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OPTIMAL STABILITY OF RECONSTRUCTION OF PLANE LIPSCHITZ CRACKS*

LUCA RONDI†

Abstract. We establish an optimal stability estimate for the determination of a finite number of Lipschitz perfectly insulating cracks inside a planar conductor by performing two suitably chosen electrostatic boundary measurements.

Key words. inverse problems, stability, Lipschitz cracks, corkscrew condition

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1. Introduction.

We study the inverse problem of determining a finite number of unknown perfectly insulating cracks σ_j, $j = 1, \dots, N$, whose union is denoted with Σ, inside a known, possibly inhomogeneous and anisotropic, planar conductor Ω, whose known background conductivity is given by A, through voltage and current electrostatic measurements at the boundary.

We prescribe two current densities ψ_1, ψ_2, and we measure on Γ_0, a subarc of the boundary of Ω, $\partial\Omega$, the corresponding electrostatic potentials u_i. We recall that the electrostatic potential u_i satisfies the following Neumann-type boundary value problem:

$$
\begin{cases} \operatorname{div}(A\nabla u_i) = 0 & \text{in } \Omega\backslash\Sigma, \\ A\nabla u_i \cdot \nu = 0 & \text{on either side of } \sigma_j,\ j = 1, \dots, N, \\ A\nabla u_i \cdot \nu = \psi_i & \text{on } \partial\Omega, \end{cases} \tag{1.1}
$$

where ν denotes the unit normal, with the outward orientation when on $\partial\Omega$.

If ψ_1 and ψ_2 are suitably chosen—for example, they can model a two-electrode configuration where the positive electrode is kept fixed whereas the negative one is moved in a different position as we change the current density from ψ_1 to ψ_2—then the measurements $u_i|_{\Gamma_0}$, $i = 1, 2$, uniquely determine the unknown multiple crack Σ.

This inverse problem was introduced in [8], where the first uniqueness result in two dimensions was proved. Since then, many results concerning uniqueness and stability have been obtained; we refer to [5] and the references therein for a detailed account of these issues in two and three dimensions.

We are interested in estimating the error, in the Hausdorff distance, on the determination of Σ from an estimate of the error on the measurements $u_i|_{\Gamma_0}$. It has already been proven that

(a) if the components of Σ are a priori known to be Lipschitz regular, then the stability estimate is of log-log type (see [12, Theorem 4.1, part (I)]);

(b) if we a priori know either the coordinate system with respect to which the components of Σ are Lipschitz regular, or that the components of Σ are $C^{1,\alpha}$

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†Dipartimento di Scienze Matematiche, Università degli Studi di Trieste, via Valerio, 12/1 34127 Trieste, Italy (rondi@mathsun1.univ.trieste.it).

regular, with $0 < \alpha \leq 1$, then the stability estimate is of log type (see [12, Theorem 4.1, parts (II) and (III)]).

These results were proved first in the case of a single crack. More precisely, part (a) in [3] and part (b), at least for what concerns the $C^{1,\alpha}$ case, in [11]. Their extension to the case of multiple cracks is essentially based on arguments developed in [4] for the treatment of the multiple cavities case.

The aim of the present paper is to fill the gap between cases (a) and (b). In fact, we prove that, under the same assumptions as those of [12, Theorem 4.1, part (I)], Lipschitz regularity is enough to obtain a stability estimate of log type. We remark that single log estimates are usually obtained through a two-step procedure; see [1], where this argument was developed for the first time. For example, the proof of (b) relies on (a), as the first step, and, as the second step, on carefully studying the relation between two unknown multiple cracks Σ and Σ' (corresponding to two different sets of measurements) if they are close enough in the Hausdorff distance, in particular, on proving a uniform interior cone property for the open set $\Omega\backslash(\Sigma \cup \Sigma')$. However, if we consider case (a), it might happen that no kind of uniform interior cone property holds for all the points of $\Sigma \cup \Sigma'$, no matter how close the two multiple cracks are in the Hausdorff distance. On the other hand, if we consider the proof of [12, Theorem 4.1], it is clear that the arguments, at a given stage, in particular in the proof of Proposition 4.9 in [12] (or of Proposition 5.1 in [3] for the case of a single crack), are developed only locally in a suitable neighborhood of the point z where the Hausdorff distance between Σ and Σ' is reached. In this paper we establish that if Σ and Σ' are Lipschitz and close enough, then the points in $\Sigma \cup \Sigma'$ belonging to such a suitable neighborhood of the point z can be reached through a suitable sequence of discs contained in $\Omega\backslash(\Sigma \cup \Sigma')$; see Lemma 3.3. Such a condition, which allows us to carry over the second step of the procedure, is similar to the so-called *corkscrew condition* used in [9] to define nontangentially accessible domains.

