A weak comparison principle for solutions of very degenerate elliptic equations

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Abstract

We prove a comparison principle for weak solutions of elliptic quasi-linear equations in divergence form whose ellipticity constants degenerate at every point where $\nabla u \in K$, where $K \subset \mathbb{R}^N$ is a Borel set containing the origin.

1 Introduction

Let $K \subset \mathbb{R}^N$, $N \geq 2$, be a Borel set containing the origin $O$. We consider a vector function $A : \mathbb{R}^N \to \mathbb{R}^N$, $A \in L^\infty_{\text{loc}}(\mathbb{R}^N)$, such that

\[
\begin{cases}
A(\xi) = 0, & \text{if } \xi \in K,

[ A(\xi) - A(\eta) ] \cdot (\xi - \eta) > 0, & \forall \eta \in \mathbb{R}^N \setminus \{ \xi \}, \text{ if } \xi \not\in K,
\end{cases}
\]  

(1.1)

where $\cdot$ denotes the scalar product in $\mathbb{R}^N$. In this note we prove a comparison principle for Lipschitz weak solutions of

\[
\begin{aligned}
-\text{div } A(\nabla u) &= g, & \text{in } \Omega, \\
u &= \psi, & \text{on } \partial \Omega,
\end{aligned}
\]  

(1.2)

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $\psi \in W^{1,\infty}(\Omega)$ and $g \in L^1(\Omega)$. As usual, $u \in W^{1,\infty}(\Omega)$ is a weak solution of (1.2) if $u - \psi \in W^{1,\infty}_0(\Omega)$ and $u$ satisfies

\[
\int_\Omega A(\nabla u) \cdot \nabla \phi \, dx = \int_\Omega g \phi \, dx, \quad \text{for every } \phi \in C_0^1(\Omega).
\]  

(1.3)

For weak comparison principle we mean the following: if $u_1, u_2$ are two solutions of (1.2) with $u_1 \leq u_2$ on $\partial \Omega$, then $u_1 \leq u_2$ in $\Omega$. Clearly, the weak comparison principle implies the uniqueness of the solution.

It is well known that if $K$ is the singleton $\{ O \}$, then (1.1) guarantees the validity of the weak comparison principle (see for instance [11] and [18]). For this reason, from now on $K$ will be a set containing the origin and at least another point of $\mathbb{R}^N$.

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Our interest in this kind of equations comes from recent studies in traffic congestion problems (see [2] and [3]), complex-valued solutions of the eikonal equation (see [13]–[16]) and in variational problems which are relaxations of non-convex ones (see for instance [4] and [10]).

As an example, we can think to \( f : [0, +\infty) \to [0, +\infty) \) given by

\[
  f(s) = \frac{1}{p} (s - 1)^p,
\]

(1.4)

where \( p > 1 \) and \((\cdot)_+\) stands for the positive part, and consider the functional

\[
  I(u) = \int_{\Omega} \left[ f(|\nabla u(x)|) - g(x)u(x) \right] dx, \quad u \in \psi W^{1,\infty}_0(\Omega).
\]

(1.5)

As it is well-known, \((\ref{eq:1.3})\) is the Euler-Lagrange equation associated to \((\ref{eq:1.5})\) with \( A \) given by

\[
  A(\nabla u) = \frac{f'(|\nabla u|)}{|\nabla u|} \nabla u,
\]

(1.6)

and it is easy to verify that \( A \) satisfies \((\ref{eq:1.1})\) with \( K = \{ \xi \in \mathbb{R}^N : |\xi| \leq 1 \} \). It is clear that in this case the monotonicity condition in \((\ref{eq:1.1})\) can be read in terms of the convexity of \( f \). Indeed, \( f \) is not strictly convex in \([0, +\infty)\) since it vanishes in \([0, 1]\); however, if \( s_1 > 1 \) then

\[
  f((1 - t)s_0 + ts_1) < (1 - t)f(s_0) + tf(s_1), \quad t \in [0, 1],
\]

for any \( s_0 \in [0, +\infty) \) and \( s_0 \neq s_1 \); the convexity holds in the strict sense whenever a value greater than 1 is considered.

Coming back to our original problem we notice that, since \( A \) vanishes in \( K \), \((\ref{eq:1.2})\) is strongly degenerate and no more than Lipschitz regularity of the solution can be expected. It is clear that if \( g = 0 \), then every function with gradient in \( K \) will satisfy the equation. Besides the papers cited before, we mention \([1, 5, 9, 17]\) where regularity issues were tackled and \([6]\) where it is proven that solutions to \((\ref{eq:1.2})\) satisfy an obstacle problem for the gradient in the viscosity sense. Here, we will not specify the assumptions on \( A \) and \( g \) that guarantee the existence of a Lipschitz solution and we refer to the mentioned papers for this interesting issue.

