Reducibility of non-resonant transport equation on $\mathbb{T}^d$ with unbounded perturbations

Dario Bambusi, Beatrice Langella, Riccardo Montalto *

Abstract

We prove reducibility of a transport equation on the d-dimensional torus $\mathbb{T}^d$ with a time quasi-periodic unbounded perturbation. As far as we know this is the first example of a reducibility result for an equation in more than one dimensions with unbounded perturbations. Furthermore the unperturbed problem has eigenvalues whose differences are dense on the real axis.

1 Introduction

In this paper we obtain reducibility for a transport equation on the d-dimensional torus $\mathbb{T}^d$, $\mathbb{T} := \mathbb{R}/(2\pi\mathbb{Z})$, $d \geq 1$ of the form

$$\partial_t u = \Big(\nu + \varepsilon V(\omega t, x)\Big) \cdot \nabla u + \varepsilon \mathcal{W}(\omega t)[u], \tag{1.1}$$

where the frequencies $\omega \in \mathbb{R}^{\mathtt{n}}$, and $\nu \in \mathbb{R}^d$ play the role of parameters, $\varepsilon > 0$ is a small parameter, $V \in \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}} \times \mathbb{T}^d, \mathbb{R}^d)$ is a real function and $\mathcal{W}(\varphi)$, $\varphi \in \mathbb{T}^{\mathtt{n}}$ is a pseudo-differential operator of order $1 - \mathfrak{e}$, for some $\mathfrak{e} > 0$. More precisely our aim is to show that for ε small enough and for *most* values of $\widetilde{\omega} = (\omega, \nu) \in \Omega := [1, 2]^{\mathtt{n}+d}$, there exists a bounded and invertible transformation (acting on the scale of Sobolev spaces) which transforms the PDE (1.1) into another one whose vector field is a time independent diagonal operator.

This is the first example of a reducibility result for unbounded perturbations of a Hamiltonian PDE in more than one space dimension. Furthermore, the unperturbed problem has eigenvalues whose differences are dense on the real axis, a case which is usually considered as particular difficult to deal with.

Following [BBM14] (see also [BM16, Bam18, Bam17, Mon17a, BBHM17]), the proof consists of two steps: first we use pseudo-differential calculus in order to transform the original system to a system with a smoothing perturbation (smoothing theorem) and then we apply a KAM scheme in order to actually obtain reducibility. The smoothing theorem is obtained through a variant of the theory developed in [BGMR17] and the KAM theory is a variant of the one developed in [BBHM17]. The main purpose of the present paper is to show that it is possible to glue together such tools in order to deal with a nontrivial

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higher dimensional problem. The main technical difficulty consists in showing that the frequencies (ω, ν) can be used to tune the small divisors and to fulfill some second Melnikov type nonresonance conditions.

A further novelty is that, in the equation (1.1), it is natural to consider perturbations $\mathcal{W}$ s.t. $\mathrm{i}\mathcal{W}$ is not a symmetric operator, so we consider the case where $\mathrm{i}\mathcal{W}$ is only symmetric hyperbolic (namely that $\mathcal{W} + \mathcal{W}^*$ is an operator of order 0, see Definition 2.3 below) and, in order to get information on the behavior of the solutions, we study also the case where it has some additional structures, namely reality and reversibility (see Definition 2.3 below). In this case we also get the stability, namely all the Sobolev norms of the solutions of the equation (1.1) stay bounded for all times. Note that by Corollary 2.5, in the non-reversible case, one can construct solutions whose Sobolev norms go to infinity.

There is a wide literature on the dynamics of time periodic or quasiperiodic Schrödinger type equations, starting from the pioneering works [Bel85, Com87] (see also [DS96]). Concerning the problem of reducibility, we just mention [Kuk93, BG01, LY10], in which the classical methods developed in KAM theory (in particular [Kuk87, Kuk97]) have been adapted and extended in order to deal with the case where the unperturbed equation has order n and the perturbation is of order $\delta \leq n-1$. All these results are for equations in one space dimension.

The breakthrough for further developments was obtained in [BBM14], developing ideas introduced in [IPT05]. The strategy introduced in [BBM14] is based on the usage of pseudo-differential calculus, which allows to reduce the order of the perturbation, before applying reducibility schemes based on KAM theory. In particular their method allows to reduce the original problem to a problem in which the perturbation is a smoothing operator of arbitrary order. These ideas have been applied in the field of KAM theory for one dimensional PDEs by several authors (see [BBM16a, BBM16b, FP15, BM16, Mon17a, Bam18, Bam17]) and the extension to some particular models in more than one dimension has also been obtained [BGMR18, Mon17a].

The idea of using pseudo-differential calculus in order to conjugate the original system to another one with a smoothing perturbation has shown to be very useful, also in control theory, see [ABHK18, BFH17, BHM18] and in the problem of estimating the growth of the Sobolev norms [BGMR17, Mon18, Mon18a].

Actually, the methods developed in [BGMR17] are the starting point of the present paper.

The second kind of ideas on which we rely were developed in [BBHM17] (and extended in [Mon17a]) where the authors developed a reducibility scheme for smoothing perturbation of a system whose frequencies fulfill very bad nonresonance conditions (see eq. (1.3) below). The idea is that the smoothing character of the nonlinearity can be used to recover a smoothness loss due to the small denominators. In [BBHM17], the method was applied to the case where the frequencies of the linear system grow at infinity like $\omega_j = j^{1/2}$, $j \in \mathbb{N}$. Here we adapt the scheme to the case where the differences between couples of frequencies are dense on the real axis.

We recall that previous reducibility results in higher dimensional systems have been obtained only in cases where the frequencies of the unperturbed system have a very particular structure [EK09, GP16] so that the more or less standard second order Melnikov conditions can be imposed *blockwise*.

The paper is organized as follows. In Section 2 we state precisely our main theorem. In Section 3 we conjugate the vector field of the equation (1.1) to another one which is an arbitrarily smoothing perturbation of a diagonal operator. The reduction to constant coefficients of the highest order is implemented in Section 3.1 (following [FGMP18]).

In Section 3.2 we reduce to constant coefficients the lower order terms up to an arbitrarily smoothing remainder (following [BGMR17]). In the present paper, such a procedure is implemented by assuming only that the remainders arising at each step are symmetric hyperbolic.

In Section 4 we perform a KAM-reducibility scheme for vector fields which are smoothing perturbations of a diagonal one, by imposing second order Melnikov conditions with loss of derivatives in space (see Theorem 4.8). Note that the *final eigenvalues* $\lambda_j^{(\infty)}$, appearing in the definition of the set (4.54) (on which you get the diagonalization) have an asymptotic expansion of the form

$$\lambda_j^{(\infty)} = \mathrm{i}\nu^{(0)} \cdot j + z(j) + O(\varepsilon \langle j \rangle^{-2m}) \tag{1.2}$$

for some $m > 0$ large enough, where $\nu^{(0)}$ is a constant vector, z is a Fourier multiplier of order $1 - \mathfrak{e}$. The fact that z is a pseudo-differential operator is used in the measure estimate of Section 4.5, in particular, in Lemma 4.15 to obtain the estimate $|z(j) - z(j')| \lesssim \varepsilon |j - j'|$ for any $j, j' \in \mathbb{Z}^d$. In (1.2) all the quantities at r.h.s. also depend on the parameters $(\omega, \nu, \varepsilon)$.

We point out that the nonresonance condition we assume is

$$|\mathrm{i}\omega \cdot l + \lambda_j^{(\infty)} - \lambda_{j'}^{(\infty)}| \geq \frac{2\gamma}{\langle l \rangle^\tau \langle j \rangle^\tau \langle j' \rangle^\tau} \ , \quad \forall (l, j, j') \neq (0, j, j) \ , \tag{1.3}$$

correspondingly the set of the parameters in which we are able to prove reducibility is the set of the (ω, ν) s.t. (1.3) holds.

Finally, in the appendix A, we collect some properties on flows of Pseudo-PDEs, Egorov type theorems and norms that we shall use along our reduction procedure.

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2 Statement of the main result

In order to state precisely the main results of the paper, we introduce some notations.

For any $s \in \mathbb{R}$ we consider the Sobolev space $\mathcal{H}^s(\mathbb{T}^d)$ endowed by the norm

$$\|u\|_{\mathcal{H}^s} := \Big(\sum_{\xi \in \mathbb{Z}^d} \langle \xi \rangle^{2s} |\widehat{u}(\xi)|^2 \Big)^{\frac{1}{2}}$$

where $\langle \xi \rangle := (1 + |\xi|^2)^{\frac{1}{2}}$ and $\widehat{u}(\xi)$ are the Fourier coefficients of u. Given two Banach spaces X, Y we denote by $\mathcal{B}(X, Y)$ the space of bounded linear operators $X \to Y$ equipped by the standard operator norm. If $X = Y$, we simply write

$\mathcal{B}(X)$ instead of $\mathcal{B}(X,X)$.
In the following, given $\alpha,\ \beta\in\mathbb{R}$, we will write $\alpha\lesssim\beta$ if there exists $C>0$ (independent of all the relevant quantities) such that $\alpha\leq C\beta$. Sometimes we will write $\alpha\lesssim_{s_1,\dots,s_n}\beta$ if C depends on parameters $s_1,\cdots,s_n$,
We will use the following classes of pseudo-differential operators:

Definition 2.1. *Let $m\in\mathbb{R}$. We say that a $\mathcal{C}^\infty$ function $a:\mathbb{T}^d\times\mathbb{R}^d\to\mathbb{C}$ is a symbol of class S^m if for any multiindex $\alpha,\ \beta\in\mathbb{N}^d$ there exists a constant $C_{\alpha,\beta}>0$ such that*

$$|\partial_x^\alpha\partial_\xi^\beta a(x,\xi)|\leq C_{\alpha,\beta}\langle\xi\rangle^{m-|\beta|}\,,\quad\forall(x,\xi)\in\mathbb{T}^d\times\mathbb{R}^d\,. \tag{2.1}$$

A symbol a defines univocally a linear operator A acting as

$$A[u](x):=\sum_{\xi\in\mathbb{Z}^d}a(x,\xi)\widehat{u}(\xi)e^{\mathrm{i}x\cdot\xi}\,,\qquad\forall u\in\mathcal{C}^\infty(\mathbb{T}^d)\,,$$

that we denote by $A=\mathrm{Op}\left(a\right)$.

Definition 2.2. *An operator A is called a pseudo-differential operator of order m, namely $A\in OPS^m$, if there exists $a\in S^m$ such that*

$$A=Op(a).$$

The constants $C_{\alpha,\beta}$ of Definition 2.1 form a family of seminorms for S^m and for OPS^m.

In the following, we will consider pseudo-differential operators depending in a smooth way on the angles $\varphi\in\mathbb{T}^{\mathfrak{n}}$ and in a Lipschitz way on the frequencies $\widetilde{\omega}=(\omega,\nu)\in\Omega_0\subseteq\Omega$. We will denote them by $\mathcal{L}ip\left(\Omega_0;\mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}};OPS^m\right)\right)$.

We finally state some properties that we will assume to hold on our system (1.1):

Definition 2.3 (Structural hypotheses)**.** *(i) We say that $\mathcal{R}\in\mathcal{B}(L^2(\mathbb{T}^d))$ is a real operator if it maps real valued functions into real valued functions, namely*

$$u\in L^2(\mathbb{T}^d;\mathbb{R})\Rightarrow\mathcal{R}[u]\in L^2(\mathbb{T}^d;\mathbb{R}).$$

Equivalently, we can say that $\mathcal{R}$ is a real operator if $\mathcal{R}=\overline{\mathcal{R}}$ where the operator $\overline{\mathcal{R}}$ is defined by $\overline{\mathcal{R}}[u]:=\overline{\mathcal{R}[\overline{u}]}$, $u\in L^2(\mathbb{T}^d)$.

(ii) Let $\varphi\mapsto\mathcal{R}(\varphi),\mathcal{Q}(\varphi)$ be smooth φ-dependent families of real operators $\mathbb{T}^{\mathfrak{n}}\to\mathcal{B}\big(L^2(\mathbb{T}^d)\big)$; we say that $\mathcal{R}$ is reversible if

$$\mathcal{R}(\varphi)\circ S=-S\circ\mathcal{R}(-\varphi)\,,\quad\forall\varphi\in\mathbb{T}^{\mathfrak{n}}, \tag{2.2}$$

where S is the involution defined by

$$S:L^2(\mathbb{T}^d)\to L^2(\mathbb{T}^d)\,,\quad u(x)\mapsto u(-x). \tag{2.3}$$

On the other hand, we say that $\mathcal{Q}$ is reversibility preserving if

$$\mathcal{Q}(\varphi)\circ S=S\circ\mathcal{Q}(-\varphi)\,,\quad\forall\varphi\in\mathbb{T}^{\mathfrak{n}}. \tag{2.4}$$

(iii) We say that $\mathcal{R} \in OPS^1$ is symmetric hyperbolic if $\mathcal{R} + \mathcal{R}^* \in OPS^0$.

We will also consider the case where V is even, namely one has

$$V(-\varphi, -x) = V(\varphi, x) \ .$$

Define the constant

$$s_0 := \left[\frac{\mathfrak{n}}{2}\right] + 1 \, . \tag{2.5}$$

This paper is devoted to the proof of the following result.

Theorem 2.4. *Let $V \in \mathcal{C}^\infty(\mathbb{T}^\mathfrak{n} \times \mathbb{T}^d, \mathbb{R}^d)$, $\mathcal{W} \in \mathcal{C}^\infty\left(\mathbb{T}^\mathfrak{n}; OPS^{1-\mathfrak{e}}\right)$ and assume that $\mathcal{W}$ is symmetric hyperbolic. Then for any $s \geq s_0$, $\sigma \geq 0$ there exists $\varepsilon^* > 0$ such that $\forall \varepsilon < \varepsilon^*$ there exists a closed set $\Omega_\varepsilon \subseteq \Omega$ of asymptotically full Lebesgue measure, i.e. $\lim_{\varepsilon \to 0} |\Omega \setminus \Omega_\varepsilon| = 0$, such that the following holds: $\forall\ \widetilde{\omega} = (\omega, \nu) \in \Omega_\varepsilon$ there exists a linear bounded and invertible operator $\mathcal{U}(\varphi) = \mathcal{U}(\varphi; \widetilde{\omega}) \in \mathcal{B}(\mathcal{H}^\sigma)$, $\varphi \in \mathbb{T}^\mathfrak{n}$ such that, if u solves (1.1), then v defined by $u = \mathcal{U}(\omega t) v$ solves*

$$\partial_t v = H_\infty v, \tag{2.6}$$

where

$$H_\infty = \operatorname{diag}(\lambda_j^{(\infty)}(\tilde{\omega}, \varepsilon)) \tag{2.7}$$

Furthermore, the eigenvalues $\{\lambda_j^{(\infty)}(\tilde{\omega}, \varepsilon)\}_{j \in \mathbb{Z}^d}$ have the structure

$$\lambda_j^{(\infty)}(\tilde{\omega}, \varepsilon) = \mathrm{i}\nu^{(0)} \cdot j + z(j) + \mathcal{O}(\epsilon j^{-m}) \ , \tag{2.8}$$

with $z(.) \in S^{1-\mathfrak{e}}$ which is also dependent in a Lipschitz way on $\tilde{\omega}$, and $\nu^{(0)} = \nu^{(0)}(\tilde{\omega})$ which fulfills

$$\left|\nu^{(0)} - \nu\right| \leq C\varepsilon \ .$$

Finally, if the following assumption holds

(Sym) V is even and $\mathcal{W}$ is real and reversible ,

then $\lambda_j^{(\infty)} \in \mathrm{i}\mathbb{R} \quad \forall\ j \in \mathbb{Z}^d$.

From the theorem above we can deduce information concerning the dynamics of the PDE (1.1).

Corollary 2.5. *Under the same assumptions of Theorem 2.4, but not (Sym) only one of the following two possibilities occurs*

(1) All the solutions of (1.1) are almost periodic and

$$u_0 \in \mathcal{H}^\sigma \implies \|u(t, \cdot)\|_{\mathcal{H}^\sigma} \lesssim \|u_0\|_{\mathcal{H}^\sigma} \tag{2.9}$$

uniformly w.r. to $t \in \mathbb{R}$.

(2) There exist $a, C > 0$ and some initial data u_0 s.t.

$$\|u(t, \cdot)\|_{\mathcal{H}^\sigma} \geq Ce^{a|t|} \|u_0\|_{\mathcal{H}^\sigma} \tag{2.10}$$

either for $t > 0$ or for $t < 0$ or for $t \in \mathbb{R}$.

We remark that under the assumption (Sym) only possibility (1) occurs.

3 Regularization up to smoothing remainders

In this section we conjugate the vector field

$$H(\varphi) := \big(\nu + \varepsilon V(\varphi, x)\big) \cdot \nabla + \varepsilon \mathcal{W}^{(0)}(\varphi)\,, \quad \mathcal{W} \in OPS^{1-\mathfrak{e}} \tag{3.1}$$

to another one which is a smoothing perturbation of a time independent diagonal operator.
First remark that a time dependent linear invertible transformation $u = \Phi(\omega t)u'$ transforms the equation $\dot{u} = Hu$ into the equation $\dot{u}' = H'u'$, where

$$H' = \Phi_{\omega *} H := \Phi(\varphi)^{-1}[H\Phi(\varphi) - \omega \cdot \partial_\varphi \Phi(\varphi)]\ .$$

Definition 3.1 (Lipschitz norm)**.** *Given a Banach space $(X, \|\cdot\|_X)$, a set $\Omega_0 \subset \Omega = [1,2]^{\mathtt{n}+d}$, $\gamma > 0$ and a Lipschitz function $f : \Omega_0 \to X$, we denote by $\|\cdot\|_X^{\mathrm{Lip}(\gamma)}$ the Lipschitz norm defined by*

$$\begin{aligned} &\|f\|_X^{\mathrm{Lip}(\gamma)} := \|f\|_X^{\sup} + \gamma \|f\|_X^{\mathrm{lip}}\,, \\ &\|f\|^{\sup} := \sup_{\widetilde{\omega} \in \Omega_0} \|f(\widetilde{\omega})\|_X\,, \quad \|f\|_X^{\mathrm{lip}} := \sup_{\substack{\widetilde{\omega}_1, \widetilde{\omega}_2 \in \Omega_0 \\ \widetilde{\omega}_1 \neq \widetilde{\omega}_2}} \frac{\|f(\widetilde{\omega}_1) - f(\widetilde{\omega}_2)\|_X}{|\widetilde{\omega}_1 - \widetilde{\omega}_2|}\,. \end{aligned} \tag{3.2}$$

In the case where $\gamma = 1$, we simply write $\|\cdot\|_X^{\mathrm{Lip}}$ for $\|\cdot\|_X^{\mathrm{Lip}(1)}$. If $X = \mathbb{C}$ we write $|\cdot|^{\mathrm{Lip}(\gamma)}, |\cdot|^{\sup}, |\cdot|^{\mathrm{lip}}$ for $\|\cdot\|_{\mathbb{C}}^{\mathrm{Lip}(\gamma)}, \|\cdot\|_{\mathbb{C}}^{\sup}, \|\cdot\|_{\mathbb{C}}^{\mathrm{lip}}$.

