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Ultrafilters maximal for finite embeddability

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Abstract: In this paper we study a notion of preorder that arises in combinatorial number theory, namely the finite embeddability between sets of natural numbers, and its generalization to ultrafilters, which is related to the algebraical and topological structure of the Stone-Čech compactification of the discrete space of natural numbers. In particular, we prove that there exist ultrafilters maximal for finite embeddability, and we show that the set of such ultrafilters is the closure of the minimal bilateral ideal in the semigroup $(\beta\mathbb{N},\oplus)$, namely $\overline{K(\beta\mathbb{N},\oplus)}$. By combining this characterization with some known combinatorial properties of certain families of sets we easily derive some combinatorial properties of ultrafilters in $\overline{K(\beta\mathbb{N},\oplus)}$. We also give an alternative proof of our main result based on nonstandard models of arithmetic.

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1 Introduction

In this paper we study some properties of a notion of preorder that arises in combinatorial number theory, the finite embeddability between sets of natural numbers. (See Di Nasso [5] and Ruzsa [10], where this notion was implicitly used, and Blass and Di Nasso [1], where many basic properties of this notion and its generalization to ultrafilters are studied.) We recall its definition.

Definition 1.1 *For A,B subsets of $\mathbb{N}$, we say that A is finitely embeddable in B and we write $A\leq_{fe}B$ if each finite subset F of A has a rightward translate $n+F$ included in B.*

We use the standard notation $n+F=F+n=\{n+a \mid a\in F\}$ and we use the standard convention that $\mathbb{N}=\{0,1,2,\ldots\}$. We also study the following generalization of $\leq_{fe}$ to ultrafilters.

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Definition 1.2 *For ultrafilters $\mathcal{U}, \mathcal{V}$ on $\mathbb{N}$, we say that $\mathcal{U}$ is finitely embeddable in $\mathcal{V}$ and we write $\mathcal{U} \leq_{fe} \mathcal{V}$ if, for each set $B \in \mathcal{V}$, there is some $A \in \mathcal{U}$ such that $A \leq_{fe} B$.*

Many basic properties of these preorders are proved in Blass and Di Nasso [1] by using standard and nonstandard techniques. In this present paper we use similar techniques to study some different properties of these preorders and some easy applications to combinatorial number theory. Our main result is that there exist ultrafilters maximal for finite embeddability and that the set of such maximal ultrafilters is the closure of the minimal bilateral ideal in $(\beta\mathbb{N}, \oplus)$, namely $\overline{K(\beta\mathbb{N}, \oplus)}$. By combining this characterization with some known combinatorial properties of certain families of sets we easily deduce many combinatorial properties of ultrafilters in $\overline{K(\beta\mathbb{N}, \oplus)}$, eg that for every ultrafilter $\mathcal{U} \in \overline{K(\beta\mathbb{N}, \oplus)}$, for every $A \in \mathcal{U}$, A has positive upper Banach density, it contains arbitrarily long arithmetic progressions and it is piecewise syndetic[1]. We will also show that there do not exist minimal sets in $(\mathcal{P}_{\aleph_0}(\mathbb{N}), \leq_{fe})$ nor there exist minimum ultrafilters in $(\beta\mathbb{N} \setminus \mathbb{N}, \leq_{fe})$, where $\mathcal{P}_{\aleph_0}(\mathbb{N})$ is the set of infinite subsets of $\mathbb{N}$ and $\beta\mathbb{N} \setminus \mathbb{N}$ is the set of nonprincipal ultrafilters. These topics are studied in sections 2 and 3. In section 4 we reprove our main result by nonstandard methods; nevertheless, this is the only section in which nonstandard methods are used, so the rest of the paper is accessible also to readers unfamiliar with nonstandard methods.

We refer to Hindman and Strauss [6] for all the notions about combinatorics and ultrafilters that we will use, to Chang and Keisler [3, §4.4] for the foundational aspects of nonstandard analysis and to Davis [4] for all the nonstandard notions and definitions. Finally, we refer the interested reader to Luperi Baglini [7, Chapter 4] for other properties and characterizations of the finite embeddability.

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2 Some basic properties of finite embeddability between sets

Let n be a natural number. Throughout this section we will denote by $\mathcal{P}_{\geq n}(\mathbb{N})$ the following set:

$$\mathcal{P}_{\geq n}(\mathbb{N}) = \{A \subseteq \mathbb{N} \mid |A| \geq n\};$$

[1] Let us note that many of these combinatorial properties of ultrafilters in $\overline{K(\beta\mathbb{N}, \oplus)}$ were already known.

similarly, we will denote by $\mathcal{P}_{\aleph_0}(\mathbb{N})$ the following set:

$$\mathcal{P}_{\aleph_0}(\mathbb{N}) = \{A \subseteq \mathbb{N} \mid |A| = \aleph_0\}.$$

Moreover, we will denote by $\equiv_{fe}$ the equivalence relation such that for every $A, B \subseteq \mathbb{N}$,

$$A \equiv_{fe} B \Leftrightarrow A \leq_{fe} B \text{ and } B \leq_{fe} A.$$

We will write $A <_{fe} B$ if $A \leq_{fe} B$ and $A \not\equiv_{fe} B$.