We wish to emphasize that logarithmic stability estimates are optimal for this inverse problem. In fact, the abstract method developed in [6] from an idea of Mandache [10] provides the instability character of the problem; see [7], an expanded version of [6], for details.

Finally we wish to remark that, with a completely analogous procedure, we can extend this stability result to the inverse problem of multiple cavities. That is, if we perform two measurements of this kind, corresponding to prescribed current densities ψ_1 and ψ_2 as above, then we can obtain a stability estimate of log type for the determination of Lipschitz multiple cavities. We notice that we have uniqueness and stability results for the determination of multiple cavities with a single measurement, which can be of the most general type; see [3] and the references therein for the two-dimensional case and [2] for the higher-dimensional one. However, in the planar case, with a single measurement the stability estimate is of log-log type if the cavities are assumed to be Lipschitz, and it is of log type if the cavities satisfy the conditions described in case (b) above. Unfortunately, the technique used to prove the stability results with a single measurement is quite different, even if it has many common features with the one used for the inverse crack problem. In particular, in order to exploit the fact that the defects Σ and Σ' are the closures of open sets, we need to study the stability of a Cauchy-type problem up to the whole boundary of $\Omega\backslash(\Sigma \cup \Sigma')$; thus we cannot restrict our analysis to a neighborhood of the point where the Hausdorff distance is reached. This can be clearly observed once we notice that, locally, we are not able to distinguish between a portion of a crack and a portion of the boundary of a cavity. Therefore the approach developed in this paper cannot be directly applied to improve the stability with a single measurement. It remains an interesting open problem to establish log-type estimates for the determination of Lipschitz multiple cavities by a single measurement.

The plan of the paper is as follows. In section 2 we precisely state the stability result Theorem 2.3 and make some preliminary considerations. In section 3 the proof of Theorem 2.3 is developed.

2. Statement of the stability result. We begin with the definition of a quantitative notion of smoothness for open and closed curves in $\mathbb{R}^2$. The following standard notation will be used. For every $z = x + \mathrm{i}y \in \mathbb{C}$, $x = \Re z$ and $y = \Im z$ being the real and imaginary parts of z, respectively, and for every $r > 0$, we denote with $B_r(z)$ the open disc with center z and radius r. As usual, we shall identify complex numbers $z = x + \mathrm{i}y \in \mathbb{C}$ with points $(x, y) \in \mathbb{R}^2$. We shall use the following notation for complex derivatives:

$$f_{\overline{z}} = (f_x + \mathrm{i}f_y)/2, \quad f_z = (f_x - \mathrm{i}f_y)/2.$$

We denote by $J = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ the counterclockwise rotation of $90°$ and by $(\cdot)^T$ transpose.

DEFINITION 2.1. *Let $\gamma \subset \mathbb{R}^2$ be a bounded, simple curve, either open or closed. Then, with two fixed positive constants δ and M, we say that γ is* Lipschitz *with constants δ, M if for every $z \in \gamma$ there exists a coordinate system (x, y) with origin in z such that with respect to these coordinates $\gamma \cap B_\delta(z)$ is a* Lipschitz graph *with constant M, that is, $\gamma \cap B_\delta(z) = \{y = \phi(x) : \ a \leq x \leq b\} \cap B_\delta(z)$, where ϕ is a Lipschitz function on $[-\delta, \delta]$ such that $\|\phi\|_{C^{0,1}[-\delta,\delta]} \leq M$, a and b satisfy $-\delta \leq a \leq 0 \leq b \leq \delta$, and at least one of them has modulus equal to δ.*

Let $\Omega \subset \mathbb{R}^2$ be a bounded, simply connected domain. We say that $\sigma \subset \Omega$ is a *crack* in Ω if it is a closed set in Ω, which can be represented as the image of a simple open curve. We say that $\Sigma \subset \Omega$ is a *multiple crack* in Ω if it is the finite union of pairwise disjoint cracks in Ω.

We suppose that the following assumptions on the data of the inverse problem and the following a priori information on the unknown multiple crack present in Ω hold. We wish to remark that these assumptions and a priori information are essentially minimal and coincide with those used in previous papers, and we repeat them here for the convenience of the reader.

Assumptions on the domain. Let Ω be a bounded, simply connected domain in $\mathbb{R}^2$. We assume that the diameter of Ω is bounded by a given positive constant L and that its boundary $\partial\Omega$ is a simple closed curve which is Lipschitz with given positive constants δ, M.

From these assumptions we may deduce the following properties of Ω. We may find a constant L_1 depending on δ, M, and L only such that

$$0 < \delta \leq \mathrm{length}(\partial\Omega) \leq L_1.$$

Furthermore, there exists a constant M_1, depending on δ, M, and L only, such that

$$\mathrm{length}_{\partial\Omega}(z_0, z_1) \leq M_1|z_0 - z_1| \quad \text{for any } z_0, z_1 \in \partial\Omega. \tag{2.1}$$

Here $\mathrm{length}_{\partial\Omega}(z_0, z_1)$ is the length of the smallest arc in $\partial\Omega$ connecting z_0 to z_1. Moreover the measure of Ω, $|\Omega|$, is bounded from below and above by positive constants depending on δ, M, and L only.

