Optimal Stability for the Inverse Problem of Multiple Cavities

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We deal with the determination of finitely many cavities in a planar heterogeneous conductor from one current and voltage measurement collected on the exterior boundary. We prove stability estimates under essentially minimal a priori regularity assumptions. We construct an explicit example showing the optimality of such stability estimates.

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1. INTRODUCTION

Consider a simply connected bounded open set $\Omega$ of the plane and a closed subset $\Sigma$ which is the union of finitely many, pairwise disjoint, closed simply connected subset $\sigma_i$, $i = 1, \ldots, n$, each of them coinciding with the closure of its interior (that is for any $i = 1, \ldots, n$, $\sigma_i = \overline{\sigma_i}$ where $\overline{\sigma_i}$ denotes the interior part of $\sigma_i$).

The Neumann problem

$$\begin{align*}
\text{div}(A \nabla u) &= 0 & \text{in} & \Omega \setminus \Sigma, \\
A \nabla u \cdot n &= 0 & \text{on} & \partial \sigma_i, i = 1, \ldots, n, \\
A \nabla u \cdot n &= \psi & \text{on} & \partial \Omega
\end{align*}$$

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provides a model for the electrostatic potential $u$ in the conductor $\Omega$ when each $\sigma_i$, $i = 1, \ldots, n$, represents a cavity inside it, $\Sigma$ being therefore a multiple cavity, $\psi$ is the applied current density and $A$ is the, possibly anisotropic and inhomogeneous, conductivity tensor. Here, we assume: $\int_{\Omega} \psi = 0$, $\psi \not\equiv 0$, $A \in L^\infty(\Omega)$ is uniformly elliptic, and we denote by $n$ the exterior normal to $\Omega \setminus \Sigma$.

We shall deal with the inverse problem of determining the multiple cavity $\Sigma$, when, given $\Omega$, $A$ and $\psi$, the potential $u$ is measured on an open portion $\Gamma$ of the exterior boundary $\partial \Omega$.

Such a problem presents some similarities with other well-known inverse boundary value problems.

(I) The so-called inverse problem of cracks is the one in which each component of $\Sigma$, instead of having interior points, is just a simple arc. For this case, it is well known that, either when $\Sigma$ is assumed to have only one component, [15], or finitely many ones, [6], two distinct, suitably chosen measurements are sufficient and necessary to uniquely determine the multiple crack. Moreover, stability estimates for the determination of a single crack have been obtained. See [8] for an updated account on such results.

(II) Consider for simplicity $A \equiv I$, then (1.1) can be viewed as the limit as $k \to 0$ of the problems

$$
\begin{align*}
\text{div}((1 + (k-1) \chi_\Sigma) \nabla u_k) &= 0 \quad \text{in } \Omega, \\
\nabla u_k \cdot n &= \psi \quad \text{on } \partial \Omega.
\end{align*}
$$

Here $\chi_\Sigma$ is the characteristic function of $\Sigma$. In this case $\Sigma$ represents an inclusion in $\Omega$, whose conductivity gets smaller as $k \to 0$. When $k \neq 1$ is fixed, the relative inverse problem of determining $\Sigma$ is known as the inverse conductivity problem with one measurement. Plenty of papers have been devoted to this problem but, still, the uniqueness question remains open. For references, see, for instance, [7].

Contrary to the above stated inverse problems, in the case of cavities the uniqueness with a single measurement is nearly straightforward. Let us outline a proof, suited to the two-dimensional setting, which has the advantage of requiring very little about the regularity of the conductivity $A$ and of the boundaries of the conductor $\Omega$ and of the cavities. Let $v$ be a stream function associated to $u$, (a notion that generalizes the one of harmonic conjugate), namely a function satisfying

$$
\nabla v = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} A \nabla u \quad \text{almost everywhere in } \Omega \setminus \sigma.
$$
It can be seen that, due to (1.1), such a function exists, it is single valued, and it satisfies, for some unknown constants \( c_i, i = 1, ..., n \),

\[
\begin{align*}
\text{div}(B \nabla v) &= 0 \quad \text{in } \Omega \setminus \Sigma, \\
v &= c_i \quad \text{on } \partial \sigma_i, \ i = 1, ..., n, \\
v &= \Psi \quad \text{on } \partial \Omega,
\end{align*}
\]

(1.3)

and also, in a weak sense,

\[
\int_{\beta} B \nabla v \cdot \nu = 0 \quad \text{for every smooth Jordan curve } \beta \subset \Omega \setminus \Sigma.
\]

(1.4)

Here \( B = (\det A)^{-1} A^T \), \( (\cdot)^T \) denoting transpose, and \( \Psi \) is an antiderivative of \( \psi \) along \( \partial \Omega \). See, for details, \([6]\) and also \([8]\).

Notice also that \( v \) can be continuously extended to \( \Omega \) by setting \( v|_{\partial \rho_i} = v|_{\partial \sigma_i} = c_i \) for any \( i = 1, ..., n \).

Suppose now \( \Sigma' \) is another multiple cavity and let \( u', v' \) be the solution to (1.1), (1.3) respectively when \( \Sigma \) is replaced with \( \Sigma' \). If \( \Sigma, \Sigma' \) give rise to the same boundary measurement, that is \( u|_{\rho_i} = u'|_{\rho_i} \), then \( v, v' \) have the same Cauchy data on \( \Gamma \). By the unique continuation property, and the maximum principle, one obtains \( v \equiv v' \) in \( \Omega \) (see again \([6]\) for details).

Hence, if we had \( \Sigma' \setminus \Sigma \neq \emptyset \) then \( \Sigma' \) would have some of its interior points inside \( \Omega \setminus \Sigma \) and we would obtain \( v \equiv v' \equiv \text{const.} \) on an open subset of \( \Omega \setminus \Sigma \). Again by unique continuation, we obtain \( v \equiv \text{const.} \) which is impossible since \( \psi \neq 0 \).

The aim of this paper is to prove stability estimates for the inverse problem of cavities under very general assumptions on the conductivity, on the regularity of the boundaries of \( \Omega \) and of the cavities and on the prescribed current density \( \psi \). We shall prove our stability results under different type of regularity assumptions on \( \Sigma \), see Theorem 2.1. Moreover we shall show their optimality by an explicit example, Theorem 4.1. See also \([5]\), where an example, different in various respects, but of the same nature, was presented for the so-called material loss (or corrosion) problem.

Our approach has some common features with the one used in \([8]\) for the stability estimates in the determination of a single crack from two measurements and it will require a sequence of intermediate steps: first of all we shall prove an inverse Hölder estimate on the function \( f = u + iv \), see Theorem 3.3. Then, according to the different a priori regularity assumptions on \( \Sigma \), we shall derive stability estimates for a Cauchy type problem, see Proposition 3.5 and 3.6, which, along with the inverse Hölder estimate previously recalled, will allow us to conclude the proof of Theorem 2.1.
However, there are various relevant additional difficulties. First, the possible presence of multiple cavities introduces technical complications in the treatment of quasiconformal mappings between multiply connected domains, a crucial step in this treatment being Lemma 3.2 which provides estimates on the size deformation of a circular domain (that is a multiply connected domain bounded by finitely many circles) under the effect of a $k$-quasiconformal mapping.

Second, we recall that in the stability estimates obtained in [4], [8] for the crack problem, and, more specifically, in [20], for the problem of a cavity, the prescribed Neumann data $\psi$ was assumed to satisfy certain conditions on its sign changes which enabled to show that, in a generalized sense, the corresponding potential $u$ had no interior critical points. Here no such assumption on $\psi$ will be made, in fact any nontrivial data $\psi$ will suit our purpose. On the other hand, we shall obtain, as it is reasonably expected, that the constants in the stability estimates depend on the oscillation character of $\psi$. That is, the less is the oscillation of $\psi$ the better is the stability. Roughly speaking, such an oscillation character will be controlled by the quantity $H_2$, appearing in (2.6) below, which dominates a ratio of two different norms for $\psi$. We shall prove that, under such a bound on the oscillation of $\psi$, taking $f = u + iv$ where $v$ is the above mentioned stream function associated to $u$, and fixing any $z^0 \in \Omega \setminus \Sigma$ then, locally, $|f - f(z^0)|$ can be dominated from below by an explicit function vanishing at finitely many points and with finite order (see Theorem 3.3).

We believe that such type of estimate, which to the authors' knowledge is new in the context of elliptic equations in divergence form and measurable coefficients in two variables, may prove to be useful also for other purposes and especially for other inverse boundary value problems.

We wish to mention that stability estimates for a strictly related problem of determination of an interior boundary have been obtained in [13], where they consider the case of a single cavity $\sigma$, they assume the conductivity $A$ to be homogeneous, $A \equiv I$, and the regularity assumptions on the boundaries are slightly different.

Let us also mention that a companion problem, in a different topological setting, when $\sigma \subset \Omega$ intersects $\partial \Omega$ on a nontrivial arc in such a way that $\Omega \setminus \sigma$ remains simply connected, has already been treated by various authors, [5], [9], [10], [12, 14], [17] and [20, 21].

An extended account on these and other related topics can be found in the doctoral dissertation [22] of which the present research constitutes a part.

The plan of the paper is the following. In Section 2, we present our basic assumptions and state our main Theorem 2.1, which contains our stability results under various type of a priori assumptions on $\Sigma$. Section 3 contains the proof of Theorem 2.1 in its various steps. Finally, in Section 4, we illustrate an example showing the optimality of the stability estimates (II) and (III).
in Theorem 2.1. In fact such an example provides a much stronger statement, Theorem 4.1, showing that logarithmic stability is the best possible also when all pairs of boundary measurements \( \{ u_{\|\alpha}, A v_{\|\alpha} \} \) are available.

2. THE STABILITY THEOREM

Given \( z \in \mathbb{C} \) and \( r > 0 \), we denote by \( B_r(z) \) the open disc with center \( z \) and radius \( r \) and by \( B_r[z] \) its closure, that is \( B_r[z] = B_r(z) \).

We shall need, in several places, quantitative notions of smoothness for the boundary of \( \Omega \) and the boundaries of the cavities. Such assumptions can be summarized as follows.

Given an integer \( k = 0, 1, 2, \ldots \), a number \( \alpha, 0 < \alpha \leq 1 \), and a finite family of simple closed curves \( \gamma_i, i = 1, \ldots, n \), such that the domains bounded by each \( \gamma_i \) are pairwise disjoint, we shall say that this family is \( C^{k,\alpha} \) with constants \( d, M > 0 \) if for any \( z \in \bigcup_{i=1}^n \gamma_i \), \( \bigcup_{i=1}^n \gamma_i \cap B_r(z) \) is given, up to a rigid transformation, by the graph \( \{ y = \phi(x), x^2 + y^2 < \delta^2 \} \) of a \( C^{k,\alpha} \) function \( \phi \) on \( [-\delta, \delta] \) such that \( ||\phi||_{C^{k,\alpha}([-\delta, \delta])} \leq M \).

