# SOME OVERDETERMINED PROBLEMS RELATED TO THE ANISOTROPIC CAPACITY

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Abstract. We characterize the Wulff shape of an anisotropic norm in terms of solutions to overdetermined problems for the Finsler $p$-capacity of a convex set $\Omega \subset \mathbb{R}^N$, with $1 < p < N$. In particular we show that if the Finsler $p$-capacitary potential $u$ associated to $\Omega$ has two homothetic level sets then $\Omega$ is Wulff shape. Moreover, we show that the concavity exponent of $u$ is $\mathsf{q} = -(p-1)/(N-p)$ if and only if $\Omega$ is Wulff shape.

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**Key words.** Wulff shape. Overdetermined problems. Capacity. Concavity exponent.

## 1. Introduction

The aim of this paper is to study some unconventional overdetermined problems for the Finsler $p$-capacity of a bounded convex set $\Omega$ associated to a norm $H$ of $\mathbb{R}^N$, $N \geq 3$.

Given a bounded convex domain $\Omega \subset \mathbb{R}^N$, the $p$-capacity of $\Omega$ is defined by

$$\mathrm{Cap_p}(\Omega) = \inf \left\{ \frac{1}{p} \int_{\mathbb{R}^N} |D\varphi|^p \, dx, \ \varphi \in C_0^\infty(\mathbb{R}^N), \varphi(x) \geq 1 \text{ for } x \in \Omega \right\}$$

with $1 < p < N$. When the Euclidean norm $|\cdot|$ is replaced by a more general norm $H(\cdot)$, one can consider the so called *Finsler* $p$-capacity $\mathrm{Cap_{H,p}}(\Omega)$, which is defined by

$$\mathrm{Cap_{H,p}}(\Omega) = \inf \left\{ \frac{1}{p} \int_{\mathbb{R}^N} H^p(D\varphi) \, dx, \ \varphi \in C_0^\infty(\mathbb{R}^N), \varphi(x) \geq 1 \text{ for } x \in \Omega \right\}, \tag{1.1}$$

for $1 < p < N$. Under suitable assumptions on the norm $H$ and on the set $\Omega$, the above infimum is attained and

$$\mathrm{Cap_{H,p}}(\Omega) = \frac{1}{p} \int_{\mathbb{R}^N} H^p(Du_\Omega) \, dx \,,$$

where $u_\Omega$ is the solution of the Finsler $p$-capacity problem

$$\begin{cases} \Delta_p^H u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \\ u = 1 & \text{on } \partial\Omega \,, \\ u \to 0 & \text{as } H(x) \to +\infty \,. \end{cases} \tag{1.2}$$

Here $\Delta_p^H$ denotes the Finsler $p$-Laplace operator, i.e. $\Delta_p^H u = \mathrm{div}(H^{p-1}(Du)\nabla H(Du))$. The function $u_\Omega$ is named *(Finsler) p-capacitary potential* of $\Omega$.

When $\Omega$ is Wulff shape, i.e. it is a sublevel set of the dual norm $H_0$

$$\Omega = B_{H_0}(r) = \{x \in \mathbb{R}^N : \ H_0(x - \bar{x}) < r\}$$

(see Section 2 for definitions), the solution to (1.2) can be explicitly computed and it is given by

$$v_r(x) = \Big( \frac{H_0(x - \bar{x})}{r} \Big)^{\frac{1}{\mathsf{q}}} , \tag{1.3}$$

with

$$\mathsf{q} = -\frac{p-1}{N-p} \,. \tag{1.4}$$

It is straightforward to verify that the potential $v_r$ in (1.3) enjoys the following properties:

(i) the function $v_r^{\mathsf{q}}$ is convex, i.e. $v_r$ is $\mathsf{q}$-*concave*;
(ii) the superlevel sets of $v_r$ are homothetic sets and they are Wulff shapes;
(iii) $H(Dv_r)$ is constant on the level sets of $v_r$.