3.1 Reduction to constant coefficients of the highest order term

We consider a diffeomorphism of the torus $\mathbb{T}^d$ of the form

$$\mathbb{T}^d \to \mathbb{T}^d, \quad x \mapsto x + \alpha(\varphi, x)$$

where $\alpha \in \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}} \times \mathbb{T}^d, \mathbb{R}^d)$ is a function to be determined. It is well known that for $\|\alpha\|_{C^1}$ small enough such a diffeomorphism is invertible and its inverse has the form

$$\mathbb{T}^d \to \mathbb{T}^d, \quad y \mapsto y + \widetilde{\alpha}(\varphi, y)$$

with $\widetilde{\alpha} \in \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}} \times \mathbb{T}^d, \mathbb{R}^d)$. We then consider the transformation

$$\mathcal{A}(\varphi) : u(x) \mapsto u(x + \alpha(\varphi, x))\,, \quad \varphi \in \mathbb{T}^{\mathtt{n}} \tag{3.3}$$

whose inverse is given by

$$\mathcal{A}(\varphi)^{-1} : u(y) \mapsto u(y + \widetilde{\alpha}(\varphi, y))\,, \quad \varphi \in \mathbb{T}^{\mathtt{n}}\,. \tag{3.4}$$

A direct calculation shows that the *quasi-periodic push-forward* of the vector field $H(\varphi)$ is given by

$$H^{(0)}(\varphi) = \mathcal{A}_{\omega *} H(\varphi) = V^{(0)}(\varphi, x) \cdot \nabla + \varepsilon \mathcal{W}^{(0)}(\varphi) \tag{3.5}$$

where

$$\begin{aligned} V^{(0)}(\varphi, x) &:= \mathcal{A}(\varphi)^{-1}\Big(\omega \cdot \partial_\varphi \alpha + \nu + \varepsilon V + \big(\nu + \varepsilon V\big) \cdot \nabla \alpha\Big) \\ \mathcal{W}^{(0)}(\varphi) &:= \mathcal{A}(\varphi)^{-1} \mathcal{W}(\varphi) \mathcal{A}(\varphi)\,. \end{aligned} \tag{3.6}$$

The following proposition is a direct consequence of Proposition 3.4 in [FGMP18] to which we refer for the proof. It allows to choose the function $\alpha(\varphi, x)$ so that the highest order term $V^{(0)}(\varphi, x) \cdot \nabla$ in (3.5) is reduced to constant coefficients.

Proposition 3.2. *Let $\gamma \in (0,1)$ and $\tau > \mathtt{n}+d$. There exists a Lipschitz function $\nu^{(0)} : \Omega \to \mathbb{R}^d, \tilde{\omega} \mapsto \nu^{(0)}(\tilde{\omega})$ (where we recall that $\Omega := [1,2]^{\mathtt{n}+d}$) such that*

$$|\nu^{(0)}(\tilde{\omega}) - \nu|^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon, \tag{3.7}$$

and, in the set

$$\Omega_{0,\gamma} := \Big\{ \tilde{\omega} \in \Omega : \ |\omega \cdot l + \nu^{(0)}(\tilde{\omega}) \cdot j| > \frac{\gamma}{\langle l, j \rangle^{\tau}} \, , \ \forall (l,j) \in \mathbb{Z}^{\nu+d} \setminus \{0\} \Big\} \, , \tag{3.8}$$

the following holds. There exists a map

$$\alpha : \mathbb{T}^{\nu+d} \times \Omega_{0,\gamma} \to \mathbb{R}^d \, , \tag{3.9}$$

so that the map $\mathbb{T}^{\mathtt{n}+d} \to \mathbb{T}^{\mathtt{n}+d}$, $(\varphi, x) \mapsto (\varphi, x + \alpha(\varphi, x))$ is a diffeomorphism with inverse given by $(\varphi, y) \mapsto (\varphi, y + \widetilde{\alpha}(\varphi, y))$, furthermore

$$\|\alpha\|_s^{\mathrm{Lip}(\gamma)} \lesssim_s \varepsilon\gamma^{-1}, \quad \|\widetilde{\alpha}\|_s^{\mathrm{Lip}(\gamma)} \lesssim_s \varepsilon\gamma^{-1} \, , \quad \forall s \geq 0 \, . \tag{3.10}$$

Moreover for any $\tilde{\omega} \in \Omega_{0,\gamma}$ $V^{(0)}$ reduces to a constant (as a function of x and φ), namely

$$V^{(0)} = \mathcal{A}^{-1}(\varphi)\Big(\omega \cdot \partial_\varphi \alpha + \nu + \varepsilon V + \big(\nu + \varepsilon V\big) \cdot \nabla \alpha\Big) = \nu^{(0)}(\tilde{\omega}). \tag{3.11}$$

Finally, if V is even, then α and $\widetilde{\alpha}$ are odd.

Remark 3.3. *By standard arguments one has $|\Omega \setminus \Omega_{0,\gamma}| \lesssim \gamma$. More precisely, on the one side one has that vectors which are Diophantine with constant γ have complement with measure of order γ, and on the other, Lipschitz maps preserve the order of magnitude of the measure of sets.*

Remark 3.4. *Using the definitions* (3.3), (3.4) *and the estimates* (3.9), (3.10), *a direct calculation shows that the map $\mathbb{T}^{\mathtt{n}} \mapsto \mathcal{B}(\mathcal{H}^s)$, $\varphi \mapsto \mathcal{A}(\varphi)^{\pm 1}$ is bounded for any $s \geq 0$ and*

$$\sup_{\varphi \in \mathbb{T}^{\mathtt{n}}} \|\mathcal{A}(\varphi)^{\pm 1} - \mathrm{Id}\|_{\mathcal{B}(\mathcal{H}^{s+1}, \mathcal{H}^s)} \lesssim_s \varepsilon\gamma^{-1}, \quad \forall s \geq 0 \, ,$$
$$\sup_{\varphi \in \mathbb{T}^{\mathtt{n}}} \|\partial_\varphi^\alpha \mathcal{A}(\varphi)^{\pm 1}\|_{\mathcal{B}(\mathcal{H}^{s+|\alpha|}, \mathcal{H}^s)} \lesssim_{s,\alpha} \varepsilon\gamma^{-1}, \quad \forall s \geq 0, \quad \forall \alpha \in \mathbb{N}^{\mathtt{n}} \, .$$

Recalling (3.5), (3.6) and applying Proposition 3.2 one gets that the vector field $H^{(0)}(\varphi)$ takes the form

$$H^{(0)}(\varphi) = \nu^{(0)} \cdot \nabla + \varepsilon \mathcal{W}^{(0)}(\varphi) \tag{3.12}$$

We now study the properties of $\mathcal{W}^{(0)}$.

Lemma 3.5. *One has that $\mathcal{W}^{(0)} \in \mathcal{L}ip\Big(\Omega_{0,\gamma}, \mathcal{C}^\infty\Big(\mathbb{T}^{\mathtt{n}}, OPS^{1-\mathfrak{e}}\Big)\Big)$. Moreover $\mathcal{W}^{(0)}$ is symmetric hyperbolic. Furthermore, if V is even and $\mathcal{W}$ real and reversible, then $\mathcal{W}^{(0)}$ is real and reversible.*

Proof. Let $\Phi(\varphi) := \mathcal{A}(\varphi)^{-1}$, i.e. $\Phi(\varphi)[u](y) = u(y + \widetilde{\alpha}(\varphi, y))$ and for any $\tau \in [0, 1]$ we consider $\Phi(\tau, \varphi)[u](y) := u(y + \tau\widetilde{\alpha}(\varphi, y))$. Let $\psi(\tau, \varphi, y) := \Phi(\tau, \varphi)[u](y)$, then $\psi(0, \varphi, y) = u(y)$ and

$$\partial_\tau \psi = a(\tau, \varphi, y) \cdot \nabla\psi , \quad a(\tau, \varphi, y) := \big(\mathrm{Id} + \tau\nabla\widetilde{\alpha}(\varphi, y)\big)^{-1}\widetilde{\alpha}(\varphi, y) . \tag{3.13}$$

Then by the Egorov theorem (see Theorem A.0.9 in [Tay91]) it follows that $\mathcal{W}^{(0)} \in \mathcal{L}ip\Big(\Omega_{0,\gamma}, \mathcal{C}^\infty\Big(\mathbb{T}^{\mathfrak{n}}, OPS^{1-\mathfrak{e}}\Big)\Big)$.

We now show that $\mathcal{W}^{(0)}$ is symmetric hyperbolic. Since by (3.9), (3.10) the functions $\alpha, \widetilde{\alpha} = O(\varepsilon\gamma^{-1})$ one has that

$$\det\big(\mathrm{Id} + \nabla\alpha\big) , \det\big(\mathrm{Id} + \nabla\widetilde{\alpha}\big) > 0$$

for $\varepsilon\gamma^{-1}$ small enough. Moreover, using that $y \mapsto y + \widetilde{\alpha}(y)$ is the inverse diffeomorphism of $x \mapsto x + \alpha(x)$ one gets that

$$\det\big(\mathrm{Id} + \nabla\widetilde{\alpha}(y)\big) = \frac{1}{\det\big(\mathrm{Id} + \nabla\alpha\big)|_{x=y+\widetilde{\alpha}(y)}} . \tag{3.14}$$

A direct calculation shows that

$$\mathcal{A}^* = \det\big(\mathrm{Id} + \nabla\widetilde{\alpha}\big)\mathcal{A}^{-1} , \quad (\mathcal{A}^{-1})^* = \det\big(\mathrm{Id} + \nabla\alpha\big)\mathcal{A} .$$

Then

$$\begin{aligned}(\mathcal{W}^{(0)})^* &= (\mathcal{A}^{-1}\mathcal{W}\mathcal{A})^* = \mathcal{A}^*\mathcal{W}^*(\mathcal{A}^{-1})^* \\ &= \det\big(\mathrm{Id} + \nabla\widetilde{\alpha}\big)\mathcal{A}^{-1}\mathcal{W}^*\det\big(\mathrm{Id} + \nabla\alpha\big)\mathcal{A} \\ &= \det\big(\mathrm{Id} + \nabla\widetilde{\alpha}\big)\mathcal{A}^{-1}\det\big(\mathrm{Id} + \nabla\alpha\big)\mathcal{W}^*\mathcal{A} \\ &\quad + \det\big(\mathrm{Id} + \nabla\widetilde{\alpha}\big)\mathcal{A}^{-1}[\mathcal{W}^* , \det\big(\mathrm{Id} + \nabla\alpha\big)]\mathcal{A} .\end{aligned} \tag{3.15}$$

Since $\mathcal{W}^* \in OPS^{1-\mathfrak{e}}$ one has that the commutator $[\mathcal{W}^* , \det\big(\mathrm{Id} + \nabla\alpha\big)] \in OPS^{-\mathfrak{e}} \subset OPS^0$. Using that $\mathcal{A}(\varphi)^{-1} = \Phi(\varphi)$ is the time 1 flow map of the PDE (3.13), by applying the Egorov Theorem A.0.9 in [Tay91], one gets that $\det\big(\mathrm{Id} + \nabla\widetilde{\alpha}\big)\mathcal{A}^{-1}[\mathcal{W}^* , \det\big(\mathrm{Id} + \nabla\alpha\big)]\mathcal{A} \in OPS^0$. hence

$$\begin{aligned}(\mathcal{W}^{(0)})^* &= \det\big(\mathrm{Id} + \nabla\widetilde{\alpha}\big)\mathcal{A}^{-1}\det\big(\mathrm{Id} + \nabla\alpha\big)\mathcal{W}^*\mathcal{A} + OPS^0 \\ &= \det\big(\mathrm{Id} + \nabla\widetilde{\alpha}\big)\det\big(\mathrm{Id} + \nabla\alpha\big)|_{x=y+\widetilde{\alpha}(y)}\mathcal{A}^{-1}\mathcal{W}^*\mathcal{A} + OPS^0 \\ &\overset{(3.14)}{=} \mathcal{A}^{-1}\mathcal{W}^*\mathcal{A} + OPS^0 .\end{aligned} \tag{3.16}$$

Finally, using that $\mathcal{W}$ is symmetric hyperbolic, i.e. $\mathcal{W} + \mathcal{W}^* \in OPS^0$, by (3.14) and applying again the Egorov Theorem A.0.9 in [Tay91] to deduce that $\mathcal{A}^{-1}(\mathcal{W} + \mathcal{W}^*)\mathcal{A} \in OPS^0$ one gets that $\mathcal{W}^{(0)} + (\mathcal{W}^{(0)})^* \in OPS^0$. In the real and reversible case, one has that $\mathcal{W}$ is a reversible operator. By Proposition 3.2, one has that $\alpha, \widetilde{\alpha}$ are odd functions, implying that $\mathcal{A}$, $\mathcal{A}^{-1}$ are reversibility preserving operators. Hence one concludes that $\mathcal{W}^{(0)} = \mathcal{A}^{-1}\mathcal{W}\mathcal{A}$ is a reversible operator. $\square$

3.2 Reduction of the lower order terms

The reduction of the lower order terms is contained in the following result, which is an adaptation of Theorem 3.8 of [BGMR17] to a symmetric hyperbolic context.

Theorem 3.6. $\forall\ M > 0$ *there exists a sequence of symmetric hyperbolic maps* $\{G_j(\varphi, \tilde{\omega})\}_{j=1}^M$ *with* $G_j(\varphi, \tilde{\omega}) \in \mathcal{L}ip\left(\Omega_{0,\gamma}; \mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}}; OPS^{1-j\mathfrak{e}}\right)\right)$ *such that the change of variables* $\psi = e^{-\varepsilon G_1(\varphi,\tilde{\omega})} \cdots\ e^{-\varepsilon G_M(\varphi,\tilde{\omega})}\phi$ *transforms* $H_0 + \varepsilon\mathcal{W}^{(0)}(\varphi)$ *into the operator*

$$H^{(M)}(\varphi) = H_0 + \varepsilon Z^{(M)}(\tilde{\omega}) + \varepsilon\mathcal{W}^{(M)}(\varphi, \tilde{\omega}), \tag{3.17}$$

where $Z^{(M)}$ *is a time independent Fourier multiplier, which in particular fulfills*

$$[Z^{(M)}, K_m] = 0, \quad m = 1 \ldots, d, \tag{3.18}$$

and

$$\begin{aligned} Z^{(M)}(\tilde{\omega}) &\in \mathcal{L}ip\left(\Omega_{0,\gamma}; OPS^{1-\mathfrak{e}}\right), \\ \mathcal{W}^{(M)}(\varphi, \tilde{\omega}) &\in \mathcal{L}ip\left(\Omega_{0,\gamma}; \mathcal{C}^\infty(\mathbb{T}^{\mathfrak{n}}; OPS^{1-M\mathfrak{e}})\right). \end{aligned} \tag{3.19}$$

Furthermore, if $\mathcal{W}^{(0)}$ *is real and reversible, then* $Z^{(M)}$, $\mathcal{W}^{(M)}$ *are real and reversible too.*

We now prove such theorem.

Denote $K_j = i\partial_j, \quad j = 1, \ldots, d$, then $K_1, \ldots, K_d$ are self-adjoint commuting operators such that $K_m \in OPS^1\ \forall m = 1, \ldots, d$. Define $K = (K_1, \ldots, K_d)$. The main step for the proof of Theorem 3.6 is the following lemma, which is a variant of Lemma 3.7 of [BGMR17]:

Lemma 3.7. *Let* $W \in \mathcal{L}ip\left(\Omega_{0,\gamma}; \mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}}; OPS^\eta\right)\right)$, *be given and consider the homological equation*

$$\omega \cdot \partial_\varphi G + [H_0,\ G] = W - \langle W \rangle \tag{3.20}$$

with

$$\langle W \rangle := \frac{1}{(2\pi)^{\mathfrak{n}+d}} \int_{\mathbb{T}^d} \int_{\mathbb{T}^{\mathfrak{n}}} e^{i\tau\cdot K} W e^{-i\tau\cdot K}\ d\varphi\ d\tau\ ;$$

then (3.20) *has a solution* $G \in \mathcal{L}ip\left(\Omega_{0,\gamma}; \mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}}; OPS^\eta\right)\right)$.
If W *is symmetric hyperbolic,* G *is symmetric hyperbolic. Moreover, if* W *is real and reversible,* G *is real and reversibility preserving; if* W *is anti self-adjoint,* G *is anti self-adjoint.*

Proof. Define $\forall \tau \in \mathbb{T}^d$

$$W(\tau) := e^{i\tau\cdot K} W e^{-i\tau\cdot K},$$

then we look for G s.t.

$$G(\tau) := e^{i\tau\cdot K} G e^{-i\tau\cdot K}$$

solves

$$\omega \cdot \partial_\varphi G(\tau) + [H_0,\ G(\tau)] = W(\tau) - \langle W \rangle \quad \forall\ \tau \in \mathbb{T}^d, \tag{3.21}$$

observing that since $G = G(0),\ W = W(0)$, solving equation (3.21) $\forall\ \tau$ implies having solved (3.20).
Note that $\forall\ \eta \in \mathbb{R},\ \forall\ A \in OPS^\eta$ the map

$$[-1,\ 1] \ni \tau \mapsto e^{-i\tau\cdot K} A e^{i\tau\cdot K} \in \mathcal{C}^\infty\left(\mathbb{T}^d; OPS^\eta\right) \tag{3.22}$$

(see Remark A.5 of the appendix). We make a Fourier expansion both in φ and τ variables, namely

$$W_{\tilde{\omega}}(\varphi,\tau) = \sum_{k\in\mathbb{Z}^d}\sum_{l\in\mathbb{Z}^{\mathtt{n}}} \widehat{W}_{kl}(\tilde{\omega}) e^{i\varphi\cdot l} e^{i\tau\cdot k}, \tag{3.23}$$

and similarly for G. A direct calculation shows that

$$[H_0,\ G(\tau)] = \sum_{k,\ l} i\left(\nu^{(0)}\cdot k\right)\ \widehat{G}_{kl} e^{i\tau\cdot k} e^{i\varphi\cdot l}\,.$$

Thus, taking the $(k,l)-$th Fourier coefficient of equation (3.21), one has

$$i\left(\omega\cdot l + \nu^{(0)}\cdot k\right)\widehat{G}_{kl} = \widehat{W}_{kl} \quad \text{ if } (k,l)\neq(0,0), \quad \widehat{G}_{00}=0.$$