We shall especially focus on the case \( k = 0, \alpha = 1 \), in which case we shall speak of Lipschitz curves.

Use will be also made of the following notion. Given two finite families of simple closed curves, \( \gamma_i, i = 1, \ldots, n \), and \( \gamma'_j, j = 1, \ldots, m \), both satisfying the assumption that the domains bounded by each of the curves of the same family are pairwise disjoint, we shall say that they are Relative Lipschitz Graphs (RLG for short) with constants \( d, M \) if for every \( z \in \bigcup_{i=1}^n \gamma_i \cup \bigcup_{j=1}^m \gamma'_j \), there exists a coordinate system \((x, y)\) with origin in \( z \) such that with respect to these coordinates \( \bigcup_{i=1}^n \gamma_i \cap B_r(z) = \{ y = \phi(x), x^2 + y^2 < \delta^2 \} \) and \( \bigcup_{j=1}^m \gamma'_j \cap B_r(z) = \{ y = \phi'(x), x^2 + y^2 < \delta^2 \} \) where \( \phi \) and \( \phi' \) are Lipschitz on \( [-\delta, \delta] \) with Lipschitz norm bounded by \( M \). Moreover we assume that \( \{ y < \phi(x), x^2 + y^2 < \delta^2 \} \) and \( \{ y < \phi'(x), x^2 + y^2 < \delta^2 \} \) are contained in one of the domains bounded by \( \gamma_i, i = 1, \ldots, n \), and \( \gamma'_j, j = 1, \ldots, m \), respectively. We remark that either \( \bigcup_{i=1}^n \gamma_i \cap B_r(z) \) or \( \bigcup_{j=1}^m \gamma'_j \cap B_r(z) \) might be empty. In this case it is enough to have \( \phi \) (or respectively \( \phi' \)) larger than or equal to \( \delta \), if \( B_r(z) \) is contained in one of the domains bounded by a curve belonging to the first (or respectively to the second) family, or otherwise \( \phi \leq \delta \) (or \( \phi' \leq \delta \) respectively).

Before stating our main Theorem, let us illustrate the main a priori assumptions.

Prior Information on the Domain

Given positive constants \( \delta, M \), and \( L \), let \( \Omega \) be a bounded, simply connected domain in \( \mathbb{R}^2 \) whose boundary \( \partial \Omega \) is a simple, closed Lipschitz curve with constants \( \delta, M \) and length bounded by \( L \).
Let us observe that by the a priori information we may deduce that the length of $W$ is greater than or equal to $d$ and there exists a positive constant $M_1$ depending on $d$, $M$ and $L$ only such that

$$\text{length}_{\partial W}(z_0, z_1) \leq M_1 |z_0 - z_1|, \quad \text{for every } z_0, z_1 \in \partial W,$$

where $\text{length}_{\partial W}(z_0, z_1)$ denotes the length of the smallest arc in $\partial W$ connecting $z_0$ to $z_1$.

The bound on the length of $\partial W$ allows us to obtain an upper bound on the diameter, and consequently on the measure, of $\Omega$ by a constant depending on $L$ only. Finally, the measure of $\Omega$ can be bounded from below by a positive constant depending on $d$ and $M$ only.

**Prior Information on the Conductivity**

Given $\lambda$, $\Delta > 0$, let $A = A(z)$, $z \in \Omega$, be a $2 \times 2$ matrix with bounded measurable entries verifying

$$\begin{align*}
(a) & \quad A(z) \xi \cdot \xi \geq \lambda |\xi|^2 \text{ for every } \xi \in \mathbb{R}^2 \text{ and for a.e. } z \in \Omega; \\
(b) & \quad |a_{ij}(z)| \leq A \text{ for every } i, j = 1, 2 \text{ and for a.e. } z \in \Omega.
\end{align*}$$

**Prior Information on the Multiple Cavity**

We shall assume that $\Sigma \subset \Omega$ is the union of finitely many, pairwise disjoint, closed and not empty sets $\sigma_i$, $i = 1, \ldots, n$, $n \geq 1$, each of them bounded by a simple closed curve $\gamma_i$. Concerning the regularity of the curves $\gamma_i$, we shall pose various alternative assumptions in the statement of Theorem 2.1.

Moreover we shall assume

$$\text{dist}(z, \partial \Omega) \geq \delta \quad \text{for any } z \in \Sigma.$$

We remark that this kind of definition guarantees that $\Omega \setminus \Sigma$ is a connected open set.

**Prior Information on the Boundary Data**

The current density on the boundary will be given by a non trivial function $\psi \in L^2(\partial \Omega)$ with zero mean, that is $\int_{\partial \Omega} \psi = 0$.

We define the antiderivative along $\partial \Omega$ of $\psi$ as

$$\Psi(s) = \int_{s_0}^s \psi(t) \, dt,$$

where the indefinite integral is taken with respect to the arclength on $\partial \Omega$ oriented in the counterclockwise direction.
We recall that the function $\mathcal{Y}$ is defined up to an additive constant. For the time being, we normalize $\mathcal{Y}$ in such a way that $\int_{\Omega} \mathcal{Y} = 0$ and for this choice of the additive constant we prescribe that, for given constants $H$, $H_1 > 0$, we have

\begin{align}
(2.5) & \quad \|\mathcal{Y}\|_{L^2(\partial \Omega)} \leq H; \\
(2.5)(b) & \quad \|\mathcal{Y}\|_{L^2(\partial \Omega)} \geq H_1.
\end{align}

From (2.5)(a) and (2.5)(b) we immediately infer

\begin{align}
(2.6) & \quad \frac{\|\mathcal{Y}\|_{L^2(\partial \Omega)}}{\|\mathcal{Y}\|_{L^2(\partial \Omega)}} \leq H_2,
\end{align}

where $H_2 = H / H_1$ and $\mathcal{Y}$ has zero average.

Furthermore, by (2.5)(a) and (2.1), $\mathcal{Y}$ verifies for any $z_0, z_1 \in \partial \Omega$

\begin{align}
(2.7) & \quad |\mathcal{Y}(z_0) - \mathcal{Y}(z_1)| \leq H(\text{length}_{\partial \Omega}(z_0, z_1))^{1/2} \leq H_3 |z_0 - z_1|^{1/2},
\end{align}

where $H_3 = H M_1^{1/2}$.

**Prior Information on the Measurements**

Let $\Gamma \subset \partial \Omega$ be a subarc whose length is greater than $\delta$.

The set of constants $\{\delta, M, L, \lambda, A, H, H_1\}$ will be referred to as the a priori data.

Let us finally recall that, under the stated assumptions, a weak solution $u \in W^{1,2}(\Omega \setminus \Sigma)$ satisfying

\begin{align}
(2.8) & \quad \int_{\Omega \setminus \Sigma} A \nabla u \cdot \nabla \varphi = \int_{\partial \Omega} \psi \varphi \quad \text{for every} \quad \varphi \in W^{1,2}(\Omega \setminus \Sigma),
\end{align}

exists and it is unique up to an additive constant.

Given another multiple cavity $\Sigma'$, satisfying the a priori assumptions, with components $\sigma_j', \ j = 1, \ldots, m, \ m \geq 1$, whose boundaries are simple closed curves denoted by $\gamma_j', \ j = 1, \ldots, m, \ m \geq 1$, we shall denote by $u'$ a solution to (2.8) when $\Sigma$ is replaced with $\Sigma'$.

**Theorem 2.1.** Let the above prior assumptions be satisfied. Suppose

\begin{align}
(2.9) & \quad \|u - u'|_{L^2(\Gamma)} \leq \epsilon.
\end{align}
We have:

(I) If the two families of boundaries $\gamma_i$ and $\gamma_j$ are Lipschitz with constant $\delta, M$, then

\[
d_H(S, S') \leq \omega(\varepsilon),
\]

where $\omega: (0, +\infty) \mapsto (0, +\infty)$ satisfies

\[
\omega(\varepsilon) \leq K (\log |\log \varepsilon|)^{-\beta} \quad \text{for every } \varepsilon, \quad 0 < \varepsilon < 1/e
\]

and $K, \beta > 0$ depend on the a priori data only.

Furthermore there exists a constant $\varepsilon_0 > 0$, depending on the a priori data only, so that if $\varepsilon \leq \varepsilon_0$ then the number of connected components of $\Sigma$ and $\Sigma'$ is the same, for instance equal to $n$, and, up to rearranging their order, we have

\[
d_H(\sigma_i, \sigma'_j) \leq \omega(\varepsilon), \quad \text{for every } i = 1, \ldots, n,
\]

$\omega$ as in (2.11).

(II) If $\gamma_i$ and $\gamma_j$ are RLG with constants $\delta, M$, then (2.10) holds where in this case $\omega: (0, +\infty) \mapsto (0, +\infty)$ satisfies

\[
\omega(\varepsilon) \leq K_1 |\log \varepsilon|^{-\beta_1} \quad \text{for every } \varepsilon, \quad 0 < \varepsilon < 1/e
\]

and $K_1, \beta_1 > 0$ depend on the a priori data only.

Also in this case, if $\varepsilon \leq \varepsilon_0$, $\varepsilon_0 > 0$ depending on the a priori data only, $\Sigma$ and $\Sigma'$ have the same number $n$ of connected components, and, again after rearranging their order, (2.12) holds with $\omega$ as in (2.13).

(III) If, for some $k = 1, 2, \ldots$ and some $\alpha, 0 < \alpha \leq 1$, $\gamma_i$ and $\gamma_j$ are $C^{k, \alpha}$ with constants $\delta, M$ then $\Sigma$ and $\Sigma'$ verify (2.10) where $\omega$ is as above in (2.13) with $K_1, \beta_1 > 0$ depending on the a priori data and on $k$ and $\alpha$ only.

As before, we may find $\varepsilon_0 > 0$ depending on the a priori data, on $k$ and on $\alpha$ only, such that if $\varepsilon \leq \varepsilon_0$ both $\Sigma$ and $\Sigma'$ have $n$ connected components, which ordered in a suitable way verify (2.12) with $\omega$ as in (2.13), $K_1, \beta_1 > 0$ depending on the a priori data and on $k$ and $\alpha$ only. Moreover, for any $i = 1, \ldots, n$, there exist regular parametrisations $z_i = z_i(t)$ and $z'_i = z'_i(t)$, $0 \leq t \leq 1$, of $\gamma_i$ and $\gamma'_i$ respectively such that for every $\bar{x}$, $0 < \bar{x} < \alpha$,

\[
\|z_i - z'_i\|_{C^{k,\alpha}[0, 1]} \leq K_2 \omega(\varepsilon)^{(k-\alpha)/(k+\alpha)},
\]
where $\omega$ still verifies (2.13) and $K_2$ depends on the a priori data, on $k$, on $\alpha$ and on $\bar{\alpha}$ only.