The aim of this paper is to show that each of the properties (i)-(iii) characterizes the Wulff shape under some regularity assumptions on the norm $H$ and on $\Omega$. In particular, we assume that $H \in \mathcal{J}_p$ where

$$\mathcal{J}_p = \{H \in C_+^2(\mathbb{R}^N \setminus \{0\}) \,, H^p \in C^{2,1}(\mathbb{R}^N \setminus \{0\}) \ \} \,. \tag{1.5}$$

Our first main result is related to property (i) and it is about concavity properties of the solution to (1.2). We recall that a nonnegative function $v$ with convex support is *$\alpha$-concave*, for some $\alpha \in [-\infty, +\infty]$, if

- $v$ is a positive constant in its support set, in case $\alpha = +\infty$;
- $v^\alpha$ is concave, in case $\alpha > 0$;
- $\log v$ is concave, in case $\alpha = 0$ (and $v$ is called *log-concave*);
- $v^\alpha$ is convex, in case $\alpha < 0$;
- all its super level sets $\{v > t\}$ are convex, in case $\alpha = -\infty$ (and $v$ is called *quasi-concave*).

Notice that if $v$ is $\alpha$-concave for some $\alpha > -\infty$, then it is $\beta$-concave for every $\beta \in [-\infty, \alpha]$. Then quasi-concavity is the weakest among concavity properties.

Concavity properties of solutions to elliptic and parabolic equations are a popular field of investigation. Classical results in this framework are for instance the log-concavity of the first Dirichlet eigenfunction of the Laplacian (see [2]), the preservation of concavity by the heat flow (see again [2]), the $\frac{1}{2}$-concavity (i.e. the concavity of the square root) of the torsion function (see [16, 6, 7, 14]), the quasi-concavity of the Newton potential and of the $p$-capacitary potential (see [10, 15]). The latter results are especially related to the situation we consider in this paper. Indeed, it is proved in [1] (and it can be also obtained with the methods of [3]) that when $\Omega$ is a convex domain, its $p$-capacitary potential $u_\Omega$ is a *quasi-concave function*, i.e. all its superlevel sets are convex. Moreover, as we have seen, quasi-concavity is the weakest property in this context and one may expect and ask more than this. Then, following [14, 17, 13], it is natural to define the *concavity exponent* associated to the solution to (1.2) as

$$\alpha(\Omega, p) = \sup\{\beta \le 1 \,:\, u_\Omega \text{ is } \beta\text{-concave } \}\,. \tag{1.6}$$

In the Euclidean case, it was proved in [17] that the concavity exponent attains its maximum when $\Omega$ is a ball (and only in this case). In the following theorem, we characterize the Wulff shape in terms of property (i) above. More precisely, we generalize the results of [17] to the anisotropic setting and we prove that the exponent $\mathsf{q}$ characterizes the Wulff shape.

**Theorem 1.1.** *Let $H$ be a norm of $\mathbb{R}^N$ in the class (1.5) and let $\Omega$ be a bounded convex domain of $\mathbb{R}^N$ of class $C^2$. Then*

$$\alpha(\Omega, p) \le \mathsf{q},$$

*with $\mathsf{q}$ given by (1.4), and equality holds if and only if $\Omega$ is Wulff shape.*

The proof of Theorem 1.1 is based on the Brunn-Minkowski inequality for Finsler $p$-capacity, recently proved in [1] (and here recalled in Proposition 2.1), and upon the fact $u$ can have a level set homothetic to $\Omega$ if and only if $\Omega$ is a ball. Clearly this property is related to property (ii) above. And indeed the characterization of the Wulff shape is achieved whenever *just two* superlevel sets of $u_\Omega$ are homothetic, as expressed in the following theorem.

**Theorem 1.2.** *Let $H$ be a norm of $\mathbb{R}^N$ in the class (1.5). Let $\Omega \subset \mathbb{R}^N$, $N \ge 3$, be a bounded convex domain with boundary of class $C^2_+$. If there exists a solution to (1.2) having two homothetic superlevel sets, then $\Omega$ is Wulff shape.*

The Euclidean counterpart of Theorem 1.2 was proved in [17]. The proof of Theorem 1.2 in the anisotropic setting passes through the following theorem, which is related to property (iii).