We stress that some regularity may be expected if we look at \( A(\nabla u) \). In \([3\text{ and }4]\) the authors prove some Sobolev regularity results for \( A(\nabla u) \) under more restrictive assumptions on \( A \) and \( g \). We also mention that results on the continuity of \( A(\nabla u) \) can be found in \([8\text{ and }17]\).

In Section 2, we prove a weak comparison principle for Lipschitz solutions of \((\ref{eq:1.3})\) by assuming the following: (i) one of the two solutions satisfies a Sobolev regularity assumption on \( A(\nabla u) \); (ii) the Lebesgue measure of the set where \( g \) vanishes is zero. As we shall prove, the former guarantees that the set where \( \nabla u \in K \) and \( g \) does not vanish has measure zero. The latter seems to be optimal for proving our result. Indeed, if we assume that \( g = 0 \), then any Lipschitz function with gradient in \( K \) would be a solution and we cannot have a comparison between any two of such solutions. For instance, if we consider \( A \) as in \((\ref{eq:1.6})\) with \( f \) given by \((\ref{eq:1.4})\), then a simple example of functions that satisfy \((\ref{eq:1.2})\) is given by \( u_\sigma(x) = \sigma \operatorname{dist}(x, \partial \Omega) \), with \( \sigma \in [-1, 1] \). Since every \( u_\sigma = 0 \) on \( \partial \Omega \), \((\ref{eq:1.2})\) does not have a unique solution and a comparison principle
can not hold. Generally speaking, any region where \( g \) vanishes will be source of problems for proving a comparison principle. We mention that, for \( A \) as in (1.6) and \( g = 1 \), a comparison principle for minimizers of (1.5) was proven in [7].

2 Main result

Before proving our main result, we need the following lemma which generalizes a result obtained in [12] for the p-Laplacian. In what follows, \( |D| \) denotes the Lebesgue measure of a set \( D \subset \mathbb{R}^N \).

Lemma 2.1. Let \( u \in W^{1,\infty}(\Omega) \) be a solution of (1.3), with \( A \) satisfying (1.1) and let

\[
Z = \{x \in \Omega : \nabla u(x) \in K\}.
\]  

(2.1)

If \( A(\nabla u) \in W^{1,p}(\Omega) \) for some \( p \geq 1 \), then

\[
|Z \setminus G_0| = 0,
\]  

(2.2)

where

\[
G_0 = \{x \in \Omega : g(x) = 0\}.
\]  

(2.3)

In particular, if \( |G_0| = 0 \) then \( |Z| = 0 \).

Proof. Since \( A(\nabla u) \in W^{1,p}(\Omega) \), then the function

\[
\frac{|A(\nabla u)|}{\varepsilon + |A(\nabla u)|} \in W^{1,p}(\Omega),
\]

for any \( \varepsilon > 0 \). Let \( \psi \in C^0_0(\Omega) \), set

\[
\phi(x) = \frac{|A(\nabla u(x))|}{\varepsilon + |A(\nabla u(x))|} \psi(x),
\]

and notice that \( \phi \in L^{\infty}(\Omega) \cap W^{1,p}_0(\Omega) \). Since \( u \) is Lipschitz continuous and \( A \in L^{\infty}_c(\mathbb{R}^N) \), we have that \( A(\nabla u) \in L^{\infty}(\Omega) \). Hence, by an approximation argument, \( \phi \) can be used as a test function in (1.3), yielding

\[
\int_{\Omega} \frac{|A(\nabla u)|}{\varepsilon + |A(\nabla u)|} A(\nabla u) \cdot \nabla \psi dx + \varepsilon \int_{\Omega} \psi \frac{A(\nabla u) \cdot \nabla |A(\nabla u)|}{(\varepsilon + |A(\nabla u)|)^2} dx = \int_{\Omega} \frac{|A(\nabla u)|}{\varepsilon + |A(\nabla u)|} \psi g dx.
\]  

(2.4)

It is clear that

\[
\int_{\Omega} \frac{|A(\nabla u)|}{\varepsilon + |A(\nabla u)|} \psi g dx = \int_{\Omega \setminus Z} \frac{|A(\nabla u)|}{\varepsilon + |A(\nabla u)|} \psi g dx,
\]  