First, we recall that $d_H(\cdot, \cdot)$ denotes the Hausdorff distance between bounded closed sets.

Next, we observe that the assumption made at point (II) can be viewed as a non-trivial closeness condition between Lipschitz curves. In fact there are examples of pairs of Lipschitz curves which are arbitrarily close in the sense of the Hausdorff distance but are not RLG, see [21].

The key step of (III) will indeed be the following. If $\Sigma$ and $\Sigma'$ are a priori known to be $C^{k, \alpha}$, $k \geq 1$, $\alpha > 0$, with given constants $\delta$, $M$ and they are sufficiently closed in the Hausdorff sense then they are RLG.

3. PROOF OF THEOREM 2.1

For the time being, we shall assume that $\Sigma$ and $\Sigma'$ satisfy the assumptions stated in (I) of Theorem 2.1. It is easy to observe that if $\Sigma$ and $\Sigma'$ verify the assumptions (II) or (III) of Theorem 2.1, then they verify also (I) of the same Theorem. In view of assumption (I), let us remark some properties of $\Sigma$. The same properties are clearly shared also by $\Sigma'$. We have that the boundary $\gamma_1$ of any of the components $\sigma_i$ of $\Sigma$ has a length bounded by a constant depending on the a priori data only. Furthermore there exist a positive constant $\delta_1$ and an integer $N$, depending on the a priori data only, such that

\begin{equation}
\text{dist}(\sigma_i, \sigma_j) \geq \delta_1, \quad \text{for every } i \neq j, \tag{3.1}
\end{equation}

and

\begin{equation}
n = \text{number of connected components of } \Sigma \leq N. \tag{3.2}
\end{equation}

In the Introduction we have considered, (1.2), the notion of stream function and we have stated that there exists a single valued stream function $v$ associated to $u$, $u$ weak solution to (1.1). Let us recall that $v$ satisfies the Dirichlet type boundary value problem (1.3) with condition (1.4), where the constants $c_i$ are unknown, $B = (\det A)^{-1}A^T$ and $\Psi$ is defined as in (2.4). We shall always assume that $v$ is extended to $\Omega$ by setting $v|_{\sigma_i} = v|_{\sigma_j} = c_i$ for any $i = 1, \ldots, n$.

Then the complex valued function $f = u + iv$, defined in $\Omega \setminus \Sigma$, satisfies the following first order Beltrami type equation

\begin{equation}
f_z = \mu f_z + v\overline{f_z} \quad \text{almost everywhere in } \Omega \setminus \Sigma, \tag{3.3}
\end{equation}
where $\mu$ and $v$ are bounded measurable, complex valued coefficients, satisfying

$$|\mu| + |v| \leq k < 1 \quad \text{almost everywhere in } \Omega \setminus \Sigma$$

with $k$ depending on $\lambda$, $A$ only.

For any $k$, $0 \leq k < 1$, we say that a function $f$ is a $k$-quasiconformal function in a domain $D$ if it satisfies (3.3), (3.4). A univalent solution to (3.3), (3.4) is said a $k$-quasiconformal mapping. A function $f$ is a quasiconformal function, respectively mapping, if it is a $k$-quasiconformal function, respectively mapping, for some $k$, $0 \leq k < 1$. Concerning quasiconformal functions, their properties and characterizations we refer to [18].

A circular domain $D$ will be, by definition, a bounded domain whose boundary is composed by a finite number of circles, that is $D = B_R(z) \setminus \bigcup_{i=1}^{n} B_{r_i}(z_i)$, where $n$ is a positive integer, for any $i = 1, \ldots, n$, $B_{r_i}(z_i) \subset B_R(z)$ and the cavities $B_{r_i}(z_i)$ are pairwise disjoint. We call $\partial B_R(z)$ the exterior boundary and $\bigcup_{i=1}^{n} B_{r_i}(z_i)$ the multiple cavity of the circular domain $D$. Furthermore we introduce the following notations. For any cavity $B_{r_i}(z_i)$, $i = 1, \ldots, n$, let us denote

$$d_i = \text{dist} \left( B_{r_i}(z_i), \bigcup_{j \neq i} B_{r_j}(z_j) \cup \partial B_R(z) \right).$$

We shall say minimal radius (of the multiple cavity) the number $\min \{ r_i | i = 1, \ldots, n \}$ and separation distance (of the multiple cavity) the number $\min \{ d_i | i = 1, \ldots, n \}$.

**Proposition 3.1.** Under the assumptions of Part (I) of Theorem 2.1, let $u$ be a weak solution to (1.1) and $v$ be its stream function, solution to (1.3) with condition (1.4). Then the following representation holds

$$f = F \circ \chi,$$

where $\chi : \Omega \setminus \Sigma \mapsto D$ is a quasiconformal mapping satisfying

$$|\chi(x) - \chi(y)| \leq C_{1} |x - y|^{\alpha} \quad \text{for any } x, y \in \Omega \setminus \Sigma$$

and

$$||\chi^{-1}(x) - \chi^{-1}(y)|| \leq C_{1} |x - y|^{\alpha} \quad \text{for any } x, y \in D,$$
Let $D = B_1(0) \setminus \bigcup_{i=1}^{n-1} B_{r_i}(z_i)$ be a circular domain such that its exterior boundary is $\partial B_1(0)$ and is the image through $\chi$ of $\partial \Omega$ and the minimal radius and the separation distance of its multiple cavity are greater than $\delta_1 > 0$ and $F = U + iV$ is a holomorphic function on $D$. Here $C_1 > 0$, $\alpha_1$, $0 < \alpha_1 < 1$, and $\delta_2 > 0$ depend on the a priori data only.

**Proof.** We may find a bi-Lipschitz transformation $\chi_1$ from $\mathbb{C}$ onto itself such that the image through $\chi_1$ of $\Omega \setminus \Sigma$ is a circular domain $\tilde{D}$ such that $0 \in \tilde{D}$, its exterior boundary $\partial B_1(0) = \chi_1(\partial \Omega)$ and the minimal radius and separation distance of its multiple cavity are greater than $\delta_1 > 0$, $\delta_1$ depending on the a priori data only. The Lipschitz constants of such a transformation and of its inverse are dominated by constants only depending on $M, \delta$ and $L$.

The function $f = f \circ \chi_1^{-1}$ is $k_1$-quasiconformal, where $k_1$ depends only on $k$ and on the Lipschitz constants of $\chi_1$ and $\chi_1^{-1}$. Then by a Representation Theorem proved by L. Bers and L. Nirenberg, [11], there exist a $k_1$-quasiconformal mapping $\chi_2$ from $B_1(0)$ onto itself, with $\chi_2(0) = 0$, and a holomorphic function $\tilde{F} = U + i\tilde{V}$ on $\chi_2(\tilde{D})$ such that the representation $\tilde{f} = \tilde{F} \circ \chi_2$ holds.

By [24, Chapter 3, Theorem 5.2], we may find a conformal mapping $\chi_3$ from $\chi_2(\tilde{D})$ onto a circular domain $D$ still satisfying $\chi_3(0) = 0$ and $\partial B_1(0) = \chi_3(\partial B_1(0))$, $\partial B_1(0)$ being the exterior boundary of $D$. Then picking $\chi = \chi_3 \circ \chi_2 \circ \chi_1$ and $F = U + iV = \tilde{F} \circ \chi_3^{-1}$ the conclusion is immediate once the following Lemma is available.

**Lemma 3.2.** Let $D_0$ be a circular domain such that $0 \in D_0$, its exterior boundary is $\partial B_1(0)$ and the minimal radius and separation distance of its multiple cavity are greater than a positive constant $d_0$. Fixed $k$, $0 \leq k < 1$, there exist constants $d_1 > 0$, $C_2 > 0$ and $\alpha_2$, $0 < \alpha_2 < 1$, depending on $d_0$ and $k$ only such that if $\chi$ is a $k$-quasiconformal mapping from $D_0$ onto another circular domain $D_1$ whose exterior boundary is $\partial B_1(0)$ such that $\chi(0) = 0$ and $\partial B_1(0) = \chi(\partial B_1(0))$, then the minimal radius and separation distance of the multiple cavity of $D_1$ are greater than $d_1$ and $\chi$ verifies

\[
|\chi(x) - \chi(y)| \leq C_2 |x - y|^{\alpha_2} \quad \text{for any} \quad x, y \in D_0
\]

and

\[
|\chi^{-1}(x) - \chi^{-1}(y)| \leq C_2 |x - y|^{\alpha_2} \quad \text{for any} \quad x, y \in D_1.
\]

We defer the rather technical proof of this Lemma to the Appendix.
Let $D$, $F$ and $q$ be as in the thesis of Proposition 3.1. Then, by the regularity properties of $D$ and $q$, by (2.5)(a) and (2.7) and standard regularity theory we immediately infer

$$|F(z_1) - F(z_2)| \leq C_3 |z_1 - z_2|^\alpha_3$$

for every $z_1, z_2 \in \tilde{D}$,

and consequently

$$|f(z_1) - f(z_2)| \leq C_4 |z_1 - z_2|^\alpha_4$$

for every $z_1, z_2 \in \Omega \setminus \Sigma$, where $C_3$, $C_4$ and $\alpha_3$, $\alpha_4$, $0 < \alpha_3, \alpha_4 < 1$, depend on the a priori data only.

We remark that if as usual we extend $v$ on $\Omega$ in such a way that $v|_{\partial \Omega} = v|_{\partial \Omega_0} = c_i$ for any $i = 1, \ldots, n$, then it is easy to show that we have

$$|v(z_1) - v(z_2)| \leq C_5 |z_1 - z_2|^\alpha_4$$

for every $z_1, z_2 \in \Omega_0$, $C_5$ depending on the a priori data only.

Before stating the following Theorem let us recall that the Kelvin transform with respect to the ball $B = B_n(z_0)$ is given by

$$T_B(z) = r_0^2/(z - z_0) + z_0, \quad z \in \mathbb{C}.$$

**Theorem 3.3.** Under the assumptions of Part (I) of Theorem 2.1, there exists a positive constant $d_0$, depending on the a priori data only, such that for every $z^0 \in \Omega \setminus \Sigma$ and for every $d \leq d_0$ there exist finitely many points $z_k \in \Omega$ such that for every $z \in \Omega \setminus \Sigma$ satisfying $\text{dist}(z, \partial \Omega) \geq d$ we have

$$|f(z) - f(z^0)| \geq c(d) \prod_k \left( \frac{|z - z_k|}{C_6} \right)^{b_k},$$

where $b_k$ are positive integers satisfying

$$\sum_k b_k \leq C(d),$$

$C_6$ depending on the a priori data only, $\alpha_1$ as in (3.7) and $c(d) > 0$ and $C(d)$ depending on the a priori data and on $d$ only.

