z_{\perp}\right) \mapsto z_{S}, \quad \pi_{\perp}: h_{S}^{0} \times h_{\perp}^{0} \rightarrow h_{\perp}^{0}, \quad z=\left(z_{S}, z_{\perp}\right) \mapsto z_{\perp} \tag{3.10}
\end{equation*}
$$

Furthermore, let $J_{S}:=\pi_{S} J \pi_{S}, J_{\perp}:=\pi_{\perp} J \pi_{\perp}$. Then $J_{S}^{-1}=\pi_{S} J^{-1} \pi_{S}, J_{\perp}^{-1}=\pi_{\perp} J^{-1} \pi_{\perp}$, or, more explicitly,

$$
\left\langle J_{S}^{-1} \widehat{z}_{S}, \widehat{w}_{S}\right\rangle=\sum_{n \in S} \frac{1}{\mathrm{i} 2 \pi n} \widehat{z}_{n} \widehat{w}_{-n}, \forall \widehat{z}_{S}, \widehat{w}_{S} \in h_{S}^{0}, \quad\left\langle J_{\perp}^{-1} \widehat{z}_{\perp}, \widehat{w}_{\perp}\right\rangle=\sum_{n \in S^{\perp}} \frac{1}{\mathrm{i} 2 \pi n} \widehat{z}_{n} \widehat{w}_{-n}, \quad \forall \widehat{z}_{\perp}, \widehat{w}_{\perp} \in h_{\perp}^{0}
$$

Lemma 3.5. For any $n \geq 1, z=\left(z_{S}, z_{\perp}\right) \in \mathcal{V},(J \mathcal{L}(z))^{n}$ has an expansion of arbitrary order $N \geq 1$,

$$
\left(\begin{array}{cc}
0 & 0 \\
\mathcal{F}_{\perp} \circ \sum_{k=1}^{N} \mathcal{A}_{k}^{S}\left(z ;(J \mathcal{L})^{n}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]
\end{array} \quad \mathcal{F}_{\perp} \circ \sum_{k=1}^{N} \mathcal{A}_{k}^{\perp}\left(z ;(J \mathcal{L})^{n}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N}\left(z ;(J \mathcal{L})^{n}\right)
$$

where for any integers $s \geq 0,1 \leq k \leq N$, the maps

$$
\begin{gathered}
\mathcal{V} \rightarrow \mathcal{B}\left(h_{S}^{0}, H^{s}\right), z \mapsto \mathcal{A}_{k}^{S}\left(z ;(J \mathcal{L})^{n}\right), \quad \mathcal{V} \rightarrow \mathcal{B}\left(h_{\perp}^{0}, H^{s}\right), z \mapsto \mathcal{A}_{k}^{\perp}\left(z ;(J \mathcal{L})^{n}\right), \\
\mathcal{V} \cap h_{0}^{s} \rightarrow \mathcal{B}\left(h_{0}^{0}, h_{0}^{s+N+1}\right), z \mapsto \mathcal{R}_{N}\left(z ;(J \mathcal{L})^{n}\right)
\end{gathered}
$$

are real analytic. For any $z=\left(z_{S}, z_{\perp}\right) \in \mathcal{V}, \widehat{z}=\left(\widehat{z}_{S}, \widehat{z}_{\perp}\right) \in h_{0}^{0}$, $\mathcal{A}_{k}^{S}\left(z ;(J \mathcal{L})^{n}\right)\left[\widehat{z}_{S}\right]$ and $\mathcal{A}_{k}^{\perp}\left(z ;(J \mathcal{L})^{n}\right)\left[\widehat{z}_{\perp}\right]$ satisfy the estimates

$$
\begin{align*}
& \left\|\mathcal{A}_{k}^{S}\left(z ;(J \mathcal{L})^{n}\right)\left[\widehat{z}_{S}\right]\right\|_{s} \lesssim_{s, k}\left\|\widehat{z}_{S}\right\|\left(C(k)\left\|z_{\perp}\right\|_{0}\right)^{n-1}, \\
& \left\|\mathcal{A}_{k}^{\perp}\left(z ;(J \mathcal{L})^{n}\right)\left[\widehat{z}_{\perp}\right]\right\|_{s} \lesssim_{s, k}\left\|\widehat{z}_{\perp}\right\|_{0}\left(C(k)\left\|z_{\perp}\right\|_{0}\right)^{n-1} \tag{3.11}
\end{align*}
$$

whereas for any integer $s \geq 0, z=\left(z_{S}, z_{\perp}\right) \in \mathcal{V} \cap h_{0}^{s}, \widehat{z} \in h_{0}^{0}$

$$
\begin{equation*}
\left\|\mathcal{R}_{N}\left(z ;(J \mathcal{L})^{n}\right)[\widehat{z}]\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}\|\widehat{z}\|_{0}\left(C_{0}(N)\left\|z_{\perp}\right\|_{0}\right)^{n-1} \tag{3.12}
\end{equation*}
$$

for some constants $C(k), C_{0}(N) \geq 1$. Furthermore, the following estimates hold for the derivatives of these maps: for $k, l \geq 1$, there exists a constant $C(k, l) \geq 1$, so that for any $z_{S} \in \mathcal{V}_{S}, \widehat{z}_{S} \in h_{S}^{0}, \widehat{z}_{\perp} \in h_{\perp}^{0}$, $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{0}$,

$$
\begin{aligned}
& \left\|d^{l}\left(\mathcal{A}_{k}^{S}\left(z ;(J \mathcal{L})^{n}\right)\left[\widehat{z}_{S}\right]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s} \lesssim_{s, l, k}\left\|\widehat{z}_{S}\right\|\left(\prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}\right)\left(C(k, l)\left\|z_{\perp}\right\|_{0}\right)^{0 \vee(n-1-l)}, \\
& \left\|d^{l}\left(\mathcal{A}_{k}^{\perp}\left(z ;(J \mathcal{L})^{n}\right)\left[\widehat{z}_{\perp}\right]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s} \lesssim_{s, l, k}\left\|\widehat{z}_{\perp}\right\|_{0}\left(\prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}\right)\left(C(k, l)\left\|z_{\perp}\right\|_{0}\right)^{0 \vee(n-1-l)} .
\end{aligned}
$$

Finally, there exist constants $C_{0}(N, l) \geq 1, l \geq 1$, so that for any $z \in \mathcal{V} \cap h_{0}^{s}, \widehat{z} \in h_{0}^{0}, \widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s}$,

$$
\begin{aligned}
&\left\|d^{l}\left(\mathcal{R}_{N}\left(z ;(J \mathcal{L})^{n}\right)[\widehat{z}]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+N+1} \\
& \quad \lesssim_{s, l, N}\|\widehat{z}\|_{0}\left(\sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{0}+\left\|z_{\perp}\right\|_{s} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}\right)\left(C_{0}(N, l)\left\|z_{\perp}\right\|_{0}\right)^{0 \vee(n-l)} .
\end{aligned}
$$

We refer to Remark 2.2 where we comment on the notation introduced for the coefficients in such expansions. Furthermore, we recall that $\mathcal{V}_{\perp}$ denotes the open ball in $h_{\perp}^{0}$, centered at 0 , whose radius is smaller than one and downscaled according to our needs.

Proof. We begin by proving the claimed statements for $n=1$. By (2.21), the operator $\mathcal{L}(z)$, $z \in \mathcal{V}$, can be written as

$$
\mathcal{L}(z)=\left(\begin{array}{cc}
0 & 0 \\
\mathcal{L}_{\perp}^{S}(z) & 0
\end{array}\right)+\left(\begin{array}{cc}
\mathcal{L}_{S}^{S}(z) & \mathcal{L} \frac{1}{S}(z) \\
0 & 0
\end{array}\right)
$$

Using that $J_{\perp} \circ \mathcal{F}_{\perp}=\mathcal{F}_{\perp} \circ \partial_{x}$, it then follows from Lemma 2.1 that

$$
J \mathcal{L}(z)=\left(\begin{array}{cc}
0 & 0  \tag{3.13}\\
\mathcal{F}_{\perp} \circ \sum_{k=1}^{N} \mathcal{A}_{k}^{S}\left(z_{S} ; J \mathcal{L}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right] & 0
\end{array}\right)+\mathcal{R}_{N}(z ; J \mathcal{L})
$$

where $\mathcal{A}_{k}^{S}\left(z_{S} ; J \mathcal{L}\right)$ and $\mathcal{R}_{N}(z ; J \mathcal{L})$ are obtained from $\mathcal{A}_{k}\left(z_{S} ; \mathcal{L}_{\perp}^{S}\right)$ and $\mathcal{R}_{N}\left(z ; \mathcal{L}_{\perp}^{S}\right)$, given by Lemma 2.1 in a straightforward way. It then follows that for any $s \geq 0$ and $1 \leq k \leq N$, the maps

$$
\mathcal{V}_{S} \rightarrow \mathcal{B}\left(h_{S}^{0}, H^{s}\right), z_{S} \mapsto \mathcal{A}_{k}^{S}\left(z_{S} ; J \mathcal{L}\right), \quad \mathcal{V} \cap h_{0}^{s} \rightarrow \mathcal{B}\left(h_{0}^{0}, h_{0}^{s+N+1}\right), z \mapsto \mathcal{R}_{N}(z ; J \mathcal{L})
$$

are real analytic and that $\mathcal{R}_{N}(z ; J \mathcal{L})$ is of the form

$$
\mathcal{R}_{N}(z ; J \mathcal{L})=\left(\begin{array}{cc}
\mathcal{R}_{N}(z ; J \mathcal{L})_{S}^{S} & \mathcal{R}_{N}(z ; J \mathcal{L})_{S}^{\perp} \\
\mathcal{R}_{N}(z ; J \mathcal{L})_{\perp}^{S} & 0
\end{array}\right)
$$

For any integer $s \geq 0, z_{S} \in \mathcal{V}_{S}, \widehat{z}_{S} \in h_{S}^{0}, \widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{0}, l \geq 1$, one has

$$
\begin{equation*}
\left\|\mathcal{A}_{k}^{S}\left(z_{S} ; J \mathcal{L}\right)\left[\widehat{z}_{S}\right]\right\|_{s} \lesssim_{s, k}\left\|\widehat{z}_{S}\right\|, \quad\left\|d^{l}\left(\mathcal{A}_{k}^{S}\left(z_{S} ; J \mathcal{L}\right)\left[\widehat{z}_{S}\right]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s} \lesssim_{s, k, l}\left\|\widehat{z}_{S}\right\| \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0} \tag{3.14}
\end{equation*}
$$

Furthermore, for any integer $s \geq 0, z=\left(z_{S}, z_{\perp}\right) \in \mathcal{V} \cap h_{0}^{s}, \widehat{z} \in h_{0}^{0}, \widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s}, l \geq 1$, the remainder term satisfies $\left\|\mathcal{R}_{N}(z ; J \mathcal{L})[\widehat{z}]\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}\|\widehat{z}\|_{0}$ and

$$
\begin{equation*}
\left\|d^{l}\left(\mathcal{R}_{N}(z ; J \mathcal{L})[\widehat{z}]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+N+1} \lesssim_{s, N, l}\|\widehat{z}\|_{0}\left(\sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{0}+\left\|z_{\perp}\right\|_{s} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}\right) \tag{3.15}
\end{equation*}
$$

To prove the claimed statements for $n+1, n \geq 1$, write $(J \mathcal{L}(z))^{n+1}=J \mathcal{L}(z)(J \mathcal{L}(z))^{n}$. By the expansion (3.13) it follows that $(J \mathcal{L}(z))^{n+1}$ is of the form

$$
\left(\begin{array}{cc}
0 & 0  \tag{3.16}\\
\mathcal{F}_{\perp} \circ \sum_{k=1}^{N} \mathcal{A}_{k}^{S}(z) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right] & \mathcal{F}_{\perp} \circ \sum_{k=1}^{N} \mathcal{A}_{k}^{\perp}(z) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]
\end{array}\right)+\mathcal{R}_{N}(z)
$$

where $\mathcal{A}_{k}^{S}(z) \equiv \mathcal{A}_{k}^{S}\left(z ;(J \mathcal{L})^{n+1}\right), \mathcal{A}_{k}^{\perp}(z) \equiv \mathcal{A}_{k}^{\perp}\left(z ;(J \mathcal{L})^{n+1}\right)$, and $\mathcal{R}_{N}(z) \equiv \mathcal{R}_{N}\left(z ;(J \mathcal{L})^{n+1}\right)$ are given by

$$
\begin{gathered}
\left.\mathcal{A}_{k}^{S}\left(z ;(J \mathcal{L})^{n+1}\right):=\mathcal{A}_{k}^{S}\left(z_{S} ; J \mathcal{L}\right) \circ(J \mathcal{L}(z))^{n}\right)_{S}^{S}, \quad \mathcal{A}_{k}^{\perp}\left(z ;(J \mathcal{L})^{n+1}\right):=\mathcal{A}_{k}^{S}\left(z_{S} ; J \mathcal{L}\right) \circ\left((J \mathcal{L}(z))^{n}\right)_{S}^{\perp}, \\
\mathcal{R}_{N}\left(z ;(J \mathcal{L})^{n+1}\right):=\mathcal{R}_{N}(z ; J \mathcal{L}) \circ(J \mathcal{L}(z))^{n} .
\end{gathered}
$$

It then follows that these maps are real analytic as claimed in the statement of the lemma. Furthermore, for any $s \geq 0, z=\left(z_{S}, z_{\perp}\right) \in \mathcal{V}, \widehat{z}_{S} \in h_{S}^{0}$ one has by (3.14)

$$
\left\|\mathcal{A}_{k}^{S}\left(z ;(J \mathcal{L})^{n+1}\right)\left[\widehat{z}_{S}\right]\right\|_{s} \lesssim_{s, k}\left\|\left((J \mathcal{L}(z))^{n}\right)_{S}^{S}\left[\widehat{z}_{S}\right]\right\| \lesssim_{s, k}\left(C_{0}\left\|z_{\perp}\right\|_{0}\right)^{n}\left\|\widehat{z}_{S}\right\|
$$

where $C_{0}:=C(0 ; \mathcal{L})$ is given by (2.25). Similarly, again by (3.14), for any $z=\left(z_{S}, z_{\perp}\right) \in \mathcal{V}, \widehat{z}_{\perp} \in h_{\perp}^{0}$ one has

$$
\left\|\mathcal{A}_{k}^{\perp}\left(z ;(J \mathcal{L})^{n+1}\right)\left[\widehat{z}_{\perp}\right]\right\|_{s} \lesssim_{s, k}\left\|\left((J \mathcal{L}(z))^{n}\right)_{S}^{\perp}\left[\widehat{z}_{\perp}\right]\right\| \lesssim_{s, k}\left(C_{0}\left\|z_{\perp}\right\|_{0}\right)^{n}\left\|\widehat{z}_{\perp}\right\|_{0},
$$

whereas for any $s \geq 0, z=\left(z_{S}, z_{\perp}\right) \in \mathcal{V} \cap h_{0}^{s}, \widehat{z}=\left(\widehat{z}_{S}, \widehat{z}_{\perp}\right) \in h_{0}^{0}$

$$
\left\|\mathcal{R}_{N}\left(z ;(J \mathcal{L})^{n+1}\right)[\widehat{z}]\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}\left\|(J \mathcal{L}(z))^{n}[\widehat{z}]\right\|_{0} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}\left(C_{0}\left\|z_{\perp}\right\|_{0}\right)^{n}\|\widehat{z}\|_{0}
$$

Next we estimate the derivatives of $\mathcal{A}_{k}^{S}\left(z ;(J \mathcal{L})^{n+1}\right)\left[\widehat{z}_{S}\right]$. For any $s \geq 0, z_{S} \in \mathcal{V}_{S}, \widehat{z}_{S} \in h_{S}^{0}, \widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{0}$, $l \geq 1$, the estimate of $\left\|d^{l}\left(\mathcal{A}_{k}^{S}\left(z ;(J \mathcal{L})^{n+1}\right)\left[\widehat{z}_{S}\right]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s}$ is obtained from the estimates of

$$
\left\|d^{m}\left(\mathcal{A}_{k}^{S}(z ; J \mathcal{L})\left[\widehat{w}_{l-m}\right]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{m}\right]\right\|_{s}, \quad \widehat{w}_{l-m}:=d^{l-m}\left(\left((J \mathcal{L}(z))^{n}\right)_{S}^{S}\left[\widehat{z}_{S}\right]\right)\left[\widehat{z}_{m+1}, \ldots, \widehat{z}_{l}\right] \in h_{S}^{0}
$$

and $\left\|\widehat{w}_{l-m}\right\|$ where $0 \leq m \leq l$. By (3.14), one has

$$
\left\|d^{m}\left(\mathcal{A}_{k}^{S}(z ; J \mathcal{L})\left[\widehat{w}_{l-m}\right]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{m}\right]\right\|_{s} \lesssim_{s, m}\left\|\widehat{w}_{l-m}\right\| \prod_{j=1}^{m}\left\|\widehat{z}_{j}\right\|_{0}
$$

Note that we introduced the element $\widehat{w}_{l-m}$ to indicate that in $\left\|d^{m}\left(\mathcal{A}_{k}^{S}(z ; J \mathcal{L})\left[\widehat{w}_{l-m}\right]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{m}\right]\right\|_{s}$ the derivative $d^{m}$ does not act on $\widehat{w}_{l-m}$. Increasing the constant $C_{0}$ and/or decreasing the radius of the ball $\mathcal{V}_{\perp}$ depending on the size of $l$ it follows from (2.25) - (2.26) that

$$
\left\|\widehat{w}_{l-m}\right\| \lesssim_{s, l-m}\left\|\widehat{z}_{S}\right\|\left(\prod_{j=m+1}^{l}\left\|\widehat{z}_{j}\right\|_{0}\right)\left(C_{0}\left\|z_{\perp}\right\|_{0}\right)^{0 \vee(n-l+m)} .
$$

Combining these estimates implies that

$$
\left\|d^{l}\left(\mathcal{A}_{k}^{S}\left(z ;(J \mathcal{L})^{n+1}\right)\left[\widehat{z}_{S}\right]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s} \lesssim_{s, l, k}\left\|\widehat{z}_{S}\right\|\left(\prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}\right)\left(C_{0}\left\|z_{\perp}\right\|_{0}\right)^{0 \vee(n-l)}
$$

In the same way one shows that for any $s \geq 0, z_{S} \in \mathcal{V}_{S}, \widehat{z}_{\perp} \in h_{\perp}^{0}, \widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{0}, l \geq 1$,

$$
\left\|d^{l}\left(\mathcal{A}_{k}^{\perp}\left(z ;(J \mathcal{L})^{n+1}\right)\left[\widehat{z}_{\perp}\right]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s} \lesssim s, l, k\left\|\widehat{z}_{\perp}\right\|_{0}\left(\prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}\right)\left(C_{0}\left\|z_{\perp}\right\|_{0}\right)^{0 \vee(n-l)}
$$

Finally, the claimed estimate for $\left\|d^{l}\left(\mathcal{R}_{N}\left(z ;(J \mathcal{L})^{n}\right)[\widehat{z}]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z_{l}}\right]\right\|_{s+N+1}$ where $s \geq 0, z \in \mathcal{V} \cap h_{0}^{s}, \widehat{z} \in h_{0}^{0}$, $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s}, l \geq 1$, follows from the estimates of

$$
\left\|d^{m}\left(\mathcal{R}_{N}(z ; J \mathcal{L})\left[\widehat{v}_{l-m}\right]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{m}\right]\right\|_{s+N+1}, \quad \widehat{v}_{l-m}:=d^{l-m}\left((J \mathcal{L}(z))^{n}[\widehat{z}]\right)\left[\widehat{z}_{m+1}, \ldots, \widehat{z}_{l}\right]
$$

and $\left\|\widehat{v}_{l-m}\right\|_{0}$ where $0 \leq m \leq l$. Indeed, by (3.15),

$$
\left\|d^{m}\left(\mathcal{R}_{N}(z ; J \mathcal{L})\left[\widehat{v}_{l-m}\right]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{m}\right]\right\|_{s+N+1} \lesssim_{s, N, m}\left\|\widehat{v}_{l-m}\right\|_{0}\left(\sum_{j=1}^{m}\left\|\widehat{z}_{j}\right\|_{s} \prod_{i \neq j, 1 \leq i \leq m}\left\|\widehat{z}_{i}\right\|_{0}+\left\|z_{\perp}\right\|_{s} \prod_{j=1}^{m}\left\|\widehat{z}_{j}\right\|_{0}\right)
$$

Increasing the constant $C_{0}$ and/or decreasing the radius of the ball $\mathcal{V}_{\perp}$ depending on the size of $l$, it follows from (2.25) - (2.26) that

$$
\left\|\widehat{v}_{l-m}\right\|_{0} \leq\left\|d^{l-m}\left((J \mathcal{L}(z))^{n}[\widehat{z}]\right)\left[\widehat{z}_{m+1}, \ldots, \widehat{z}_{l}\right]\right\|_{0} \lesssim_{s, l-m}\|\widehat{z}\|_{0}\left(\prod_{j=m+1}^{l}\left\|\widehat{z}_{j}\right\|_{0}\right)\left(C_{0}\left\|z_{\perp}\right\|_{0}\right)^{0 \vee(n-l+m)}
$$

Combining these estimates yields the claimed estimate for $\left\|d^{l}\left(\mathcal{R}_{N}\left(z ;(J \mathcal{L})^{n}\right)[\widehat{z}]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+N+1}$, with constants chosen appropriately.

Recall that $\mathcal{V}_{\perp}$ denotes the open ball in $h_{\perp}^{0}$, centered at 0 , whose radius is smaller than one and downscaled at several instances in the course of our analysis.

Lemma 3.6. For any $N \geq 1, X(\tau, z)=-\mathcal{L}_{\tau}(z)^{-1}[J \mathcal{E}(z)](0 \leq \tau \leq 1, z \in \mathcal{V})$ has an expansion of the form

$$
\begin{equation*}
X(\tau, z)=\left(0, \mathcal{F}_{\perp} \circ \sum_{k=1}^{N} a_{k}(\tau, z ; X) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N}(\tau, z ; X) \tag{3.17}
\end{equation*}
$$

where for any $s \geq 0$ and $k \geq 1$, the maps

$$
[0,1] \times \mathcal{V} \rightarrow H^{s},(\tau, z) \mapsto a_{k}(\tau, z ; X), \quad[0,1] \times\left(\mathcal{V} \cap h_{0}^{s}\right) \rightarrow h_{0}^{s+N+1},(\tau, z) \mapsto \mathcal{R}_{N}(\tau, z ; X)
$$

are real analytic. Furthermore, for any $0 \leq \tau \leq 1, z \in \mathcal{V}, \widehat{z} \in h_{0}^{0}$,

$$
\left\|a_{k}(\tau, z ; X)\right\|_{s} \lesssim_{s, k}\left\|z_{\perp}\right\|_{0}^{2}, \quad\left\|d a_{k}(\tau, z ; X)[\widehat{z}]\right\|_{s} \lesssim_{s, k}\left\|z_{\perp}\right\|_{0}\|\widehat{z}\|_{0}
$$

and for any $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{0}, l \geq 2$,

$$
\left\|d^{l} a_{k}(\tau, z ; X)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s} \lesssim_{s, k, l} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
$$

For any $z \in \mathcal{V} \cap h_{0}^{s}, 0 \leq \tau \leq 1, \widehat{z} \in h_{0}^{s}$, the remainder term $\mathcal{R}_{N}(\tau, z ; X)$ satisfies

$$
\begin{gathered}
\left\|\mathcal{R}_{N}(\tau, z ; X)\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{0}^{2}+\left\|z_{\perp}\right\|_{0}^{2}\left\|z_{\perp}\right\|_{s} \lesssim_{s, N}\left\|z_{\perp}\right\|_{0}\left\|z_{\perp}\right\|_{s} \\
\left\|d \mathcal{R}_{N}(\tau, z ; X)[\widehat{z}]\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{0}^{2}\|\widehat{z}\|_{s}+\left\|z_{\perp}\right\|_{0}\|\widehat{z}\|_{0}\left(1+\left\|z_{\perp}\right\|_{s}\right) \lesssim_{s, N}\left\|z_{\perp}\right\|_{0}\left(\|\widehat{z}\|_{s}+\left\|z_{\perp}\right\|_{s}\|\widehat{z}\|_{0}\right)
\end{gathered}
$$

whereas for any $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s}, l \geq 2$,

$$
\left\|d^{l} \mathcal{R}_{N}(\tau, z ; X)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+N+1} \lesssim_{s, N, l} \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{0}+\left\|z_{\perp}\right\|_{s} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
$$

Proof. By the Neumann series expansion, one has for any $z \in \mathcal{V}, 0 \leq \tau \leq 1$,

$$
\begin{equation*}
-\mathcal{L}_{\tau}(z)^{-1}[J \mathcal{E}(z)]=-J \mathcal{E}(z)+\sum_{n \geq 1}(-1)^{n+1} \tau^{n}(J \mathcal{L}(z))^{n}[J \mathcal{E}(z)] \tag{3.18}
\end{equation*}
$$

Since $J \mathcal{E}(z)=\left(J_{S} \mathcal{E}_{S}(z), 0\right)$, Lemma 3.5 yields that for any $n \geq 1$,

$$
(-1)^{n+1} \tau^{n}(J \mathcal{L}(z))^{n}[J \mathcal{E}(z)]=\left(0, \mathcal{F}_{\perp} \circ \sum_{k=1}^{N} a_{k}\left(\tau, z ;(J \mathcal{L})^{n}[J \mathcal{E}]\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N}\left(\tau, z ;(J \mathcal{L})^{n}[J \mathcal{E}]\right)
$$

where

$$
\begin{aligned}
& a_{k}\left(\tau, z ;(J \mathcal{L})^{n}[J \mathcal{E}]\right):=(-1)^{n+1} \tau^{n} \mathcal{A}_{k}^{S}\left(z ;(J \mathcal{L})^{n}\right)\left[J_{S} \mathcal{E}_{S}(z)\right] \\
& \mathcal{R}_{N}\left(\tau, z ;(J \mathcal{L})^{n}[J \mathcal{E}]\right):=(-1)^{n+1} \tau^{n} \mathcal{R}_{N}\left(z ;(J \mathcal{L})^{n}\right)[J \mathcal{E}(z)]
\end{aligned}
$$

By applying the estimates of Lemma 3.2 and Lemma 3.5, one gets for any $s \geq 0, z \in \mathcal{V}, 0 \leq \tau \leq 1$,

$$
\begin{equation*}
\left\|a_{k}\left(\tau, z ;(J \mathcal{L})^{n}[J \mathcal{E}]\right)\right\|_{s} \lesssim_{s, k}\left\|z_{\perp}\right\|_{0}^{2}\left(C(k)\left\|z_{\perp}\right\|_{0}\right)^{n-1} \tag{3.19}
\end{equation*}
$$

and for any $s \geq 0, z \in \mathcal{V} \cap h_{0}^{s}, 0 \leq \tau \leq 1$,

$$
\begin{equation*}
\left\|\mathcal{R}_{N}\left(\tau, z ;(J \mathcal{L})^{n}[J \mathcal{E}]\right)\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{0}^{2}\left\|z_{\perp}\right\|_{s}\left(C_{0}(N)\left\|z_{\perp}\right\|_{0}\right)^{n-1} \tag{3.20}
\end{equation*}
$$

In view of (3.18) we define for any $1 \leq k \leq N$,

$$
\begin{equation*}
a_{k}(\tau, z ; X):=\sum_{n \geq 1} a_{k}\left(\tau, z ;(J \mathcal{L})^{n}[J \mathcal{E}]\right), \quad \mathcal{R}_{N}(\tau, z ; X):=-J \mathcal{E}(z)+\sum_{n \geq 1} \mathcal{R}_{N}\left(\tau, z ;(J \mathcal{L})^{n}[J \mathcal{E}]\right) \tag{3.21}
\end{equation*}
$$

Shrinking the radius of the ball $\mathcal{V}_{\perp}$, if needed, one can assume that $C(k)\left\|z_{\perp}\right\|_{0}<1, C_{0}(N)\left\|z_{\perp}\right\|_{0}<1$ for any $1 \leq k \leq N, z_{\perp} \in \mathcal{V}_{\perp}$. The expansion (3.17), the analyticity statement, and the claimed estimates for $\left\|a_{k}(\tau, z ; X)\right\|_{s}$ and $\left\|\mathcal{R}_{N}(\tau, z ; X)\right\|_{s+N+1}$ then follow from the estimates (3.19), (3.20), and Lemma 3.2, Lemma 3.5. The estimates for the derivatives of $a_{k}(\tau, z ; X)$ and $\mathcal{R}_{N}(\tau, z ; X)$ follow by similar arguments. Indeed, it follows from Lemma 3.5 that for any for $1 \leq k \leq N, z \in \mathcal{V}, 0 \leq \tau \leq 1, \widehat{z} \in h_{0}^{0}$,

$$
\left.\| d a_{k}\left(\tau, z ;(J \mathcal{L})^{n}[J \mathcal{E}]\right)\right)[\widehat{z}]\left\|_{s} \lesssim_{s, k}\right\| d\left(\mathcal{A}_{k}^{S}\left(z ;(J \mathcal{L})^{n}\right)[\widehat{w}]\right)[\widehat{z}]\left\|\left._{s}\right|_{\widehat{w}=J_{S} \mathcal{E}_{S}(z)}+\right\| \mathcal{A}_{k}^{S}\left(z ;(J \mathcal{L})^{n}\right)\left[d\left(J_{S} \mathcal{E}_{S}(z)\right)[\widehat{z}]\right] \|_{s}
$$

Using that by Lemma 3.2, $\left\|J_{S} \mathcal{E}_{S}(z)\right\| \lesssim\left\|z_{\perp}\right\|_{0}^{2}$ and $\left\|d\left(J_{S} \mathcal{E}_{S}(z)\right)[\widehat{z}]\right\|_{0} \lesssim\left\|z_{\perp}\right\|_{0}\|\widehat{z}\|_{0}$ one then concludes from Lemma 3.5

$$
\left.\left\|d\left(\mathcal{A}_{k}^{S}\left(z ;(J \mathcal{L})^{n}\right)[\widehat{w}]\right)[\widehat{z}]\right\|_{s}\right|_{\widehat{w}=J_{S} \mathcal{E}_{S}(z)} \lesssim s, k\left\|z_{\perp}\right\|_{0}^{2}\|\widehat{z}\|_{0}\left(C(k, 1)\left\|z_{\perp}\right\|_{0}\right)^{0 \vee(n-2)}
$$

and

$$
\left\|\mathcal{A}_{k}^{S}\left(z ;(J \mathcal{L})^{n}\right)\left[d\left(J_{S} \mathcal{E}_{S}(z)\right)[\widehat{z}]\right]\right\|_{s} \lesssim_{s, k}\left\|z_{\perp}\right\|_{0}\|\widehat{z}\|_{0}\left(C(k, 1)\left\|z_{\perp}\right\|_{0}\right)^{n-1} .
$$

In view of the definition (3.21) of $a_{k}(\tau, z ; X)$ one then obtains the claimed estimate $\left\|d a_{k}(\tau, z ; X)[\widehat{z}]\right\|_{s} \lesssim_{s, k}$ $\left\|z_{\perp}\right\|_{0}\|\widehat{z}\|_{0}$. The estimates for $\left\|d^{l} a_{k}(\tau, z ; X)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s}$ with $l \geq 2$ are derived in a similar fashion. Finally let us consider the estimates of the derivatives of the remainder term. For any $n \geq 1, z \in \mathcal{V} \cap h_{0}^{s}, 0 \leq \tau \leq 1$, $\widehat{z} \in h_{0}^{s}$, it follows from Lemma 3.5 and the product rule that

$$
\begin{aligned}
\left\|d\left(\mathcal{R}_{N}\left(z ;(J \mathcal{L})^{n}\right)[J \mathcal{E}(z)]\right)[\widehat{z}]\right\|_{s+N+1} \lesssim_{s, N} \| & \|\mathcal{E}(z)\|_{0}\left(\|\widehat{z}\|_{s}+\left\|z_{\perp}\right\|_{s}\|\widehat{z}\|_{0}\right)\left(C_{0}(N)\left\|z_{\perp}\right\|_{0}\right)^{n-1} \\
& +\|d J \mathcal{E}(z)[\widehat{z}]\|_{0} \mid\|z\|_{s}\left(C_{0}(N)\left\|z_{\perp}\right\|_{0}\right)^{n-1}
\end{aligned}
$$

Using again that by Lemma $3.2,\|J \mathcal{E}(z)\|_{0} \lesssim\left\|z_{\perp}\right\|_{0}^{2}$ and $\|d J \mathcal{E}(z)[\widehat{z}]\|_{0} \lesssim\left\|z_{\perp}\right\|_{0}\|\widehat{z}\|_{0}$ and taking into account the definition (3.21) of $\mathcal{R}_{N}(\tau, z ; X)$ one sees that

$$
\left\|d \mathcal{R}_{N}(\tau, z ; X)[\widehat{z}]\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{0}\|\widehat{z}\|_{0}+\left\|z_{\perp}\right\|_{0}^{2}\left(\|\widehat{z}\|_{s}+\left\|z_{\perp}\right\|_{s}\|\widehat{z}\|_{0}\right)+\left\|z_{\perp}\right\|_{0}\left\|z_{\perp}\right\|_{s}\|\widehat{z}\|_{0},
$$

yielding the claimed estimate for $\left\|d \mathcal{R}_{N}(\tau, z ; X)[\widehat{z}]\right\|_{s+N+1}$. The ones for $\left\|d^{l} \mathcal{R}_{N}(\tau, z ; X)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+N+1}$ with $l \geq 2$ are derived in a similar fashion.

After these preliminary considerations we can now state the main result of this section, saying that for any $\tau_{0}, \tau \in[0,1]$, the flow map $\Psi_{X}^{\tau_{0}, \tau}$, defined on $\mathcal{V}^{\prime}$ and with values in $\mathcal{V}$, admits an expansion, referred to as parametrix for the solution of the initial value problem of $\partial_{\tau} z=X(\tau, z)$.

Theorem 3.1. (i) For any $\tau_{0}, \tau \in[0,1], N \in \mathbb{N}$, and $z=\left(z_{S}, z_{\perp}\right) \in \mathcal{V}^{\prime}$,

$$
\begin{equation*}
\Psi_{X}^{\tau_{0}, \tau}(z)=\left(z_{S}, z_{\perp}\right)+\left(0, \mathcal{F}_{\perp} \circ \sum_{k=1}^{N} a_{k}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right) \tag{3.22}
\end{equation*}
$$

where for any $\tau_{0}, \tau \in[0,1], 1 \leq k \leq N$, and $s \geq 0$, the maps

$$
\mathcal{V}^{\prime} \rightarrow H^{s}, z \mapsto a_{k}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right), \quad \mathcal{V}^{\prime} \cap h_{0}^{s} \rightarrow h_{0}^{s+N+1}, z \mapsto \mathcal{R}_{N}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right)
$$

are real analytic. Furthermore, for any $z \in \mathcal{V}^{\prime}, \widehat{z} \in h_{0}^{0}$,

$$
\left\|a_{k}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right)\right\|_{s} \lesssim s, k^{\|} z_{\perp}\left\|_{0}^{2}, \quad\right\| d a_{k}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right)[\widehat{z}]\left\|_{s} \lesssim_{s, k}\right\| z_{\perp}\left\|_{0}\right\| \widehat{z} \|_{0}
$$

and for any $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{0}, l \geq 2$,

$$
\left\|d^{l} a_{k}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s} \lesssim_{s, k, l} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
$$

The remainder term satisfies the following estimates: for any $z \in \mathcal{V}^{\prime} \cap h_{0}^{s}, \widehat{z} \in h_{0}^{s}$

$$
\left\|\mathcal{R}_{N}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right)\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}\left\|z_{\perp}\right\|_{0}, \quad\left\|d \mathcal{R}_{N}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right)[\widehat{z}]\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}\|\widehat{z}\|_{0}+\left\|z_{\perp}\right\|_{0}\|\widehat{z}\|_{s}
$$

and for any $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s}, l \geq 2$,

$$
\left\|d^{l} \mathcal{R}_{N}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+N+1} \lesssim_{s, N, l} \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{0}+\left\|z_{\perp}\right\|_{s} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
$$

(ii) In particular, the statements of item (i) hold for $\Psi_{C}:=\Psi_{X}^{0,1}: \mathcal{V}^{\prime} \rightarrow \Psi_{X}^{0,1}\left(\mathcal{V}^{\prime}\right)$, referred to as symplectic corrector, and $\Psi_{X}^{1,0}: \mathcal{V}^{\prime} \rightarrow \Psi_{X}^{1,0}\left(\mathcal{V}^{\prime}\right)$, which by a slight abuse of terminology with respect to its domain of definition we refer to as the inverse of $\Psi_{C}$ and denote by $\Psi_{C}^{-1}$. The expansion of the map $\Psi_{C}$ is then written as $\left(z \in \mathcal{V}^{\prime}\right)$

$$
\Psi_{C}(z)=z+\left(0, \mathcal{F}_{\perp} \circ \sum_{k=1}^{N} a_{k}\left(z ; \Psi_{C}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N}\left(z ; \Psi_{C}\right)
$$

where

$$
a_{k}\left(z ; \Psi_{C}\right):=a_{k}\left(z ; \Psi_{X}^{0,1}\right), \quad \mathcal{R}_{N}\left(z ; \Psi_{C}\right):=\mathcal{R}_{N}\left(z ; \Psi_{X}^{0,1}\right)
$$

Similarly, the expansion for the inverse $\Psi_{C}^{-1}(z), z \in \mathcal{V}^{\prime}$, is written as

$$
\Psi_{C}(z)^{-1}=z+\left(0, \mathcal{F}_{\perp} \circ \sum_{k=1}^{N} a_{k}\left(z ; \Psi_{C}^{-1}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N}\left(z ; \Psi_{C}^{-1}\right)
$$

where

$$
a_{k}\left(z ; \Psi_{C}^{-1}\right):=a_{k}\left(z ; \Psi_{X}^{1,0}\right), \quad \mathcal{R}_{N}\left(z ; \Psi_{C}^{-1}\right):=\mathcal{R}_{N}\left(z ; \Psi_{X}^{1,0}\right)
$$

Proof. Clearly, item (ii) is a direct consequence of (i). Since the proof of item (i) is quite lengthy, we divide it up into several steps. First note that the flow map $\Psi^{\tau_{0}, \tau} \equiv \Psi_{X}^{\tau_{0}, \tau}$ is a bounded nonlinear operator acting on $\mathcal{V}^{\prime} \cap h_{0}^{s}, s \geq 0$, satisfying the integral equation

$$
\begin{equation*}
\Psi^{\tau_{0}, \tau}(z)=z+\int_{\tau_{0}}^{\tau} X\left(t, \Psi^{\tau_{0}, t}(z)\right) d t \tag{3.23}
\end{equation*}
$$

Using the latter equation, the coefficients $a_{k}\left(z ; \Psi^{\tau_{0}, \tau}\right), k \geq 1$, and the remainder term $\mathcal{R}_{N}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right)$ of the parametrix (3.22) are determined inductively. By (3.17), one obtains for any $0 \leq \tau_{0}, t \leq 1, z \in \mathcal{V}^{\prime}$,

$$
\begin{equation*}
X\left(t, \Psi^{\tau_{0}, t}(z)\right)=\left(0, \mathcal{F}_{\perp} \circ \sum_{k=1}^{N} a_{k}\left(t, \Psi^{\tau_{0}, t}(z) ; X\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[\pi_{\perp} \Psi^{\tau_{0}, t}(z)\right]\right)+\mathcal{R}_{N}\left(t, \Psi^{\tau_{0}, t}(z) ; X\right) \tag{3.24}
\end{equation*}
$$