It is immediate to see that the relation $\leq_{fe}$ on $\mathcal{P}(\mathbb{N})$ is reflexive and transitive but it is not antisymmetric (eg, we have $\{2n \mid n \in \mathbb{N}\} \equiv_{fe} \{2n+1 \mid n \in \mathbb{N}\}$). So $\leq_{fe}$ is a partial preorder.

Di Nasso [5] points out that $\leq_{fe}$ has the following properties (for the relevant definitions, see Hindman and Strauss [6]).

Proposition 2.1 *[5, Propositions 4.1 and 4.2] Let A, B be sets of natural numbers. Then:*

(i) *A is maximal with respect to $\leq_{fe}$ if and only if it is thick;*

(ii) *if $A \leq_{fe} B$ and A is piecewise syndetic then B is also piecewise syndetic;*

(iii) *if $A \leq_{fe} B$ and A contains a k-term arithmetic progression then also B contains a k-term arithmetic progression;*

(iv) *if $A \leq_{fe} B$ then the upper Banach densities satisfy $BD(A) \leq BD(B)$;*

(v) *if $A \leq_{fe} B$ then $A - A \subseteq B - B$;*

(vi) *if $A \leq_{fe} B$ then $\bigcap_{t \in G}(A - t) \leq_{fe} \bigcap_{t \in G}(B - t)$ for every finite $G \subseteq \mathbb{N}$.*

We will use Proposition 2.1 to (re)prove some combinatorial properties of ultrafilters in $\overline{K(\beta\mathbb{N}, \oplus)}$ in Section 3. Before that, we want to prove one more basic property of $\leq_{fe}$, namely that for every set A there does not exist a set B such that $A <_{fe} B <_{fe} A+1$. To prove this result we need the following lemma.

Lemma 2.2 *For every $A, B \subseteq \mathbb{N}$ the following two properties hold:*

(i) *if $B \not\leq_{fe} A$ and $B \leq_{fe} A+1$ then $B \subseteq A+1$;*

(ii) *if $A \leq_{fe} B$ and $A+1 \not\leq_{fe} B$ then $A \subseteq B$.*

Proof We prove only (i), since (ii) can be proved similarly. Let $F \subseteq B$ be a finite subset of B such that $n+F \nsubseteq A$ for every $n \in \mathbb{N}$. In particular, for every finite $H \subseteq B$ such that $F \subseteq H$ and for every $n \in \mathbb{N}$ we have that $n+H \nsubseteq A$. By hypothesis there exists $n \in \mathbb{N}$ such that $n+H \subseteq A+1$. If $n \geq 1$ we have a contradiction, therefore it must be $n=0$, ie $H \subseteq A+1$. Since this holds for every finite $H \subseteq B$ (with $F \subseteq H$) we deduce that $B \subseteq A+1$. □

Theorem 2.3 *Let $A, B \subseteq \mathbb{N}$. If $A \leq_{fe} B \leq_{fe} A+1$ then $A \equiv_{fe} B$ or $A+1 \equiv_{fe} B$.*

Proof If $A=\emptyset$ then $A+1=\emptyset$ so also $B=\emptyset$, therefore $A \equiv_{fe} B$. Let now $A \neq \emptyset$. Let us suppose that $A+1 \nleq_{fe} B \nleq_{fe} A$. Then, since $A \leq_{fe} B \leq_{fe} A+1$, by Lemma 2.2 we deduce that $A \subseteq B \subseteq A+1$, so $A \subseteq A+1$. This is absurd since $\min A \in A \setminus (A+1)$ if $A \neq \emptyset$. □

We now consider the problem of existence of minimal elements in various subsets of $\mathcal{P}(\mathbb{N})$. Two immediate observations are that the empty set is the minimum in $(\mathcal{P}(\mathbb{N}), \leq_{fe})$ and that $\{0\}$ is the minimum in $(\mathcal{P}(\mathbb{N})_{\geq 1}, \leq_{fe})$. Moreover, if we identify each natural number n with the singleton $\{n\}$, it is immediate to see that $(\mathbb{N}, \leq)$ forms an initial segment of $(\mathcal{P}_{\geq 1}(\mathbb{N}), \leq_{fe})$ and that, more in general, the following easy result holds.

Proposition 2.4 *A set A is minimal in $(\mathcal{P}_{\geq n}(\mathbb{N}), \leq_{fe})$ if and only if $0 \in A$ and $|A|=n$.*

The proof follows easily from the definitions. Let us note that, in particular, the following facts follow:

(i) for every natural number $m \geq n-1$ there are $\binom{m}{n-1}$ inequivalent minimal elements in $(\mathcal{P}_{\geq n}(\mathbb{N}), \leq_{fe})$ that are subsets of $\{0, \dots, m\}$;

(ii) if $n \geq 2$ then $(\mathcal{P}_{\geq n}(\mathbb{N}), \leq_{fe})$ does not have a minimum element.

If we consider only infinite subsets of $\mathbb{N}$ the situation is different: there are no minimal elements in $(\mathcal{P}_{\aleph_0}(\mathbb{N}), \leq_{fe})$, as we are now going to show.

Definition 2.5 *Let $A, B \subseteq \mathbb{N}$. We say that A is strongly non f.e. in B, and we write $A \nleq_{fe}^{S} B$, if for every set $C \subseteq A$ with $|C|=2$ we have that $C \nleq_{fe} B$. If both $A \nleq_{fe}^{S} B$ and $B \nleq_{fe}^{S} A$ we say that A, B are strongly mutually unembeddable, and we write $A \not\equiv_{S} B$.*

Let us observe that, in the previous definition, we can equivalently substitute the condition "$|C|=2$" with "$|C| \geq 2$".