**Proof.** We recall the bi-Lipschitz mapping $\chi_1: \mathbb{C} \to \mathbb{C}$ we considered at the beginning of the proof of Proposition 3.1 which verifies $\chi_1(\Omega \setminus \Sigma) = \tilde{D}$, where $\tilde{D}$ is a circular domain. We have that $\tilde{D} = B_1(0) \setminus \bigcup_{i=1}^n B_{r_i}[x_i]$, $0 \in \tilde{D}$, and there exists $\delta_1 > 0$ depending on the a priori data only such that for any $i = 1, \ldots, n$, $r_i \geq \delta_1$ and $B_{r_i}[x_i] \setminus B_{r_i}[x_i]$ is contained in $\tilde{D}$. The function $f = f \circ \chi_1^{-1}$ which is $k_1$-quasiconformal, $k_1$ depending on the a priori data...
only, may be extended to another \( k_{1} \)-quasiconformal function, still denoted by \( \tilde{f} \), on the circular domain \( \tilde{D}_{1} = B_{i}(0) \setminus \bigcup_{i=1}^{n} B_{s_{i}}[x_{i}] \), where \( 0 < l < 1 \), depends on \( \delta_{1} \) only, in the following way

\[
\tilde{f}(z) = \tilde{f}(T_{B_{s_{i}}}(z)) + 2c_{i} \quad \text{for any } \quad z \in B_{s_{i}} \setminus B_{s_{i}}[x_{i}], \quad i = 1, \ldots, n,
\]

where \( c_{i} = v|_{B_{s_{i}}} = \tilde{v}|_{B_{s_{i}}[x_{i}]} \).

As in the proof of Proposition 3.1 we apply the Representation Theorem, [11], and Lemma 3.2 to obtain a circular domain \( D, F \), a holomorphic function on \( D \), and a quasiconformal mapping \( \chi_{2} : \tilde{D}_{1} \mapsto D \) such that \( \tilde{f} = F \circ \chi_{2} \). We recall that we may assume \( D = B_{i}(0) \setminus \bigcup_{i=1}^{n} B_{s_{i}}[x_{i}] \) and that for any \( i = 1, \ldots, n \), \( s_{i} \geq \delta_{2} > 0, \delta_{2} \) depending on the \emph{a priori} data only. Moreover, for any \( i = 1, \ldots, n \), we have that \( B_{s_{i}+d_{2}}(y_{i}) \setminus B_{s_{i}}[y_{i}] \) is contained in \( D \). We denote \( \chi = \chi_{2} \circ \chi_{1} : \chi_{1}^{-1}(\tilde{D}) \mapsto D \) and we remark that \( \chi \) verifies (3.6), (3.7) on \( \chi_{1}^{-1}(\tilde{D}) \) and \( D \) respectively and on \( \Omega \setminus \Sigma \) we have \( f = F \circ \chi \).

It is easy to see that we also have

\[
|F(z_{1}) - F(z_{2})| \leq C_{7} |z_{1} - z_{2}|^{\alpha_{3}} \quad \text{for every } \quad z_{1}, z_{2} \in \tilde{D},
\]

\( C_{7} \) depending on the \emph{a priori} data only.

We take \( z^{0} \in \overline{\Omega \setminus \Sigma} \). Letting \( w^{0} = \chi(z^{0}) \), we set \( F_{0} = F(w^{0}) = f(z^{0}) \). Let \( Z = \{ w_{k} \} \) be the countable set of the zeroes of \( F - F_{0} \) in \( D \). We have that setting \( \phi = \log |F - F_{0}| \)

\[
A \phi = 0 \quad \text{in} \quad D \setminus Z,
\]

and since \( \phi \) has negatively diverging isolated singularities at each \( w_{k} \), there exist positive integers \( b_{k} \) such that, in the sense of distributions,

\[
A \phi = 2\pi \sum_{k} b_{k} \delta(\cdot - w_{k}) \quad \text{in} \quad D.
\]

Fixed a positive \( d \) we denote

\[
D_{d} = \{ z \in D \mid \text{dist}(z, \partial D) > d \}.
\]

Then, by arguments in [3] based on Harnack’s inequality and the comparison principle, there exist positive constants \( C_{k} \) and \( C_{k}' \) depending on \( \delta_{2} \) only such that

\[
\sum_{w_{k} \in D_{d}} b_{k} \leq C_{k} d^{-c_{4}} \left[ 1 + \log \frac{\max_{D} |F - F_{0}|}{\max_{D_{d}} |F - F_{0}|} \right].
\]
Moreover, there exist positive constants $C_9, C_{-9}$ and $C_{10}$ also depending on $d_2$ only such that if we set $c_1(d) = C_9 d - C_{-9}$, which is greater than 1 if $d$ is small enough, we have for any $w \in D_d$

$$|F(w) - F_0| \geq e^{-c_1(d)} \left[ \frac{\max_{D_d} |F-F_0|}{\max_{D_d} |F-F_0|} \right] \prod_{w_k \in D_d} \left( \frac{|w-w_k|}{C_{10}} \right)^{b_k}.$$ 

By (3.16) we readily observe that

$$\max_{\Omega} |F-F_0| \leq C_{11},$$

where $C_{11}$ depends on the a priori data only. Moreover, if we denote $V_0 = V(z^0)$, we have the following estimate

$$\max_{\Omega} |F-F_0| \geq \max_{\Omega} |V-V_0| \geq \frac{1}{2} \text{osc}_D |V|.$$ 

Then we infer that $\text{osc}_D |V| \geq \text{osc}_{\partial\Omega(0)} |V|$ and also $\text{osc}_{\partial\Omega(0)} |V| = \text{osc}_{\partial\Omega} |v| = \text{osc}_{\partial\Omega} |\mathcal{P}|$.

Hence, since $\text{osc}_{\partial\Omega} |\mathcal{P}| \geq \|\mathcal{P}\|_{L^\infty(\partial\Omega)}/|\partial\Omega|$, by (2.5)(b) and the a priori information on the domain $\Omega$, we can find a positive constant $C_{12}$ depending on the a priori data only such that

$$\max_{\Omega} |F-F_0| \geq C_{12}.$$ 

Again by (3.16) we may find $\tilde{d}_0 > 0$ depending on the a priori data only such that for any $d$, $0 < d \leq \tilde{d}_0$, we have

$$\max_{D_d} |F-F_0| \geq C_{12}/2.$$ 

Then by the Hölder continuity properties of $\chi$ and its inverse, (3.6) and (3.7), we may find a constant $\tilde{d}_0$ depending on the a priori data only such that for any $d$, $0 < d \leq \tilde{d}_0$, there exists $\tilde{d}_0 < \tilde{d}_0$, depending on the Hölder constants of $\chi$ and $\chi^{-1}$ and on $d$ only, such that for every $z \in \Omega \setminus \tilde{\Omega}$ satisfying $\text{dist}(z, \partial\Omega) \geq d$ we have $w = \chi(z) \in D_{\tilde{d}_0}$.

Then, since $|f(z) - f(z^0)| = |F(w) - F_0|$, taking $z_k = \chi^{-1}(w_k)$, by (3.18), (3.19), (3.20) and by (3.7) it follows

$$|f(z) - f(z^0)| \geq e^{-c_1(d)} \left[ \frac{(C_{12}/2) c_1(d)}{(C_{11}) c_1(d)-1} \right] \prod_{k} \left( \frac{|z-z_k|}{C_{11}} \right)^{b_k/s_1},$$ 

which completes the proof.
where $C_{13}$ depends on the *a priori data* only and, by (3.17), we clearly have

\begin{equation}
\sum_{k} b_k \leq C_{13} d^{-\epsilon_i}\left[1 + \log \left(\frac{C_{11}}{C_{12}/2}\right)\right].
\end{equation}

This clearly concludes the proof. 

**Proposition 3.4.** Let all the hypotheses of Part (I) of Theorem 2.1, with the exception of (2.9), be satisfied. Let $v$ and $v'$ be the stream functions associated to $u$ and $u'$ respectively. If we have

\begin{equation}
\|v - v'\|_{L^\infty(\Omega)} \leq \eta,
\end{equation}

then the two multiple cavities $\Sigma$ and $\Sigma'$ satisfy

\begin{equation}
d_H(\Sigma, \Sigma') \leq K_{3}\eta^{b_{3}},
\end{equation}

$K_{3} > 0$, $\beta_{3}$, $0 < \beta_{3} < 1$, depending on the *a priori data* only.

**Proof.** Let $p = d_H(\Sigma, \Sigma')$. Let us assume, without losing the generality, that $p = \sup_{z \in \Sigma} \text{dist}(z, \Sigma)$.

Then there exist positive constants $C_{14}$ and $C_{15}$, depending on the *a priori data* only, and a point $z^{0} \in \Sigma'$ such that $B_{C_{14}p}(z^{0}) \subset \Sigma'$ and for any $w \in B_{C_{15}p}(z^{0})$ we have $\text{dist}(w, \Sigma) \geq C_{15}p$. Since $B_{C_{14}p}(z^{0}) \subset \Sigma'$, recalling (2.3), clearly we also have $\text{dist}(w, \partial \Omega) \geq \delta$ for any $w \in B_{C_{14}p}(z^{0})$.

By the maximum principle, the level set $\{u = u(z^{0})\}$ contains a continuum containing $z^{0}$ and intersecting $\partial B_{C_{14}p}(z^{0})$ in at least two different points. Let us fix $d = \min \{d_{0}, \delta\}$, $d_{0}$ as in Theorem 3.3. Let us consider the points $z_{k}$ obtained in Theorem 3.3 with respect to the point $z^{0}$ and the positive number $d$. Their number, by (3.14), is bounded by a constant $N$ depending on the *a priori data* only. There exists a constant $C_{16} > 0$ depending on $N$ and on $C_{14}$ only such that we may find $N+1$ pairwise disjoint open discs with radius $C_{16}p$ that are contained in $B_{C_{14}p}(z^{0})$ and whose center belongs to $\{u = u(z^{0})\}$. Therefore at least one of these discs has none of the points $z_{k}$ in its interior. Let $z^{1}$ be the center of this disc. Clearly for any $z_{k}$ we have $|z^{1} - z_{k}| \geq C_{16}p$.

Then by (3.13) we have

\[ |f(z^{1}) - f(z^{0})| \geq c(d) \prod_{k} \left(\frac{|z^{1} - z_{k}|}{C_{k}}\right)^{b_{1}/s_{1}}. \]
hence, by (3.14) and since $|z^1 - z_k| \geq C_{16} p$,

$$|f(z^1) - f(z^0)| \geq c(d) \left( \frac{C_{16} p}{C_6} \right)^{C(d)/\alpha_1}.$$  

Since we have that $u(z^1) = u(z^0)$ and, obviously $v(z^1) = v(z^0)$, we deduce

$$|f(z^0) - f(z^1)| = |v(z^0) - v(z^1)| \leq |v(z^0) - v'(z^0)| + |v'(z^1) - v'(z')| \leq 2\eta.$$  

Putting together the last two equations the conclusion easily follows.

