# A weak comparison principle for solutions of very degenerate elliptic equations 

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#### Abstract

We prove a comparison principle for weak solutions of elliptic quasilinear equations in divergence form whose ellipticity constants degenerate at every point where $\nabla u \in K$, where $K \subset \mathbb{R}^{N}$ is a Borel set containing the origin.


## 1 Introduction

Let $K \subset \mathbb{R}^{N}, N \geq 2$, be a Borel set containing the origin $O$. We consider a vector function $A: \mathbb{R}^{N} \rightarrow \mathbb{R}^{N}, A \in L_{\text {loc }}^{\infty}\left(\mathbb{R}^{N}\right)$, such that

$$
\begin{cases}A(\xi)=0, & \text { if } \xi \in K  \tag{1.1}\\ {[A(\xi)-A(\eta)] \cdot(\xi-\eta)>0, \quad \forall \eta \in \mathbb{R}^{N} \backslash\{\xi\},} & \text { if } \xi \notin K\end{cases}
$$

where $\cdot$ denotes the scalar product in $\mathbb{R}^{N}$. In this note we prove a comparison principle for Lipschitz weak solutions of

$$
\begin{cases}-\operatorname{div} A(\nabla u)=g, & \text { in } \Omega  \tag{1.2}\\ u=\psi, & \text { on } \partial \Omega\end{cases}
$$

where $\Omega$ is a bounded domain in $\mathbb{R}^{N}, \psi \in W^{1, \infty}(\Omega)$ and $g \in L^{1}(\Omega)$. As usual, $u \in W^{1, \infty}(\Omega)$ is a weak solution of (1.2) if $u-\psi \in W_{0}^{1, \infty}(\Omega)$ and $u$ satisfies

$$
\begin{equation*}
\int_{\Omega} A(\nabla u) \cdot \nabla \phi d x=\int_{\Omega} g \phi d x, \quad \text { for every } \phi \in C_{0}^{1}(\Omega) \tag{1.3}
\end{equation*}
$$

For weak comparison principle we mean the following: if $u_{1}, u_{2}$ are two solutions of (1.3) with $u_{1} \leq u_{2}$ on $\partial \Omega$, then $u_{1} \leq u_{2}$ in $\bar{\Omega}$. Clearly, the weak comparison principle implies the uniqueness of the solution.

It is well known that if $K$ is the singleton $\{O\}$, then (1.1) guarantees the validity of the weak comparison principle (see for instance [11] and [18]). For this reason, from now on $K$ will be a set containing the origin and at least another point of $\mathbb{R}^{N}$.

[^0]Our interest in this kind of equations comes from recent studies in traffic congestion problems (see [2] and [3]), complex-valued solutions of the eikonal equation (see [13-16]) and in variational problems which are relaxations of non-convex ones (see for instance [4] and [10]).

As an example, we can think to $f:[0,+\infty) \rightarrow[0,+\infty)$ given by

$$
\begin{equation*}
f(s)=\frac{1}{p}(s-1)_{+}^{p}, \tag{1.4}
\end{equation*}
$$

where $p>1$ and $(\cdot)_{+}$stands for the positive part, and consider the functional

$$
\begin{equation*}
I(u)=\int_{\Omega}[f(|\nabla u(x)|)-g(x) u(x)] d x, \quad u \in \psi+W_{0}^{1, \infty}(\Omega) \tag{1.5}
\end{equation*}
$$

As it is well-known, (1.3) is the Euler-Lagrange equation associated to (1.5) with $A$ given by

$$
\begin{equation*}
A(\nabla u)=\frac{f^{\prime}(|\nabla u|)}{|\nabla u|} \nabla u \tag{1.6}
\end{equation*}
$$

and it is easy to verify that $A$ satisfies (1.1) with $K=\left\{\xi \in \mathbb{R}^{N}:|\xi| \leq 1\right\}$. It is clear that in this case the monotonicity condition in (1.1) can be read in terms of the convexity of $f$. Indeed, $f$ is not strictly convex in $[0,+\infty)$ since it vanishes in $[0,1]$; however, if $s_{1}>1$ then

$$
f\left((1-t) s_{0}+t s_{1}\right)<(1-t) f\left(s_{0}\right)+t f\left(s_{1}\right), \quad t \in[0,1]
$$

for any $s_{0} \in[0,+\infty)$ and $s_{0} \neq s_{1}$ : the convexity holds in the strict sense whenever a value greater than 1 is considered.

Coming back to our original problem we notice that, since $A$ vanishes in $K$, (1.2) is strongly degenerate and no more than Lipschitz regularity of the solution can be expected. It is clear that if $g=0$, then every function with gradient in $K$ will satisfy the equation. Besides the papers cited before, we mention [1, 5, 9, 17] where regularity issues were tackled and [6] where it is proven that solutions to (1.2) satisfy an obstacle problem for the gradient in the viscosity sense. Here, we will not specify the assumptions on $A$ and $g$ that guarantee the existence of a Lipschitz solution and we refer to the mentioned papers for this interesting issue.

