# A weak comparison principle for solutions of very degenerate elliptic equations

Giulio Ciraolo\*

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#### Abstract

We prove a comparison principle for weak solutions of elliptic quasilinear equations in divergence form whose ellipticity constants degenerate at every point where  $\nabla u \in K$ , where  $K \subset \mathbb{R}^N$  is a Borel set containing the origin.

### 1 Introduction

Let  $K \subset \mathbb{R}^N$ ,  $N \geq 2$ , be a Borel set containing the origin O. We consider a vector function  $A: \mathbb{R}^N \to \mathbb{R}^N$ ,  $A \in L^{\infty}_{loc}(\mathbb{R}^N)$ , such that

$$\begin{cases} A(\xi) = 0, & \text{if } \xi \in K, \\ [A(\xi) - A(\eta)] \cdot (\xi - \eta) > 0, & \forall \, \eta \in \mathbb{R}^N \setminus \{\xi\}, & \text{if } \xi \notin K, \end{cases}$$
 (1.1)

where  $\cdot$  denotes the scalar product in  $\mathbb{R}^N$ . In this note we prove a comparison principle for Lipschitz weak solutions of

$$\begin{cases}
-\operatorname{div} A(\nabla u) = g, & \text{in } \Omega, \\
u = \psi, & \text{on } \partial\Omega,
\end{cases}$$
(1.2)

where  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ ,  $\psi \in W^{1,\infty}(\Omega)$  and  $g \in L^1(\Omega)$ . As usual,  $u \in W^{1,\infty}(\Omega)$  is a weak solution of (1.2) if  $u - \psi \in W_0^{1,\infty}(\Omega)$  and u satisfies

$$\int_{\Omega} A(\nabla u) \cdot \nabla \phi dx = \int_{\Omega} g \phi dx, \quad \text{for every } \phi \in C_0^1(\Omega). \tag{1.3}$$

For weak comparison principle we mean the following: if  $u_1, u_2$  are two solutions of (1.3) with  $u_1 \leq u_2$  on  $\partial\Omega$ , then  $u_1 \leq u_2$  in  $\overline{\Omega}$ . Clearly, the weak comparison principle implies the uniqueness of the solution.

It is well known that if K is the singleton  $\{O\}$ , then (1.1) guarantees the validity of the weak comparison principle (see for instance [11] and [18]). For this reason, from now on K will be a set containing the origin and at least another point of  $\mathbb{R}^N$ .

<sup>\*</sup>Department of Mathematics and Informatics, Università di Palermo, Via Archirafi 34, 90123, Italy. E-mail: g.ciraolo@math.unipa.it

Our interest in this kind of equations comes from recent studies in traffic congestion problems (see [2] and [3]), complex-valued solutions of the *eikonal* equation (see [13]–[16]) and in variational problems which are relaxations of non-convex ones (see for instance [4] and [10]).

As an example, we can think to  $f:[0,+\infty)\to[0,+\infty)$  given by

$$f(s) = \frac{1}{p}(s-1)_{+}^{p},\tag{1.4}$$

where p > 1 and  $(\cdot)_+$  stands for the positive part, and consider the functional

$$I(u) = \int_{\Omega} [f(|\nabla u(x)|) - g(x)u(x)]dx, \quad u \in \psi + W_0^{1,\infty}(\Omega).$$
 (1.5)

As it is well-known, (1.3) is the Euler-Lagrange equation associated to (1.5) with A given by

$$A(\nabla u) = \frac{f'(|\nabla u|)}{|\nabla u|} \nabla u, \tag{1.6}$$

and it is easy to verify that A satisfies (1.1) with  $K = \{\xi \in \mathbb{R}^N : |\xi| \le 1\}$ . It is clear that in this case the monotonicity condition in (1.1) can be read in terms of the convexity of f. Indeed, f is not strictly convex in  $[0, +\infty)$  since it vanishes in [0, 1]; however, if  $s_1 > 1$  then

$$f((1-t)s_0 + ts_1) < (1-t)f(s_0) + tf(s_1), \quad t \in [0,1],$$

for any  $s_0 \in [0, +\infty)$  and  $s_0 \neq s_1$ : the convexity holds in the strict sense whenever a value greater than 1 is considered.