Assumptions on the background conductivity. Let $A = A(z)$, $z \in \Omega$, be a conductivity tensor with bounded measurable entries satisfying, for given positive constants λ and Λ,

$$\begin{aligned} &A(z)\xi \cdot \xi \geq \lambda|\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^2 \text{ and for a.e. } z \in \Omega, \\ &|a_{ij}(z)| \leq \Lambda \quad \text{for every } i,j = 1,2 \text{ and for a.e. } z \in \Omega. \end{aligned} \tag{2.2}$$

Assumptions on the boundary data. Let γ_0, γ_1, γ_2 be three fixed simple arcs in $\partial\Omega$, pairwise internally disjoint.

Given $H > 0$, let us fix three functions η_0, η_1, $\eta_2 \in L^2(\partial\Omega)$ such that for every $i = 0,1,2$

$$\begin{aligned} &\eta_i \geq 0 \text{ on } \partial\Omega, \quad \operatorname{supp}(\eta_i) \subset \gamma_i, \\ &\int_{\partial\Omega} \eta_i = 1, \quad \|\eta_i\|_{L^2(\partial\Omega)} \leq H. \end{aligned} \tag{2.3}$$

Then we prescribe the current densities on the boundary ψ_1, ψ_2 to be given by

$$\psi_1 = \eta_0 - \eta_1, \quad \psi_2 = \eta_0 - \eta_2. \tag{2.4}$$

We have

$$\int_{\partial\Omega} \psi_i = 0, \quad \|\psi_i\|_{L^2(\partial\Omega)} \leq 2H \quad \text{for every } i = 1,2. \tag{2.5}$$

We shall consider also the antiderivatives along $\partial\Omega$ of ψ_1, ψ_2,

$$\Psi_i(s) = \int \psi_i(s)\mathrm{d}s, \quad i = 1,2, \tag{2.6}$$

where the indefinite integral is taken, as usual, with respect to arclength on $\partial\Omega$ in the counterclockwise direction. The functions Ψ_1, Ψ_2 are defined up to an additive constant.

We remark that from the assumptions on Ω, through (2.1), we have that, for every $i = 1,2$, Ψ_i satisfies the following Hölder continuity property for any z_0, $z_1 \in \partial\Omega$:

$$|\Psi_i(z_0) - \Psi_i(z_1)| \leq 2H(\text{length}_{\partial\Omega}(z_0,z_1))^{1/2} \leq H_1|z_0 - z_1|^{1/2}, \tag{2.7}$$

where $H_1 = 2HM_1^{1/2}$, M_1 as in (2.1).

Assumptions on the measurements. Let $\Gamma_0 \subset \partial\Omega$ be a subarc whose length is greater than or equal to δ.

A priori information on the multiple interior crack. We assume that an admissible multiple crack $\Sigma \subset \Omega$ is the union of finitely many, pairwise disjoint cracks σ_j, $j = 1,\ldots,N$, $N \geq 1$.

We suppose that each crack σ_j, $j = 1,\ldots,N$, is Lipschitz with constants δ, M. Moreover we suppose that

$$\operatorname{dist}(\Sigma, \partial\Omega) \geq \delta \tag{2.8}$$

and that

$$\operatorname{dist}(\sigma_j, \sigma_l) \geq \delta \quad \text{for any } j \neq l. \tag{2.9}$$

Let us make some remarks about the properties of the admissible multiple cracks. First we notice that Σ is not empty and each component of Σ is a simple open curve whose length is bounded from below and above by positive constants depending on δ, M, and L only.

Let Σ and $\Sigma' = \bigcup_{l=1}^{N'} \sigma'_l$, $N' \geq 1$, be two multiple interior cracks satisfying the a priori information. Then the following lemma is easy to prove. We recall that we denote the Hausdorff distance with d_H and that throughout the paper we set

$$p = \mathrm{d}_H(\Sigma, \Sigma').$$

LEMMA 2.2. *There exists a constant $p_0 > 0$, depending on δ, M, and L only, such that if $p \leq p_0$, then these two properties hold.*

First, the number of connected components of Σ and Σ' is the same, for instance, equal to N, and, up to rearranging their order and swapping Σ with Σ', we can assume that

$$\mathrm{d}_H(\sigma_j, \sigma'_j) \leq \mathrm{d}_H(\Sigma, \Sigma') \quad \textit{for every } j = 1, \ldots, N \tag{2.10}$$

and that there exists $z'_0 \in \sigma'_1$ so that

$$\mathrm{dist}(z'_0, \sigma_1) = \mathrm{d}_H(\sigma_1, \sigma'_1) = p. \tag{2.11}$$

Furthermore, $\Sigma \cup \Sigma' \subset \partial G$, where G is the connected component of $\Omega \backslash (\Sigma \cup \Sigma')$ whose boundary contains $\partial\Omega$.