(2.5)

and that Cauchy-Schwarz inequality yields

\[
\left| \frac{\varepsilon A(\nabla u) \cdot \nabla |A(\nabla u)|}{(\varepsilon + |A(\nabla u)|)^2} \right| \leq |\nabla (|A(\nabla u)|)|
\]  

(2.6)
uniformly for $\varepsilon > 0$. Since $\nabla(|A(\nabla u)|) \in L^p(\Omega)$, from (2.3)–(2.6) and by letting $\varepsilon$ go to zero, we obtain from Lebesgue’s dominated convergence Theorem that

$$\int_{\Omega} A(\nabla u) \cdot \nabla \psi \, dx = \int_{\Omega \setminus Z} g \psi \, dx,$$

for any $\psi \in C^1_0(\Omega)$. From (1.3) we have

$$\int_{\Omega} g \psi \, dx = \int_{\Omega \setminus Z} g \psi \, dx \quad \text{for any } \psi \in C^1_0(\Omega),$$

that is

$$g(x) = 0 \quad \text{for almost every } x \in Z,$$

which implies (2.2).

Our main result is the following.

**Theorem 2.2.** Let $u_j \in W^{1,\infty}(\Omega)$, $j = 1, 2$, be two solutions of (1.3) with $A$ satisfying (1.1) and $g$ such that $|G_0| = 0$, with $G_0$ given by (2.3). Furthermore, let us assume that $A(\nabla u_j) \in W^{1,p}(\Omega)$ for some $p \geq 1$ and $j \in \{1, 2\}$.

If $u_1 \leq u_2$ on $\partial \Omega$ then $u_1 \leq u_2$ in $\Omega$.

**Proof.** We proceed by contradiction. Let us assume that $U = \{x \in \Omega : u_1 > u_2\}$ is nonempty. Since $u_1$ and $u_2$ are continuous, then $U$ is open and we can assume that it is connected (otherwise we repeat the argument for each connected component). Without loss of generality, we can assume that $A(\nabla u_1) \in W^{1,p}(\Omega)$ and we define $E_1 = \{x \in \Omega : \nabla u_1 \notin K\}$.

Let $\phi = (u_1 - u_2)_+$. Since $u_1 \leq u_2$ on $\partial \Omega$, then $\phi \in W^{1,\infty}_0(\Omega)$ and (1.3) yields:

$$\int_U A(\nabla u_j) \cdot \nabla (u_1 - u_2) \, dx = \int_U g(u_1 - u_2) \, dx, \quad j = 1, 2.$$

By subtracting the two identities, we have

$$\int_U [A(\nabla u_1) - A(\nabla u_2)] \cdot (\nabla u_1 - \nabla u_2) \, dx = 0. \quad (2.7)$$

We notice that Lemma 2.1 yields $|\{\nabla u_1 \in K\}| = 0$ and thus

$$\int_U [A(\nabla u_1) - A(\nabla u_2)] \cdot (\nabla u_1 - \nabla u_2) \, dx =$$

$$= \int_{U \cap E_1} [A(\nabla u_1) - A(\nabla u_2)] \cdot (\nabla u_1 - \nabla u_2) \, dx; \quad (2.7)$$

and the monotonicity condition in (1.1) imply that

$$\nabla u_1 = \nabla u_2 \quad \text{a.e. in } U \cap E_1. \quad (2.8)$$

Since $|\{\nabla u_1 \in K\}| = 0$, we obtain that $\nabla u_1 = \nabla u_2$ a.e. in $U$. Being $u_1 = u_2$ on $\partial U$, we have that $u_1 = u_2$ in $U$, which gives a contradiction.

It is clear that Theorem 2.2 implies the uniqueness of a solution for (1.2). Moreover, from Theorem 2.2 we also obtain the following comparison principle.
Corollary 2.3. Let \( u_j, \ j = 1, 2 \), \( A \) and \( g \) be as in Theorem 2.2. If \( u_1 < u_2 \) on \( \partial \Omega \) then \( u_1 < u_2 \) in \( \Omega \).

Proof. Since \( \partial \Omega \) is compact and \( u_1 \) and \( u_2 \) are continuous in \( \Omega \), there exists a constant \( c > 0 \) such that \( u_1 + c \leq u_2 \) on \( \partial \Omega \). Being \( u_1 + c \) a solution of \( \text{[16]} \), Theorem 2.2 yields \( u_1 + c \leq u_2 \) in \( \Omega \) and, since \( c \) is positive, we conclude. \( \square \)

References