**Theorem 1.3.** *Let $\Omega \subset \mathbb{R}^N$ be a convex domain containing the origin and with boundary of class $C^{2,\alpha}$. Let $H \in \mathcal{J}_p$ and let $R > 0$ be such that $\overline{\Omega} \subset B_{H_0}(R)$. There exists a solution to*

$$\begin{cases} \Delta_p^H u = 0 & \text{in } B_{H_0}(R) \setminus \overline{\Omega} \\ u = 1 & \text{on } \partial\Omega \\ u = 0 & \text{on } \partial B_{H_0}(R) \\ H(Du) = C & \text{on } \partial B_{H_0}(R)\,, \end{cases} \tag{1.7}$$

*for some constant $C > 0$ if and only if $\Omega = B_{H_0}(r)$, with*

$$r\,C = \frac{N-p}{p-1}\,. \tag{1.8}$$

Since two boundary conditions (Dirichlet and Neumann) are imposed on a prescribed part of the boundary, Theorem 1.3 clearly falls in the realm of *overdetermined problems*: since the domain $\Omega$ is not prescribed, the unknown of the problem is in fact the couple $(\Omega, u)$, and by imposing that $u$ has some peculiar property (which is not commonly shared by all the solution of the involved PDE), one ask whether this is sufficient to uniquely determine the domain $\Omega$. In this sense, also Theorem 1.1 and Theorem 1.2 can be considered as overdetermined problems, since we ask for a solution of a Dirichlet problem satisfying some extra special condition ($\mathbf{q}$-concavity or homothety of level sets, respectively).

The paper is organized as follows. In Section 2 we introduce some notation and basic properties of Finsler norms; then we recall some known fact about the Finsler capacity $\mathrm{Cap_{H,p}}$ of a convex set and, in particular, the Brunn-Minkowski inequality from [1]. Theorems 1.3, 1.2 and 1.1 are proved in Sections 3, 4 and 5, respectively.

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## 2. Notations

2.1. **Norms of $\mathbb{R}^N$.** We consider the space $\mathbb{R}^N$ endowed with a generic norm $H : \mathbb{R}^N \to \mathbb{R}$ such that:

(i) $H$ is convex;
(ii) $H(\xi) \geq 0$ for $\xi \in \mathbb{R}^N$ and $H(\xi) = 0$ if and only if $\xi = 0$;
(iii) $H(t\xi) = |t|H(\xi)$ for $\xi \in \mathbb{R}^N$ and $t \in \mathbb{R}$.

Then we identify the dual space of $\mathbb{R}^N$ with $\mathbb{R}^N$ itself via the scalar product $\langle \cdot ; \cdot \rangle$. Accordingly the space $\mathbb{R}^N$ turns out to be endowed also with the dual norm $H_0$ given by

$$H_0(x) = \sup_{\xi \neq 0} \frac{\langle x; \xi \rangle}{H(\xi)} \quad \text{for } x \in \mathbb{R}^N . \tag{2.1}$$

We denote by $B_{H_0}(r)$ the anisotropic ball centered at $O$ with radius $r$ in the norm $H_0$, i.e.

$$B_{H_0}(r) = \{x \in \mathbb{R}^N : \ H_0(x) < r\}.$$

Analogously, we define

$$B_H(r) = \{\xi \in \mathbb{R}^N : \ H(\xi) < r\}.$$

The sets $B_{H_0}(r)$ and $B_H(r)$ are called *Wulff shape* of $H$ and $H_0$, respectively; in the special case $r = 1$ they are indicated by $B_{H_0}, B_H$, respectively. Notice that, in the language of the theory of convex bodies, $H$ is *the support function* of $B_{H_0}$ and $H_0$ is in turn the support function of $B_H$.

For a regular convex domain $\Omega$ the Finsler perimeter is defined by

$$P_H(\partial\Omega) = \int_{\partial\Omega} H(\nu)\, d\sigma,$$

where $\nu$ is the outer unit normal to $\partial\Omega$.