For any $i = 1, 2$, let $u_i \in W^{1,2}(\Omega \backslash \Sigma)$ be the weak solution to (1.1). That is, we understand that u_i satisfies

$$\int_{\Omega\backslash\Sigma} A\nabla u_i \cdot \nabla\varphi = \int_{\partial\Omega} \psi_i \varphi \quad \text{for any } \varphi \in W^{1,2}(\Omega\backslash\Sigma).$$

We remark that u_i is unique up to additive constants. We denote by u'_i the solution to (1.1) when Σ is replaced with Σ'.

The set of constants δ, M, L, λ, Λ, and H will be referred to as the *a priori data*. We are now in position to state the main result.

THEOREM 2.3. *Under the previously stated assumptions, let $\varepsilon > 0$ be such that*

$$\max_{i=1,2} \|u_i - u'_i\|_{L^\infty(\Gamma_0)} \leq \varepsilon. \tag{2.12}$$

Then

$$\mathrm{d}_H(\Sigma, \Sigma') \leq \omega(\varepsilon), \tag{2.13}$$

where $\omega : (0, +\infty) \mapsto (0, +\infty)$ satisfies

$$\omega(\varepsilon) \leq K|\log\varepsilon|^{-\beta} \quad \textit{for every } \varepsilon,\ 0 < \varepsilon < 1/\mathrm{e}, \tag{2.14}$$

and K, $\beta > 0$ depend on the a priori data only.

We conclude this section by describing some properties of the solution to (1.1) and its stream function, assuming that the hypotheses of Theorem 2.3 are satisfied. For details and proofs we refer to [12, Chapter 4].

Let $i = 1, 2$ and let u_i solve (1.1). Then there exists a global single-valued function $v_i \in W^{1,2}(\Omega\backslash\Sigma)$ which satisfies

$$\nabla v_i = JA\nabla u_i \quad \text{almost everywhere in } \Omega\backslash\Sigma. \tag{2.15}$$

Such a function is referred to as the *stream function* associated to u_i. Moreover, letting $f_i = u_i + \mathrm{i}v_i$, we have

$$(f_i)_{\overline{z}} = \mu_1 (f_i)_z + \mu_2 \overline{(f_i)_z} \quad \text{almost everywhere in } \Omega\backslash\Sigma, \tag{2.16}$$

where μ_1 and μ_2 are bounded, measurable, complex-valued coefficients which depend on f_i and satisfy

$$|\mu_1| + |\mu_2| \leq k < 1 \quad \text{almost everywhere in } \Omega\backslash\Sigma, \tag{2.17}$$

where k is a constant depending on λ, Λ only.

Moreover, v_i satisfies in the weak sense the following Dirichlet-type boundary value problem:

$$\begin{cases} \operatorname{div}(B\nabla v_i) = 0 & \text{in } \Omega\backslash\Sigma, \\ v_i = d_j & \text{on } \sigma_j,\ j = 1, \dots, N, \\ v_i = \Psi_i & \text{on } \partial\Omega, \\ \int_\gamma B\nabla v_i \cdot \nu = 0 & \text{for any smooth Jordan curve } \gamma \subset \Omega\backslash\Sigma, \end{cases} \tag{2.18}$$

where $B = (\det A)^{-1} A^T$. We remark that the constants d_j are unknown and depend on $i = 1, 2$.

The weak formulation of (2.18) is the following. We want to find $v_i \in W^{1,2}(\Omega)$ such that v_i is constant in the trace sense on any crack σ_j, its trace on $\partial\Omega$ equals Ψ_i, and it satisfies

$$\int_{\Omega\backslash\Sigma} B\nabla v_i \cdot \nabla\varphi = 0 \quad \text{for any } \varphi \in W_0^{1,2}(\Omega) : \varphi = \text{const. on any crack.}$$

Let us finally remark that the stream function v_i is unique up to additive constants.

For any $i = 1, 2$, the following Hölder estimates hold (see [12, Proposition 4.6]):

$$|v_i(z_1) - v_i(z_2)| \leq C_1 |z_1 - z_2|^{\alpha_1} \quad \text{for every } z_1, z_2 \in \overline{\Omega}, \tag{2.19}$$

$$|u_i(z_1) - u_i(z_2)| \leq C_1 (\tilde{d}(z_1, z_2))^{\alpha_1} \quad \text{for every } z_1, z_2 \in \tilde{\Omega}. \tag{2.20}$$

Here C_1 and $\alpha_1 > 0$ depend on the a priori data only. We denote with $\tilde{\Omega}$ the compact manifold obtained by the appropriate gluing of $\overline{\Omega}\backslash\Sigma$ to the degenerate simple closed curve $\tilde{\sigma}_j$ obtained by overlapping two copies of σ_j, $j = 1, \dots, N$, and with $\tilde{d}$ the geodesic distance on $\tilde{\Omega}$.