For $|k|+|l|\neq 0$, define

$$\widehat{G}_{kl} := \frac{\widehat{W}_{kl}}{i(\omega\cdot l + \nu^{(0)}\cdot k)}\ ,$$

then, by regularity of the map $(\varphi,\tau)\mapsto W(\varphi,\tau)$ all the seminorms of the operator $\widehat{W}_{kl}$ decay faster than any power of $(|k|+|l|)$, and since the frequencies belong to $\Omega_{0,\gamma}$ (cf. (3.8)), it follows that the seminorms of the operator $\widehat{G}_{kl}$ exhibit the same decay; hence the series defining $G(\tau)$ converges absolutely and $G = G(0)\in\mathcal{C}^\infty\left(\mathbb{T}^{\mathtt{n}}; OPS^\eta\right)$.
Lipschitz regularity with respect to $\tilde{\omega}=(\omega,\nu)\in\Omega_{0,\gamma}$ follows observing that given $(\omega_1,\nu_1),(\omega_2,\nu_2)\in\Omega_{0,\gamma}$, one has that

$$\begin{aligned}\widehat{G}_{kl}(\omega_1)-\widehat{G}_{kl}(\omega_2) = \widehat{G}_{kl}(\omega_1)&\frac{(\omega_1-\omega_2)\cdot l + \left(\nu^{(0)}(\omega_1,\nu_1)-\nu^{(0)}(\omega_2,\nu_2)\right)\cdot k}{(\omega_1\cdot l+\nu^{(0)}(\omega_1,\nu_1)\cdot k)(\omega_2\cdot l+\nu^{(0)}(\omega_2,\nu_2)\cdot k)}\\ &+\frac{\widehat{G}_{kl}(\omega_1)-\widehat{G}_{kl}(\omega_2)}{\mathrm{i}(\omega_2\cdot l+\nu^{(0)}(\omega_2,\nu_2)\cdot k)}\end{aligned}$$

using the fact that the map $(\omega,\nu)\mapsto\nu^{(0)}(\omega,\nu)$ is Lipschitz (see Proposition 3.2) and the diophantine estimate required in (3.8).
Symmetric hyperbolicity: We observe that

$$W+W^* = e^{-i\tau\cdot K}\left(W(\tau)+W^*(\tau)\right)e^{i\tau\cdot K}, \quad G+G^* = e^{-i\tau\cdot K}\left(G(\tau)+G^*(\tau)\right)e^{i\tau\cdot K}.$$

Hence W (resp., G) is symmetric hyperbolic if and only if $W(\tau)$ (resp., $G(\tau)$) is symmetric hyperbolic.
Thus, arguing as before and being

$$\widehat{(W^*)}_{k,l} = \overline{\widehat{W}_{-k,-l}} \quad \forall\ k\in\mathbb{Z}^d,\ l\in\mathbb{Z}^{\mathtt{n}},$$

it follows that if $\forall\ k\in\mathbb{Z}^d,\ l\in\mathbb{Z}^{\mathtt{n}}$ $\widehat{W}_{k,l}+\overline{\widehat{W}_{-k,-l}}$ are the Fourier coefficients of an operator in OPS^0, then

$$\widehat{G}_{k,l}+\overline{\widehat{G}_{-k,-l}} = \frac{\widehat{W}_{k,l}+\overline{\widehat{W}_{-k,-l}}}{i\left(\omega\cdot l+\nu\cdot k\right)}$$

are again Fourier coefficients of an operator in OPS^0.
Reversibility: We apply Lemma A.6 of the Appendix to deduce reversibility of W and we observe that an operator $A(\tau,\varphi)$ is reversible (resp. reversibility preserving) if and only if, developing in Fourier series as in (3.23), its coefficients satisfy

$$\widehat{A}_{kl}\circ S = -S\circ \widehat{A}_{-k-l} \qquad \Big(\text{resp. } \widehat{A}_{kl}\circ S = S\circ \widehat{A}_{-k-l}\Big),$$

so that $\forall\ k\in\mathbb{Z}^d,\ l\in\mathbb{Z}^{\mathfrak{n}}$,

$$\widehat{G}_{kl}\circ S = \frac{\widehat{W}_{kl}\circ S}{i(\omega\cdot l+\nu\cdot k)} = \frac{-S\circ\widehat{W}_{-k-l}}{-(\omega\cdot(-l)+\nu\cdot(-k))} = S\circ\widehat{G}_{-k-l}.$$

Hence G, and thus $\mathcal{G}$, is reversibility preserving. (See Lemma A.6.)
Reality: Reality condition in Fourier coefficients reads

$$\widehat{A}_{lk} = \overline{\widehat{A}_{-l-k}}.$$

We apply Lemma A.6 again to deduce that reality of W (resp, G) is equivalent to reality of $\mathcal{W}$ (resp, $\mathcal{G}$) and we compute

$$\widehat{G}_{kl} = \frac{\widehat{W}_{kl}}{i(\omega\cdot l+\nu\cdot k)} = \frac{\overline{\widehat{W}_{-k-l}}}{-i(\omega\cdot(-l)+\nu\cdot(-k))} = \overline{\widehat{G}_{-k-l}}.$$

$\square$

Proof of Theorem 3.6. Fix $M>0$. We prove by induction that $\forall j=0,\dots,N-1$

$$H^{(j)}(\varphi) = H_0+\varepsilon Z^{(j)}(\tilde\omega)+\varepsilon\mathcal{W}^{(j)}(\varphi,\tilde\omega)$$

is mapped by the change of variables

$$u = e^{-\varepsilon G_j(\varphi,\tilde\omega)}v \tag{3.24}$$

into

$$H^{(j+1)}(\varphi) = H_0+\varepsilon Z^{(j+1)}(\tilde\omega)+\varepsilon\mathcal{W}^{(j+1)}(\varphi,\tilde\omega),$$

with

$$\begin{aligned} Z^{(j+1)}(\tilde\omega) &\in \mathcal{L}ip\Big(\Omega_{0,\gamma};\mathcal{C}^\infty(\mathbb{T}^{\mathfrak{n}};OPS^{1-\mathfrak{e}})\Big),\\ \mathcal{W}^{(j+1)} &\in \mathcal{L}ip\left(\Omega_{0,\gamma};\mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}};OPS^{1-(j+1)\mathfrak{e}}\right)\right), \end{aligned}\tag{3.25}$$

$\mathcal{W}^{(j+1)}$ symmetric hyperbolic and $Z^{(j+1)}(\tilde\omega)$ a Fourier multiplier commuting with all the K_m.
If $j=0$, the hypotheses are satisfied for $Z^{(0)}=0$, $\mathcal{W}^{(0)}=\mathcal{W}\in\mathcal{L}ip\left(\Omega_{0,\gamma};\mathcal{C}^\infty(\mathbb{T}^{\mathfrak{n}};OPS^{1-\mathfrak{e}})\right)$.
Suppose now that $H^{(j)}$ satisfies the required hypotheses; the change of coordi-

nates (3.24) maps $H^{(j)}$ into

$$
\begin{aligned}
H^{(j+1)}(\varphi,\tilde\omega) = {} & H_0+\varepsilon Z^{(j)}(\tilde\omega)+\varepsilon\langle\mathcal{W}^{(j)}\rangle && (3.26)\\
&+\varepsilon\left(-\omega\cdot\partial_\varphi G_j+[H_0,G_j]+\mathcal{W}^{(j)}(\varphi,\tilde\omega)-\langle\mathcal{W}^{(j)}\rangle\right) && (3.27)\\
&+e^{\varepsilon G_j(\varphi,\tilde\omega)}H_0e^{-\varepsilon G_j(\varphi,\tilde\omega)}-H_0-\varepsilon[H_0,G_j] && (3.28)\\
&+\varepsilon e^{\varepsilon G_j(\varphi,\tilde\omega)}Z^{(j)}(\tilde\omega)e^{-\varepsilon G_j(\varphi,\tilde\omega)}-\varepsilon Z^{(j)}(\tilde\omega) && (3.29)\\
&+\varepsilon e^{\varepsilon G_j(\varphi,\tilde\omega)}\mathcal{W}^{(j)}(\varphi,\tilde\omega)e^{-\varepsilon G_j(\varphi,\tilde\omega)}-\varepsilon\mathcal{W}^{(j)}(\varphi,\tilde\omega) && (3.30)\\
&-\varepsilon\int_0^1 e^{-\varepsilon sG_j(\varphi,\tilde\omega)}\omega\cdot\partial_\varphi G_j(\varphi,\tilde\omega)e^{\varepsilon sG_j(\varphi,\tilde\omega)}\,ds+\varepsilon\omega\cdot\partial_\varphi G_j. && (3.31)
\end{aligned}
$$

From Lemma 3.7 it is possible to find an operator $G_j\in\mathcal{L}ip\left(\Omega_{0,\gamma};\mathcal{C}^\infty(\mathbb{T}^{\mathfrak{n}};OPS^{1-j\mathfrak{e}})\right)$ such that G_j is symmetric hyperbolic and (3.27) equals zero. Since Lemma A.4 of the Appendix entails that

$$
\begin{aligned}
(3.28)&\in\mathcal{L}ip\left(\Omega_{0,\gamma};\mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}};OPS^{1-2j\mathfrak{e}}\right)\right),\\
(3.29)&\in\mathcal{L}ip\left(\Omega_{0,\gamma};\mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}};OPS^{1-(j+1)\mathfrak{e}}\right)\right),\\
(3.30)&\in\mathcal{L}ip\left(\Omega_{0,\gamma};\mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}};OPS^{1-2j\mathfrak{e}}\right)\right),\\
(3.31)&\in\mathcal{L}ip\left(\Omega_{0,\gamma};\mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}};OPS^{1-2j\mathfrak{e}}\right)\right),
\end{aligned}
$$

if we define

$$
\begin{aligned}
Z^{(j+1)}(\tilde\omega)&:=Z^{(j)}(\tilde\omega)+\langle\mathcal{W}^{(j)}\rangle,\\
\varepsilon\mathcal{W}^{(j+1)}(\varphi,\tilde\omega)&=(3.28)+(3.29)+(3.30)+(3.31),
\end{aligned}
\tag{3.32}
$$

we have $\mathcal{W}^{(j+1)}(\varphi,\tilde\omega)\in\mathcal{L}ip\left(\Omega_{0,\gamma};\mathcal{C}^\infty\left(\mathbb{T}^{\mathfrak{n}};OPS^{1-(j+1)\mathfrak{e}}\right)\right)$.
We observe that (3.28) is of order ε, as can be seen performing a Taylor expansion of the operator $e^{-\varepsilon G_j(\varphi,\tilde\omega)}H_0e^{\varepsilon G_j(\varphi,\tilde\omega)}$ as in Lemma A.4 of the Appendix. Reality and reversibility of $\mathcal{W}^{(j+1)}(\varphi,\tilde\omega)$ follow from Lemma A.1, whereas symmetric hyperbolicity of $\mathcal{W}^{(j+1)}(\varphi,\tilde\omega)$ follows from Lemma A.7. □

Remark 3.8. *For all $j=1,\ \dots,\ M$ we have $e^{\varepsilon G_j}\in\mathcal{B}(\mathcal{H}^\sigma)\ \forall\ \sigma$, and*

$$
\|e^{\varepsilon G_j}-\mathrm{Id}\|_{\mathcal{B}(H^\sigma,H^{\sigma-(1-j\mathfrak{e})})}\lesssim\varepsilon\|G_j\|_{\mathcal{B}(H^\sigma,H^{\sigma-(1-j\mathfrak{e})})}.
$$

Furthermore, from Lemma A.1, $\forall\ \alpha\in\mathbb{N}$ we have

$$
\partial_\varphi^\alpha e^{\varepsilon G_j}\in\mathcal{B}\left(\mathcal{H}^\sigma,\mathcal{H}^{\sigma-(1-j\mathfrak{e})|\alpha|}\right).
$$

Note that, since $Z^{(M)}\in\mathcal{L}ip\Big(\Omega_{0,\gamma};\mathcal{C}^\infty(\mathbb{T}^{\mathfrak{n}};OPS^{1-\mathfrak{e}})\Big)$ then $Z^{(M)}=\mathrm{Op}(z(\xi))$ with $z\in\mathcal{L}ip\Big(\Omega_{0,\gamma};\mathcal{C}^\infty(\mathbb{T}^{\mathfrak{n}};S^{1-\mathfrak{e}})\Big)$. Hence $\partial_\xi z\in\mathcal{L}ip\Big(\Omega_{0,\gamma};\mathcal{C}^\infty(\mathbb{T}^{\mathfrak{n}};S^{-\mathfrak{e}})\Big)$ and the following estimate holds

$$
\sup_{\xi\in\mathbb{R}^d}\langle\xi\rangle^{\mathfrak{e}-1}|z|^{\mathrm{Lip}},\ \sup_{\xi\in\mathbb{R}^d}\langle\xi\rangle^{1-\mathfrak{e}}|\partial_\xi z(\xi,\cdot)|^{\mathrm{Lip}}\lesssim\varepsilon;
\tag{3.33}
$$

Concerning the second of (3.33), we remark that we will only use the fact that $|\partial_\xi z(\xi,\cdot)|^{\mathrm{Lip}}$ is bounded.

4 Reducibility

4.1 Functional Setting

Given a linear operator $R : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$, we denote by $R_j^{j'}$ its matrix elements with respect to the exponential basis $\{e^{\mathrm{i}j\cdot x} : j \in \mathbb{Z}^d\}$, namely

$$R_j^{j'} := \int_{\mathbb{T}^d} R[e^{\mathrm{i}j'\cdot x}]\, e^{-\mathrm{i}j\cdot x}\, dx\,, \quad \forall j, j' \in \mathbb{Z}^d\,.$$

We define some families of operators related to $R \in \mathcal{B}(L^2(\mathbb{T}^d))$ that will be useful in our estimates:

Definition 4.1. *Given $\beta \geq 0$ and $R \in \mathcal{B}(L^2(\mathbb{T}^d))$, we define the operator $\langle\nabla\rangle^\beta R$ as*

$$(\langle\nabla\rangle^\beta R)_j^{j'} := \langle j - j'\rangle^\beta R_j^{j'}\,.$$

We remark that this operator is useful since, for any operator R and any function u, one has

$$\nabla R u = R\nabla u + [R;\nabla]u\ ,$$

and

$$[R;\nabla] \simeq \langle\nabla\rangle R\ .$$

Definition 4.2. *We consider the space*

$$\mathcal{B}^{HS}(\mathcal{H}^{\sigma_1}, \mathcal{H}^{\sigma_2}) := \left\{R \in \mathcal{B}\left(\mathcal{H}^{\sigma_1}, \mathcal{H}^{\sigma_2}\right) \,\middle|\, \|R\|^{HS}_{\sigma_1,\sigma_2} < +\infty\right\},$$

with

$$\left(\|R\|^{HS}_{\sigma_1,\sigma_2}\right)^2 := \sum_{k\in\mathbb{Z}^d}\sum_{k'\in\mathbb{Z}^d} \langle k\rangle^{2\sigma_2} |R_k^{k'}|^2 \langle k'\rangle^{-2\sigma_1}.$$

We consider operators $R(\varphi)$ depending on the angles $\varphi \in \mathbb{T}^{\mathtt{n}}$, with $R \in \mathcal{H}^s\left(\mathbb{T}^{\mathtt{n}};\ \mathcal{B}^{HS}(\mathcal{H}^{\sigma_1}, \mathcal{H}^{\sigma_2})\right)$. Thus we define the time Fourier coefficients of R : $\forall\, l \in \mathbb{Z}^{\mathtt{n}}$ $\hat{R}(l)$ is the operator with matrix elements

$$(\hat{R}(l))_j^{j'} := \frac{1}{(2\pi)^{\mathtt{n}}} \int_{\mathbb{T}^{\mathtt{n}}} R_j^{j'}\, e^{-\mathrm{i}l\varphi}\ \ d\varphi. \tag{4.1}$$

Definition 4.3 (Class of operators)**.** *Given $s, \sigma \geq 0$, we consider the space*

$$\mathcal{M}^s_{\sigma_1,\sigma_2} := \mathcal{H}^s\left(\mathbb{T}^{\mathtt{n}};\ \mathcal{B}^{HS}(\mathcal{H}^{\sigma_1}, \mathcal{H}^{\sigma_2})\right), \tag{4.2}$$

endowed with the norm

$$\|R\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}} := \Big(\sum_{l\in\mathbb{Z}^{\mathtt{n}}} \langle l\rangle^{2s} \left(\|\widehat{R}(l)\|^{HS}_{\sigma_1,\sigma_2}\right)^2\Big)^{\frac{1}{2}}\,. \tag{4.3}$$

Definition 4.4 (Higher regularity norm)**.** *Let $\Omega_0 \subseteq \Omega$ and $R \in \mathcal{L}ip\left(\Omega_0; \mathcal{M}^s_{\sigma_1,\sigma_2}\right)$. Given $\beta > 0$, if $R(\tilde\omega)$ is such that*

$$R(\tilde\omega) \in \mathcal{L}ip\left(\Omega_0; \mathcal{M}^{s+\beta}_{\sigma_1,\sigma_2}\right), \quad \langle\nabla\rangle^\beta R(\tilde\omega) \in \mathcal{L}ip\left(\Omega_0; \mathcal{M}^s_{\sigma_1,\sigma_2}\right),$$

we define

$$\|R\|^{\mathrm{Lip}}_{\mathcal{W}^{s,\beta}_{\sigma_1,\sigma_2}} := \|R\|^{\mathrm{Lip}}_{\mathcal{M}^{s+\beta}_{\sigma_1,\sigma_2}} + \|\langle\nabla\rangle^\beta R\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_1,\sigma_2}}\,. \tag{4.4}$$

Definition 4.5 (Cutoffs). *Given an operator $R : L^2(\mathbb{T}^d) \to L^2(\mathbb{T}^d)$, for any $N \in \mathbb{N}$, we define the projector $\pi_N R$ as*

$$(\pi_N R)_j^{j'} := \begin{cases} R_j^{j'} & \text{if } |j-j'| < N \\ 0 & \text{if } |j-j'| \geq N \end{cases} \tag{4.5}$$

and we set $\pi_N^\perp R := R - \pi_N R$. For $R : \mathbb{T}^{\mathtt{n}} \to \mathcal{B}(L^2(\mathbb{T}^d))$, $\varphi \mapsto R(\varphi)$, we define $\Pi_N R$ as

$$\Pi_N R(\varphi) := \sum_{|l| \leq N} \pi_N \widehat{R}(l)\, e^{\mathrm{i} l \cdot \varphi} . \tag{4.6}$$

We then set $\Pi_N^\perp R := R - \Pi_N R$.