Let us denote $\Phi = W + iZ = u - u' + i(v - v') : \Omega \setminus (\Sigma \cup \Sigma') \mapsto C$.

We can normalize $Z$ in order to have that it is identically zero on $\partial \Omega$.

Moreover by (2.9) we obtain $|W| \leq \epsilon$ on $\Gamma$.

Recalling (3.11) there exists a constant $D_1$ depending on the a priori data only such that

$$|\Phi(z)| \leq D_1 \quad \text{for any } z \in \Omega \setminus (\Sigma \cup \Sigma').$$  

We shall consider the following Cauchy type problem

\begin{align}
\Phi_z = \mu \Phi_x + v \Phi_y & \quad \text{in } \Omega \setminus (\Sigma \cup \Sigma'), \\
|\Phi| & \leq \epsilon \quad \text{on } \Gamma, \\
3\Phi &= 0 \quad \text{on } \partial \Omega,
\end{align}

where $|\mu| + |v| \leq k < 1$.

Recalling Proposition 3.4, the stability estimate on the inverse problem of cavities has been reduced to a stability estimate for the Cauchy type problem (3.26), that is obtaining an upper bound for $|Z|$ on $\partial \Omega$ in terms of the boundary error $\epsilon$.

We shall obtain different kinds of stability estimates for the Cauchy type problem (3.26), depending on the assumptions stated in the different parts of Theorem 2.1.

**Proposition 3.5.** Let the assumptions of Part (I) of Theorem 2.1 be satisfied and let $v$ and $v'$ be the stream functions associated to $u$ and $u'$ respectively. Then we have

$$\|v - v'\|_{L^\infty(\partial \Omega)} \leq \eta(\epsilon),$$
where \( \eta: (0, +\infty) \mapsto (0, +\infty) \) satisfies
\[
\eta(\varepsilon) \leq K_4 (\log |\log \varepsilon|)^{-2}\beta, \quad \text{for every } \varepsilon, \quad 0 < \varepsilon < 1/e,
\]
where \( K_4 \) and \( \beta > 0 \) depend on the a priori data only.

Proof. We give a sketch of the proof which is based on a technique developed in [4] (see also [8]). The main difference here is the presence of a multiple cavity, instead of a single crack.

First of all we define, as in [4], the following kind of so-called \( h \)-tubes. If \( z_0 \in \Gamma \), let \( l \) be the segment bisecting the open angular sector \( S \subset \Omega \) whose vertex is \( z_0 \), whose radius is \( d \) and whose amplitude depends on \( M \) only. We know that \( \text{dist}(z_1, \partial \Omega) \geq M_2 |z_0 - z_1| \), for any \( z_1 \in l \), \( M_2 < 1 \) depending on \( M \) only.

Let \( \gamma \) be a smooth curve contained in \( \Omega \setminus (\Sigma \cup \Sigma') \) so that its first endpoint \( z_0 \) belongs to \( \Gamma \), \( \gamma \) coincide with \( l \) for a length of at least \( h \) and thereafter the distance of any point of \( \gamma \) from \( \partial \Omega \) is greater than \( M_2 h \). An \( h \)-tube will be the \( M_2 h \) neighbourhood of any curve \( \tilde{\gamma} \) obtained by removing from such a curve \( \gamma \) its linear part of length \( h \) which is contained in \( l \).

An \( h \)-accessible point will be a point belonging to the closure of an \( h \)-tube which is contained in \( \Omega \setminus (\Sigma \cup \Sigma') \). We denote with \( G_h \) the set of \( h \)-accessible points.

If we apply the method used in [4] together with Theorem 4.5 in [8], we obtain for every \( z \in G_h \) and every \( h, 0 < h \leq h_0 \),
\[
|v(z) - v'(z)| \leq D_2 h^{a_4} + (D_3 + \varepsilon) \left( \frac{\varepsilon}{D_1 + \varepsilon} \right)^{\exp(-D_4/\varepsilon^2)},
\]
with constants \( D_2, D_3, D_4, h_0 \) depending on the a priori data only and \( a_4 \) as in (3.12).

Given the Hölder continuity of \( v \) and of \( v' \), which is stated in (3.12), and the maximum principle, we may extend the estimate (3.29) to any \( z \in \bar{Q} \) applying the method described in the proof of Theorem 3.1 in [4] with few modifications. In particular the main difference is that for any connected component \( Q \) of \( \Omega \setminus G_h \) more than two connected components of \( \Sigma \) or \( \Sigma' \) may be involved. However, in this case, there exists a constant \( D_7 \) depending on the a priori data only such that for any connected component \( \sigma \) of \( \Sigma \) or \( \Sigma' \) contained in \( Q \) with at least one point belonging to \( \partial G_h \), we may find a point \( w_0 \in \sigma \cap \partial G_h \) and a point \( w_1 \in \partial G_h \) belonging to another connected component of \( \Sigma \) or \( \Sigma' \) contained in \( Q \), such that \( |w_0 - w_1| \leq D_7 h \). For analogous considerations see [4, Lemma 3.6]. Then, by an iterated use of the above inequality and by the maximum principle, we find that there exists a constant \( c \) depending on \( Q \) such that if \( \bar{c} \) is the constant value of \( v \
(or respectively \(v'\)) on any connected component of \(\Sigma\) (respectively \(\Sigma'\)) contained in \(Q\) then we have

\[
|\tilde{c} - c| \leq D_6 h^a + \left( D_8 + \varepsilon \right) \left( \frac{e}{D_8 + \varepsilon} \right)^{\exp(-D_4/h^2)}, \quad \text{for every } h, \ 0 < h \leq h_0,
\]

\(D_6\) depending on the \(a\) priori data only. Obtained this result we conclude as in [4].

Once (3.29) is available for any \(z \in \tilde{\Omega}\) the thesis easily follows.

**Proposition 3.6.** Let the hypothesis of Part (II) of Theorem 2.1 be satisfied. Then \(v\) and \(v'\), the stream functions associated to \(u\) and \(u'\) respectively, verify (3.27) where \(\eta: (0, +\infty) \mapsto (0, +\infty)\) satisfies

\[
\eta(e) \leq K_3 |\log e|^{-\beta_4} \quad \text{for every } e, \ 0 < e < 1/e,
\]

\(K_3\) and \(\beta_4 > 0\) depending on the \(a\) priori data only.

**Proof.** Let \(G\) be the connected component of \(\Omega \setminus (\Sigma \cup \Sigma')\) such that \(\Gamma \subset \partial G\). Since \(\gamma_i\) and \(\gamma'_i\) are RLG then it is not difficult to show that \(G\) satisfies a uniform interior cone condition, that is for any point \(z \in \partial G\) there exists an angular sector \(S\) contained in \(G\), with vertex in \(z\) and whose positive radius and amplitude depend on the \(a\) priori data only and do not depend on \(z\).

Therefore by the technique developed in [21] we are able to obtain

\[
|\phi(z) - \phi'(z)| \leq D_9 |\log \varepsilon|^{-\alpha_5} \quad \text{for every } z \in G,
\]

where \(D_9\) and \(\alpha_5 > 0\) depend on the \(a\) priori data only. Then, again with the help of the maximum principle, the conclusion follows.

**Lemma 3.7.** Let us fix a positive integer \(k\) and a constant \(\alpha, 0 < \alpha \leq 1\), and let \(\Gamma_0 = \bigcup_{i=1}^n \gamma_i\) and \(\Gamma'_0 = \bigcup_{i=1}^m \gamma'_i\) be two finite families of simple closed curves, such that the domains bounded by each of the curves of one of the two families are pairwise disjoint. We assume that the two families are both \(C^{k,\alpha}\) with constants \(\delta, M\), and the length of any curve belonging to one of the two families is bounded by a constant \(L\).

Then there exists \(p_0 > 0\) depending on \(\delta, M, L, k\) and \(\alpha\) only such that if \(p = d_H(\Gamma_0, \Gamma'_0) \leq p_0\) then both \(\Gamma_0\) and \(\Gamma'_0\) have \(n\) connected components, which ordered in a suitable way verify

\[
d_H(\gamma_i, \gamma'_i) \leq p \quad \text{for any } i = 1, \ldots, n.
\]
Furthermore for any \( i = 1, \ldots, n \), there exist regular parametrisations \( z_i = z_i(t) \), \( 0 \leq t \leq 1 \), of \( \gamma_i \) and \( \gamma'_i \) respectively such that for every \( \bar{\alpha}, 0 < \bar{\alpha} < \alpha \),

\[
\|z_i - z'_i\|_{C^{k,\alpha}[0,1]} \leq K_6 (\delta_0, \delta_0')^{(\alpha - \bar{\alpha})/(\bar{\alpha} + 1)},
\]

where \( K_6 \) depends on \( \delta, M, L, k, \alpha \) and on \( \bar{\alpha} \) only.

\textbf{Proof.} By our assumptions we have that both the families verify

\[
\text{dist}(\gamma_i, \gamma_j) \geq \delta_0, \quad \text{for every} \quad i \neq j,
\]

with a constant \( \delta_0 > 0 \) depending on \( \delta, M, L, k \) and \( \alpha \) only.

Therefore the first part of the Lemma is obvious. Once the first part is established, the second one may be obtained following a procedure analogous to the one used to prove Lemma 2.1 in [21].

\textbf{Proof of Theorem 2.1.} Concerning Part (I) of Theorem 2.1, (2.10) and (2.11) are a direct consequence of Proposition 3.4 and of Proposition 3.5, whereas (2.12) may be deduced from (2.10) and (2.11) by taking into account (3.1).

The Part (II) may be obtained through Proposition 3.4 and Proposition 3.6.

For what concerns Part (III) the proof is an easy consequence of the previous part of Theorem 2.1 and of Lemma 3.7. In fact we have that the two families of curves \( \Gamma_0 = \bigcup_{i=1}^{n} \gamma_i \) and \( \Gamma'_0 = \bigcup_{j=1}^{m} \gamma'_j \) which consist of the boundaries of the connected components of \( \Sigma \) and \( \Sigma' \) respectively satisfy the assumptions of Lemma 3.7.

Then if \( \epsilon \) is small enough we have, by Part (I) of Theorem 2.1, \( d_H(\Gamma_0, \Gamma'_0) \leq p_0 \) and hence the number of connected components of \( \Sigma \) and \( \Sigma' \) is the same.

Given (3.32), (3.33), it is not difficult to show that there exists \( \epsilon_1 > 0 \), depending on the \textit{a priori data} and on \( k \) and \( \alpha \) only such that if \( \epsilon \leq \epsilon_1 \), then \( \gamma_i, i = 1, \ldots, n \), and \( \gamma'_j, j = 1, \ldots, m \), are RLG with constants \( \delta_3 > 0, M_3 > 0 \) with \( \delta_3 \) and \( M_3 \) depending on \( \delta, M, L, k \) and \( \alpha \) only and not depending on \( \epsilon \).