It is useful to stress the difference between the estimates (2.19), (2.20). In fact, since v_i is constant on each σ_j, it is expected that v_i is continuous across each σ_j. Instead u_i may have different one-sided limits on σ_j. This is the main motivation for the introduction of the metric $\tilde{d}$.

For any $i = 1, 2$, let v_i' be the stream function associated to u_i' and $f_i' = u_i' + \mathrm{i}v_i'$. In what follows, we shall always normalize v_i and v_i' in such a way that $v_i = v_i'$ on $\partial\Omega$. Then we have that, for any $i = 1, 2$,

$$\|f_i - f_i'\|_{L^\infty(\Gamma_0)} \leq \varepsilon, \tag{2.21}$$

and, by (2.19), (2.20) and by assuming that $\varepsilon \leq 1/\mathrm{e}$,

$$\|f_i - f_i'\|_{L^\infty(\Omega)} \leq C_2, \tag{2.22}$$

where C_2 depends on the a priori data only.

Furthermore (see [12, Proposition 4.11]),

$$|v_i(z) - v_i'(z)| \leq \eta(\varepsilon) \quad \text{for any } z \in \overline{\Omega}, \tag{2.23}$$

where η is a positive function defined on $(0, +\infty)$ such that

$$\eta(\varepsilon) \leq C_3(\log|\log \varepsilon|)^{-\alpha_2} \quad \text{for every } \varepsilon,\ 0 < \varepsilon < 1/\mathrm{e}. \tag{2.24}$$

Here C_3 and α_2 are positive constants depending on the a priori data only.

3. Proof of Theorem 2.3. We begin with the following two results.

THEOREM 3.1. *Theorem* 2.3 *holds true if we replace* (2.14) *with*

$$\omega(\varepsilon) \leq K_1(\log|\log(\varepsilon)|)^{-\beta_1} \quad \textit{for every } \varepsilon,\ 0 < \varepsilon < 1/\mathrm{e}, \tag{3.1}$$

K_1, $\beta_1 > 0$ *depending on the a priori data only.*

PROPOSITION 3.2. *Suppose that the assumptions of Theorem* 2.3, *with the exception of* (2.12), *are satisfied. Let us further assume that* $p \leq p_0$, *and hence* (2.10) *and* (2.11) *are satisfied.*

If there exist positive constants c_0 *and* η *such that for every* r, $0 \leq r \leq c_0 p$, *there exists* $z' \in \sigma_1' \cap \partial B_r(z_0')$ *such that*

$$|v_i(z') - v_i'(z')| \leq \eta \quad \textit{for any } i = 1, 2, \tag{3.2}$$

then we have

$$p \leq K_2 \eta^{\beta_2}, \tag{3.3}$$

where K_2 *and* β_2 *are positive constants depending on* c_0 *and the a priori data only.*

Theorem 3.1 is the first part of Theorem 4.1 in [12]. The proof of Proposition 3.2 follows exactly the same argument as that used to prove Proposition 4.9 in [12]. It appears clear that our aim is to improve the estimate (2.23)–(2.24) at least for points which are near to the point where the Hausdorff distance is reached.

The following geometric construction is crucial. Let us recall that G is the connected component of $\Omega \backslash (\Sigma \cup \Sigma')$ whose boundary contains $\partial\Omega$ and that whenever $p \leq p_0$, we assume that the conclusions of Lemma 2.2 hold.

LEMMA 3.3. *Let* Ω, Σ, *and* Σ' *be as in Theorem* 2.3. *Then there exist positive constants* p_1, $0 < p_1 \leq p_0$, c_0, δ_0, C_4, *and* C_5, $0 < C_5 < 1$, *depending on* δ, M, *and* L *only, such that if* $p \leq p_1$, *then for every* r, $0 \leq r \leq c_0 p$, *there exists* $z' \in \sigma_1' \cap \partial B_r(z_0')$ *satisfying the following condition*

(a) *there exists a sequence of discs* $D_n = B_{r_n}(z_n)$ *such that, for any* $n \in \mathbb{N}$,

$$\begin{aligned} &\overline{D_n} \cap \overline{D_{n+1}} \neq \emptyset, \quad 2D_n = B_{2r_n}(z_n) \subset G, \\ &|z_n - z'| \leq C_4 r_n, \quad r_{n+1} \leq C_5 r_n, \end{aligned} \tag{3.4}$$

and, moreover,

$$\operatorname{dist}(D_1, \partial G) \geq \delta_0. \tag{3.5}$$

Let us remark that a point $z \in \Sigma \cup \Sigma'$ satisfies condition (a) provided there exists an open sector of a cone with vertex in z which is contained in G. In fact, in such a case, we can construct the discs satisfying condition (a) as follows. We take discs D_n

centered on the bisecting line of the sector so that $2D_n$ is contained in the sector and D_n is tangent to D_{n+1}, which is obviously chosen to be closer to z; see [1] for details. However, such a cone condition might not be satisfied if Σ and Σ' are only Lipschitz regular, even if they are very close in the Hausdorff distance; see Example 2.14 in [12]. Nevertheless, we show that, roughly speaking, in a neighborhood of the point where the Hausdorff distance is reached, condition (a) is satisfied, even if we might still lack the cone condition.

