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A remark on a paper by Alessandrini and Vessella ☆

Luca Rondi

Dipartimento di Matematica e Informatica, Università degli Studi di Trieste, Italy

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Abstract

We prove that the Lipschitz constant of the Lipschitz stability result for the inverse conductivity problem proved in [G. Alessandrini, S. Vessella, Lipschitz stability for the inverse conductivity problem, Adv. in Appl. Math. 35 (2005) 207–241], behaves exponentially with respect to the number N of regions considered.

Let $\Omega = B_1(0) \subset \mathbb{R}^n$, where $n \geqslant 2$ denotes the space dimension. Let $D = [-1/2, 1/2]^n$ be the cube of side 1 centred at the origin. We have that D is compactly contained inside Ω. Let us consider the class of admissible conductivities

$$\mathcal{A} = \big\{\gamma \in L^\infty(\Omega)\colon\ 1/2 < \gamma < 3/2 \text{ a.e. in } \Omega \text{ and } \gamma = 1 \text{ a.e. in } \Omega \setminus D\big\}.$$

For any $\gamma \in \mathcal{A}$, we set the *Dirichlet-to-Neumann map* associated to γ as the operator $\Lambda_\gamma : H^{1/2}(\partial\Omega) \mapsto H^{-1/2}(\partial\Omega)$ given by

$$H^{1/2}(\partial\Omega) \ni \varphi \mapsto \gamma \nabla u \cdot \nu|_{\partial\Omega} \in H^{-1/2}(\partial\Omega),$$

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E-mail address: rondi@units.it.

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where $u \in H^1(\Omega)$ solves the elliptic Dirichlet problem

$$\begin{cases} \operatorname{div}(\gamma \nabla u) = 0 & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

Let us fix a positive integer N and let N_1 be the smallest integer such that $N \leqslant N_1^n$. We divide each side of the cube D into N_1 equal parts of length $h = 1/N_1$ and we let $\mathcal{S}_{N_1}$ be the set of all the open cubes of the type $D' = (-1/2 + (j_1' - 1)h, -1/2 + j_1'h) \times \cdots \times (-1/2 + (j_n' - 1)h, -1/2 + j_n'h)$, where $j_1', \ldots, j_n'$ are integers belonging to $\{1, \ldots, N_1\}$. We order such cubes as follows. For any two different D' and D'' belonging to $\mathcal{S}_{N_1}$ we say that $D \prec D''$ if and only if there exists $i_0 \in \{1, \ldots, n\}$ such that $j_i' = j_i''$ for any $i < i_0$ and $j_{i_0}' < j_{i_0}''$.

Let D_j, $j = 1, \ldots, N$, be the first N cubes, with respect to the order described above, of the set $\mathcal{S}_{N_1}$ and let $D_0 = \Omega \setminus \bigcup_{j=1}^{N} \overline{D_j}$. We consider the following set of admissible conductivities

$$\mathcal{A}_N = \left\{ \gamma \in \mathcal{A}\colon\ \gamma(x) = \sum_{j=1}^{N} \gamma_j \chi_{D_j}(x) + \chi_{D_0}(x) \right\},$$

where χ denotes the characteristic function and γ_j, $j = 1, \ldots, N$, are not prescribed constants belonging to $[1/2, 3/2]$.

It is not difficult to show that, for suitable constants A, r_0, L, M, α and λ, depending at most on n and N, the hypotheses of Theorem 7 of [1] are satisfied, therefore there exists a constant C_N, depending on n and N only, such that for any $\gamma^{(1)}$, $\gamma^{(2)}$ belonging to $\mathcal{A}_N$ we have

$$\left\|\gamma^{(1)} - \gamma^{(2)}\right\|_{L^\infty(\Omega)} \leqslant C_N \|\Lambda_{\gamma^{(1)}} - \Lambda_{\gamma^{(2)}}\|_{\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))}.$$

Our aim is to estimate from below the Lipschitz constant C_N in terms of N. In the sequel we shall always omit the dependence of the constants from the space dimension n. We have the following result, essentially based on arguments developed in [2].