Coming back to our original problem we notice that, since A vanishes in K, (1.2) is strongly degenerate and no more than Lipschitz regularity of the solution can be expected. It is clear that if g=0, then every function with gradient in K will satisfy the equation. Besides the papers cited before, we mention [1, 5, 9, 17] where regularity issues were tackled and [6] where it is proven that solutions to (1.2) satisfy an obstacle problem for the gradient in the viscosity sense. Here, we will not specify the assumptions on A and g that guarantee the existence of a Lipschitz solution and we refer to the mentioned papers for this interesting issue.

We stress that some regularity may be expected if we look at  $A(\nabla u)$ . In [3] and [4] the authors prove some Sobolev regularity results for  $A(\nabla u)$  under more restrictive assumptions on A and g. We also mention that results on the continuity of  $A(\nabla u)$  can be found in [8] and [17].

In Section 2, we prove a weak comparison principle for Lipschitz solutions of (1.3) by assuming the following: (i) one of the two solutions satisfies a Sobolev regularity assumption on  $A(\nabla u)$ ; (ii) the Lebesgue measure of the set where g vanishes is zero. As we shall prove, the former guarantees that the set where  $\nabla u \in K$  and g does not vanish has measure zero. The latter seems to be optimal for proving our result. Indeed, if we assume that g=0, then any Lipschitz function with gradient in K would be a solution and we can not have a comparison between any two of such solutions. For instance, if we consider A as in (1.6) with f given by (1.4), then a simple example of functions that satisfy (1.2) is given by  $u_{\sigma}(x) = \sigma \operatorname{dist}(x, \partial \Omega)$ , with  $\sigma \in [-1, 1]$ . Since every  $u_{\sigma} = 0$  on  $\partial \Omega$ , (1.2) does not have a unique solution and a comparison principle

can not hold. Generally speaking, any region where g vanishes will be source of problems for proving a comparison principle. We mention that, for A as in (1.6) and g = 1, a comparison principle for minimizers of (1.5) was proven in [7].

# 2 Main result

Before proving our main result, we need the following lemma which generalizes a result obtained in [12] for the p-Laplacian. In what follows, |D| denotes the Lebesgue measure of a set  $D \subset \mathbb{R}^N$ .

**Lemma 2.1.** Let  $u \in W^{1,\infty}(\Omega)$  be a solution of (1.3), with A satisfying (1.1) and let

$$Z = \{ x \in \Omega : \ \nabla u(x) \in K \}. \tag{2.1}$$

If  $A(\nabla u) \in W^{1,p}(\Omega)$  for some  $p \geq 1$ , then

$$|Z \setminus G_0| = 0, (2.2)$$

where

$$G_0 = \{ x \in \Omega : g(x) = 0 \}.$$
 (2.3)

In particular, if  $|G_0| = 0$  then |Z| = 0.

*Proof.* Since  $A(\nabla u) \in W^{1,p}(\Omega)$ , then the function

$$\frac{|A(\nabla u)|}{\varepsilon + |A(\nabla u)|} \in W^{1,p}(\Omega),$$

for any  $\varepsilon > 0$ . Let  $\psi \in C_0^1(\Omega)$ , set

$$\phi(x) = \frac{|A(\nabla u(x))|}{\varepsilon + |A(\nabla u(x))|} \psi(x),$$

and notice that  $\phi \in L^{\infty}(\Omega) \cap W_0^{1,p}(\Omega)$ . Since u is Lipschitz continuous and  $A \in L^{\infty}_{loc}(\mathbb{R}^N)$ , we have that  $A(\nabla u) \in L^{\infty}(\Omega)$ . Hence, by an approximation argument,  $\phi$  can be used as a test function in (1.3), yielding

$$\int_{\Omega} \frac{|A(\nabla u)|}{\varepsilon + |A(\nabla u)|} A(\nabla u) \cdot \nabla \psi dx + \varepsilon \int_{\Omega} \psi \frac{A(\nabla u) \cdot \nabla |A(\nabla u)|}{(\varepsilon + |A(\nabla u)|)^{2}} dx = 
= \int_{\Omega} \frac{|A(\nabla u)|}{\varepsilon + |A(\nabla u)|} \psi g dx. \quad (2.4)$$