We begin with some preliminaries and by fixing some notation. Without loss of generality, by (2.8) and (2.9), we can restrict ourselves to the case of single cracks, σ_1 and σ_1', and take as G the unbounded connected component of $\mathbb{R}^2\backslash(\sigma_1 \cup \sigma_1')$.

Let us assume that $p \le \min\{p_0, \delta/4\}$ and that $z_0' \in \sigma_1'$ satisfies $\mathrm{dist}(z_0', \sigma_1) = p$, as in Lemma 2.2. Furthermore, let $z_0 \in \sigma_1$ be the point where this distance is reached, that is, such that $|z_0' - z_0| = p$.

Let us consider the coordinate system (x, y) with origin in z_0' such that with respect to these coordinates $\sigma_1 \cap B_\delta(z_0)$ is a *Lipschitz graph* with constant M and $z_0 = (x_0, y_0)$ with $x_0 \ge 0$ and $y_0 \le 0$. For any $i = 1, 2$, let $z_i = (x_i, y_i) \in \partial B_p(z_0')$ be such that $y_i \le 0$ and the tangent line to $\partial B_p(z_0')$ at the point z_i has slope $(-1)^i M$. Clearly we have $x_2 > 0$, $x_1 = -x_2$, and $y_1 = y_2$.

With the notation $S_r(\theta_1, \theta_2)$, where $r > 0$ and $\theta_1 < \theta_2$, we denote the open sector of a cone so defined:

$$S_r(\theta_1, \theta_2) = \{z = (x, y) : \ |z| < r \text{ and } \theta_1 < \arg z < \theta_2\}.$$

We say that r is the radius and $\theta_2 - \theta_1$ is the amplitude of such a sector. Let θ, $0 < \theta < \pi/2$, be the angle between a line of slope M and the y-axis and let

$$S_0 = S_{\delta/2}(\pi/2 - \theta, \pi/2 + \theta).$$

The proof of Lemma 3.3 is rather technical. Before entering into details, let us sketch its main features. Let us consider, for simplicity, the case in which $x_1 \le x_0 \le x_2$. Let us consider the function

$$f(x) = \begin{cases} -M(x - x_1) + y_1 & \text{if } x \le x_1, \\ -\sqrt{1 - x^2} & \text{if } x_1 \le x \le x_2, \\ M(x - x_2) + y_2 & \text{if } x \ge x_2. \end{cases}$$

We have that since $\sigma_1 \cap B_\delta(z_0)$ is a *Lipschitz graph* with constant M, then σ_1 must be, locally, below the graph of f. In particular, $S_0(z_i) = z_i + S_0$, for any $i = 0, 1, 2$, and $S_0(z_0') = z_0' + S_0$ do not contain points of σ_1. Moreover, by construction, any point z belonging to $S_0(z_0')$ has distance greater than p from the graph of f and, consequently, also from σ_1. Since p is the Hausdorff distance between σ_1 and σ_1', we infer that $S_0(z_0') \cap \sigma_1'$ is empty.

We have shown that there exists an open sector of a cone with vertex in z_0' which is contained in G, and thus condition (a) is satisfied for $r = 0$. For $r > 0$, we proceed as follows. Let $z \in \sigma_1' \cap \partial B_r(z_0')$, and let us consider the two opposite sectors with vertex in z which do not intersect σ_1', which exist by the Lipschitz character of σ_1'. Such sectors, at least for small r and near z, are contained in the epigraph of f, and thus do not intersect σ_1, too. Then the main idea is the following. We proceed from z along the bisecting line of one of these two sectors until we meet the bisecting line of one among the sectors $S_0(z_i)$, $i = 1, 2$, or $S_0(z_0')$. Then we turn and continue along this other bisecting line. This piecewise linear curve will be the direction along which we approach z with the sequence of balls contained in G that provides condition (a).

Proof of Lemma 3.3. Under the previous hypotheses and notation, we have two cases: Either

(i) the tangent line to $\partial B_p(z_0')$ at the point z_0 has slope m less than or equal to M; or

(ii) the tangent line to $\partial B_p(z_0')$ at the point z_0 has slope m greater than M (including the extreme case of a vertical tangent, when $z_0 = (p, 0)$).