So the conclusion follows.

\textbf{4. INSTABILITY EXAMPLE}

Let \( \Omega = B_1(0) \) and let \( \sigma_0 = B_{1/2}(0) \). Let \( D_0 = \Omega \setminus \sigma_0 \). The two connected components of the boundary of \( D_0 \) are the two simple closed curves \( \beta = \partial \Omega = \partial B_1(0) \) and \( \gamma_0 = \partial \sigma_0 = \partial B_{1/2}(0) \).
For any $n = 1, 2, \ldots$, let us denote by $f_n$ the holomorphic function so defined

\begin{equation}
 f_n(z) = z \exp[\epsilon_n(z^n - z^{-n})], \quad z \in \mathbb{C} \setminus \{0\}, \ n = 1, 2, \ldots,
\end{equation}

where $\epsilon_n$ is the following positive real constant

\begin{equation}
 \epsilon_n = \frac{C_0}{n^{2k}},
\end{equation}

where $k$ is a fixed positive integer and $C_0$ is a positive constant to be chosen later.

The first derivative of $f_n$ is given by

\begin{equation}
 f'_n(z) = [1 + \epsilon_n(z^n - z^{-n})] \exp[\epsilon_n(z^n - z^{-n})], \quad z \in \mathbb{C} \setminus \{0\}, \ n = 1, 2, \ldots,
\end{equation}

hence we may find a positive constant $C_0, C_0$ not depending on $n$ and on $k$, such that if (4.2) holds then we have

\begin{equation}
 |f'_n(z) - 1| \leq 1/4, \quad \text{for any} \quad z \in \mathbb{D}_0, \ n = 1, 2, \ldots,
\end{equation}

and therefore $f_n$ is invertible on a neighbourhood (which may depend on $n$) of $\mathbb{D}_0$.

From now on we shall assume that this condition is satisfied. For any $n = 1, 2, \ldots$, we call $D_n = f_n(D_0)$. The boundary of $D_n$ has two connected components, the image through $f_n$ of $\beta$ and $\gamma_0$ respectively. It is easily seen that $f_n(\beta) = \beta$ and we shall denote by $\gamma_n$ the image through $f_n$ of $\gamma_0$. We remark that $\gamma_n$ is a Jordan curve and we denote by $\sigma_n$ the closed region bounded by $\gamma_n$. Therefore we have that $D_n = \Omega \setminus \sigma_n$.

By switching to polar coordinates, we shall characterize more precisely the behaviour of $f_n$ along $\beta$ and $\gamma_0$ and hence the regularity properties of $\gamma_n$ and, consequently, of $\sigma_n$.

Let us introduce polar coordinates in the following way. Given $z \in \mathbb{C} \setminus \{0\}$ let $(\rho, \theta)$, $\rho > 0$, satisfy $z = \rho \exp(i\theta)$. We have that $\rho = |z|$ and $\theta$ is defined up to equivalence modulus $2\pi$. We call $(\rho, \theta)$ the polar coordinates of $z$. Then, in these coordinates, $f_n$ can be written as

\begin{equation}
 f_n(\rho, \theta) = (\varphi_n(\rho, \theta), \phi_n(\rho, \theta)),
\end{equation}

where

\begin{equation}
 \varphi_n(\rho, \theta) = \rho \exp[\epsilon_n(\rho^n - \rho^{-n}) \cos n\theta]
\end{equation}
and
\[ \phi_n(\rho, \theta) = \theta + \varepsilon_n(\rho^n + \rho^{-n}) \sin n\theta. \]

First of all we notice that if \( \rho = 1 \) then \( \phi_n(1, \theta) = 1 \) for any \( \theta \in \mathbb{R} \) and we have
\[ |\phi_n(1, \theta) - \theta| \leq 2\varepsilon_n, \quad \text{for any } \theta \in \mathbb{R}. \]  

(4.4)

Then we want to estimate the Hausdorff distance between \( \sigma_n \) and \( \sigma_0 \). It is easy to observe that
\[ d_H(\sigma_n, \sigma_0) = \max_{[0, 2\pi]} |\phi_n(1/2, \theta) - 1/2|. \]

We may find two constants \( C_1 \) and \( C_2 \), \( 0 < C_1 < C_2 \), such that
\[ 0 < C_1 \varepsilon_n 2^n \leq d_H(\sigma_n, \sigma_0) \leq C_2 \varepsilon_n 2^n. \]  

(4.5)

Without loss of generality, changing \( C_0 \) in (4.2) if necessary, we may assume \( C_2 \varepsilon_n 2^n \leq 1/4. \)

Let us fix \( \rho, 1/2 \leq \rho \leq 1 \), and let us consider the function \( \phi_n(\rho, \cdot) : [0, 2\pi] \mapsto \mathbb{R} \). Then we can find \( C_0 > 0 \) not depending on \( n \) and on \( k \) such that if (4.2) holds then \( \left| \frac{\partial}{\partial \theta} \phi_n(\rho, \theta) - 1 \right| \leq 1/3 \) for any \( \rho \in [1/2, 1] \) and \( \theta \in [0, 2\pi] \). By this estimate we infer that \( \phi_n(\rho, \cdot) : [0, 2\pi] \mapsto [0, 2\pi] \) is bi-Lipschitzian with Lipschitz constants not depending on \( n \), on \( k \) and on \( \rho \).

Moreover, for any integer \( i \geq 2 \) we notice that
\[ \left| \frac{\partial^i}{\partial \theta^i} \phi_n(\rho, \theta) \right| \leq \varepsilon_n(\rho^n + \rho^{-n}) n^i. \]

If we fix the positive integer \( k \) and we define \( \varepsilon_n \) as in (4.2) with \( C_0 \) satisfying the previously stated conditions, it is straightforward to prove that for any \( n = 1, 2, \ldots \), \( \gamma_n \) is a \( C^k \) simple closed curve with constants \( \delta, M \) not depending on \( n \). Here the notion of a \( C^k \) curve with constants \( \delta, M \) is in the sense specified at the beginning of Section 2, with the obvious modification of replacing the \( C^k \) norm with the one in \( C^{k^*} \).

For any \( n = 0, 1, 2, \ldots \), let us consider, as usual, the following Sobolev spaces \( H^1(D_n) = \{ u \in L^2(D_n) \mid \nabla u \in L^2(D_n) \} \). We denote by \( H^{1/2}(\beta) \) its corresponding trace space on \( \beta \). By \( H^{-1/2}(\beta) \) we shall denote the dual space to \( H^{1/2}(\beta) \). With \( \oplus H^{1/2}(\beta) \) and \( \ominus H^{-1/2}(\beta) \) the corresponding subspaces of elements with zero means are considered. We remark that \( \ominus H^{1/2}(\beta) \) and \( \oplus H^{-1/2}(\beta) \) are dual to each other. With \( \ominus L^2(\beta) \) we denote the \( L^2 \) functions on \( \beta \) with zero average. We remark that the dual of \( \ominus L^2(\beta) \) is the space itself. Finally, if \( X \) and \( Y \) are two Banach spaces we shall denote
by $B(X, Y)$ the space of all bounded linear operators from $X$ to $Y$, with the usual norm.

Concerning trace spaces, fractional Sobolev spaces and interpolation inequalities, which will be used several times in the sequel, we refer to [1] and [19].

Let $\eta \in H^{-1/2}(\beta)$. Then for any $n = 0, 1, 2, \ldots$, let us consider the following Neumann type boundary value problem

$$
\begin{align*}
\Delta u_n &= 0 \quad \text{in } D_n, \\
\nabla u_n \cdot v &= 0 \quad \text{on } \gamma_n, \\
\nabla u_n \cdot v &= \eta \quad \text{on } \beta, \\
\n\left. u_n \right|_{\beta} &\in H^{1/2}(\beta).
\end{align*}
$$

The weak formulation of the problem is the following. To find $u_n \in H^1(D_n)$ such that $\left. u_n \right|_{\beta} \in H^{1/2}(\beta)$ and the following holds

$$
\int_{D_n} \nabla u_n \nabla \phi = \eta \left[ \phi \right]_{\beta}, \quad \text{for any } \phi \in H^1(D_n).
$$

We have that the solution to (NP$_n$) exists and is unique and we may find a constant $C$ not depending on $n$ such that if $D = B_1 \setminus B_{d/2}$ then

$$
\|u_n\|_{H^1(D_n)} \leq C \|\eta\|_{H^{-1/2}(\beta)}.
$$

(4.6)

For any $n = 0, 1, 2, \ldots$, let $N_n : H^{-1/2}(\beta) \mapsto H^{1/2}(\beta)$ be the Neumann-to-Dirichlet map defined in the following way

$$
N_n(\eta) = \left. u_n \right|_{\beta} \quad \text{for any } \eta \in H^{-1/2}(\beta),
$$

(4.7)

where $u_n$ is the solution to (NP$_n$).

From (4.6) we have that

$$
\|N_n(\eta)\|_{H^{1/2}(\beta)} \leq C \|\eta\|_{H^{-1/2}(\beta)}, \quad \text{for any } \eta \in H^{-1/2}(\beta),
$$

where $C$ is a positive constant which does not depend on $n$.

Let us state our instability result.

**Theorem 4.1.** Let us fix a positive integer $k$. Then there exists a constant $C_0 > 0$ such that if (4.2) holds then for any $n = 0, 1, 2, \ldots$, $\gamma_n$ is a $C^k$ simple
closed curve with positive constants $\delta$, $M$ not depending on $n$ and the following inequality holds

\begin{equation}
\|d_H(\sigma_n, \sigma_0)\| \geq C \log \|N_n - N_0\|_{H^{1/2}(\partial_0, s H^{1/2}(\beta))}^{-k},
\end{equation}

where $C$ is a positive constant which does not depend on $n$.

**Remark.** Let us observe that in inequality (4.9) some kind of dependence on $k$, the number of derivatives of the curves $\gamma_n$ which are *a priori* uniformly bounded, should be expected. In fact, in a similar setting, [13], Hölder type dependence on a suitably chosen boundary measurement was proved if an analyticity condition on the unknown curve $\gamma$ holds.

The proof of Theorem 4.1 will be obtained through three lemmas.