We stress that some regularity may be expected if we look at $A(\nabla u)$. In [3] and [4] the authors prove some Sobolev regularity results for $A(\nabla u)$ under more restrictive assumptions on $A$ and $g$. We also mention that results on the continuity of $A(\nabla u)$ can be found in [8] and [17].

In Section 2, we prove a weak comparison principle for Lipschitz solutions of (1.3) by assuming the following: (i) one of the two solutions satisfies a Sobolev regularity assumption on $A(\nabla u)$; (ii) the Lebesgue measure of the set where $g$ vanishes is zero. As we shall prove, the former guarantees that the set where $\nabla u \in K$ and $g$ does not vanish has measure zero. The latter seems to be optimal for proving our result. Indeed, if we assume that $g=0$, then any Lipschitz function with gradient in $K$ would be a solution and we can not have a comparison between any two of such solutions. For instance, if we consider $A$ as in (1.6) with $f$ given by (1.4), then a simple example of functions that satisfy (1.2) is given by $u_{\sigma}(x)=\sigma \operatorname{dist}(x, \partial \Omega)$, with $\sigma \in[-1,1]$. Since every $u_{\sigma}=0$ on $\partial \Omega$, (1.2) does not have a unique solution and a comparison principle
can not hold. Generally speaking, any region where $g$ vanishes will be source of problems for proving a comparison principle. We mention that, for $A$ as in (1.6) and $g=1$, a comparison principle for minimizers of (1.5) was proven in [7.

## 2 Main result

Before proving our main result, we need the following lemma which generalizes a result obtained in [12] for the p-Laplacian. In what follows, $|D|$ denotes the Lebesgue measure of a set $D \subset \mathbb{R}^{N}$.

Lemma 2.1. Let $u \in W^{1, \infty}(\Omega)$ be a solution of (1.3), with $A$ satisfying (1.1) and let

$$
\begin{equation*}
Z=\{x \in \Omega: \nabla u(x) \in K\} . \tag{2.1}
\end{equation*}
$$

If $A(\nabla u) \in W^{1, p}(\Omega)$ for some $p \geq 1$, then

$$
\begin{equation*}
\left|Z \backslash G_{0}\right|=0 \tag{2.2}
\end{equation*}
$$

where

$$
\begin{equation*}
G_{0}=\{x \in \Omega: g(x)=0\} \tag{2.3}
\end{equation*}
$$

In particular, if $\left|G_{0}\right|=0$ then $|Z|=0$.
Proof. Since $A(\nabla u) \in W^{1, p}(\Omega)$, then the function

$$
\frac{|A(\nabla u)|}{\varepsilon+|A(\nabla u)|} \in W^{1, p}(\Omega)
$$

for any $\varepsilon>0$. Let $\psi \in C_{0}^{1}(\Omega)$, set

$$
\phi(x)=\frac{|A(\nabla u(x))|}{\varepsilon+|A(\nabla u(x))|} \psi(x),
$$

and notice that $\phi \in L^{\infty}(\Omega) \cap W_{0}^{1, p}(\Omega)$. Since $u$ is Lipschitz continuous and $A \in L_{\text {loc }}^{\infty}\left(\mathbb{R}^{N}\right)$, we have that $A(\nabla u) \in L^{\infty}(\Omega)$. Hence, by an approximation argument, $\phi$ can be used as a test function in (1.3), yielding

$$
\begin{array}{r}
\int_{\Omega} \frac{|A(\nabla u)|}{\varepsilon+|A(\nabla u)|} A(\nabla u) \cdot \nabla \psi d x+\varepsilon \int_{\Omega} \psi \frac{A(\nabla u) \cdot \nabla|A(\nabla u)|}{(\varepsilon+|A(\nabla u)|)^{2}} d x= \\
\quad=\int_{\Omega} \frac{|A(\nabla u)|}{\varepsilon+|A(\nabla u)|} \psi g d x \tag{2.4}
\end{array}
$$