Expansion of $\partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[\pi_{\perp} \Psi^{\tau_{0}, t}(z)\right], 1 \leq k \leq N$ : To find candidates for the coeffcients $a_{k}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right)$ we argue formally and substitute the expansion (3.22) into the expression $\partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[\pi_{\perp} \Psi^{\tau_{0}, t}(z)\right]$ yielding

$$
\begin{align*}
\partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[\pi_{\perp} \Psi^{\tau_{0}, t}(z)\right] & =\partial_{x}^{-k}\left(\mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]+\sum_{j=1}^{N} a_{j}\left(z ; \Psi^{\tau_{0}, t}\right) \partial_{x}^{-j} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]+\mathcal{F}_{\perp}^{-1} \pi_{\perp} \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, t}\right)\right) \\
& =\partial_{x}^{-k}\left(\mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]+\sum_{j=1}^{N-k} a_{j}\left(z ; \Psi^{\tau_{0}, t}\right) \partial_{x}^{-j} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N, k}^{(1)}\left(t, z ; \tau_{0}\right) \tag{3.25}
\end{align*}
$$

where

$$
\begin{equation*}
\mathcal{R}_{N, k}^{(1)}\left(t, z ; \tau_{0}\right):=\partial_{x}^{-k}\left(\sum_{j=N-k+1}^{N} a_{j}\left(z ; \Psi^{\tau_{0}, t}\right) \partial_{x}^{-j} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]+\mathcal{F}_{\perp}^{-1} \pi_{\perp} \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, t}\right)\right) . \tag{3.26}
\end{equation*}
$$

Using Lemma E.2(i) and the notation established there, one has

$$
\partial_{x}^{-k}\left(a_{j}\left(z ; \Psi^{\tau_{0}, t}\right) \partial_{x}^{-j} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)=\sum_{i=0}^{N-k-j} C_{i}(k, j)\left(\partial_{x}^{i} a_{j}\left(z ; \Psi^{\tau_{0}, t}\right)\right) \partial_{x}^{-k-j-i} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]+\mathcal{R}_{N, k, j}^{(2)}\left(t, z ; \tau_{0}\right)
$$

where

$$
\begin{equation*}
\mathcal{R}_{N, k, j}^{(2)}\left(t, z ; \tau_{0}\right):=\mathcal{R}_{N, k, j}^{\psi d o}\left(a_{j}\left(z ; \Psi^{\tau_{0}, t}\right)\right) \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right] \tag{3.27}
\end{equation*}
$$

By Lemma E. 2 for any $z \in \mathcal{V}^{\prime} \cap h_{0}^{s}, s \geq 0,0 \leq t, \tau_{0} \leq 1,1 \leq j \leq N$

$$
\begin{equation*}
\left\|\mathcal{R}_{N, k, j}^{(2)}\left(t, z ; \tau_{0}\right)\right\|_{s+N+1} \lesssim_{s, N} \max _{1 \leq i \leq N}\left\|a_{i}\left(z ; \Psi^{\tau_{0}, t}\right)\right\|_{s+2 N}\left\|z_{\perp}\right\|_{s} \tag{3.28}
\end{equation*}
$$

Hence, (3.25) reads

$$
\partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[\pi_{\perp} \Psi^{\tau_{0}, t}(z)\right]=\partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]+\sum_{j=1}^{N-k} \sum_{i=0}^{N-k-j} C_{i}(k, j)\left(\partial_{x}^{i} a_{j}\left(z ; \Psi^{\tau_{0}, t}\right)\right) \partial_{x}^{-k-j-i} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]+\mathcal{R}_{N, k}^{(3)}\left(t, z ; \tau_{0}\right)
$$

where

$$
\begin{equation*}
\mathcal{R}_{N, k}^{(3)}\left(t, z ; \tau_{0}\right):=\mathcal{R}_{N, k}^{(1)}\left(t, z ; \tau_{0}\right)+\sum_{j=1}^{N-k} \mathcal{R}_{N, k, j}^{(2)}\left(t, z ; \tau_{0}\right) . \tag{3.29}
\end{equation*}
$$

Changing in the double sum $\sum_{j=1}^{N-k} \sum_{i=0}^{N-k-j}$ the index $i$ of summation to $n:=i+j$ and then interchanging the order of summation, one obtains

$$
\sum_{j=1}^{N-k} \sum_{i=0}^{N-k-j} C_{i}(k, j)\left(\partial_{x}^{i} a_{j}\right) \partial_{x}^{-k-j-i}=\sum_{n=1}^{N-k} \sum_{j=1}^{n} C_{n-j}(k, j)\left(\partial_{x}^{n-j} a_{j}\right) \partial_{x}^{-k-n}
$$

implying that $\partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[\pi_{\perp} \Psi^{\tau_{0}, t}(z)\right]$ equals

$$
\begin{equation*}
\partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]+\sum_{n=1}^{N-k}\left(\sum_{j=1}^{n} C_{n-j}(k, j)\left(\partial_{x}^{n-j} a_{j}\left(z ; \Psi^{\tau_{0}, t}\right)\right)\right) \partial_{x}^{-k-n} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]+\mathcal{R}_{N, k}^{(3)}\left(t, z ; \tau_{0}\right) \tag{3.30}
\end{equation*}
$$

Expansion of $\sum_{k=1}^{N} a_{k}\left(t, \Psi^{\tau_{0}, t}(z) ; X\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[\pi_{\perp} \Psi^{\tau_{0}, t}(z)\right]$ : To simplify notation, introduce

$$
\begin{equation*}
a_{k}\left(t, z ; \tau_{0}\right):=a_{k}\left(t, \Psi^{\tau_{0}, t}(z) ; X\right) \tag{3.31}
\end{equation*}
$$

and then substitute (3.30) into $\sum_{k=1}^{N} a_{k}\left(t, \Psi^{\tau_{0}, t}(z) ; X\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[\pi_{\perp} \Psi^{\tau_{0}, t}(z)\right]$ to get

$$
\begin{align*}
& \sum_{k=1}^{N} a_{k}\left(t, z ; \tau_{0}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[\pi_{\perp} \Psi^{\tau_{0}, t}(z)\right]=\sum_{k=1}^{N} a_{k}\left(t, z ; \tau_{0}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]  \tag{3.32}\\
& \quad+\sum_{k=1}^{N} \sum_{n=1}^{N-k} \sum_{j=1}^{n} C_{n-j}(k, j) a_{k}\left(t, z ; \tau_{0}\right)\left(\partial_{x}^{n-j} a_{j}\left(z ; \Psi^{\tau_{0}, t}\right)\right) \partial_{x}^{-k-n} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]+\sum_{k=1}^{N} a_{k}\left(t, z ; \tau_{0}\right) \mathcal{R}_{N, k}^{(3)}\left(t, z ; \tau_{0}\right)
\end{align*}
$$

Changing the index of summation $n$ to $l:=k+n$ and then interchanging the sum with respect to $k$ and $l$ and in turn with respect to $k$ and $j$, the triple sum in (3.32) becomes

$$
\begin{align*}
& \sum_{k=1}^{N} \sum_{l=k+1}^{N} \sum_{j=1}^{l-k} C_{l-k-j}(k, j) a_{k}\left(t, z ; \tau_{0}\right)\left(\partial_{x}^{l-k-j} a_{j}\left(z ; \Psi^{\tau_{0}, t}\right)\right) \partial_{x}^{-l} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right] \\
& =\sum_{l=2}^{N}\left(\sum_{k=1}^{l-1} \sum_{j=1}^{l-k} C_{l-k-j}(k, j) a_{k}\left(t, z ; \tau_{0}\right)\left(\partial_{x}^{l-k-j} a_{j}\left(z ; \Psi^{\tau_{0}, t}\right)\right)\right) \partial_{x}^{-l} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right] \\
& =\sum_{l=2}^{N}\left(\sum_{j=1}^{l-1} \sum_{k=1}^{l-j} C_{l-k-j}(k, j) a_{k}\left(t, z ; \tau_{0}\right)\left(\partial_{x}^{l-k-j} a_{j}\left(z ; \Psi^{\tau_{0}, t}\right)\right)\right) \partial_{x}^{-l} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right] \tag{3.33}
\end{align*}
$$

Expansion of $X\left(t, \Psi^{\tau_{0}, t}(z)\right)$ : Writing $k$ for $l$ and $n$ for $k$, the expansion (3.24) of $X\left(t, \Psi^{\tau_{0}, t}(z)\right)$ takes the form

$$
\begin{equation*}
\left(0, \mathcal{F}_{\perp}\left(a_{1}\left(t, z ; \tau_{0}\right) \partial_{x}^{-1} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{F}_{\perp} \circ \sum_{k=2}^{N}\left(a_{k}\left(t, z ; \tau_{0}\right)+b_{k}\left(t, z ; \tau_{0}\right)\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N}^{(4)}\left(t, z ; \tau_{0}\right) \tag{3.34}
\end{equation*}
$$

where

$$
\begin{equation*}
b_{k}\left(t, z ; \tau_{0}\right)=\sum_{j=1}^{k-1} \sum_{n=1}^{k-j} C_{k-n-j}(n, j) a_{n}\left(t, z ; \tau_{0}\right) \partial_{x}^{k-n-j} a_{j}\left(z ; \Psi^{\tau_{0}, t}\right) \tag{3.35}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathcal{R}_{N}^{(4)}\left(t, z ; \tau_{0}\right):=\left(0, \mathcal{F}_{\perp} \circ \sum_{k=1}^{N} a_{k}\left(t, z ; \tau_{0}\right) \mathcal{R}_{N, k}^{(3)}\left(t, z ; \tau_{0}\right)\right)+\mathcal{R}_{N}\left(t, \Psi^{\tau_{0}, t}(z) ; X\right) \tag{3.36}
\end{equation*}
$$

Definition and estimates of $a_{k}\left(z ; \Psi^{\tau_{0}, t}\right), 1 \leq k \leq N$ : The fact that for any given $1 \leq k \leq N$, the coefficient $b_{k}\left(t, z ; \tau_{0}\right)$ only depends on the unknown coefficients $a_{j}\left(z ; \Psi^{\tau_{0}, t}\right)$ with $1 \leq j \leq k-1$, but not on $a_{k}\left(z ; \Psi^{\tau_{0}, t}\right)$ allows to determine $a_{k}\left(z ; \Psi^{\tau_{0}, t}\right)$ recursively by using equation (3.23). Indeed, substituting (3.34) for $X\left(t, \Psi^{\tau_{0}, t}(z)\right)$ into equation (3.23) and combining it with (3.22), leads, up to remainder terms, to the equations

$$
\mathcal{F}_{\perp}\left(a_{1}\left(z ; \Psi^{\tau_{0}, \tau}\right) \partial_{x}^{-1} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)=\mathcal{F}_{\perp}\left(\int_{\tau_{0}}^{\tau} a_{1}\left(t, z ; \tau_{0}\right) d t \partial_{x}^{-1} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)
$$

and for any $2 \leq k \leq N$,

$$
\mathcal{F}_{\perp}\left(a_{k}\left(z ; \Psi^{\tau_{0}, \tau}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)=\mathcal{F}_{\perp}\left(\int_{\tau_{0}}^{\tau}\left(a_{k}\left(t, z ; \tau_{0}\right)+b_{k}\left(t, z ; \tau_{0}\right)\right) d t \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)
$$

We then define for any $z \in \mathcal{V}^{\prime}, 0 \leq \tau_{0}, \tau \leq 1$,

$$
a_{1}\left(z ; \Psi^{\tau_{0}, \tau}\right):=\int_{\tau_{0}}^{\tau} a_{1}\left(t, z ; \tau_{0}\right) d t
$$

and for any $k \geq 2, a_{k}\left(z ; \Psi^{\tau_{0}, \tau}\right):=\int_{\tau_{0}}^{\tau}\left(a_{k}\left(t, z ; \tau_{0}\right)+b_{k}\left(t, z ; \tau_{0}\right)\right) d t$, or more explicitly,

$$
\begin{equation*}
a_{k}\left(z ; \Psi^{\tau_{0}, \tau}\right)=\int_{\tau_{0}}^{\tau}\left(a_{k}\left(t, z ; \tau_{0}\right)+\sum_{j=1}^{k-1} \sum_{n=1}^{k-j} C_{k-n-j}(n, j) a_{n}\left(t, z ; \tau_{0}\right) \partial_{x}^{k-n-j} a_{j}\left(z ; \Psi^{\tau_{0}, t}\right)\right) d t \tag{3.37}
\end{equation*}
$$

To prove the claimed estimates for $a_{k}\left(z ; \Psi^{\tau_{0}, \tau}\right)$, we first estimate $a_{k}\left(t, z ; \tau_{0}\right)$. Recall that by (3.31), $a_{k}\left(t, z ; \tau_{0}\right)=$ $a_{k}\left(t, \Psi^{\tau_{0}, t}(z) ; X\right)$. By Lemma 3.6 and Lemma 3.4 one has for any $0 \leq \tau_{0}, t \leq 1, z \in \mathcal{V}^{\prime}$, and $s \geq 0$,

$$
\begin{equation*}
\left\|a_{k}\left(t, z ; \tau_{0}\right)\right\|_{s} \lesssim_{s, k}\left\|\pi_{\perp} \Psi^{\tau_{0}, t}(z)\right\|_{0}^{2} \lesssim_{s, k}\left\|z_{\perp}\right\|_{0}^{2} \tag{3.38}
\end{equation*}
$$

It then follows from the definition of $a_{1}\left(z ; \Psi^{\tau_{0}, \tau}\right)$ that for any $s \geq 0$,

$$
\left\|a_{1}\left(z ; \Psi^{\tau_{0}, \tau}\right)\right\|_{s} \lesssim s \quad\left\|z_{\perp}\right\|_{0}^{2} \quad \forall 0 \leq \tau_{0}, \tau \leq 1, \forall z \in \mathcal{V}^{\prime}
$$

To prove corresponding estimates for $a_{k}\left(z ; \Psi^{\tau_{0}, \tau}\right)$ with $2 \leq k \leq N$, we argue by induction. Assume that for any $1 \leq j \leq k-1$ and $s \geq 0$,

$$
\begin{equation*}
\left\|a_{j}\left(z ; \Psi^{\tau_{0}, \tau}\right)\right\|_{s} \lesssim_{s, j}\left\|z_{\perp}\right\|_{0}^{2}, \quad \forall 0 \leq \tau_{0}, \tau \leq 1, \forall z \in \mathcal{V}^{\prime} \tag{3.39}
\end{equation*}
$$

By the estimate (3.38), the definition (3.37) of $a_{k}\left(z ; \Psi^{\tau_{0}, \tau}\right)$, and the interpolation Lemma E. 3 , one then concludes that the estimate (3.39) is also satisfied for $j=k$. Using the analyticity properties established for $a_{k}\left(\tau, z ; \tau_{0}\right)$ and $\Psi^{\tau_{0}, \tau}(z)$, one verifies the ones stated for the coefficients $a_{k}\left(z ; \Psi^{\tau_{0}, \tau}\right)$.
Estimates of the derivatives of $a_{k}\left(z ; \Psi^{\tau_{0}, \tau}\right)$ : By Lemma 3.6, Lemma 3.4 and the chain rule one has for any $0 \leq \tau_{0}, t \leq 1, z \in \mathcal{V}^{\prime}, \widehat{z} \in h_{0}^{0}, s \geq 0$,
and if in addition, $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{0}, l \geq 2$,

$$
\begin{equation*}
\left\|d^{l} a_{k}\left(t, z ; \tau_{0}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s} \lesssim s \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0} \tag{3.41}
\end{equation*}
$$

By the definition of $a_{1}\left(z ; \Psi^{\tau_{0}, \tau}\right),(3.40)$ and (3.41) yield the claimed estimate for $\left\|d^{l} a_{1}\left(z ; \Psi^{\tau_{0}, \tau}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s}$ for any $l \geq 1$. To prove corresponding estimates for the derivatives of $a_{k}\left(z ; \Psi^{\tau_{0}, \tau}\right)$ with $2 \leq k \leq N$, we again argue by induction. Assume that for any $1 \leq j \leq k-1$ and $s \geq 0$,

$$
\begin{equation*}
\| d a_{j}\left(z ; \Psi^{\tau_{0}, \tau}\right)\left[\widehat{z}\left\|_{s} \lesssim_{s, j}\right\| z_{\perp}\left\|_{0}\right\| \widehat{z} \|_{0}, \quad \forall 0 \leq \tau_{0}, \tau \leq 1, \forall z \in \mathcal{V}^{\prime}, \widehat{z} \in h_{0}^{0}\right. \tag{3.42}
\end{equation*}
$$

By the definition (3.37) of $a_{k}\left(z ; \Psi^{\tau_{0}, \tau}\right)$, the estimate (3.40) and the product rule it then follows that (3.42) also holds for $j=k$. The estimates for $\left\|d^{l} a_{k}\left(z ; \Psi^{\tau_{0}, \tau}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s}$ with $l \geq 2$ are derived in a similar fashion. Definition and estimate of $\mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)$ : The remainder term $\mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)$ is defined so that the identity (3.22) holds,

$$
\mathcal{R}_{N}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right):=\Psi_{X}^{\tau_{0}, \tau}(z)-z-\left(0, \mathcal{F}_{\perp} \circ \sum_{k=1}^{N} a_{k}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)
$$

where $a_{k}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right)$ are given by (3.37). By (3.23) and the expansion (3.34) of $X, \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)$ satisfies

$$
\mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)=\int_{\tau_{0}}^{\tau} \mathcal{R}_{N}^{(4)}\left(t, z ; \tau_{0}\right) d t
$$

where by (3.36)

$$
\begin{equation*}
\int_{\tau_{0}}^{\tau} \mathcal{R}_{N}^{(4)}\left(t, z ; \tau_{0}\right) d t=\left(0, \mathcal{F}_{\perp} \circ \sum_{k=1}^{N} \int_{\tau_{0}}^{\tau} a_{k}\left(t, z ; \tau_{0}\right) \mathcal{R}_{N, k}^{(3)}\left(t, z ; \tau_{0}\right) d t\right)+\int_{\tau_{0}}^{\tau} \mathcal{R}_{N}\left(t, \Psi^{\tau_{0}, t}(z) ; X\right) d t \tag{3.43}
\end{equation*}
$$

We estimate the two components $\pi_{S} \int_{\tau_{0}}^{\tau} \mathcal{R}_{N}^{(4)}\left(t, z ; \tau_{0}\right) d t$ and $\pi_{\perp} \int_{\tau_{0}}^{\tau} \mathcal{R}_{N}^{(4)}\left(t, z ; \tau_{0}\right) d t$ of $\int_{\tau_{0}}^{\tau} \mathcal{R}_{N}^{(4)}\left(t, z ; \tau_{0}\right) d t$ separately. By (3.43)

$$
\pi_{S} \int_{\tau_{0}}^{\tau} \mathcal{R}_{N}^{(4)}\left(t, z ; \tau_{0}\right) d t=\int_{\tau_{0}}^{\tau} \pi_{S} \mathcal{R}_{N}\left(t, \Psi^{\tau_{0}, t}(z) ; X\right) d t
$$

and

$$
\pi_{\perp} \int_{\tau_{0}}^{\tau} \mathcal{R}_{N}^{(4)}\left(t, z ; \tau_{0}\right) d t=\int_{\tau_{0}}^{\tau} \pi_{\perp} \mathcal{R}_{N}\left(t, \Psi^{\tau_{0}, t}(z) ; X\right) d t+\mathcal{F}_{\perp} \circ \sum_{k=1}^{N} \int_{\tau_{0}}^{\tau} a_{k}\left(t, z ; \tau_{0}\right) \mathcal{R}_{N, k}^{(3)}\left(t, z ; \tau_{0}\right) d t .
$$

By Lemma 3.6 and Lemma 3.4 for any $z \in \mathcal{V}^{\prime}, 0 \leq \tau_{0}, t \leq 1$,

$$
\begin{equation*}
\left\|\pi_{S} \mathcal{R}_{N}\left(t, \Psi^{\tau_{0}, t}(z) ; X\right)\right\| \lesssim_{N}\left\|\mathcal{R}_{N}\left(t, \Psi^{\tau_{0}, t}(z) ; X\right)\right\|_{0} \lesssim_{N}\left\|z_{\perp}\right\|_{0}^{2}, \tag{3.44}
\end{equation*}
$$

implying that

$$
\left\|\int_{\tau_{0}}^{\tau} \pi_{S} \mathcal{R}_{N}\left(t, \Psi^{\tau_{0}, t}(z) ; X\right) d t\right\| \lesssim_{s, N}\left\|z_{\perp}\right\|_{0}^{2}
$$

By the definition (3.29) of $\mathcal{R}_{N, k}^{(3)}\left(t, z ; \tau_{0}\right)$, one has

$$
\int_{\tau_{0}}^{\tau} a_{k}\left(t, z ; \tau_{0}\right) \mathcal{R}_{N, k}^{(3)}\left(t, z ; \tau_{0}\right) d t=\int_{\tau_{0}}^{\tau} a_{k}\left(t, z ; \tau_{0}\right) \mathcal{R}_{N, k}^{(1)}\left(t, z ; \tau_{0}\right) d t+\sum_{j=1}^{N-k} \int_{\tau_{0}}^{\tau} a_{k}\left(t, z ; \tau_{0}\right) \mathcal{R}_{N, k, j}^{(2)}\left(t, z ; \tau_{0}\right) d t .
$$

Furthermore, by the definition (3.26) of $\mathcal{R}_{N, k}^{(1)}\left(t, z ; \tau_{0}\right)$, the term $\int_{\tau_{0}}^{\tau} a_{k}\left(t, z ; \tau_{0}\right) \mathcal{R}_{N, k}^{(1)}\left(t, z ; \tau_{0}\right) d t$ equals

$$
\begin{equation*}
\int_{\tau_{0}}^{\tau} a_{k}\left(t, z ; \tau_{0}\right) \partial_{x}^{-k}\left(\sum_{j=N-k+1}^{N} a_{j}\left(z ; \Psi^{\tau_{0}, t}\right) \partial_{x}^{-j} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right) d t+\int_{\tau_{0}}^{\tau} a_{k}\left(t, z ; \tau_{0}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1} \pi_{\perp} \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, t}\right) d t \tag{3.45}
\end{equation*}
$$

Altogether, one concludes that $\pi_{\perp} \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)$ satisfies the integral equation

$$
\begin{equation*}
\pi_{\perp} \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)=B_{N}\left(\tau, z ; \tau_{0}\right)+\mathcal{F}_{\perp} \circ \int_{\tau_{0}}^{\tau}\left(\sum_{k=1}^{N} a_{k}\left(t, z ; \tau_{0}\right) \partial_{x}^{-k}\right) \mathcal{F}_{\perp}^{-1} \pi_{\perp} \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, t}\right) d t \tag{3.46}
\end{equation*}
$$

where

$$
\begin{align*}
B_{N}\left(\tau, z ; \tau_{0}\right):= & \int_{\tau_{0}}^{\tau} \pi_{\perp} \mathcal{R}_{N}\left(t, \Psi^{\tau_{0}, t}(z) ; X\right) d t+\mathcal{F}_{\perp} \circ \sum_{k=1}^{N} \sum_{j=1}^{N-k} \int_{\tau_{0}}^{\tau} a_{k}\left(t, z ; \tau_{0}\right) \mathcal{R}_{N, k, j}^{(2)}\left(t, z ; \tau_{0}\right) d t \\
& +\mathcal{F}_{\perp} \circ \sum_{k=1}^{N} \sum_{j=N-k+1}^{N} \int_{\tau_{0}}^{\tau} a_{k}\left(t, z ; \tau_{0}\right) \partial_{x}^{-k}\left(a_{j}\left(z ; \Psi^{\tau_{0}, t}\right) \partial_{x}^{-j} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right) d t \tag{3.47}
\end{align*}
$$

By the estimates (3.28) of $\mathcal{R}_{N, k, j}^{(2)}\left(t, z ; \tau_{0}\right)$, the ones of $a_{k}(\tau, z ; X)$ and $\mathcal{R}_{N}(\tau, z ; X)$ given by Lemma 3.6. the estimates of $\Psi^{\tau_{0}, t}(z)$, given by Lemma 3.4 and the ones of $a_{k}\left(z ; \Psi^{\tau_{0}, \tau}\right)$, given by (3.39), and using the interpolation Lemma E. 3 one obtains for any $s \geq 0$,

$$
\left\|B_{N}\left(\tau, z ; \tau_{0}\right)\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}\left\|_{z_{\perp}}\right\|_{0}, \quad \forall z \in \mathcal{V}^{\prime} \cap h_{0}^{s}, \quad \forall 0 \leq \tau_{0}, \tau \leq 1 .
$$

Note that $\sum_{k=1}^{N} a_{k}\left(t, z ; \tau_{0}\right) \partial_{x}^{-k}$ is a pseudodifferential operator of order -1 where by (3.38) the coefficients $a_{k}\left(t, z ; \tau_{0}\right)$ satisfy $\left\|a_{k}\left(t, z ; \tau_{0}\right)\right\|_{s} \lesssim_{s, k}\left\|z_{\perp}\right\|_{0}^{2}$. Hence for any $z \in \mathcal{V}^{\prime} \cap h_{0}^{s}, 0 \leq \tau_{0}, \tau \leq 1$,

$$
\left\|\sum_{k=1}^{N} a_{k}\left(t, z ; \tau_{0}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1} \pi_{\perp} \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, t}\right)\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{0}^{2}\left\|\mathcal{F}_{\perp}^{-1} \pi_{\perp} \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, t}\right)\right\|_{s+N+1}
$$

By Gronwall's inequality and since $\mathcal{V}_{\perp}^{\prime}$ is a ball of sufficiently small radius, the integral equation (3.46) yields that for any $s \geq 0$,

$$
\begin{equation*}
\left\|\pi_{\perp} \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}\left\|_{z_{\perp}}\right\|_{0}, \quad \forall z \in \mathcal{V}^{\prime} \cap h_{0}^{s}, \quad \forall 0 \leq \tau_{0}, t \leq 1 . \tag{3.48}
\end{equation*}
$$

The estimates (3.44), (3.48) imply the claimed estimate of $\mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)$. The stated analyticity property of $\mathcal{R}_{N}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right)$ then follows from the already established analyticity properties of $\Psi_{X}^{\tau_{0}, \tau}(z), a_{k}\left(\tau, z ; \tau_{0}\right)$, and $a_{k}\left(z ; \Psi^{\tau_{0}, \tau}\right)$ (cf e.g. [7] Theorem A.5]).
Estimates of the derivatives of $\mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)$ : The estimates of the derivatives of $\mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)$ can be obtained in a similar way as the ones for $\mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)$. Indeed, for any $s \geq 0, z \in \mathcal{V}^{\prime} \cap h_{0}^{s}, 0 \leq \tau_{0}, \tau \leq 1, \widehat{z} \in h_{0}^{s}$, one has

$$
d \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)[\widehat{z}]=\left(0, \mathcal{F}_{\perp} \circ \sum_{k=1}^{N} \int_{\tau_{0}}^{\tau} d\left(a_{k}\left(t, z ; \tau_{0}\right) \mathcal{R}_{N, k}^{(3)}\left(t, z ; \tau_{0}\right)\right)[\widehat{z}] d t\right)+\int_{\tau_{0}}^{\tau} d\left(\mathcal{R}_{N}\left(t, \Psi^{\tau_{0}, t}(z) ; X\right)\right)[\widehat{z}] d t
$$

Again, we estimate $\pi_{S}\left(d \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)[\widehat{z}]\right)=\int_{\tau_{0}}^{\tau} \pi_{S}\left(d\left(\mathcal{R}_{N}\left(t, \Psi^{\tau_{0}, t}(z) ; X\right)\right)[\widehat{z}]\right) d t$ and $\pi_{\perp}\left(d \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)[\widehat{z}]\right)$ separately. By Lemma 3.6, Lemma 3.4, and the chain rule, one has

$$
\begin{equation*}
\left\|\int_{\tau_{0}}^{\tau} \pi_{S}\left(d\left(\mathcal{R}_{N}\left(t, \Psi^{\tau_{0}, t}(z) ; X\right)\right)[\widehat{z}]\right) d t\right\| \lesssim_{N}\left\|z_{\perp}\right\|_{0}\|\widehat{z}\|_{0} \tag{3.49}
\end{equation*}
$$

whereas by (3.46), $\pi_{\perp}\left(d \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)[\widehat{z}]\right)$ satisfies

$$
\begin{equation*}
\pi_{\perp}\left(d \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)[\widehat{z}]\right)=B_{N}^{(1)}\left(\tau, z ; \tau_{0}\right)[\widehat{z}]+\mathcal{F}_{\perp} \circ \int_{\tau_{0}}^{\tau}\left(\sum_{k=1}^{N} a_{k}\left(t, z ; \tau_{0}\right) \partial_{x}^{-k}\right) \mathcal{F}_{\perp}^{-1} \pi_{\perp}\left(d \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, t}\right)[\widehat{z}]\right) d t \tag{3.50}
\end{equation*}
$$

with $B_{N}^{(1)}\left(\tau, z ; \tau_{0}\right)[\widehat{z}]$ given by

$$
B_{N}^{(1)}\left(\tau, z ; \tau_{0}\right)[\widehat{z}]=d\left(B_{N}\left(\tau, z ; \tau_{0}\right)\right)[\widehat{z}]+\mathcal{F}_{\perp} \circ \int_{\tau_{0}}^{\tau}\left(\sum_{k=1}^{N}\left(d a_{k}\left(t, z ; \tau_{0}\right)[\widehat{z}]\right) \partial_{x}^{-k}\right) \mathcal{F}_{\perp}^{-1} \pi_{\perp} \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, t}\right) d t
$$

Since

$$
\left\|B_{N}^{(1)}\left(\tau, z ; \tau_{0}\right)[\widehat{z}]\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}\|\widehat{z}\|_{0}+\left\|z_{\perp}\right\|_{0}\|\widehat{z}\|_{s}, \quad \forall z \in \mathcal{V}^{\prime} \cap h_{0}^{s}, \widehat{z} \in h_{0}^{s}, 0 \leq \tau_{0}, \tau \leq 1
$$

we conclude from (3.50) by Gronwall's inequality that

$$
\begin{equation*}
\left\|\pi_{\perp}\left(d \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)[\widehat{z}]\right)\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}\|\widehat{z}\|_{0}+\left\|z_{\perp}\right\|_{0}\|\widehat{z}\|_{s}, \quad \forall z \in \mathcal{V}^{\prime} \cap h_{0}^{s}, \widehat{z} \in h_{0}^{s}, 0 \leq \tau_{0}, \tau \leq 1 \tag{3.51}
\end{equation*}
$$

The estimates (3.49) and (3.51) imply the claimed estimate for $d \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)[\widehat{z}]$. In a similar fashion, one derives the estimates for $\left\|d^{l} \mathcal{R}_{N}\left(z ; \Psi^{\tau_{0}, \tau}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+N+1}$ with $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s}, l \geq 2$.

It turns out that the flow maps $\Psi_{X}^{\tau_{0}, \tau}$ and hence the symplectic corrector $\Psi_{C}$ and its inverse $\Psi_{C}^{-1}$ preserve the reversible structures, introduced in Section $\mathbb{1}$, acts on. To state the result in more detail, note that without loss of generality, we may assume that the neighborhood $\mathcal{V}^{\prime}=\mathcal{V}_{S}^{\prime} \times \mathcal{V}_{\perp}^{\prime}$ (cf Lemma 3.4) is invariant under the map $\mathcal{S}_{\text {rev }}$.
Addendum to Theorem 3.1 (i) For any $0 \leq \tau_{0}, \tau \leq 1$, $\Psi_{X}^{\tau_{0}, \tau} \circ \mathcal{S}_{\text {rev }}=\mathcal{S}_{\text {rev }} \circ \Psi_{X}^{\tau_{0}, \tau}$ on $\mathcal{V}^{\prime}$ and for any $z \in \mathcal{V}^{\prime}, x \in \mathbb{R}, N \in \mathbb{N}$, and $1 \leq k \leq N$,

$$
a_{k}\left(\mathcal{S}_{\text {rev }} z ; \Psi_{X}^{\tau_{0}, \tau}\right)(x)=(-1)^{k} a_{k}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right)(-x), \quad \mathcal{R}_{N}\left(\mathcal{S}_{\text {rev }} z ; \Psi_{X}^{\tau_{0}, \tau}\right)=\mathcal{S}_{\text {rev }}\left(\mathcal{R}_{N}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right)\right)
$$

(ii) As a consequence, $\Psi_{C}$ and $\Psi_{C}^{-1}$ are invariant under $\mathcal{S}_{\text {rev }}$ on $\mathcal{V}^{\prime}$,

$$
\begin{equation*}
\Psi_{C} \circ \mathcal{S}_{r e v}=\mathcal{S}_{r e v} \circ \Psi_{C}, \quad \Psi_{C}^{-1} \circ \mathcal{S}_{r e v}=\mathcal{S}_{r e v} \circ \Psi_{C}^{-1} \tag{3.52}
\end{equation*}
$$

and for any $z \in \mathcal{V}^{\prime}, x \in \mathbb{R}, N \in \mathbb{N}$, and $1 \leq k \leq N$,

$$
a_{k}\left(\mathcal{S}_{r e v} z ; \Psi_{C}\right)(x)=(-1)^{k} a_{k}\left(z ; \Psi_{C}\right)(-x), \quad \mathcal{R}_{N}\left(\mathcal{S}_{r e v} z ; \Psi_{C}\right)=\mathcal{S}_{\text {rev }}\left(\mathcal{R}_{N}\left(z ; \Psi_{C}\right)\right)
$$

Proof of Addendum to Theorem 3.1 Clearly, item (ii) is a direct consequence of item (i). By the Addendum to Lemma 2.1, the operator $\mathcal{L}(z)$, introduced in (2.20), satisfies $\mathcal{L}\left(\mathcal{S}_{\text {rev }} z\right) \circ \mathcal{S}_{\text {rev }}=-\mathcal{S}_{\text {rev }} \circ \mathcal{L}(z)$ on $\mathcal{V}$. It implies that for any $z \in \mathcal{V}, \mathcal{E}\left(\mathcal{S}_{\text {rev }} z\right)=-\mathcal{S}_{\text {rev }} \mathcal{E}(z)$ where $\mathcal{E}(z)$ has been introduced in (3.4). Altogether we then conclude that the vector field $X(\tau, z)$, introduced in (3.7), satisfies

$$
X\left(\tau, \mathcal{S}_{\text {rev }} z\right)=\mathcal{S}_{\text {rev }} X(\tau, z), \quad \forall z \in \mathcal{V}, 0 \leq \tau \leq 1
$$

and hence by the uniqueness of the initial value problem of $\partial_{\tau} z=X(\tau, z)$, the solution map satisfies

$$
\Psi_{X}^{\tau_{0}, \tau}\left(\mathcal{S}_{\text {rev }} z\right)=\mathcal{S}_{\text {rev }} \Psi_{X}^{\tau_{0}, \tau}(z), \quad \forall z \in \mathcal{V}^{\prime}, 0 \leq \tau_{0}, \tau \leq 1
$$

The claimed identities for $a_{k}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right)$ and $\mathcal{R}_{N}\left(z ; \Psi_{X}^{\tau_{0}, \tau}\right)$ then follow from the expansion (3.22).
We now discuss two applications of Theorem 3.1. The first one concerns the expansion of the transpose $d \Psi_{X}^{0, \tau}(z)^{t}$ of the differential $d \Psi_{X}^{0, \tau}(z)$ which will be used in Section 4 in the proof of Lemma 4.10. Recall that for any $z \in \mathcal{V}^{\prime}$, and $\widehat{z}, \widehat{w} \in h_{0}^{0}$

$$
\Lambda_{\tau}(z)[\widehat{z}, \widehat{w}]=\left\langle J^{-1} \mathcal{L}_{\tau}(z)[\widehat{z}], \widehat{w}\right\rangle, \quad \mathcal{L}_{\tau}(z)=\operatorname{Id}+\tau J \mathcal{L}(z), \quad 0 \leq \tau \leq 1
$$

and that the flow $\Psi_{X}^{0, \tau}$ satisfies $\partial_{\tau}\left(\left(\Psi_{X}^{0, \tau}\right)^{*} \Lambda_{\tau}\right)=0$ and hence $\left(\Psi_{X}^{0, \tau}\right)^{*} \Lambda_{\tau}=\Lambda_{0}$. By the definition of the pullback this means that for any $z \in V^{\prime}, 0 \leq \tau \leq 1, \widehat{z}, \widehat{w} \in h_{0}^{0}$,

$$
\left\langle J^{-1} \mathcal{L}_{\tau}\left(\Psi_{X}^{0, \tau}(z)\right)\left[d \Psi_{X}^{0, \tau}(z)[\widehat{z}]\right], d \Psi_{X}^{0, \tau}(z)[\widehat{w}]\right\rangle=\left\langle J^{-1} \widehat{z}, \widehat{w}\right\rangle \quad \text { or } \quad d \Psi_{X}^{0, \tau}(z)^{t} J^{-1} \mathcal{L}_{\tau}\left(\Psi_{X}^{0, \tau}(z)\right) d \Psi_{X}^{0, \tau}(z)=J^{-1}
$$

Using that $d \Psi_{X}^{0, \tau}(z)^{-1}=d \Psi_{X}^{\tau, 0}\left(\Psi_{X}^{0, \tau}(z)\right)$ one obtains the following formula for $d \Psi_{X}^{0, \tau}(z)^{t}$,

$$
\begin{equation*}
d \Psi_{X}^{0, \tau}(z)^{t}=J^{-1} d \Psi_{X}^{\tau, 0}\left(\Psi_{X}^{0, \tau}(z)\right) \mathcal{L}_{\tau}\left(\Psi_{X}^{0, \tau}(z)\right)^{-1} J \tag{3.53}
\end{equation*}
$$

Note that $d \Psi_{X}^{\tau, 0}\left(\Psi_{X}^{0, \tau}(z)\right)$ and $\mathcal{L}_{\tau}\left(\Psi_{X}^{0, \tau}(z)\right)^{-1}$ are bounded linear operators on $h_{0}^{0}$, implying that $d \Psi_{X}^{0, \tau}(z)^{t}$ is one on $h_{0}^{1}$, and that these operators and their derivatives depend continuously on $0 \leq \tau \leq 1$.