Proposition 2.6 *Let X be an infinite subset of $\mathbb{N}$. Then there are $A, B \subseteq X$, A, B infinite, such that $A \cap B = \emptyset$ and $A \not\equiv_S B$.*

Proof Let $X = \{x_n \mid n \in \mathbb{N}\}$, with $x_n < x_{n+1}$ for every $n \in \mathbb{N}$. We set

$$a_0 = x_0, b_0 = x_1$$

and, recursively, we set

$$a_{n+1} = \min\{x \in X \mid x > a_n + b_n + 1\},\ b_{n+1} = \min\{x \in X \mid x > b_n + a_{n+1} + 1\}.$$

Finally, we set $A = \{a_n \mid n \in \mathbb{N}\}$ and $B = \{b_n \mid n \in \mathbb{N}\}$. Clearly $A \cap B = \emptyset$, and both A, B are infinite subsets of X. Now we let $a_{n_1} < a_{n_2}$ be any elements in A. Let us suppose that there are $b_{m_1} < b_{m_2}$ in B with $a_{n_2} - a_{n_1} = b_{m_2} - b_{m_1}$ and let us assume that $b_{n_2} > a_{n_2}$ (if the converse hold, we can just exchange the roles of $a_{n_1}, a_{n_2}, b_{m_1}, b_{m_2}$). By construction, since $b_{m_2} > a_{n_2}$, we have $b_{m_2} - b_{m_1} \geq a_{n_2} + 1 > a_{n_2}$, while $a_{n_2} - a_{n_1} \leq a_{n_2}$. So $A \not\equiv_S B$. □

Proposition 2.6 has a few easy consequences that we now prove.

Corollary 2.7 *For every infinite set $X \subseteq \mathbb{N}$ there is an infinite set $A \subseteq X$ such that $X \not\leq_{fe} A$.*

Proof Let A, B be infinite subsets of X such that $A \not\equiv_S B$. Then X cannot be finitely embeddable in A and B; otherwise $B \leq_{fe} X \leq_{fe} A$, so $B \leq_{fe} A$; a contradiction. □

Corollary 2.8 *For every infinite set $X \subseteq \mathbb{N}$ there is an infinite descending chain $X = X_0 \supset X_1 \supset X_2 \ldots$ in $\mathcal{P}_{\aleph_0}(\mathbb{N})$ such that $X_{i+1} \not\leq_{fe} X_i$ for every $i \in \mathbb{N}$.*

Proof The result follows immediately by Corollary 2.7. □

Corollary 2.9 *There are no minimal elements in $(\mathcal{P}_{\aleph_0}(\mathbb{N}), \leq_{fe})$.*

Proof The result follows immediately by Corollary 2.8. □

3 Properties of finite embeddability between ultrafilters

In this section we want to prove some basic properties of $(\beta\mathbb{N}, \leq_{fe})$, in particular an analogue of Theorem 2.3 for ultrafilters, and to characterize the maximal ultrafilters with respect to $\leq_{fe}$. We fix some notations: we will denote by $\equiv_{fe}$ the equivalence relation such that, for every $\mathcal{U}, \mathcal{V}$ ultrafilters on $\mathbb{N}$,

$$\mathcal{U} \equiv_{fe} \mathcal{V} \Leftrightarrow \mathcal{U} \leq_{fe} \mathcal{V} \text{ and } \mathcal{U} \leq_{fe} \mathcal{V}.$$

3.1 Some basic properties of finite embeddability between ultrafilters

The first result that we prove is that Theorem 2.3 has an analogue in the setting of ultrafilters.

Theorem 3.1 *For every $\mathcal{U}, \mathcal{V} \in \beta\mathbb{N}$ if $\mathcal{U} \leq_{fe} \mathcal{V} \leq_{fe} \mathcal{U} \oplus 1$ then $\mathcal{U} \equiv_{fe} \mathcal{V}$ or $\mathcal{U} \oplus 1 \equiv_{fe} \mathcal{V}$.*

Proof Let us suppose that $\mathcal{U} \oplus 1 \not\leq_{fe} \mathcal{V} \not\leq_{fe} \mathcal{U}$. In particular, $\mathcal{U} \oplus 1 \neq \mathcal{V}$, so there exists $A \in \mathcal{U}$ such that $A + 1 \notin \mathcal{V}$. Since $\mathcal{V} \not\leq_{fe} \mathcal{U}$ there exists $B \in \mathcal{U}$ such that $K \not\leq_{fe} B$ for every $K \in \mathcal{V}$. In particular, $K \not\leq_{fe} A \cap B$ for every $K \in \mathcal{V}$.

Moreover, since $(A \cap B) + 1 \in \mathcal{U} \oplus 1$ we derive that there exists $C \in \mathcal{V}$ such that $C \leq_{fe} (A \cap B) + 1$. So we have that

$$C \not\leq_{fe} (A \cap B) \text{ and } C \leq_{fe} (A \cap B) + 1;$$

by Lemma 2.2 we conclude that $C \subseteq (A \cap B) + 1$. But $C \in \mathcal{V}$, so $(A \cap B) + 1 \in \mathcal{V}$ and, since $(A \cap B) + 1 \subseteq A + 1$, this entails that $A + 1 \in \mathcal{V}$, which is absurd. □

Another result that we want to prove is that $(\beta\mathbb{N}, \leq_{fe})$ is not a total preorder.