It is clear that

$$
\begin{equation*}
\int_{\Omega} \frac{|A(\nabla u)|}{\varepsilon+|A(\nabla u)|} \psi g d x=\int_{\Omega \backslash Z} \frac{|A(\nabla u)|}{\varepsilon+|A(\nabla u)|} \psi g d x \tag{2.5}
\end{equation*}
$$

and that Cauchy-Schwarz inequality yields

$$
\begin{equation*}
\left|\varepsilon \frac{A(\nabla u) \cdot \nabla|A(\nabla u)|}{(\varepsilon+|A(\nabla u)|)^{2}}\right| \leq|\nabla(|A(\nabla u)|)| \tag{2.6}
\end{equation*}
$$

uniformly for $\varepsilon>0$. Since $\nabla(|A(\nabla u)|) \in L^{p}(\Omega)$, from (2.4)-(2.6) and by letting $\varepsilon$ go to zero, we obtain from Lebesgue's dominated convergence Theorem that

$$
\int_{\Omega} A(\nabla u) \cdot \nabla \psi d x=\int_{\Omega \backslash Z} g \psi d x
$$

for any $\psi \in C_{0}^{1}(\Omega)$. From (1.3) we have

$$
\int_{\Omega} g \psi d x=\int_{\Omega \backslash Z} g \psi d x \quad \text { for any } \psi \in C_{0}^{1}(\Omega)
$$

that is

$$
g(x)=0 \text { for almost every } x \in Z
$$

which implies (2.2).
Our main result is the following.
Theorem 2.2. Let $u_{j} \in W^{1, \infty}(\Omega), j=1,2$, be two solutions of (1.3), with $A$ satisfying (1.1) and $g$ such that $\left|G_{0}\right|=0$, with $G_{0}$ given by (2.3). Furthermore, let us assume that $A\left(\nabla u_{j}\right) \in W^{1, p}(\Omega)$ for some $p \geq 1$ and $j \in\{1,2\}$.

If $u_{1} \leq u_{2}$ on $\partial \Omega$ then $u_{1} \leq u_{2}$ in $\bar{\Omega}$.
Proof. We proceed by contradiction. Let us assume that $U=\left\{x \in \Omega: u_{1}>u_{2}\right\}$ is nonempty. Since $u_{1}$ and $u_{2}$ are continuous, then $U$ is open and we can assume that it is connected (otherwise we repeat the argument for each connected component). Without loss of generality, we can assume that $A\left(\nabla u_{1}\right) \in W^{1, p}(\Omega)$ and we define $E_{1}=\left\{x \in \Omega: \nabla u_{1} \notin K\right\}$.

Let $\phi=\left(u_{1}-u_{2}\right)_{+}$. Since $u_{1} \leq u_{2}$ on $\partial \Omega$, then $\phi \in W_{0}^{1, \infty}(\Omega)$ and (1.3) yields:

$$
\int_{U} A\left(\nabla u_{j}\right) \cdot \nabla\left(u_{1}-u_{2}\right) d x=\int_{U} g\left(u_{1}-u_{2}\right) d x, \quad j=1,2 .
$$

By subtracting the two identities, we have

$$
\begin{equation*}
\int_{U}\left[A\left(\nabla u_{1}\right)-A\left(\nabla u_{2}\right)\right] \cdot\left(\nabla u_{1}-\nabla u_{2}\right) d x=0 \tag{2.7}
\end{equation*}
$$

We notice that Lemma 2.1] yields $\left|\left\{\nabla u_{1} \in K\right\}\right|=0$ and thus

$$
\begin{aligned}
\int_{U}\left[A\left(\nabla u_{1}\right)-A\left(\nabla u_{2}\right)\right] \cdot(\nabla & \left.u_{1}-\nabla u_{2}\right) d x= \\
& =\int_{U \cap E_{1}}\left[A\left(\nabla u_{1}\right)-A\left(\nabla u_{2}\right)\right] \cdot\left(\nabla u_{1}-\nabla u_{2}\right) d x
\end{aligned}
$$

(2.7) and the monotonicity condition in (1.1) imply that

$$
\begin{equation*}
\nabla u_{1}=\nabla u_{2} \text { a.e. in } U \cap E_{1} . \tag{2.8}
\end{equation*}
$$

Since $\left|\left\{\nabla u_{1} \in K\right\}\right|=0$, we obtain that $\nabla u_{1}=\nabla u_{2}$ a.e. in $U$. Being $u_{1}=u_{2}$ on $\partial U$, we have that $u_{1}=u_{2}$ in $U$, which gives a contradiction.

It is clear that Theorem [2.2 implies the uniqueness of a solution for (1.2). Moreover, from Theorem 2.2, we also obtain the following comparison principle.

Corollary 2.3. Let $u_{j}, j=1,2, A$ and $g$ be as in Theorem 2.2. If $u_{1}<u_{2}$ on $\partial \Omega$ then $u_{1}<u_{2}$ in $\bar{\Omega}$.

Proof. Since $\partial \Omega$ is compact and $u_{1}$ and $u_{2}$ are continuous in $\bar{\Omega}$, there exists a constant $c>0$ such that $u_{1}+c \leqq u_{2}$ on $\partial \Omega$. Being $u_{1}+c$ a solution of (1.3), Theorem 2.2 yields $u_{1}+c \leq u_{2}$ in $\bar{\Omega}$ and, since $c$ is positive, we conclude.

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