Corollary 3.1. For any $0 \leq \tau \leq 1, z \in \mathcal{V}^{\prime}$, the transpose $d \Psi_{X}^{0, \tau}(z)^{t}$ (with respect to the standard inner product) of the differential $d \Psi_{X}^{0, \tau}(z)$ is a bounded linear operator $d \Psi_{X}^{0, \tau}(z)^{t}: h_{0}^{1} \rightarrow h_{0}^{1}$ and for any $N \in \mathbb{N}$ and $\widehat{z} \in h_{0}^{1}, d \Psi_{X}^{0, \tau}(z)^{t}[\widehat{z}]$ admits an expansion of the form

$$
\left(0, \widehat{z}_{\perp}+\mathcal{F}_{\perp} \circ \sum_{k=1}^{N} a_{k}\left(z ;\left(d \Psi_{X}^{0, \tau}\right)^{t}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[\widehat{z}_{\perp}\right]+\mathcal{F}_{\perp} \circ \sum_{k=1}^{N} \mathcal{A}_{k}\left(z ;\left(d \Psi_{X}^{0, \tau}\right)^{t}\right)[\widehat{z}] \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1} z_{\perp}\right)+\mathcal{R}_{N}\left(z ;\left(d \Psi_{X}^{0, \tau}\right)^{t}\right)[\widehat{z}]
$$

where for any integer $s \geq 0$ and $1 \leq k \leq N$,

$$
\begin{gathered}
a_{k}\left(\cdot ;\left(d \Psi_{X}^{0, \tau}\right)^{t}\right): \mathcal{V}^{\prime} \rightarrow H^{s}, z \mapsto a_{k}\left(z ;\left(d \Psi_{X}^{0, \tau}\right)^{t}\right), \quad \mathcal{A}_{k}\left(\cdot ;\left(d \Psi_{X}^{0, \tau}\right)^{t}\right): \mathcal{V}^{\prime} \rightarrow \mathcal{B}\left(h_{0}^{1}, H^{s}\right), z \mapsto \mathcal{A}_{k}\left(z ;\left(d \Psi_{X}^{0, \tau}\right)^{t}\right), \\
\mathcal{R}_{N}\left(\cdot ;\left(d \Psi_{X}^{0, \tau}\right)^{t}\right): \mathcal{V}^{\prime} \cap h_{0}^{s} \rightarrow \mathcal{B}\left(h_{0}^{s+1}, h_{0}^{s+1+N+1}\right), z \mapsto \mathcal{R}_{N}\left(z ;\left(d \Psi_{X}^{0, \tau}\right)^{t}\right)
\end{gathered}
$$

are real analytic maps. Furthermore, for any $z \in \mathcal{V}^{\prime}, 1 \leq k \leq N, \widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{0}, l \geq 2$,

$$
\begin{gathered}
\left\|a_{k}\left(z ;\left(d \Psi_{X}^{0, \tau}\right)^{t}\right)\right\|_{s} \lesssim_{s, k}\left\|z_{\perp}\right\|_{0}^{2}, \quad\left\|d a_{k}\left(z ;\left(d \Psi_{X}^{0, \tau}\right)^{t}\right)\left[\widehat{z}_{1}\right]\right\|_{s} \lesssim_{s, k}\left\|z_{\perp}\right\|_{0}\left\|\widehat{z}_{1}\right\|_{0} \\
\left\|d^{l} a_{k}\left(z ;\left(d \Psi_{X}^{0, \tau}\right)^{t}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s} \lesssim_{s, k, l} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
\end{gathered}
$$

and for any $z \in \mathcal{V}^{\prime}, \widehat{z} \in h_{0}^{1}, \widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{0}, l \geq 1$,

$$
\left\|\mathcal{A}_{k}\left(z ;\left(d \Psi_{X}^{0, \tau}\right)^{t}\right)[\widehat{z}]\right\|_{s} \lesssim_{s, k}\left\|z_{\perp}\right\|_{0}\|\widehat{z}\|_{1}, \quad\left\|d^{l}\left(\mathcal{A}_{k}\left(z ;\left(d \Psi_{X}^{0, \tau}\right)^{t}\right)[\widehat{z}]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s} \lesssim_{s, k, l}\|\widehat{z}\|_{1} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
$$

The remainder $\mathcal{R}_{N}\left(z ;\left(d \Psi_{X}^{0, \tau}\right)^{t}\right)$ satisfies for any $z \in \mathcal{V}^{\prime} \cap h_{0}^{s}, \widehat{z} \in h_{0}^{s+1}$, and $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s}, l \in \mathbb{N}$,

$$
\begin{aligned}
& \left\|\mathcal{R}_{N}\left(z ;\left(d \Psi_{X}^{0, \tau}\right)^{t}\right)[\widehat{z}]\right\|_{s+1+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{0}\|\widehat{z}\|_{s+1}+\left\|z_{\perp}\right\|_{s}\|\widehat{z}\|_{1} \\
& \qquad\left\|d^{l}\left(\mathcal{R}_{N}\left(z ;\left(d \Psi_{X}^{0, \tau}\right)^{t}\right)[\widehat{z}]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}\right]\right\|_{s+1+N+1} \\
& \quad \lesssim_{s, N, l}\|\widehat{z}\|_{s+1} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}+\|\widehat{z}\|_{1} \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{0}+\left\|z_{\perp}\right\|_{s}\|\widehat{z}\|_{1} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
\end{aligned}
$$

Remark 3.1. Corollary 3.1 holds in particular for $d \Psi_{C}(z)^{t}=d \Psi_{X}^{0,1}(z)^{t}$.
Proof. The starting point is the formula (3.53) for $d \Psi_{X}^{0, \tau}(z)^{t}$. Note that $\|J \widehat{z}\|_{s} \lesssim_{s}\|\widehat{z}\|_{s+1}$ and $\left\|J^{-1} \widehat{z}\right\|_{s+1} \lesssim_{s}$ $\|\widehat{z}\|_{s}$. Hence it suffices to derive corresponding estimates for the operator $d \Psi_{X}^{\tau, 0}\left(\Psi_{X}^{0, \tau}(z)\right) \mathcal{L}_{\tau}\left(\Psi_{X}^{0, \tau}(z)\right)^{-1}$. By Theorem 3.1 for any $\widehat{w} \in h_{0}^{0}, d \Psi_{X}^{\tau, 0}(w)[\widehat{w}]$ admits an expansion of the form

$$
\widehat{w}+\left(0, \mathcal{F}_{\perp} \circ \sum_{k=1}^{N} a_{k}\left(w ; \Psi_{X}^{\tau, 0}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[\widehat{w}_{\perp}\right]+\mathcal{F}_{\perp} \circ \sum_{k=1}^{N} d a_{k}\left(w ; \Psi_{X}^{\tau, 0}\right)[\widehat{w}] \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[w_{\perp}\right]\right)+d \mathcal{R}_{N}\left(w ; \Psi_{X}^{\tau, 0}\right)[\widehat{w}]
$$

where in the situation at hand

$$
w=\Psi_{X}^{0, \tau}(z)=z+\left(0, \mathcal{F}_{\perp} \circ \sum_{k=1}^{N} a_{k}\left(z ; \Psi_{X}^{0, \tau}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N}\left(z ; \Psi_{X}^{0, \tau}\right)
$$

By writing $\mathcal{L}_{1}(w)^{-1}=(\operatorname{Id}+J \mathcal{L}(w))^{-1}$ as a Neumann series, its asymptotic expansion is then obtained from Lemma 3.5 (cf also proof of Theorem 3.1). Combining these results, one obtains the corresponding asymptotic expansion of $d \Psi_{X}^{\tau, 0}\left(\Psi_{X}^{0, \tau}(z)\right) \mathcal{L}_{\tau}\left(\Psi_{X}^{0, \tau}(z)\right)^{-1}$. The claimed estimate then follow from Lemma 3.5 and Theorem 3.1

As a second application of Theorem 3.1 we compute the Taylor expansion of the symplectic corrector $\Psi_{C}\left(z_{S}, z_{\perp}\right)$ in $z_{\perp}$ at 0 . This expansion will be needed in the subsequent section to show that the KdV Hamiltonian, when expressed in the new coordinates provided by the map $\Psi_{L} \circ \Psi_{C}$, is in Birkhoff normal form up to order three. Note that by Theorem 3.1 for any $z_{S} \in \mathcal{V}_{S}^{\prime}, \widehat{z}_{\perp} \in h_{\perp}^{0}, 1 \leq k \leq N$,

$$
a_{k}\left(\left(z_{S}, 0\right) ; \Psi_{C}\right)=0, \quad d_{\perp} a_{k}\left(\left(z_{S}, 0\right) ; \Psi_{C}\right)\left[\widehat{z}_{\perp}\right]=0, \quad \mathcal{R}_{N}\left(\left(z_{S}, 0\right) ; \Psi_{C}\right)=0, \quad d_{\perp}\left(\mathcal{R}_{N}\left(\left(z_{S}, 0\right) ; \Psi_{C}\right)\right)\left[\widehat{z}_{\perp}\right]=0
$$

Hence the Taylor expansion of $\mathcal{R}_{N}\left(z ; \Psi_{C}\right)$ in $z_{\perp}$ of order three at 0 reads

$$
\begin{equation*}
\mathcal{R}_{N}\left(z ; \Psi_{C}\right)=\mathcal{R}_{N, 2}\left(z ; \Psi_{C}\right)+\mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right), \quad \mathcal{R}_{N, 2}\left(z ; \Psi_{C}\right):=\frac{1}{2} d_{\perp}^{2} \mathcal{R}_{N}\left(\left(z_{S}, 0\right) ; \Psi_{C}\right)\left[z_{\perp}, z_{\perp}\right] \tag{3.54}
\end{equation*}
$$

with the Taylor remainder term $\mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right)$ given by

$$
\begin{equation*}
\mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right)=\int_{0}^{1} d_{\perp}^{3} \mathcal{R}_{N}\left(\left(z_{S}, t z_{\perp}\right) ; \Psi_{C}\right)\left[z_{\perp}, z_{\perp}, z_{\perp}\right] \frac{1}{2}(1-t)^{2} d t \tag{3.55}
\end{equation*}
$$

whereas for any $1 \leq k \leq N, \mathcal{F}_{\perp}\left(a_{k}\left(z ; \Psi_{C}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)$ vanishes in $z_{\perp}$ at 0 up to order two. Furthermore, according to Corollary 3.1, for any $\left(z_{S}, 0\right) \in \mathcal{V}^{\prime}, \overline{\mathcal{R}}_{N}\left(\left(z_{S}, 0\right) ; d \Psi_{C}^{t}\right)=0$ and hence for any $\widehat{z} \in h_{0}^{1}$, the Taylor expansion of $\mathcal{R}_{N}\left(z ; d \Psi_{C}^{t}\right)[\widehat{z}]$ of order 2 in $z_{\perp}$ around 0 reads

$$
\begin{equation*}
\mathcal{R}_{N}\left(z ; d \Psi_{C}^{t}\right)[\widehat{z}]=\mathcal{R}_{N, 1}\left(z ; d \Psi_{C}^{t}\right)[\widehat{z}]+\mathcal{R}_{N, 2}\left(z ; d \Psi_{C}^{t}\right)[\widehat{z}] \tag{3.56}
\end{equation*}
$$

where $\mathcal{R}_{N, 2}\left(z ; d \Psi_{C}^{t}\right)[\widehat{z}]$ denotes the Taylor remainder term of order 2 .

Corollary 3.2. (i) For any integer $N \geq 1$, the Taylor expansion of the symplectic corrector $\Psi_{C}\left(z_{S}, z_{\perp}\right)$ in $z_{\perp}$ around 0 reads

$$
\Psi_{C}(z)=\left(z_{S}, 0\right)+\left(0, z_{\perp}\right)+\mathcal{R}_{N, 2}\left(z ; \Psi_{C}\right)+\Psi_{C, 3}(z)
$$

where $\Psi_{C, 3}(z) \equiv \Psi_{C, N, 3}(z)$ is given by

$$
\begin{equation*}
\Psi_{C, N, 3}(z):=\left(0, \mathcal{F}_{\perp} \circ \sum_{k=1}^{N} a_{k}\left(z ; \Psi_{C}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right) \tag{3.57}
\end{equation*}
$$

For any $s \geq 0$, the map $\mathcal{V}^{\prime} \cap h_{0}^{s} \rightarrow h_{0}^{s+N+1}, z \mapsto \mathcal{R}_{N, 2}\left(z ; \Psi_{C}\right)$ is real analytic and the following estimates hold: for any $z \in \mathcal{V}^{\prime} \cap h_{0}^{s}, \widehat{z} \in h_{0}^{s}$,

$$
\left\|\mathcal{R}_{N, 2}\left(z ; \Psi_{C}\right)\right\|_{s+N+1} \lesssim s, N^{\left\|z_{\perp}\right\|_{s}\left\|z_{\perp}\right\|_{0}, \quad\left\|d \mathcal{R}_{N, 2}\left(z ; \Psi_{C}\right)[\widehat{z}]\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{0}\|\widehat{z}\|_{s}+\left\|z_{\perp}\right\|_{s}\|\widehat{z}\|_{0}, ~, ~ . ~}
$$

and, if in addition $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s}, l \geq 2$,

$$
\left\|d^{l} \mathcal{R}_{N, 2}\left(z ; \Psi_{C}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+N+1} \lesssim_{s, N, l} \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{0}+\left\|z_{\perp}\right\|_{s} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
$$

Similarly, for any $s \geq 0$, the map $\mathcal{V}^{\prime} \cap h_{0}^{s} \rightarrow h_{0}^{s+N+1}, z \mapsto \mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right)$ is real analytic and the following estimates hold: for any $z \in \mathcal{V}^{\prime} \cap h_{0}^{s}, \widehat{z}_{1}, \widehat{z}_{2} \in h_{0}^{s}$,
$\left\|\mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right)\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}\left\|z_{\perp}\right\|_{0}^{2}, \quad\left\|d \mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right)\left[\widehat{z}_{1}\right]\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}\left\|z_{\perp}\right\|_{0}\left\|\widehat{z}_{1}\right\|_{0}+\left\|z_{\perp}\right\|_{0}^{2}\left\|\widehat{z}_{1}\right\|_{s}$, $\left\|d^{2} \mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right)\left[\widehat{z}_{1}, \widehat{z}_{2}\right]\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{0}\left(\left\|\widehat{z}_{1}\right\|_{s}\left\|\widehat{z}_{2}\right\|_{0}+\left\|\widehat{z}_{1}\right\|_{0}\left\|\widehat{z}_{2}\right\|_{s}\right)+\left\|z_{\perp}\right\|_{s}\left\|\widehat{z}_{1}\right\|_{0}\left\|\widehat{z}_{2}\right\|_{0}$,
and if in addition $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s}, l \geq 3$,

$$
\left\|d^{l} \mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+N+1} \lesssim_{s, N, l} \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{0}+\left\|z_{\perp}\right\|_{s} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
$$

(ii) For any integer $N \geq 1$ and $\widehat{z} \in h_{0}^{1}$, the Taylor expansion of $d \Psi_{C}\left(z_{S}, z_{\perp}\right)^{t}[\widehat{z}]$ in $z_{\perp}$ around 0 is given by

$$
d \Psi_{C}(z)^{t}[\widehat{z}]=\widehat{z}+\Psi_{C, 1}^{t}(z)[\widehat{z}]+\Psi_{C, 2}^{t}(z)[\widehat{z}]
$$

where $\Psi_{C, 1}^{t}(z)=\mathcal{R}_{N, 1}\left(z ; d \Psi_{C}^{t}\right)(c f(3.56))$ and $\Psi_{C, 2}^{t}(z)[\widehat{z}]$ has an expansion of the form
$\Psi_{C, 2}^{t}(z)[\widehat{z}]=\left(0, \mathcal{F}_{\perp} \circ \sum_{k=1}^{N} a_{k}\left(z ; d \Psi_{C}^{t}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[\widehat{z}_{\perp}\right]+\mathcal{F}_{\perp} \circ \sum_{k=1}^{N} \mathcal{A}_{k}\left(z ; d \Psi_{C}^{t}\right)[\widehat{z}] \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N, 2}\left(z ; d \Psi_{C}^{t}\right)[\widehat{z}]$
with $a_{k}\left(z ; d \Psi_{C}^{t}\right), \mathcal{A}_{k}\left(z ; d \Psi_{C}^{t}\right)$ given as in Corollary[3.1, and $\mathcal{R}_{N, 2}\left(z ; d \Psi_{C}^{t}\right)[\widehat{z}]$ given by (3.56). For any $i=1,2$, $s \geq 0$,

$$
\mathcal{R}_{N, i}\left(\cdot ; d \Psi_{C}^{t}\right): \mathcal{V}^{\prime} \cap h_{0}^{s} \rightarrow \mathcal{B}\left(h_{0}^{s+1}, h_{0}^{s+1+N+1}\right), z \mapsto \mathcal{R}_{N, i}\left(z ; d \Psi_{C}^{t}\right)
$$

is a real analytic maps. Furthermore, for any $z \in \mathcal{V}^{\prime} \cap h_{0}^{s}, \widehat{z} \in h_{0}^{s+1}$,

$$
\left\|\mathcal{R}_{N, 1}\left(z ; d \Psi_{C}^{t}\right)[\widehat{z}]\right\|_{s+1+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{0}\|\widehat{z}\|_{s+1}+\left\|z_{\perp}\right\|_{s}\|\widehat{z}\|_{1}
$$

and if in addition $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s}, l \in \mathbb{N}$,

$$
\begin{aligned}
& \left\|d^{l}\left(\mathcal{R}_{N, 1}\left(z ; d \Psi_{C}^{t}\right)[\widehat{z}]\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+1+N+1} \\
& \quad \lesssim_{s, N, l}\|\widehat{z}\|_{s+1} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}+\|\widehat{z}\|_{1} \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{0}+\left\|z_{\perp}\right\|_{s}\|\widehat{z}\|_{1} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
\end{aligned}
$$

and for any $z \in \mathcal{V}^{\prime} \cap h_{0}^{s}, \widehat{z} \in h_{0}^{s+1}, \widehat{z}_{1} \in h_{0}^{s}$,

$$
\begin{gathered}
\| \mathcal{R}_{N, 2}\left(z ; d \Psi_{C}^{t}\right)\left[\left(\hat{z}\left\|_{s+1+N+1} \lesssim_{s, N}\right\| z_{\perp}\left\|_{0}^{2}\right\| \widehat{z}\left\|_{s+1}+\right\| z_{\perp}\left\|_{s}\right\| z_{\perp}\left\|_{0}\right\| \widehat{z} \|_{1}\right.\right. \\
\| d\left(\mathcal{R}_{N, 2}\left(z ; d \Psi_{C}^{t}\right)[(\widehat{z}))\left[\widehat{z}_{1}\right]\left\|_{s+1+N+1} \lesssim_{s, N}\right\| z_{\perp}\left\|_{0}\right\| \widehat{z}_{1}\left\|_{0}\right\| \widehat{z}_{s+1}+\left\|z_{\perp}\right\|_{0}\left\|\widehat{z}_{1}\right\|_{s}\|\vec{z}\|_{1}+\left\|z_{\perp}\right\|_{s}\left\|\widehat{z}_{1}\right\|_{0} \| \widehat{z}_{1}\right.
\end{gathered}
$$ and if in addition $\widehat{z}_{2}, \ldots, \widehat{z}_{l} \in h_{0}^{s}, l \geq 2$,

$$
\begin{aligned}
& \| d^{l}\left(\mathcal { R } _ { N , 2 } ( z ; d \Psi _ { C } ^ { t } ) [ \widehat { z } ) \left(\widehat{z}_{1}, \ldots, \widehat{z}_{l} \|_{s+1+N+1}\right.\right. \\
& \searrow_{s, N, l}\|\widehat{z}\|_{s+1} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}+\|\vec{z}\|_{1} \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{0}+\|\widehat{z}\|_{1}\left\|z_{\perp}\right\|_{s} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0} .
\end{aligned}
$$

Proof. (i) The claimed properties of $\mathcal{R}_{N, 2}\left(z ; \Psi_{C}\right)$ follow directly from Theorem 3.1 In view of the formula (3.55) the same is true for the ones of $\mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right)$. Item (ii) is a direct consequence of Corollary 3.1]

## 4 The KdV Hamiltonian in new coordinates

In this section we provide an expansion of the transformed KdV Hamiltonian $\mathcal{H}=H^{k d v} \circ \Psi$ where the map $\Psi=\Psi_{L} \circ \Psi_{C}$ is the composition of $\Psi_{L}$ (cf Section (2) with the symplectic corrector $\Psi_{C}$ (cf Section (3) and $H^{k d v}$ is the KdV Hamiltonian given by

$$
\begin{equation*}
H^{k d v}(u)=\frac{1}{2} \int_{0}^{1} u_{x}^{2} d x+\int_{0}^{1} u^{3} d x . \tag{4.1}
\end{equation*}
$$

First we need to make some preliminary considerations. Recall that for any finite subset $S_{+} \subset \mathbb{N}$, the Birkhoff map $\Psi^{k d v}$ establishes a one to one correspondance between $\mathcal{M}_{S}$ and the set $M_{S}$ of $S$-gap potentials where $S=S_{+} \cup\left(-S_{+}\right)$. For any $S-$ gap potential $q$, the corresponding KdV actions $I=\left(I_{S}, I_{\perp}\right)$, defined in terms of the Birkhoff coordinates $\Phi^{k d v}(q)$, satisfy $I_{\perp}=0$. Denote by $\Omega_{\perp}\left(I_{S}\right) \equiv \Omega_{\perp}^{k d v}\left(I_{S}\right)$ and $\Omega_{S}\left(I_{S}\right) \equiv \Omega_{S}^{k d v}\left(I_{S}\right)$ the diagonal linear operators defined by

$$
\begin{gather*}
\Omega_{S}\left(I_{S}\right):=\operatorname{diag}\left(\left(\Omega_{n}\left(I_{S}\right)\right)_{n \in S}\right): h_{S}^{0} \rightarrow h_{S}^{0},\left(z_{n}\right)_{n \in S} \rightarrow\left(\Omega_{n}\left(I_{S}\right) z_{n}\right)_{n \in S}  \tag{4.2}\\
\Omega_{\perp}\left(I_{S}\right):=\operatorname{diag}\left(\left(\Omega_{n}\left(I_{S}\right)\right)_{n \in S^{\perp}}\right): h_{\perp}^{2} \rightarrow h_{\perp}^{0},\left(z_{n}\right)_{n \in S^{\perp}} \rightarrow\left(\Omega_{n}\left(I_{S}\right) z_{n}\right)_{n \in S^{\perp}} \tag{4.3}
\end{gather*}
$$

where for any $n \geq 1$,

$$
\begin{equation*}
\Omega_{n}\left(I_{S}\right) \equiv \Omega_{n}^{k d v}\left(I_{S}\right):=\frac{1}{2 \pi n} \omega_{n}\left(\left(I_{S}, 0\right)\right), \quad \Omega_{-n}\left(I_{S}\right) \equiv \Omega_{-n}^{k d v}\left(I_{S}\right):=\Omega_{n}^{k d v}\left(I_{S}\right) \tag{4.4}
\end{equation*}
$$

and $\omega_{n}(I) \equiv \omega_{n}^{k d v}(I)$ is the $n$th KdV frequency, viewed as a function of the actions. By Lemma C.7 one has:

Lemma 4.1. For any finite gap potential $q \in M_{S}, n \geq 1$, and $N \geq 1$ one has

$$
\begin{equation*}
\Omega_{n}\left(I_{S}\right)=(2 \pi n)^{2}+\sum_{k=1}^{N} \frac{\Omega_{2 k}^{a e}\left(I_{S}\right)}{(2 \pi n)^{2 k}}+\frac{\mathcal{R}_{2 N}^{\Omega_{n}}\left(I_{S}\right)}{(2 \pi n)^{2 N+1}} \tag{4.5}
\end{equation*}
$$

where $\Omega_{2 k}^{a e}\left(I_{S}\right)=\omega_{2 k-1}^{a e}\left(I_{S}, 0\right), \mathcal{R}_{2 N}^{\Omega_{n}}\left(I_{S}\right)=\mathcal{R}_{2 N}^{\omega_{n}}\left(I_{S}, 0\right)$ and $\omega_{2 k-1}^{a e}\left(I_{S}, 0\right), \mathcal{R}_{2 N}^{\omega_{n}}\left(I_{S}, 0\right)$ are given by Lemma C. 7.

Assume that $q(t)$ is a solution of the $\operatorname{KdV}$ equation (1.1) in $M_{S}$ with $z(t):=\Phi^{k d v}(q(t)) \in \mathcal{V}$ for any $t$. Note that $z(t)$ is of the form $\left(z_{S}(t), 0\right)$, the actions $I=\left(I_{n}\right)_{n \geq 1}$ of $q(t)$ are independent of $t$, and $I=\left(I_{S}, 0\right)$ where $I_{S}=\left(\frac{1}{2 \pi n} z_{n}(0) z_{-n}(0)\right)_{n \in S_{+}}$. Furthermore, $\partial_{t} z_{S}(t) \Longrightarrow J_{S} \Omega_{S}\left(I_{S}\right)\left[z_{S}(t)\right]$, or in more detail, for any $n \in S$,

$$
\partial_{t} z_{n}(t)=2 \pi \mathrm{i} n \Omega_{n}\left(I_{S}\right) z_{n}(t) .
$$

Denote by $\widehat{q}(t)$ the solution of the equation, obtained by linearizing the KdV equation along $q(t)$,

$$
\begin{equation*}
\partial_{t} \widehat{q}(t)=\partial_{x} d \nabla H^{k d v}(q(t))[\widehat{q}(t)] \tag{4.6}
\end{equation*}
$$

We need to investigate $\partial_{x} d \nabla H^{k d v}(q(t))[\widehat{q}(t)]$ further. If $\widehat{q}(0)$ is of the form $d \Psi_{L}\left(z_{S}(0), 0\right)\left[0, \widehat{z}_{\perp}(0)\right] \quad(=$ $\left.\Psi_{1}\left(z_{S}(0)\right)\left[\widehat{z}_{\perp}(0)\right]\right)$ with $\widehat{z}_{\perp}(0) \in h_{\perp}^{3}$, then by (2.2) (definition of $\Psi_{L}$ ) and (2.3) (formula of the differential $\left.d \Psi_{L}\right), \widehat{z}_{\perp}(t)$, defined by $\widehat{q}(t)=\Psi_{1}\left(z_{S}(t)\right)\left[\widehat{z}_{\perp}(t)\right]$, solves the equation

$$
\partial_{t} \widehat{z}_{\perp}(t)=J_{\perp} \Omega_{\perp}\left(I_{S}\right)\left[\widehat{z}_{\perp}(t)\right]
$$

or more explicitly, for any $n \in S_{+}^{\perp}$,

$$
\partial_{t} \widehat{z}_{n}(t)=2 \pi \mathrm{i} n \Omega_{n}\left(I_{S}\right) \widehat{z}_{n}(t), \quad \partial_{t} \widehat{z}_{-n}(t)=2 \pi \mathrm{i}(-n) \Omega_{-n}\left(I_{S}\right) \widehat{z}_{-n}(t)
$$

By differentiating $\widehat{q}(t)=\Psi_{1}\left(z_{S}(t)\right)\left[\widehat{z}_{\perp}(t)\right]$ with respect to $t$, one gets

$$
\begin{align*}
\partial_{t} \widehat{q}(t) & =\Psi_{1}\left(z_{S}(t)\right)\left[\partial_{t} \widehat{z}_{\perp}(t)\right]+d_{S}\left(\Psi_{1}\left(z_{S}(t)\right)\left[\widehat{z}_{\perp}(t)\right]\right)\left[\partial_{t} z_{S}(t)\right] \\
& =\Psi_{1}\left(z_{S}(t)\right)\left[J_{\perp} \Omega_{\perp}\left(I_{S}\right)\left[\widehat{z}_{\perp}(t)\right]\right]+d_{S}\left(\Psi_{1}\left(z_{S}(t)\right)\left[\widehat{z}_{\perp}(t)\right]\right)\left[\partial_{t} z_{S}(t)\right] \tag{4.7}
\end{align*}
$$

Comparing (4.6) and (4.7) and using that $\partial_{t} z_{S}(t)=J_{S} \Omega_{S}\left(I_{S}\right)\left[z_{S}(t)\right]$, one gets

$$
\begin{gather*}
\partial_{x} d \nabla H^{k d v}(q(t))\left[\Psi_{1}\left(z_{S}(t)\right)\left[\widehat{z}_{\perp}(t)\right]\right]=\Psi_{1}\left(z_{S}(t)\right)\left[J_{\perp} \Omega_{\perp}\left(I_{S}\right)\left[\widehat{z}_{\perp}(t)\right]\right] \\
+d_{S}\left(\Psi_{1}\left(z_{S}(t)\right)\left[\widehat{z}_{\perp}(t)\right]\right)\left[J_{S} \Omega_{S}\left(I_{S}\right)\left[z_{S}(t)\right]\right] \tag{4.8}
\end{gather*}
$$

Now apply $\Psi_{1}\left(z_{S}(t)\right)^{-1}$ to both sides of the latter equality yielding

$$
\begin{align*}
\Psi_{1}\left(z_{S}(t)\right)^{-1} \partial_{x} d \nabla & H^{k d v}(q(t))\left[\Psi_{1}\left(z_{S}(t)\right)\left[\widehat{z}_{\perp}(t)\right]\right]=J_{\perp} \Omega_{\perp}\left(I_{S}\right)\left[\widehat{z}_{\perp}(t)\right] \\
& +\Psi_{1}\left(z_{S}(t)\right)^{-1} d_{S}\left(\Psi_{1}\left(z_{S}(t)\right)[\widehat{z}(t)]\right)\left[J_{S} \Omega_{S}\left(I_{S}\right)\left[z_{S}(t)\right]\right] \tag{4.9}
\end{align*}
$$

Since $\Psi_{1}\left(z_{S}\right)$ is symplectic one has $\Psi_{1}\left(z_{S}\right)^{t} \partial_{x}^{-1} \Psi_{1}\left(z_{S}\right)=J_{\perp}^{-1}$ or $\Psi_{1}\left(z_{S}\right)^{-1} \partial_{x}=J_{\perp} \Psi_{1}\left(z_{S}\right)^{t}$, implying that

$$
\begin{align*}
J_{\perp} \Psi_{1}\left(z_{S}(t)\right)^{t} d \nabla H^{k d v}(q(t))[ & \left.\Psi_{1}\left(z_{S}(t)\right)\left[\widehat{z}_{\perp}(t)\right]\right]=J_{\perp} \Omega_{\perp}\left(I_{S}\right)\left[\widehat{z}_{\perp}(t)\right] \\
& +\Psi_{1}\left(z_{S}(t)\right)^{-1} d_{S}\left(\Psi_{1}\left(z_{S}(t)\right)\left[\widehat{z}_{\perp}(t)\right]\right)\left[J_{S} \Omega_{S}\left(I_{S}\right)\left[z_{S}(t)\right]\right] \tag{4.10}
\end{align*}
$$

The latter identity implies that for any $z_{S} \in \mathcal{V}_{S}, I_{S}=\left(\frac{1}{2 \pi n} z_{n} z_{-n}\right)_{n \in S_{+}}, q=\Psi^{k d v}\left(z_{S}, 0\right), \widehat{z}_{\perp} \in h_{\perp}^{3}$,

$$
\begin{equation*}
\Psi_{1}\left(z_{S}\right)^{t} d \nabla H^{k d v}(q)\left[\Psi_{1}\left(z_{S}\right)\left[\widehat{z}_{\perp}\right]\right]=\Omega_{\perp}\left(I_{S}\right)\left[\widehat{z}_{\perp}\right]+\mathcal{G}\left(z_{S}\right)\left[\widehat{z}_{\perp}\right] \tag{4.11}
\end{equation*}
$$

where $\mathcal{G}\left(z_{S}\right): h_{\perp}^{0} \rightarrow h_{\perp}^{0}$ is given by

$$
\begin{equation*}
\mathcal{G}\left(z_{S}\right)\left[\widehat{z}_{\perp}\right]:=J_{\perp}^{-1} \Psi_{1}\left(z_{S}\right)^{-1} d_{S}\left(\Psi_{1}\left(z_{S}\right)\left[\widehat{z}_{\perp}\right]\right)\left[J_{S} \Omega_{S}\left(I_{S}\right)\left[z_{S}\right]\right] . \tag{4.12}
\end{equation*}
$$

In the next lemma we record an expansion for the operator $\mathcal{G}\left(z_{S}\right)$.
Lemma 4.2. For any integer $N \geq 1$, the operator $\mathcal{G}\left(z_{S}\right): h_{\perp}^{0} \rightarrow h_{\perp}^{0}$ admits an expansion of the form

$$
\mathcal{G}\left(z_{S}\right)=\mathcal{F}_{\perp} \circ \sum_{k=1}^{N} a_{k}\left(z_{S} ; \mathcal{G}\right) \partial_{x}^{-k} \circ \mathcal{F}_{\perp}^{-1}+\mathcal{R}_{N}\left(z_{S} ; \mathcal{G}\right)
$$

where for any $1 \leq k \leq N, s \geq 0$, the maps

$$
\mathcal{V}_{S} \rightarrow H^{s}, z_{S} \mapsto a_{k}\left(z_{S} ; \mathcal{G}\right), \quad \mathcal{V}_{S} \mapsto \mathcal{B}\left(h_{\perp}^{s}, h_{\perp}^{s+N+1}\right), z_{S} \mapsto \mathcal{R}_{N}\left(z_{S} ; \mathcal{G}\right)
$$

are real analytic.
Proof. In view of the definition (4.12) of $\mathcal{G}$, the lemma follows from item (ii) of Theorem 2.1 (expansion of $\left.\Psi_{1}\left(z_{S}\right)\right)$ and Lemma E. 2

After this preliminary discussion, we can now study the transformed Hamiltonian $H^{k d v} \circ \Psi$ where $\Psi=$ $\Psi_{L} \circ \Psi_{C}$. We split the analysis into two parts. First we expand $\mathcal{H}^{(1)}:=H^{k d v} \circ \Psi_{L}$ and then we analyze $\mathcal{H}^{(2)}=\mathcal{H}^{(1)} \circ \Psi_{C}$.

Expansion of $\mathcal{H}^{(1)}:=H^{k d v} \circ \Psi_{L}$
To expand $H^{k d v} \circ \Psi_{L}$, it is useful to write $H^{k d v}(u)$ as $H^{k d v}(u)=H_{2}^{k d v}(u)+H_{3}^{k d v}(u)$ where

$$
\begin{equation*}
H_{2}^{k d v}(u):=\frac{1}{2}\left\langle\left(-\partial_{x}^{2}\right) u, u\right\rangle, \quad H_{3}^{k d v}(u):=\int_{0}^{1} u^{3} d x \tag{4.13}
\end{equation*}
$$

The $L^{2}$-gradient $\nabla H^{k d v}$ of $H^{k d v}$ and its derivative are then given by

$$
\begin{equation*}
\nabla H^{k d v}(u)=-\partial_{x}^{2} u+3 u^{2}, \quad d \nabla H^{k d v}(u)=-\partial_{x}^{2}+6 u \tag{4.14}
\end{equation*}
$$

Let $z_{S} \in \mathcal{V}_{S}, q=\Psi^{k d v}\left(z_{S}, 0\right)$. The Taylor expansion of $H^{k d v}(q+v)$ around $q$ in direction $v=\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]$ with $z_{\perp} \in \mathcal{V}_{\perp} \cap h_{\perp}^{1}$ reads

$$
\begin{equation*}
H^{k d v}(q+v)=H^{k d v}(q)+\left\langle\nabla H^{k d v}(q), v\right\rangle+\frac{1}{2}\left\langle d \nabla H^{k d v}(q)[v], v\right\rangle+\int_{0}^{1} v^{3} d x \tag{4.15}
\end{equation*}
$$

Since $v=d \Psi^{k d v}\left(z_{S}, 0\right)\left[0, z_{\perp}\right]$ one has $\left\langle\nabla H^{k d v}(q), v\right\rangle=\left.\partial_{y}\right|_{y=0} H^{k d v}\left(\Psi^{k d v}\left(z_{S}, y z_{\perp}\right)\right)$. Recall that $\mathcal{H}^{k d v}=$ $H^{k d v} \circ \Psi^{k d v}$ is a function of the actions alone and $I_{n}=\frac{1}{2 \pi n} z_{n} z_{-n}, n \geq 1$. It implies that

$$
\left.\partial_{y}\right|_{y=0} H^{k d v}\left(\Psi^{k d v}\left(z_{S}, y z_{\perp}\right)\right)=\left.\sum_{n \in S_{+}^{\perp}} \omega_{n}\left(I_{S}, 0\right) \partial_{y}\right|_{y=0} y^{2} I_{n}=0
$$

and hence $\left\langle\nabla H^{k d v}(q), v\right\rangle=0$. Since $\Psi_{L}(z)=q+\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]$ one then gets

$$
\mathcal{H}^{(1)}(z)=H^{k d v}\left(\Psi_{L}(z)\right)=H^{k d v}(q)+\frac{1}{2}\left\langle d \nabla H^{k d v}(q)\left[\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right], \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right\rangle+\int_{0}^{1}\left(\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)^{3} d x
$$

By formula (4.11),

$$
\left\langle d \nabla H^{k d v}(q)\left[\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right], \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right\rangle=\left\langle\Omega_{\perp}^{k d v}\left(I_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle+\left\langle\mathcal{G}\left(z_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle .
$$

Since $\Psi_{1}\left(z_{S}\right)^{t} d \nabla H^{k d v}(q) \Psi_{1}\left(z_{S}\right)$ and $\Omega_{\perp}^{k d v}\left(I_{S}\right)$ are symmetric, so is the operator $\mathcal{G}\left(z_{S}\right)$. In summary,

$$
\begin{equation*}
\mathcal{H}^{(1)}(z)=\mathcal{H}_{S}^{k d v}(z)+\frac{1}{2}\left\langle\Omega_{\perp}^{k d v}\left(I_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle+\mathcal{P}_{2}^{(1)}(z)+\mathcal{P}_{3}^{(1)}(z) \tag{4.16}
\end{equation*}
$$

where for any $z=\left(z_{S}, z_{\perp}\right) \in \mathcal{V} \cap h_{0}^{1}$,

$$
\begin{equation*}
\mathcal{H}_{S}^{k d v}(z):=H^{k d v}\left(\Psi^{k d v}\left(z_{S}, 0\right)\right), \quad \mathcal{P}_{2}^{(1)}(z):=\frac{1}{2}\left\langle\mathcal{G}\left(z_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle, \quad \mathcal{P}_{3}^{(1)}(z):=\int_{0}^{1}\left(\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)^{3} d x \tag{4.17}
\end{equation*}
$$

Note that $\mathcal{H}_{S}^{k d v}(z)=\mathcal{H}_{S}^{k d v}\left(\Pi_{S} z\right)$ where $\Pi_{S}: h_{S}^{0} \times h_{\perp}^{0} \rightarrow h_{S}^{0} \times h_{\perp}^{0}$ denotes the projection, given by $\left(\widehat{z}_{S}, \widehat{z}_{\perp}\right) \mapsto$ $\left(\widehat{z}_{S}, 0\right)$ (cf (3.9)). We record that $\mathcal{P}_{2}^{(1)}(z)$ is quadratic and $\mathcal{P}_{3}^{(1)}(z)$ cubic in $z_{\perp}$ where the superscript (1) refers to the Hamiltonian $\mathcal{H}^{(1)}$. Recall from (E.3) that for any $a \in H^{1}$, the paraproduct $T_{a} u$ of the function $a$ with $u \in L^{2}$ with respect to the cut-off function $\chi$ is defined as $\left(T_{a} u\right)(x)=\sum_{k, n \in \mathbb{Z}} \chi(k, n) a_{k} u_{n} e^{\mathrm{i} 2 \pi(k+n) x}$ with $u_{n}, n \in \mathbb{Z}$, denoting the Fourier coefficients of $u$.

Lemma 4.3. For any integer $N \geq 1$, there exists an integer $\sigma_{N} \geq N$ (loss of regularity) so that on $\mathcal{V} \cap h_{0}^{\sigma_{N}}$, the $L^{2}$-gradient $\nabla \mathcal{P}_{3}^{(1)}$ of $\mathcal{P}_{3}^{(1)}$ admits the asymptotic expansion of the form

$$
\begin{equation*}
\nabla \mathcal{P}_{3}^{(1)}(z)=\left(0, \mathcal{F}_{\perp} \circ \sum_{k=0}^{N} T_{a_{k}\left(z ; \nabla \mathcal{P}_{3}^{(1)}\right)} \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(1)}\right) \tag{4.18}
\end{equation*}
$$

where for any $s \geq 0,1 \leq k \leq N$, the maps

$$
\mathcal{V} \cap h_{0}^{s+\sigma_{N}} \rightarrow H^{s}, z \mapsto a_{k}\left(z ; \nabla \mathcal{P}_{3}^{(1)}\right), \quad \mathcal{V} \cap h_{0}^{s \vee \sigma_{N}} \rightarrow h_{0}^{s+N+1}, z \mapsto \mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(1)}\right)
$$

are real analytic. Furthermore, for any $z \in \mathcal{V} \cap h_{0}^{s+\sigma_{N}}$ with $\|z\|_{\sigma_{N}} \leq 1,\left\|a_{k}\left(z ; \nabla \mathcal{P}_{3}^{(1)}\right)\right\|_{s} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s+\sigma_{N}}$ and if in addition $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s+\sigma_{N}}, l \geq 1$,

$$
\begin{equation*}
\left\|d^{l} a_{k}\left(z ; \nabla \mathcal{P}_{3}^{(1)}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s} \lesssim_{s, k, l} \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s+\sigma_{N}} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{\sigma_{N}}+\left\|z_{\perp}\right\|_{s+\sigma_{N}} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{\sigma_{N}} \tag{4.19}
\end{equation*}
$$

Similarly, for any $z \in \mathcal{V} \cap h_{0}^{s \vee \sigma_{N}}$ with $\|z\|_{\sigma_{N}} \leq 1, \widehat{z} \in h_{0}^{s \vee \sigma_{N}},\left\|\mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(1)}\right)\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s \vee \sigma_{N}}\left\|z_{\perp}\right\|_{\sigma_{N}}$ and

$$
\begin{equation*}
\left\|d \mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(1)}\right)[\widehat{z}]\right\|_{s+N+1} \lesssim s, N\left\|z_{\perp}\right\|_{s \vee \sigma_{N}}\|\widehat{z}\|_{\sigma_{N}}+\left\|z_{\perp}\right\|_{\sigma_{N}}\|\widehat{z}\|_{s \vee \sigma_{N}} \tag{4.20}
\end{equation*}
$$

and if in addition $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s \vee \sigma_{N}}, l \geq 2$, then

$$
\begin{equation*}
\left\|d^{l} \mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(1)}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+N+1} \lesssim_{s, N, l} \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s \vee \sigma_{N}} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{\sigma_{N}}+\left\|z_{\perp}\right\|_{s \vee \sigma_{N}} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{\sigma_{N}} \tag{4.21}
\end{equation*}
$$

Proof. By a straightforward calculation, one has $\nabla_{\perp} \mathcal{P}_{3}^{(1)}(z)=3 \Psi_{1}\left(z_{S}\right)^{t}\left(\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)^{2}$. By the Bony decomposition given in Lemma E.1 (ii),

$$
\left(\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)^{2}=2 T_{\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]+\mathcal{R}^{(B)}\left(\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right], \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right)
$$

The expansion (4.18) and the stated estimates follow from Theorem 2.1(ii), Corollary 2.2, and Lemmata E. 1 E. 2 E. 3 .