If (i) holds, that is, when $x_1 \leq x_0 \leq x_2$, then $S_0(z_0') = z_0' + S_0$ satisfies $\mathrm{dist}(S_0(z_0'), \sigma_1) \geq p$. Therefore, since $p = \mathrm{d}_H(\sigma_1 \cup \sigma_1')$, we have $S_0(z_0') \cap (\sigma_1 \cup \sigma_1') = \emptyset$.

On the other hand, if (ii) holds, then we pick $\tilde{S}_0(z_0')$ as $\tilde{S}_0(z_0') = z_0' + \tilde{S}_0 = z_0' + S_{\delta/2}(\pi/2, 3\pi/2 - \theta)$, and we still have that $\tilde{S}_0(z_0') \cap (\sigma_1 \cup \sigma_1') = \emptyset$. We wish to remark that (ii) can hold only if z_0 is an endpoint of σ_1.

The construction of $S_0(z_0')$ or $\tilde{S}_0(z_0')$, respectively, already proves that the lemma is true for $r = 0$, that is, for z_0'.

We have that, with respect to a coordinate system which is rotated in the counterclockwise sense of an angle $\tilde{\theta}$, $0 \leq \tilde{\theta} < \pi$, with respect to the system (x, y), $\sigma_1' \cap B_\delta(z_0')$ is a Lipschitz graph with constant M, which implies that for any $z' \in \sigma_1' \cap \overline{B}_p(z_0')$, the sectors $S_0'(z')^{\pm} = z' \pm S_{\delta/2}(\pi/2 - \theta + \tilde{\theta}, \pi/2 + \theta + \tilde{\theta})$ do not intersect σ_1'. Furthermore, $S_0(z') = z' + S_0$ does not contain points of σ_1.

If $\tilde{\theta} \in [0, 15\theta/8] \cup [\pi - 15\theta/8, \pi)$, then for any $z' \in \sigma_1' \cap \overline{B}_p(z_0')$, we have that the intersection of $S_0(z')$ either with $S_0'(z')^+$ or with $S_0'(z')^-$ contains a sector of a cone of radius $\delta/2$ and amplitude at least $\theta/8$. Such a sector has empty intersection with $\sigma_1 \cup \sigma_1'$; thus a uniform cone property holds and the lemma easily follows.

We now consider the case in which $\tilde{\theta} \in (15\theta/8, \pi - 15\theta/8)$. We restrict ourselves to the case in which (i) holds; the case in which (ii) holds can be treated in a completely analogous way.

Let us consider the ball $B_{\delta/2}(z_0')$. We have that $B_{\delta/2}(z_0') \backslash (S_0'(z_0')^+ \cup S_0'(z_0')^-)$ consists of two closed sectors $\tilde{S}^+$ and $\tilde{S}^-$, where the first one is the only one whose intersection with $S_0(z_0')$ is not empty. If we further subtract $S_0(z_0')$, then we obtain at most three closed sectors, $\tilde{S}^-$, $\hat{S}_1$, and $\hat{S}_2$, the last ones being contained in $\tilde{S}^+$. We order $\hat{S}_1$ and $\hat{S}_2$ in the counterclockwise direction; that is, we take $\hat{S}_1$ as the one contained in $\{x \geq 0\}$ and $\hat{S}_2$ as the one contained in $\{x \leq 0\}$, keeping in mind that one or both of them can be empty or have an empty interior. We observe that at most one between $\hat{S}_1$ and $\hat{S}_2$ contains points of σ_1'.

Assume that $\sigma_1' \cap \hat{S}_1$ is not empty. Then there exist constants c_1, $0 < c_1 < 1$, and $c_2 > 0$, depending on δ and M only, such that for every $z' \in \sigma_1' \cap \hat{S}_1 \cap \overline{B}_{c_1 p}(z_0')$ we have that $S_0'(z')^+ \cap S_0(z_0')$ is not empty, $S_0'(z')^+ \cap S_0(z_0') \subset z_0' + (S_{\delta/4}(\pi/2 - \theta, \pi/2 + \theta) \backslash S_{c_2|z' - z_0'|}(\pi/2 - \theta, \pi/2 + \theta))$, and the angle between the bisecting lines of $S_0'(z')^+$ and $S_0(z_0')$ is greater than a positive constant depending on M only. Then we can prove that for every $z' \in \sigma_1' \cap \hat{S}_1 \cap \overline{B}_{c_1 p}(z_0')$, condition (a) holds. We take a sequence of discs in $S_0(z_0')$, each one so that its center is on the bisecting line of $S_0(z_0')$, it is tangential to the next one, and the disc with double radius and same center is still contained in $S_0(z_0')$, till we reach the intersection of the bisecting lines of $S_0'(z')^+$ and $S_0(z_0')$. From that point on, we continue the construction by taking discs, with analogous properties as before, along the sector $S_0'(z')^+$.