In the following lemma we point out a key estimate for the remainder $\Pi_N^\perp R$ of an operator R :

Lemma 4.6. *Let $R(\tilde{\omega}) \in \mathcal{M}^s_{\sigma_1,\sigma_2}$, $\tilde{\omega} \in \Omega_0 \subseteq \Omega$. Then for any $N > 0$,*

$$\|\Pi_N R\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_1,\sigma_2}}, \ \|\Pi_N^\perp R\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_1,\sigma_2}} \leq \|R\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_1,\sigma_2}} . \tag{4.7}$$

Moreover, let $\beta > 0$ and assume that $R(\tilde{\omega}) \in \mathcal{M}^{s+\beta}_{\sigma_1,\sigma_2}$, $\langle \nabla \rangle^\beta R(\tilde{\omega}) \in \mathcal{M}^s_{\sigma_1,\sigma_2}$, $\tilde{\omega} \in \tilde{\Omega}$. Then, for any $N \in \mathbb{N}$, one has $\Pi_N^\perp R(\tilde{\omega}) \in \mathcal{M}^s_{\sigma_1,\sigma_2}$ and

$$\|\Pi_N^\perp R\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_1,\sigma_2}} \leq N^{-\beta} \|R\|^{\mathrm{Lip}}_{\mathcal{W}^{s,\beta}_{\sigma_1,\sigma_2}} \tag{4.8}$$

Proof. Estimate (4.7) is a direct consequence of the definitions (4.3)-(4.6). We prove estimate (4.8). By (4.5), (4.6), one has

$$\begin{aligned} &\Pi_N^\perp R(\varphi) = R_{1,N}(\varphi) + R_{2,N}(\varphi) , \\ &R_{1,N}(\varphi) := \sum_{|l| \leq N} \pi_N^\perp \widehat{R}(l) e^{\mathrm{i} l \cdot \varphi} , \quad R_{2,N}(\varphi) := \sum_{|l| > N} \widehat{R}(l) e^{\mathrm{i} l \cdot \varphi} . \end{aligned} \tag{4.9}$$

We estimate separately the two terms in the above formula.
Estimate of $R_{1,N}$. For any $\ell \in \mathbb{Z}^{\mathtt{n}}$, one has

$$\begin{aligned} \Big(\|\pi_N^\perp \widehat{R}(l)\|^{HS}_{\sigma_1,\sigma_2} \Big)^2 &= \sum_{\substack{k,k' \in \mathbb{Z}^d \\ |k-k'| > N}} |\widehat{R}(l)_k^{k'}|^2 \langle k \rangle^{2\sigma_2} \langle k' \rangle^{-2\sigma_1} \\ &\leq N^{-2\beta} \sum_{k,k' \in \mathbb{Z}^d} \langle k-k' \rangle^{2\beta} |\widehat{R}(l)_k^{k'}|^2 \langle k \rangle^{2\sigma_2} \langle k' \rangle^{-2\sigma_1} \\ &= N^{-2\beta} \Big(\|\langle \nabla \rangle^\beta \widehat{R}(l)\|^{HS}_{\sigma_1,\sigma_2} \Big)^2 . \end{aligned}$$

Therefore, recalling (4.3), one gets the estimate

$$\|R_{1,N}\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}} \leq N^{-\beta} \|\langle \nabla \rangle^\beta R\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}} . \tag{4.10}$$

Estimate of $R_{2,N}$. The operator $R_{2,N}$ can be estimated as

$$\begin{aligned} \Big(\|R_{2,N}\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}} \Big)^2 &= \sum_{|l| > N} \langle l \rangle^{2s} \Big(\| \widehat{R}(l)\|^{HS}_{\sigma_1,\sigma_2} \Big)^2 \\ &\leq N^{-2\beta} \sum_{l \in \mathbb{Z}^{\mathtt{n}}} \langle l \rangle^{2(s+\beta)} \Big(\| \widehat{R}(l)\|^{HS}_{\sigma_1,\sigma_2} \Big)^2 \\ &= N^{-2\beta} \Big(\|R\|_{\mathcal{M}^{s+\beta}_{\sigma_1,\sigma_2}} \Big)^2 , \end{aligned}$$

implying that

$$\|R_{2,N}\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}} \le N^{-\beta}\|R\|_{\mathcal{M}^{s+\beta}_{\sigma_1,\sigma_2}}\,. \tag{4.11}$$

The claimed inequality then follows by (4.4), (4.9), (4.10) and (4.11). $\square$

4.2 Diagonalization

Fix $M>0$ and consider the matrix representation of the regularized operator $H^{(M)}$ of Theorem 3.6, namely

$$A_0 + P_0(\varphi), \quad A_0 := D_0 + Z \tag{4.12}$$

where D_0, Z and P_0 are the matrix representations of $\nu^{(0)}(\tilde\omega)\cdot\nabla$, $\varepsilon Z^{(M)}$ and $\mathcal{W}^{(M)}$ respectively.
Since $\nu^{(0)}\cdot\nabla$ and $Z^{(M)}$ depend only on ∇ and not on the x variable, their associated operators D_0 and Z remain diagonal if we pass to Fourier variables, so that we deal with the sum of a diagonal operator $A_0 = D_0 + Z$ and a perturbative term $P_0(\varphi)$ whose dependence on the angle φ we want to eliminate. More precisely

$$A_0 = \mathrm{diag}_{j\in\mathbb{Z}^d}\lambda_j^{(0)}, \quad \lambda_j^{(0)} := \mathrm{i}\nu^{(0)}\cdot j + z(j) \tag{4.13}$$

where we recall that $z\in\mathcal{L}ip(\Omega_{0,\gamma};OPS^{1-\mathfrak{e}})$. Before to state the reducibility theorem, we fix some constants. Given $\tau>0$ we define

$$\alpha := 12\tau + 7\,, \quad \beta := \alpha + 1\,, \quad m := 2\tau + 2 \tag{4.14}$$

Moreover, we fix the scale on which we perform the reducibility scheme as

$$N_k = N_0^{\left(\frac{3}{2}\right)^k} \quad \forall k\in\mathbb{N}, \quad N_{-1} := 1 \tag{4.15}$$

where for convenience we link N_0 and γ as

$$N_0 = \gamma^{-1} \tag{4.16}$$

where γ is the constant appearing in the definition (3.8) of the set $\Omega_{0,\gamma}$ (see also (4.22) in the theorem below). We also fix the number M of regularization steps in Theorem 3.6 as

$$M := 2m + 2\beta + [d/2] + 1\,. \tag{4.17}$$

Remark 4.7. *By Theorem 3.6 one has that $P_0 = \varepsilon\mathcal{W}^{(M)} \in \mathcal{C}^\infty(\mathbb{T}^{\mathfrak{n}};OPS^{-M})$. Since by (4.17), $M > 2m + 2\beta + \frac{d}{2}$, by applying Lemma A.14, one has that*

$$\|P_0\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\,,\ \|P_0\|^{\mathrm{Lip}}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} \lesssim_{s,\sigma} \varepsilon\,, \quad \forall s\ge 0\,, \quad \forall\sigma>0\,. \tag{4.18}$$

Theorem 4.8. (KAM reducibility) *Consider the system (3.17). Let $\gamma\in(0,1)$, $\tau>0$. Then for any $s>[\mathfrak{n}/2]+1, \sigma\ge 0$ there exist constants $C_0 = C_0(s,\sigma,\tau)>0$ large enough and $\delta=\delta(s,\sigma,\tau)\in(0,1)$ small enough such that, if*

$$N_0^{C_0}\varepsilon \le \delta \tag{4.19}$$

then, for all $k\ge 0$:

$(\mathbf{S1})_k$ *There exists a vector field*

$$H_k(\varphi) := A_k + P_k(\varphi)\,, \qquad \varphi \in \mathbb{T}^\nu\,, \tag{4.20}$$

$$A_k = \mathrm{diag}_{j\in\mathbb{Z}^d}\lambda_j^{(k)}, \quad \lambda_j^{(k)}(\tilde\omega) = \lambda_j^{(0)}(\tilde\omega) + \rho_j^{(k)}(\tilde\omega) \tag{4.21}$$

defined for all $\widetilde\omega \in \mathcal{O}_{k,\gamma}$*, where we set* $\mathcal{O}_{0,\gamma} := \Omega_{0,\gamma}$ *(see (3.8)) and for* $k \geq 1$*,*

$$\mathcal{O}_{k,\gamma} := \Big\{\widetilde\omega = (\omega,\nu) \in \mathcal{O}_{k-1,\gamma} : |\mathrm{i}\omega\cdot l + \lambda_j^{(k-1)}(\widetilde\omega) - \lambda_{j'}^{(k-1)}(\widetilde\omega)| \geq \frac{\gamma}{\langle l\rangle^\tau\langle j\rangle^\tau\langle j'\rangle^\tau}$$
$$\forall (l,j,j') \neq (0,j,j), \quad |l|, |j-j'| \leq N_{k-1}\Big\}\,. \tag{4.22}$$

For $k \geq 0$*, the Lipschitz functions* $\mathcal{O}_{k,\gamma} \to \mathbb{C}$*,* $\tilde\omega \mapsto \rho_j^{(k)}(\tilde\omega)$*,* $j \in \mathbb{Z}^d$ *satisfy*

$$\sup_{j\in\mathbb{Z}^d} \langle j\rangle^{2m}|\rho_j^{(k)}|^{\mathrm{Lip}} \lesssim_{s,\sigma} \varepsilon\,. \tag{4.23}$$

There exist a constant $C_* = C_*(s,\sigma,\beta,\tau,m) > 0$ *such that*

$$\|P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \leq C_* N_{k-1}^{-\alpha}\varepsilon, \quad \|P_k\|^{\mathrm{Lip}}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} \leq C_* N_{k-1}\varepsilon\,. \tag{4.24}$$

Moreover, for $k \geq 1$*,*

$$H_k(\varphi) = (\Phi_{k-1})_{\omega *}H_{k-1}(\varphi)\,, \quad \Phi_{k-1} := \mathrm{Id} + X_{k-1} \tag{4.25}$$

where the map X_{k-1} *satisfies the estimates*

$$\|X_{k-1}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma\pm m,\sigma\pm m}} \lesssim_s N_k^{4\tau+2}N_{k-1}^{-\alpha}\varepsilon\,. \tag{4.26}$$

Moreover, if $P_0(\varphi)$ *is real and reversible, for any* $k \geq 1$*,* $P_k(\varphi)$ *is real and reversible and*

$$\lambda_j^{(k)} \in \mathrm{i}\mathbb{R} \quad \forall j \in \mathbb{Z}^d. \tag{4.27}$$

$(\mathbf{S2})_k$ *For all* $j \in \mathbb{Z}^d$*, there exists a Lipschitz extension to the set* $\Omega_{0,\gamma}$ *defined in (3.8), that we denote by* $\widetilde\lambda_j^{(k)} : \Omega_{0,\gamma} \to \mathbb{C}$ *of* $\lambda_j^{(k)} : \mathcal{O}_{k,\gamma} \to \mathbb{C}$ *satisfying, for* $k \geq 1$*,*

$$|\widetilde\lambda_j^{(k)} - \widetilde\lambda_j^{(k-1)}|^{\mathrm{Lip}} \lesssim \langle j\rangle^{-2m}\|P_{k-1}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \lesssim_{s,\sigma} \langle j\rangle^{-2m}N_{k-2}^{-\alpha}\varepsilon\,. \tag{4.28}$$

We remark that $(\mathbf{S2})_k$ will be used to construct the final eigenvalues $\lambda_j^{(\infty)}$. The procedure will be to show that as $k \to \infty$, the sequence $\lambda_j^{(k)}$ admits a limit on $\Omega_{0,\gamma}$ and then to use the final value $\lambda_j^{(\infty)}$ in order to define the set in which reducibility holds (c.f. eq. (4.54)).

4.3 Proof of Theorem 4.8

Proof of $(\mathbf{Si})_0$, $i = 1,2$. Properties (4.20)-(4.24) hold by setting $\rho_j^{(0)} = 0$ for any $j \in \mathbb{Z}^d$, $N_{-1} := 1$ and recalling the estimate (4.18).

$(\mathbf{S2})_0$ holds, since the constant $\lambda_j^{(0)}$ is already defined for all $\tilde\omega \in \Omega_{0,\gamma}$ and in the real and reversible case it satisfies $\lambda_j^{(0)} \in \mathrm{i}\mathbb{R}$ in force of Proposition 3.2. Thus we simply set $\rho_j^{(0)} = 0$ for any $j \in \mathbb{Z}^d$.

4.3.1 The reducibility step: proof of $(\mathbf{Si})_{k+1}$, $i=1,2$.

PROOF OF $(\mathbf{S1})_{k+1}$.
We now describe the inductive step, showing how to define a transformation $\Phi_k := \mathrm{Id} + X_k$ so that the transformed vector field $H_{k+1}(\varphi) = (\Phi_k)_{\omega *} H_k(\varphi)$ has the desired properties. If we perform a change of coordinates of the form $u' := \Phi_k(\varphi)u$, $\Phi_k(\varphi) = \mathrm{Id} + X_k(\varphi)$ one has that $H_{k+1}(\varphi) = (\Phi_k)_{\omega *} H_k(\varphi)$ takes the form

$$\begin{aligned} H_{k+1}(\varphi) = A_k &+ \Phi_k(\varphi)^{-1}\big(\Pi_{N_k} P_k(\varphi) + [X_k(\varphi), A_k] - \omega\cdot\partial_\varphi X_k(\varphi)\big) \\ &+ \Phi_k(\varphi)^{-1}\big(\Pi_{N_k}^{\perp} P_k(\varphi) + P_k(\varphi) X_k(\varphi)\big) \end{aligned}$$

We look for a transformation $X_k(\varphi)$ solving the *homological equation*

$$\Pi_{N_k} P_k(\varphi) + [X_k(\varphi), A_k] - \omega\cdot\partial_\varphi X_k(\varphi) = \overline{P_k} \tag{4.29}$$

where $\overline{P_k}$ is a diagonal operator. Then we set

$$\begin{aligned} &A^{k+1} = A_k + \overline{P_k}, \quad P^{k+1} = \Pi_{N_k}^{\perp} P_k + P_k X_k + (\Phi_k^{-1} - \mathrm{Id})\big(\overline{P}_k + \Pi_{N_k}^{\perp} P_k + P_k X_k\big)\,, \\ &\overline{P}_k := \mathrm{diag}_{j\in\mathbb{Z}} (\widehat{P}_k)_j^j(0)\,. \end{aligned} \tag{4.30}$$

By formula (4.30) one obtains that

$$A_{k+1} := \mathrm{diag}_{j\in\mathbb{Z}^d} \lambda_j^{(k+1)}$$

where for any $j \in \mathbb{Z}^d$

$$\begin{aligned} \lambda_j^{(k+1)} &:= \lambda_j^{(k)} + \widehat{P}_k(0)_j^j = \mathrm{i}\nu^{(0)}\cdot j + \varepsilon z(j) + \rho_j^{(k+1)} \\ \rho_j^{(k+1)} &:= \rho_j^{(k)} + \widehat{P}_k(0)_j^j\,. \end{aligned} \tag{4.31}$$

In the real and reversible case, since P_k is real and reversible, by Lemma A.8 one has $\widehat{P}_k(0)_j^j \in \mathrm{i}\mathbb{R}$, and since $\lambda_j^{(k)}, \rho_j^{(k)} \in \mathrm{i}\mathbb{R}$ then one has that $\lambda_j^{(k+1)}, \rho_j^{(k+1)} \in \mathrm{i}\mathbb{R}$.

By the definition (4.31), applying Lemma A.15 and using the estimate (4.24), one gets that for any $j \in \mathbb{Z}^d$ for any $i \in \{0, 1, \ldots, k\}$

$$\begin{aligned} |\lambda_j^{(i+1)} - \lambda_j^{(i)}|^{\mathrm{Lip}} = |\rho_j^{(i+1)} - \rho_j^{(i)}|^{\mathrm{Lip}} &= |(\widehat{P}_i)_{jj}(0)|^{\mathrm{Lip}} \\ &\lesssim \langle j\rangle^{-2m} \|P_i\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \lesssim_{s,\sigma} \langle j\rangle^{-2m} N_{i-1}^{-\alpha}\varepsilon\,. \end{aligned} \tag{4.32}$$

We now verify the estimate (4.23) at the step $k+1$. By using a telescoping argument, recalling that $\rho_j^{(0)} = 0$ for any $j \in \mathbb{Z}^d$, one gets that

$$|\rho_j^{(k+1)}|^{\mathrm{Lip}} \le \sum_{i=0}^{k} |\rho_j^{(i+1)} - \rho_j^{(i)}|^{\mathrm{Lip}} \overset{(4.32)}{\lesssim_{s,\sigma}} \langle j\rangle^{-2m}\varepsilon \sum_{i=0}^{\infty} N_{i-1}^{-\alpha} \lesssim \langle j\rangle^{-2m}\varepsilon \tag{4.33}$$

since the series $\sum_{i=0}^{\infty} N_{i-1}^{-\alpha}$ is convergent (see (4.15)). Hence (4.23) is verified at the step $k+1$.

In the next lemma we will show how to solve the homological equation (4.29). This is the main lemma of the section.

Lemma 4.9. *Let $m > 2\tau + 1$. Then for any $\tilde{\omega} \in \mathcal{O}_{k+1,\gamma}$ (recall (4.22)), the homological equation*

$$[A_k, X_k] + \omega \cdot \partial_\varphi X_k = \Pi_{N_k} P_k - \overline{P}_k, \tag{4.34}$$

with

$$\overline{P}_k = \mathrm{diag}_{j \in \mathbb{Z}^d} \widehat{P_k}(0)_j^j \tag{4.35}$$

has a solution X_k defined on $\mathcal{O}_{k,\gamma}$ and satisfying the estimates

$$\|X_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma\pm m,\sigma\pm m}} \lesssim N_k^{4\tau+2} \|P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}, \tag{4.36}$$

$$\|\langle\nabla\rangle^\beta X_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma\pm m,\sigma\pm m}} \lesssim N_k^{4\tau+2} \|\langle\nabla\rangle^\beta P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}. \tag{4.37}$$

Furthermore, if P_k is real and reversible then X_k is real and reversibility preserving.