**Lemma 4.2.** There exists a positive constant $C$ such that for any $\eta \in _0L^2(\beta)$ we have

\begin{equation}
\|N_0(\eta)\|_{H^{1/2}(\beta)} \leq C \|\eta\|_{L^2(\beta)}.
\end{equation}

**Proof.** We have already observed, (4.8), that

\begin{equation}
\|N_0(\eta)\|_{H^{1/2}(\beta)} \leq C \|\eta\|_{H^{-1/2}(\beta)}, \quad \text{for any } \eta \in _0H^{-1/2}(\beta).
\end{equation}

Moreover it is not difficult to show that if $u_0$ is the solution to (NP$_0$) then we have, for a positive constant $C$,

\begin{equation}
\|u_0\|_{H^1(D_0)} \leq C \|\eta\|_{H^{-1/2}(\beta)}, \quad \text{for any } \eta \in _0H^{-1/2}(\beta).
\end{equation}

By standard regularity results, see for instance [23], we have that if $\eta \in _0H^{1/2}(\beta)$ then $u_0$ belongs to $H^2(D_0)$ and the following estimate holds

\begin{equation}
\|u_0\|_{H^2(D_0)} \leq C(\|\eta\|_{H^{1/2}(\beta)} + \|u_0\|_{H^1(D_0)}), \quad \text{for any } \eta \in _0H^{1/2}(\beta).
\end{equation}

Then we immediately deduce

\begin{equation}
\|N_0(\eta)\|_{H^{1/2}(\beta)} \leq C \|\eta\|_{H^{1/2}(\beta)}, \quad \text{for any } \eta \in _0H^{1/2}(\beta).
\end{equation}

Therefore the thesis may be obtained through (4.11) and (4.12) by using standard interpolation inequalities.

**Lemma 4.3.** There exists a positive constant $C$ not depending on $n$ such that

\begin{equation}
\|(N_n - N_0)(\eta)\|_{L^2(\beta)} \leq C \|\eta\|_{L^2(\beta)}, \quad \text{for any } \eta \in _0L^2(\beta), n = 1, 2, \ldots.
\end{equation}
Proof. For any \( n = 0, 1, 2, \ldots \), let us consider the linear operator 
\( N_n: L^2(\beta) \mapsto L^2(\beta) \). We have that \( N_n \), with respect to these two spaces, 
is bounded and self-adjoint. This can be easily deduced by the weak 
formulation of our boundary value problem, \((NP_n^*)\).

Let \( h_n: D_n \mapsto D_0 \) be the inverse map of \( f_n \), then \( h_n \) can be extended to the 
closure of \( D_0 \) and let us recall some properties of the restriction of \( h_n \) to \( \beta \).

We have that \( h_n|_\beta: \beta \mapsto \beta \) is invertible, bi-Lipschitz with constants not 
depending on \( n \) and the following estimates holds

\[
|h_n(z) - z| \leq C\epsilon_n,
\]

where \( C \) does not depend on \( n \).

For any \( n = 1, 2, \ldots \), let us define the linear operator 
\( T_n: L^2(\beta) \mapsto L^2(\beta) \) in the following way

\[
T_n(\eta)(z) = \eta(h_n(z)), \quad \text{for any } z \in \beta, \eta \in L^2(\beta).
\]

These linear operators are continuous with norm independent on \( n \), that is

\[
\|T_n(\eta)\|_{L^2(\beta)} \leq C\|\eta\|_{L^2(\beta)}, \quad \text{for any } \eta \in L^2(\beta), n = 1, 2, \ldots,
\]

they are invertible,

\[
(T_n)^{-1}(\eta)(z) = \eta(f_n(z)), \quad \text{for any } z \in \beta, \eta \in L^2(\beta),
\]

and their inverses are continuous with norm independent on \( n \).

Let \( T^*_n \) be the adjoint operator to \( T_n \), \( n = 1, 2, \ldots \), then \( T^*_n: L^2(\beta) \mapsto L^2(\beta) \) is defined

\[
T^*_n(\eta) = (T_n)^{-1}\left(\eta \frac{1}{|h_n|}\right), \quad \text{for any } \eta \in L^2(\beta).
\]

Finally let us observe that if \( \eta \in \mathcal{O} L^2(\beta) \) then also \( T^*_n(\eta) \in \mathcal{O} L^2(\beta) \).

Let \( P: L^2(\beta) \mapsto \mathcal{O} L^2(\beta) \) be the projection of \( L^2(\beta) \) onto \( \mathcal{O} L^2(\beta) \) given by

\[
P(\eta) = \eta - \frac{1}{2\pi} \int_\beta \eta, \quad \text{for any } \eta \in L^2(\beta).
\]

Clearly \( P \) is a linear bounded operator with norm 1.

We claim that the following representation holds

\[
N_n(\eta) = P[T_n N_0 T^*_n](\eta), \quad \text{for any } \eta \in \mathcal{O} L^2(\beta).
\]
Let \( u_n \) be the solution to \((\text{NP}_n)\) with Neumann datum \( \eta \in \mathcal{L}^2(\beta) \). Let us denote \( v_n = u_n \circ f_n \). Then \( v_n \) solves

\[
\begin{align*}
    \Delta v_n &= 0 & \text{in } \mathcal{D}_0, \\
    \nabla v_n \cdot n &= 0 & \text{on } \gamma_0, \\
    \nabla v_n \cdot v &= T_n^* \eta & \text{on } \beta.
\end{align*}
\]

(4.17)

Therefore we have \( u_n|_\beta = T_n(v_n|_\beta) \) and \( v_n|_\beta \) is equal to \( \mathcal{N}_0 T_n^* (\eta) \) up to an additive constant. Hence \( \mathcal{N}_n(\eta) = u_n|_\beta = T_n \mathcal{N}_0 T_n^* (\eta) + c_n \).

By the fact that \( \mathcal{N}_n(\eta) \in \mathcal{L}^2(\beta) \) we can immediately infer that \( c_n = - \frac{1}{2\pi} \int_\beta T_n \mathcal{N}_0 T_n^* (\eta) \) and hence (4.16) follows.

Now let us take \( \psi \in \mathcal{H}^1(\beta) \). We want to estimate \( \| (T_n - I)(\psi) \|_{\mathcal{L}^2(\beta)} \). We have that

\[
\| (T_n - I)(\psi) \|_{\mathcal{L}^2(\beta)} = \int_0^{2\pi} |\psi(h_n(\theta)) - \psi(\theta)|^2 \, d\theta.
\]

Then by (4.14) we deduce that \( |\psi(h_n(\theta)) - \psi(\theta)| \leq C \epsilon_n^{1/2} \| \psi \|_{\mathcal{H}^1(\beta)} \) and hence we obtain

\[
\| (T_n - I)(\psi) \|_{\mathcal{L}^2(\beta)} \leq C \epsilon_n^{1/2} \| \psi \|_{\mathcal{H}^1(\beta)}, \quad \text{for any } \eta \in \mathcal{H}^1(\beta),
\]

(4.18)

\( C \) not depending on \( n \).

Therefore by Lemma 4.2 we may find a constant \( C \) which does not depend on \( n \) such that

\[
\| (T_n \mathcal{N}_0 - \mathcal{N}_0)(\eta) \|_{\mathcal{L}^2(\beta)} \leq C \epsilon_n^{1/2} \| \eta \|_{\mathcal{L}^2(\beta)}, \quad \text{for any } \eta \in \mathcal{L}^2(\beta).
\]

(4.19)

By duality we have, with the same constant \( C \),

\[
\| (\mathcal{N}_0 T_n^* - \mathcal{N}_0)(\eta) \|_{\mathcal{L}^2(\beta)} \leq C \epsilon_n^{1/2} \| \eta \|_{\mathcal{L}^2(\beta)}, \quad \text{for any } \eta \in \mathcal{L}^2(\beta).
\]

(4.20)

Obviously \( P \mathcal{N}_0 = \mathcal{N}_0 \), then \( \mathcal{N}_n - \mathcal{N}_0 = P(T_n \mathcal{N}_0 T_n^* - \mathcal{N}_0) \) and hence for any \( \eta \in \mathcal{L}^2(\beta) \) we have

\[
\| (\mathcal{N}_n - \mathcal{N}_0)(\eta) \|_{\mathcal{L}^2(\beta)} \leq \| (T_n \mathcal{N}_0 T_n^* - \mathcal{N}_0)(\eta) \|_{\mathcal{L}^2(\beta)}.
\]

Since

\[
\| (T_n \mathcal{N}_0 T_n^* - \mathcal{N}_0)(\eta) \|_{\mathcal{L}^2(\beta)} \leq \| T_n (\mathcal{N}_0 T_n^* - \mathcal{N}_0)(\eta) \|_{\mathcal{L}^2(\beta)} + \| (T_n \mathcal{N}_0 - \mathcal{N}_0)(\eta) \|_{\mathcal{L}^2(\beta)}
\]

the thesis follows from (4.15), (4.19) and (4.20).
Lemma 4.4. \( N_n - N_0 \) is an infinitely smoothing operator, that is for any positive integer \( i \) there exists a constant \( C = C(i) \) not depending on \( n \) such that we have
\[
\| (N_n - N_0)(\eta) \|_{H^i(\beta)} \leq C(i) \| \eta \|_{H^{-1/2}(\beta)}, \quad \text{for any} \quad \eta \in \delta H^{-1/2}(\beta).
\]

Proof. Let us fix \( \eta \in \delta H^{-1/2}(\beta) \) and let \( u_n \) and \( u_0 \) be the solutions to (NP\( _n \)) and (NP\( _0 \)) respectively. By (4.6) and the mean value property of harmonic functions it is clear that for any \( z \) such that \( |z| = \frac{7}{8} \) there exists a constant \( C \) not depending on \( n \) and on \( \eta \) such that
\[
| (u_n - u_0)(z) | \leq C \| \eta \|_{H^{-1/2}(\beta)}.
\]
(4.21)

Then we notice that along \( \beta \) \( u_n - u_0 \) satisfies a homogeneous Neumann condition. Therefore we may extend \( u_n - u_0 \) on \( B_{\frac{7}{8}} \setminus \overline{B}_{\frac{7}{8}} \) according to the following reflection rule
\[
(u_n - u_0)(z) = (u_n - u_0)(1/z), \quad \text{for any} \quad z \in B_{\frac{7}{8}} \setminus \overline{B}_{\frac{7}{8}}.
\]

We have that \( u_n - u_0 \) is harmonic in \( B_{\frac{7}{8}} \setminus \overline{B}_{\frac{7}{8}} \), by the maximum principle and (4.21), on the same domain is bounded by \( C \| \eta \|_{H^{-1/2}(\beta)} \), therefore the thesis easily follows.

Proof of Theorem 4.1. By Lemma 4.3 and Lemma 4.4 applied with \( i = 2 \) and standard interpolation results we immediately infer
\[
\| (N_n - N_0)(\eta) \|_{H^1(\beta)} \leq C e_n^{1/4} \| \eta \|_{L^2(\beta)}, \quad \text{for any} \quad \eta \in \delta L^2(\beta).
\]

By duality we have
\[
\| (N_n - N_0)(\eta) \|_{L^2(\beta)} \leq C e_n^{1/4} \| \eta \|_{H^{-1}(\beta)}, \quad \text{for any} \quad \eta \in \delta H^{-1}(\beta).
\]

Then, again by interpolation inequalities, we deduce
\[
\| (N_n - N_0)(\eta) \|_{H^{-1/2}(\beta)} \leq C e_n^{1/4} \| \eta \|_{H^{-1/2}(\beta)}, \quad \text{for any} \quad \eta \in \delta H^{-1/2}(\beta),
\]
with \( C \) a constant not depending on \( n \).