2.2. **Finsler capacity.** For a bounded convex domain $\Omega$ in $\mathbb{R}^N$ its *Finsler p-capacity*, denoted by $\mathrm{Cap_{H,p}}(\Omega)$, is defined as follows:

$$\mathrm{Cap_{H,p}}(\Omega) = \inf \left\{ \frac{1}{p} \int_{\mathbb{R}^N} H^p(D\varphi)\, dx, \ \varphi \in C_0^\infty(\mathbb{R}^N), \varphi(x) \geq 1 \text{ for } x \in \Omega \right\},$$

for $N \geq 3$ and $1 < p < N$. If $H$ is a norm in the class (1.5), the integral operator is strictly convex and hence (1.1) admits a unique solution $u_\Omega$, which satisfies

$$\begin{cases} \Delta_p^H u = 0 & \text{in } \mathbb{R}^N \setminus \overline{\Omega}, \\ u = 1 & \text{on } \partial\Omega\,, \\ u \to 0 & \text{as } H(x) \to +\infty\,. \end{cases}$$

The function $u_\Omega$ is called the *Finsler p-capacitary potential* of $\Omega$. As already noticed when $\Omega$ is a convex set the potential $u_\Omega$ is at least quasi-concave, that is its superlevel sets are convex sets (see Lemma 4.4 [1]).

In the special case $\Omega = B_{H_0}(r)$ the capacitary potential is easily computed and is given by (1.3), but this is not possible for general convex domain. However, when $\Omega$ is a convex set, asymptotic estimates for $u_\Omega$ are known. In particular, it has recently been proved in [1] the following:

$$\lim_{|x|\to\infty} u_\Omega(x) H_0(x)^{\frac{N-p}{p-1}} = \mathsf{C}\ \mathrm{Cap_{H,p}}^{\frac{1}{p-1}}(\Omega), \tag{2.2}$$

where $\mathsf{C} = (N-2)P_H^{\frac{-1}{p-1}}(\partial B_{H_0})$. Moreover, one can prove that there exists a positive constant $\gamma$ such that

$$\gamma^{-1} H(x)^{\frac{1}{\mathsf{q}}-1} \le H(Du(x)) \le \gamma H(x)^{\frac{1}{\mathsf{q}}-1}\,, \tag{2.3}$$

(see [5], [4]).

The $p$-capacity operator satisfies a Brunn-Minkowski inequality. In the Euclidean setting, this was proved in [8], such result has been recently extended to quite general operators in divergence form in [1]. Here, we recall the following from [1].

**Proposition 2.1** ([1]). *Let $K, D$ be compact convex sets in $\mathbb{R}^N$ satisfying*

$$\mathrm{Cap_{H,p}}(K), \mathrm{Cap_{H,p}}(D) > 0\,.$$

*For $1<p<N$ and $\lambda \in [0,1]$ it holds*

$$\mathrm{Cap_{H,p}}^{\frac{1}{N-p}}((1-\lambda)K + \lambda D) \ge (1-\lambda)\mathrm{Cap_{H,p}}^{\frac{1}{N-p}}(K) + \lambda \mathrm{Cap_{H,p}}^{\frac{1}{N-p}}(D), \tag{2.4}$$

*and equality holds if and only if $K$ and $D$ are homothetic sets.*

## 3. Proof of Theorem 1.3

Let

$$v(x) = \frac{H_0(x)^{\frac{1}{\mathsf{q}}} - R^{\frac{1}{\mathsf{q}}}}{r^{\frac{1}{\mathsf{q}}} - R^{\frac{1}{\mathsf{q}}}}\,, \quad x \in \mathbb{R}^n \setminus \{O\}\,,$$

with $r$ and $\mathsf{q}$ given by (1.8) and (1.4), respectively. When $\Omega = B_{H_0}(r)$ is Wulff shape of radius $r$, a direct check shows that $v$ is the solution to (1.7).

Now we prove the reverse assertion. Let

$$\tilde u(x) = \begin{cases} u(x) & \text{if } x \in \overline{B_{H_0}(R)} \setminus \Omega\,, \\ v(x) & \text{if } x \in \mathbb{R}^n \setminus \overline{B_{H_0}(R)}\,. \end{cases}$$

We notice that $\tilde u \in C^1(\mathbb{R}^n \setminus \Omega)$ and it satisfies $\Delta_p^H \tilde u = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$ (which follows from the weak formulation of the equation).