Proof. To simplify notations, here we drop the index k, namely we write A, P, X, λ_j, ρ_j instead of A_k, P_k, X_k, $\lambda_j^{(k)}$, $\rho_j^{(k)}$. Taking the (j, j') matrix element and the $l-$th Fourier coefficient of (4.34) we get:

$$\begin{aligned}
&(\mathrm{i}\omega \cdot l + \lambda_j - \lambda_{j'}) \; \widehat{X}(l)_j^{j'} = \widehat{P}(l)_j^{j'} \quad \text{if } 0 < |j - j'| < N, \; 0 < |l| < N \\
&\widehat{X}(l)_j^{j'} = 0 \quad \text{otherwise}
\end{aligned}$$

Since $\tilde{\omega} \in \mathcal{O}_{k+1,\gamma}$ one has

$$|\widehat{X}(l)_j^{j'}| \leq \frac{|\widehat{P}(l)_j^{j'}| |j|^\tau |j'|^\tau |l|^\tau}{\gamma}, \tag{4.38}$$

hence

$$\begin{aligned}
|\widehat{X}(l)_j^{j'}| &\lesssim \gamma^{-1} |\widehat{P}(l)_j^{j'}| |l|^\tau \langle j'\rangle^\tau \Big(\langle j'\rangle^\tau + |j - j'|^\tau\Big) \\
&\leq \gamma^{-1} |\widehat{P}(l)_j^{j'}| N^\tau \langle j'\rangle^\tau \Big(\langle j'\rangle^\tau + N^\tau\Big) \\
&\lesssim \gamma^{-1} |\widehat{P}(l)_j^{j'}| N^{2\tau} \langle j'\rangle^{2\tau},
\end{aligned} \tag{4.39}$$

Similarly, one gets

$$|\widehat{X}(l)_j^{j'}| \lesssim \gamma^{-1} |\widehat{P}(l)_j^{j'}| N^{2\tau} \langle j\rangle^{2\tau}. \tag{4.40}$$

Thus, recalling that $\tau < m$, (see (4.14)) the norm $\|X\|_{\mathcal{M}^s_{\sigma+m,\sigma+m}}$ is estimated by:

$$\begin{aligned}
\Big(\|X\|_{\mathcal{M}^s_{\sigma+m,\sigma+m}}\Big)^2 &= \sum_{l \in \mathbb{Z}^{\mathfrak{n}}} \langle l\rangle^{2s} \sum_{j,j' \in \mathbb{Z}^d} \langle j\rangle^{2(\sigma+m)} |\widehat{X}(l)_j^{j'}(l)|^2 \langle j'\rangle^{-2(\sigma+m)} \\
&\lesssim \gamma^{-2} N^{4\tau} \sum_{l \in \mathbb{Z}^{\mathfrak{n}}} \langle l\rangle^{2s} \sum_{j,j' \in \mathbb{Z}^d} \langle j\rangle^{2(\sigma+m)} |\widehat{P}(l)_j^{j'}|^2 \langle j'\rangle^{4\tau} \langle j'\rangle^{-2(\sigma+m)} \\
&\leq \gamma^{-2} N^{4\tau} \sum_{l \in \mathbb{Z}^{\mathfrak{n}}} \langle l\rangle^{2s} \sum_{j,j' \in \mathbb{Z}^d} \langle j\rangle^{2(\sigma+m)} |\widehat{P}(l)_j^{j'}|^2 \langle j'\rangle^{-2(\sigma-m)} \\
&= \gamma^{-2} N^{4\tau} \Big(\|P\|_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\Big)^2.
\end{aligned} \tag{4.41}$$

Similarly, one obtains

$$\Big(\|X\|_{\mathcal{M}^s_{\sigma-m,\sigma-m}}\Big)^2 \lesssim \gamma^{-2} N^{4\tau} \Big(\|P\|_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\Big)^2 . \tag{4.42}$$

To estimate the norm of the operator $\langle \nabla \rangle^\beta X$, we argue as in (4.39), (4.40) to get

$$\begin{aligned} \langle j-j'\rangle^\beta |\widehat{X}(l)_j^{j'}| &\lesssim N^{2\tau} \langle j\rangle^{2\tau} \langle j-j'\rangle^\beta |\widehat{P}(l)_j^{j'}|, \\ \langle j-j'\rangle^\beta |\widehat{X}(l)_j^{j'}| &\lesssim N^{2\tau} \langle j'\rangle^{2\tau} \langle j-j'\rangle^\beta |\widehat{P}(l)_j^{j'}|; \end{aligned} \tag{4.43}$$

hence we repeat the same argument of (4.41), (4.42) to get (4.37). Concerning Lipschitz estimates, recall that the eigenvalues λ_j, $j \in \mathbb{Z}^d$ have the expansion

$$\lambda_j(\tilde{\omega}) = \lambda_j^{(0)}(\tilde{\omega}) + \rho_j(\tilde{\omega}) = \mathrm{i}\nu^{(0)}(\tilde{\omega}) \cdot j + z(\tilde{\omega}, j) + \rho_j(\tilde{\omega}) .$$

By (3.7), (3.33) and the induction hypotheses (4.23) one has that for any $\widetilde{\omega}_1, \widetilde{\omega}_2 \in \Omega_\gamma$ and any $j, j' \in \mathbb{Z}^d$, one has

$$|(\lambda_j - \lambda_{j'})(\widetilde{\omega}_1) - (\lambda_j - \lambda_{j'})(\widetilde{\omega}_2)| \lesssim \varepsilon \gamma^{-1} \langle j-j'\rangle |\widetilde{\omega}_1 - \widetilde{\omega}_2| . \tag{4.44}$$

Hence, one uses $|l|, |j-j'| \le N$, (4.38), (4.44) and the inequality

$$|l|^{2\tau+1} |j|^{2\tau} |j'|^{2\tau} \lesssim_\tau N^{2\tau+1} |j|^{2\tau} \left(|j|^{2\tau} + N^{2\tau}\right) \lesssim N^{4\tau+1} \langle j\rangle^{4\tau}$$

to deduce the Lipschitz estimates as usual. By Lemma A.8 of the Appendix, if $A = \mathrm{diag}_{j\in\mathbb{Z}^d} \lambda_j$ and P are real and reversible one easily get that X is real and reversible too. $\square$

The estimate (4.26) follows from (4.36) and (4.24). Moreover, using that by (4.14), $\alpha > 6\tau + 3$ and by using the smallness condition (4.19), one gets that

$$\|X_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma\pm m,\sigma\pm m}} \le \delta(s) \tag{4.45}$$

for some $\delta(s) \in (0,1)$ small enough. Therefore, one can apply Lemma A.11 implying that

$$\begin{aligned} \|\Phi_k^{-1} - \mathrm{Id}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma\pm m,\sigma\pm m}} &\lesssim_{s,\sigma} \|X_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma\pm m,\sigma\pm m}} \overset{(4.36)}{\lesssim_s} N_k^{4\tau+2} \|P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \\ \|\langle\nabla\rangle^\beta (\Phi_k^{-1} - \mathrm{Id})\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma\pm m,\sigma\pm m}} &\lesssim_{s,\beta} \|\langle\nabla\rangle^\beta X_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma\pm m,\sigma\pm m}} \overset{(4.37)}{\lesssim_{s,\beta}} N_k^{4\tau+2} \|\langle\nabla\rangle^\beta P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \end{aligned} \tag{4.46}$$

In the next lemma, we obtain key estimates for the remainder term P_{k+1} defined in (4.30).

Lemma 4.10. *There exists a constant $C = C(s,\sigma,\tau) > 0$ such that the operator $P_{k+1}(\varphi)$ defined in* (4.30) *fulfills*

$$\begin{aligned} \|P_{k+1}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} &\le C\Big(N_k^{4\tau+2} \big(\|P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\big)^2 + N_k^{-\beta} \|P_k\|^{\mathrm{Lip}}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}}\Big), \\ \|P_{k+1}\|^{\mathrm{Lip}}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} &\le C \|P_k\|^{\mathrm{Lip}}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} . \end{aligned} \tag{4.47}$$

Furthermore, if $P_k(\varphi)$ is real and reversible then $P_{k+1}(\varphi)$ is real and reversible too.

Proof. By recalling the definition of P_{k+1} given in (4.30), using the inductive estimates (4.36), (4.37), and the estimate (4.46), by applying Lemma 4.6 and Lemma A.10 in the appendix, which gives an estimate of the product of operators, we get

$$\begin{aligned} \|P_{k+1}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} &\lesssim_{s,\sigma} N_k^{4\tau+2}\big(\|P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\big)^2 \\ &\quad + N_k^{-\beta}\left(\|P_k\|^{\mathrm{Lip}}_{\mathcal{M}^{s+\beta}_{\sigma-m,\sigma+m}} + \|\langle\nabla\rangle^\beta P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\right), \end{aligned} \tag{4.48}$$

$$\|P_{k+1}\|^{\mathrm{Lip}}_{\mathcal{M}^{s+\beta}_{\sigma-m,\sigma+m}} \lesssim_{s,\sigma} N_k^{4\tau+2}\|P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\|P_k\|^{\mathrm{Lip}}_{\mathcal{M}^{s+\beta}_{\sigma-m,\sigma+m}} + \|P_k\|^{\mathrm{Lip}}_{\mathcal{M}^{s+\beta}_{\sigma-m,\sigma+m}}, \tag{4.49}$$

$$\begin{aligned} \|\langle\nabla\rangle^\beta P_{k+1}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} &\lesssim_{s,\sigma} \|\langle\nabla\rangle^\beta P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \\ &\quad + N_k^{4\tau+2}\|P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\|\langle\nabla\rangle^\beta P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}. \end{aligned} \tag{4.50}$$

Recalling that $\|\cdot\|^{\mathrm{Lip}}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} = \|\cdot\|^{\mathrm{Lip}}_{\mathcal{M}^{s+\beta}_{\sigma-m,\sigma+m}} + \|\langle\nabla\rangle^\beta\cdot\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}$ and summing up the contribution of (4.49), (4.50), we get

$$\begin{aligned} \|P_{k+1}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} &\lesssim N_k^{4\tau+2}\big(\|P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\big)^2 + N_k^{-\beta}\|P_k\|^{\mathrm{Lip}}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}}, \\ \|P_{k+1}\|^{\mathrm{Lip}}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} &\lesssim N_k^{4\tau+2}\|P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\|P_k\|^{\mathrm{Lip}}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} + \|P_k\|^{\mathrm{Lip}}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}}. \end{aligned} \tag{4.51}$$

Furthermore, by using the smallness condition (4.19), recalling the definition (4.15), using that $\alpha > 6\tau+3$, taking N_0 large enough and ε small enough one gets that

$$N_k^{4\tau+2}\|P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \lesssim N_k^{4\tau+2}N_{k-1}^{-\alpha}\varepsilon \le 1$$

and then (4.51) implies the claimed estimate (4.47).
Finally, if P_k is real and reversible, then by Lemma 4.9, the operator X_k (and hence $\Phi_k = \mathrm{Id} + X_k$ and Φ_k^{-1}) is real and reversibility preserving. By the definition (4.30), one concludes that P_{k+1} is real and reversible. ☐

By Lemma 4.10 one has

$$\|P_{k+1}\|^{\mathrm{Lip}}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} \le C\|P_k\|^{\mathrm{Lip}}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} \overset{(4.24)}{\le} CC_*\varepsilon N_{k-1} \le C_*\varepsilon N_k$$

provided $CN_{k-1} \le N_k$ for any $k \ge 0$. This latter condition is verified by taking $N_0 > 0$ large enough. Furthermore

$$\begin{aligned} \|P_{k+1}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} &\le CN_k^{4\tau+2}\big(\|P_k\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}}\big)^2 + CN_k^{-\beta}\|P_k\|^{\mathrm{Lip}}_{\mathcal{W}^{s,\beta}_{\sigma-m,\sigma+m}} \\ &\overset{(4.24)}{\le} CN_k^{4\tau+2}C_*^2\varepsilon^2N_{k-1}^{-2\alpha} + CN_k^{-\beta}C_*N_{k-1}\varepsilon \le C_*\varepsilon N_k^{-\alpha} \end{aligned}$$

provided

$$2CN_k^{\alpha+4\tau+2}N_{k-1}^{-2\alpha}\varepsilon \le 1\,, \quad ,2CN_k^{\alpha-\beta}N_{k-1} \le 1 \quad \forall k \ge 0\,.$$

The above conditions are verified by (4.14), the smallness condition (4.19), recalling the definition (4.15) and taking ε small enough and N_0 large enough.

Hence the estimate (4.24) is proved at the step $k+1$. The proof of $(\mathbf{S1})_{k+1}$ is then concluded.

Proof of $(\mathbf{S2})_{k+1}$. By the estimate (4.32), on the set $\mathcal{O}_{k,\gamma}$, $\delta_j^{(k)} := \rho_j^{(k+1)} - \rho_j^{(k)}$ satisfies $|\delta_j^{(k)}|^{\mathrm{Lip}} \lesssim \langle j \rangle^{-2m} \| P_k \|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \lesssim_{s,\sigma} \langle j \rangle^{-2m} N_{k-1}^{-\alpha} \varepsilon$ for any $j \in \mathbb{Z}^d$. By the Kirszbraun Theorem (see Lemma M.5 in [KP03]), we extend the function $\delta_j^{(k)} : \mathcal{O}_{k,\gamma} \to \mathbb{C}$ to a function $\widetilde{\delta}_j^{(k)} : \Omega_{0,\gamma} \to \mathbb{C}$ which still satisfies the estimate $|\widetilde{\delta}_j^{(k)}|^{\mathrm{Lip}} \lesssim \langle j \rangle^{-2m} \| P_k \|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \lesssim_{s,\sigma} \langle j \rangle^{-2m} N_{k-1}^{-\alpha} \varepsilon$. Therefore, $(\mathbf{S2})_{k+1}$ follows by defining $\widetilde{\rho}_j^{(k+1)} := \widetilde{\rho}_j^{(k)} + \widetilde{\delta}_j^{(k)}$ and $\widetilde{\lambda}_j^{(k+1)} = \lambda_j^{(0)} + \widetilde{\rho}_j^{(k+1)}$ (note that $\lambda_j^{(0)}$ is already defined on $\Omega_{0,\gamma}$). Note that in the real and reversible case, one has that $\rho_j^{(k)}, \lambda_j^{(k)} : \mathcal{O}_{\gamma,k-1} \to \mathrm{i}\mathbb{R}$, $\widetilde{\rho}_j^{(k)}, \widetilde{\lambda}_j^{(k)} : \Omega_{0,\gamma} \to \mathrm{i}\mathbb{R}$, $\delta_j^{(k)} : \mathcal{O}_{k,\gamma} \to \mathrm{i}\mathbb{R}$ and hence $\widetilde{\lambda}_j^{(k+1)}$, $\widetilde{\rho}_j^{(k+1)} : \Omega_{0,\gamma} \to \mathrm{i}\mathbb{R}$.

4.4 Passing to the limit and completing the diagonalization procedure

By Theorem 4.8-$(\mathbf{S2})_k$, using a telescoping argument, for any $j \in \mathbb{Z}^d$, the sequence $(\widetilde{\rho}_j^{(k)})_{k \geq 0}$ is a Cauchy sequence w.r. to the norm $| \cdot |^{\mathrm{Lip}}$ in $\Omega_{0,\gamma}$, and hence it converges to $\rho_j^{(\infty)}$. The following estimates hold:

$$
|\widetilde{\rho}_j^{(k)} - \rho_j^{(\infty)}|^{\mathrm{Lip}} \lesssim_{s,\sigma} \langle j \rangle^{-2m} N_{k-1}^{-\alpha} \varepsilon \,, \qquad |\rho_j^{(\infty)}|^{\mathrm{Lip}} \lesssim_{s,\sigma} \langle j \rangle^{-2m} \varepsilon \,. \tag{4.52}
$$

Note that in the real and reversible case, $\rho_j^{(\infty)} : \Omega_{0,\gamma} \to \mathrm{i}\mathbb{R}$ for any $j \in \mathbb{Z}^d$.
We then define the *final eigenvalues* $\lambda_j^{(\infty)} : \Omega_{0,\gamma} \to \mathbb{C}$ as

$$
\lambda_j^{(\infty)} := \lambda_j^{(0)} + \rho_j^{(\infty)} \overset{(4.13)}{=} \mathrm{i}\nu^{(0)} \cdot j + z(j) + \rho_j^{(\infty)} \,, \quad j \in \mathbb{Z}^d \,. \tag{4.53}
$$

We then define

$$
\begin{aligned}
\mathcal{O}_{\infty,\gamma} := \Big\{ & \widetilde{\omega} = (\omega, \nu) \in \Omega_{0,\gamma} \ : \ |\mathrm{i}\omega \cdot l + \lambda_j^{(\infty)}(\widetilde{\omega}) - \lambda_{j'}^{(\infty)}(\widetilde{\omega})| \geq \frac{2\gamma}{\langle l \rangle^\tau \langle j \rangle^\tau \langle j' \rangle^\tau} \\
& \forall (l, j, j') \neq (0, j, j) \quad \Big\} \,.
\end{aligned} \tag{4.54}
$$

The following lemma holds.

Lemma 4.11. *One has* $\mathcal{O}_{\infty,\gamma} \subseteq \cap_{k \geq 0} \mathcal{O}_{k,\gamma}$.

Proof. We prove by induction that for any $k \geq 0$ one has $\mathcal{O}_{\infty,\gamma} \subseteq \mathcal{O}_{k,\gamma}$. For $k = 0$, it follows by definition that $\mathcal{O}_{\infty,\gamma} \subseteq \mathcal{O}_{0,\gamma}$ since $\mathcal{O}_{0,\gamma} = \Omega_{0,\gamma}$. Then assume that $\mathcal{O}_{\infty,\gamma} \subseteq \mathcal{O}_{k,\gamma}$ for some $k \geq 0$ and let us show that $\mathcal{O}_{\infty,\gamma} \subseteq \mathcal{O}_{k+1,\gamma}$. Let $\widetilde{\omega} = (\omega, \nu) \in \mathcal{O}_{\infty,\gamma}$. Since by the induction hypothesis $\widetilde{\omega} \in \mathcal{O}_{k,\gamma}$ one has that by Theorem 4.8-$(\mathbf{S1})_k$, $\lambda_j^{(k)}(\widetilde{\omega})$ is well defined and by Theorem 4.8-$(\mathbf{S2})_k$ one has that $\widetilde{\lambda}_j^{(k)}(\widetilde{\omega}) = \lambda_j^{(k)}(\widetilde{\omega})$ and $\widetilde{\rho}_j^{(k)}(\widetilde{\omega}) = \rho_j^{(k)}(\widetilde{\omega})$ (recall that $\lambda_j^{(k)} = \lambda_j^{(0)} + \rho_j^{(k)}$ and

$\widetilde{\lambda}_j^{(k)} = \lambda_j^{(0)} + \widetilde{\rho}_j^{(k)}$). We then have that for any $(l, j, j') \neq (0, j, j)$, $|l|, |j-j'| \leq N_k$,

$$\begin{aligned}|\mathrm{i}\omega \cdot l + \lambda_j^{(k)}(\widetilde{\omega}) - \lambda_{j'}^{(k)}(\widetilde{\omega})| &\geq |\mathrm{i}\omega \cdot l + \lambda_j^{(\infty)}(\widetilde{\omega}) - \lambda_{j'}^{(\infty)}(\widetilde{\omega})| - |\widetilde{\rho}_j^{(k)}(\widetilde{\omega}) - \rho_j^{(\infty)}(\widetilde{\omega})| \\ &\quad - |\widetilde{\rho}_{j'}^{(k)}(\widetilde{\omega}) - \rho_{j'}^{(\infty)}(\widetilde{\omega})| \\ &\overset{(4.52)}{\geq} \frac{2\gamma}{\langle l\rangle^\tau \langle j\rangle^\tau \langle j'\rangle^\tau} - \frac{C\varepsilon}{N_{k-1}^\alpha \min\{\langle j\rangle, \langle j'\rangle\}^{2m}} \\ &\geq \frac{\gamma}{\langle l\rangle^\tau \langle j\rangle^\tau \langle j'\rangle^\tau}\end{aligned}$$

provided

$$\frac{C\varepsilon \langle l\rangle^\tau \langle j\rangle^\tau \langle j'\rangle^\tau}{\gamma N_{k-1}^\alpha \min\{\langle j\rangle, \langle j'\rangle\}^{2m}} \leq 1\,. \tag{4.55}$$

Using that $|l|, |j-j'| \leq N_k$, $m > \tau$ and since

$$\langle j\rangle\langle j'\rangle \leq \big(\langle j-j'\rangle + \min\{\langle j\rangle, \langle j'\rangle\}\big)^2 \lesssim \langle j-j'\rangle^2 + \min\{\langle j\rangle, \langle j'\rangle\}^2 \lesssim N_k^2 + \min\{\langle j\rangle, \langle j'\rangle\}^2$$

one gets that

$$\frac{\langle l\rangle^\tau \langle j\rangle^\tau \langle j'\rangle^\tau}{\min\{\langle j\rangle, \langle j'\rangle\}^{2m}} \lesssim N_k^{3\tau}\,. \tag{4.56}$$

Therefore

$$\frac{C\varepsilon \langle l\rangle^\tau \langle j\rangle^\tau \langle j'\rangle^\tau}{\gamma N_{k-1}^\alpha \min\{\langle j\rangle, \langle j'\rangle\}^{2m}} \leq C'\varepsilon\gamma^{-1} N_k^{3\tau} N_{k-1}^{-\alpha} \leq 1$$

since $\alpha > \frac{9}{2}\tau$ (see (4.14)) and by taking ε small enough (see the smallness condition (4.19) and recall that $\gamma^{-1} = N_0$). Condition (4.55) is then verified and hence $\widetilde{\omega} \in \mathcal{O}_{k+1,\gamma}$. This concludes the proof of the lemma. $\square$

For any $k \geq 0$, $\widetilde{\omega} \in \mathcal{O}_{\infty,\gamma}$ we define the map

$$\mathcal{V}_k(\varphi, \widetilde{\omega}) \equiv \mathcal{V}_k(\varphi) := \Phi_0(\varphi) \circ \Phi_1(\varphi) \circ \ldots \circ \Phi_k(\varphi)\,. \tag{4.57}$$

Note that by Lemma 4.11 and Theorem 4.8 all the maps $\Phi_k(\varphi)$ are well defined for $\widetilde{\omega} \in \mathcal{O}_{\infty,\gamma}$.
The following lemma holds

Lemma 4.12. *The sequence $(\mathcal{V}_k)_{k\geq 0}$ converges to an invertible operator $\mathcal{V}_\infty$ in $\mathcal{L}ip\Big(\mathcal{O}_{\infty,\gamma}; \mathcal{H}^s\big(\mathbb{T}^{\mathtt{n}}; \mathcal{B}(\mathcal{H}^{\sigma\pm m}, \mathcal{H}^{\sigma\pm m})\big)\Big)$ and the operator $\mathcal{V}_\infty^{\pm 1} - \mathrm{Id}$ satisfies the estimate*

$$\|\mathcal{V}_\infty^{\pm 1} - \mathrm{Id}\|^{\mathrm{Lip}}_{H^s\left(\mathbb{T}^{\mathtt{n}}, \mathcal{B}(\mathcal{H}^{\sigma\pm m}, \mathcal{H}^{\sigma\pm m}\right)} \lesssim_{s,\sigma} N_0^{4\tau+2}\varepsilon\,.$$

Moreover in the real and reversible case, $\mathcal{V}_\infty^{\pm 1}$ is real and reversibility preserving.