Expansion of $\mathcal{H}^{(2)}:=\mathcal{H}^{(1)} \circ \Psi_{C}$
To compute the expansion of $\mathcal{H}^{(2)}(z)=\mathcal{H}^{(1)}\left(\Psi_{C}(z)\right)$ on $\mathcal{V}^{\prime} \cap h_{0}^{1}$, we study the composition of each of the terms in (4.16) with the symplectic corrector $\Psi_{C}$ separately. Recall that $\Psi_{C}$ is defined on $\mathcal{V}^{\prime}$ and takes values in $\mathcal{V}$.
Term $\mathcal{H}_{S}^{k d v}$ : By Corollary 3.2, $\Psi_{C}(z)$ has a Taylor expansion in $z_{\perp}$ around 0 of the form

$$
\Psi_{C}(z)=\left(z_{S}, 0\right)+\left(0, z_{\perp}\right)+\tilde{\Psi}_{C}(z), \quad \tilde{\Psi}_{C}(z):=\mathcal{R}_{N, 2}\left(z ; \Psi_{C}\right)+\Psi_{C, 3}(z), \quad \Psi_{C, 3}(z) \equiv \Psi_{C, N, 3}(z)
$$

where $R_{N, 2}\left(z ; \Psi_{C}\right)$ is the term of order two, given by $R_{N, 2}\left(z ; \Psi_{C}\right)=\frac{1}{2} d_{\perp}^{2} \mathcal{R}_{N}\left(\left(z_{S}, 0\right) ; \Psi_{C}\right)\left[z_{\perp}, z_{\perp}\right]$ (cf (3.54) $)$, and $\Psi_{C, 3}(z)$ is given by (3.57)

$$
\Psi_{C, 3}(z)=\left(0, \mathcal{F}_{\perp} \circ \sum_{k=1}^{N} a_{k}\left(z ; \Psi_{C}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right)
$$

with $\mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right)$ denoting the Taylor remainder term (3.55). Since $\mathcal{H}_{S}^{k d v}(z)=\mathcal{H}_{S}^{k d v}\left(\Pi_{S} z\right)$ (cf (4.17)), the Taylor expansion of $\mathcal{H}_{S}^{k d v}\left(\Psi_{C}(z)\right)=\mathcal{H}_{S}^{k d v}\left(z+\tilde{\Psi}_{C}(z)\right)$ reads

$$
\begin{equation*}
\mathcal{H}_{S}^{k d v}\left(\Psi_{C}(z)\right)=\mathcal{H}_{S}^{k d v}(z)+\left\langle\nabla_{S} \mathcal{H}_{S}^{k d v}(z), \pi_{S} \mathcal{R}_{N, 2}\left(z ; \Psi_{C}\right)\right\rangle+\mathcal{P}_{3}^{(2 a)}(z) \tag{4.22}
\end{equation*}
$$

where $\mathcal{P}_{3}^{(2 a)}(z)$ is the Taylor remainder term of order three, given by

$$
\left\langle\nabla_{S} \mathcal{H}_{S}^{k d v}(z), \pi_{S} \Psi_{C, 3}(z)\right\rangle+\int_{0}^{1}(1-y)\left\langle d_{S}\left(\nabla_{S} \mathcal{H}_{S}^{k d v}\left(z+y \tilde{\Psi}_{C}(z)\right)\right)\left[\pi_{S} \tilde{\Psi}_{C}(z)\right], \pi_{S} \tilde{\Psi}_{C}(z)\right\rangle d y
$$

and $\pi_{S}: h_{S}^{0} \times h_{\perp}^{0} \rightarrow h_{S}^{0}$ denotes the map given by $z=\left(z_{S}, z_{\perp}\right) \mapsto z_{S}$ (cf (3.10) ). Since $\pi_{S} \Psi_{C, 3}(z)=$ $\pi_{S} \mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right)$ and $\pi_{S} \tilde{\Psi}_{C}(z)=\pi_{S} \mathcal{R}_{N}\left(z ; \Psi_{C}\right)$, one has

$$
\begin{align*}
\mathcal{P}_{3}^{(2 a)}(z) & =\left\langle\nabla_{S} \mathcal{H}_{S}^{k d v}(z), \pi_{S} R_{N, 3}\left(z ; \Psi_{C}\right)\right\rangle \\
& +\int_{0}^{1}(1-y)\left\langle d_{S}\left(\nabla_{S} \mathcal{H}_{S}^{k d v}\left(z+y \tilde{\Psi}_{C}(z)\right)\right)\left[\pi_{S} \mathcal{R}_{N}\left(z ; \Psi_{C}\right)\right], \pi_{S} \mathcal{R}_{N}\left(z ; \Psi_{C}\right)\right\rangle d y \tag{4.23}
\end{align*}
$$

In the next lemma we show that $\nabla \mathcal{P}_{3}^{(2 a)}(z)$ is in $h_{0}^{s+N+1}$ for any $z \in \mathcal{V}^{\prime} \cap h_{0}^{s}$.

Lemma 4.4. The Hamiltonian $\mathcal{P}_{3}^{(2 a)}: \mathcal{V}^{\prime} \rightarrow \mathbb{R}$ is real analytic and for any integers $s \geq 0, N \geq 1$, the map $\mathcal{V}^{\prime} \cap h_{0}^{s} \rightarrow h_{0}^{s+N+1}, z \mapsto \nabla \mathcal{P}_{3}^{(2 a)}(z)$ is real analytic. Furthermore, for any $z \in \mathcal{V}^{\prime} \cap h_{0}^{s}$, and $\widehat{z} \in h_{0}^{s}$,

$$
\left\|\nabla \mathcal{P}_{3}^{(2 a)}(z)\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}\left\|z_{\perp}\right\|_{0}, \quad\left\|d \nabla \mathcal{P}_{3}^{(2 a)}(z)\left[\widehat{z}_{1}\right]\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}\left\|\widehat{z}_{1}\right\|_{0}+\left\|z_{\perp}\right\|_{0}\left\|\widehat{z}_{1}\right\|_{s}
$$

and if in addition $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s}, l \geq 2$,

$$
\left\|d^{l} \nabla \mathcal{P}_{3}^{(2 a)}(z)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+N+1} \lesssim_{s, N, l} \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{0}+\left\|z_{\perp}\right\|_{s} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
$$

Proof. We begin by analyzing the first term $\left\langle\nabla_{S} \mathcal{H}_{S}^{k d v}(z), \pi_{S} \Psi_{C, 3}(z)\right\rangle$ on the right hand side of (4.23). It is given by the finite sum $\sum_{n \in S} h_{n}(z)$ where

$$
h_{n}(z):=\left(\nabla \mathcal{H}_{S}^{k d v}(z)\right)_{n}\left(\Psi_{C, 3}(z)\right)_{-n}=\left(\partial_{z_{-n}} \mathcal{H}_{S}^{k d v}(z)\right)\left\langle\mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right), e_{n}\right\rangle, \quad \forall n \in S
$$

and $\left(e_{n}\right)_{n \in S}$ denotes the standard basis of $h_{S}^{0}$. The derivative of $h_{n}$ in direction $\widehat{z} \in h_{0}^{0}$ then reads

$$
\begin{aligned}
\left\langle\nabla h_{n}(z), \widehat{z}\right\rangle & =\left\langle\nabla \partial_{z_{-n}} \mathcal{H}_{S}^{k d v}(z), \widehat{z}\right\rangle\left\langle\mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right), e_{n}\right\rangle+\partial_{z_{-n}} \mathcal{H}_{S}^{k d v}(z)\left\langle d\left(\mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right)\right)[\widehat{z}], e_{n}\right\rangle \\
& =\left\langle\nabla \partial_{z_{-n}} \mathcal{H}_{S}^{k d v}(z), \widehat{z}\right\rangle\left\langle\mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right), e_{n}\right\rangle+\partial_{z_{-n}} \mathcal{H}_{S}^{k d v}(z)\left\langle\left(d \mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right)\right)^{t}\left[e_{n}\right], \widehat{z}\right\rangle
\end{aligned}
$$

implying that

$$
\nabla h_{n}(z)=\left\langle\mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right), e_{n}\right\rangle \nabla \partial_{z_{-n}} \mathcal{H}_{S}^{k d v}(z)+\partial_{z_{-n}} \mathcal{H}_{S}^{k d v}(z)\left(d \mathcal{R}_{N, 3}\left(z ; \Psi_{C}\right)\right)^{t}\left[e_{n}\right]
$$

By Corollary 3.2, for any $s \geq 0, \mathcal{V}^{\prime} \cap h_{0}^{s} \rightarrow h_{0}^{s+N+1}, z \mapsto \nabla h_{n}(z)$ is real analytic and satisfies the estimates $\left\|\nabla h_{n}(z)\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}\left\|z_{\perp}\right\|_{0}$. The estimates for the higher order derivatives of $h_{n}, n \in S$, are obtained by differentiating the expression for $\nabla h_{n}(z)$ and using the estimates of Corollary 3.2.
In order to analyze the second term on the right hand side of (4.23) it suffices to study the functions $h_{n, k}(z)$, $n, k \in S$, given by

$$
h_{n, k}(z ; y):=\partial_{z_{-n}} \partial_{z_{-k}} \mathcal{H}_{S}^{k d v}\left(z+y \tilde{\Psi}_{C}(z)\right)\left\langle\mathcal{R}_{N}\left(z ; \Psi_{C}\right), e_{n}\right\rangle\left\langle\mathcal{R}_{N}\left(z ; \Psi_{C}\right), e_{k}\right\rangle
$$

where $0 \leq y \leq 1$. Clearly, $h_{n, k}(z ; y)$ depends continuously on $y$ as do all the derivatives with respect to the variable $z$. Since $\mathcal{H}_{S}^{k d v}\left(z+y \tilde{\Psi}_{C}(z)\right)$ only depends on $\pi_{S}\left(z+y \tilde{\Psi}_{C}(z)\right)$ one sees that

$$
\begin{aligned}
\left\langle\nabla h_{n, k}(z ; y), \widehat{z}\right\rangle= & \left\langle\nabla_{S}\left(\partial_{z_{-n}} \partial_{z_{-k}} \mathcal{H}_{S}^{k d v}\left(z+y \tilde{\Psi}_{C}(z)\right)\right), \pi_{S}\left(\operatorname{Id}+\mathrm{yd} \tilde{\Psi}_{\mathrm{C}}(\mathrm{z})\right)\langle\widehat{\mathrm{z}}]\right\rangle\left\langle\mathcal{R}_{\mathrm{N}}\left(\mathrm{z} ; \Psi_{\mathrm{C}}\right), \mathrm{e}_{\mathrm{n}}\right\rangle\left\langle\mathcal{R}_{\mathrm{N}}\left(\mathrm{z} ; \Psi_{\mathrm{C}}\right), \mathrm{e}_{\mathrm{k}}\right\rangle \\
& +\partial_{z_{-n}} \partial_{z_{-k}} \mathcal{H}_{S}^{k d v}\left(z+y \tilde{\Psi}_{C}(z)\right)\left\langle\left(d \mathcal{R}_{N}\left(z ; \Psi_{C}\right)\right)^{t}\left[e_{n}\right], \widehat{z}\right\rangle\left\langle\mathcal{R}_{N}\left(z ; \Psi_{C}\right), e_{k}\right\rangle \\
& +\partial_{z_{-n}} \partial_{z_{-k}} \mathcal{H}_{S}^{k d v}\left(z+y \tilde{\Psi}_{C}(z)\right)\left\langle\mathcal{R}_{N}\left(z ; \Psi_{C}\right), e_{n}\right\rangle\left\langle\left(d \mathcal{R}_{N}\left(z ; \Psi_{C}\right)\right)^{t}\left[e_{k}\right], \widehat{z}\right\rangle
\end{aligned}
$$

implying that

$$
\begin{aligned}
\nabla h_{n, k}(z ; y)= & \left(\mathrm{Id}+\mathrm{sd} \tilde{\Psi}_{\mathrm{C}}(\mathrm{z})\right)^{\mathrm{t}}\left[\Pi_{\mathrm{S}} \nabla_{\mathrm{S}}\left(\partial_{\mathrm{z}_{-\mathrm{n}}} \partial_{\mathrm{z}_{-\mathrm{k}}} \mathcal{H}_{\mathrm{S}}^{\mathrm{kdv}}\left(\mathrm{z}+\mathrm{y} \tilde{\Psi}_{\mathrm{C}}(\mathrm{z})\right)\right)\right]\left\langle\mathcal{R}_{\mathrm{N}}\left(\mathrm{z} ; \Psi_{\mathrm{C}}\right), \mathrm{e}_{\mathrm{n}}\right\rangle\left\langle\mathcal{R}_{\mathrm{N}}\left(\mathrm{z} ; \Psi_{\mathrm{C}}\right), \mathrm{e}_{\mathrm{k}}\right\rangle \\
& +\partial_{z_{-n}} \partial_{z_{-k}} \mathcal{H}_{S}^{k d v}\left(z+y \tilde{\Psi}_{C}(z)\right)\left\langle\mathcal{R}_{N}\left(z ; \Psi_{C}\right), e_{k}\right\rangle\left(d \mathcal{R}_{N}\left(z ; \Psi_{C}\right)\right)^{t}\left[e_{n}\right] \\
& +\partial_{z_{-n}} \partial_{z_{-k}} \mathcal{H}_{S}^{k d v}\left(z+y \tilde{\Psi}_{C}(z)\right)\left\langle\mathcal{R}_{N}\left(z ; \Psi_{C}\right), e_{n}\right\rangle\left(d \mathcal{R}_{N}\left(z ; \Psi_{C}\right)\right)^{t}\left[e_{k}\right]
\end{aligned}
$$

By Corollary 3.2. for any $s \geq 0$, the map $\mathcal{V}^{\prime} \cap h_{0}^{s} \rightarrow h_{0}^{s+N+1}, z \mapsto \nabla h_{n, k}(z ; y)$ is real analytic and satisfies the estimate $\left\|\nabla h_{n, k}(z ; y)\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}\left\|z_{\perp}\right\|_{0}^{2}$. The estimates for the higher order derivatives are obtained by differentiating $\nabla h_{n, k}$ and applying again Corollary 3.2.

Term $\mathcal{H}_{\Omega}(z):=\frac{1}{2}\left\langle\Omega_{\perp}\left(I_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle$ : According to (4.3), the operator $\Omega_{\perp}\left(I_{S}\right)$ reads

$$
\begin{equation*}
\Omega_{\perp}\left(I_{S}\right)=D_{\perp}^{2}+\Omega_{\perp}^{(0)}\left(I_{S}\right) \tag{4.24}
\end{equation*}
$$

where

$$
\begin{equation*}
D_{\perp}:=\operatorname{diag}_{n \in S^{\perp}}(2 \pi n), \quad \Omega_{\perp}^{(0)}\left(I_{S}\right):=\operatorname{diag}_{n \in S^{\perp}}\left(\Omega_{n}\left(I_{S}\right)-4 \pi^{2} n^{2}\right) \tag{4.25}
\end{equation*}
$$

By Lemma 4.1, the following holds.

Lemma 4.5. For any integer $N \geq 2$, the operator $\Omega_{\perp}^{(0)}\left(I_{S}\right)$ admits the expansion

$$
\Omega_{\perp}^{(0)}\left(I_{S}\right)=\mathcal{F}_{\perp} \circ \sum_{k=2}^{N} a_{k}\left(I_{S} ; \Omega_{\perp}^{(0)}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}+\mathcal{R}_{N}\left(I_{S} ; \Omega_{\perp}^{(0)}\right)
$$

where for any $2 \leq k \leq N$ and $s \geq 0$, the maps

$$
\mathbb{R}_{\geq 0}^{S_{+}} \rightarrow \mathbb{R}, I_{S} \mapsto a_{k}\left(I_{S} ; \Omega_{\perp}^{(0)}\right), \quad \mathbb{R}_{\geq 0}^{S_{+}} \rightarrow \mathcal{B}\left(h_{\perp}^{s}, h_{\perp}^{s+N+1}\right), I_{S} \mapsto \mathcal{R}_{N}\left(I_{S} ; \Omega_{\perp}^{(0)}\right)
$$

are real analytic.
To analyze $\mathcal{H}_{\Omega}\left(\Psi_{C}(z)\right)$, we write the quadratic form $\left\langle\Omega_{\perp}\left(I_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle, z_{\perp} \in h_{0}^{1}$, as a sum

$$
\left\langle\Omega_{\perp}\left(I_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle=\left\langle D_{\perp}^{2}\left[z_{\perp}\right], z_{\perp}\right\rangle+\left\langle\Omega_{\perp}^{(0)}\left(I_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle
$$

and consider $\left\langle D_{\perp}^{2}\left[z_{\perp}\right], z_{\perp}\right\rangle$ and $\left\langle\Omega_{\perp}^{(0)}\left(I_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle$ separately. When substituting for $z_{\perp}$ in $\left\langle D_{\perp}^{2}\left[z_{\perp}\right], z_{\perp}\right\rangle$ the expression $\pi_{\perp} \Psi_{C}(z)=z_{\perp}+\pi_{\perp} \tilde{\Psi}_{C}(z)$, one gets

$$
\begin{align*}
\left\langle D_{\perp}^{2}\left[z_{\perp}+\pi_{\perp} \tilde{\Psi}_{C}(z)\right],\right. & \left.z_{\perp}+\pi_{\perp} \tilde{\Psi}_{C}(z)\right\rangle=\left\langle D_{\perp}^{2}\left[z_{\perp}\right], z_{\perp}\right\rangle+\left\langle D_{\perp}^{2}\left[z_{\perp}\right], \pi_{\perp} \tilde{\Psi}_{C}(z)\right\rangle \\
& +\left\langle D_{\perp}^{2}\left[\pi_{\perp} \tilde{\Psi}_{C}(z)\right], z_{\perp}\right\rangle+\left\langle D_{\perp}^{2}\left[\pi_{\perp} \tilde{\Psi}_{C}(z)\right], \pi_{\perp} \tilde{\Psi}_{C}(z)\right\rangle \tag{4.26}
\end{align*}
$$

where the map $\pi_{\perp}$ is defined in (3.10). With a view towards the expansion of $H^{k d v} \circ \Psi$, stated in Theorem 1.1. we treat the difference

$$
\frac{1}{2}\left\langle D_{\perp}^{2}\left[z_{\perp}+\pi_{\perp} \tilde{\Psi}_{C}(z)\right], z_{\perp}+\pi_{\perp} \tilde{\Psi}_{C}(z)\right\rangle-\frac{1}{2}\left\langle D_{\perp}^{2}\left[z_{\perp}\right], z_{\perp}\right\rangle
$$

as part of the error term $\mathcal{P}_{3}(z)$. It needs special care since the two terms

$$
\left\langle D_{\perp}^{2}\left[z_{\perp}\right], \pi_{\perp} \tilde{\Psi}_{C}(z)\right\rangle, \quad\left\langle D_{\perp}^{2}\left[\pi_{\perp} \tilde{\Psi}_{C}(z)\right], z_{\perp}\right\rangle
$$

could have the property that the associated Hamiltonian vector field is unbounded. We write

$$
\begin{equation*}
\mathcal{H}_{\Omega}\left(\Psi_{C}(z)\right)=\mathcal{H}_{\Omega}(z)+\mathcal{P}_{3}^{(2 b)}(z), \quad \mathcal{P}_{3}^{(2 b)}(z):=\mathcal{H}_{\Omega}\left(\Psi_{C}(z)\right)-\mathcal{H}_{\Omega}(z) \tag{4.27}
\end{equation*}
$$

and note that by the mean value theorem,

$$
\begin{equation*}
\mathcal{P}_{3}^{(2 b)}(z)=\int_{0}^{1} \mathcal{P}_{\Omega}\left(\tau, \Psi_{X}^{0, \tau}(z)\right) d \tau \tag{4.28}
\end{equation*}
$$

where for $\tau \in[0,1], z \in \mathcal{V}, \mathcal{P}_{\Omega}(\tau, z)$ is defined by

$$
\begin{equation*}
\mathcal{P}_{\Omega}(\tau, z):=\left\langle\nabla \mathcal{H}_{\Omega}(z), X(\tau, z)\right\rangle . \tag{4.29}
\end{equation*}
$$

In a first step we analyze $\mathcal{P}_{\Omega}(\tau, z)$. One has

$$
\begin{equation*}
\left\langle\nabla \mathcal{H}_{\Omega}(z), X(\tau, z)\right\rangle=\left\langle\nabla_{S} \mathcal{H}_{\Omega}(z), \pi_{S} X(\tau, z)\right\rangle+\frac{1}{2}\left\langle\Omega_{\perp}\left(I_{S}\right) z_{\perp}, \pi_{\perp} X(\tau, z)\right\rangle . \tag{4.30}
\end{equation*}
$$

Since $\mathcal{H}_{\Omega}=\frac{1}{2}\left\langle\Omega_{\perp}\left(I_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle$ and $\Omega_{\perp}\left(I_{S}\right)=D_{\perp}^{2}+\Omega_{\perp}^{(0)}\left(I_{S}\right)$ one has

$$
\begin{equation*}
\left\langle\nabla_{S} \mathcal{H}_{\Omega}(z), \pi_{S} X(\tau, z)\right\rangle=\frac{1}{2} \sum_{j \in S}\left(\partial_{z_{-j}} \mathcal{H}_{\Omega}(z)\right)\left\langle X(\tau, z), e_{j}\right\rangle=\frac{1}{2} \sum_{j \in S}\left\langle\partial_{z_{-j}} \Omega_{\perp}^{(0)}\left(I_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle\left\langle X(\tau, z), e_{j}\right\rangle \tag{4.31}
\end{equation*}
$$

Concerning the term $\frac{1}{2}\left\langle\Omega_{\perp}\left(I_{S}\right) z_{\perp}, \pi_{\perp} X(\tau, z)\right\rangle$ in (4.30), recall that $X(\tau, z)=-\mathcal{L}_{\tau}(z)^{-1}[J \mathcal{E}(z)]$ (cf (3.7)), $\mathcal{L}_{\tau}(z)=\operatorname{Id}+\tau \mathrm{J} \mathcal{L}(\mathrm{z})(\mathrm{cf}(3.2))$ and hence

$$
\begin{equation*}
X(\tau, z)=-J \mathcal{E}(z)-\tau J \mathcal{L}(z)[X(\tau, z)] \tag{4.32}
\end{equation*}
$$

Since $\mathcal{E}(z)=\left(\mathcal{E}_{S}(z), 0\right)$ and $J^{t}=-J$, the term $\left\langle\Omega_{\perp}\left(I_{S}\right) z_{\perp}, \pi_{\perp} X(\tau, z)\right\rangle$ becomes

$$
\begin{equation*}
-\left\langle\Omega_{\perp}\left(I_{S}\right) z_{\perp}, \pi_{\perp} \tau J \mathcal{L}(z) X(\tau, z)\right\rangle=\tau\left\langle J_{\perp} \Omega_{\perp}\left(I_{S}\right) z_{\perp}, \pi_{\perp} \mathcal{L}(z) X(\tau, z)\right\rangle \tag{4.33}
\end{equation*}
$$

By (2.21) the component $\mathcal{L}_{\perp}^{\perp}(z)$ of $\mathcal{L}(z)$ vanishes, implying that $\pi_{\perp} \mathcal{L}(z) X(\tau, z)=\mathcal{L}_{\perp}^{S}(z) \pi_{S} X(\tau, z)$. Substituting the latter expression into (4.33) and using that $\mathcal{L}_{\perp}^{S}(z)^{t}=-\mathcal{L} \frac{\perp}{\perp}(z)$ since $\mathcal{L}(z)$ is skew adjoint (cf $(2.20)$ ) then leads to

$$
\begin{equation*}
\left\langle\Omega_{\perp}\left(I_{S}\right) z_{\perp}, \pi_{\perp} X(\tau, z)\right\rangle=\tau\left\langle J_{\perp} \Omega_{\perp}\left(I_{S}\right) z_{\perp}, \mathcal{L}_{\perp}^{S}(z) \pi_{S} X(\tau, z)\right\rangle=-\tau\left\langle\mathcal{L}_{S}^{\perp}(z) J_{\perp} \Omega_{\perp}\left(I_{S}\right) z_{\perp}, \pi_{S} X(\tau, z)\right\rangle \tag{4.34}
\end{equation*}
$$

Furthermore, by (2.23),

$$
\begin{equation*}
\mathcal{L}_{S}^{\perp}(z)\left[J_{\perp} \Omega_{\perp}\left(I_{S}\right) z_{\perp}\right]=\left(\left\langle\partial_{x}^{-1} \Psi_{1}\left(z_{S}\right)\left[J_{\perp} \Omega_{\perp}\left(I_{S}\right) z_{\perp}\right], \partial_{z_{n}} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right\rangle\right)_{n \in S} \tag{4.35}
\end{equation*}
$$

Since $\Omega_{\perp}\left(I_{S}\right)=D_{\perp}^{2}+\Omega_{\perp}^{(0)}\left(I_{S}\right)$ and $J_{\perp} D_{\perp}^{2}=\mathrm{i} D_{\perp}^{3}$ we need to analyze $\Psi_{1}\left(z_{S}\right) \mathrm{i} D_{\perp}^{3}$. By Remark $2.2, \Psi_{1}\left(z_{S}\right) \mathrm{i} D_{\perp}^{3}$ is a bounded linear operator $h_{\perp}^{s} \rightarrow \stackrel{H}{0}_{s-\overline{3}}$ for any $s \geq 0$.
Lemma 4.6. For any integer $N \geq 0$, the operator $\mathcal{T}\left(z_{S}\right):=\Psi_{1}\left(z_{S}\right) \mathrm{i} D_{\perp}^{3}-\mathcal{F}_{\perp}^{-1} \mathrm{i} D_{\perp}^{3} \mathcal{F}_{\perp} \Psi_{1}\left(z_{S}\right)$ admits the expansion

$$
\mathcal{T}\left(z_{S}\right)=\sum_{k=-1}^{N} a_{k}\left(z_{S} ; \mathcal{T}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}+\mathcal{R}_{N}\left(z_{S} ; \mathcal{T}\right)
$$

where for any $s \geq 0,-1 \leq k \leq N$, the maps

$$
\mathcal{V}_{S} \rightarrow H^{s}, z_{S} \mapsto a_{k}\left(z_{S} ; \mathcal{T}\right), \quad \mathcal{V}_{S} \rightarrow \mathcal{B}\left(h_{\perp}^{s}, H^{s+N+1}\right), z_{S} \mapsto \mathcal{R}_{N}\left(z_{S} ; \mathcal{T}\right)
$$

are real analytic. A similar statement holds for the transpose $\mathcal{T}\left(z_{S}\right)^{t}$ of $\mathcal{T}\left(z_{S}\right)$.
Proof. First note that the expression obtained from $\Psi_{1}\left(z_{S}\right) \mathrm{i} D_{\perp}^{3}-\mathcal{F}_{\perp}^{-1} \mathrm{i} D_{\perp}^{3} \mathcal{F}_{\perp} \Psi_{1}\left(z_{S}\right)$ by replacing $\Psi_{1}\left(z_{S}\right)$ by its highest order part $\mathcal{F}_{\perp}^{-1}$ (cf Theorem [2.1), vanishes. Since the order of the commutator of two scalar pseudodifferential operators of order one is again of order one, it follows that the operator $\mathcal{T}\left(z_{S}\right)$ is of order 1 , meaning that the expansion of $\mathcal{T}\left(z_{S}\right)$ is of the form as stated. Taking into account that $\mathrm{i} D_{\perp}^{3}=-\mathcal{F}_{\perp} \partial_{x}^{3} \mathcal{F}_{\perp}^{-1}$, the claimed statements follow from Theorem 2.1 (expansion of $\Psi_{1}\left(z_{S}\right)$ ) and Corollary 2.2 (expansion of $\left.\Psi_{1}\left(z_{S}\right)^{t}\right)$.

Taking into account that $J_{\perp} \Omega_{\perp}\left(I_{S}\right)=\mathrm{i} D_{\perp}^{3}+J_{\perp} \Omega_{\perp}^{(0)}\left(I_{S}\right)$, the operator $\partial_{x}^{-1} \Psi_{1}\left(z_{S}\right) J_{\perp} \Omega_{\perp}\left(I_{S}\right)$, appearing in formula 4.35) reads

$$
\partial_{x}^{-1} \Psi_{1}\left(z_{S}\right) J_{\perp} \Omega_{\perp}\left(I_{S}\right)=\partial_{x}^{-1} \Psi_{1}\left(z_{S}\right) \mathrm{i} D_{\perp}^{3}+\partial_{x}^{-1} \Psi_{1}\left(z_{S}\right) J_{\perp} \Omega_{\perp}^{(0)}\left(I_{S}\right)
$$

By the definition of $\mathcal{T}\left(z_{S}\right)$ and using that $\partial_{x}^{-1} \mathcal{F}_{\perp}^{-1} \mathrm{i} D_{\perp}^{3}=-\partial_{x}^{2} \mathcal{F}_{\perp}^{-1}$, one then gets

$$
\partial_{x}^{-1} \Psi_{1}\left(z_{S}\right) \mathrm{i} D_{\perp}^{3}=\partial_{x}^{-1} \mathcal{F}_{\perp}^{-1} \mathrm{i} D_{\perp}^{3} \mathcal{F}_{\perp} \Psi_{1}\left(z_{S}\right)+\partial_{x}^{-1} \mathcal{T}\left(z_{S}\right)=-\partial_{x}^{2} \Psi_{1}\left(z_{S}\right)+\partial_{x}^{-1} \mathcal{T}\left(z_{S}\right)
$$

Altogether we thus have shown that the nth component $\left\langle\partial_{x}^{-1} \Psi_{1}\left(z_{S}\right)\left[J_{\perp} \Omega_{\perp}\left(I_{S}\right) z_{\perp}\right], \partial_{z_{n}} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right\rangle, n \in S$, of $\mathcal{L} \frac{\perp}{S}(z)\left[J_{\perp} \Omega_{\perp}\left(I_{S}\right) z_{\perp}\right]$ is given by

$$
\begin{equation*}
\left(\mathcal{L}_{S}^{\perp}(z)\left[J_{\perp} \Omega_{\perp}\left(I_{S}\right) z_{\perp}\right]\right)_{n}=-\left\langle\partial_{x}^{2} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right], \partial_{z_{n}} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right\rangle+\left\langle\mathcal{T}_{1, n}\left(z_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle \tag{4.36}
\end{equation*}
$$

where for any $z_{S} \in \mathcal{V}_{S}$, the operator $\mathcal{T}_{1, n}\left(z_{S}\right)$ is given by

$$
\begin{equation*}
\mathcal{T}_{1, n}\left(z_{S}\right):=\left(\partial_{z_{n}} \Psi_{1}\left(z_{S}\right)\right)^{t} \partial_{x}^{-1} \mathcal{T}\left(z_{S}\right)+\left(\partial_{z_{n}} \Psi_{1}\left(z_{S}\right)\right)^{t} \partial_{x}^{-1} \Psi_{1}\left(z_{S}\right) J_{\perp} \Omega_{\perp}^{(0)}\left(I_{S}\right) \tag{4.37}
\end{equation*}
$$

Since $\left(\partial_{z_{n}} \Psi_{1}\left(z_{S}\right)\right)^{t}$ is one smoothing (cf Corollary 2.2) and $\partial_{x}^{-1} \mathcal{T}\left(z_{S}\right)$ is of order zero (cf Lemma 4.6), one sees that $\mathcal{T}_{1, n}\left(z_{S}\right)$ maps $h_{\perp}^{0}$ into $h_{\perp}^{1}$. More precisely, the following result holds.

Lemma 4.7. For any $n \in S$ and $N \in \mathbb{N}$, the operator $\mathcal{T}_{1, n}\left(z_{S}\right)$, defined by 4.37) for $z_{S} \in \mathcal{V}_{S}$, admits the expansion

$$
\mathcal{T}_{1, n}\left(z_{S}\right)=\mathcal{F}_{\perp} \circ \sum_{k=1}^{N} a_{k}\left(z_{S} ; \mathcal{T}_{1, n}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}+\mathcal{R}_{N}\left(z_{S} ; \mathcal{T}_{1, n}\right)
$$

where for any $s \geq 0,1 \leq k \leq N$, the maps

$$
\mathcal{V}_{S} \rightarrow H^{s}, z_{S} \mapsto a_{k}\left(z_{S} ; \mathcal{T}_{1, n}\right), \quad \mathcal{V}_{S} \rightarrow \mathcal{B}\left(h_{\perp}^{s}, h_{\perp}^{s+N+1}\right), z_{S} \mapsto \mathcal{R}_{N}\left(z_{S} ; \mathcal{T}_{1, n}\right)
$$

are real analytic.
Proof. The claimed statements follow from Corollary 2.2, Lemmata 4.5, 4.6, and Lemma E. 2,
We now turn our attention to the term $-\left\langle\partial_{x}^{2} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right], \partial_{z_{n}} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right\rangle$ in (4.36). By (4.13)

$$
d \nabla H^{k d v}(q)=-\partial_{x}^{2}+d \nabla H_{3}^{k d v}(q)=-\partial_{x}^{2}+6 q, \quad H_{3}^{k d v}(q):=\int_{0}^{1} q^{3} d x
$$

Hence using that $-\partial_{x}^{2}=d \nabla H^{k d v}(q)-6 q$ one obtains for any $n \in S$

$$
\begin{align*}
& -\left\langle\partial_{x}^{2} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right], \partial_{z_{n}} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right\rangle=\frac{1}{2} \partial_{z_{n}}\left\langle-\partial_{x}^{2} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right], \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right\rangle \\
& \quad=\frac{1}{2} \partial_{z_{n}}\left\langle d \nabla H^{k d v}(q)\left[\Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right], \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right\rangle-\frac{1}{2} \partial_{z_{n}}\left\langle 6 q \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right], \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right\rangle . \tag{4.38}
\end{align*}
$$

Since by (4.11),

$$
\Psi_{1}\left(z_{S}\right)^{t} d \nabla H^{k d v}(q) \Psi_{1}\left(z_{S}\right)=\Omega_{\perp}\left(I_{S}\right)+\mathcal{G}\left(z_{S}\right)
$$

we conclude that $-\left\langle\partial_{x}^{2} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right], \partial_{z_{n}} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right\rangle$ equals

$$
\begin{equation*}
\frac{1}{2}\left\langle\partial_{z_{n}} \Omega_{\perp}\left(I_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle+\frac{1}{2}\left\langle\partial_{z_{n}} \mathcal{G}\left(z_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle-\frac{1}{2}\left\langle\partial_{z_{n}}\left(6 \Psi_{1}\left(z_{S}\right)^{t} q \Psi_{1}\left(z_{S}\right)\right)\left[z_{\perp}\right], z_{\perp}\right\rangle \tag{4.39}
\end{equation*}
$$

Using again that for any $n \in S, \partial_{z_{n}} \Omega_{\perp}\left(I_{S}\right)=\partial_{z_{n}} \Omega_{\perp}^{(0)}\left(I_{S}\right)$, one thus obtains

$$
-\left\langle\partial_{x}^{2} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right], \partial_{z_{n}} \Psi_{1}\left(z_{S}\right)\left[z_{\perp}\right]\right\rangle=\left\langle\mathcal{T}_{2, n}\left(z_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle
$$

where

$$
\begin{equation*}
\mathcal{T}_{2, n}\left(z_{S}\right):=\frac{1}{2} \partial_{z_{n}}\left(\Omega_{\perp}^{(0)}\left(I_{S}\right)+\mathcal{G}\left(z_{S}\right)-6 \Psi_{1}\left(z_{S}\right)^{t} q \Psi_{1}\left(z_{S}\right)\right) \tag{4.40}
\end{equation*}
$$

Lemma 4.8. For any $n \in S$ and any integer $N \geq 0$, the operator $\mathcal{T}_{2, n}\left(z_{S}\right): h_{\perp}^{0} \rightarrow h_{\perp}^{0}$, defined by (4.40) for $z_{S} \in \mathcal{V}_{S}$, admits the expansion

$$
\mathcal{T}_{2, n}\left(z_{S}\right)=\mathcal{F}_{\perp} \circ \sum_{k=0}^{N} a_{k}\left(z_{S} ; \mathcal{T}_{2, n}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}+\mathcal{R}_{N}\left(z_{S} ; \mathcal{T}_{2, n}\right)
$$

where for any $s \geq 0,0 \leq k \leq N$, the maps

$$
\mathcal{V}_{S} \rightarrow H^{s}, z_{S} \mapsto a_{k}\left(z_{S} ; \mathcal{T}_{2, n}\right), \quad \mathcal{V}_{S} \rightarrow \mathcal{B}\left(h_{\perp}^{s}, h_{\perp}^{s+N+1}\right), z_{S} \mapsto \mathcal{R}_{N}\left(z_{S} ; \mathcal{T}_{2, n}\right)
$$

are real analytic. A similar statement holds for the transpose $\mathcal{T}_{2, n}\left(z_{S}\right)^{t}$ of the operator $\mathcal{T}_{2, n}\left(z_{S}\right)$.
Proof. The lemma follows by Lemmata 2.2, 4.2, 4.5, and Lemma E.2,

By (4.29) - (4.31), (4.34) - (4.36), and (4.40) the Hamiltonian $\mathcal{P}_{\Omega}(\tau, z)$, defined in (4.29), is given by

$$
\begin{align*}
\mathcal{P}_{\Omega}(\tau, z) & =\frac{1}{2}\left\langle\nabla_{S} \mathcal{H}_{\Omega}(z), \pi_{S} X(\tau, z)\right\rangle+\frac{\tau}{2} \sum_{j \in S}\left\langle\left(\mathcal{T}_{1, j}\left(z_{S}\right)+\mathcal{T}_{2, j}\left(z_{S}\right)\right)\left[z_{\perp}\right], z_{\perp}\right\rangle \cdot\left\langle X(\tau, z), e_{j}\right\rangle \\
& =\frac{1}{2} \sum_{j \in S}\left\langle\mathcal{T}_{3, j}\left(\tau, z_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle \cdot\left\langle X(\tau, z), e_{j}\right\rangle \tag{4.41}
\end{align*}
$$

where for any $j \in S, z_{S} \in \mathcal{V}_{S}$, and $0 \leq \tau \leq 1$, the operator $\mathcal{T}_{3, j}\left(\tau, z_{S}\right): h_{\perp}^{0} \rightarrow h_{\perp}^{0}$ is defined by

$$
\begin{equation*}
\mathcal{T}_{3, j}\left(\tau, z_{S}\right):=\partial_{z_{-j}} \Omega_{\perp}^{(0)}\left(I_{S}\right)+\tau \mathcal{T}_{1, j}\left(z_{S}\right)+\tau \mathcal{T}_{2, j}\left(z_{S}\right) \tag{4.42}
\end{equation*}
$$