Proposition 3.2 *There are nonprincipal ultrafilters $\mathcal{U}, \mathcal{V}$ such that $\mathcal{U}$ is not finitely embeddable in $\mathcal{V}$ and $\mathcal{V}$ is not finitely embeddable in $\mathcal{U}$.*

Proof Let A, B be strongly mutually unembeddable infinite sets (which existence is a consequence of Proposition 2.6). Let $\mathcal{U}, \mathcal{V}$ be nonprincipal ultrafilters such that $A \in \mathcal{U}, B \in \mathcal{V}$ and let us suppose that $\mathcal{U} \leq_{fe} \mathcal{V}$. Let $C \in \mathcal{U}$ be such that $C \leq_{fe} B$. Since $C \in \mathcal{U}$, $A \cap C$ is in $\mathcal{U}$ and it is infinite (since $\mathcal{U}$ is nonprincipal). So we have that

- $A \cap C \leq_{fe} B$, since $A \cap C \subseteq C$;
- $A \cap C \not\leq_{fe} B$, since $A \not\equiv_S B$.

This is absurd, so $\mathcal{U}$ is not finitely embeddable in $\mathcal{V}$. In the same way we can prove that $\mathcal{V}$ is not finitely embeddable in $\mathcal{U}$. □

It is easy to show that if we identify each natural number n with the principal ultrafilter $\mathcal{U}_n = \{A \in \mathcal{P}(\mathbb{N}) \mid n \in A\}$ then $(\mathbb{N}, \leq)$ is an initial segment in $(\beta\mathbb{N}, \leq_{fe})$. In particular, $\mathcal{U}_0$ is the minimum element in $\beta\mathbb{N}$. One may wonder if there is a minimum element in $(\beta\mathbb{N} \setminus \mathbb{N}, \leq_{fe})$; the answer is no. In the following proposition, by Θ_X we mean the clopen set

$$\Theta_X = \{\mathcal{U} \in \beta\mathbb{N} \mid X \in \mathcal{U}\}.$$

Proposition 3.3 *Let X be an infinite subset of $\mathbb{N}$. Then there is not a minimum in $(\Theta_X \setminus \mathbb{N}, \leq_{fe})$.*

Proof Let us suppose that such a minimum exists, and let $\mathcal{U} \in \Theta_X$ be the minimum. Let $A, B \subseteq X$ be mutually unembeddable subsets of X and let $\mathcal{V}_1, \mathcal{V}_2$ be nonprincipal ultrafilters such that $A \in \mathcal{V}_1$ and $B \in \mathcal{V}_2$ (in particular, $\mathcal{V}_1, \mathcal{V}_2 \in \Theta_X$). Since, by hypothesis, $\mathcal{U}$ is the minimum, there are $C_1, C_2 \in \mathcal{U}$ such that $C_1 \leq_{fe} A$ and $C_2 \leq_{fe} B$. Let us consider $C_1 \cap C_2 \in \mathcal{U}$. By construction, $C_1 \cap C_2$ is finitely embeddable in A and in B. But this is absurd: in fact, let $c_1 < c_2$ be any two elements in $C_1 \cap C_2$. Then there exist n, m such that $n + \{c_1, c_2\} = \{a_1, a_2\} \subset A$ and $m + \{c_1, c_2\} = \{b_1, b_2\} \subset B$, and this cannot happen, because in this case we would have $b_2 - b_1 = c_2 - c_1 = a_2 - a_1$, while $A \not\equiv_S B$. □

In particular, by taking $X = \mathbb{N}$, we obtain the following result.

Corollary 3.4 *There is not a minimum in $(\beta\mathbb{N} \setminus \mathbb{N}, \leq_{fe})$.*

3.2 Maximal ultrafilters

To study maximal ultrafilters in $(\beta\mathbb{N}, \leq_{fe})$ we need to recall three results that have been proved in Blass and Di Nasso [1]. To do that we need to introduce the notion of upward cone generated by an ultrafilter.

Definition 3.5 *For any $\mathcal{U} \in \beta\mathbb{N}$ the upward cone generated by $\mathcal{U}$ is the set*

$$\mathcal{C}(\mathcal{U}) = \{\mathcal{V} \in \beta\mathbb{N} \mid \mathcal{U} \leq_{fe} \mathcal{V}\}.$$

The following are the results proved in Blass and Di Nasso [1] that we will need.