Proof. The proof is based on standard arguments and therefore it is omitted (see for instance the proof of Corollary 4.1 in [Mon17a]). The presence of $N_0^{4\tau+2}$ in front of ε in the claimed inequality is due to the fact that (4.26) for $k = 0$ gives $\|\Phi_0 - \mathrm{Id}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma\pm m,\sigma\pm m}} \lesssim_{s,\sigma} N_0^{4\tau+2}\varepsilon$. $\square$

Lemma 4.13. *For any $\widetilde{\omega} \in \mathcal{O}_{\infty,\gamma}$, one has that $(\mathcal{V}_\infty)_{\omega *}(A_0 + P_0) = H_\infty$ (recall (4.12)) where the operator H_∞ is given by $H_\infty = \mathrm{diag}_{j\in\mathbb{Z}^d} \lambda_j^{(\infty)}$. Furthermore in the real and reversible case, the eigenvalues $\lambda_j^{(\infty)}$ are purely imaginary.*

Proof. By (4.25) and recalling the definition (4.57), one gets that for any $k \geq 1$

$$(\mathcal{V}_{k-1})_{\omega *}(A_0 + P_0(\varphi)) = H_k(\varphi) = A_k + P_k(\varphi)\,.$$

The claimed statement then follows by passing to the limit in the above identity, recalling the definition of A_k given in (4.21), the definition (4.53), the estimates (4.24), (4.52) and Lemma 4.12. ☐

4.5 Measure Estimates

In this section we show that the set $\mathcal{O}_{\infty,\gamma}$ defined in (4.54) has *large* Lebesgue measure. We prove the following

Proposition 4.14. *One has* $|\Omega \setminus \mathcal{O}_{\infty,\gamma}| \lesssim \gamma$.

Since $\Omega \setminus \mathcal{O}_{\infty,\gamma} = (\Omega \setminus \Omega_{0,\gamma}) \cup (\Omega_{0,\gamma} \setminus \mathcal{O}_{\infty,\gamma})$ and by Remark 3.3 one has that $|\Omega \setminus \Omega_{0,\gamma}| \lesssim \gamma$, it is enough to estimate the measure of the set $\Omega_{0,\gamma} \setminus \mathcal{O}_{\infty,\gamma}$. By the definition (4.54), one has that

$$\Omega_{0,\gamma} \setminus \mathcal{O}_{\infty,\gamma} = \bigcup_{\substack{(l,j,j') \in \mathbb{Z}^{n} \times \mathbb{Z}^d \times \mathbb{Z}^d \\ (l, j-j') \neq (0,0)}} \mathcal{R}_{ljj'}(\gamma)$$

$$\mathcal{R}_{ljj'}(\gamma) := \Big\{ \tilde{\omega} = (\omega, \nu) \in \Omega_{0,\gamma} : |\mathrm{i}\omega \cdot l + \lambda_j^{(\infty)}(\omega,\nu) - \lambda_{j'}^{(\infty)}(\omega,\nu)| < \frac{2\gamma}{\langle l \rangle^\tau \langle j \rangle^\tau \langle j' \rangle^\tau} \Big\} \tag{4.58}$$

Lemma 4.15. *One has* $|\mathcal{R}_{ljj'}(\gamma)| \lesssim \gamma \langle l \rangle^{-\tau} \langle j \rangle^{-\tau} \langle j' \rangle^{-\tau}$

Proof. By (4.53), one has that for any $j \in \mathbb{Z}^d$

$$\lambda_j^{(\infty)}(\omega,\nu) = \mathrm{i}\nu^{(0)}(\omega,\nu) \cdot j + z(j,\omega,\nu) + \rho_j^{(\infty)}(\omega,\nu)$$

where by the estimates (3.7), (3.33), one has $|\nu^{(0)} - \nu|^{\mathrm{Lip}(\gamma)} \lesssim \varepsilon$, $\sup_{j \in \mathbb{Z}^d} |\partial_\xi z(j)|^{\mathrm{lip}} \lesssim \varepsilon$. Then the map

$$\Psi : \Omega_{0,\gamma} \to \Psi(\Omega_{0,\gamma}), \quad (\omega,\nu) \mapsto (\omega, \nu^{(0)}(\omega,\nu))$$

is a Lipschitz homeomorphism with inverse given by $\Psi^{-1} : \Psi(\Omega_{0,\gamma}) \to \Omega_{0,\gamma}\,, \quad (\omega,\zeta) \mapsto \Psi^{-1}(\omega,\zeta)$ and satisfying

$$|\Psi^{-1} - \mathrm{Id}|^{\mathrm{sup}} \lesssim \varepsilon\,, \quad |\Psi^{-1} - \mathrm{Id}|^{\mathrm{lip}} \lesssim \varepsilon\gamma^{-1}\,. \tag{4.59}$$

Defining

$$a_j^{(\infty)}(\omega,\zeta) := \lambda_j^{(\infty)}(\Psi^{-1}(\omega,\zeta)), \quad j \in \mathbb{Z}^d$$

and

$$\widetilde{\mathcal{R}}_{ljj'}(\gamma) := \Big\{ (\omega,\zeta) \in \Psi(\Omega_{0,\gamma}) : |\mathrm{i}\omega \cdot l + a_j^{(\infty)}(\omega,\zeta) - a_{j'}^{(\infty)}(\omega,\zeta)| < \frac{2\gamma}{\langle l \rangle^\tau \langle j \rangle^\tau \langle j' \rangle^\tau} \Big\}$$

one has that

$$|\mathcal{R}_{ljj'}(\gamma)| \simeq |\widetilde{\mathcal{R}}_{ljj'}(\gamma)|, \tag{4.60}$$

then we estimate the measure of the set $\widetilde{\mathcal{R}}_{ljj'}(\gamma)$. The functions $a_j^{(\infty)}$ admit the expansion

$$a_j^{(\infty)}(\omega,\zeta) = \mathrm{i}\zeta \cdot j + z_\Psi(j,\omega,\zeta) + r_j^{(\infty)}(\omega,\zeta)$$

where

$$z_\Psi(j,\omega,\zeta) := z(j,\Psi^{-1}(\omega,\zeta)), \quad r_j^{(\infty)}(\omega,\zeta) := \rho_j^{(\infty)}(\Psi^{-1}(\omega,\zeta))\,.$$

By the estimate (4.59) and using the estimates (3.33), (4.52) on z and $\rho_j^{(\infty)}$, for $\varepsilon\gamma^{-1}$ small enough, one can easily deduce that

$$\sup_{j\in\mathbb{Z}^d} |\partial_\xi z_\Psi(j,\cdot)|^{\mathrm{Lip}} \lesssim \varepsilon, \quad \sup_{j\in\mathbb{Z}^d} \langle j\rangle^{2m} |r_j^{(\infty)}|^{\mathrm{Lip}} \lesssim \varepsilon\,. \tag{4.61}$$

Since $(l, j-j') \neq (0,0)$, we write

$$(\omega,\zeta) = (\omega(s),\zeta(s))) = \frac{(l,j-j')}{|(l,j-j')|} s + w, \quad w\in\mathbb{R}^{\mathfrak{n}+d}, \quad w\cdot(l,j-j') = 0$$

and we consider

$$\begin{aligned} f_{ljj'}(s) &:= \mathrm{i}\omega(s)\cdot l + a_j^{(\infty)}(\omega(s),\zeta(s)) - a_{j'}^{(\infty)}(\omega(s),\zeta(s)) \\ &= \mathrm{i}|(l,j-j')|s + z_\Psi(j,\omega(s),\zeta(s)) - z_\Psi(j',\omega(s),\zeta(s)) + r_j^{(\infty)}(\omega(s),\zeta(s)) - r_{j'}^{(\infty)}(\omega(s),\zeta(s))\,. \end{aligned}$$

Using the estimates (4.61) one obtains that

$$\begin{aligned} |f_{ljj'}(s_1) - f_{ljj'}(s_2)| &\geq \Big(|(l,j-j')| - C\varepsilon|j-j'| - C\varepsilon\Big)|s_1-s_2| \\ &\overset{|j-j'|\leq|(l,j-j')|}{\geq} \Big((1-C\varepsilon)|(l,j-j')| - C\varepsilon\Big)|s_1-s_2| \geq \frac{1}{2}|s_1-s_2| \end{aligned} \tag{4.62}$$

by taking ε small enough. This implies that

$$\Big|\Big\{s : |f_{ljj'}(s)| < \frac{2\gamma}{\langle l\rangle^\tau\langle j\rangle^\tau\langle j'\rangle^\tau}\Big\}\Big| \lesssim \frac{\gamma}{\langle l\rangle^\tau\langle j\rangle^\tau\langle j'\rangle^\tau}\,.$$

By a Fubini argument one gets that $|\widetilde{\mathcal{R}}_{ljj'}(\gamma)| \lesssim \gamma\langle l\rangle^{-\tau}\langle j\rangle^{-\tau}\langle j'\rangle^{-\tau}$. The claimed statement then follows by recalling (4.60). □

Proof of Proposition 4.14. By (4.58) and Lemma 4.15 one gets that

$$|\Omega_{0,\gamma}\setminus\mathcal{O}_{\infty,\gamma}| \lesssim \gamma \sum_{l\in\mathbb{Z}^{\mathfrak{n}}, j,j'\in\mathbb{Z}^d} \langle l\rangle^{-\tau}\langle j\rangle^{-\tau}\langle j'\rangle^{-\tau} \lesssim \gamma$$

since $\tau > \max\{\mathfrak{n}, d\}$. The claimed statement then follows by recalling that $|\Omega\setminus\Omega_{0,\gamma}| \lesssim \gamma$ and that $\Omega\setminus\mathcal{O}_{\infty,\gamma} = (\Omega\setminus\Omega_{0,\gamma}) \cup (\Omega_{0,\gamma}\setminus\mathcal{O}_{\infty,\gamma})$.

4.6 Proof of Theorem 2.4

We consider the composition

$$\mathcal{U}(\varphi) = \mathcal{V}(\varphi)\circ\mathcal{V}_\infty(\varphi), \quad \mathcal{V}(\varphi) := \mathcal{A}(\varphi)\circ e^{-\varepsilon\mathcal{G}_1(\varphi,\tilde\omega)}\circ\cdots\circ\ e^{-\varepsilon\mathcal{G}_M(\varphi,\tilde\omega)},$$

where $\mathcal{A}(\varphi)$ is defined in Section 3.1, the maps $e^{-\varepsilon\mathcal{G}_K}$ are constructed in Section 3.2 (see Theorem 3.6) and $\mathcal{V}_\infty$ is given in Lemma 4.12. By Section 3.1, Theorem 3.6 and Lemma 4.13, for any $\widetilde{\omega}\in\mathcal{O}_{\infty,\gamma}$, the map $\mathcal{U}(\varphi)$ conjugates the equation

(1.1) to the equation $\partial_t u = H_\infty u$ where H_∞ is the diagonal operator with eigenvalues $(\lambda_j^{(\infty)})_{j\in\mathbb{Z}^d}$. Let $0 < \mathfrak{a} < \frac{1}{C_0}$ and $N_0 := \frac{1}{\varepsilon^{\mathfrak{a}}}$ so that the smallness condition (4.19), i.e. $N_0^{C_0}\varepsilon \le \delta$ becomes

$$N_0^{C_0}\varepsilon = \varepsilon^{1-C_0\mathfrak{a}} \le \delta\,,$$

which is satisfied for ε small enough. Since $\gamma = N_0^{-1} = \varepsilon^{\mathfrak{a}}$, setting $\Omega_\varepsilon := \mathcal{O}_{\infty,\gamma}$, Proposition 4.14 implies that $\lim_{\varepsilon\to 0}|\Omega\setminus\Omega_\varepsilon| = 0$. The proof is therefore concluded.

4.7 Proof of Corollary 2.5

By Theorem 2.4, for any $\widetilde{\omega} = (\omega,\nu) \in \Omega_\varepsilon$ under the change of coordinates $u = \mathcal{U}(\omega t)v$, the Cauchy problem

$$\begin{cases}\partial_t u = \Big(\nu + \varepsilon V(\omega t, x)\Big)\cdot\nabla u + \varepsilon\mathcal{W}(\omega t)[u] \\ u(0,x) = u_0(x),\end{cases} \qquad u_0 \in \mathcal{H}^\sigma(\mathbb{T}^d) \tag{4.63}$$

is transformed into

$$\begin{cases}\partial_t v = H_\infty v \\ v(0) = v_0,\end{cases} \qquad v_0 := \mathcal{U}(0)^{-1}u_0\,. \tag{4.64}$$

Using that for any $\widetilde{\omega} = (\omega,\nu)\in\Omega_\varepsilon$, $\mathcal{U}(\varphi)$ is bounded and invertible on $\mathcal{H}^\sigma$ one gets that

$$\|\psi\|_{\mathcal{H}^\sigma} \lesssim_\sigma \|\mathcal{U}(\varphi)^{\pm 1}\psi\|_{\mathcal{H}^\sigma} \lesssim_\sigma \|\psi\|_{\mathcal{H}^\sigma}, \quad \forall \psi\in\mathcal{H}^\sigma(\mathbb{T}^d) \tag{4.65}$$

uniformly w.r.t. to $\varphi\in\mathbb{T}^{\mathtt{n}}$.
Case (1). If all the eigenvalues $\lambda_j^{(\infty)}$, $j\in\mathbb{Z}^d$ of the operator H_∞ are purely imaginary, the solution of the Cauchy problem (4.64) satisfies $\|v(t,\cdot)\|_{\mathcal{H}^\sigma} = \|v_0\|_{\mathcal{H}^\sigma}$ for any $t\in\mathbb{R}$. By the estimate (4.65) and recalling that $u = \mathcal{U}(\omega t)v$ one obtains the desired bound on the solution $u(t,x)$ of (4.63).
Case (2) Let $j\in\mathbb{Z}^d$ so that $\mathrm{Re}(\lambda_j^{(\infty)}) \neq 0$. Then for any $\alpha\in\mathbb{C}$, the solution v of the Cauchy problem (4.64) with initial datum $v_0(x) = \alpha e^{\mathrm{i}j\cdot x}$ is given by

$$v(t,x) = \alpha e^{\lambda_j^{(\infty)}t}e^{\mathrm{i}j\cdot x}\,.$$

Hence, setting $u_0 := \mathcal{U}(0)[\alpha e^{\mathrm{i}j\cdot x}] = \alpha\mathcal{U}(0)[e^{\mathrm{i}j\cdot x}]$, one has that the solution of the Cauchy problem (4.63) with such an initial datum u_0 is given by

$$u(t,x) = \mathcal{U}(\omega t)[\alpha e^{\lambda_j^{(\infty)}t}e^{\mathrm{i}j\cdot x}] = \alpha e^{\lambda_j^{(\infty)}t}\mathcal{U}(\omega t)[e^{\mathrm{i}j\cdot x}]\,.$$

Recalling (4.65) one gets that

$$\|u(t,\cdot)\|_{\mathcal{H}^\sigma} \simeq_\sigma C_j e^{\mathrm{Re}(\lambda_j^{(\infty)})t}\,.$$

This gives the growth for $t>0$ if $\mathrm{Re}\lambda_j^{(\infty)} > 0$ or for $t<0$ if $\mathrm{Re}\lambda_j^{(\infty)} > 0$. If there exists $\lambda_j^{(\infty)}$ with $\mathrm{Re}\lambda_j^{(\infty)} > 0$ and $\lambda_{j'}^{(\infty)}$ with $\mathrm{Re}\lambda_{j'}^{(\infty)} < 0$ then the solution with initial datum $\alpha e^{\mathrm{i}j\cdot x} + \beta e^{\mathrm{i}j'\cdot x}$ grows both as $t>0$ and as $t<0$.

A Appendix

To regularize (1.1), we make use of operators that are the flow at time $\tau \in [-1,\ 1]$ of the PDE

$$\partial_\tau u = G(\varphi) u$$

for a given pseudo differential operator $G(\varphi) \in OPS^\eta,\ \eta \leq 1$. An operator of this sort is denoted by $e^{\tau G}$. Thus, we state some of its main properties. The proof is a variant of Proposition A.2 of [MR17].

Lemma A.1. *Let $\eta < 1$ and $G \in \mathcal{C}^\infty(\mathbb{T}^\mathtt{n}; OPS^\eta)$ be such that $G(\varphi) + G(\varphi)^* \in OPS^0$ and let $e^{\tau G}$ be the flow of the autonomous PDE $\partial_\tau u = G(\varphi)u, \quad \tau \in [-1,1]$.*
(i) Then $e^{\tau G}(\varphi) \in \mathcal{B}(\mathcal{H}^\sigma, \mathcal{H}^\sigma)\ \forall \sigma > 0$.
(ii) $\forall \sigma > 0,\ \forall\ \alpha \in \mathbb{N}^\mathtt{n},\ \ \partial_\varphi^\alpha e^{\tau G}(\varphi) \in \mathcal{B}\left(\mathcal{H}^\sigma, \mathcal{H}^{\sigma - \eta|\alpha|}\right)$.
(iii) If $G \in \mathcal{L}ip(\Omega;\ \mathcal{C}^\infty(\mathbb{T}^\mathtt{n}; OPS^\eta))$, $\partial_\varphi^\alpha e^{\tau G}(\varphi,\omega) \in \mathcal{L}ip\left(\Omega;\ \mathcal{B}\left(\mathcal{H}^\sigma, \mathcal{H}^{\sigma-\eta|\alpha|-\eta}\right)\right) \quad \forall \sigma > 0,\ \forall\ \alpha \in \mathbb{N}^\mathtt{n}$.
Furthermore, if G is reversibility preserving (or real), $e^{\tau G}$ is reversibility preserving (resp. real) too.