The Hamiltonian $\mathcal{P}_{\Omega}(\tau, z)$ has the following properties.
Lemma 4.9. For any $0 \leq \tau \leq 1$ and any integer $N \geq 0$, the Hamiltonian $\mathcal{P}_{\Omega}(\tau, \cdot): \mathcal{V} \rightarrow \mathbb{R}$ is real analytic and $\nabla \mathcal{P}_{\Omega}(\tau, z)$ admits the expansion

$$
\nabla \mathcal{P}_{\Omega}(\tau, z)=\left(0, \mathcal{F}_{\perp} \circ \sum_{k=0}^{N} a_{k}\left(\tau, z ; \nabla \mathcal{P}_{\Omega}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N}\left(\tau, z ; \nabla \mathcal{P}_{\Omega}\right)
$$

where for any $s \geq 0,0 \leq k \leq N$, the maps

$$
\mathcal{V} \rightarrow H^{s}, z \mapsto a_{k}\left(\tau, z ; \nabla \mathcal{P}_{\Omega}\right), \quad \mathcal{V} \cap h_{0}^{s} \rightarrow h_{0}^{s+N+1}, z \mapsto \mathcal{R}_{N}\left(\tau, z ; \nabla \mathcal{P}_{\Omega}\right)
$$

are real analytic. Furthermore, for any $0 \leq \tau \leq 1, z \in \mathcal{V}, \widehat{z} \in h_{0}^{0}$

$$
\left\|a_{k}\left(\tau, z ; \nabla \mathcal{P}_{\Omega}\right)\right\|_{s} \lesssim s^{\|} z_{\perp}\left\|_{0}^{2}, \quad\right\| d a_{k}\left(\tau, z ; \nabla \mathcal{P}_{\Omega}\right)[\widehat{z}]\left\|_{s} \lesssim_{s}\right\| z_{\perp}\left\|_{0}\right\| \widehat{z} \|_{0}
$$

and if in addition, $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{0}, l \geq 2$,

$$
\left\|d^{l} a_{k}\left(\tau, z ; \nabla \mathcal{P}_{\Omega}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s} \lesssim_{s, l} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
$$

Similarly, for any $0 \leq \tau \leq 1, z \in \mathcal{V} \cap h_{0}^{s}, \widehat{z}_{1}, \widehat{z}_{2} \in h_{0}^{s}$,
$\left\|\mathcal{R}_{N}\left(\tau, z ; \nabla \mathcal{P}_{\Omega}\right)\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}\left\|z_{\perp}\right\|_{0}^{2}, \quad\left\|d \mathcal{R}_{N}\left(\tau, z ; \nabla \mathcal{P}_{\Omega}\right)\left[\widehat{z}_{1}\right]\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{0}^{2}\left\|\widehat{z}_{1}\right\|_{s}+\left\|z_{\perp}\right\|_{s}\left\|z_{\perp}\right\|_{0}\left\|\widehat{z}_{1}\right\|_{0}$,

$$
\left\|d^{2} \mathcal{R}_{N}\left(\tau, z ; \nabla \mathcal{P}_{\Omega}\right)\left[\widehat{z}_{1}, \widehat{z}_{2}\right]\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{0}\left(\left\|\widehat{z}_{1}\right\|_{s}\left\|\widehat{z}_{2}\right\|_{0}+\left\|\widehat{z}_{1}\right\|_{0}\left\|\widehat{z}_{2}\right\|_{s}\right)+\left\|z_{\perp}\right\|_{s}\left\|\widehat{z}_{1}\right\|_{0}\left\|\widehat{z}_{2}\right\|_{0},
$$

and if in addition $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s}, l \geq 3$,

$$
\left\|d^{l} \mathcal{R}_{N}\left(\tau, z ; \nabla \mathcal{P}_{\Omega}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+N+1} \lesssim_{s, N, l} \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{0}+\left\|z_{\perp}\right\|_{s} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
$$

Proof. One has $\nabla_{S} \mathcal{P}_{\Omega}(\tau, z)=\left(\partial_{z_{-n}} \mathcal{P}_{\Omega}(\tau, z)\right)_{n \in S}$ with
$\partial_{z_{-n}} \mathcal{P}_{\Omega}(\tau, z)=\frac{1}{2} \sum_{j \in S}\left\langle\partial_{z_{-n}} \mathcal{T}_{3, j}\left(\tau, z_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle \cdot\left\langle X(\tau, z), e_{j}\right\rangle+\frac{1}{2} \sum_{j \in S}\left\langle\mathcal{T}_{3, j}\left(\tau, z_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle \cdot\left\langle\partial_{z_{-n}} X(\tau, z), e_{j}\right\rangle$,
whereas $\nabla_{\perp} \mathcal{P}_{\Omega}(\tau, z)$ can be computed to be

$$
\nabla_{\perp} \mathcal{P}_{\Omega}(\tau, z)=\sum_{j \in S}\left\langle X(\tau, z), e_{j}\right\rangle \mathcal{T}_{3, j}\left(\tau, z_{S}\right)\left[z_{\perp}\right]+\frac{1}{2} \sum_{j \in S}\left\langle\mathcal{T}_{3, j}\left(\tau, z_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle\left(d_{\perp} X(\tau, z)\right)^{t}\left[e_{j}\right]
$$

The claimed statements then follow by Lemmata 3.6, 4.5, 4.7, 4.8

We are now ready to analyze the gradient of the Hamiltonian $\mathcal{P}_{3}^{(2 b)}(z):=\int_{0}^{1} \mathcal{P}_{\Omega}\left(\tau, \Psi_{X}^{0, \tau}(z)\right) d \tau(\operatorname{cf}(4.28))$.
Lemma 4.10. The Hamiltonian $\mathcal{P}_{3}^{(2 b)}: \mathcal{V}^{\prime} \rightarrow \mathbb{R}$ is real analytic and for any integer $N \geq 0$, its gradient $\nabla \mathcal{P}_{3}^{(2 b)}(z)$ admits the expansion

$$
\nabla \mathcal{P}_{3}^{(2 b)}(z)=\left(0, \mathcal{F}_{\perp} \circ \sum_{k=0}^{N} a_{k}\left(z ; \nabla \mathcal{P}_{3}^{(2 b)}\right) \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(2 b)}\right)
$$

where for any $s \geq 0,0 \leq k \leq N$, the maps

$$
\mathcal{V}^{\prime} \rightarrow H^{s}, z \mapsto a_{k}\left(z ; \nabla \mathcal{P}_{3}^{(2 b)}\right), \quad \mathcal{V}^{\prime} \cap h_{0}^{s} \rightarrow h_{0}^{s+N+1}, z \mapsto \mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(2 b)}\right)
$$

are real analytic. Furthermore, the following estimates hold: for any $z \in \mathcal{V}^{\prime}, \widehat{z} \in h_{0}^{0}$,

$$
\left\|a_{k}\left(z ; \nabla \mathcal{P}_{3}^{(2 b)}\right)\right\|_{s} \lesssim_{s}\left\|z_{\perp}\right\|_{0}^{2}, \quad\left\|d a_{k}\left(z ; \nabla \mathcal{P}_{3}^{(2 b)}\right)[\widehat{z}]\right\|_{s} \lesssim_{s}\left\|z_{\perp}\right\|_{0}\|\widehat{z}\|_{0}
$$

and if in addition $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{0}, l \geq 2$, then $\left\|d^{l} a_{k}\left(z ; \nabla \mathcal{P}_{3}^{(2 b)}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s} \lesssim_{s, l} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}$.
Similarly, for any $z \in \mathcal{V}^{\prime} \cap h_{0}^{s}, \widehat{z}_{1}, \widehat{z}_{2} \in h_{0}^{s}$,

$$
\begin{gathered}
\left\|\mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(2 b)}\right)\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s}\left\|z_{\perp}\right\|_{0}^{2}, \quad\left\|d \mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(2 b)}\right)\left[\widehat{z}_{1}\right]\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{0}^{2}\left\|\widehat{z}_{1}\right\|_{s}+\left\|z_{\perp}\right\|_{s}\left\|z_{\perp}\right\|_{0}\left\|\widehat{z}_{1}\right\|_{0}, \\
\left\|d^{2} \mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(2 b)}\right)\left[\widehat{z}_{1}, \widehat{z}_{2}\right]\right\|_{s+N+1} \lesssim s, N\left\|z_{\perp}\right\|_{0}\left(\left\|\widehat{z}_{1}\right\|_{s}\left\|\widehat{z}_{2}\right\|_{0}+\left\|\widehat{z}_{1}\right\|_{0}\left\|\widehat{z}_{2}\right\|_{s}\right)+\left\|z_{\perp}\right\|_{s}\left\|\widehat{z}_{1}\right\|_{0}\left\|\widehat{z}_{2}\right\|_{0},
\end{gathered}
$$

and if in addition $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s}, l \geq 2$, then

$$
\left\|d^{l} \mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(2 b)}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+N+1} \lesssim_{s, N, l} \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{0}+\left\|z_{\perp}\right\|_{s} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{0}
$$

Proof. By a straightforward computation, one has for any $z \in \mathcal{V}^{\prime}$,

$$
\nabla \mathcal{P}_{3}^{(2 b)}(z)=\int_{0}^{1}\left(d \Psi_{X}^{0, \tau}(z)\right)^{t} \nabla \mathcal{P}_{\Omega}\left(\tau, \Psi_{X}^{0, \tau}(z)\right) d \tau
$$

The claimed statements then follow by applying Corollary 3.1 (expansion of $d \Psi_{X}^{0, \tau}(z)^{t}$ ), Lemma 4.9 (expansion of $\nabla \mathcal{P}_{\Omega}(\tau, z)$ ), Theorem 3.1 (expansion of $\left.\Psi_{X}^{0, \tau}(z)\right)$, and Lemma E.2.

Terms $\mathcal{P}_{2}^{(1)}, \mathcal{P}_{3}^{(1)}$ : Recall that the Hamiltonians $\mathcal{P}_{2}^{(1)}$ and $\mathcal{P}_{3}^{(1)}$ were introduced in (4.17). We write

$$
\begin{equation*}
\mathcal{P}_{2}^{(1)}\left(\Psi_{C}(z)\right)+\mathcal{P}_{3}^{(1)}\left(\Psi_{C}(z)\right)=\mathcal{P}_{2}^{(1)}(z)+\mathcal{P}_{3}^{(2 c)}(z), \quad \mathcal{P}_{3}^{(2 c)}(z):=\mathcal{P}_{2}^{(1)}\left(\Psi_{C}(z)\right)-\mathcal{P}_{2}^{(1)}(z)+\mathcal{P}_{3}^{(1)}\left(\Psi_{C}(z)\right) \tag{4.43}
\end{equation*}
$$

where by the mean value theorem

$$
\mathcal{P}_{2}^{(1)}\left(\Psi_{C}(z)\right)-\mathcal{P}_{2}^{(1)}(z)=\int_{0}^{1}\left\langle\nabla \mathcal{P}_{2}^{(1)}\left(z+y\left(\Psi_{C}(z)-z\right)\right), \Psi_{C}(z)-z\right\rangle d y
$$

The Hamiltonian $\mathcal{P}_{3}^{(2 c)}\left(\Psi_{C}(z)\right)$ has the following properties.
Lemma 4.11. The Hamiltonian $\mathcal{P}_{3}^{(2 c)}: \mathcal{V}^{\prime} \cap h_{0}^{1} \rightarrow \mathbb{R}$ is real analytic and for any integer $N \geq 0$ its gradient $\nabla \mathcal{P}_{3}^{(2 c)}(z)$ admits the expansion

$$
\nabla \mathcal{P}_{3}^{(2 c)}(z)=\left(0, \mathcal{F}_{\perp} \circ \sum_{k=0}^{N} T_{a_{k}\left(z ; \nabla \mathcal{P}_{3}^{(2 c)}\right)} \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(2 c)}\right)
$$

with the property that there exists an integer $\sigma_{N} \geq N$ (loss of regularity) such that for any $s \geq 0,0 \leq k \leq N$, the maps

$$
\mathcal{V}^{\prime} \cap h^{s+\sigma_{N}} \rightarrow H^{s}, z \mapsto a_{k}\left(z ; \nabla \mathcal{P}_{3}^{(2 c)}\right), \quad \mathcal{V}^{\prime} \cap h_{0}^{s \vee \sigma_{N}} \rightarrow h_{0}^{s+N+1}, z \mapsto \mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(2 c)}\right)
$$

are real analytic. Furthermore, for any $s \geq 0, z \in \mathcal{V}^{\prime} \cap h_{0}^{s+\sigma_{N}}$ with $\|z\|_{\sigma_{N}} \leq 1, \widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s+\sigma_{N}}, l \geq 1$,

$$
\begin{aligned}
& \left\|a_{k}\left(z ; \nabla \mathcal{P}_{3}^{(2 c)}\right)\right\|_{s} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s+\sigma_{N}} \\
& \left\|d^{l} a_{k}\left(z ; \nabla \mathcal{P}_{3}^{(2 c)}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s} \lesssim_{s, N, l} \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s+\sigma_{N}} \prod_{i \neq j}\left\|\widehat{z}_{j}\right\|_{\sigma_{N}}+\left\|z_{\perp}\right\|_{s+\sigma_{N}} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{\sigma_{N}}
\end{aligned}
$$

Similarly, for any $s \geq 0, z \in \mathcal{V}^{\prime} \cap h_{0}^{s \vee \sigma_{N}}$ with $\|z\|_{\sigma_{N}} \leq 1, \widehat{z} \in h_{0}^{s \vee \sigma_{N}}$,

$$
\begin{aligned}
& \left\|\mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(2 c)}\right)\right\|_{s+N+1} \lesssim s, N\left\|z_{\perp}\right\|_{s \vee \sigma_{N}}\left\|z_{\perp}\right\|_{\sigma_{N}} \\
& \left\|d \mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(2 c)}\right)[\widehat{z}]\right\|_{s+N+1} \lesssim s, N\left\|z_{\perp}\right\|_{\sigma_{N}}\|\widehat{z}\|_{s \vee \sigma_{N}}+\left\|z_{\perp}\right\|_{s \vee \sigma_{N}}\|\widehat{z}\|_{\sigma_{N}}
\end{aligned}
$$

and if in addition $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s \vee \sigma_{N}}, l \geq 2$,

$$
\left\|d^{l} \mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(2 c)}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+N+1} \lesssim_{s, N, l} \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s \vee \sigma_{N}} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{\sigma_{N}}+\left\|z_{\perp}\right\|_{s \vee \sigma_{N}} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{\sigma_{N}}
$$

Proof. The lemma follows by differentiating the Hamiltonian $\mathcal{P}_{3}^{(2 c)}$, defined in (4.43) and then applying Corollary 3.2, Lemmata 4.2, 4.3 and using Lemmata E.1, E.2

By (4.16), (4.22), (4.27), (4.43) it follows that for $z=\left(z_{S}, z_{\perp}\right) \in \mathcal{V}^{\prime}$, the Hamiltonian $\mathcal{H}^{(2)}(z)$ is given by

$$
\begin{equation*}
\mathcal{H}^{(2)}(z)=\mathcal{H}^{k d v}\left(I_{S}\right)+\frac{1}{2}\left\langle\Omega_{\perp}\left(I_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle+\mathcal{P}_{2}^{(2)}(z)+\mathcal{P}_{3}^{(2)}(z) \tag{4.44}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{P}_{2}^{(2)}(z):=\left\langle\nabla_{S} \mathcal{H}_{S}^{k d v}(z), \pi_{S} \mathcal{R}_{N, 2}\left(z ; \Psi_{C}\right)\right\rangle+\mathcal{P}_{2}^{(1)}(z), \quad \mathcal{P}_{3}^{(2)}(z):=\mathcal{P}_{3}^{(2 a)}(z)+\mathcal{P}_{3}^{(2 b)}(z)+\mathcal{P}_{3}^{(2 c)}(z) \tag{4.45}
\end{equation*}
$$

and where we recall that by (3.54) and (4.17)

$$
\begin{equation*}
\mathcal{R}_{N, 2}\left(z ; \Psi_{C}\right)=\frac{1}{2} d_{\perp}^{2} \mathcal{R}_{N}\left(\left(z_{S}, 0\right) ; \Psi_{C}\right)\left[z_{\perp}, z_{\perp}\right], \quad \mathcal{P}_{2}^{(1)}(z)=\frac{1}{2}\left\langle\mathcal{G}\left(z_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle \tag{4.46}
\end{equation*}
$$

Note that $\mathcal{P}_{2}^{(2)}$ is quadratic with respect to $z_{\perp}$, whereas $\mathcal{P}_{3}^{(2)}$ is a remainder term of order three in $z_{\perp}$. Being quadratic with respect to $z_{\perp}, \mathcal{P}_{2}^{(2)}$ can be written as

$$
\begin{equation*}
\mathcal{P}_{2}^{(2)}(z)=\frac{1}{2}\left\langle d_{\perp} \nabla_{\perp} \mathcal{P}_{2}^{(2)}\left(z_{S}, 0\right)\left[z_{\perp}\right], z_{\perp}\right\rangle \tag{4.47}
\end{equation*}
$$

The following vanishing lemma is due to Kuksin [19]. Since our setup is different from the one in [19], we include its proof for the convenience of the reader.
Lemma 4.12. The Hamiltonian $\mathcal{P}_{2}^{(2)}$ vanishes on $\mathcal{V}^{\prime}$.
Proof. In view of (4.47), it suffices to prove that for any $z_{S} \in \mathcal{V}_{S}^{\prime}$, the operator $d_{\perp} \nabla_{\perp} \mathcal{P}_{2}^{(2)}\left(z_{S}, 0\right)$ vanishes. We establish that $d_{\perp} \nabla_{\perp} \mathcal{P}_{2}^{(2)}\left(z_{S}, 0\right)=0$ by studying the linearization of $\partial_{t} w=J \nabla \mathcal{H}^{(2)}(w)$ along an arbitrary solution $w(t)$ of the form $w(t)=\left(w_{S}(t), 0\right)$. First we need to make some preliminary considerations. Let $t \mapsto q(t) \in M_{S}$ be a solution of the KdV equation $\partial_{t} q=\partial_{x} \nabla H^{k d v}(q)$ and denote by $t \mapsto z(t):=\left(z_{S}(t), 0\right)$ the corresponding solution in Birkhoff coordinates, defined by $q(t)=\Psi^{k d v}(z(t))$. It satisfies $\partial_{t} z(t)=$
$J \Omega\left(I_{S}\right)[z(t)]$. Furthermore, let $\widehat{q}(t)$ be the solution of the equation, obtained by linearizing the KdV equation along $q(t)$,

$$
\partial_{t} \widehat{q}(t)=\partial_{x} d \nabla H^{k d v}(q(t))[\widehat{q}(t)],
$$

with initial data $\widehat{q}^{0}:=d \Psi^{k d v}\left(z_{S}(0), 0\right)\left[0, \widehat{z}_{\perp}^{0}\right]$ and $\widehat{z}_{\perp}^{0} \in h_{\perp}^{3}$. Similarly, denote by $\widehat{z}(t)$ the solution of the equation, obtained by linearizing $\partial_{t} z=J \Omega\left(I_{S}\right)[z]$ along the solution $z(t)$ with initial data $\widehat{z}^{0}=\left(0, \widehat{z}_{\perp}^{0}\right)$, $\partial_{t} \widehat{z}(t)=J d \nabla \mathcal{H}^{k d v}(z(t))[\widehat{z}(t)]$. Since $\partial_{t} z(t)=J \Omega\left(I_{S}\right)[z(t)]$ one concludes that

$$
\begin{equation*}
\widehat{z}(t)=\left(0, \widehat{z}_{\perp}(t)\right), \quad \partial_{t} \widehat{z}_{\perp}(t)=J_{\perp} \Omega_{\perp}\left(I_{S}\right)\left[\widehat{z}_{\perp}(t)\right] . \tag{4.48}
\end{equation*}
$$

and since $\Psi^{k d v}$ is symplectic and $\mathcal{H}^{k d v}=H^{k d v} \circ \Psi^{k d v}$, one has $\widehat{q}(t)=d \Psi^{k d v}\left(z_{S}(t), 0\right)[\widehat{z}(t)]$. Recall that for any $z_{S} \in \mathcal{V}_{S}^{\prime}, \Psi_{L}\left(z_{S}, 0\right)=\Psi^{k d v}\left(z_{S}, 0\right)$ (cf definition (2.2) of $\Psi_{L}$ ) and $\Psi_{C}\left(z_{S}, 0\right)=\left(z_{S}, 0\right)$ (cf Corollary 3.2), implying that $\Psi\left(z_{S}, 0\right)=\Psi^{k d v}\left(z_{S}, 0\right)$ and hence $q(t)=\Psi\left(z_{S}(t), 0\right)$ for any $t$. Since $\Psi: \mathcal{V}^{\prime} \rightarrow H_{0}^{0}$ is symplectic and $\mathcal{H}^{(2)}=H^{k d v} \circ \Psi$, one sees that $z(t)=\left(z_{S}(t), 0\right)$ is also a solution of the equation $\partial_{t} w=J \nabla \mathcal{H}^{(2)}(w)$. With these preliminary considerations made, we are now ready to prove that $d_{\perp} \nabla_{\perp} \mathcal{P}_{2}^{(2)}\left(z_{S}, 0\right)$ vanishes. To this end consider the solution $\widehat{w}(t)$ of the equation obtained by linearizing $\partial_{t} w=J \nabla \mathcal{H}^{(2)}(w)$ along the solution $z(t)=$ $\left(z_{S}(t), 0\right)$ with initial data $\widehat{w}^{0}=\left(0, \widehat{z}_{\perp}^{0}\right)$. Again using that the map $\Psi$ is symplectic and $\mathcal{H}^{(2)}=H^{k d v} \circ \Psi$, it follows that $d \Psi(z(t))[\widehat{w}(t)]$ solves the linearized KdV equation. Since $d \Psi(z(0))=d \Psi^{k d v}(z(0))$ and $\widehat{w}^{0}=\widehat{z}^{0}$, one then concludes from the uniqueness of the initial value problem that $d \Psi(z(t))[\widehat{w}(t)]=d \Psi^{k d v}(z(t))[\widehat{z}(t)]$ and hence $\widehat{w}(t)=\widehat{z}(t)$ for any $t$. It means that $\widehat{z}(t)$ satisfies also the linear equation

$$
\partial_{t} \widehat{z}(t)=J d \nabla \mathcal{H}^{(2)}(z(t))[\widehat{z}(t)] .
$$

In view of the expansion (4.44) of $\mathcal{H}^{(2)}$ one then obtains

$$
\partial_{t} \widehat{z}_{\perp}(t)=J_{\perp} \Omega_{\perp}\left(I_{S}\right)\left[\widehat{z}_{\perp}(t)\right]+J_{\perp} d_{\perp} \nabla_{\perp} \mathcal{P}_{2}\left(z_{S}(t), 0\right)\left[\widehat{z}_{\perp}(t)\right] .
$$

Comparing the latter identity with (4.48) one concludes that in particular, $d_{\perp} \nabla_{\perp} \mathcal{P}_{2}\left(z_{S}(0), 0\right)=0$. Since the initial data $z_{S}(0) \in \mathcal{V}_{S}^{\prime}$ can be chosen arbitrarily, we thus have $d_{\perp} \nabla_{\perp} \mathcal{P}_{2}\left(z_{S}, 0\right)=0$ for any $z_{S} \in \mathcal{V}_{S}^{\prime}$ as claimed.

In summary, we have proved the following results on the Hamiltonian $\mathcal{H}^{(2)}=H^{k d v} \circ \Psi$.
Theorem 4.1. The Hamiltonian $\mathcal{H}^{(2)}: \mathcal{V}^{\prime} \cap h_{0}^{1} \rightarrow \mathbb{R}$ has an expansion of the form

$$
\begin{equation*}
\mathcal{H}^{(2)}(z)=H^{k d v}(q)+\frac{1}{2}\left\langle\Omega_{\perp}\left(I_{S}\right)\left[z_{\perp}\right], z_{\perp}\right\rangle+\mathcal{P}_{3}^{(2)}(z) \tag{4.49}
\end{equation*}
$$

where $\Omega_{\perp}\left(I_{S}\right)$ is given by (4.3) and the remainder term $\mathcal{P}_{3}^{(2)}$, defined by (4.45), satisfies the following: $\mathcal{P}_{3}^{(2)}: \mathcal{V}^{\prime} \cap h_{0}^{1} \rightarrow \mathbb{R}$ is real analytic and for any integer $N \geq 1$, its gradient $\nabla \mathcal{P}_{3}^{(2)}(z)$ admits the asymptotic expansion

$$
\nabla \mathcal{P}_{3}^{(2)}(z)=\left(0, \mathcal{F}_{\perp} \circ \sum_{k=0}^{N} T_{a_{k}\left(z ; \nabla \mathcal{P}_{3}^{(2)}\right)} \partial_{x}^{-k} \mathcal{F}_{\perp}^{-1}\left[z_{\perp}\right]\right)+\mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(2)}\right)
$$

with the property that there there exists an integer $\sigma_{N} \geq N$ (loss of regularity) so that for any $s \geq 0$, $0 \leq k \leq N$, the maps

$$
\mathcal{V}^{\prime} \cap h_{0}^{s+\sigma_{N}} \rightarrow H^{s}, z \mapsto a_{k}\left(z ; \nabla \mathcal{P}_{3}^{(2)}\right), \quad \mathcal{V}^{\prime} \cap h_{0}^{s \vee \sigma_{N}} \rightarrow h_{0}^{s+N+1}, z \mapsto \mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(2)}\right)
$$

are real analytic and satisfy the following estimates: for any $s \geq 0, z \in \mathcal{V}^{\prime} \cap h_{0}^{s+\sigma_{N}}$ with $\|z\|_{\sigma_{N}} \leq 1$, $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s+\sigma_{N}}, l \geq 1$,

$$
\begin{aligned}
& \left\|a_{k}\left(z ; \nabla \mathcal{P}_{3}^{(2)}\right)\right\|_{s} \lesssim s, N\left\|z_{\perp}\right\|_{s+\sigma_{N}}, \\
& \left\|d^{l} a_{k}\left(z ; \nabla \mathcal{P}_{3}^{(2)}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s} \lesssim s, N, l \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s+\sigma_{N}} \prod_{i \neq j}\left\|\widehat{z}_{j}\right\|_{\sigma_{N}}+\left\|z_{\perp}\right\|_{s+\sigma_{N}} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{\sigma_{N}} .
\end{aligned}
$$

Similarly, for any $s \geq 1, z \in \mathcal{V}^{\prime} \cap h_{0}^{s \vee \sigma_{N}}$ with $\left\|z_{\perp}\right\|_{\sigma_{N}} \leq 1, \widehat{z} \in h_{0}^{s \vee \sigma_{N}}$,

$$
\begin{aligned}
& \left\|\mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(2)}\right)\right\|_{s+N+1} \lesssim_{s, N}\left\|z_{\perp}\right\|_{s \vee \sigma_{N}}\left\|z_{\perp}\right\|_{\sigma_{N}} \\
& \| d \mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(2)}\right)\left[\widehat{z}\left\|_{s+N+1} \lesssim_{s, N}\right\| z_{\perp}\left\|_{\sigma_{N}}\right\| \widehat{z}\left\|_{s \vee \sigma_{N}}+\right\| z_{\perp}\left\|_{s \vee \sigma_{N}}\right\| \widehat{z} \|_{\sigma_{N}}\right.
\end{aligned}
$$

and if in addition $\widehat{z}_{1}, \ldots, \widehat{z}_{l} \in h_{0}^{s \vee \sigma_{N}}, l \geq 2$,

$$
\left\|d^{l} \mathcal{R}_{N}\left(z ; \nabla \mathcal{P}_{3}^{(2)}\right)\left[\widehat{z}_{1}, \ldots, \widehat{z}_{l}\right]\right\|_{s+N+1} \lesssim_{s, N, l} \sum_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{s \vee \sigma_{N}} \prod_{i \neq j}\left\|\widehat{z}_{i}\right\|_{\sigma_{N}}+\left\|z_{\perp}\right\|_{s \vee \sigma_{N}} \prod_{j=1}^{l}\left\|\widehat{z}_{j}\right\|_{\sigma_{N}}
$$

Proof. The identity (4.49) folllows from formula (4.44) and Lemma 4.12. The claimed asymptotic expansion of the gradient of $\mathcal{P}_{3}^{(2)}$ and its properties follow from Lemmata 4.4, 4.10, 4.11 and Lemma E.1.

## 5 Summary of the proofs of Theorem 1.1 and Theorem 1.2

In this section we summarize the proofs of Theorem 1.1, of its addendum, and of Theorem 1.2, First recall that in view of the envisioned applications, these theorems are formulated in terms of action angle coordinates on the submanifold $M_{S}^{o}$ of proper $S$-gap potentials. Denote by $\Xi$ the map relating action angle variables and complex Birkhoff coordinates,

$$
\Xi: \mathbb{T}^{S_{+}} \times \mathbb{R}_{>0}^{S_{+}} \times h_{\perp}^{0} \rightarrow h_{S}^{0} \times h_{\perp}^{0},\left(\theta_{S}, I_{S}, z_{\perp}\right) \mapsto\left(z_{S}\left(\theta_{S}, I_{S}\right), z_{\perp}\right), \quad z_{ \pm n}=\sqrt{2 \pi n I_{n}} e^{\mp \mathrm{i} \theta_{n}}, n \in S_{+}
$$

Clearly, $\Xi$ is symplectic and, for any $s \geq 0$, the map $\Xi: \mathbb{T}^{S_{+}} \times \mathbb{R}_{>0}^{S_{+}} \times h_{\perp}^{s} \rightarrow h_{S}^{0} \times h_{\perp}^{s}$, is real analytic. Furthermore, in view of the definition (1.9), the map $\Xi$ preserves the reversible structure. Hence the claimed results for the map $\Psi_{L} \circ \Psi_{C} \circ \Xi$ follow from the corresponding ones for the map $\Psi_{L} \circ \Psi_{C}$. In what follows we summarize the proofs of the results for $\Psi_{L} \circ \Psi_{C}$ corresponding to the ones claimed for $\Psi_{L} \circ \Psi_{C} \circ \Xi$.
Proof of Theorem [1.1. By a slight abuse of notation, the map $\Psi$ of Theorem 1.1 is defined to be the composition $\Psi_{L} \circ \Psi_{C}$. By (3.8), it is defined on the neighborhood $\mathcal{V}^{\prime}=\mathcal{V}_{S}^{\prime} \times \mathcal{V}_{\perp}^{\prime}$ where $\mathcal{V}_{S}^{\prime}$ is a bounded neighborhood of any given compact subset $\mathcal{K} \subset h_{S}^{0}$ and $\mathcal{V}_{\perp}^{\prime}$ is a ball in $h_{\perp}^{0}$ of radius smaller than 1 , centered at 0 . The expansion of $\Psi$, corresponding to the one of (AE1), follows from the expansion for the map $\Psi_{L}$, provided by Theorem 2.1 and the one for the map $\Psi_{C}$, provided by Theorem 3.1.
The expansion of the transpose $d \Psi(z)^{t}$ of the derivative $d \Psi(z)$, corresponding to the one of (AE2), follows from the fact that $\Psi: \mathcal{V}^{\prime} \rightarrow L_{0}^{2}$ is symplectic, meaning that for any $z \in \mathcal{V}^{\prime}$, the operator $d \Psi(z)^{t}: H_{0}^{1} \rightarrow h_{0}^{1}$ satisfies $d \Psi(z)^{t}=J^{-1}(d \Psi(z))^{-1} \partial_{x}$. The expansion of $\Psi(z)$ in (AE1) then leads to an expansion of $d \Psi(z)$ and in turn of $(d \Psi(z))^{-1}$ and hence of $d \Psi(z)^{t}$. In addition, the identity $d \Psi(z)^{t}=J^{-1}(d \Psi(z))^{-1} \partial_{x}$ implies that the coefficient $a_{1}\left(z ; d \Psi^{t}\right)$ in the expansion of $d \Psi^{t}$ satisfies $a_{1}\left(z ; d \Psi^{t}\right)=-a_{1}(z ; \Psi)$.
The expansion of the Hamiltonian $\mathcal{H}^{(2)}$ and of the remainder term $\mathcal{P}_{3}^{(2)}$, corresponding to the one in (AE3), are provided in Theorem 4.1.
Proof of Addendum to Theorem 1.1. Clearly, the Fourier transform $\mathcal{F}$ and its inverse preserve the reversible structure and by Proposition D.1 so do the Birkhoff map $\Phi^{k d v}$ and its inverse $\Psi^{k d v}$. Furthermore, by the Addendum to Theorem 2.1 and the Addendum to Theorem 3.1 also the maps $\Psi_{L}$ and $\Psi_{C}$ and hence $\Psi_{L} \circ \Psi_{C}$ preserve the reversible structure, as do the coefficients and the remainder terms in their expansions as well as the transpose of their derivatives.
Clearly, the KdV Hamiltonian $H^{k d v}$ is reversible and therefore so is $\mathcal{H}^{(2)}=H^{k d v} \circ \Psi$. By (4.49) one then concludes that also the remainder $\mathcal{P}_{3}^{(2)}$ is reversible.
Proof of Theorem 1.2. The estimates of the coefficients and the remainder in the expansion of $\Psi=\Psi_{L} \circ \Psi_{C}$, corresponding to the ones of (Est1), follow from the estimates of the coefficients and the remainder in the expansion of the map $\Psi_{L}$, provided by Theorem [2.1] and the ones of the coefficients and the remainder in the expansion of the map $\Psi_{C}$, provided by Theorem 3.1.
The estimates of the coefficients and the remainder in the expansion of $d \Psi(z)^{t}$, corresponding to the one of (Est2), follow from the fact that $\Psi: \mathcal{V}^{\prime} \rightarrow L_{0}^{2}$ is symplectic, meaning that for any $z \in \mathcal{V}^{\prime}, d \Psi(z)^{t}: H_{0}^{1} \rightarrow h_{0}^{1}$
satisfies $d \Psi(z)^{t}=J^{-1}(d \Psi(z))^{-1} \partial_{x}$ and the estimates (Est1) of the coefficients and the remainder in the expansion of $\Psi(z)$ which lead to corresponding estimates of the coefficients and the remainder in the expansion of $d \Psi(z)$ and in turn of $(d \Psi(z))^{-1}$.
The estimates of the remainder term $\mathcal{P}_{3}^{(2)}$ in the expansion of the Hamiltonian $\mathcal{H}^{(2)}=H^{k d v} \circ \Psi$, corresponding to (Est3), are provided by Theorem 4.1

## A Birkhoff map

In this appendix we review the Birkhoff map and properties of it, relevant for our purposes. We refer to 13 and 9], [14, [16] for more details in these matters.

Theorem A. 1 ([13], [16]). There exists an open neighborhood $W$ of $L_{0}^{2}$ in $L_{0, \mathbb{C}}^{2}$ and a real analytic map

$$
\Phi^{k d v}: W \rightarrow h_{0, \mathbb{C}}^{0}, \quad q \mapsto z(q)=\left(z_{n}(q)\right)_{n \neq 0}
$$

with $\Phi^{k d v}(0)=0$ so that the following holds:
(B0) For any $n \in \mathbb{N}$, the complex Birkhoff coordinates $z_{n}(q), z_{-n}(q)$ are related to the Birkhoff coordinates $x_{n}(q), y_{n}(q)$ as introduced in [13] by the formulas (1.4).
(B1) For any $s \in \mathbb{Z}_{\geq 0}$, the restriction of $\Phi^{k d v}$ to $H_{0}^{s}$ gives rise to a map $\Phi^{k d v}: H_{0}^{s} \rightarrow h_{0}^{s}$ which is a bi-analytic diffeomorphism.
(B2) The map $\Phi^{k d v}$ is canonical, meaning that on $W,\left\{z_{n}, z_{-n}\right\}=\mathrm{i} 2 \pi n$ for any $n \in \mathbb{N}$ and the brackets between all other coordinate functions vanish.
(B3) The Hamiltonian $H^{k d v} \circ\left(\Phi^{k d v}\right)^{-1}$, defined on $h_{0}^{1}$, only depends on the actions $\left(I_{n}\right)_{n \in \mathbb{N}}$ (cf (1.5)). More precisely, it can be viewed as a real analytic map $\mathcal{H}^{k d v}$ on a complex neighborhood of the positive quadrant $\ell_{+}^{1,3}$ in $\ell^{1,3}(\mathbb{N}, \mathbb{C})(c f(1.6))$.
(B4) The differential $d_{0} \Phi^{k d v}$ of $\Phi^{k d v}$ at 0 is the Fourier transform $\mathcal{F}$ (cf (1.7)).
(B5) The nonlinear part of the Birkhoff map, $A^{k d v}:=\Phi^{k d v}-\mathcal{F}$, and the one of its inverse, $B^{k d v}:=$ $\left(\Phi^{k d v}\right)^{-1}-\mathcal{F}^{-1}$, are one smoothing. More precisely, for any $s \in \mathbb{N}, A^{k d v}: H_{0}^{s} \rightarrow h_{0}^{s+1}$ and $B^{k d v}:$ $h_{0}^{s} \rightarrow H_{0}^{s+1}$ are real analytic.

The inverse of $\Phi^{k d v}$ is denoted by $\Psi^{k d v}$.
To continue we first need to introduce some more notations and review properties of the Schrödinger operator $-\partial_{x}^{2}+q$. For any $s \in \mathbb{Z}_{\geq 0}$, denote by $H_{\mathbb{C}}^{s}[0,1] \equiv H^{s}([0,1], \mathbb{C})$ the Sobolev space of functions $f:[0,1] \rightarrow \mathbb{C}$ with the property that for any $0 \leq j \leq s$, the distributional derivative $\partial_{x}^{j} f$ is in $L_{\mathbb{C}}^{2}[0,1] \equiv$ $L^{2}([0,1], \mathbb{C})$. Similarly, $H_{0, \mathbb{C}}^{s} \equiv H_{0}^{s}(\mathbb{T}, \mathbb{C})$ denotes the Sobolev space of functions $q: \mathbb{T} \rightarrow \mathbb{C}$ in $H^{s}(\mathbb{T}, \mathbb{C})$ with $\int_{0}^{1} q(x) d x=0$. For any $q \in L_{0, \mathbb{C}}^{2} \equiv H_{0, \mathbb{C}}^{0}$ and $\lambda \in \mathbb{C}$, we denote by $y_{j}(x, \lambda) \equiv y_{j}(x, \lambda, q), j=1,2$, the fundamental solutions of $-y^{\prime \prime}+q y=\lambda y$. These are the solutions satisfying the initial conditions $y_{1}(0, \lambda)=1$, $y_{1}^{\prime}(0, \lambda)=0$ and $y_{2}(0, \lambda)=0, y_{2}^{\prime}(0, \lambda)=1$. It is well known that for any $s \in \mathbb{Z}_{\geq 0}$ and $1 \leq j \leq 2$, the map

$$
\mathbb{C} \times H_{0, \mathbb{C}}^{s} \rightarrow H_{\mathbb{C}}^{s+2}[0,1],(\lambda, q) \mapsto y_{j}(\cdot, \lambda, q)
$$

is analytic (cf [24]). For $q$ in $L_{0, \mathbb{C}}^{2}$, the Schrödinger operator $-\partial_{x}^{2}+q$, considered on the interval [0, 2] with periodic boundary conditions, has a discrete spectrum. It consists of a sequence of complex numbers bounded from below. We list them lexicographically and with algebraic multiplicities, i.e., $\lambda_{0}^{+} \preceq \lambda_{1}^{-} \preceq \lambda_{1}^{+} \preceq \lambda_{2}^{-} \preceq \ldots$ where $\lambda_{n}^{ \pm} \equiv \lambda_{n}^{ \pm}(q)$ (cf [13]). They are referred to as periodic eigenvalues of $q$ and satisfy the asymptotic estimates $\lambda_{n}^{+}, \lambda_{n}^{-}=n^{2} \pi^{2}+\ell_{n}^{2}$, valid uniformly on bounded subsets of $L_{0, \mathbb{C}}^{2}$ (cf [13]). For real valued $q$, the periodic eigenvalues are real and come in isolated pairs, meaning that $\lambda_{0}^{+}<\lambda_{1}^{-} \leq \lambda_{1}^{+}<\lambda_{2}^{-} \leq \lambda_{2}^{+}<\ldots$. We remark that for any given finite subset $S_{+} \subseteq \mathbb{N}$, the manifold $M_{S}$ of $S$-gap potentials defined in the introduction, coincides with the set $\left\{q \in L_{0}^{2}: \lambda_{n}^{-}(q)=\lambda_{n}^{+}(q) \forall n \in S_{+}^{\perp}\right\}$.