Then the thesis may be obtained through a straightforward computation by recalling the definition of \( e_n \), (4.2), and the lower bound on \( d_H(\sigma_n, \sigma_0) \), (4.5).
APPENDIX

Proof of Lemma 3.2. During the proof of this Lemma we shall make use of the notion of capacity. Concerning its definition and its basic properties we refer to [16]. Here let us simply state some notations and the definition. Given a bounded domain $D$ and $E$ a subset of $D$, the pair $(E, D)$ will be called a condenser and we denote by $\text{cap}(E, D)$ the capacity of the condenser $(E, D)$. If $E$ is compact then

$$\text{cap}(E, D) = \inf_{u \in W(E, D)} \int_D |\nabla u|^2,$$

where

$$W(E, D) = \{ u \in C_0^\infty(D) \mid u \geq 1 \text{ on } E \}.$$

Then for any subset $E$ the capacity is defined as

$$\text{cap}(E, D) = \inf_{E \subset G \subset D} \sup_{K \subset G, \text{ compact}} \text{cap}(K, D).$$

We note also that the capacity may be computed explicitly if the condenser is an annulus. In fact, see again [16, page 35], we have for $0 < r < R$

(A1) \begin{equation}
\text{cap}(B_r(x), B_R(x)) = 2\pi \left( \frac{R}{r} \right)^{-1}.
\end{equation}

Let $D_0 = B_1(0) \setminus \bigcup_{i=1}^n B_{r_i}(z_i)$ and $D_1 = B_1(0) \setminus \bigcup_{i=1}^n B_{s_i}(w_i)$. We recall that $\chi(\partial B_r(0)) = \partial B_r(0)$ and we have ordered the cavities in such a way that $\chi(\partial B_{r_i}(z_i)) = \partial B_{s_i}(w_i)$ for any $i = 1, \ldots, n$. We note also that, since the minimal radius is bounded from below by $d_0 > 0$, if $n$ denotes the number of connected components of the multiple cavity of $D_0$ (and obviously also of the one of $D_1$), we have

(A2) \begin{equation}
n \leq N,
\end{equation}

$N$ depending only on $d_0$.

We denote $I = \{1, \ldots, n\}$. Then, by the lower bound on the minimal radius and on the separation distance of the multiple cavity of $D_0$, by (A1) and by elementary properties of capacity, we may find two constants $0 < C_1 < C_2$ depending on $d_0$ only such that for every $I_1$, nonempty subset of $I$, we have

(A3) \begin{equation}0 < C_1 \leq \text{cap} \left( \bigcup_{i \in I_1} B_{r_i}(z_i), B_1(0) \setminus \bigcup_{j \in I \setminus I_1} B_{s_j}(w_j) \right) \leq C_2.
\end{equation}
Since \( \chi \) is \( k \)-quasiconformal then there exists a constant \( C_1 > 0 \) depending on \( k \) only such that

\[
0 < C_1/C_3 \leq \text{cap} \left( \bigcup_{i \neq j} B_i[w_i], B_i(0) \right) \leq C_1C_2.
\]

holds for any \( I_i \subset I, I_i \neq \emptyset \), see [16, page 288].

We now claim the following result.

Claim. Given \( k, 0 \leq k < 1 \), and \( \delta_0 > 0 \), let \( f \) be a \( k \)-quasiconformal mapping from the annulus \( B_i(0) \setminus B_i−d_0(0) \) onto \( B_i(0) \setminus \sigma, \sigma \) being a closed subset of \( B_i(0) \), satisfying \( f(\partial B_i(0)) = \partial B_i(0) \) and \( 0 \in \sigma \). Then

\[
\text{dist}(\sigma, \partial B_i(0)) \geq \delta_1
\]

where \( \delta_1 > 0 \) depends on \( k \) and \( \delta_0 \) only.

By the Representation Theorem in [11, page 116] it is enough to prove the Claim when \( f = u + iv \) is conformal. Since \( 0 \in \sigma \), by (A1) and the invariance of capacity through conformal mapping, we may find \( \delta_i > 0 \) small enough such that for any \( 0 < \delta_i \leq \delta_2 \) either the oscillation of \( u \) or of \( v \) on \( \partial B_i−d_0(0) \) is greater than \( 1/4 \). Then by [2, Theorem 1.3] (see also [8, page 336]) there exist a constant \( \delta_i > 0 \) and a constant \( C > 0 \), both depending on \( k \) and \( \delta_0 \) only, such that if \( 0 < \delta_i \leq \delta_3 \) we have

\[
|f'(z)| \geq C, \quad \text{for any } z \in B_i−d_0/2 \setminus B_i−d_2/2(0),
\]

and from this the conclusion of the proof of the Claim follows very easily.

By the Claim we may immediately infer that there exists a constant \( d_2 \) depending on \( k \) and \( \delta_0 \) only such that we have

\[
\text{dist}(B_i[w_i], \partial B_i(0)) \geq d_2 \quad \text{for any } i = 1, \ldots, n.
\]

Let us denote as before

\[
\delta_i = \text{dist} \left( B_i[w_i], \bigcup_{j \neq i} B_j[w_j] \cup \partial B_i(0) \right), \quad i = 1, \ldots, n.
\]

Then, for any \( i = 1, \ldots, n \), we consider the following change of coordinates

\[
T_i(z) = r_i/(z−z_i), \quad S_i(z) = s_i/(z−w_i)
\]

and we take the function \( f_i : T_i(D_i) \mapsto S_i(D_i) \) given by

\[
f_i = S_i \circ \chi \circ T_i^{-1}.
\]
We have that there exists a \( d_0 > 0 \) depending on \( d_0 \) only such that \( T(D_0) \) contains the annulus \( B_1(0) \setminus B_{1-d_0}(0) \). Since \( 0 \notin S_i(D_1) \), \( f_i \) satisfies the hypothesis of the previous Claim, hence we may find \( d_i > 0 \) depending on \( k \) and \( d_0 \) only such that \( B_1(0) \setminus B_{1-d_i}(0) \subset S_i(D_1) \) and this implies that there exists a constant \( C_4 > 0 \) depending on \( k \) and \( d_0 \) only such that

\[
\delta_i \geq C_4 s_i \quad \text{for any} \quad i = 1, \ldots, n. \tag{A8}
\]

Let us remark that, by (A4), we have, for any \( i = 1, \ldots, n \),

\[
0 < C_1/C_3 \leq \operatorname{cap} \left( \bigcup_{i \in I_1} B_i[w_i], B_j(0) \right) \leq \operatorname{cap} \left( \bigcup_{j \in I_2} B_j[w_j] \right);
\]

hence, using (A1), we deduce that there exists a constant \( C_5 > 0 \) depending on \( d_0 \) and \( k \) only such that

\[
\delta_i \leq C_5 s_i \quad \text{for any} \quad i = 1, \ldots, n. \tag{A9}
\]

For any \( s_0, 0 < s_0 < 1 \), let us split the interval \((0, s_0)\) into the subintervals \((s_0^l, s_0^{l-1})\), \( l = 1, 2, \ldots \). Due to (A2), the bound on the number of connected components of the multiple cavity of \( D_1 \), there exists \( l \leq N + 1 \) such that \( s_i \notin (s_0^l, s_0^{l-1}) \) for every \( i \). Hence there exists \( s, 0 < s \leq s_0 \), depending on \( d_0 \) and \( s_0 \) only, such that if we set

\[
I_1 = \{ i \in I : s_i \leq s \}, \quad I_2 = \{ i \in I : s_i \geq s \},
\]

then \( I = I_1 \cup I_2 \).

Let us show \( I_1 = \emptyset \) when \( s_0 \) is sufficiently small. By contradiction let us assume \( I_1 \neq \emptyset \).

We take the condenser \((\bigcup_{i \in I_1} B_i[w_i], B_j(0) \setminus \bigcup_{j \in I_2} B_j[w_j])\) and we want to estimate its capacity. By subadditivity of capacity we have

\[
\operatorname{cap} \left( \bigcup_{i \in I_1} B_i[w_i], B_j(0) \setminus \bigcup_{j \in I_2} B_j[w_j] \right)
\]

\[
\leq \sum_{i \in I_1} \operatorname{cap} \left( B_i[w_i], B_j(0) \setminus \bigcup_{j \in I_2} B_j[w_j] \right).
\]

So let us fix \( i \in I_1 \) and let us evaluate \( \operatorname{cap}(B_i[w_i], B_j(0) \setminus \bigcup_{j \in I_2} B_j[w_j]) \). Assuming without loss of generality \( s_0 \leq d_2 \), by (A7) and (A8) applied to any \( B_j[w_j] \) with \( j \in I_2 \), we have that

\[
\operatorname{dist} \left( B_i[w_i], \bigcup_{j \in I_2} B_j[w_j] \cup \partial B_j(0) \right) \geq C_0 s_i \quad \text{for any} \quad i \in I_1, \tag{A11}
\]
where $C_6$ depends on $d_0$ and on $k$ only. Then

$$\cap\left( B_{r_i[w_i]}, B_{r}(0) \bigcup_{j \in I_2} B_{r_j[w_j]} \right)$$

$$\leq \cap(B_{r_i[w_i]}, B_{r_i+C_6s}(w_j)) \quad \text{for any } i \in I_1.$$

By (A1), since $s_i \leq s^2$,

$$\cap(B_{r_i[w_i]}, B_{r_i+C_6s}(w_i)) = 2\pi \left( \log \frac{s_i + C_6s}{s_i} \right)^{-1}$$

(A12)

$$\leq 2\pi \left( \log \frac{C_6}{s} \right)^{-1} \leq 2\pi \left( \log \frac{C_6}{s_0} \right)^{-1}.$$

Let us pick $s_0$ depending on $k$ and $d_0$ only such that

(A13)

$$2\pi \left( \log \frac{C_6}{s_0} \right)^{-1} \leq C_1/(2C_3N).$$

Then the combination of (A2), (A10), (A12) and (A13) violates the lower bound in (A4).

Hence we have found a positive constant $s$ depending on $k$ and $d_0$ only such that the minimal radius of the multiple cavity of $D_1$ is greater than $s$. Then, again by (A7) and (A8), also the separation distance may be bounded from below by a positive constant $s$ depending on $k$ and $d_0$ only. It remains to prove the Hölder continuity of $\chi$ and $\chi^{-1}$. Given the bounds on the minimal radius and the separation distance of the multiple cavities of $D_0$ and $D_0$ respectively, this may be obtained by standard reflection arguments, see [18], with the help of our Claim to control the reflection around $\partial B(0)$.

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REFERENCES