Theorem 3.6 *([1, Theorem 11]) Let $\mathcal{U}, \mathcal{V}$ be ultrafilters on $\mathbb{N}$. Then $\mathcal{U} \leq_{fe} \mathcal{V}$ if and only if $\mathcal{V} \in \overline{\{\mathcal{U} \oplus \mathcal{W} \mid \mathcal{W} \in \beta\mathbb{N}\}}$.*

Corollary 3.7 *([1, Corollary 13]) The ordering $\leq_{fe}$ on ultrafilters on $\mathbb{N}$ is upward directed.*

Corollary 3.8 *([1, Corollary 14]) For any $\mathcal{U} \in \beta\mathbb{N}$, the upward cone $\mathcal{C}(\mathcal{U})$ is a closed, two-sided ideal in $\beta\mathbb{N}$. It is the smallest closed right ideal containing $\mathcal{U}$ and therefore it is also the smallest two-sided ideal containing $\mathcal{U}$.*

For completeness, even if we will not use this fact, we also recall that Corollary 3.7 can be improved: in fact (as proved by Blass and Di Nasso in [1] and by Luperi Baglini in [7]) for every $\mathcal{U}, \mathcal{V} \in \beta\mathbb{N}$ we have

$$\mathcal{U}, \mathcal{V} \leq_{fe} \mathcal{U} \oplus \mathcal{V}.$$

Let us note that from Theorem 3.6 it easily follows that the relation $\leq_{fe}$ is not antisymmetric: in fact, if R is a minimal right ideal in $(\beta\mathbb{N}, \oplus)$ and $\mathcal{U} \in R$ then $\mathcal{C}(\mathcal{U}) = \mathcal{C}(\mathcal{U} \oplus 1)$, so $\mathcal{U} \leq_{fe} \mathcal{U} \oplus 1$ and $\mathcal{U} \oplus 1 \leq_{fe} \mathcal{U}$.

We want to prove that there exist maximum ultrafilters in $(\beta\mathbb{N}, \leq_{fe})$. Due to Corollary 3.8, since $(\beta\mathbb{N}, \leq_{fe})$ is upward directed then to prove that there are maximum ultrafilters it is sufficient[2] to prove that there are maximal elements. To prove the existence of maximal elements we use Zorn's Lemma. A technical lemma that we need is the following.

Lemma 3.9 *Let I be a totally ordered set. Then there is an ultrafilter $\mathcal{V}$ on I such that, for every element $i \in I$, the set*

$$G_i = \{j \in I \mid j \geq i\}.$$

is included in $\mathcal{V}$.

[2] Every maximal element in an upward directed preordered set $(A, \leq)$ is a maximum.

Proof We simply observe that $\{G_i\}_{i\in I}$ is a filter and we recall that every filter can be extended to an ultrafilter. □

The key property of these ultrafilters is the following:

Proposition 3.10 *Let I be a totally ordered set and let $\mathcal{V}$ be given as in Lemma 3.9. Then for every $A \in \mathcal{V}$ and $i \in I$ there exists $j \in A$ such that $i \leq j$.*

We omit the straightforward proof.

In the next Theorem we use the notion of limit ultrafilter. We recall that, given an ordered set I, an ultrafilter $\mathcal{V}$ on I and a family $\mathcal{U}_i$ of ultrafilters on $\mathbb{N}$, the $\mathcal{V}$-limit of the family $\langle \mathcal{U}_i \mid i \in I \rangle$ (denoted by $\mathcal{V} - \lim_{i\in I} \mathcal{U}_i$) is the ultrafilter such that for every $A \subseteq \mathbb{N}$,

$$A \in \mathcal{V} - \lim_{i\in I} \mathcal{U}_i \Leftrightarrow \{i \in I \mid A \in \mathcal{U}_i\} \in \mathcal{V}.$$

Let us introduce the notion of $\leq_{fe}$-chain.

Definition 3.11 *Let $(I, <)$ be an ordered set. We say that $\langle \mathcal{U}_i \mid i \in I \rangle$ is a $\leq_{fe}$-chain in $\beta\mathbb{N}$ if for every $i < j \in I$ we have $\mathcal{U}_i \leq_{fe} \mathcal{U}_j$.*

Theorem 3.12 *Every $\leq_{fe}$-chain $\langle \mathcal{U}_i \mid i \in I \rangle$ has an $\leq_{fe}$-upper bound $\mathcal{U}$.*

Proof Let $\mathcal{V}$ be an ultrafilter on I with the property expressed in Lemma 3.9. We claim that the ultrafilter

$$\mathcal{U} = \mathcal{V} - \lim_{i\in I} \mathcal{U}_i$$

is a $\leq_{fe}$-upper bound for the $\leq_{fe}$-chain $\langle \mathcal{U}_i \mid i \in I \rangle$. We have to prove that $\mathcal{U}_i \leq_{fe} \mathcal{U}$ for every index i. Let A be an element of $\mathcal{U}$. By definition,

$$A \in \mathcal{U} \Leftrightarrow I_A = \{i \in I \mid A \in \mathcal{U}_i\} \in \mathcal{V}.$$

I_A is a set in $\mathcal{V}$ so by Proposition 3.10 there exists an element $j > i$ in I_A. Therefore $A \in \mathcal{U}_j$ and, since $\mathcal{U}_i \leq_{fe} \mathcal{U}_j$, there exists an element B in $\mathcal{U}_i$ with $B \leq_{fe} A$. Hence $\mathcal{U}_i \leq_{fe} \mathcal{U}$ and the theorem is proved. □

Being an upward directed set with maximal elements, $(\beta\mathbb{N}, \leq_{fe})$ contains maximum ultrafilters. We denote by $\mathcal{M}$ the set of maximum ultrafilters. By definition, for every ultrafilter $\mathcal{U}$ we have that

$$\mathcal{U} \in \mathcal{M} \Leftrightarrow \mathcal{V} \leq_{fe} \mathcal{U} \text{ for every } \mathcal{V} \in \beta\mathbb{N}.$$

In particular, we can characterize $\mathcal{M}$ in terms of the $\leq_{fe}$-cones.