By shrinking the neigbourhood $W$ of Theorem A.1 if needed, one can ensure that for any $q \in W$, the closed intervals

$$
G_{n}=\left\{(1-t) \lambda_{n}^{-}+t \lambda_{n}^{+} \mid 0 \leq t \leq 1\right\}, \quad n \geq 1, \quad G_{0}=\left\{t+\lambda_{0}^{+} \mid-\infty<t \leq 0\right\}
$$

are disjoint from each other. By a slight abuse of terminology, we refer to the closed interval $G_{n}, n \geq 1$, as the $n$ 'th gap and to $\gamma_{n} \equiv \gamma_{n}(q)$ as the $n$ 'th gap length, $\gamma_{n}:=\lambda_{n}^{+}-\lambda_{n}^{-}$and denote by $\tau_{n} \equiv \tau_{n}(q)$ the middle point of $G_{n}, \tau_{n}=\left(\lambda_{n}^{+}+\lambda_{n}^{-}\right) / 2$. Due to the asymptotic behaviour of the periodic eigenvalues, $\left(G_{n}\right)_{n \geq 1}$ admit mutually disjoint neighbourhoods $U_{n} \subseteq \mathbb{C}, n \geq 1$, with $G_{n} \subseteq U_{n}$, referred to as isolating neighbourhoods. They can be chosen locally independently of $q$ (cf [13]). Denote by $F(\lambda) \equiv F(\lambda, q)$ the Floquet matrix

$$
F(\lambda)=\left(\begin{array}{ll}
m_{1}(\lambda) & m_{2}(\lambda)  \tag{A.1}\\
m_{1}^{\prime}(\lambda) & m_{2}^{\prime}(\lambda)
\end{array}\right), \quad m_{j}(\lambda)=y_{j}(1, \lambda), \quad m_{j}^{\prime}(\lambda)=y_{j}^{\prime}(1, \lambda), \quad j=1,2
$$

and introduce the discriminant $\Delta(\lambda) \equiv \Delta(\lambda, q):=\operatorname{Tr}(F(\lambda, q))$, its derivative $\dot{\Delta}(\lambda) \equiv \dot{\Delta}(\lambda, q):=\partial_{\lambda} \Delta(\lambda, q)$, and the anti-discriminant $\delta(\lambda) \equiv \delta(\lambda, q)=m_{1}(\lambda, q)-m_{2}^{\prime}(\lambda, q)$. The functions $m_{j}(\lambda) \equiv m_{j}(\lambda, q)$ and $m_{j}^{\prime}(\lambda) \equiv$ $m_{j}^{\prime}(\lambda, q)$ are analytic on $\mathbb{C} \times L_{0, \mathbb{C}}^{2}$. The entire functions $\Delta^{2}(\lambda)-4$ and $\dot{\Delta}(\lambda)$ have product representations (see [13, Proposition B.10, Proposition B.13])

$$
\begin{equation*}
\Delta^{2}(\lambda)-4=4\left(\lambda_{0}^{+}-\lambda\right) \prod_{n \geq 1} \frac{\left(\lambda_{n}^{+}-\lambda\right)\left(\lambda_{n}^{-}-\lambda\right)}{\pi_{n}^{4}}, \quad \dot{\Delta}(\lambda)=-\prod_{n \geq 1} \frac{\dot{\lambda}_{n}-\lambda}{\pi_{n}^{2}} \tag{A.2}
\end{equation*}
$$

where $\pi_{n}=n \pi$ for any $n \geq 1$ and where the zeroes $\dot{\lambda}_{n} \equiv \dot{\lambda}_{n}(q)$ satisfy the asymptotic estimate $\dot{\lambda}_{n}=n^{2} \pi^{2}+\ell_{n}^{2}$. We also need to consider the operator $-\partial_{x}^{2}+q$ on $[0,1]$ with Dirichlet and Neumann boundary conditions. For any $q$ in $L_{0, \mathbb{C}}^{2}$, the corresponding spectra are again discrete, consisting of sequences of complex numbers, bounded from below. They are referred to as Dirichlet and respectively, Neumann eigenvalues of $q$. We list them lexicographically and with their algebraic multiplicities $\mu_{1} \preceq \mu_{2} \preceq \mu_{3} \preceq \ldots$ and $\nu_{0} \preceq \nu_{1} \preceq \nu_{2} \preceq \ldots$. The $\mu_{n} \equiv \mu_{n}(q)$ and $\nu_{n} \equiv \nu_{n}(q)$ satisfy the asymptotics $\mu_{n}, \nu_{n}=n^{2} \pi^{2}+\ell_{n}^{2}$, valid uniformly on bounded subsets of $L_{0, \mathbb{C}}^{2}$. For real valued $q$, the Dirichlet and the Neumann eigenvalues are real and satisfy

$$
\lambda_{1}^{-} \leq \mu_{1} \leq \lambda_{1}^{+}<\lambda_{2}^{-} \leq \mu_{2} \leq \lambda_{2}^{+}<\ldots, \quad \nu_{0} \leq \lambda_{0}^{+}<\lambda_{1}^{-} \leq \nu_{1} \leq \lambda_{1}^{+}<\ldots
$$

By shrinking the neighbourhood $W$ of Theorem A.1 if needed, one can assure that for any $q \in W$ there exist isolating neighbourhoods $\left(U_{n}\right)_{n \geq 1}$ so that for any $n \geq 1, \mu_{n}, \nu_{n}, \tau_{n}, \dot{\lambda}_{n} \in U_{n}$, whereas $\lambda_{0}^{+}$and $\nu_{0}$ are not contained in any of the $U_{n}$ 's (cf [13]). Isolating neighbourhoods with this additional property can also be chosen locally independently of $q$. Note that for $q \in W$, the Dirichlet und Neumann eigenvalues are all simple and are analytic functions on $W$. Similarly, $\tau_{n}$ and $\dot{\lambda}_{n}, n \in \mathbb{N}$, are analytic on $W$. In addition, $m_{2}(\lambda)$ and $m_{1}^{\prime}(\lambda)$ admit the product representations (cf [13, Proposition B.6])

$$
m_{2}(\lambda)=\prod_{n \geq 1} \frac{\mu_{n}-\lambda}{\pi_{n}^{2}}, \quad m_{1}^{\prime}(\lambda)=-\left(\nu_{0}-\lambda\right) \prod_{n \geq 1} \frac{\nu_{n}-\lambda}{\pi_{n}^{2}}
$$

Let $S_{+} \subset \mathbb{N}$ be finite and set $S=S_{+} \cup\left(-S_{+}\right)$. For any $s \in \mathbb{Z}_{\geq 0}$, we identify $h_{0}^{s}$ with $h_{S}^{0} \times h_{\perp}^{s}$ and $h_{0, \mathbb{C}}^{s}$ with $\mathbb{C}^{S} \times h_{\perp, \mathbb{C}}^{s}$. The manifold $M_{S}$ of $S$-gap potentials is given by $\Psi^{k d v}\left(h_{S}^{0} \times\{0\}\right)$. By item $(\stackrel{\rightharpoonup}{B} 1)$ of Theorem A. 1 it then follows that $M_{S} \subset \cap_{s \geq 0} H_{0}^{s}$. Actually, potentials in $M_{S}$ are real analytic functions. For our purposes, it is useful to consider the Hilbert spaces $H_{0, \mathbb{C}}^{w}:=\left\{q \in L_{0, \mathbb{C}}^{2}:\|q\|_{w} \equiv\left\|\left(q_{n}\right)_{n \neq 0}\right\|_{w}<\infty\right\}$ and $h_{0, \mathbb{C}}^{w}:=\left\{\left(z_{n}\right)_{n \neq 0} \in \ell_{0, \mathbb{C}}^{2}:\left\|\left(z_{n}\right)_{n \neq 0}\right\|_{w}<\infty\right\}$ where

$$
\left\|\left(z_{n}\right)_{n \neq 0}\right\|_{w}:=\left(\sum_{n \neq 0} w_{n}^{2}\left|z_{n}\right|^{2}\right)^{1 / 2}, \quad w_{n}:=\langle n\rangle^{r} e^{a|n|^{\sigma}}, n \in \mathbb{Z}, \quad r \geq 0, a>0,0<\sigma<1
$$

The weight $w=\left(w_{n}\right)_{n \in \mathbb{Z}}$ is referred to as Gevrey weight and the Hilbert space $H_{0, \mathbb{C}}^{w}$ as weighted Sobolev space. Functions in $H_{0, \mathbb{C}}^{w}$ are Gevrey smooth. Correspondingly, we define the real Hilbert spaces $H_{0}^{w}:=\{q \in$ $H_{0, \mathbb{C}}^{w}: q$ real valued $\}$ and $h_{0}^{w}:=\left\{\left(z_{n}\right)_{n \neq 0} \in h_{0, \mathbb{C}}^{w}: z_{-n}=\overline{z_{n}} \forall n \geq 1\right\}$. To fix ideas we will only consider the Gevrey weight with parameters $r=0, a=1$, and $\sigma=1 / 2$ und denote it by $w_{*}$, but any other choice of a

Gevrey weight would also be possible. Note that $M_{S} \subset H_{0}^{w_{*}}$ and that $H_{0, \mathbb{C}}^{w_{*}}$ naturally embeds into $H_{0, \mathbb{C}}^{s}$ for any $s \in \mathbb{Z}_{\geq 0}$. According to [14, Addendum 1 to Theorem 5], we have the following
Addendum to Theorem A. 1 The restriction of $\Phi^{k d v}$ to $H_{0}^{w_{*}}$ gives rise to a map $\Phi^{k d v}: H_{0}^{w_{*}} \rightarrow h_{0}^{w_{*}}$ which is a bi-analytic diffeomorphism. In particular, for any $q \in M_{S}$, there exists a neighborhood $V_{q}^{*}$ of $q$ in $H_{0, \mathbb{C}}^{w_{*}} \cap W$ and a neighborhood $\mathcal{V}_{z(q)}^{*}$ of $z(q)=\Phi^{k d v}(q)$ in $h_{0, \mathbb{C}}^{w_{*}^{*}}$ so that the restriction of $\Phi^{k d v}$ to $V_{q}^{*}$ gives rise to a real analytic diffeomorphism $\Phi^{k d v}: V_{q}^{*} \rightarrow \mathcal{V}_{z(q)}^{*}$. The neighborhood $\mathcal{V}_{z(q)}^{*}$ can be chosen to be of the form $\mathcal{V}_{z_{S}(q)}^{*} \times \mathcal{V}_{0, \perp}^{*}$ where $\mathcal{V}_{z_{S}(q)}^{*}$ is a neighborhood of $z_{S}(q)=\left(z_{n}(q)\right)_{n \in S}$ in $\mathbb{C}^{S}$ and $\mathcal{V}_{0, \perp}^{*}$ is a ball in $h_{\perp, \mathbb{C}}^{w}$, centered at 0 with radius depending on $q$. We denote the set $\Psi^{k d v}\left(\mathcal{V}_{z_{S}(q)}^{*} \times\{0\}\right)$ by $V_{q, S}^{*}$. It consists of complex valued $S$-gap potentials near $q$.

## B Floquet solutions

In this appendix we obtain formulas for Floquet solutions $f_{ \pm n}(x, q), n \in S_{+}^{\perp}$, for potentials $q$ in $M_{S}$ which will be used in Proposition 2.2 of Section 2 to relate these solutions to the differentials of the complex Birkhoff coordinates $z_{ \pm n}(q)$. The resulting formulas are a key ingredient for proving the asymptotic expansion of the $\operatorname{map} \Psi_{L}$ (cf Section 2). Without further reference, we will use the notations estabished in Appendix A.
We begin by recalling the notion of Floquet solutions of $-y^{\prime \prime}+q y=\lambda y$ for any given $q \in L_{0}^{2}$ and $\lambda \in \mathbb{R}$. The eigenvalues $\kappa_{ \pm}(\lambda) \equiv \kappa_{ \pm}(\lambda, q)$ of the Floquet matrix $F(\lambda)$ (cf (A.1)) are given by the roots of $\operatorname{det}(F(\lambda)-$ $\left.\kappa \operatorname{Id}_{2 \times 2}\right)=\kappa^{2}-\Delta(\lambda) \kappa+1$. For $\lambda$ in $\left(\lambda_{0}^{+}, \infty\right) \backslash\left(\bigcup_{n \geq 1}\left[\lambda_{n}^{-}, \lambda_{n}^{+}\right]\right)$one has $\kappa_{ \pm}(\lambda)=\frac{\Delta(\lambda)}{2} \mp \frac{1}{2} \sqrt[c]{\Delta^{2}(\lambda)-4} \in \mathbb{C}$ where $\sqrt[c]{\Delta^{2}(\lambda)-4}$ denotes the canonical root determined by $\operatorname{sign}\left(\sqrt[c]{\Delta^{2}(\lambda)-4}\right)=-\mathrm{i}$ for $\lambda_{0}^{+}<\lambda<\lambda_{1}^{-}$ (cf [13, definition (6.10)]). For $\lambda \in\left(\lambda_{0}^{+}, \infty\right) \backslash\left(\bigcup_{n \geq 1}\left[\lambda_{n}^{-}, \lambda_{n}^{+}\right]\right), m_{2}(\lambda) \neq 0$ ( $\lambda$ is not a Dirichlet eigenvalue), $m_{1}^{\prime}(\lambda) \neq 0$ ( $\lambda$ is not a Neumann eigenvalue) and $\left(1, a_{+}(\lambda)\right) \in \mathbb{C}^{2}$ is an eigenvector of $F(\lambda)$ corresponding to the eigenvalue $\kappa_{+}(\lambda)$

$$
F(\lambda)\binom{1}{a_{+}(\lambda)}=\left(\begin{array}{ll}
m_{1}(\lambda) & m_{2}(\lambda) \\
m_{1}^{\prime}(\lambda) & m_{2}^{\prime}(\lambda)
\end{array}\right)\binom{1}{a_{+}(\lambda)}=\kappa_{+}(\lambda)\binom{1}{a_{+}(\lambda)}
$$

where $a_{+}(\lambda) \equiv a_{+}(\lambda, q)$ is given by

$$
\begin{equation*}
a_{+}(\lambda)=\frac{\kappa_{+}(\lambda)-m_{1}(\lambda)}{m_{2}(\lambda)} \quad \text { or, equivalently, } \quad a_{+}(\lambda)=\frac{m_{1}^{\prime}(\lambda)}{\kappa_{+}(\lambda)-m_{2}^{\prime}(\lambda)} . \tag{B.1}
\end{equation*}
$$

Similarly, $\left(1, a_{-}(\lambda)\right) \in \mathbb{C}^{2}, a_{-}(\lambda) \equiv a_{-}(\lambda, q)$, is an eigenvector of $F(\lambda)$ corresponding to the eigenvalue $\kappa_{-}(\lambda)$,

$$
\begin{equation*}
F(\lambda)\binom{1}{a_{-}(\lambda)}=\kappa_{-}(\lambda)\binom{1}{a_{-}(\lambda)}, \quad a_{-}(\lambda)=\frac{\kappa_{-}(\lambda)-m_{1}(\lambda)}{m_{2}(\lambda)} \quad \text { or } \quad a_{-}(\lambda)=\frac{m_{1}^{\prime}(\lambda)}{\kappa_{-}(\lambda)-m_{2}^{\prime}(\lambda)} . \tag{B.2}
\end{equation*}
$$

If $\lambda_{n}^{+}$is a double periodic eigenvalue, one has $\lambda_{n}^{-}=\lambda_{n}^{+}=\tau_{n}$ and $F\left(\tau_{n}\right)=(-1)^{n} \operatorname{Id}_{2 \times 2}$. By de l'Hospital's rule, the formulas in (B.1) - (B.2) admit limits at such eigenvalues. Recall that $S_{+} \subseteq \mathbb{N}$ is finite, $S_{+}^{\perp}=\mathbb{N} \backslash S_{+}$ and $S=S_{+} \cup\left(-S_{+}\right)$. We denote by ${ }^{\circ}$ the derivative with respect to $\lambda$.

Lemma B.1. For any $q \in M_{S}$ and $n \in S_{+}^{\perp}$, the following holds:
(i) $(-1)^{n} \dot{m}_{2}\left(\tau_{n}\right)>0,(-1)^{n+1} \ddot{\Delta}\left(\tau_{n}\right)>0$.
(ii) The limit $a_{ \pm n} \equiv a_{ \pm n}(q):=\lim _{\lambda \rightarrow \tau_{n}} a_{ \pm}(\lambda, q)$ exists and

$$
\begin{equation*}
a_{ \pm n}=-\frac{\dot{m}_{1}\left(\tau_{n}\right)}{\dot{m}_{2}\left(\tau_{n}\right)} \pm \mathrm{i} \frac{\sqrt[+]{(-1)^{n+1} \ddot{\Delta}\left(\tau_{n}\right) / 2}}{(-1)^{n} \dot{m}_{2}\left(\tau_{n}\right)} \tag{B.3}
\end{equation*}
$$

(iii) One has $\dot{m}_{1}{ }^{\prime}\left(\tau_{n}\right) \neq 0$ and

$$
\begin{equation*}
a_{ \pm n}=\frac{\dot{m}_{1}{ }^{\prime}\left(\tau_{n}\right)}{-\dot{m}_{2}{ }^{\prime}\left(\tau_{n}\right) \pm \mathrm{i}(-1)^{n} \sqrt[+]{(-1)^{n+1} \ddot{\Delta}\left(\tau_{n}\right) / 2}} \tag{B.4}
\end{equation*}
$$

Remark B.1. Recall that for any given $q \in M_{S}, V_{q, S}^{*}$ is the set of $S$-gap potentials, introduced at the end of Appendix A. By shrinking $W$ of Theorem A.1, if needed, the expressions in the formulas for $a_{ \pm n}, n \in S_{+}^{\perp}$, of item (ii) and (iii) of Lemma B.1 are well defined, real analytic functions on $V_{q, S}^{*}$ and the formulas for $a_{ \pm n}$ continue to hold for any potential in $V_{q, S}^{*}(c f$ Appendix (A)).
Proof. (i) For any $q \in M_{S}$ and $n \in S_{+}^{\perp}, \tau_{n}=\mu_{n}, m_{1}\left(\tau_{n}\right)=(-1)^{n}, m_{2}^{\prime}\left(\tau_{n}\right)=(-1)^{n}$, and $\dot{m}_{2}\left(\mu_{n}\right)=$ $(-1)^{n} \int_{0}^{1} y_{2}\left(x, \mu_{n}\right)^{2} d x$ (cf [13, Proposition B.4]). Since $\tau_{n}$ is a nondegenerate critical point of $\Delta$, one has $\dot{\Delta}\left(\tau_{n}\right)=0$ and $\ddot{\Delta}\left(\tau_{n}\right) \neq 0$. Furthermore, one has $\Delta\left(\tau_{n}\right)=(-1)^{n} 2$ and $(-1)^{n+1} \ddot{\Delta}\left(\tau_{n}\right)>0$. (ii) It is well known that $F(\lambda)$ is real analytic in $\lambda$ (cf Appendix A). Expanding $m_{1}(\lambda)$ and $m_{2}(\lambda)$ at $\tau_{n}, n \in S_{+}^{\perp}$, one has

$$
m_{1}(\lambda)=(-1)^{n}+\dot{m}_{1}\left(\tau_{n}\right)\left(\lambda-\tau_{n}\right)+O\left(\left(\lambda-\tau_{n}\right)^{2}\right), \quad m_{2}(\lambda)=\dot{m}_{2}\left(\tau_{n}\right)\left(\lambda-\tau_{n}\right)+O\left(\left(\lambda-\tau_{n}\right)^{2}\right)
$$

and $\Delta(\lambda)=(-1)^{n} 2+\frac{\ddot{\Delta}\left(\tau_{n}\right)}{2}\left(\lambda-\tau_{n}\right)^{2}+O\left(\left(\lambda-\tau_{n}\right)^{3}\right)$. It follows that for $\lambda_{n-1}^{+}<\lambda<\lambda_{n+1}^{-}$,

$$
\sqrt[c]{\Delta^{2}(\lambda)-4}=\mathrm{i}(-1)^{n+1} \sqrt[+]{(-1)^{n+1} 2 \ddot{\Delta}\left(\tau_{n}\right)}\left(\lambda-\tau_{n}\right)+O\left(\left(\lambda-\tau_{n}\right)^{2}\right)
$$

(where we set $\sqrt[c]{\Delta^{2}\left(\tau_{n}\right)-4}=0$ ). Combining these asymptotic estimates, the claimed formula (B.3) then follows from (B.1) - (B.2) (iii) Since $\tau_{n}$ is a Neumann eigenvalue and Neumann eigenvalues are simple, it follows that $\dot{m}_{1}^{\prime}\left(\tau_{n}\right) \neq 0$. Expanding $m_{1}^{\prime}(\lambda), m_{2}^{\prime}(\lambda)$ at $\tau_{n}$ one gets

$$
m_{1}^{\prime}(\lambda)=\dot{m}_{1}^{\prime}\left(\tau_{n}\right)\left(\lambda-\tau_{n}\right)+O\left(\left(\lambda-\tau_{n}\right)^{2}\right), \quad m_{2}^{\prime}(\lambda)=(-1)^{n}+\dot{m}_{2}^{\prime}\left(\tau_{n}\right)\left(\lambda-\tau_{n}\right)+O\left(\left(\lambda-\tau_{n}\right)^{2}\right)
$$

Furthermore, one has

$$
\kappa_{ \pm}(\lambda)=(-1)^{n} \pm \mathrm{i} \frac{(-1)^{n}}{2} \sqrt[+]{(-1)^{n+1} 2 \ddot{\Delta}\left(\tau_{n}\right) / 2}\left(\lambda-\tau_{n}\right)+O\left(\left(\lambda-\tau_{n}\right)^{2}\right)
$$

Formula ( $\overline{\mathrm{B} .4})$ then follows from ( $(\overline{\mathrm{B} .1})-(\overline{\mathrm{B} .2)}$.
For $q \in M_{S}$ and $n \in S_{+}^{\perp}$, we define the Floquet solutions $f_{ \pm n}(x) \equiv f_{ \pm n}(x, q)$ at $\tau_{n}$ by

$$
f_{ \pm n}(x, q):=y_{1}\left(x, \tau_{n}, q\right)+a_{ \pm n}(q) y_{2}\left(x, \tau_{n}, q\right)
$$

By Lemma B. 1 and Remark B. 1 they are welldefined for any potential in $V_{q, S}^{*}$. Furthermore, consider the normalized solutions $H_{n}(x) \equiv H_{n}(x, q)$ and $G_{n}(x) \equiv G_{n}(x, q)$ of $-y^{\prime \prime}+q y=\tau_{n} y$,

$$
\begin{align*}
& H_{n}(x):=\left(-\frac{2 \dot{m}_{2}\left(\tau_{n}\right)}{\ddot{\Delta}\left(\tau_{n}\right)}\right)^{\frac{1}{2}}\left(y_{1}\left(x, \tau_{n}\right)-\frac{\dot{m}_{1}\left(\tau_{n}\right)}{\dot{m}_{2}\left(\tau_{n}\right)} y_{2}\left(x, \tau_{n}\right)\right)  \tag{B.5}\\
& G_{n}(x):=\left(-\frac{2 \dot{m}_{2}\left(\tau_{n}\right)}{\ddot{\Delta}\left(\tau_{n}\right)}\right)^{\frac{1}{2}} \frac{+\sqrt{(-1)^{n+1} \ddot{\Delta}\left(\tau_{n}\right) / 2}}{(-1)^{n} \dot{m}_{2}\left(\tau_{n}\right)} y_{2}\left(x, \tau_{n}\right) \tag{B.6}
\end{align*}
$$

One then has $G_{n}(0)=0$ and

$$
\begin{equation*}
H_{n}(x)+\mathrm{i} G_{n}(x)=\left(-\frac{2 \dot{m}_{2}\left(\tau_{n}\right)}{\ddot{\Delta}\left(\tau_{n}\right)}\right)^{\frac{1}{2}} f_{n}(x), \quad H_{n}(x)-\mathrm{i} G_{n}(x)=\left(-\frac{2 \dot{m}_{2}\left(\tau_{n}\right)}{\ddot{\Delta}\left(\tau_{n}\right)}\right)^{\frac{1}{2}} f_{-n}(x) \tag{B.7}
\end{equation*}
$$

Note that for any $q \in M_{S}, f_{-n}(x, q)=\overline{f_{n}(x, q)}$, and $H_{n}(x, q)$ and $G_{n}(x, q)$ are the normalized real and respectively imaginary parts of $f_{n}(x, q)$. In addition, they satisfy $H_{n}(0, q)>0$ and $G_{n}^{\prime}(0, q)>0$ by the formulas above. Hence given $q \in M_{S}$, by shrinking $V_{q, S}^{*}$, if needed, we can assume that $\operatorname{Re} H_{n}(0)>0$ and $\operatorname{Re} G_{n}^{\prime}(0)>0$ on $V_{q, S}^{*}$ for any $n \in S_{+}^{\perp}$.

Proposition B.1. For any $q \in M_{S}$ and $n \in S_{+}^{\perp}$ the following holds: (i) For any $s \in \mathbb{Z}_{\geq 0}, H_{n}(\cdot, p), G_{n}(\cdot, p)$, and $f_{ \pm n}(\cdot, p)$ are analytic maps in $p \in V_{q, S}^{*}$ with values in $H_{\mathbb{C}}^{s}[0,1]$.
(ii) For any $p \in V_{q, S}^{*}$, the Floquet solutions

$$
\left(-\frac{2 \dot{m}_{2}\left(\tau_{n}\right)}{\ddot{\Delta}\left(\tau_{n}\right)}\right)^{\frac{1}{2}} f_{ \pm n}(x)=H_{n}(x) \pm \mathrm{i} G_{n}(x)
$$

have the property that $H_{n}, G_{n}$ are the unique solutions of $-y^{\prime \prime}+p y=\tau_{n} y$, satisfying the following normalization conditions: $\quad(i i 1) \int_{0}^{1} H_{n}(x) G_{n}(x) d x=0 \quad$ and
(ii2)

$$
\int_{0}^{1} G_{n}^{2}(x) d x=1, \quad G_{n}(0)=0, \quad \operatorname{Re} G_{n}^{\prime}(0)>0
$$

(ii3) $\int_{0}^{1} H_{n}^{2}(x) d x=1, \quad \operatorname{Re} H_{n}(0)>0$.

Proof. (i) Since for any $s \in \mathbb{Z}_{\geq 0}, H_{0, \mathbb{C}}^{w_{*}}$ embeds into the Sobolev space $H_{0, \mathbb{C}}^{s}$ the claimed analyticity statements follow from the results recorded in Appendix A (ii) Clearly, the statement on uniqueness follows from the uniqueness of the initial value problem for $-y^{\prime \prime}+q y=\tau_{n} y$. Hence it remains to prove that $G_{n}$ and $H_{n}$ satisfy items (ii1) - (ii3). In view of item (i), it suffices to verify these normalisation conditions on $M_{S}$. By [24], Theorem 6, page 21, one has

$$
\begin{gather*}
\dot{m}_{1}^{\prime}\left(\tau_{n}\right)=(-1)^{n+1} \int_{0}^{1} y_{1}\left(x, \tau_{n}\right)^{2} d x, \quad \dot{m}_{2}\left(\tau_{n}\right)=(-1)^{n} \int_{0}^{1} y_{2}\left(x, \tau_{n}\right)^{2} d x  \tag{B.8}\\
\dot{m}_{2}^{\prime}\left(\tau_{n}\right)=(-1)^{n+1} \int_{0}^{1} y_{1}\left(x, \tau_{n}\right) y_{2}\left(x, \tau_{n}\right) d x=-\dot{m}_{1}\left(\tau_{n}\right) \tag{B.9}
\end{gather*}
$$

To prove (ii1) it is to show that $J:=\int_{0}^{1}\left(\operatorname{Re} f_{n}(x)\right) y_{2}\left(x, \tau_{n}\right) d x=0$. By (B.5)-( (B.9),

$$
J=\int_{0}^{1}\left(y_{1}\left(x, \tau_{n}\right)-\frac{\dot{m}_{1}\left(\tau_{n}\right)}{\dot{m}_{2}\left(\tau_{n}\right)} y_{2}\left(x, \tau_{n}\right)\right) y_{2}\left(x, \tau_{n}\right) d x=(-1)^{n+1} \dot{m}_{2}^{\prime}\left(\tau_{n}\right)-\frac{\dot{m}_{1}\left(\tau_{n}\right)}{\dot{m}_{2}\left(\tau_{n}\right)}(-1)^{n} \dot{m}_{2}\left(\tau_{n}\right)
$$

Since $\dot{\Delta}\left(\tau_{n}\right)=0$ and at the same time $\dot{\Delta}\left(\tau_{n}\right)=\dot{m}_{1}\left(\tau_{n}\right)+\dot{m}_{2}^{\prime}\left(\tau_{n}\right)$ one concludes that $J=0$. (ii2) By the definition of $G_{n}$, one has $G_{n}(0)=0$ and $G_{n}^{\prime}(0)>0$. To see that $\int_{0}^{1} G_{n}(x)^{2} d x=1$, note that

$$
\int_{0}^{1}(\operatorname{Imf}(x))^{2} d x=\frac{(-1)^{n+1} \ddot{\Delta}\left(\tau_{n}\right) / 2}{\left(\dot{m}_{2}\left(\tau_{n}\right)\right)^{2}}(-1)^{n} \dot{m}_{2}\left(\tau_{n}\right)=-\frac{\ddot{\Delta}\left(\tau_{n}\right)}{2 \dot{m}_{2}\left(\tau_{n}\right)}
$$

implying that $\int_{0}^{1} G_{n}(x)^{2} d x=1$. (ii3) Since $\operatorname{Re} f_{n}(0)=y_{1}\left(0, \tau_{n}\right)=1$, one has $H_{n}(0)>0$ whereas

$$
J:=\int_{0}^{1} \operatorname{Re} f_{n}(x)^{2} d x=\int_{0}^{1} y_{1}\left(x, \tau_{n}\right)^{2}-2 \frac{\dot{m}_{1}\left(\tau_{n}\right)}{\dot{m}_{2}\left(\tau_{n}\right)} \int_{0}^{1} y_{1}\left(x, \tau_{n}\right) y_{2}\left(x, \tau_{n}\right) d x+\left(\frac{\dot{m}_{1}\left(\tau_{n}\right)}{\dot{m}_{2}\left(\tau_{n}\right)}\right)^{2} \int_{0}^{1} y_{2}\left(x, \tau_{n}\right)^{2} d x
$$

is given by (cf ( $\overline{\mathrm{B} .8})-(\overline{\mathrm{B} .9})$ )
$J=(-1)^{n+1} \dot{m}_{1}^{\prime}\left(\tau_{n}\right)-2 \frac{\dot{m}_{1}\left(\tau_{n}\right)}{\dot{m}_{2}\left(\tau_{n}\right)}(-1)^{n} \dot{m}_{1}\left(\tau_{n}\right)+\left(\frac{\dot{m}_{1}\left(\tau_{n}\right)}{\dot{m}_{2}\left(\tau_{n}\right)}\right)^{2}(-1)^{n} \dot{m}_{2}\left(\tau_{n}\right)=(-1)^{n+1} \dot{m}_{1}^{\prime}\left(\tau_{n}\right)+(-1)^{n+1} \frac{\dot{m}_{1}\left(\tau_{n}\right)^{2}}{\dot{m}_{2}\left(\tau_{n}\right)}$.
By the definition (B.5), it is therefore to show that

$$
(-1)^{n} \frac{\ddot{\Delta}\left(\tau_{n}\right)}{2 \dot{m}_{2}\left(\tau_{n}\right)}=\dot{m}_{1}^{\prime}\left(\tau_{n}\right)+\frac{\dot{m}_{1}\left(\tau_{n}\right)^{2}}{\dot{m}_{2}\left(\tau_{n}\right)} \quad \text { or } \quad \dot{m}_{1}^{\prime}\left(\tau_{n}\right) \dot{m}_{2}\left(\tau_{n}\right)=(-1)^{n} \ddot{\Delta}\left(\tau_{n}\right) / 2-\dot{m}_{1}\left(\tau_{n}\right)^{2}
$$

This latter identity follows by combining the two formulas for $a_{n}$, given in Lemma B. 1 .

## C Asymptotic expansions

The main purpose of this appendix is to provide for any $S$-gap potential $q$ an asymptotic expansion of the Floquet solutions $f_{ \pm n}(x) \equiv f_{ \pm n}(x, q)$ as $n \rightarrow \infty$. These expansions are a key ingredient for proving the asymptotic expansion of the map $\Psi_{L}$, stated in Theorem [2.1] in Section 2, At the end of this appendix we provide an asymptotic expansion for the KdV frequencies for $S$-gap potentials, needed for the expansion of the KdV Hamiltonian in the new coordinates.
Throughout this appendix, if not mentioned otherwise, we assume that $q \in M_{S}$ where $S=S_{+} \cup\left(-S_{+}\right)$and $S_{+} \subset \mathbb{N}$ is a finite subset. Furthermore, $V_{q, S}^{*}$ is the neighborhood of $q$, introduced at the end of Appendix A. If not mentioned otherwise, $\sqrt{\mu}$ denotes the principal branch $\sqrt[+]{\mu}$ of the square root, defined for $\mu \in \mathbb{C} \backslash(-\infty, 0]$. Recall that for any $p \in V_{q, S}^{*}$, one has $\int_{0}^{1} p(x) d x=0$ and

$$
\begin{equation*}
f_{ \pm n}(x)=y_{1}\left(x, \tau_{n}\right)+a_{ \pm n} y_{2}\left(x, \tau_{n}\right), \quad \forall n \in S_{+}^{\perp} \tag{C.1}
\end{equation*}
$$

where $y_{j}(x, \lambda), j=1,2$, denote the fundamental solutions of $-y^{\prime \prime}+p y=\lambda y, a_{ \pm n}$ are the complex numbers given by (B.3), and $\tau_{n}=\left(\lambda_{n}^{+}+\lambda_{n}^{-}\right) / 2$. Note that if $p$ is real valued, then

$$
\tau_{n}(p) \in \mathbb{R}, \quad a_{-n}(p)=\overline{a_{n}(p)}, \quad \text { and } \quad f_{-n}(x, p)=\overline{f_{n}(x, p)}, \quad \forall n \in S_{+}^{\perp}
$$

Theorem C.1. Let $q \in M_{S}$ and $N \in \mathbb{Z}_{\geq 0}$. Then for any $p \in V_{q, S}^{*}$, the Floquet solutions $f_{n}, n \in S^{\perp}$, have an expansion as $|n| \rightarrow \infty$ of the form

$$
\begin{equation*}
f_{n}(x, p)=e^{\mathrm{i} \pi n x}\left(1+\sum_{k=1}^{N} \frac{f_{k}^{a e}(x, p)}{(2 \pi \mathrm{i} n)^{k}}+\frac{\mathcal{R}_{N}^{f_{n}}(x, p)}{(2 \pi \mathrm{i} n)^{N+1}}\right) \tag{C.2}
\end{equation*}
$$

where for any $s \geq 0$, the coefficients $V_{q, S}^{*} \rightarrow H_{\mathbb{C}}^{s}, p \mapsto f_{k}^{a e}(\cdot, p), k \geq 1$, are real analytic and the remainder $V_{q, S}^{*} \rightarrow H_{\mathbb{C}}^{s}, p \mapsto \mathcal{R}_{N}^{f_{n}}(\cdot, p)$, is analytic. In addition, for any given $j \geq 0$,

$$
\begin{equation*}
\sup _{\substack{0 \leq x \leq 1 \\ n \in S^{\perp}}}\left|\partial_{x}^{j} \mathcal{R}_{N}^{f_{n}}(x, p)\right| \leq C_{N, j} \tag{C.3}
\end{equation*}
$$

where the constant $C_{N, j} \geq 1$ can be chosen locally uniformly for $p$ in $V_{q, S}^{*}$.
To prove Theorem C.1, we first need to establish some auxiliary results.
Lemma C.1. For any $q \in M_{S}$ and any integers $N, M \geq 0$, the following holds: for any $p \in V_{q, S}^{*}$ and $\nu \in \mathbb{C} \backslash\{0\}$, there exist solutions $y_{N, M}(x, \nu) \equiv y_{N, M}(x, \nu, p)$ of $-y^{\prime \prime}+p y=\nu^{2} y$ of the form

$$
\begin{equation*}
y_{N, M}(x, \nu, p)=e^{\mathrm{i} \nu x}\left(1+\sum_{k=1}^{N} \frac{y_{k}^{a e}(x, p)}{(2 \mathrm{i} \nu)^{k}}+\frac{\tilde{y}_{N, M}(x, \nu, p)}{(2 \mathrm{i} \nu)^{N+1}}\right) . \tag{C.4}
\end{equation*}
$$

The functions $y_{k}^{a e}(x) \equiv y_{k}^{a e}(x, p), k \geq 1$, are defined inductively by

$$
\begin{equation*}
y_{k}^{a e}(x)=\int_{0}^{x}\left(-\partial_{t}^{2}+p\right) y_{k-1}^{a e}(t) d t, \quad y_{0}^{a e}(x) \equiv 1 \tag{C.5}
\end{equation*}
$$

and for any $s \in \mathbb{Z}_{\geq 0}$, the maps $V_{q, S}^{*} \rightarrow H_{\mathbb{C}}^{s}[0,1]$, $p \mapsto y_{k}^{a e}(\cdot, p)$, are real analytic. The remainder $\tilde{y}_{N, M}(x, \nu, p)$ satisfies $\tilde{y}_{N, M}(0, \nu, p)=0, \partial_{x} \tilde{y}_{N, M}(0, \nu, p)=\sum_{k=1}^{M} \frac{\partial_{x} y_{N+k}^{a e}(0, p)}{(2 i \nu)^{k}}$ and has the property that for any $s \in \mathbb{Z}_{\geq 0}$, $(\mathbb{C} \backslash\{0\}) \times V_{q, S}^{*} \rightarrow H_{\mathbb{C}}^{s}[0,1],(\nu, p) \mapsto \tilde{y}_{N, M}(\cdot, \nu, p)$ is analytic. Furthermore, if $\nu \in \mathbb{R} \backslash\{0\}$ and $p$ is real valued then $y_{N, M}(x,-\nu, p)=\overline{y_{N, M}(x, \nu, p)}$. In addition, for any given $c>0$, the remainder $\tilde{y}_{N, M}(\cdot, \nu, p)$ satisfies

$$
\begin{equation*}
\sup _{|\operatorname{|im} \nu| \leq \mathrm{c},|\nu| \geq 1}^{\substack{\leq x \leq 1,0 \leq j \leq M}}\left|\partial_{x}^{j} \tilde{y}_{N, M}(x, \nu, p)\right| \leq C_{N, M} \tag{C.6}
\end{equation*}
$$

where the constant $C_{N, M}$ can be chosen locally uniformly in $p \in V_{q, S}^{*}$.