Fix any $t > 1$ and set $E = B_{H_0}(tR) \setminus \overline{\Omega}$. For $\tau \in [0,1]$, we define

$$u_\tau = \tau \tilde u + (1-\tau) v$$

in $\overline{E}$; notice that $u_1 = \tilde u$ and $u_0 = v$.

The function $\tilde u - v$ satisfies an elliptic equation. Indeed, for any $\phi \in C_0^1(E)$ we have

$$\begin{aligned} 0 &= \int_E H(Du_1)^{p-1} \langle \nabla H(Du_1); D\phi\rangle\, dx - \int_E H(Du_0)^{p-1}\langle \nabla H(Du_0); D\phi\rangle\, dx \\ &= \int_E \langle \left( \int_0^1 \frac{d}{d\tau} \left( H(Du_\tau)^{p-1} \nabla H(Du_\tau)\right) d\tau \right); D\phi\rangle\, dx \\ &= \int_E \mathbf{A}(x) \langle D(\tilde u - v); D\phi\rangle\, dx\,, \end{aligned}$$

where $\mathbf{A}(x) = a_{ij}(x)$ is given by

$$\begin{aligned} a_{ij}(x) &= \int_0^1 \left( (p-1) H(Du_\tau)^{p-2} H_{\xi_j}(Du_\tau) H_{\xi_i}(Du_\tau) + H(Du_\tau)^{p-1} H_{\xi_i\xi_j}(Du_\tau)\right) d\tau \\ &= \frac{1}{p}\int_0^1 (\nabla^2 H^p(Du_\tau))_{ij} d\tau\,. \end{aligned}$$

From (2.3) we have that

$$A \le H(Du) \le B \tag{3.1}$$

in $E$ for some constant $A, B > 0$. Notice that, since the super level sets of $u$ are convex sets (see [1]) and those of $v$ are Wulff shapes centered at the origin, we have

$$\langle Du; \frac{x}{|x|}\rangle \ge \varepsilon > 0, \qquad \varepsilon \le \langle Dv; \frac{x}{|x|}\rangle \le M, \tag{3.2}$$

in $E$ for some positive constants $\varepsilon$ and $M$. Hence, we can find $\tau_0 \in (0,1)$ such that

$$(1-\tau_0)|Dv| \le \tau_0 |Du|/2\,,$$

which implies that $|Du_\tau| \geq \tau_0|Du|/2$ for every $\tau \in [\tau_0, 1]$. From (3.2) and (3.1) we obtain

$$|Du_\tau| \geq \min\Big((1-\tau_0)\varepsilon,\, \frac{\tau_0}{2}A\Big) > 0 \quad \text{in } E\,,$$

which finally gives that $a \leq |Du_\tau| \leq b$ in $E$ for some constants $a, b > 0$ . Such estimates imply that the operator

$$Lw = \operatorname{div}(\mathbf{A}(\mathbf{x})Dw)$$

is uniformly elliptic. Moreover, since $H^p \in C^{2,1}(\mathbb{R}^n \setminus \{O\})$, we also have that $a_{ij}$ are locally Lipschitz. Hence, $L$ satisfies the assumptions of Theorem 1.1 in [12] (see also [11]) and we have the analytic continuation for $\tilde{u} - v$ in $E$, whence $\tilde{u} - v \equiv 0$ in $E$ which implies that $u \equiv v$ in $\overline{B_{H_0}(R)} \setminus \Omega$ and we conclude.

## 4. Proof of Theorem 1.2

For any $t \in (0,1)$ we set $U(t) = \{x \in \mathbb{R}^N \;:\; u(x) \geq t\}$ and let $u_{U(t)}$ be the $p$-capacitary potential of $U(t)$. Hence

$$u_{U(t)}(x) = \frac{1}{t}u(x), \tag{4.1}$$

for every $x \in \mathbb{R}^N \setminus U(t)$, where $u$ is the Finsler $p$-capacitary potential of $\Omega$.