Corollary 3.13 $\mathcal{M} = \bigcap_{\mathcal{U} \in \beta\mathbb{N}} \mathcal{C}(\mathcal{U})$.

Proof We simply observe that $\mathcal{M} \subseteq \mathcal{C}(\mathcal{U})$ for every ultrafilter $\mathcal{U}$ and that, if $\mathcal{U}$ is a maximum ultrafilter, then $\mathcal{C}(\mathcal{U}) = \mathcal{M}$. □

We can now prove our main result.

Theorem 3.14 $\mathcal{M} = \overline{K(\beta\mathbb{N}, \oplus)}$.

Proof Given any ultrafilter $\mathcal{U}$, by Proposition 3.6 we know that $\mathcal{C}(\mathcal{U})$ is the minimal closed bilateral ideal containing $\mathcal{U}$. By Corollary 3.13 we know that $\mathcal{M} = \bigcap_{\mathcal{U} \in \beta\mathbb{N}} \mathcal{C}(\mathcal{U})$ so, in particular, being the intersection of a family of closed bilateral ideal $\mathcal{M}$ itself is a closed bilateral ideal. So if $\mathcal{U}$ is any ultrafilter in $K(\beta\mathbb{N}, \oplus)$, we know that:

(i) $\mathcal{M} \subseteq \mathcal{C}(\mathcal{U})$;

(ii) $\mathcal{C}(\mathcal{U}) = \overline{K(\beta\mathbb{N}, \oplus)}$.

Therefore $\mathcal{M}$ is a closed bilateral ideal included in $\overline{K(\beta\mathbb{N}, \oplus)}$, and the only such ideal is $\overline{K(\beta\mathbb{N}, \oplus)}$ itself. □

The previous result has a few interesting consequences.

Corollary 3.15 *An ultrafilter $\mathcal{U}$ is a maximum in $(\beta\mathbb{N}, \leq_{fe})$ if and only if every element A of $\mathcal{U}$ is piecewise syndetic.*

Proof This follows from a well known characterization of $\overline{K(\beta\mathbb{N}, \oplus)}$: an ultrafilter $\mathcal{U}$ is in $\overline{K(\beta\mathbb{N}, \oplus)}$ if and only if every element A of $\mathcal{U}$ is piecewise syndetic (see, eg, Hindman and Strauss [6]). □

We thank one of the anonymous referees for pointing out the following property of piecewise syndetic sets.

Corollary 3.16 *If A is piecewise syndetic then for every infinite $B \subseteq \mathbb{N}$ there exists an infinite $C \subseteq B$ such that $C \leq_{fe} A$.*

Proof Let $\mathcal{U}$ be a maximum ultrafilter such that $A \in \mathcal{U}$ and $\mathcal{V}$ be a nonprincipal ultrafilter such that $B \in \mathcal{V}$. Since $\mathcal{U}$ is a maximum there exists $D \in \mathcal{V}$ such that $D \leq_{fe} A$. The result follows by setting $C = D \cap B$. □

As mentioned in the introduction, the notion of finite embeddability is related to some properties that arise in combinatorial number theory. A particularity of maximum ultrafilters in $(\beta\mathbb{N}, \leq_{fe})$ is that every set in a maximum ultrafilter satisfies many of these combinatorial properties.

Definition 3.17 *We say that a property P is $\leq_{fe}$-upward invariant if the following holds: for every $A, B \subseteq \mathbb{N}$, if $P(A)$ holds and $A \leq_{fe} B$ then $P(B)$ holds.*

We way that P is partition regular if the family $S_P = \{A \subseteq \mathbb{N} \mid P(A) \text{ holds}\}$ contains an ultrafilter (ie, if for every finite partition $\mathbb{N} = A_1 \cup ... \cup A_n$ there exists at least one index $i \leq n$ such that $A_i \in S_P$).

Clearly, the following properties are $\leq_{fe}$-upward invariant:

(i) A is thick;

(ii) A is piecewise syndetic;

(iii) A contains arbitrarily long arithmetic progressions;

(iv) $BD(A) > 0$, where $BD(A)$ is the upper Banach density of A.

In particular, properties (ii), (iii), (iv) are also partition regular: that (ii) is partition regular was originally proved by T. Brown in [2], property (iii) is the content of Van der Waerden's Theorem (see [11]) and (iv) is due to the subadditivity of the upper Banach density. We now want to show that these properties are important in relation to maximal ultrafilters.

Proposition 3.18 *Let P be a partition regular $\leq_{fe}$-upward invariant property of sets. Then for every piecewise syndetic set A $P(A)$ holds.*

Proof Let P be given, let $S_P = \{B \subseteq \mathbb{N} \mid P(B) \text{ holds}\}$ and let $\mathcal{V} \subseteq S_P$ (such an ultrafilter exists because P is partition regular). Let A be piecewise syndetic, and let $\mathcal{U}$ be a maximum ultrafilter such that $A \in \mathcal{U}$. Since $\mathcal{U}$ is a maximum, $\mathcal{V} \leq_{fe} \mathcal{U}$. Let $B \in \mathcal{V}$ be such that $B \leq_{fe} A$. Since P is $\leq_{fe}$-upward invariant and $P(B)$ holds, we deduce that $P(A)$ holds. □

For example, the results expressed in the following corollary can be seen also as a consequence of Proposition 3.18 (and the known facts that "being AP-rich" and "having positive upper Banach density" are partition regular properties).