Remark C.1. By (C.5), for any $k \geq 1$ and $s \geq k-1$, the map $y_{k}^{a e}: H_{0, \mathbb{C}}^{s} \rightarrow H_{\mathbb{C}}^{s-k+2}[0,1], q \mapsto y_{k}^{a e}(\cdot, q)$ is real analytic. Writing $Q(x):=\int_{0}^{x} q(t) d t$, the formulas for $y_{1}^{a e}, y_{2}^{a e}$, and $y_{3}^{a e}$ read as follows:

$$
\begin{gathered}
y_{1}^{a e}(x, q)=Q(x), \quad y_{2}^{a e}(x, q)=-(q(x)-q(0))+\frac{1}{2} Q(x)^{2}, \\
y_{3}^{a e}(x, q)=\left(q^{\prime}(x)-q^{\prime}(0)\right)-\int_{0}^{x} q(t)^{2} d t+q(0) Q(x)-q(x) Q(x)+\frac{1}{6} Q(x)^{3} .
\end{gathered}
$$

Proof. For any $p \in V_{q, S}^{*}$ and $\nu \neq 0$, the solutions $y_{N, M}(x, \nu) \equiv y_{N, M}(x, \nu, p)$ of $-y^{\prime \prime}+p y=\nu^{2} y$ are obtained from solutions of the form

$$
\begin{equation*}
y_{N_{1}}(x, \nu)=e^{\mathrm{i} \nu x}\left(1+\sum_{k=1}^{N_{1}} \frac{y_{k}^{a e}(x)}{(2 \mathrm{i} \nu)^{k}}+\frac{\widetilde{y}_{N_{1}}(x, \nu)}{(2 \mathrm{i} \nu)^{N_{1}+1}}\right) \tag{C.7}
\end{equation*}
$$

with an appropriate choice of $N_{1}$. Solutions of the form (C.7) were studied in [21, Chapter 1, Section 4]. Here for any $k \geq 1, y_{k}^{a e}(x) \equiv y_{k}^{a e}(x, p)$ is defined by (C.5) and $\widetilde{y}_{N_{1}}(x, \nu) \equiv \widetilde{y}_{N_{1}}(x, \nu, p)$ is the unique solution of the inhomogeneous ODE

$$
\begin{equation*}
\left(-\partial_{x}^{2}+p-\nu^{2}\right)\left[e^{\mathrm{i} \nu x} \widetilde{y}_{N_{1}}(x, \nu)\right]=-2 \mathrm{i} \nu e^{\mathrm{i} \nu x} \partial_{x} y_{N_{1}+1}^{a e}(x) \tag{C.8}
\end{equation*}
$$

satisfying the initial conditions

$$
\begin{equation*}
\widetilde{y}_{N_{1}}(0, \nu)=0, \quad \partial_{x} \widetilde{y}_{N_{1}}(0, \nu)=0 \tag{C.9}
\end{equation*}
$$

Clearly, for any $s \geq 0$ and $k \geq 1$, the maps $V_{q, S}^{*} \rightarrow H_{\mathbb{C}}^{s}[0,1], p \mapsto y_{k}^{a e}(\cdot, p)$, are real analytic. Arguing as in [13, Chapter 1], one sees that for any $s \geq 0$, the map $(\mathbb{C} \backslash\{0\}) \times V_{q, S}^{*} \rightarrow H_{\mathbb{C}}^{s}[0,1],(\nu, p) \mapsto \tilde{y}_{N_{1}}(\cdot, \nu, p)$ is analytic. Furthermore, note that $y_{N_{1}}(0, \nu)=0$ and $\partial_{x} y_{N_{1}}(0, \nu)=\mathrm{i} \nu+\sum_{k=1}^{N_{1}} \frac{\partial_{x} y_{k}^{a e}(0)}{(2 \mathrm{i} \nu)^{k}}$. It then follows that for any $\nu \neq 0, y_{N_{1}}(x, \nu)$ is a solution of $-y^{\prime \prime}+p y=\nu^{2} y$. Indeed

$$
\begin{aligned}
&\left(-\partial_{x}^{2}+p-\nu^{2}\right) y_{N_{1}}=e^{\mathrm{i} \nu x} \sum_{k=1}^{N_{1}} \frac{-\partial_{x} y_{k}^{a e}-\partial_{x}^{2} y_{k-1}^{a e}+p y_{k-1}^{a e}}{(2 \mathrm{i} \nu)^{k-1}} \\
&+e^{\mathrm{i} \nu x} \frac{-\partial_{x}^{2} y_{N_{1}}^{a e}+p y_{N_{1}}^{a e}}{(2 \mathrm{i} \nu)^{N_{1}}}+\left(-\partial_{x}^{2}+p-\nu^{2}\right)\left[e^{\mathrm{i} \nu x} \frac{\widetilde{y}_{N_{1}}(x, \nu)}{(2 \mathrm{i} \nu)^{N_{1}+1}}\right]
\end{aligned}
$$

Hence by (C.5) and (C.8), $\left(-\partial_{x}^{2}+p-\nu^{2}\right) y_{N_{1}}=0$. The solution $y_{N, M}(x, \nu)$ is then defined as follows

$$
y_{N, M}(x, \nu)=e^{\mathrm{i} \nu x}\left(1+\sum_{k=1}^{N} \frac{y_{k}^{a e}(x)}{(2 \mathrm{i} \nu)^{k}}+\frac{\widetilde{y}_{N, M}(x, \nu)}{(2 \mathrm{i} \nu)^{N+1}}\right), \quad \widetilde{y}_{N, M}(x, \nu)=\sum_{k=1}^{M} \frac{y_{N+k}^{a e}(x)}{(2 \mathrm{i} \nu)^{k-1}}+\frac{\widetilde{y}_{N+M}(x, \nu)}{(2 \mathrm{i} \nu)^{M}}
$$

Since by the definition (C.5), $y_{k}^{a e}(0)=0$ for any $k \geq 1$ and by (C.9), $\widetilde{y}_{N+M}(0, \nu)=0$ one concludes that $\widetilde{y}_{N, M}(0, \nu)=0$. Furthermore, since by (C.9) $\partial_{x} \widetilde{y}_{N+M}(0, \nu)=0$, one has

$$
\partial_{x} \widetilde{y}_{N, M}(0, \nu)=\sum_{k=1}^{M} \frac{\partial_{x} y_{N+k}^{a e}(0)}{(2 \mathrm{i} \nu)^{k-1}}
$$

and by the arguments above, for any $s \geq 0$, the map $(\mathbb{C} \backslash\{0\}) \times V_{q, S}^{*} \rightarrow H_{\mathbb{C}}^{s}[0,1],(\nu, p) \mapsto \tilde{y}_{N, M}(\cdot, \nu, p)$ is analytic. Furthermore, if $\nu \in \mathbb{R} \backslash\{0\}$ and $p$ is real valued then

$$
y_{N, M}^{\prime}(0,-\nu)=-\mathrm{i} \nu+\sum_{k=1}^{N+M} \frac{\partial_{x} y_{k}^{a e}(0)}{(-2 \mathrm{i} \nu)^{k}}=\overline{y_{N, M}^{\prime}(0,-\nu)}
$$

 of the initial value problem that $\left.y_{N, M}(\cdot,-\nu)=\overline{y_{N, M}(\cdot, \nu}\right)$. It remains to show the estimate (C.6). Note that
for any $p \in V_{q, S}^{*}$, the terms $\frac{y_{N+k}^{a e}(x)}{(2 i \nu)^{k-1}}, 1 \leq k \leq M$, satisfy an estimate of the type (C.6). Hence it suffices to show that

$$
\begin{equation*}
\sup _{\substack{|\operatorname{lm} \nu| \leq \mathrm{c},|\nu| \geq 1 \\ 0 \leq x \leq 1,0 \leq j \leq M}} \frac{\left|\partial_{x}^{j} \widetilde{y}_{N+M}(x, \nu)\right|}{|\nu|^{j}} \leq \widetilde{C}_{N, M} \tag{C.10}
\end{equation*}
$$

for some constant $\widetilde{C}_{N, M}>0$. By (C.8), (C.9), $\widetilde{y}_{N+M}$ solves the initial value problem

$$
\begin{align*}
& \left(-\partial_{x}^{2}+p-\nu^{2}\right)\left[e^{\mathrm{i} \nu x} \widetilde{y}_{N+M}(x, \nu)\right]=-2 \mathrm{i} \nu e^{\mathrm{i} \nu x} \partial_{x} y_{N+M+1}^{a e}(x), \\
& \widetilde{y}_{N+M}(0, \nu)=0, \quad \partial_{x} \widetilde{y}_{N+M}(0, \nu)=0 \tag{C.11}
\end{align*}
$$

By the method of the variation of the constants it is given by

$$
\begin{equation*}
\widetilde{y}_{N+M}(x, \nu)=-2 \mathrm{i} \nu \int_{0}^{x} K\left(x, t, \nu^{2}\right) e^{\mathrm{i} \nu(t-x)} \partial_{t} y_{N+M+1}^{a e}(t) d t \tag{C.12}
\end{equation*}
$$

where $K\left(x, t, \nu^{2}\right)=y_{1}\left(x, \nu^{2}\right) y_{2}\left(t, \nu^{2}\right)-y_{1}\left(t, \nu^{2}\right) y_{2}\left(x, \nu^{2}\right)$, satisfying the estimates (cf [13, Chapter 1])

$$
\begin{aligned}
& \sup _{\substack{0 \leq x \leq 1 \\
|\operatorname{Im} \nu| \leq c,|\nu| \geq 1}}\left|y_{1}\left(x, \nu^{2}\right)\right| \leq C, \\
& \sup _{\substack{0 \leq x \leq 1 \\
0 \leq x \leq 1 \\
|\operatorname{Im} \nu| \leq c,|\nu| \geq 1}} \frac{\left|\partial_{x} y_{1}\left(x, \nu^{2}\right)\right|}{|\nu|} \leq C, \\
& \left|\sup _{\substack{0 \leq x \leq 1 \\
|\operatorname{Im} \nu| \leq c,|\nu| \geq 1}}\right| \nu\left|\left|y_{2}\left(x, \nu^{2}\right)\right| \leq C\right. \\
& |\operatorname{Im} \nu| \leq c,|\nu| \geq 1
\end{aligned},
$$

for some constant $C>0$. It then follows from (C.12) and (C.5) that $\sup _{|\operatorname{|Im} \nu| \leq c,|\nu| \geq 1}^{0 \leq x \leq 1}\left|\widetilde{y}_{N+M}(x, \nu)\right| \leq C$. Since $K\left(x, x, \nu^{2}\right)=0$ one has that

$$
\partial_{x} \widetilde{y}_{N+M}(x, \nu)=-\mathrm{i} \nu \widetilde{y}_{N+M}(x, \nu)-2 \mathrm{i} \nu \int_{0}^{x} \partial_{x} K\left(x, t, \nu^{2}\right) e^{\mathrm{i} \nu(t-x)} \partial_{t} y_{N+M+1}^{a e}(t) d t
$$

implying that $\sup _{\substack{0 \leq x \leq 1 \\|\operatorname{Im} \nu| \leq c,|\nu| \geq 1}} \frac{\left|\partial_{x} \widetilde{y}_{N+M}(x, \nu)\right|}{|\nu|} \leq C$. Using equation (C.11) one gets

$$
\begin{equation*}
-\partial_{x}^{2} \widetilde{y}_{N+M}(x, \nu)=2 \mathrm{i} \nu \partial_{x} \widetilde{y}_{N+M}(x, \nu)-p \widetilde{y}_{N+M}(x, \nu)-2 \mathrm{i} \nu \partial_{x} y_{N+M+1}^{a e}(x) \tag{C.13}
\end{equation*}
$$

yielding sup $\underset{\mid \operatorname{Im} \nu \leq x \leq 1}{0 \leq c,|\nu| \geq 1}<\frac{\left|\partial_{x}^{2} \widetilde{y}_{N+M}(x, \nu)\right|}{|\nu|^{2}} \leq C$. By taking derivatives of (C.13), one then concludes that there exists a constant $C_{N, M}>0$ so that for any $0 \leq j \leq M$, $\sup _{|\operatorname{Im} \nu| \leq c,|\nu| \geq 1}^{0 \leq x \leq 1} \frac{\left|\partial_{x}^{j} \widetilde{y}_{N+M}(x, \nu)\right|}{|\nu|^{j}} \leq \widetilde{C}_{N, M}$. Going through the arguments of the proof one concludes that the constant $\bar{C}_{N, M}$ can be chosen locally uniformly for $p \in V_{q, S}^{*}$.

Let $q \in M_{S}$ and $N, M \geq 0$. Then for any $p \in V_{q, S}^{*}$ and $\nu \neq 0$ with $|\nu|$ sufficiently large, the solutions $y_{N, M}(x, \nu)$ and $y_{N, M}(x,-\nu)$ of $-y^{\prime \prime}+p y=\nu^{2} y$, considered in LemmaC. 1 are linearly independent. Indeed by Lemma C. 1 for any $\nu \neq 0$,

$$
y_{N, M}(0, \nu)=1, \quad \partial_{x} y_{N, M}(0, \nu)=\mathrm{i} \nu+\sum_{k=1}^{N+M} \frac{\partial_{x} y_{k}^{a e}(0)}{(2 \mathrm{i} \nu)^{k}}
$$

implying that the Wronskian of $y_{N, M}(\cdot,-\nu)$ and $y_{N, M}(\cdot, \nu)$ equals

$$
\operatorname{det}\left(\begin{array}{cc}
y_{N, M}(0,-\nu) & y_{N, M}(0, \nu) \\
\partial_{x} y_{N, M}(0,-\nu) & \partial_{x} y_{N, M}(0, \nu)
\end{array}\right)=\partial_{x} y_{N, M}(0, \nu)-\partial_{x} y_{N, M}(0,-\nu)=2 \mathrm{i} \nu+\sum_{k=1}^{N+M} \frac{\partial_{x} y_{k}^{a e}(0)}{(2 \mathrm{i} \nu)^{k}}\left(1-(-1)^{k}\right)
$$

Hence there exists $\nu_{b} \geq 1$ so that

$$
\begin{equation*}
\left|2 \mathrm{i} \nu+\sum_{k=1}^{N+M} \frac{\partial_{x} y_{k}^{a e}(0)}{(2 \mathrm{i} \nu)^{k}}\left(1-(-1)^{k}\right)\right| \geq 1, \quad \forall \nu \text { with }|\nu| \geq \nu_{b} \tag{C.14}
\end{equation*}
$$

The bound $\nu_{b}$ can be chosen locally uniformly in $p \in V_{q, S}^{*}$. It then follows that for $|\nu| \geq \nu_{b}, y_{1}\left(x, \nu^{2}\right)$, $y_{2}\left(x, \nu^{2}\right)$ are linear combinations of $y_{N, M}(x, \nu)$ and $y_{N, M}(x,-\nu)$,

$$
\begin{gather*}
y_{1}\left(x, \nu^{2}\right)=\alpha_{N, M}(-\nu) y_{N, M}(x, \nu)+\alpha_{N, M}(\nu) y_{N, M}(x,-\nu)  \tag{C.15}\\
\alpha_{N, M}(\nu)=\frac{\partial_{x} y_{N, M}(0, \nu)}{\partial_{x} y_{N, M}(0, \nu)-\partial_{x} y_{N, M}(0,-\nu)} \tag{C.16}
\end{gather*}
$$

and

$$
\begin{gather*}
y_{2}\left(x, \nu^{2}\right)=\beta_{N, M}(\nu) y_{N, M}(x, \nu)+\beta_{N, M}(-\nu) y_{N, M}(x,-\nu),  \tag{C.17}\\
\beta_{N, M}(\nu)=\frac{1}{\partial_{x} y_{N, M}(0, \nu)-\partial_{x} y_{N, M}(0,-\nu)} . \tag{C.18}
\end{gather*}
$$

We note that $\beta_{N, M}(-\nu)=-\beta_{N, M}(\nu)$ and for $|\nu|$ sufficiently large, $\alpha_{N, M}(\nu) \simeq \frac{1}{2}$ and $\beta_{N, M}(\nu) \simeq \frac{1}{2 i \nu}$. Furthermore, in case $p$ is real valued and $\nu \in \mathbb{R}$ with $|\nu| \geq \nu_{b}$, one has

$$
\alpha_{N, M}(-\nu)=\overline{\alpha_{N, M}(\nu)} \quad \text { and } \quad \beta_{N, M}(-\nu)=\overline{\beta_{N, M}(\nu)}=-\beta_{N, M}(\nu)
$$

implying that $\beta_{N, M}(\nu)$ is purely imaginary. Finally, to prove Theorem C.1, the following two additional results are needed. It is well know that $\tau_{n}=\left(\lambda_{n}^{+}+\lambda_{n}^{-}\right) / 2$ admits an asymptotic expansion as $n \rightarrow \infty$ (cf e.g. [15, Theorem 1.3]). More precisely, we have the following

Lemma C.2. Let $q \in M_{S}$ and $N \in \mathbb{Z}_{\geq 0}$. Then for any $p \in V_{q, S}^{*}, \tau_{n}(p), n \in S_{+}^{\perp}$, has an expansion of the form

$$
\begin{equation*}
\tau_{n}(p)=n^{2} \pi^{2}+\sum_{k=1}^{N} \frac{\tau_{2 k}^{a e}(p)}{(2 \pi \mathrm{i} n)^{2 k}}+\frac{\mathcal{R}_{2 N}^{\tau_{n}}(p)}{(2 \pi \mathrm{i} n)^{2 N+2}} \tag{C.19}
\end{equation*}
$$

where $V_{q, S}^{*} \rightarrow \mathbb{C}, p \mapsto \tau_{2 k}^{a e}(p), k \geq 1$, and $V_{q, S}^{*} \rightarrow \mathbb{C}, p \mapsto \mathcal{R}_{2 N}^{\tau_{n}}(p)$ are real analytic. As a consequence, choosing $n_{0} \geq m_{S}:=1+\max \{n \in S\}$ so that $\operatorname{Re}\left(\tau_{n}(p)\right)>0$ for any $n \geq n_{0}$, it follows that $2 \mathrm{i} \sqrt{\tau_{n}(p)}$, $n \geq n_{0}$, admits an expansion of the form

$$
\begin{equation*}
2 \mathrm{i} \sqrt{\tau_{n}(p)}=2 \pi \mathrm{i} n\left(1+\sum_{k=2}^{N} \frac{\sqrt{\tau}_{2 k}^{a e}(p)}{(2 \pi \mathrm{i} n)^{2 k}}+\frac{\mathcal{R}_{2 N}^{\sqrt{\tau_{n}}}(p)}{(2 \pi \mathrm{i} n)^{2 N+2}}\right) \tag{C.20}
\end{equation*}
$$

where $V_{q, S}^{*} \rightarrow \mathbb{C}, p \mapsto \sqrt{\tau}_{2 k}^{a e}(p), k \geq 2$, and $V_{q, S}^{*} \rightarrow \mathbb{C}, p \mapsto \mathcal{R}_{2 N}^{\sqrt{\tau_{n}}}(p)$ are real analytic. In addition, the remainders $\mathcal{R}_{2 N}^{\tau_{n}}(p)$ and $\mathcal{R}_{2 N}^{\sqrt{\tau_{n}}}(p)$ satisfy

$$
\sup _{n \in S_{+}^{\perp}}\left|\mathcal{R}_{2 N}^{\tau_{n}}(p)\right| \leq C_{N}, \quad \sup _{n \geq n_{0}}\left|\mathcal{R}_{2 N}^{\sqrt{\tau_{n}}}(p)\right| \leq C_{N}
$$

where the constants $C_{N}>0$ and $n_{0}>m_{S}$ can be chosen locally uniformly for $p \in V_{q, S}^{*}$.
Proof. The functions $\tau_{2 k}^{a e}(q), k \geq 2$, are given by polynomial expressions of integrals of densities, involving $q$ and its derivatives up to order $2 k$ (cf [15]) and are real analytic maps, $H_{0, \mathbb{C}}^{k} \rightarrow \mathbb{C}, q \mapsto \tau_{2 k}^{a e}(q)$. Since $\tau_{n}$ is real analytic on $V_{q, S}^{*}$ so is

$$
\mathcal{R}_{2 N}^{\tau_{n}}(p)=(2 \pi \mathrm{i} n)^{2 N+2}\left(\tau_{n}(p)-n^{2} \pi^{2}-\sum_{k=1}^{N} \frac{\tau_{2 k}^{a e}(p)}{(2 \pi \mathrm{i} n)^{2 k}}\right) .
$$

The claimed bounds for $\mathcal{R}_{2 N}^{\tau_{n}}(p)$ were established in [15]. In view of the asymptotics of $\tau_{n}(p)$, one finds $n_{0} \geq m_{S}$ with the claimed properties and then obtains the coefficients $\sqrt{\tau}_{2 k}^{a e}(p)$ from the expansion of $\tau_{n}(p)$ in a recursive way and concludes that they are real analytic. Since $\sqrt{\tau_{n}(p)}$ is real analytic one again concludes that the remainder term $\mathcal{R}_{2 N}^{\sqrt{\tau_{n}}}(p)$ is real analytic as well and deduces the claimed bounds.

The second result concerns the asymptotic expansion of the coefficients $a_{n}$, defined in ( $\overline{\mathrm{B} .3}$ ).
Lemma C.3. Let $q \in M_{S}$ and $N \in \mathbb{Z}_{\geq 0}$. Then for any $p \in V_{q, S}^{*}, a_{n}(p), n \in S^{\perp}$, has an expansion of the form

$$
\begin{equation*}
a_{n}(p)=\mathrm{i} n \pi+\sum_{k=0}^{N} \frac{a_{k}^{a e}(p)}{(2 \pi \mathrm{i} n)^{k}}+\frac{\mathcal{R}_{N}^{a_{n}}(p)}{(2 \pi \mathrm{i} n)^{N+1}} \tag{C.21}
\end{equation*}
$$

where $V_{q, S}^{*} \rightarrow \mathbb{C}, p \mapsto a_{k}^{a e}(p), k \geq 0$, are real analytic and $V_{q, S}^{*} \rightarrow \mathbb{C}, p \mapsto \mathcal{R}_{N}^{a_{n}}(p)$ is analytic. In addition, the remainders $\mathcal{R}_{N}^{a_{n}}(p)$ satisfy

$$
\sup _{n \in S^{\perp}}\left|\mathcal{R}_{N}^{a_{n}}(p)\right| \leq C_{N}
$$

where the constant $C_{N}>0$ can be chosen locally uniformly for $p \in V_{q, S}^{*}$.
Proof. To start with, we compute the leading term in the expansion of $a_{n}$. Recall that by (B.3),

$$
a_{ \pm n}=-\frac{\dot{m}_{1}\left(\tau_{n}\right)}{\dot{m}_{2}\left(\tau_{n}\right)} \pm \mathrm{i} \frac{\sqrt{(-1)^{n+1} \ddot{\Delta}\left(\tau_{n}\right) / 2}}{(-1)^{n} \dot{m}_{2}\left(\tau_{n}\right)}, \quad n \in S_{+}^{\perp}
$$

By (B.8) - (B.9), for any $n \in S_{+}^{\perp}, \dot{m}_{1}\left(\tau_{n}\right)=(-1)^{n} \int_{0}^{1} y_{1}\left(t, \tau_{n}\right) y_{2}\left(t, \tau_{n}\right) d t$ and $\dot{m}_{2}\left(\tau_{n}\right)=(-1)^{n} \int_{0}^{1} y_{2}\left(x, \tau_{n}\right)^{2} d x$, yielding

$$
\begin{equation*}
a_{ \pm n}=-\frac{\int_{0}^{1} y_{1}\left(t, \tau_{n}\right) y_{2}\left(t, \tau_{n}\right) d t}{\int_{0}^{1} y_{2}\left(t, \tau_{n}\right)^{2} d t} \pm \mathrm{i} \frac{1}{\left(2 \int_{0}^{1} y_{2}\left(t, \tau_{n}\right)^{2} d t\right)^{\frac{1}{2}}} \sqrt{-\frac{\ddot{\Delta}\left(\tau_{n}\right)}{\dot{m}_{2}\left(\tau_{n}\right)}} . \tag{C.22}
\end{equation*}
$$

Since $\sqrt{\tau_{n}}=n \pi\left(1+O\left(\frac{1}{n^{4}}\right)\right)\left(\right.$ Lemma C.2) and hence $y_{1}\left(t, \tau_{n}\right)=\cos (n \pi t)+O\left(\frac{1}{n}\right)$ and $y_{2}\left(t, \tau_{n}\right)=\frac{\sin (n \pi t)}{n \pi}+$ $O\left(\frac{1}{n^{2}}\right)(\operatorname{cf}[24$, Chapter 1]) one has

$$
\begin{equation*}
\int_{0}^{1} y_{1}\left(t, \tau_{n}\right) y_{2}\left(t, \tau_{n}\right) d t=O\left(\frac{1}{n^{2}}\right), \quad 2 \int_{0}^{1} y_{2}\left(t, \tau_{n}\right)^{2} d t=\frac{1}{n^{2} \pi^{2}}\left(1+O\left(\frac{1}{n}\right)\right) \tag{C.23}
\end{equation*}
$$

To analyze the quotient $-\frac{\ddot{\Delta}\left(\tau_{n}\right)}{\dot{m}_{2}\left(\tau_{n}\right)}$ we use the product representation of $\dot{\Delta}(\lambda)$ and $m_{2}(\lambda)$ (cf Appendix A),

$$
\dot{\Delta}(\lambda)=-\prod_{k \geq 1} \frac{\dot{\lambda}_{k}-\lambda}{\pi^{2} k^{2}}, \quad m_{2}(\lambda)=\prod_{k \geq 1} \frac{\mu_{k}-\lambda}{\pi^{2} k^{2}}
$$

Since for $n \in S_{+}^{\perp}, \lambda_{n}^{+}=\dot{\lambda}_{n}=\mu_{n}=\tau_{n}$,

$$
\ddot{\Delta}\left(\tau_{n}\right)=\frac{1}{n^{2} \pi^{2}} \prod_{k \neq n} \frac{\dot{\lambda}_{k}-\tau_{n}}{\pi^{2} k^{2}}, \quad \dot{m}_{2}\left(\tau_{n}\right)=-\frac{1}{\pi^{2} n^{2}} \prod_{k \neq n} \frac{\mu_{k}-\tau_{n}}{\pi^{2} k^{2}}
$$

and one concludes that

$$
-\frac{\ddot{\Delta}\left(\tau_{n}\right)}{\dot{m}_{2}\left(\tau_{n}\right)}=\prod_{k \in S_{+}} \frac{\dot{\lambda}_{k}-\tau_{n}}{\mu_{k}-\tau_{n}}=\prod_{k \in S_{+}} \frac{1-\frac{\dot{\lambda}_{k}}{\tau_{n}}}{1-\frac{\mu_{k}}{\tau_{n}}} .
$$

Altogether we thus have proved that for any $n \in S_{+}^{\perp}$,

$$
a_{ \pm n}= \pm \mathrm{i} n \pi+O(1) \quad \text { and } \quad a_{ \pm n}=-\frac{\int_{0}^{1} y_{1}\left(t, \tau_{n}\right) y_{2}\left(t, \tau_{n}\right) d t}{\int_{0}^{1} y_{2}\left(t, \tau_{n}\right)^{2} d t} \pm \mathrm{i} \frac{1}{\left(2 \int_{0}^{1} y_{2}\left(t, \tau_{n}\right)^{2} d t\right)^{\frac{1}{2}}}\left(\prod_{k \in S_{+}} \frac{1-\frac{\dot{\lambda}_{k}}{\tau_{n}}}{1-\frac{\mu_{k}}{\tau_{n}}}\right)^{\frac{1}{2}}
$$

Expressing $y_{1}\left(t, \tau_{n}\right)$ and $y_{2}\left(t, \tau_{n}\right)$ in terms of $y_{N, M}\left(x, \pm \sqrt{\tau_{n}}\right)$ (cf (C.15), (C.17), one obtains an expansion of the form (C.21) where the coefficients $a_{k}^{a e}$ can be explicitly computed by using the expansion of $y_{N, M}(x, \nu)$, obtained in LemmaC.1 and the one of $\sqrt{\tau_{n}}$ of LemmaC.2. It follows that for any $k \geq 0$, the map $V_{q, S}^{*} \rightarrow \mathbb{C}$,
$p \mapsto a_{k}^{a e}(p)$ is analytic. In case $p$ is real valued one has $a_{-n}=\overline{a_{n}}$ (cf definition (B.3) of $a_{ \pm n}$ ). By an inductive argument it then follows from the expansion of $a_{n}+a_{-n}$ that for any $k \geq 0$, the coefficient $a_{k}^{a e}$ is real valued. With regard to the remainder term, since for any $n \in S^{\perp}, a_{n}$ is analytic on $V_{q, S}^{*}$ (cf Remark B.1) one sees that $\mathcal{R}_{N}^{a_{n}}$ is analytic on $V_{q, S}^{*}$. The claimed estimates are obtained from the corresponding estimates of Lemma C. 1 and Lemma C. 2 .

Proof of Theorem C.1. Let $q \in M_{S}$ and $N \geq 0$. To prove that for $p \in V_{q, S}^{*}, f_{n}(x, p)$ has an expansion of the form (C.2) we first note that since $y_{1}\left(x, \tau_{n}\right)=\cos (n \pi x)+O\left(\frac{1}{n}\right), y_{2}\left(x, \tau_{n}\right)=\frac{1}{n \pi} \sin (n \pi x)+O\left(\frac{1}{n^{2}}\right)$, and $a_{ \pm n}=\mathrm{i} n \pi+O(1)$, one has $f_{ \pm n}(x)=e^{ \pm \mathrm{i} \pi n x}+O\left(\frac{1}{n}\right)$. To obtain the expansion as claimed, we want to apply Lemma C.1 Choose $M \geq 0$ (arbitrarily large) and $n_{0} \geq m_{S}$ (cf Lemma C.2) so that $\operatorname{Re} \tau_{n}>0$ and in addition $\left|\sqrt{\tau_{n}}\right| \geq \nu_{b}$ for any $n \geq n_{0}$ where $\nu_{b} \geq 1$ is given by (C.14). We then substitute the formulas (C.15) and (C.17) with $\nu^{2}=\tau_{n}$ into the expression (C.1) for $f_{ \pm n}(x) \equiv f_{ \pm n}(x, p)$ to get for $n \geq n_{0}$

$$
\begin{align*}
& f_{ \pm n}(x)=\alpha_{N, M}\left(-\sqrt{\tau_{n}}\right) y_{N, M}\left(x, \sqrt{\tau_{n}}\right)+\alpha_{N, M}\left(\sqrt{\tau_{n}}\right) y_{N, M}\left(x,-\sqrt{\tau_{n}}\right) \\
& \quad+a_{ \pm n}\left(\beta_{N, M}\left(\sqrt{\tau_{n}}\right) y_{N, M}\left(x, \sqrt{\tau_{n}}\right)+\beta_{N, M}\left(-\sqrt{\tau_{n}}\right) y_{N, M}\left(x,-\sqrt{\tau_{n}}\right)\right) \tag{C.24}
\end{align*}
$$

Using the expansions of $\sqrt{\tau_{n}}$ (Lemma C.2), $a_{ \pm n}$ (Lemma C.3), and $y_{N, M}(x, \nu)$ (Lemma C.1) one gets an expansion of $f_{n},|n| \geq n_{0}$, of the form (C.2) where the coefficients $f_{k}^{a e}, k \geq 1$, and the remainder $\mathcal{R}_{N}^{f_{n}}$ can be explicitly computed. One verifies that for any $s \geq 0$ and $k \geq 1, f_{k}^{a e}: V_{q, S}^{*} \rightarrow H_{\mathbb{C}}^{s}[0,1]$ is analytic. Furthermore, by choosing $M$ sufficiently large and using the estimates of the lemmas referred to above, one obtains the claimed estimate (C.3) of $\mathcal{R}_{N}^{f_{n}}$ for any $|n| \geq n_{0}$. Note that at this point, we only know that $f_{k}^{a e}(\cdot, p)$ is an element in $H_{\mathbb{C}}^{s}[0,1]$ for any $s \geq 0$. But since $e^{-\mathrm{i} \pi n x} f_{n}(x)$ is one periodic in $x$, it follows by induction that for any $k \geq 1, f_{k}^{a e}(x)$ is one periodic in $x$ as well. Since in case $p$ is real valued, $e^{-\mathrm{i} \pi n x} f_{n}(x)=\overline{e^{\mathrm{i} \pi n x} f_{-n}(x)}$ one reads off from the expansions of $e^{-\mathrm{i} \pi n x} f_{n}(x)$ and $\overline{e^{\mathrm{i} \pi n x} f_{-n}(x)}$ that $f_{k}^{a e}$, $k \geq 1$, are real valued. Altogether this shows that for any $s \geq 0$ and $k \geq 1, f_{k}^{a e}: V_{q, S}^{*} \rightarrow H_{\mathbb{C}}^{s}$ is real analytic. For $n \in S^{\perp}$ with $|n|<n_{0}$, we define $\mathcal{R}_{N}^{f_{n}}(x)$ by

$$
\mathcal{R}_{N}^{f_{n}}(x)=(2 \pi \mathrm{i} n)^{N+1}\left(e^{-\mathrm{i} \pi n x} f_{n}(x)-1-\sum_{k=1}^{N} \frac{f_{k}^{a e}(x)}{(2 \pi \mathrm{i} n)^{k}}\right) .
$$

We then conclude that for any $n \in S^{\perp}, \mathcal{R}_{N}^{f_{n}}(x)$ is one periodic in $x$. Furthermore, since for any $n \in S^{\perp}$ and $s \geq 0, e^{-\mathrm{i} \pi n x} f_{n}: V_{q, S}^{*} \rightarrow H_{\mathbb{C}}^{s}$ is analytic (cf(C.1)) it follows that $\mathcal{R}_{N}^{f_{n}}: V_{q, S}^{*} \rightarrow H_{\mathbb{C}}^{s}$ is analytic as well. Going through the arguments of the proof one sees that the estimate (C.3) holds for any $n \in S^{\perp}$ and that the constant $C_{N, j}$ in (C.3) can be chosen locally uniformly for $p \in V_{q, S}^{*}$.

The next result states how the map $S_{r e v}$, defined in Section acts on the functions $f_{n}(x, q)$ and how on the coefficients and the remainder of its expansion.
Addendum to Theorem C.1, For any $q \in M_{S}$ and $n \in S^{\perp}$

$$
f_{n}\left(x, S_{r e v} q\right)=f_{-n}(-x, q)\left(=\left(S_{\text {rev }} f_{-n}\right)(x, q)\right), \quad \forall x \in \mathbb{R}
$$

and as a consequence,

$$
\begin{equation*}
f_{k}^{a e}\left(x, S_{\text {rev }} q\right)=(-1)^{k} f_{k}^{a e}(-x, q), k \geq 1, \quad \mathcal{R}_{N}^{f_{n}}\left(x, S_{\text {rev }} q\right)=(-1)^{N+1} \mathcal{R}_{N}^{f_{-n}}(-x, q) \tag{C.25}
\end{equation*}
$$

Proof of Addendum to Theorem C.1, Let $q \in M_{S}$ and $n \in S_{+}^{\perp}$. By Lemma D.1, one knows that $\tau_{n}\left(S_{\text {rev }}(q)\right)=\tau_{n}(q)$ and

$$
\begin{equation*}
f_{ \pm n}\left(x, S_{\text {rev }} q\right)=\left.\left(y_{1}(x)+a_{ \pm n} y_{2}(x)\right)\right|_{\tau_{n}, S_{r e v} q}=y_{1}\left(-x, \tau_{n}, q\right)-a_{ \pm n}\left(S_{\text {rev }} q\right) y_{2}\left(-x, \tau_{n}, q\right) \quad \forall x \in \mathbb{R} \tag{C.26}
\end{equation*}
$$

where $\tau_{n} \equiv \tau_{n}(q)$. Recall that $a_{ \pm n}$ is given by $a_{ \pm n}=-\frac{\dot{m}_{1}\left(\tau_{n}\right)}{\dot{m}_{2}\left(\tau_{n}\right)} \pm \mathrm{i} \frac{\sqrt{(-1)^{n+1} \ddot{\Delta}\left(\tau_{n}\right) / 2}}{(-1)^{n} \dot{m}_{2}\left(\tau_{n}\right)}$. Note that again by Lemma D.1 one has

$$
m_{2}\left(\lambda, S_{\text {rev }} q\right)=m_{2}(\lambda, q), \quad \Delta\left(\lambda, S_{\text {rev }} q\right)=\Delta(\lambda, q), \quad m_{1}\left(\lambda, S_{\text {rev }} q\right)=m_{2}^{\prime}(\lambda, q), \quad \forall \lambda \in \mathbb{R} .
$$

Since $\dot{\Delta}\left(\tau_{n}, q\right)=0$ and hence $\dot{m}_{2}^{\prime}\left(\tau_{n}, q\right)=-\dot{m}_{1}\left(\tau_{n}, q\right)$ it then follows that $\dot{m}_{1}\left(\tau_{n}, S_{\text {rev }} q\right)=-\dot{m}_{1}\left(\tau_{n}, q\right)$. Combining all these identities one concludes that $a_{ \pm n}\left(S_{\text {rev }} q\right)=-a_{\mp n}(q)$ and hence by (C.26)

$$
f_{ \pm n}\left(x, S_{r e v} q\right)=y_{1}\left(-x, \tau_{n}, q\right)+a_{\mp n}\left(S_{\text {rev }} q\right) y_{2}\left(-x, \tau_{n}, q\right)=f_{\mp n}(-x, q)
$$

as claimed. Considering the expansions of the latter identities one obtains (C.25).
To obtain the asymptotic expansion for $\Psi_{L}$, presented in Section 2 we need to establish such an expansion for each of the factors appearing in the definition (2.7) of $W_{ \pm n}(q)$ for a finite gap potential $q$. First we consider the factor $\xi_{n}$ which compares the square root of the n'th action with the n'th gap length. For any $q \in W$ (cf Theorem A.1) with $\gamma_{n}(q) \neq 0$, it is given by $\sqrt{8 I_{n}(q) / \gamma_{n}^{2}(q)}$. In case $\gamma_{n}(q)=0$, it can be computed by a limiting argument. By a slight abuse of terminology, we denote this limit also by $\sqrt{8 I_{n}(q) / \gamma_{n}^{2}(q)}$.

Lemma C.4. Let $q \in M_{S}$ and $N \in \mathbb{Z}_{\geq 0}$. Then for any $p \in V_{q, S}^{*}, \xi_{n}(q):=\sqrt{8 I_{n}(q) / \gamma_{n}^{2}(q)}, n \in S_{+}^{\perp}$, has an expansion of the form

$$
\sqrt{n \pi} \xi_{n}(p)=1+\sum_{k=1}^{N} \frac{\xi_{2 k}^{a e}(p)}{(2 \pi \mathrm{i} n)^{2 k}}+\frac{\mathcal{R}_{2 N}^{\xi_{n}}(p)}{(2 \pi \mathrm{i} n)^{2 N+2}}
$$

where $V_{q, S}^{*} \rightarrow \mathbb{C}, p \mapsto \xi_{2 k}^{a e}(p), k \geq 1$, and $V_{q, S}^{*} \rightarrow \mathbb{C}, p \mapsto \mathcal{R}_{2 N}^{\xi_{n}}(p)$, are real analytic. In addition, the remainders $\mathcal{R}_{2 N}^{\xi_{n}}(p)$ satisfy

$$
\begin{equation*}
\sup _{n \in S_{+}^{\perp}}\left|\mathcal{R}_{2 N}^{\xi_{n}}(p)\right| \leq C_{N} \tag{C.27}
\end{equation*}
$$

where the constant $C_{N}>0$ can be chosen locally uniformly for $p \in V_{q, S}^{*}$.
Proof. Let $q \in M_{S}$ and $N \in \mathbb{Z}_{\geq 0}$. Following the proof of [13, Theorem 7.3], for any $p \in V_{q, S}^{*}$ and $n \in S_{+}^{\perp}$, $8 I_{n}(p) / \gamma_{n}^{2}(p)$ can be computed by considering a sequence of $S_{n}$-gap potentials $\left(p_{j}\right)_{j \geq 1}$ in $W$ (cf Theorem A.1) with $\gamma_{n}\left(p_{j}\right)>0$ so that $p_{j} \rightarrow p$ as $j \rightarrow \infty$ where $S_{n}:=S \cup\{-n, n\}$. One then obtains in the limit the formula $8 I_{n}(p) / \gamma_{n}^{2}(p)=\chi\left(\tau_{n}(p), p\right)$ where $\chi(\lambda) \equiv \chi(\lambda, p)$ is given by $\frac{1}{\sqrt{\lambda-\lambda_{0}}} \prod_{k \in S_{+}} \frac{\lambda-\dot{\lambda}_{k}}{\sqrt{\left(\lambda_{k}^{+}-\lambda\right)\left(\lambda_{k}^{-}-\lambda\right)}}$, implying that for $n \geq n_{0}$ with $n_{0} \geq m_{S}$ chosen so that $\operatorname{Re}\left(\tau_{n}(p)\right)>0$ for any $n \geq n_{0}$ (cf Lemma C.2)

$$
\begin{equation*}
\chi\left(\tau_{n}\right)=\frac{1}{\sqrt{\tau_{n}}} \frac{1}{\sqrt{1-\frac{\lambda_{0}}{\tau_{n}}}} \prod_{k \in S_{+}} \frac{1-\frac{\dot{\lambda}_{k}}{\tau_{n}}}{\sqrt{\left(1-\frac{\lambda_{k}^{+}}{\tau_{n}}\right)\left(1-\frac{\lambda_{k}^{-}}{\tau_{n}}\right)}} \tag{C.28}
\end{equation*}
$$

Combining (C.28) with the expansion of $\tau_{n}(\mathrm{cf}$ Lemma C.2) then yields the expansion

$$
\sqrt{n \pi} \xi_{n}(p)=\sqrt{n \pi} \sqrt{\chi\left(\tau_{n}(p), p\right)}=1+\sum_{k=1}^{N} \frac{\xi_{2 k}^{a e}(p)}{(2 \pi \mathrm{i} n)^{2 k}}+\frac{\mathcal{R}_{2 N}^{\xi_{n}}(p)}{(2 \pi \mathrm{i} n)^{2 N+2}}
$$

where $V_{q, S}^{*} \rightarrow \mathbb{C}, p \mapsto \xi_{2 k}^{a e}(p), k \geq 1$, are real analytic and $\sup _{n \in S_{+}^{\perp}}\left|\mathcal{R}_{2 N}^{\xi_{n}}(p)\right|$ is bounded. For $n \in S_{+}^{\perp}$ with $n<n_{0}, \mathcal{R}_{2 N}^{\xi_{n}}(p)$ is defined by $(2 \pi \mathrm{i} n)^{2 N+2}\left(\sqrt{n \pi} \xi_{n}(p)-1-\sum_{k=1}^{N} \frac{\xi_{2 k}^{a e}(p)}{(2 \pi \mathrm{in})^{2 k}}\right)$. Since for any $n \in S_{+}^{\perp}, \sqrt{n \pi} \xi_{n}$ is real analytic on $V_{q, S}^{*}$, so is $\mathcal{R}_{N}^{\xi_{n}}(p)$. Going through the arguments of the proof one sees that the constant $C_{N}$ in (C.27) can be chosen locally uniformly for $p \in V_{q, S}^{*}$.