It is clear that

$$\int_{\Omega} \frac{|A(\nabla u)|}{\varepsilon + |A(\nabla u)|} \psi g dx = \int_{\Omega \setminus Z} \frac{|A(\nabla u)|}{\varepsilon + |A(\nabla u)|} \psi g dx, \tag{2.5}$$

and that Cauchy-Schwarz inequality yields

$$\left| \varepsilon \frac{A(\nabla u) \cdot \nabla |A(\nabla u)|}{(\varepsilon + |A(\nabla u)|)^2} \right| \le |\nabla (|A(\nabla u)|)| \tag{2.6}$$

uniformly for  $\varepsilon > 0$ . Since  $\nabla(|A(\nabla u)|) \in L^p(\Omega)$ , from (2.4)–(2.6) and by letting  $\varepsilon$  go to zero, we obtain from Lebesgue's dominated convergence Theorem that

$$\int_{\Omega} A(\nabla u) \cdot \nabla \psi dx = \int_{\Omega \setminus Z} g \psi dx,$$

for any  $\psi \in C_0^1(\Omega)$ . From (1.3) we have

$$\int_{\Omega} g\psi dx = \int_{\Omega \setminus Z} g\psi dx \quad \text{ for any } \psi \in C_0^1(\Omega),$$

that is

$$g(x) = 0$$
 for almost every  $x \in \mathbb{Z}$ ,

which implies (2.2).

Our main result is the following.

**Theorem 2.2.** Let  $u_j \in W^{1,\infty}(\Omega)$ , j = 1, 2, be two solutions of (1.3), with A satisfying (1.1) and g such that  $|G_0| = 0$ , with  $G_0$  given by (2.3). Furthermore, let us assume that  $A(\nabla u_j) \in W^{1,p}(\Omega)$  for some  $p \geq 1$  and  $j \in \{1, 2\}$ .

If  $u_1 \leq u_2$  on  $\partial \Omega$  then  $u_1 \leq u_2$  in  $\overline{\Omega}$ .

*Proof.* We proceed by contradiction. Let us assume that  $U = \{x \in \Omega : u_1 > u_2\}$  is nonempty. Since  $u_1$  and  $u_2$  are continuous, then U is open and we can assume that it is connected (otherwise we repeat the argument for each connected component). Without loss of generality, we can assume that  $A(\nabla u_1) \in W^{1,p}(\Omega)$  and we define  $E_1 = \{x \in \Omega : \nabla u_1 \notin K\}$ .

Let  $\phi = (u_1 - u_2)_+$ . Since  $u_1 \leq u_2$  on  $\partial \Omega$ , then  $\phi \in W_0^{1,\infty}(\Omega)$  and (1.3) yields:

$$\int_{U} A(\nabla u_{j}) \cdot \nabla (u_{1} - u_{2}) dx = \int_{U} g(u_{1} - u_{2}) dx, \quad j = 1, 2.$$

By subtracting the two identities, we have

$$\int_{U} \left[ A(\nabla u_1) - A(\nabla u_2) \right] \cdot (\nabla u_1 - \nabla u_2) dx = 0. \tag{2.7}$$

We notice that Lemma 2.1 yields  $|\{\nabla u_1 \in K\}| = 0$  and thus

$$\int_{U} [A(\nabla u_1) - A(\nabla u_2)] \cdot (\nabla u_1 - \nabla u_2) dx =$$

$$= \int_{U \cap E_1} [A(\nabla u_1) - A(\nabla u_2)] \cdot (\nabla u_1 - \nabla u_2) dx;$$

(2.7) and the monotonicity condition in (1.1) imply that

$$\nabla u_1 = \nabla u_2 \text{ a.e. in } U \cap E_1. \tag{2.8}$$

Since  $|\{\nabla u_1 \in K\}| = 0$ , we obtain that  $\nabla u_1 = \nabla u_2$  a.e. in U. Being  $u_1 = u_2$  on  $\partial U$ , we have that  $u_1 = u_2$  in U, which gives a contradiction.