Proof. Item (i) is a well known result. It is proved trough a Galerkin type approximation on the subspace E_N of the compact supported sequences $\{\hat{u}_k\}_{k \in \mathbb{Z}^d}$ such that $\hat{u}_k = 0 \quad$ if $|k| > N$. See [Tay91], Section 0.8, for details.
Items (ii) and (iii) follow as in Lemma A.3 in [BM16].
Reversibility preserving property: We remark that since

$$S \circ \partial_\tau = \partial_\tau \circ S,$$

one both has

$$\partial_\tau [S \circ e^{\tau G(\varphi)}] u = S \circ \partial_\tau \circ e^{\tau G(\varphi)} u = S \circ G(\varphi) e^{\tau G(\varphi)} u = G_\lambda(-\varphi) \circ S\ u$$

and

$$\partial_\tau [e^{\tau G(-\varphi)} \circ S] u = G(-\varphi) \circ S\ u.$$

Since $S \circ e^{\tau G(\varphi)}$ and $e^{\tau G(-\varphi)} \circ S$ solve the same initial value problem for all the functions $u(x)$, they must coincide. Thus we can deduce the reversibility preserving property for $e^{\tau G(\varphi)}$.
Reality: the proof of the reality can be done arguing similarly, using that since $G = \overline{G}$, then $e^{\tau G(\varphi)}$ and $\overline{e^{\tau G(\varphi)}}$ solve the same initial value problem. $\square$

Let $a : [-1,1] \times \mathbb{T}^\mathtt{n} \times \mathbb{T}^d \to \mathbb{R}^d$, $(\tau, \varphi, x) \mapsto a(\tau, \varphi, x)$ be a $\mathcal{C}^\infty$ function and let us consider the transport equation

$$\partial_\tau u = a(\tau, \varphi, x) \cdot \nabla u\,. \tag{A.1}$$

We denote by $\Phi(\tau_0, \tau, \varphi)$ the flow of the above PDE. For convenience, we set $\Phi(\tau, \varphi) \equiv \Phi(0, \tau, \varphi)$. The following lemma holds:

Lemma A.2. (i) *For any $\tau_0, \tau \in [0,1]$ the flow $\Phi(\tau_0, \tau, \varphi)$ of the equation* (A.1) *is a bounded linear operator on the Sobolev space $\mathcal{H}^s(\mathbb{T}^d)$ for any $s \geq 0$. Moreover the map $\varphi \mapsto \Phi(\tau_0, \tau, \varphi)$ is differentiable and for any $\alpha \in \mathbb{N}^\mathtt{n}$, the map $\mathbb{T}^\mathtt{n} \to \mathcal{B}(\mathcal{H}^{s+|\alpha|}, \mathcal{H}^s)$, $\varphi \mapsto \partial_\varphi^\alpha \Phi(\tau_0, \tau, \varphi)$ is bounded.*

(ii) Assume that $a = a(\widetilde{\omega}, \tau, \varphi, x)$, $(\widetilde{\omega}, \tau, \varphi, x) \in \Omega \times [0,1] \times \mathbb{T}^{\mathtt{n}} \times \mathbb{T}^d$ *is in* $\mathcal{L}ip\Big(\Omega\,, \mathcal{C}^\infty([0,1] \times \mathbb{T}^{\mathtt{n}} \times \mathbb{T}^d, \mathbb{R}^d)\Big)$*. Then for any* $\alpha \in \mathbb{N}^{\mathtt{n}}$ *the map*

$$\mathbb{T}^{\mathtt{n}} \to \mathcal{L}ip\Big(\Omega, \mathcal{B}(\mathcal{H}^{s+|\alpha|+1}, \mathcal{H}^s)\Big), \quad \varphi \mapsto \partial_\varphi^\alpha \Phi(\tau_0, \tau, \widetilde{\omega}, \varphi)$$

is bounded.

Remark A.3. *Let* $A(\varphi) \in \mathcal{L}ip\,(\Omega;\ \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}}; OPS^m))$ *and* $G \in \mathcal{L}ip\,(\Omega;\ \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}}; OPS^\eta))\,,$ *with* $\eta < 1$*. If* $\forall\ j \in \mathbb{N}$ *we define*

$$Ad_G^0 A = A, \quad Ad_G^{j+1} A = [G, Ad_G^j A], \tag{A.2}$$

then

$$Ad_G^j A \in \mathcal{L}ip\,\Big(\Omega;\ \mathcal{C}^\infty\,\Big(\mathbb{T}^{\mathtt{n}}; OPS^{m-j(1-\eta)}\Big)\Big) \quad \forall\ j \in \mathbb{N}.$$

The following simpler version of the Egorov theorem holds.

Lemma A.4. *Let* $A(\varphi) \in \mathcal{L}ip\,(\Omega;\ \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}}; OPS^m))$ *and* $G \in \mathcal{L}ip\,(\Omega;\ \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}}; OPS^\eta))\,,$ *with* $\eta < 1$ *and* G *such that* $G(\varphi) + G(\varphi)^* \in OPS^0$*. Then*

$$e^{\tau G} A e^{-\tau G} \in \mathcal{L}ip\,(\Omega;\ \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}}; OPS^m))\,.$$

Proof. This version of the Egorov theorem is actually simpler than the one stated in Theorem A.0.9 in [Tay91]. The reason is that the order of G is strictly smaller than one and hence one has the asymptotic expansion

$$e^{\tau G} A e^{-\tau G} \sim \sum_{j=0}^{\infty} A_j$$

with $A_j \in OPS^{m-j(1-\eta)}$ (see remark A.3). $\square$

Remark A.5. *Note that by Theorem A.0.9 in [Tay91] one has that if* $A \in \mathcal{L}ip\,(\Omega;\ \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}}; OPS^m))$*, then* $e^{i\tau\cdot K} A e^{-i\tau\cdot K},\ \partial_\tau^\alpha\left(e^{i\tau\cdot K} A e^{-i\tau\cdot K}\right) \in \mathcal{L}ip\,(\Omega;\ \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}}; OPS^m))$ $\forall \alpha \in \mathbb{N}^d$*.*

Lemma A.6. *Given* S *acting as* $S : u(x) \mapsto u(-x)$*, a linear operator* $A(\varphi)$ *satisfies the reversibility condition*

$$A(\varphi) \circ S = -S \circ A(-\varphi)$$

if and only if $\mathcal{A}(\tau, \varphi) := e^{i\tau\cdot K} A(\varphi) e^{-i\tau\cdot K}$ *satisfies the reversibility condition*

$$\mathcal{A}(\tau, \varphi) \circ S = -S \circ A(-\tau, -\varphi).$$

Analogously, $A(\varphi)$ *satisfies the reversibility preserving condition*

$$A(\varphi) \circ S = S \circ A(-\varphi)$$

if and only if $\mathcal{A}(\tau, \varphi) := e^{i\tau\cdot K} A(\varphi) e^{-i\tau\cdot K}$ *satisfies the reversibility preserving condition*

$$\mathcal{A}(\tau, \varphi) \circ S = S \circ A(-\tau, -\varphi).$$

Furthermore, $A(\varphi)$ *is real if and only if* $\mathcal{A}(\tau, \varphi)$ *is real.*

Proof. We only prove the statement concerning the reversibility. The statement on reality can be proved similarly.
A direct calculation shows that $e^{i\tau\cdot K}\circ S = S\circ e^{-i\tau\cdot K}$, hence, if $A(\varphi)$ is φ-reversible, one immediately gets

$$\mathcal{A}(\tau,\varphi)S = -S\mathcal{A}(-\tau,-\varphi).$$

Vice versa, $\mathcal{A}(\tau,\varphi)\circ S = -S\circ\mathcal{A}(-\tau,-\varphi)$ implies (for $\tau = 0$)

$$A(\varphi)\circ S = \mathcal{A}(0,\varphi)\circ S = -S\circ\mathcal{A}(0,-\varphi) = -S\circ A(-\varphi).$$

$\square$

Lemma A.7. *Let $\eta < 1$, $G \in \mathcal{C}^\infty(\mathbb{T}^n; OPS^\eta)$ with $G + G^* \in OPS^0$ and $A \in \mathcal{C}^\infty(\mathbb{T}^n; OPS^1)$. Then*

(i)

$$Ad_G^k A + (Ad_G^k A)^* \in OPS^{-(k-1)(1-\eta)} \quad \forall\ k \geq 1;$$

(ii) In particular,

$$\big(e^G A e^{-G} - A\big) + \big(e^G A e^{-G} - A\big)^* \in OPS^0.$$

Proof. Proof of (i).We argue by induction: if $k = 1$, one has

$$\begin{aligned}[][G,A] - [G^*,\ A^*] &= [G,A] + [A^*,G^*] \\ &= [G, A + A^*] + [A^*,\ G + G^*] \in OPS^0.\end{aligned}$$

Assume that for some $k \geq 1$

$$Ad_G^k A + (Ad_G^k A)^* \in OPS^{-(k-1)(1-\eta)}\,.$$

A direct calculation shows that

$$Ad_G^{k+1} A + (Ad_G^{k+1} A)^* = [G + G^*, Ad_G^k A] - [G^*, Ad_G^k A + (Ad_G^k A)^*]\,.$$

Since by Remark A.3 $Ad_G^k A, (Ad_G^k A)^* \in OPS^{1-k(1-\eta)}$ and using the induction hypothesis and that $G^* \in OPS^\eta$, $G + G^* \in OPS^0$, one obtains that $Ad_G^{k+1} A + (Ad_G^{k+1} A)^* \in OPS^{-k(1-\eta)}$.
Proof of (ii). $\forall\ M > 0$ one computes

$$e^{-G} A e^G - A = \sum_{k=1}^M \frac{Ad_G^k A}{k!} + \int_0^1 \frac{(1-s)^{M+1}}{(M+1)!} e^{-sG} Ad_G^{M+1} A e^{sG}.$$

By applying Remark A.3, choosing M large enough such that $\eta - (1-M)(1-\eta) < 0$, one gets that

$$e^{-G} A e^G - A + \big(e^{-G} A e^G - A\big)^* = \sum_{k=1}^M \frac{Ad_G^k A + (Ad_G^k A)^*}{k!} + OPS^0 \overset{item(i)}{\in} OPS^0\,.$$

$\square$

Lemma A.8. *Let $P \in \mathcal{M}^s_{\sigma_1,\sigma_2}$ and $\forall\ k,k' \in \mathbb{Z}^d$, $\forall\ l \in \mathbb{Z}^{\mathtt{n}}$ let $[\widehat{P}(l)]_k^{k'}$ be the $(k,k')-$th matrix element with respect to the basis $\{e^{ik\cdot x} \mid k \in \mathbb{Z}^d\}$ of the operator $\widehat{P}(l)$ defined as in (4.1). The following conditions hold:*

(a) $P(\varphi)$ is real if and only if

$$[\widehat{P}(l)]_k^{k'} = \left([\widehat{P}(-l)]_{-k}^{-k'}\right)^* ;$$

(b) $P(\varphi)$ is reversible if and only if

$$[\widehat{P}(l)]_k^{k'} = -[\widehat{P}(-l)]_{-k}^{-k'} ;$$

(c) $P(\varphi)$ is reversibility preserving if and only if

$$[\widehat{P}(l)]_k^{k'} = [\widehat{P}(-l)]_{-k}^{-k'} .$$

A.1 Tame estimates in $\mathcal{M}^s_{\sigma_1,\sigma_2}$

Lemma A.9. (i) *Let $\sigma_1,\sigma_2,\sigma_3 \in \mathbb{R}$ and let us assume that $\mathcal{R}$, $\mathcal{P}$ are linear operators such that*
$\mathcal{P} \in \mathcal{B}^{HS}(\mathcal{H}^{\sigma_1},\mathcal{H}^{\sigma_2})$, $\mathcal{R} \in \mathcal{B}^{HS}(\mathcal{H}^{\sigma_2},\mathcal{H}^{\sigma_3})$, Then $\mathcal{R}\mathcal{P} \in \mathcal{B}^{HS}(\mathcal{H}^{\sigma_1},\mathcal{H}^{\sigma_3})$ with

$$\|\mathcal{R}\mathcal{P}\|^{HS}_{\sigma_1,\sigma_3} \le \|\mathcal{R}\|^{HS}_{\sigma_2,\sigma_3}\|\mathcal{P}\|^{HS}_{\sigma_1,\sigma_2}.$$

(ii) *Let $\sigma_1,\sigma_2,\sigma_3 \in \mathbb{R}$, $\beta \ge 0$ and assume that $\langle\nabla\rangle^\beta\mathcal{P}, \mathcal{P} \in \mathcal{B}^{HS}(\mathcal{H}^{\sigma_1},\mathcal{H}^{\sigma_2})$., $\langle\nabla\rangle^\beta\mathcal{R}, \mathcal{R} \in \mathcal{B}^{HS}(\mathcal{H}^{\sigma_2},\mathcal{H}^{\sigma_3})$, Then $\langle\nabla\rangle^\beta\mathcal{R}\mathcal{P} \in \mathcal{B}^{HS}(\mathcal{H}^{\sigma_1},\mathcal{H}^{\sigma_3})$ with*

$$\|\langle\nabla\rangle^\beta\mathcal{R}\mathcal{P}\|^{HS}_{\sigma_1,\sigma_3} \lesssim_\beta \|\langle\nabla\rangle^\beta\mathcal{R}\|^{HS}_{\sigma_2,\sigma_3}\|\mathcal{P}\|^{HS}_{\sigma_1,\sigma_2} + \|\mathcal{R}\|^{HS}_{\sigma_2,\sigma_3}\|\langle\nabla\rangle^\beta\mathcal{P}\|^{HS}_{\sigma_1,\sigma_2}$$

Proof. We prove the estimate (ii). The estimate (i) can be proved by similar arguments (and it is actually simpler). We have that

$$\begin{aligned}
\left(\|\langle\nabla\rangle^\beta\mathcal{R}\mathcal{P}\|^{HS}_{\sigma_1,\sigma_3}\right)^2 &= \sum_{k,k',\in\mathbb{Z}^d}\langle k\rangle^{2\sigma_3}\langle k'\rangle^{-2\sigma_1}\big|\sum_{j\in\mathbb{Z}^d}\langle k-k'\rangle^\beta\mathcal{R}_k^j\mathcal{P}_j^{k'}\big|^2\\
&\lesssim_\beta \sum_{k,k',\in\mathbb{Z}^d}\langle k\rangle^{2\sigma_3}\langle k'\rangle^{-2\sigma_1}\big[\sum_{j\in\mathbb{Z}^d}\left(\langle k-j\rangle^\beta + \langle j-k'\rangle^\beta\right)|\mathcal{R}_k^j\mathcal{P}_j^{k'}|\big]^2\\
&\lesssim_\beta \sum_{k,k',\in\mathbb{Z}^d}\langle k\rangle^{2\sigma_3}\langle k'\rangle^{-2\sigma_1}\big[\sum_{j\in\mathbb{Z}^d}|\left(\langle\nabla\rangle^\beta\mathcal{R}\right)_k^j|\,|\mathcal{P}_j^{k'}|\big]^2\\
&+ \sum_{k,k',\in\mathbb{Z}^d}\langle k\rangle^{2\sigma_3}\langle k'\rangle^{-2\sigma_1}\big[\sum_{j\in\mathbb{Z}^d}|\mathcal{R}_k^j|\,|\left(\langle\nabla\rangle^\beta\mathcal{P}\right)_j^{k'}|\big]^2\\
&\lesssim_\beta \sum_{k,j\in\mathbb{Z}^d}\langle k\rangle^{2\sigma_3}|\left(\langle\nabla\rangle^\beta\mathcal{R}\right)_k^j|^2\langle j\rangle^{-2\sigma_2}\sum_{j,k'\in\mathbb{Z}^d}\langle j\rangle^{2\sigma_2}|\mathcal{P}_j^{k'}|^2\langle k'\rangle^{-2\sigma_1}\\
&+ \sum_{k,j\in\mathbb{Z}^d}\langle k\rangle^{2\sigma_3}|\mathcal{R}_k^j|^2\langle j\rangle^{-2\sigma_2}\sum_{j,k'\in\mathbb{Z}^d}\langle j\rangle^{2\sigma_2}|\left(\langle\nabla\rangle^\beta\mathcal{P}\right)_j^{k'}|^2\langle k'\rangle^{-2\sigma_1}\\
&\lesssim_\beta \left(\|\langle\nabla\rangle^\beta\mathcal{R}\|^{HS}_{\sigma_2,\sigma_3}\right)^2\left(\|\mathcal{P}\|^{HS}_{\sigma_1,\sigma_2}\right)^2 + \left(\|\mathcal{R}\|^{HS}_{\sigma_2,\sigma_3}\right)^2\left(\|\langle\nabla\rangle^\beta\mathcal{P}\|^{HS}_{\sigma_1,\sigma_2}\right)^2.
\end{aligned}$$

□

Lemma A.10.
(i) Let $s \geq s_0$, $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}$, $\mathcal{P}(\lambda) \in \mathcal{M}^s_{\sigma_1,\sigma_2}$, $\mathcal{R}(\lambda) \in \mathcal{M}^s_{\sigma_2,\sigma_3}$. Then $\mathcal{R}\mathcal{P}(\lambda) \in \mathcal{M}^s_{\sigma_1,\sigma_3}$ and

$$\|\mathcal{R}\mathcal{P}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_1,\sigma_3}} \lesssim_s \|\mathcal{R}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_2,\sigma_3}} \|\mathcal{P}\|^{\mathrm{Lip}}_{\mathcal{M}^{s_0}_{\sigma_1,\sigma_2}} + \|\mathcal{R}\|^{\mathrm{Lip}}_{\mathcal{M}^{s_0}_{\sigma_2,\sigma_3}} \|\mathcal{P}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_1,\sigma_2}}.$$

(ii) Let $\beta \geq 0$, $s \geq s_0$, $\sigma_1, \sigma_2, \sigma_3 \in \mathbb{R}$. Assume that $\langle \nabla \rangle^\beta \mathcal{P}(\lambda) \in \mathcal{M}^s_{\sigma_1,\sigma_2}$, $\langle \nabla \rangle^\beta \mathcal{R}(\lambda) \in \mathcal{M}^s_{\sigma_2,\sigma_3}$. Then $\langle \nabla \rangle^\beta \mathcal{R}\mathcal{P}(\lambda) \in \mathcal{M}^s_{\sigma_1,\sigma_3}$, and

$$\|\langle \nabla \rangle^\beta \mathcal{R}\mathcal{P}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_1,\sigma_3}} \lesssim_{s,\beta} \|\langle \nabla \rangle^\beta \mathcal{R}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_2,\sigma_3}} \|\mathcal{P}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_1,\sigma_2}} + \|\mathcal{R}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_2,\sigma_3}} \|\langle \nabla \rangle^\beta \mathcal{P}\|^{\mathrm{Lip}}_{\mathcal{M}^s_{\sigma_1,\sigma_2}}.$$