Next we prove an expansion for the factor $\dot{m}_{2}\left(\tau_{n}(q), q\right) / \ddot{\Delta}\left(\tau_{n}(q), q\right)$ in (2.7) for $q \in M_{S}$. More precisely, we show the following
Lemma C.5. Let $q \in M_{S}$ and $N \in \mathbb{Z}_{\geq 0}$. Then for any $p \in V_{q, S}^{*}, d_{n}(p):=-\dot{m}_{2}\left(\tau_{n}(p), p\right) / \ddot{\Delta}\left(\tau_{n}(p), p\right)$, $n \in S_{+}^{\perp}$, has an expansion of the form

$$
d_{n}(p)=1+\sum_{k=1}^{N} \frac{d_{2 k}^{a e}(p)}{(2 \pi \mathrm{i} n)^{2 k}}+\frac{\mathcal{R}_{2 N}^{d_{n}}(p)}{(2 \pi \mathrm{i} n)^{2 N+2}}
$$

where $V_{q, S}^{*} \rightarrow \mathbb{C}, p \mapsto d_{2 k}^{a e}(p), k \geq 1$, and $V_{q, S}^{*} \rightarrow \mathbb{C}, p \mapsto \mathcal{R}_{2 N}^{d_{n}}(p)$, are real analytic. In addition, the remainders $\mathcal{R}_{2 N}^{d_{n}}(p)$ satisfy

$$
\sup _{n \in S_{\ddagger}^{\perp}}\left|\mathcal{R}_{2 N}^{d_{n}}(p)\right| \leq C_{N}
$$

where the constant $C_{N}>0$ can be chosen locally uniformly for $p \in V_{q, S}^{*}$.
Proof. Let $q \in M_{S}$ and $N \in \mathbb{Z}_{\geq 0}$ be given. For any $p \in V_{q, S}^{*}, m_{2}(\lambda) \equiv m_{2}(\lambda, p)$ admits the product representation (cf Appendix (A) $m_{2}(\lambda)=\prod_{k \geq 1} \frac{\mu_{k}-\lambda}{\pi^{2} k^{2}}$ where $\left(\mu_{k}\right)_{k \geq 1}$ denote the Dirichlet eigenvalues of the operator $-\partial_{x}^{2}+p$, listed in lexicograpic order. By (A.2), $\dot{\Delta}(\lambda)$ also admits such a representation, $\dot{\Delta}(\lambda)=-\prod_{k \geq 1} \frac{\dot{\lambda}_{k}-\lambda}{\pi^{2} k^{2}}$ with $\left(\dot{\lambda}_{k}\right)_{k \geq 1}$ being listed in lexicographic order. Since $\left(\mu_{k}\right)_{k \geq 1},\left(\dot{\lambda}_{k}\right)_{k \geq 1}$ are simple

$$
\dot{m}_{2}\left(\mu_{n}\right)=-\frac{1}{\pi^{2} n^{2}} \prod_{k \neq n} \frac{\mu_{k}-\mu_{n}}{\pi^{2} k^{2}} \neq 0 \quad \text { and } \quad \ddot{\Delta}\left(\dot{\lambda}_{n}\right)=\frac{1}{\pi^{2} n^{2}} \prod_{k \neq n} \frac{\dot{\lambda}_{k}-\dot{\lambda}_{n}}{\pi^{2} k^{2}} \neq 0
$$

For any $n \in S_{+}^{\perp}$, one has $\mu_{n}=\dot{\lambda}_{n}=\tau_{n}$ and hence one concludes that for $n$ sufficiently large so that $\operatorname{Re} \tau_{n}>0$,

$$
\begin{equation*}
-\frac{\dot{m}_{2}\left(\tau_{n}\right)}{\ddot{\Delta}\left(\tau_{n}\right)}=\prod_{k \in S_{+}} \frac{\mu_{k}-\tau_{n}}{\dot{\lambda}_{k}-\tau_{n}}=\prod_{k \in S_{+}} \frac{1-\frac{\mu_{k}}{\tau_{n}}}{1-\frac{\dot{\lambda}_{k}}{\tau_{n}}} . \tag{C.29}
\end{equation*}
$$

Combining this with the results of $\tau_{n}$ (Lemma C.2) then yields the expansion of the stated form. Going through the arguments of the proof one concludes that $d_{2 k}^{a e}, k \geq 1$, and $\mathcal{R}_{2 N}^{d_{n}}$ have the claimed properties.

It remains to prove that the factors $e^{ \pm \mathrm{i} \beta_{n}(q)}$, appearing in the definition (2.7) of $W_{ \pm n}(x, q)$, admit an expansion as well. Clearly, it suffices to prove such an expansion for $\beta_{n}(q)$. Recall that for any $q \in M_{S}$ and $n \in S_{+}^{\perp}, \beta_{n}(q)$ is given by $\beta_{n}(q)=\sum_{\ell \in S_{+}} \beta_{\ell}^{n}(q)([13$, Theorem 8.5]) and by [13, page 70],

$$
\beta_{\ell}^{n}(q)=\int_{\lambda_{\ell}^{-}(q)}^{\mu_{\ell}^{*}(q)} \frac{\psi_{n}(\lambda, q)}{\sqrt{\Delta^{2}(\lambda, q)-4}} d \lambda, \quad \mu_{\ell}^{*}(q)=\left(\mu_{n}(q), \delta\left(\mu_{\ell}\right)\right) \in \mathbb{R}^{2}
$$

where $\delta(\lambda) \equiv \delta(\lambda, q)$ denotes the anti-discriminant. Here we used that for any $\ell \in S_{+}^{\perp} \backslash\{n\}$, one has $\mu_{\ell}=\lambda_{\ell}^{-}$ and hence $\beta_{\ell}^{n}(q)=0$. Furthermore recall that by [13, Theorem 8.1] $\psi_{n}(\lambda, q)$ is an entire function of $\lambda$.

Lemma C.6. Let $q \in M_{S}$ and $N \in \mathbb{Z}_{\geq 0}$. After shrinking $V_{q, S}^{*}$, if needed, it follows that for any $p \in V_{q, S}^{*}$, $\beta_{n}(p), n \in S_{+}^{\perp}$, admits an expansion of the form

$$
\beta_{n}(p)=\frac{1}{n \pi} \sum_{k=0}^{N} \frac{\beta_{2 k}^{a e}(p)}{(2 \pi \mathrm{i} n)^{2 k}}+\frac{\mathcal{R}_{2 N}^{\beta_{n}}(p)}{(2 \pi \mathrm{i} n)^{2 N+2}}
$$

where $V_{q, S}^{*} \rightarrow \mathbb{C}, p \mapsto \beta_{2 k}^{a e}(p), k \geq 1$, and $V_{q, S}^{*} \rightarrow \mathbb{C}, p \mapsto \mathcal{R}_{2 N}^{\beta_{n}}(p)$, are real analytic. In addition, the remainders $\mathcal{R}_{2 N}^{\beta_{n}}(p)$ satisfy

$$
\sup _{n \in S_{+}^{\perp}}\left|\mathcal{R}_{2 N}^{\beta_{n}}(p)\right| \leq C_{N}
$$

where the constant $C_{N}>0$ can be chosen locally uniformly for $p \in V_{q, S}^{*}$.
Proof. Let $p \in V_{q, S}^{*}$ and $n \in S_{+}^{\perp}$. Since $p$ is an $S$-gap potential, it follows from [13, Theorem 8.5] that the quotient of $\psi_{n}(\lambda) \equiv \psi_{n}(\lambda, p)$ with $\sqrt{\Delta^{2}(\lambda)-4} \equiv \sqrt{\Delta^{2}(\lambda, p)-4}$ is of the form

$$
\frac{\psi_{n}(\lambda)}{\sqrt{\Delta^{2}(\lambda)-4}}=\frac{\lambda^{M}+s_{M-1}^{n} \lambda^{M-1}+\cdots+s_{0}^{n}}{\sqrt{\mathcal{R}}} \frac{n \pi}{\tau_{n}-\lambda}
$$

where up to a sign, the complex numbers $s_{j}^{n} \equiv s_{j}^{n}(p), 0 \leq j \leq M-1$, are the symmetric functions of the roots $\sigma_{\ell}^{n}, \ell \in S_{+}$, of $\psi_{n}(\lambda), M=|S|$, and $\mathcal{R}(\lambda) \equiv \mathcal{R}(\lambda, p)$ is given by

$$
\mathcal{R}(\lambda)=\left(\lambda_{0}^{+}-\lambda\right) \prod_{j \in S_{+}}\left(\lambda_{j}^{+}-\lambda\right)\left(\lambda_{j}^{-}-\lambda\right)
$$

Here we used that for any $k \neq n$ with $\lambda_{k}^{+}=\lambda_{k}^{-}$, the eigenvalue $\lambda_{k}^{+}$is also a root of $\psi_{n}(\lambda)$ and we listed the roots $\sigma_{\ell}^{n}, \ell \in S_{+}$, in lexicographic order. Without loss of generality we thus may assume in the sequel that $\lambda_{\ell}^{+} \neq \lambda_{\ell}^{-}$for any $\ell \in S_{+}$. It then follows that for any $\ell \in S_{+}$,

$$
\beta_{\ell}^{n}=\int_{\lambda_{\ell}^{-}}^{\mu_{\ell}^{*}} \frac{\psi_{n}(\lambda)}{\sqrt{\Delta^{2}(\lambda)-4}} d \lambda=\frac{1}{n \pi} \int_{\lambda_{\ell}^{-}}^{\mu_{\ell}^{*}} \frac{\lambda^{M}}{\sqrt{\mathcal{R}}} \frac{n^{2} \pi^{2}}{\tau_{n}-\lambda} d \lambda+\frac{1}{n \pi} \sum_{j=0}^{M-1} s_{j}^{n} \int_{\lambda_{\ell}^{-}}^{\mu_{\ell}^{*}} \frac{\lambda^{j}}{\sqrt{\mathcal{R}}} \frac{n^{2} \pi^{2}}{\tau_{n}-\lambda} d \lambda
$$

Using LemmaC.2, one concludes that $\frac{n^{2} \pi^{2}}{\tau_{n}-\lambda}=1+O\left(\frac{1}{n^{2}}\right)$ and shows in a straightforward way that $\frac{n^{2} \pi^{2}}{\tau_{n}-\lambda}$ and hence the integrals $\int_{\lambda_{\ell}^{-}}^{\mu_{\ell}^{*}} \frac{\lambda^{j}}{\sqrt{\mathcal{R}}} \frac{n^{2} \pi^{2}}{\tau_{n}-\lambda} d \lambda$ admit an expansion in $\frac{1}{(2 \pi \mathrm{in})^{2 k}}, k \geq 0$, with coefficients and remainder having properties as stated. It thus remains to show that for any $0 \leq j \leq M-1, s_{j}^{n}$ also admits such an expansion. By the uniqueness statement of [13, Proposition D.7] (and after shrinking $V_{q, S}^{*}$, if needed,) it follows that $\left(s_{j}^{n}\right)_{0 \leq j \leq M-1}$ is the unique solution of the following inhomogeneous, linear $M \times M$ system

$$
\sum_{j=0}^{M-1} s_{j}^{n} \int_{\lambda_{\ell}^{-}}^{\lambda_{\ell}^{+}} \frac{\lambda^{j}}{\sqrt{\mathcal{R}}} \frac{n^{2} \pi^{2}}{\tau_{n}-\lambda} d \lambda=-\int_{\lambda_{\ell}^{-}}^{\lambda_{\ell}^{+}} \frac{\lambda^{M}}{\sqrt{\mathcal{R}}} \frac{n^{2} \pi^{2}}{\tau_{n}-\lambda} d \lambda, \quad \forall \ell \in S_{+}
$$

It then follows that $\operatorname{det}\left(E^{n}\right) \neq 0$ where $E^{n} \equiv E^{n}(p)$ denotes the $M \times M$ matrix with coefficients

$$
E_{\ell j}^{n}=\int_{\lambda_{\ell}^{-}}^{\lambda_{\ell}^{+}} \frac{\lambda^{j}}{\sqrt{\mathcal{R}}} \frac{n^{2} \pi^{2}}{\tau_{n}-\lambda} d \lambda, \quad \ell \in S_{+}, 0 \leq j \leq M-1 .
$$

Therefore,

$$
\left(s_{j}^{n}\right)_{0 \leq j \leq M-1}=-\left(E^{n}\right)^{-1}\left(b_{\ell}^{n}\right)_{\ell \in S_{+}}, \quad b_{\ell}^{n}=\int_{\lambda_{\ell}^{-}}^{\lambda_{\ell}^{+}} \frac{\lambda^{M}}{\sqrt{\mathcal{R}}} \frac{n^{2} \pi^{2}}{\tau_{n}-\lambda} d \lambda, \quad \ell \in S_{+} .
$$

Using once again the expansion of $\tau_{n}$ of Lemma C. 2 one shows that $s_{j}^{n}, 0 \leq j \leq M-1$, admit an expansion in $\frac{1}{(2 \pi \text { in })^{2 k}}, k \geq 0$, with coefficients and remainder having properties as stated. As an aside, we remark that by [13, Proposition D.7], $\lim _{n \rightarrow \infty} \sigma_{\ell}^{n}=\dot{\lambda}_{\ell}, \ell \in S_{+}$, and hence $\lim _{n \rightarrow \infty} s_{j}^{n}=s_{j}$ for any $0 \leq j \leq M-1$ where up to signs, $\left(s_{j}\right)_{0 \leq j \leq M-1}$ are the symmetric polynomials of $\dot{\lambda}_{\ell}, \ell \in S_{+}$. One then concludes that $\sigma_{\ell}^{n}=\dot{\lambda}_{\ell}+O\left(\frac{1}{n^{2}}\right), \ell \in S_{+}$, and in turn $s_{j}^{n}=s_{j}+O\left(\frac{1}{n^{2}}\right), 0 \leq j \leq M-1$.

We finish this appendix by proving an expansion of the KdV frequencies $\omega_{n} \equiv \omega_{n}^{k d v}$ (cf Section (1) at finite gap potentials. Using the Birkhoff map, we view them as functions of the potential, which by a slight abuse of notation, we denote also by $\omega_{n}$.

Lemma C.7. Let $q \in M_{S}$ and $N \in \mathbb{Z}_{\geq 0}$. Then for any $p \in V_{q, S}^{*}$, the $\operatorname{KdV}$ frequencies $\omega_{n}(p)$, $n \in S_{+}^{\perp}$, have an expansion of the form

$$
\begin{equation*}
\omega_{n}(p)=(2 \pi n)^{3}+\sum_{k=1}^{N} \frac{\omega_{2 k-1}^{a e}(p)}{(2 \pi n)^{2 k-1}}+\frac{\mathcal{R}_{2 N}^{\omega_{n}}(p)}{(2 \pi n)^{2 N+1}} \tag{C.30}
\end{equation*}
$$

where $V_{q, S}^{*} \rightarrow \mathbb{C}, p \mapsto \omega_{2 k-1}^{a e}(p), k \geq 1$, and $V_{q, S}^{*} \rightarrow \mathbb{C}, p \mapsto \mathcal{R}_{2 N}^{\omega_{n}}(p)$, are real analytic. In addition, the remainders $\mathcal{R}_{2 N}^{\omega_{n}}(p)$ satisfy

$$
\sup _{n \in S_{+}^{\perp}}\left|\mathcal{R}_{2 N}^{\omega_{n}}(p)\right| \leq C_{N}
$$

where the constant $C_{N}>0$ can be chosen locally uniformly for $p \in V_{q, S}^{*}$.

Proof. Let $q \in M_{S}$ and $N \in \mathbb{Z}_{\geq 0}$ be given. The basic ingredient into our proof of (C.30) are formulas of the frequencies in terms of periods of an Abelian differential of the second kind on the hyperelliptic Riemann surface $\Sigma_{p}$, associated to the periodic spectrum of $L_{p}=-\partial_{x}^{2}+p$ (see [5], [6, [8, [17, [22]). We follow [17, Section 2] and note that the arguments made there extend to complex valued potentials: for any $p \in V_{q, S}^{*}$, denote by $\Sigma_{p}$ the compact Riemann surface associated to the simple periodic eigenvalues of $p$,

$$
\Sigma_{p}:=\left\{(\lambda, \mu) \in \mathbb{C}^{2}: \mu^{2}=\left(\lambda-\lambda_{0}\right) \prod_{j \in J}\left(\lambda-\lambda_{j}^{+}\right)\left(\lambda-\lambda_{j}^{-}\right)\right\} \cup\{\infty\}
$$

where $J \equiv J(p):=\left\{j \in S_{+}: \lambda_{j}^{+}(p) \neq \lambda_{j}^{-}(p)\right\}$. The variable $z \in \mathbb{C}$ around the point $z=0$ gives a complex chart in a neighborhood of the branch point $\infty \in \Sigma_{p}$ via the substitution $\lambda=-\frac{1}{z^{2}}$. By construction, this chart is defined uniquely up to a change of sign of the variable $z, z \mapsto-z$, and is referred to as standard chart. Then $\Sigma_{p}$ admits an Abelian differential $\Omega_{4}$ of the second kind, uniquely determined by the following properties: (i) $\Omega_{4}$ is holomorphic on $\Sigma_{p} \backslash\{\infty\}$; (ii) near $\infty, \Omega_{4}$ is of the form

$$
\Omega_{4}=\frac{1}{z^{4}} d z+h(z) d z, \quad h \text { holomorphic near } z=0
$$

in the appropriate standard chart; (iii) $\int_{a_{j}} \Omega_{4}=0$ for any $j \in J$ where $a_{j}$ are the smooth cycles around the gap $\left[\lambda_{j}^{-}, \lambda_{j}^{+}\right]$defined in [17, Section 2]. The differential $\Omega_{4}$ is of the form

$$
\begin{equation*}
\frac{\mathrm{i}}{2} \frac{\lambda^{M+1}+c_{1} \lambda^{M}+\cdots+c_{M+1}}{\sqrt{\left(\lambda-\lambda_{0}^{+}\right) \prod_{j \in J}\left(\lambda-\lambda_{j}^{+}\right)\left(\lambda-\lambda_{j}^{-}\right)}} d \lambda \tag{C.31}
\end{equation*}
$$

where $M:=|J(p)|$ and the coefficients $c_{1}, \ldots, c_{M+1}$ are real analytic functions on $V_{q, S}^{*}$. Then by [17, formula (2.19)],

$$
\omega_{n}=12 \mathrm{i} \int_{b_{n}} \Omega_{4}, \quad \forall n \geq 1
$$

where $b_{n}, n \geq 1$, are the cycles as defined in [17, Section 2]. Let $m_{S}:=1+\max \{k \in S\}$. Then for any $n \geq m_{S}, \lambda_{n}^{-}=\lambda_{n}^{+}=\tau_{n}$. It then follows from the definition of the cycles $b_{n}$ that for any $n \geq m_{S}$

$$
\begin{equation*}
\omega_{n}=\omega_{m_{S}}+12 \mathrm{i}\left(2 \int_{\tau_{m_{S}}}^{\tau_{n}} \Omega_{4}\right) \tag{C.32}
\end{equation*}
$$

with the appropriate choice of the root in the denominator of $\Omega_{4}$. The abelian integral $\int_{\tau_{m_{S}}}^{\lambda} \Omega_{4}$ has an expansion as $\lambda \rightarrow \infty$ of the form

$$
\int_{\tau_{m_{S}}}^{\lambda} \Omega_{4}=b_{*} \lambda^{\frac{3}{2}}+b_{0} \lambda^{\frac{1}{2}}+b_{1} \frac{1}{\lambda^{\frac{1}{2}}}+\cdots+b_{* *}=\lambda^{\frac{1}{2}}\left(b_{*} \lambda+b_{0}+b_{1} \frac{1}{\lambda}+\cdots\right)+b_{* *}
$$

and hence as $n \rightarrow \infty$

$$
\begin{equation*}
\int_{\tau_{m_{S}}}^{\tau_{n}} \Omega_{4}=b_{* *}+\sqrt{\tau_{n}}\left(b_{*} \tau_{n}+b_{0}+b_{1} \frac{1}{\tau_{n}}+\cdots\right) \tag{C.33}
\end{equation*}
$$

In view of the formula [17, (2.20)] of $\Omega_{4}$, the coefficients $b_{*}, b_{* *}, b_{0}, b_{1}, \ldots$ are real analytic functions on $V_{q, S}^{*}$. Furthermore, it is well known (cf e.g. [16, Proposition 8.1]) that since $\int_{0}^{1} p(x) d x=0$

$$
\begin{equation*}
\omega_{n}=(2 \pi n)^{3}+O\left(\frac{1}{n}\right) \tag{C.34}
\end{equation*}
$$

Combining (C.32) - C.34) with the results on $\tau_{n}$ and $\sqrt{\tau_{n}}$ of Lemma C.2 one obtains an expansion of $\omega_{n}$, $n \geq m_{S}$, of the form (C.30 where $\omega_{2 k-1}^{a e}: V_{q, S}^{*} \rightarrow \mathbb{C}, k \geq 1$, and $\mathcal{R}_{2 N}^{\omega_{n}}: V_{q, S}^{*} \rightarrow \mathbb{C}, n \geq m_{S}$, are real analytic and $\mathcal{R}_{2 N}^{\omega_{n}}: V_{q, S}^{*} \rightarrow \mathbb{C}, n \geq m_{S}$, have the claimed bounds. For $n \in S_{+}^{\perp}$ with $n<m_{S}$, one defines $\mathcal{R}_{2 N}^{\omega_{n}}$ by $(\mathrm{C} .30)$ and since $\omega_{n}$ are real analytic on $V_{q, S}^{*}$, one then concludes that $\mathcal{R}_{2 N}^{\omega_{n}}, n \in S_{+}^{\perp}$, are real analytic on $V_{q, S}^{*}$ and satisfy the claimed bounds.

## D Reversibility structure

In this appendix we prove that the Birkhoff map $\Phi^{k d v}$ and hence also its inverse $\Psi^{k d v}$ preserve the reversible structure, defined by the maps

$$
S_{\text {rev }}: L_{0}^{2} \rightarrow L_{0}^{2}, \quad\left(S_{\text {rev }} q\right)(x):=q(-x) \quad \text { and } \quad \mathcal{S}_{\text {rev }}: h_{0}^{0} \rightarrow h_{0}^{0}, \quad\left(\mathcal{S}_{\text {rev }} w\right)_{n}:=w_{-n}, \forall n \neq 0
$$

Proposition D.1. One has

$$
\Phi^{k d v} \circ S_{r e v}=\mathcal{S}_{r e v} \circ \Phi^{k d v}
$$

As a consequence, $S_{\text {rev }} \circ \Psi^{k d v}=\Psi^{k d v} \circ \mathcal{S}_{\text {rev }}$ and by the chain rule, for any $q \in L_{0}^{2}(\mathbb{T})$ and $w \in h_{0}^{0}$

$$
\left(d_{S_{r e v}(q)} \Phi^{k d v}\right) \circ S_{r e v}=\mathcal{S}_{r e v} \circ d_{q} \Phi^{k d v}, \quad\left(d_{\mathcal{S}_{r e v}(w)} \Psi^{k d v}\right) \circ \mathcal{S}_{r e v}=S_{r e v} \circ d_{w} \Psi^{k d v}
$$

First we establish some preliminary results. Recall that $y_{j}(x, q) \equiv y_{j}(x, \lambda, q), j=1,2$, denote the fundamental solutions of $-y^{\prime \prime}+q y=\lambda y, \Delta(\lambda) \equiv \Delta(\lambda, q)$ the discriminant, and $\delta(\lambda) \equiv \delta(\lambda, q)$ the antidiscriminant,

$$
\Delta(\lambda)=y_{1}(1, \lambda)+y_{2}^{\prime}(1, \lambda), \quad \delta(\lambda)=y_{1}(1, \lambda)-y_{2}^{\prime}(1, \lambda) .
$$

In a straightforward way, one verifies the following
Lemma D.1. For any $q \in L_{0}^{2}, \lambda \in \mathbb{C}, x \in \mathbb{R}$,

$$
y_{1}\left(x, \lambda, S_{\text {rev }} q\right)=y_{1}(-x, \lambda, q), \quad y_{2}\left(x, \lambda, S_{\text {rev }} q\right)=-y_{2}(-x, \lambda, q)
$$

or alternatively,
$y_{1}\left(x, \lambda, S_{\text {rev }} q\right)=\left.\left(y_{2}^{\prime}(1) y_{1}(1-x)-y_{1}^{\prime}(1) y_{2}(1-x)\right)\right|_{\lambda, q}, \quad y_{2}\left(x, \lambda, S_{\text {rev }} q\right)=\left.\left(y_{2}(1) y_{1}(1-x)-y_{1}(1) y_{2}(1-x)\right)\right|_{\lambda, q}$.
The latter identities imply that

$$
\begin{equation*}
\Delta\left(\lambda, S_{r e v} q\right)=\Delta(\lambda, q), \quad \delta\left(\lambda, S_{r e v} q\right)=-\delta(\lambda, q), \quad y_{2}\left(1, \lambda, S_{r e v} q\right)=y_{2}(1, \lambda, q) \tag{D.1}
\end{equation*}
$$

An immediate consequence of the first identity in (D.1) is that

$$
\begin{equation*}
\lambda_{0}^{+}\left(S_{r e v} q\right)=\lambda_{0}^{+}(q), \quad \lambda_{n}^{ \pm}\left(S_{r e v} q\right)=\lambda_{n}^{ \pm}(q), \forall n \geq 1, \quad \gamma_{n}\left(S_{r e v} q\right)=\gamma_{n}(q), \forall n \geq 1 \tag{D.2}
\end{equation*}
$$

Moreover by (D.1),

$$
\begin{equation*}
\mu_{n}\left(S_{\text {rev }} q\right)=\mu_{n}(q), \quad \delta\left(\mu_{n}, S_{\text {rev }} q\right)=-\delta\left(\mu_{n}, q\right), \quad \forall n \geq 1 \tag{D.3}
\end{equation*}
$$

For any $q \in L_{0}^{2}$, the action variables $I_{n} \equiv I_{n}(q), n \geq 1$, are defined by contour integrals (cf. [13, p 64]),

$$
I_{n}=\frac{1}{\pi} \int_{\Gamma_{n}} \frac{\lambda \dot{\Delta}(\lambda)}{\sqrt{\Delta(\lambda)^{2}-4}} d \lambda
$$

Furthermore the normalizing factor $\xi_{n} \equiv \xi_{n}(q)$, defined for $q \in L_{0}^{2}$ with $\gamma_{n}(q)>0$ by $\xi_{n}=\sqrt{8 I_{n} / \gamma_{n}^{2}}$, extends analytically to $L_{0}^{2}$ (cf [13, Theorem 7.3]). By [13, Theorem 8.5], $\beta_{n}=\sum_{k \neq n} \beta_{k}^{n}$ is well defined on $L_{0}^{2}$ where $\beta_{k}^{n} \equiv \beta_{k}^{n}(q)$ is given by (cf [13, p 70])

$$
\beta_{k}^{n}=\int_{\lambda_{k}^{-}}^{\mu_{k}^{*}} \frac{\psi_{n}(\lambda)}{\sqrt{\Delta^{2}(\lambda)-4}} d \lambda, \quad \mu_{k}^{*}=\left(\mu_{k}, \delta\left(\mu_{k}\right)\right)
$$

with the sign of $\sqrt{\Delta^{2}(\lambda)-4}$ determined by $\sqrt[*]{\Delta\left(\mu_{k}\right)-4}=\delta\left(\mu_{k}\right)$. On the other hand, $\eta_{n} \equiv \eta_{n}(q)$ and $\theta_{n} \equiv \theta_{n}(q)$ are well defined modul $2 \pi$ on $L_{0}^{2} \backslash Z_{n}$ by

$$
\eta_{n}=\int_{\lambda_{n}^{-}}^{\mu_{n}^{*}} \frac{\psi_{n}(\lambda)}{\sqrt{\Delta^{2}(\lambda)-4}} d \lambda, \quad \theta_{n}=\eta_{n}+\beta_{n}
$$

where $Z_{n}=\left\{q \in L_{0}^{2}: \gamma_{n}(q)=0\right\}$. One then concludes from (D.1), (D.2), (D.3) that the following holds.

Corollary D.1. For any $q \in L_{0}^{2}$ and $n \geq 1$,

$$
\begin{equation*}
I_{n}\left(S_{\text {rev }} q\right)=I_{n}(q), \quad \xi_{n}\left(S_{\text {rev }} q\right)=\xi_{n}(q), \quad \beta_{n}\left(S_{\text {rev }} q\right)=-\beta_{n}(q) \tag{D.4}
\end{equation*}
$$

Furthermore on $L_{0}^{2} \backslash Z_{n}, \theta_{n}\left(S_{\text {rev }} q\right)=-\theta_{n}(q)$ modulo $2 \pi$.
With these preparations made we now prove Proposition D. 1
Proof of Proposition D.1. For any $n \geq 1$ and $q \in L_{0}^{2} \backslash Z_{n}$, the complex Birkhoff coordinates $z_{n}(q), z_{-n}(q)$ are given by $z_{n}(q)=\sqrt{n \pi} \sqrt{I_{n}(q)} e^{-\mathrm{i} \theta_{n}(q)}, z_{-n}(q)=\sqrt{n \pi} \sqrt{I_{n}(q)} e^{\mathrm{i} \theta_{n}(q)}$, whereas for $q \in Z_{n}, z_{n}(q)=0$ and $z_{-n}(q)=0$. Hence it follows from Corollary D.1 that $z_{n}\left(S_{r e v} q\right)=z_{-n}(q)$ and $z_{-n}\left(S_{r e v} q\right)=z_{n}(q)$ for any $n \geq 1$. This proves that $\Phi^{k d v} \circ S_{\text {rev }}=\mathcal{S}_{\text {rev }} \circ \Phi^{k d v}$.

## E Properties of pseudodifferential and paradifferential calculus

In this appendix we collect some well known facts about pseudodifferential and paradifferential calculus on the torus. We refer to [23] for further details. Let $\chi \in C^{\infty}\left(\mathbb{R}^{2}, \mathbb{R}\right)$ be an admissible cut-off function. It means that $\chi$ is an even function and that there exist $0<\varepsilon_{1}<\varepsilon_{2}<1$ so that for any $(\vartheta, \eta) \in \mathbb{R}^{2}$ and $\alpha, \beta \in \mathbb{Z}_{\geq 0}$,

$$
\begin{align*}
& \chi(\vartheta, \eta)=1, \quad \forall|\vartheta| \leq \varepsilon_{1}+\varepsilon_{1}|\eta|, \quad \chi(\vartheta, \eta)=0, \quad \forall|\vartheta| \geq \varepsilon_{2}+\varepsilon_{2}|\eta|  \tag{E.1}\\
&\left.\mid \partial_{\vartheta}^{\alpha} \partial_{\eta}^{\beta} \chi(\vartheta, \eta)\right) \mid \leq C_{\alpha, \beta}(1+|\eta|)^{-\alpha-\beta} \tag{E.2}
\end{align*}
$$

For any $a \in H^{1}$, the paraproduct $T_{a} u$ of the function $a$ with $u \in L^{2}$ (with respect to the cut-off function $\chi$ ) is defined as

$$
\begin{equation*}
\left(T_{a} u\right)(x):=\sum_{k, n \in \mathbb{Z}} \chi(k, n) a_{k} u_{n} e^{\mathrm{i} 2 \pi(k+n) x} \tag{E.3}
\end{equation*}
$$

where we recall that $u_{n}, n \in \mathbb{Z}$, denote the Fourier coefficients of $u, u_{n}=\int_{0}^{1} u(x) e^{-2 \pi \mathrm{i} n x} d x$. Note that since $a, u$, and $\chi$ are real valued and $\chi$ is even, $T_{a} u$ is real valued as well. Given any $s, s^{\prime} \in \mathbb{Z}$, we denote by $\mathcal{B}\left(H^{s}, H^{s^{\prime}}\right)$ the Banach space of all bounded linear operators $H^{s} \rightarrow H^{s^{\prime}}$, endowed with the operator norm $\|\cdot\|_{\mathcal{B}\left(H^{s}, H^{s^{\prime}}\right)}$. In case $s=s^{\prime}$, we also write $\mathcal{B}\left(H^{s}\right)$ instead of $\mathcal{B}\left(H^{s}, H^{s}\right)$. Given any linear operator $A \in \mathcal{B}\left(H^{s}, H^{s^{\prime}}\right)$, we denote by $A^{t}$ the transpose of $A$ with respect to the $L^{2}$-inner product. It is an element in $\mathcal{B}\left(\left(H^{s^{\prime}}\right)^{*},\left(H^{s}\right)^{*}\right)$ where $\left(H^{s}\right)^{*}$ denotes the dual of $H^{s}$.

Lemma E.1. (i) For any $s \in \mathbb{Z}_{\geq 0}$ and $a \in H^{1}$, the linear operator $T_{a}: u \mapsto T_{a} u$ is in $\mathcal{B}\left(H^{s}\right)$. Furthermore the linear map $H^{1} \rightarrow \mathcal{B}\left(H^{s}\right), a \mapsto T_{a}$, is bounded, $\left\|T_{a}\right\|_{\mathcal{B}\left(H^{s}\right)} \lesssim_{s}\|a\|_{1}$.
(ii) Let $a \in H^{s_{1}}, b \in H^{s_{2}}$ and $s_{1}, s_{2} \in \mathbb{Z}_{\geq 1}$. Then

$$
a b=T_{a} b+T_{b} a+\mathcal{R}^{(B)}(a, b)
$$

where the bilinear map $\mathcal{R}^{(B)}: H^{s_{1}} \times H^{s_{2}} \rightarrow H^{s_{1}+s_{2}-1},(a, b) \mapsto \mathcal{R}^{(B)}(a, b)$, is continuous and satisfies the estimate

$$
\left\|\mathcal{R}^{(B)}(a, b)\right\|_{s_{1}+s_{2}-1} \lesssim_{s_{1}, s_{2}}\|a\|_{s_{1}}\|b\|_{s_{2}}
$$

(iii) Let $a \in H^{\rho}$ with $\rho \in \mathbb{Z}_{\geq 2}$. Then for any $s \geq 0, T_{a}^{t}-T_{a} \in \mathcal{B}\left(H^{s}, H^{s+\rho-1}\right)$ and

$$
\left\|T_{a}^{t}-T_{a}\right\|_{\mathcal{B}\left(H^{s}, H^{s+\rho-1}\right)} \lesssim_{s, \rho}\|a\|_{\rho}
$$

(iv) Let $a, b \in H^{\rho}$ with $\rho \in \mathbb{Z}_{\geq 1}$. Then for any $s \geq 0, T_{a} \circ T_{b}-T_{a b} \in \mathcal{B}\left(H^{s}, H^{s+\rho-1}\right)$ and

$$
\left\|T_{a} \circ T_{b}-T_{a b}\right\|_{\mathcal{B}\left(H^{s}, H^{s+\rho-1}\right)} \lesssim_{s, \rho}\|a\|_{\rho}\|b\|_{\rho}
$$

Lemma E.2. (i) Let $k, j \in \mathbb{Z}_{\geq 0}$ and $a \in C^{\infty}(\mathbb{T})$. Then for any $s \in \mathbb{Z}_{\geq 0}$ and $N \in \mathbb{N}$ with $N \geq k+j$, the composition $\partial_{x}^{-k} \circ a \partial_{x}^{-j}$ is a bounded linear operator $H^{s} \rightarrow H^{s+k+j}$ which admits an expansion of the form

$$
\partial_{x}^{-k} \circ a \partial_{x}^{-j}=a \partial_{x}^{-k-j}+\sum_{i=1}^{N-k-j} C_{i}(k, j)\left(\partial_{x}^{i} a\right) \partial_{x}^{-k-j-i}+\mathcal{R}_{N, k, j}^{\psi d o}(a)
$$

where $C_{i}(k, j), 1 \leq i \leq N-k-j$, are constants depending on $k, j$ and the remainder $\mathcal{R}_{N, k, j}^{\psi d o}(a)$ is a bounded linear operator $H^{s} \rightarrow H^{s+N+1}$, satisfying the estimate

$$
\begin{equation*}
\left\|\mathcal{R}_{N, k, j}^{\psi d o}(a)\right\|_{\mathcal{B}\left(H^{s}, H^{s+N+1}\right)} \lesssim_{s, N}\|a\|_{s+2 N} \tag{E.4}
\end{equation*}
$$

(ii) Let $k, j \in \mathbb{Z}_{\geq 0}$ and $N \geq k+j$. There exists a constant $\sigma_{N}>N-k-j+1$ such that for any $a \in H^{\sigma_{N}}$ and any $s \in \mathbb{Z}_{\geq 0}$, the composition $\partial_{x}^{-k} \circ T_{a} \circ \partial_{x}^{-j}$ is a bounded linear operator $H^{s} \rightarrow H^{s+k+j}$ which admits an expansion of the form

$$
\partial_{x}^{-k} \circ T_{a} \circ \partial_{x}^{-j}=T_{a} \partial_{x}^{-k-j}+\sum_{i=1}^{N-k-j} C_{i}(k, j) T_{\partial_{x}^{i} a} \partial_{x}^{-k-j-i}+\mathcal{R}_{N, k, j}^{(B)}(a)
$$

where $C_{i}(k, j), 1 \leq i \leq N-k-j$, are constants depending on $k, j$ and for any $s \geq 0$, the remainder $\mathcal{R}_{N, k, j}^{(B)}(a)$ is a bounded linear operator $H^{s} \rightarrow H^{s+N+1}$, satisfying the estimate

$$
\begin{equation*}
\left\|\mathcal{R}_{N, n, k}^{(B)}(a)\right\|_{\mathcal{B}\left(H^{s}, H^{s+N+1}\right)} \lesssim_{s, N}\|a\|_{\sigma_{N}} . \tag{E.5}
\end{equation*}
$$

Finally, we record the following well known tame estimates of products of functions in Sobolev spaces.
Lemma E.3. For any $s \in \mathbb{Z}_{\geq 1}$,

$$
\|u v\|_{s} \gtrsim_{s}\|u\|_{s}\|v\|_{1}+\|u\|_{1}\|v\|_{s}, \quad \forall u, v \in H^{s} .
$$

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