Let $\mathsf{t}, \mathsf{s} \in (0,1)$, with $\mathsf{t} < \mathsf{s}$, be the levels of $u$ such that $U(\mathsf{t}), U(\mathsf{s})$ are homothetic, that is: there exist $\xi \in \mathbb{R}^N$ and $\rho > 1$ such that $U(\mathsf{t}) = \rho U(\mathsf{s}) + \xi$. Up to a translation we can assume $\xi = 0$ and hence

$$u_{U(\mathsf{t})}(x) = u_{U(\mathsf{s})}\Big(\frac{x}{\rho}\Big). \tag{4.2}$$

*Step 1:* $\rho^{-\frac{N-p}{p-1}} = \mathsf{t}/\mathsf{s}$.

From (2.2), (4.1) and (4.2), we have

$$\mathsf{C}\ \mathrm{Cap}_{\mathrm{H,p}}(\Omega)^{\frac{1}{p-1}} = \lim_{|x|\to\infty} \mathsf{t}\ u_{U(\mathsf{s})}\Big(\frac{x}{\rho}\Big)H_0^{\frac{N-p}{p-1}}(x)\,.$$

By using again (2.2) and (4.1), and from the homogeneity of $H_0$, we find

$$\mathsf{C}\ \mathrm{Cap}_{\mathrm{H,p}}(\Omega)^{\frac{1}{p-1}} = \frac{\mathsf{t}}{\mathsf{s}}\mathsf{C}\ \mathrm{Cap}_{\mathrm{H,p}}(\Omega)^{\frac{1}{p-1}}\rho^{\frac{N-p}{p-1}}$$

which implies that $\rho^{-\frac{N-p}{p-1}} = \mathsf{t}/\mathsf{s}$.

*Step 2: Let $r_k = \mathsf{t}^k\ \mathsf{s}^{1-k}$ for $k \geq 0$. Then $U(r_0) = U(\mathsf{s})$ and $U(r_k) = \rho^k U(\mathsf{s})$ for $k \in \mathbb{N}$.*

Indeed notice that for every $z < \mathsf{t}$ the set $U(z)$ is homothetic to $U(z\frac{\mathsf{s}}{\mathsf{t}})$ since by (4.1), (4.2) we have

$$U(z) = \{u(x) \geq z\} = \{u_{U(\mathsf{s})}(\frac{x}{\rho}) \geq \frac{z}{\mathsf{t}}\} = \{u(\frac{x}{\rho}) \geq z\frac{\mathsf{s}}{\mathsf{t}}\},$$

that is $U(z) = \rho U(z\frac{\mathsf{s}}{\mathsf{t}})$. Hence, recalling that $U(\mathsf{s}) = U(r_0)$ and that $r_k = \frac{\mathsf{t}}{\mathsf{s}}r_{k-1}$, we obtain

$$U(r_k) = \rho^k U(\mathsf{s})$$

for every $k \geq 0$.

*Step 3: $U(\mathsf{s})$ is Wulff shape.*

Let $x, y \in \partial U(\mathsf{s})$ and define

$$x_k = \rho^k x\,,$$
$$y_k = \rho^k y\,.$$

Notice that

$$\lim_{k\to\infty}|x_k| = \lim_{k\to\infty}|y_k| = +\infty\,. \tag{4.3}$$

From *Step 2* the points $x_k, y_k$ belong to $\partial U(r_k)$, so that $u(x_k) = u(y_k) = r_k$. From (5.1) and (2.2) we obtain

$$\lim_{k\to\infty} u(x_k)H_0^{\frac{N-p}{p-1}}(x_k) = \mathsf{C}\ \mathrm{Cap}_{\mathrm{H,p}}(\Omega)^{\frac{1}{p-1}} = \lim_{k\to\infty} u(y_k)H_0^{\frac{N-p}{p-1}}(y_k),$$

i.e.

$$\lim_{k\to\infty} r_k H_0^{\frac{N-p}{p-1}}(x_k) = \lim_{k\to\infty} r_k H_0^{\frac{N-p}{p-1}}(y_k)\,.$$

By recalling the definition of $x_k$ and $y_k$ and *Step 1*, we have

$$\lim_{k\to\infty} H_0^{\frac{N-p}{p-1}}(x) = \lim_{k\to\infty} H_0^{\frac{N-p}{p-1}}(y),$$

which implies that

$$H_0(x) = H_0(y)$$

for every $x, y \in \partial U(\mathsf{s})$, i.e. $U(\mathsf{s})$ is Wulff shape.