If $\sigma_1' \cap \hat{S}_2$ is not empty, then we can repeat the same reasoning using $S_0'(z')^-$ instead of $S_0'(z')^+$.

It might happen that $\sigma_1' \cap (\hat{S}_1 \cup \hat{S}_2)$ is strictly contained in $B_{c_1 p}(z_0')$, and therefore the proof is not yet concluded. In this case, we can find positive constants p_1, $0 < p_1 \leq \min\{p_0, \delta/4\}$, c_3, c_4, and θ_1, $0 < \theta_1 \leq \theta$, depending on δ and M only, such that if $p \leq p_1$, then for any r, $0 < r \leq c_3 p$, there exists $z' \in \sigma'_1 \cap \tilde{S}^-$ such that $|z' - z'_0| = r$ and the following holds. Let $\tilde{S}'_0(z')^{\pm}$ be the sector with vertex in z', radius $\delta/2$, the same bisecting line as $S'_0(z')^{\pm}$, and amplitude $2\theta_1$. Then either (i) $\tilde{S}'_0(z')^+ \cap S_0(z_1)$ is not empty, $\tilde{S}'_0(z')^+ \cap S_0(z_1) \subset z_1 + (S_{\delta/4}(\pi/2-\theta,\pi/2+\theta)\backslash S_{c_4p}(\pi/2-\theta,\pi/2+\theta))$, the angle between the bisecting lines of $\tilde{S}'_0(z')^+$ and $S_0(z_1)$ is greater than a positive constant depending on M only, and $S_0(z_1)\backslash(z_1 + S_{c_4p}(\pi/2-\theta,\pi/2+\theta))$ does not contain points of σ'_1; or (ii) the same properties are satisfied by $\tilde{S}'_0(z')^-$ and $S_0(z_2)$. Then we repeat the construction used before using either the two sectors $\tilde{S}'_0(z')^+$ and $S_0(z_1)$ or $\tilde{S}'_0(z')^-$ and $S_0(z_2)$. $\square$

We can now conclude the proof of our stability result.

Proof of Theorem 2.3. By Theorem 3.1, we can assume without loss of generality that $p \leq p_1$. Then, by Lemma 3.3, we can find c_0, δ_0, C_4, and C_5, $0 < C_5 < 1$, depending on δ, M, and L only, such that for every r, $0 \leq r \leq c_0 p$, there exists $z' \in \sigma'_1 \cap \partial B_r(z'_0)$ satisfying condition (a).

Then, for any $i = 1, 2$, and any of these z' satisfying condition (a), we have

$$|v_i(z') - v'_i(z')| \leq C_6 |\log \varepsilon|^{-\alpha_3}, \tag{3.6}$$

where C_6 and $\alpha_3 > 0$ depend on the a priori data only.

In fact, let us fix $i \in \{1,2\}$ and let us call $f = u + \mathrm{i}v = u_i - u'_i + \mathrm{i}(v_i - v'_i)$. We have that f is quasiregular inside $\Omega\backslash(\Sigma \cup \Sigma')$; that is, it satisfies a Beltrami-type equation like (2.16)–(2.17).

Let G_{δ_0} be the set of points in G whose distance from $\Sigma \cup \Sigma'$ is greater than or equal to δ_0. We assume, without loss of generality, that $\delta_0 \leq \delta/4$; thus a neighborhood of $\partial\Omega$ in G is contained in G_{δ_0}.

Let $\Omega_1 = G_{\delta_0} \cup (\bigcup_{n\in\mathbb{N}} 2D_n)$, with D_n as in condition (a) applied to z'. We have that Ω_1 is a domain contained in G such that $\partial\Omega \subset \partial\Omega_1$.

Let us observe that for any r, $0 < r \leq |z_1 - z'|$, with z_1 the center of D_1, there exists $w_r \in \bigcup_{n\in\mathbb{N}} \overline{D_n}$ such that $|w_r - z'| = r$. Furthermore, we can take such a w_r in D_n, where n satisfies $n < C_7(1 + |\log r|)$, with C_7 depending on C_4, C_5, and L only.

Then (3.6) can be obtained as follows. By recalling (2.21) and (2.22), we can estimate, in terms of ε, $|f|$ inside Ω_1 by using the method of harmonic measure, which has been generalized to operators with nonconstant and anisotropic coefficients in [3].

We can estimate $|v(z')|$ using the interior estimate of $|f|$ at the point w_r, $0 < r \leq |z_1 - z|$, and (2.19). A precise estimate of $|f(w_r)|$ is obtained through a repeated use of the Harnack inequality along the sequence of discs D_n. We refer to the proof of Proposition 4.12 in [12] for details.

Then the conclusion follows immediately from (3.6) and Proposition 3.2. $\square$

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