It is clear that Theorem 2.2 implies the uniqueness of a solution for (1.2). Moreover, from Theorem 2.2, we also obtain the following comparison principle.

**Corollary 2.3.** Let  $u_j$ , j = 1, 2, A and g be as in Theorem 2.2. If  $u_1 < u_2$  on  $\partial \Omega$  then  $u_1 < u_2$  in  $\overline{\Omega}$ .

*Proof.* Since  $\partial\Omega$  is compact and  $u_1$  and  $u_2$  are continuous in  $\overline{\Omega}$ , there exists a constant c>0 such that  $u_1+c\leq u_2$  on  $\partial\Omega$ . Being  $u_1+c$  a solution of (1.3), Theorem 2.2 yields  $u_1+c\leq u_2$  in  $\overline{\Omega}$  and, since c is positive, we conclude.  $\square$ 

## References

- [1] L. Brasco: Global  $L^{\infty}$  gradient estimates for solutions to a certain degenerate elliptic equation. Nonlinear Anal., 74 (2011), 516-531.
- [2] Brasco L., Carlier G.: On certain anisotropic elliptic equations arising in congested optimal transport: local gradient bounds. Preprint (2012). Available at http://cvgmt.sns.it/paper/1890/.
- [3] Brasco L., Carlier G., Santambrogio F.: Congested traffic dynamics, weak flows and very degenerate elliptic equations. J. Math. Pures Appl., 93 (2010), 652-671.
- [4] Carstensen C., Müller S.: Local stress regularity in scalar nonconvex variational problems. SIAM J. Math. Anal., 34 (2002), 495-509.
- [5] Celada P., Cupini G., Guidorzi M.: Existence and regularity of minimizers of nonconvex integrals with p - q growth. ESAIM Control Optim. Calc. Var., 13 (2007), 343–358.
- [6] Ciraolo G.: A viscosity equation for minimizers of a class of very degenerate elliptic functionals. To appear in Geometric Properties for Parabolic and Elliptic PDE's, Springer INdAM Series (2013).
- [7] Ciraolo G., Magnanini R., Sakaguchi S.: Symmetry of minimizers with a level surface parallel to the boundary. Preprint (2012) arXiv:1203.5295.
- [8] Colombo M., Figalli A.: Regularity results for very degenerate elliptic equations. Preprint (2012). Available at http://cvgmt.sns.it/paper/1996/.
- [9] Esposito L., Mingione G., Trombetti C.: On the Lipschitz regularity for certain elliptic problems. Forum Math. 18 (2006), 263–292.
- [10] Fonseca I., Fusco N., Marcellini P.: An existence result for a nonconvex variational problem via regularity. ESAIM Control Optim. Calc. Var. 7 (2002), 69–95.
- [11] Gilbarg D., Trudinger N.S.: Elliptic partial differential equations of second order, Springer-Verlag, Berlin-New York, 1977.
- [12] Lou H.: On singular sets of local solutions to p-Laplace equations. Chin. Ann. Math. 29B (2008), no. 5, 521-530.
- [13] Magnanini R., Talenti G.: On complex-valued solutions to a 2D eikonal equation. Part one: qualitative properties. Nonlinear Partial Differential Equations, Contemporary Mathematics 283 (1999), American Mathematical Society, 203–229.

- [14] \_\_\_\_\_: On complex-valued solutions to a 2D eikonal equation. Part two: existence theorems. SIAM J. Math. Anal. 34 (2003), 805–835.
- [15] \_\_\_\_\_: On complex-valued solutions to a 2D Eikonal Equation. Part Three: analysis of a Backlund transformation. Applic. Anal. 85 (2006), no. 1-3, 249–276.
- [16] \_\_\_\_\_\_: On complex-valued 2D eikonals. Part four: continuation past a caustic. Milan Journal of Mathematics 77 (2009), no. 1, 1–66.
- [17] Santambrogio F., Vespri V.: Continuity in two dimensions for a very degenerate elliptic equation. Nonlinear Anal., 73 (2010), 3832-3841.
- [18] Tolksdorf P.: Regularity for a more general class of quasilinear elliptic equations. J. Differential Equations, 51 (1984), 126-150.