*Conclusion:* from *Step 2.* we obtain that $U(r_k)$ is Wulff shape for any $k \geq 0$, which implies that the super level sets $U(s)$ are concentric Wulff shapes. In particular there exists $\beta > 0$ such that $u = \beta H_0(x)^{1/\mathsf{q}}$ for any $x \in \mathbb{R}^N \setminus U(\mathsf{s})$. From Theorem 1.3 we conclude.

## 5. Proof of Theorem 1.1

Let $\mathsf{q}$ be given by (1.4). Notice that for every $x_0 \in \mathbb{R}^N$ and every $R > 0$, the concavity exponent of $B_{H_0}(R, x_0)$ can be explicitly computed thanks to (1.3) and it holds $\alpha(B_{H_0}(R, x_0), p) = \mathsf{q}$.

We are going to prove that if the capacitary potential $u$ of the set $\Omega$ is $\mathsf{q}$-concave then $\Omega$ is Wulff shape and this entails the desired result. Indeed if $u$ is $q$-concave, then $u$ is $s$-concave too, for every $s < q$.

Assume that the function $u$ is $\mathsf{q}$-concave. Since $\mathsf{q} < 0$, then $u^{\mathsf{q}}$ is a convex function. We denote by $V(t)$ the sublevel sets of the function $u^{\mathsf{q}}$, i.e. $V(t) = \{u^{\mathsf{q}} \leq t\}$; the superlevel sets of $u$ will be denoted by $U(t)$. Hence

$$U(t) = V(t^{\mathsf{q}}) \,.$$

Since $u^{\mathsf{q}}$ is convex, for every $t_0, t_1 \in \mathbb{R}$ and every $\lambda \in [0, 1]$ we have

$$V((1-\lambda)t_0 + \lambda t_1) \supseteq (1-\lambda)V(t_0) + \lambda V(t_1). \tag{5.1}$$

Let $0 < r < s < 1$. By choosing $t_0 = r^{\mathsf{q}}, t_1 = s^{\mathsf{q}}$ and defining

$$t = ((1-\lambda)r^{\mathsf{q}} + \lambda s^{\mathsf{q}})^{\frac{1}{\mathsf{q}}}, \tag{5.2}$$

(5.1) can be written as

$$V(t^{\mathsf{q}}) \supseteq (1-\lambda)V(r^{\mathsf{q}}) + \lambda V(s^{\mathsf{q}}),$$

and hence

$$U(t) \supseteq (1-\lambda)U(r) + \lambda U(s).$$

From the monotonicity of the capacity and from Brunn-Minkowski inequality (2.4) it follows

$$\begin{aligned} \mathrm{Cap}_{\mathrm{H,p}}(U(t)) &\geq \mathrm{Cap}_{\mathrm{H,p}}((1-\lambda)U(r) + \lambda U(s)) \\ &\geq \Big((1-\lambda)\mathrm{Cap}_{\mathrm{H,p}}{}^{\frac{1}{N-p}}(U(r)) + \lambda \mathrm{Cap}_{\mathrm{H,p}}{}^{\frac{1}{N-p}}(U(s))\Big)^{N-p}, \end{aligned} \tag{5.3}$$

and, since for every $r \in (0, 1)$

$$\mathrm{Cap}_{\mathrm{H,p}}(U(r)) = r^{1-p}\mathrm{Cap}_{\mathrm{H,p}}(\Omega),$$

inequality (5.3) gives

$$\mathrm{Cap}_{\mathrm{H,p}}(\Omega)t^{1-p} \geq \mathrm{Cap}_{\mathrm{H,p}}(\Omega)\Big((1-\lambda)r^{\frac{1-p}{N-p}} + \lambda s^{\frac{1-p}{N-p}}\Big)^{N-p}. \tag{5.4}$$

The definition of $t$ in (5.2) implies that the equality case holds in (5.4) and this entails that the equality sign in the Brunn-Minkowski inequality (2.4) is attained. Hence the superlevel set $U(r)$ is homothetic to $U(s)$ and Theorem 1.2 yields the conclusion.

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