Theorem. *There exist $N_0 \in \mathbb{N}$ and a positive constant K_1 such that for any $N \geqslant N_0$ we have*

$$C_N \geqslant \frac{1}{8} \exp\bigl(K_1 N^{1/(2n-1)}\bigr).$$

Proof. We define the following metric spaces. For any $N \in \mathbb{N}$, we consider $(\mathcal{A}_N, d_0)$ where d_0 is the distance given by the $L^\infty(\Omega)$ norm. Let $(\mathcal{B}, d_1)$ be the metric space where $\mathcal{B} = \{\Lambda_\gamma\colon\ \gamma \in \mathcal{A}\}$ and d_1 is the distance induced by the norm in $\mathcal{L}(H^{1/2}(\partial\Omega), H^{-1/2}(\partial\Omega))$. We let $\gamma^{(0)} = \chi_\Omega$.

First, we prove that for any δ, $0 < \delta \leqslant 1/2$, and any $N \in \mathbb{N}$ there exists $\tilde{\mathcal{A}}_N \subset \mathcal{A}_N$ such that $d_0(\tilde{\gamma}, \gamma^{(0)}) \leqslant \delta$ for any $\tilde{\gamma} \in \tilde{\mathcal{A}}_N$, for any two distinct points $\gamma^{(1)}$, $\gamma^{(2)}$ in $\tilde{\mathcal{A}}_N$ we have $d_0(\gamma^{(1)}, \gamma^{(2)}) \geqslant \delta$ and $\tilde{\mathcal{A}}_N$ has 3^N elements.

In fact, it is enough to take as $\tilde{\mathcal{A}}_N$ the set of functions $\tilde{\gamma}$ which assume, on each D_j, $j = 1, \ldots, N$, a value among 1, $1-\delta$ and $1+\delta$.

We recall the following definition. For a given positive ε, $\tilde{\mathcal{B}} \subset \mathcal{B}$ is said to be an ε-*net* for $\mathcal{B}$ if for every $\Lambda \in \mathcal{B}$ there exists $\tilde{\Lambda} \in \tilde{\mathcal{B}}$ such that $d_1(\Lambda, \tilde{\Lambda}) \leqslant \varepsilon$.

By using the arguments of the proof of Proposition 3.2 in [2], it is possible to show that we can apply Lemma 2.3 in [2] to $\mathcal{B}$. Therefore we have that for any ε, $0 < \varepsilon < 1/\mathrm{e}$, there exists an ε-*net* for $\mathcal{B}$ with at most $\exp(K_2(-\log\varepsilon)^{2n-1})$ elements, K_2 being a positive absolute constant.

For any $0 < \varepsilon < 1/\mathrm{e}$ and any $N \in \mathbb{N}$, let $Q(\varepsilon, N) = \exp(K_2(-\log\varepsilon)^{2n-1})$. Let us remark that $3^N > Q(\varepsilon, N)$ if and only if $\varepsilon > \exp(-K_1 N^{1/(2n-1)}) = \varepsilon_0(N)/2$, where K_1 is a positive absolute constant. There exists $N_0 \in \mathbb{N}$ such that for any $N \geqslant N_0$ we have $\varepsilon_0(N) < 1/\mathrm{e}$. Thus, for any $N \geqslant N_0$, if we take $\varepsilon = \varepsilon_0(N)$, we have $3^N > Q(\varepsilon, N)$, then for any δ, $0 < \delta \leqslant 1/2$, there exist $\gamma^{(1)}$ and $\gamma^{(2)}$ belonging to $\mathcal{A}_N$ such that $d_0(\gamma^{(i)}, \gamma^{(0)}) \leqslant \delta$ for any $i = 1, 2$ and

$$\delta \leqslant d_0\bigl(\gamma^{(1)}, \gamma^{(2)}\bigr) \leqslant C_N d_1(\Lambda_{\gamma^{(1)}}, \Lambda_{\gamma^{(2)}}) \leqslant 2C_N \varepsilon_0(N).$$

Choosing $\delta = 1/2$, we can conclude that $C_N \geqslant \frac{1}{8}\exp(K_1 N^{1/(2n-1)})$. □

References

[1] G. Alessandrini, S. Vessella, Lipschitz stability for the inverse conductivity problem, Adv. in Appl. Math. 35 (2005) 207–241

[2] M. Di Cristo, L. Rondi, Examples of exponential instability for inverse inclusion and scattering problems, Inverse Problems 19 (2003) 685–701.