Corollary 3.19 *Let $A \subseteq \mathbb{N}$ be piecewise syndetic. Then:*

(i) $BD(A) > 0$;

(ii) *A contains arbitrarily long arithmetic progressions.*

In the forthcoming paper [8] we show how similar arguments can be used to prove combinatorial properties of other families of ultrafilters, eg, to prove that for every ultrafilter $\mathcal{U} \in \overline{K(\beta\mathbb{N}, \odot)}$, for every $A \in \mathcal{U}$, A contains arbitrarily long arithmetic progression and it contains a solution to every partition regular homogeneous equation[3].

4 A nonstandard proof of the main result

In this section we assume the reader to be familiar with the basics of nonstandard analysis. In particular, we will use the notions of nonstandard extension of subsets of $\mathbb{N}$ and the transfer principle. We refer to Chang and Keisler [3] and Davis [4] for an introduction to the foundations of nonstandard analysis and to the nonstandard tools that we are going to use.

Both in Blass and Di Nasso [1] and in Luperi Baglini [7] it has been shown that the relation of finite embeddability between sets has a very nice characterization in terms of nonstandard analysis, which allows to study some of its properties in a quite simple and elegant way. We recall the characterization (in the following proposition, it is assumed

[3] An equation $P(x_1, ..., x_n) = 0$ is partition regular if and only if for every finite coloration $\mathbb{N} = C_1 \cup ... \cup C_n$ of $\mathbb{N}$ there exists an index i and monochromatic elements $a_1, ..., a_n \in C_i$ such that $P(a_1, ..., a_n) = 0$.

for technical reasons that the nonstandard extension that we consider satisfies at least the $\mathfrak{c}^+$-enlarging property[4], where $\mathfrak{c}$ is the cardinality of $\mathcal{P}(\mathbb{N})$).

Proposition 4.1 *([1, Proposition 15]) Let A, B be subsets of $\mathbb{N}$. The following two conditions are equivalent:*

(i) *A is finitely embeddable in B;*

(ii) *there is a hypernatural number α in ${}^*\mathbb{N}$ such that $\alpha + A \subseteq {}^*B$.*

We use Proposition 4.1 to reprove directly, with nonstandard methods, Theorem 3.14.

Nonstandard proof of Theorem 3.14 Let $\mathcal{U} \in \overline{K(\beta\mathbb{N}, \oplus)}$, let A be a set in $\mathcal{U}$, and let $\mathcal{V}$ be an ultrafilter on $\mathbb{N}$. Since A is piecewise syndetic there is a natural number n such that

$$T = \bigcup_{i=1}^{n} (A + i)$$

is thick. By transfer[5] it follows that there are hypernatural numbers $\alpha \in {}^*\mathbb{N}$ and $\eta \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that the interval $[\alpha, \alpha + \eta]$ is included in ${}^*T$. In particular, since η is infinite, $\alpha + \mathbb{N} \subseteq {}^*T$.

For every $i \leq n$ we consider

$$B_i = \{n \in \mathbb{N} \mid \alpha + n \in {}^*(A + i)\}.$$

Since $\bigcup_{i=1}^{n} B_i = \mathbb{N}$, there is an index i such that $B_i \in \mathcal{V}$. We claim that $B_i \leq_{fe} A$. In fact, by construction $\alpha + B_i \subseteq {}^*A + i$, so

$$(\alpha - i) + B_i \subseteq {}^*A.$$

By Proposition 4.1, this entails that $B_i \leq_{fe} A$, and this proves that $\mathcal{V} \leq_{fe} \mathcal{U}$ for every ultrafilter $\mathcal{V}$. Hence $\mathcal{U}$ is a maximum in $(\beta\mathbb{N}, \leq_{fe})$, therefore $\overline{K(\beta\mathbb{N}, \oplus)} \subseteq \mathcal{M}$. But, since the property of being piecewise syndetic is $\leq_{fe}$-upward invariant, from Proposition 3.18 we also derive that $\mathcal{M} \subseteq \overline{K(\beta\mathbb{N}, \oplus)}$. □

[4] We recall that a nonstandard extension ${}^*\mathbb{N}$ of $\mathbb{N}$ has the $\mathfrak{c}^+$ enlarging property if, for every family $\mathcal{F}$ of subsets of $\mathbb{N}$ with the finite intersection property, the intersection $\bigcap_{A \in \mathcal{F}} {}^*A$ is nonempty.

[5] Thick sets can be characterized by mean of nonstandard analysis as follows (see, eg, Luperi Baglini [7]): a set $T \subseteq \mathbb{N}$ is thick if and only if ${}^*T$ contains an interval of infinite length.

5 Questions

We conclude the paper with two open questions:

- Does there exist a minimal ultrafilter in $\beta\mathbb{N}\setminus\mathbb{N}$?

We conjecture that the answer is no. In fact, due to some (vague) similarities between the finite embeddability and the Rudin-Keisler preorder on ultrafilters (see, eg, Rudin [9]), one might guess that selective ultrafilters are minimal with respect to finite embeddability. To prove that this is not the case we recall the following characterization of selective ultrafilters: an ultrafilter $\mathcal{S}$ on $\mathbb{N}$ is selective if and only if for every function $f:\mathbb{N}\to\mathbb{N}$ there exists $A\in\mathcal{S}$ such that f is constant or strictly increasing on A.