Proof. Estimate (i). By applying Lemma A.9-(i), one computes

$$\begin{aligned}
\left(\|(\mathcal{R}\mathcal{P})\|_{\mathcal{M}^s_{\sigma_1,\sigma_3}}\right)^2 &\leq \sum_{l \in \mathbb{Z}^n} \langle l \rangle^{2s} \left[\sum_{l' \in \mathbb{Z}^n} \|\widehat{\mathcal{R}}(l-l')\|^{HS}_{\sigma_2,\sigma_2} \|\widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_2} \right]^2 \\
&\lesssim_s \sum_{l \in \mathbb{Z}^n} \left[\sum_{l' \in \mathbb{Z}^n} \langle l' \rangle^s \|\widehat{\mathcal{R}}(l-l')\|^{HS}_{\sigma_2,\sigma_3} \|\widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_2} \right]^2 \\
&+ \sum_{l \in \mathbb{Z}^n} \left[\sum_{l' \in \mathbb{Z}^n} \langle l-l' \rangle^s \|\widehat{\mathcal{R}}(l-l')\|^{HS}_{\sigma_2,\sigma_3} \|\widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_2} \right]^2 \\
&\leq \sum_{l,l' \in Z^n} \langle l' \rangle^{2s} \langle l-l' \rangle^{2s_0} \left(\|\widehat{\mathcal{R}}(l-l')\|^{HS}_{\sigma_2,\sigma_3} \right)^2 \left(\|\widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_2} \right)^2 \\
&+ \sum_{l,l' \in Z^n} \langle l' \rangle^{2s_0} \langle l-l' \rangle^{2s} \left(\|\widehat{\mathcal{R}}(l-l')\|^{HS}_{\sigma_2,\sigma_3} \right)^2 \left(\|\widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_2} \right)^2 \\
&= \left(\|\mathcal{R}\|_{\mathcal{M}^{s_0}_{\sigma_2,\sigma_3}} \right)^2 \left(\|\mathcal{P}\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}} \right)^2 + \left(\|\mathcal{R}\|_{\mathcal{M}^s_{\sigma_2,\sigma_3}} \right)^2 \left(\|\mathcal{P}\|_{\mathcal{M}^{s_0}_{\sigma_1,\sigma_2}} \right)^2.
\end{aligned}$$

To get the required estimate in Lipschitz norm, it is sufficient to decompose

$$(\mathcal{R}\mathcal{P})(\lambda_2) - (\mathcal{R}\mathcal{P})(\lambda_1) = \mathcal{R}(\lambda_2)(\mathcal{P}(\lambda_2) - \mathcal{P}(\lambda_1)) + (\mathcal{R}(\lambda_2) - \mathcal{R}(\lambda_1))\mathcal{P}(\lambda_1)$$

and to apply the above inequality to both the terms of the right-hand side, taking respectively

$$\mathcal{R}(\lambda_2) \text{ as } \mathcal{R}, \quad \mathcal{P}(\lambda_2) - \mathcal{P}(\lambda_1) \text{ as } \mathcal{P}$$

and

$$\mathcal{R}(\lambda_2) - \mathcal{R}(\lambda_1) \text{ as } \mathcal{R}, \quad \mathcal{P}(\lambda_1) \text{ as } \mathcal{P}.$$

Estimate (ii). Arguing as before, one has

$$\begin{aligned}
\left(\|\langle \nabla \rangle^\beta (\mathcal{R}\mathcal{P})\|_{\mathcal{M}^s_{\sigma_1,\sigma_3}}\right)^2 &= \sum_{l \in Z^p} \langle l \rangle^{2s} \left(\|\langle \nabla \rangle^\beta \widehat{(\mathcal{R}\mathcal{P})}(l)\|^{HS}_{\sigma_1,\sigma_3}\right)^2 \\
&\leq \sum_{l \in \mathbb{Z}^n} \langle l \rangle^{2s} \left(\sum_{l' \in \mathbb{Z}^p} \|\langle \nabla \rangle^\beta \widehat{\mathcal{R}}(l-l') \widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_3} \right)^2 \\
&\lesssim_s \sum_{l,l' \in Zp} \langle l' \rangle^{2s} \langle l-l' \rangle^{2s} \left(\|\langle \nabla \rangle^\beta \widehat{\mathcal{R}}(l-l') \widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_3} \right)^2
\end{aligned} \tag{A.3}$$

where in the last inequality, we have used that

$$\langle l\rangle^{2s} \lesssim_s \langle l'\rangle^{2s} + \langle l-l'\rangle^{2s} \lesssim_s \langle l'\rangle^{2s}\langle l-l'\rangle^{2s}\,.$$

By applying Lemma A.9-(ii) (to estimate $\|\langle\nabla\rangle^\beta\widehat{\mathcal{R}}(l-l')\widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_3}$) one obtains that

$$\begin{aligned}
\Big(\|\langle\nabla\rangle^\beta(\mathcal{R}\mathcal{P})\|_{\mathcal{M}^s_{\sigma_1,\sigma_3}}\Big)^2 &\lesssim_{s,\beta} \sum_{l,l'\in Zp}\langle l'\rangle^{2s}\langle l-l'\rangle^{2s}\Big(\|\langle\nabla\rangle^\beta\widehat{\mathcal{R}}(l-l')\|^{HS}_{\sigma_2,\sigma_3}\Big)^2\Big(\|\widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_2}\Big)^2 \\
&\quad + \sum_{l,l'\in Zp}\langle l'\rangle^{2s}\langle l-l'\rangle^{2s}\Big(\|\widehat{\mathcal{R}}(l-l')\|^{HS}_{\sigma_2,\sigma_3}\Big)^2\Big(\|\langle\nabla\rangle^\beta\widehat{\mathcal{P}}(l')\|^{HS}_{\sigma_1,\sigma_2}\Big)^2 \\
&\lesssim_{s,\beta} \Big(\|\langle\nabla\rangle^\beta\mathcal{R}\|_{\mathcal{M}^s_{\sigma_2,\sigma_3}}\Big)^2\Big(\|\mathcal{P}\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}}\Big)^2 \\
&\quad + \Big(\|\mathcal{R}\|_{\mathcal{M}^s_{\sigma_2,\sigma_3}}\Big)^2\Big(\|\langle\nabla\rangle^\beta\mathcal{P}\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}}\Big)^2.
\end{aligned}$$

Concerning the Lipschitz estimates, as in the proof of (i) we write

$$\langle\nabla\rangle^\beta\left(\mathcal{R}\mathcal{P}(\lambda_2)-\mathcal{R}\mathcal{P}(\lambda_1)\right) = \langle\nabla\rangle^\beta\mathcal{R}(\lambda_2)\left(\mathcal{P}(\lambda_2)-\mathcal{P}(\lambda_1)\right) + \langle\nabla\rangle^\beta\left(\mathcal{R}(\lambda_2)-\mathcal{R}(\lambda_1)\right)\mathcal{P}(\lambda_1)$$

and we repeat the same argument with

$$\mathcal{R}(\lambda_2)\text{ as }\mathcal{R},\quad \mathcal{P}(\lambda_2)-\mathcal{P}(\lambda_1)\text{ as }\mathcal{P}$$

and

$$\mathcal{R}(\lambda_2)-\mathcal{R}(\lambda_1)\text{ as }\mathcal{R},\qquad \mathcal{P}(\lambda_1)\text{ as }\mathcal{P}.$$

□

Iterating the estimates of Lemma A.10, one gets for any $s\geq s_0$, $\sigma\in\mathbb{R}$, $n\geq 1$

$$\begin{aligned}
&\|\mathcal{R}^n\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}} \leq C(s)^n\|\mathcal{R}\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}}\left(\|\mathcal{R}\|^{\rm Lip}_{\mathcal{M}^{s_0}_{\sigma,\sigma}}\right)^{n-1}, \\
&\|\langle\nabla\rangle^\beta(\mathcal{R}^n)\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}} \leq C(s,\beta)^n\|\langle\nabla\rangle^\beta\mathcal{R}\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}}\left(\|\mathcal{R}\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}}\right)^{n-1}.
\end{aligned}\tag{A.4}$$

The following lemma holds:

Lemma A.11. *Let $s\geq s_0$, $\sigma\in\mathbb{R}$, $\beta\geq 0$ and $X(\lambda),\langle\nabla\rangle^\beta X(\lambda)\in\mathcal{M}^s_{\sigma,\sigma}$. Then there exists $\delta(s,\beta)\in(0,1)$ such that if $\|X\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}}\leq\delta(s,\beta)$, then $\Phi := {\rm Id}+X$ is invertible and its inverse Φ^{-1} satisfies the estimates*

$$\|\Phi^{-1}-{\rm Id}\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}} \lesssim_s \|X\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}},\quad \|\langle\nabla\rangle^\beta(\Phi^{-1}-{\rm Id})\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}} \lesssim_s \|\langle\nabla\rangle^\beta X\|^{\rm Lip}_{\mathcal{M}^s_{\sigma,\sigma}}\,.$$

Proof. By the Neumann series one has $\Phi^{-1}-{\rm Id} = \sum_{n\geq 1}(-1)^n X^n$. Then, applying the estimates (A.4) to each term X^n, the claimed statement follows. □

A.2 Other estimates in $\mathcal{M}^s_{\sigma_1,\sigma_2}$

Lemma A.12. *(i) Let $\sigma_1, \sigma_2 \in \mathbb{R}$ and $A \in \mathcal{B}\left(\mathcal{H}^{\sigma_1-\eta}, \mathcal{H}^{\sigma_2}\right)$, $\eta > \frac{d}{2}$, then*

$$\|A\|^{HS}_{\sigma_1,\sigma_2} \lesssim_\eta \|A\|_{\mathcal{B}(\mathcal{H}^{\sigma_1-\eta},\mathcal{H}^{\sigma_2})},$$

(ii) Let $\sigma_1, \sigma_2 \in \mathbb{R}$, $\beta \geq 0$, $\eta > \frac{d}{2}$. Then if $A \in \mathcal{B}(\mathcal{H}^{\sigma_1-\beta-\eta}, \mathcal{H}^{\sigma_2+\beta})$, one has

$$\|\langle \nabla \rangle^\beta A\|^{HS}_{\sigma_1,\sigma_2} \lesssim_\beta \|A\|_{\mathcal{B}(\mathcal{H}^{\sigma_1-\beta-\eta},\mathcal{H}^{\sigma_2+\beta})}.$$

Proof. Proof of (i). Let us consider $\forall\ k' \in \mathbb{Z}^d$ $u^{(k')} \in \mathcal{H}^{\sigma_1}$ defined by

$$\hat{u}^{(k')}_h = \begin{cases} \langle k' \rangle^{-(\sigma_1-\eta)} & \text{if } h = k' \\ 0 & \text{if } h \neq k'; \end{cases}$$

We have that

$$\begin{aligned}\sum_k \langle k \rangle^{2\sigma_2} |A^{k'}_k|^2 \langle k' \rangle^{-2(\sigma_1-\eta)} &= \|Au^{(k')}\|^2_{\mathcal{H}^{\sigma_2}} \\ &\leq \|A\|^2_{\mathcal{B}(\mathcal{H}^{\sigma_1-\eta},\mathcal{H}^{\sigma_2})} \|u^{(k')}\|^2_{\mathcal{H}^{\sigma_1-\eta}} \\ &= \|A\|^2_{\mathcal{B}(\mathcal{H}^{\sigma_1-\eta},\mathcal{H}^{\sigma_2})},\end{aligned}$$

since $\|u^{(k')}\|_{\mathcal{H}^{\sigma_1-\eta}} = 1$. Thus we deduce that $\forall\ k'$

$$\sum_k \langle k \rangle^{2\sigma_2} |A^{k'}_k|^2 \leq \|A\|^2_{\mathcal{B}(\mathcal{H}^{\sigma_1-\eta},\mathcal{H}^{\sigma_2})} \langle k' \rangle^{2(\sigma_1-\eta)}. \tag{A.5}$$

Let now u be a generic function in $\mathcal{H}^{\sigma_1}$: from (A.5) it follows that

$$\begin{aligned}\left(\|A\|^{HS}_{\sigma_1,\sigma_2}\right)^2 &= \sum_{k,k' \in \mathbb{Z}^d} \langle k \rangle^{2\sigma_2} |A^{k'}_k|^2 \langle k' \rangle^{-2\sigma_1} \\ &\leq \sum_{k' \in \mathbb{Z}^d} \langle k' \rangle^{2(\sigma_1-\eta)} \|A\|^2_{\mathcal{B}(\mathcal{H}^{\sigma_1-\eta},\mathcal{H}^{\sigma_2})} \langle k' \rangle^{-2\sigma_1} \\ &\lesssim_{\sigma_0} \|A\|^2_{\mathcal{B}(\mathcal{H}^{\sigma_1-\eta},\mathcal{H}^{\sigma_2})}.\end{aligned}$$

Proof of (ii). Using that for any $j, j' \in \mathbb{Z}^d$, $\langle j - j' \rangle^{2\beta} \lesssim_\beta \langle j \rangle^{2\beta} + \rangle j' \rangle^{2\beta} \lesssim_\beta \langle j \rangle^{2\beta} \langle j' \rangle^{2\beta}$, one gets that

$$\begin{aligned}\left(\|\langle \nabla \rangle^\beta A\|^{HS}_{\sigma_1,\sigma_2}\right)^2 &= \sum_{j,j' \in \mathbb{Z}^d} \langle j \rangle^{2\sigma_2} \langle j - j' \rangle^{2\beta} |A^{j'}_j|^2 \langle j' \rangle^{-2\sigma_1} \\ &\lesssim_\beta \sum_{j,j' \in \mathbb{Z}^d} \langle j \rangle^{2(\sigma_2+\beta)} |A^{j'}_j|^2 \langle j' \rangle^{-2(\sigma_1-\beta)} = \left(\|A\|^{HS}_{\sigma_1-\beta,\sigma_2+\beta}\right)^2.\end{aligned} \tag{A.6}$$

The claimed statement follows by applying item (i) (replacing σ_1 with $\sigma_1 - \beta$ and σ_2 with $\sigma_2 + \beta$). ∎

Lemma A.13. *(i) Let $A \in \mathcal{C}^s\Big(\mathbb{T}^{\mathfrak{n}}; \mathcal{B}\left(\mathcal{H}^{\sigma_1-\eta},\ \mathcal{H}^{\sigma_2}\right)\Big)$, $\sigma_1, \sigma_2 \in \mathbb{R}$, $\eta > \frac{d}{2}$ and $s \geq 0$. Then*

$$\|A\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}} \lesssim \|A\|_{\mathcal{C}^s\left(\mathbb{T}^{\mathfrak{n}};\mathcal{B}(\mathcal{H}^{\sigma_1-\eta},\ \mathcal{H}^{\sigma_2})\right)}.$$

(*ii*) *Let* $s \geq 0$, $\sigma_1, \sigma_2 \in \mathbb{R}$, $\beta \geq 0$, $\eta > \frac{d}{2}$ *and* $A \in \mathcal{C}^s\Big(\mathbb{T}^{\mathtt{n}}; \mathcal{B}(\mathcal{H}^{\sigma_1-\beta-\eta}, \mathcal{H}^{\sigma_2+\beta})\Big)$. *Then*

$$\|\langle\nabla\rangle^\beta A\|_{\mathcal{M}^s_{\sigma_1,\sigma_2}} \lesssim_\beta \|A\|_{\mathcal{C}^s\Big(\mathbb{T}^{\mathtt{n}}; \mathcal{B}(\mathcal{H}^{\sigma_1-\beta-\eta}, \mathcal{H}^{\sigma_2+\beta})\Big)}$$

Proof. The claimed statement follows recalling that $\mathcal{M}^s_{\sigma_1,\sigma_2} = \mathcal{H}^s\Big(\mathbb{T}^{\mathtt{n}}; \mathcal{B}^{HS}(\mathcal{H}^{\sigma_1}, \mathcal{H}^{\sigma_2})\Big)$, by applying Lemma A.12 and using that for every Banach space X one has that $\|\cdot\|_{H^s(\mathbb{T}^{\mathtt{n}};X)} \leq \|\cdot\|_{\mathcal{C}^s(\mathbb{T}^{\mathtt{n}};X)}$. $\square$

Lemma A.14. (*i*) *Let* $m \geq 0$, $A \in \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}}, OPS^{-\kappa})$, $\kappa > 2m + \frac{d}{2}$. *Then for any* $\sigma \in \mathbb{R}$,
$A \in \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}}; \mathcal{B}(\mathcal{H}^{\sigma+m-\kappa}, \mathcal{H}^{\sigma+m}))$ *and for any* $s \geq 0$

$$\|A\|_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \lesssim \|A\|_{\mathcal{C}^s(\mathbb{T}^{\mathtt{n}};\mathcal{B}(\mathcal{H}^{\sigma+m-\kappa},\mathcal{H}^{\sigma+m}))}$$

(*ii*) *Let* $m, \beta \geq 0$ *and* $A \in \mathcal{C}^\infty(\mathbb{T}^{\mathtt{n}}; OPS^{-\kappa})$, $\kappa > 2m + 2\beta + \frac{d}{2}$. *Then for any* $\sigma \in \mathbb{R}$,
$A \in \mathcal{C}^\infty\Big(\mathbb{T}^{\mathtt{n}}; \mathcal{B}\big(\mathcal{H}^{\sigma+m+\beta-\kappa}, \mathcal{H}^{\sigma+m+\beta}\big)\Big)$ *and for any* $s \geq 0$

$$\|\langle\nabla\rangle^\beta A\|_{\mathcal{M}^s_{\sigma-m,\sigma+m}} \lesssim_\beta \|A\|_{\mathcal{C}^s(\mathbb{T}^{\mathtt{n}};\mathcal{B}(\mathcal{H}^{\sigma+m+\beta-\kappa},\mathcal{H}^{\sigma+m+\beta}))} \,.$$

Proof. The statement (*i*) follows by applying Lemma A.13-(*i*) with $\sigma_1 = \sigma - m$, $\sigma_2 = \sigma + m$, $\eta = \kappa - 2m$.
The statement (*ii*) follows by applying Lemma A.13-(*ii*) with $\sigma_1 = \sigma - m$, $\sigma_2 = \sigma + m$, $\eta = \kappa - 2m - 2\beta$. $\square$

Lemma A.15. *Let* $\sigma \in \mathbb{R}$, $\kappa \geq 0$, $P(\lambda) \in \mathcal{B}^{HS}(\mathcal{H}^\sigma, \mathcal{H}^{\sigma+\kappa})$, $\lambda \in \Omega_o \subseteq \mathbb{R}^{\mathtt{n}+d}$. *Then* $\forall j \in \mathbb{Z}^d$ *its matrix elements* P^j_j *satisfy*

$$|P^j_j| \leq \|P\|^{HS}_{\sigma,\sigma+\kappa}\langle j\rangle^{-\kappa}, \quad |P^j_j|^{\mathrm{Lip}} \leq \|P\|^{HS,\mathrm{Lip}}_{\sigma,\sigma+\kappa}\langle j\rangle^{-\kappa}.$$

Proof. For any $j \in \mathbb{Z}^d$, one has

$$(\|P\|^{HS}_{\sigma,\sigma+\kappa})^2 = \sum_{k,k' \in \mathbb{Z}^d} \langle k\rangle^{2(\sigma+\kappa)}|P^{k'}_k|\langle k'\rangle^{-2\sigma} \geq \langle j\rangle^{2(\sigma+\kappa)}|P^j_j|\langle j\rangle^{-2\sigma} = \langle j\rangle^\kappa|P^j_j| \,.$$

The Lipschitz estimate follows arguing similarly by estimating $\frac{\|P(\lambda_1)-P(\lambda_2\|^{HS}_{\sigma,\sigma+\kappa}}{|\lambda_1-\lambda_2|}$ for any $\lambda_1, \lambda_2 \in \Omega_o$, $\lambda_1 \neq \lambda_2$. $\square$

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