Proposition 5.1 *If $\mathcal{S}$ is a non principal selective ultrafilter on $\mathbb{N}$ then $\mathcal{S}\not\leq_{fe}\mathcal{S}\ominus 1$.*

Proof Let

$$x_0=0;\ x_1=1;\ x_{n+1}=2x_n+1.$$

Let $f:\mathbb{N}\to\mathbb{N}$ be defined as follows:

$$f(x)=n\Leftrightarrow x_n\leq x<x_{n+1}.$$

Since f cannot be constant on any infinite set, there exists $A\in\mathcal{S}$ such that f is strictly increasing on A. Let $A=\{a_n\mid n\in\mathbb{N}\}$, where $a_n<a_{n+1}$ for every $n\in\mathbb{N}$. Let $A=A_1\cup A_2$, where

$$A_1=\{a_{2n}\mid n\in\mathbb{N}\};\quad A_2=\{a_{2n+1}\mid n\in\mathbb{N}\}.$$

Let us suppose that $A_1\in\mathcal{S}$ (the other case can be treated similarly). Let $A_1=\{a_{1,n}\mid n\in\mathbb{N}\}$, where $a_{1,n}<a_{1,n+1}$ for every $n\in\mathbb{N}$. Then, by construction, we have that

$$a_{1,n+1}-a_{1,n}>x_{2n+2}-x_{2n+1}>x_{2n+1}\geq a_{1,n}-a_{1,n-1}.$$

In particular, it follows that $A\not\leq_{fe}^{S}A-1$: in fact, let $a_n,a_m\in A$, $n<m$. Let us suppose that there exist $k,i,j\in\mathbb{N}$, $i<j$ such that $k+\{a_n,a_m\}=\{a_i-1,a_j-1\}$. Then $a_m-a_n=a_j-a_i$, and this by construction is possible if and only if $a_n=a_i$ and $a_m=a_j$. But then we would have $k=-1$, which is impossible.

In particular, for every $B \in \mathcal{S}$ we have that $B \not\leq_{fe} A_1 - 1$: in fact if there exists such a B then we would have that $B \cap A_1 \leq_{fe} A_1 - 1$ and this is impossible. So we have the result. □

In particular, since $\mathcal{U} \ominus 1 \leq_{fe} \mathcal{U}$ for $\mathcal{U} \in \beta\mathbb{N} \setminus \mathcal{U}_0$, we deduce that selective ultrafilters are not minimal in $\beta\mathbb{N} \setminus \mathbb{N}$. Still we do not know if there are any minimal ultrafilters.

- Is it possible to find any simple characterization of the property $\mathcal{U} \not\leq_{fe} \mathcal{V}$ and $\mathcal{V} \not\leq_{fe} \mathcal{U}$?

From the proof of Proposition 3.2 it follows that if there exist infinite sets $A \not\equiv_S B$ with $A \in \mathcal{U}$ and $B \in \mathcal{V}$ then $\mathcal{U} \not\leq_{fe} \mathcal{V}$ and $\mathcal{V} \not\leq_{fe} \mathcal{U}$. And by the definitions it follows that if $\mathcal{U} \not\leq_{fe} \mathcal{V}$ and $\mathcal{V} \not\leq_{fe} \mathcal{U}$ then there are $A \in \mathcal{U}$, $B \in \mathcal{V}$ such that $A \not\leq_{fe} B$ and $B \not\leq_{fe} A$. Is it possible to improve this last result?

References

[1] A. Blass, M. Di Nasso, *Finite Embeddability of sets and ultrafilters*, submitted, arXiv 1405.2841.

[2] T. Brown, *An interesting combinatorial method in the theory of locally finite semigroups*, Pacific J. Math. 36 (1971), 285–289, doi: 10.2140/pjm.1971.36.285.

[3] C. C. Chang, H. J. Keisler, *Model theory (3rd ed.)*, North-Holland, Amsterdam (1990), ISBN 0-444-88054-2.

[4] M. Davis, *Applied Nonstandard Analysis*, John Wiley & Sons (1977), ISBN 10: 0486442292.

[5] M. Di Nasso, *Embeddability properties of difference sets*, Integers, Volume 14 (2014), A27.

[6] N. Hindman, D. Strauss, *Algebra in the Stone-Čech Compactification* (2nd edition), de Gruyter (2012), ISBN: 978-3-11-025835-6.

[7] L. Luperi Baglini, *Hyperintegers and Nonstandard Techniques in Combinatorics of Numbers*, PhD Dissertation (2012), University of Siena, arXiv 1212.2049.

[8] L. Luperi Baglini, *$\mathcal{F}$-finite embeddabilities for sets and ultrafilters*, in preparation, arXiv 1401.6518.

[9] M. E. Rudin, *Partial orders on the types in $\beta\mathbb{N}$*, Trans. Amer. Math. Soc. 155 (1971), 353–362, doi: 10.1090/S0002-9947-1971-0273581-5.

[10] I. Z. Ruzsa, *On difference sets*, Studia Sci. Math. Hungar. 13 (1978), 319–326.

[11] B. van der Waerden, *Beweis einer Baudetschen Vermutung*, Nieuw Arch. Wiskunde 19 (1927), 212–216, JFM 53.0073.12.

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