A non-archimedean algebra and the Schwartz impossibility theorem

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Abstract

In the 1950s L. Schwartz proved his famous impossibility result: for every $k \in \mathbb{N}$ there does not exist a differential algebra $(\mathsf{A}, +, \otimes, D)$ in which the distributions can be embedded, where D is a linear operator that extends the distributional derivative and satisfies the Leibnitz rule (namely $D(u \otimes v) = Du \otimes v + u \otimes Dv$) and $\otimes$ is an extension of the pointwise product on $\mathcal{C}^0(\mathbb{R})$.

In this paper we prove that, by changing the requests, it is possible to avoid the impossibility result of Schwartz. Namely we prove that it is possible to construct an algebra of functions $(\mathsf{A},+,\otimes,D)$ such that (1) the distributions can be embedded in A in such a way that the restriction of the product to $\mathcal{C}^1(\mathbb{R})$ functions agrees with the pointwise product, namely for every $f, g \in \mathcal{C}^1(\mathbb{R})$

$$\Phi(fg) = \Phi(f) \otimes \Phi(g),$$

and (2) there exists a linear operator $D : \mathsf{A} \to \mathsf{A}$ that extends the distributional derivative and satisfies a weak form of the Leibnitz rule.

The algebra that we construct is an algebra of restricted ultrafunctions, which are generalized functions defined on a subset Σ of a non-archimedean field $\mathbb{K}$ (with $\mathbb{R} \subset \Sigma \subset \mathbb{K}$) and with values in $\mathbb{K}$. To study the restricted ultrafunctions we will use some techniques of nonstandard analysis.

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Contents

1 Introduction

There is an issue regarding distributions that is important for a variety of applications, namely the problem of defining a multiplication of distributions that satisfies some property of coherence with respect to the weak derivative and to the restriction to continuous functions (see [9], Chapter 1 for a discussion on this topic). A possible way to define such a multiplication is to embed the space of distributions in a differential algebra $(\mathfrak{A}, +, \otimes, D)$ and to use $\otimes$ to define the multiplication of distributions. A famous result that limits this approach was proved by L. Schwartz in [13]: he proved that it is impossible to construct a differential algebra $(\mathfrak{A}, +, \otimes, D)$ such that

(i) there is a linear embedding

$$\Phi : \mathcal{D}'(\mathbb{R}) \to \mathfrak{A}$$

such that $\Phi(1)$ is the unity in $\mathfrak{A}$;

(ii) there is a linear operator $D : \mathfrak{A} \to \mathfrak{A}$ such that the following diagram

$$\begin{array}{ccc} \mathcal{D}'(\mathbb{R}) & \xrightarrow{\partial} & \mathcal{D}'(\mathbb{R}) \\ \Phi \downarrow & & \Phi \downarrow \\ \mathfrak{A} & \xrightarrow{D} & \mathfrak{A} \end{array}$$

commutes, where ∂ is the usual distributional derivative;

(iii) the restriction of $\otimes$ to the continuous functions agrees with the pointwise product, namely

$$\Phi(fg) = \Phi(f) \otimes \Phi(g);$$

(iv) the Leibnitz rule holds:

$$D(uv) = Duv + uDv.$$

Actually, for every $k \in \mathbb{N}$, the impossibility result holds even if we modify (iii) as follows:

(iii)$_k$ the restriction of $\otimes$ to $\mathcal{C}^k(\mathbb{R}) \times \mathcal{C}^k(\mathbb{R})$ agrees with the pointwise product, namely

$$\Phi(fg) = \Phi(f) \otimes \Phi(g).$$

In order to embedd the distributions in a differential algebra one has to weaken at least one of the requests (i),..., (iv). A famous approach to this problem is given by Colombeau's Algebras, in which (iii) is replaced by

(iii)$_\infty$ the restriction of $\otimes$ to $\mathcal{C}^\infty(\mathbb{R}) \times \mathcal{C}^\infty(\mathbb{R})$ agrees with the pointwise product, namely

$$\Phi(fg) = \Phi(f) \otimes \Phi(g) .$$

Jean-François Colombeau proved the existence of algebras satisfying (i), (ii), (iii)$_\infty$, (iv). The central ideas of his construction were first published in [4], [5] and [7], and the foundations of his work are written in the books [7], [8]. For a more recent reference on this topic we suggest the book [9].

In this paper we prove a different existence result by relaxing the requests (i), (ii), (iii), (iv) in a different way. We slightly weaken (iii) but we weaken (iv) in a more substantial way. We substitute (iii) with (iii)$_1$, namely

(iii)$_1$: the restriction of $\otimes$ to $\mathcal{C}^1(\mathbb{R})\times\mathcal{C}^1(\mathbb{R})$ agrees with the pointwise product, namely

$$\Phi(fg) = \Phi(f) \otimes \Phi(g) .$$

Let us show how we weaken the Leibnitz rule (iv). If u and v are functions. by integrating (iv) we get

$$\int Duv + \int uDv = [uv]_{-\infty}^{+\infty} , \tag{1}$$

provided that

$$[uv]_{-\infty}^{+\infty} = \lim_{x\to+\infty} u(x)v(x) - \lim_{x\to-\infty} u(x)v(x).$$

is well defined. Clearly (1) is a weaker request than the Leibnitz rule. We make a request on the elements of $\mathfrak{A}$ which generalizes (1):

(iv)' (**Weak Leibnitz Rule**) For every $u, v \in \mathfrak{A}$

$$\langle Du, v\rangle + \langle u, Dv\rangle \ = [uv]_{-\beta}^{+\beta} , \tag{2}$$

where $\langle u, v\rangle$ is a scalar product such that, for every $f, g \in \mathcal{C}_0^1(\mathbb{R})$,

$$\langle \Phi(T_f), \Phi(T_g)\rangle = \int f(x)g(x)dx$$

where T_f, T_g are the distributions associated to f, g and β is a suitable "point at infinity".

Notice that (1) is used to define the notion of weak derivative and the duality in the theory of distribution. So, even if (1) and (2) are weaker than the Leibnitz rule, they are essential in the applications.

We will show that the requests (i), (ii), $(\text{iii})_1$, (iv)' are consistent by constructing explicitly an algebra $\mathfrak{A}$ that satisfies these properties. This construction will be done by using the space of ultrafunctions, which is a space of generalized functions that has been introduced in [1] and further studied in [2] and [3]. An interesting feature of the algebra $\mathfrak{A}$ is that there exists a non-archimedean field $\mathbb{K} \supset \mathbb{R}$ such that $\mathfrak{A}$ is a subalgebra of the algebra of functions

$$u : \Sigma \to \mathbb{K} \text{ where } \mathbb{R} \subset \Sigma \subset \mathbb{K},$$

equipped with the pointwise operations:

$$(u + v)(x) = u(x) + v(x); \quad (u \otimes v)(x) = u(x)v(x).$$

Our construction uses some tools of nonstandard analysis. In the literature, nonstandard analysis has been used many times to study questions related to Schwartz's impossibility result and to the Colombeau's algebras. For example, in [12], the field of asymptotic real numbers has been introduced, which is related to Colombeau algebras; also we recall the more recent results in [10] and [14]. However, our construction is quite different with respect to these previous nonstandard approaches.

1.1 Notations and definitions

We use this section to fix some notations and to recall some definitions:

- $\mathfrak{F}(X, Y)$ denotes the set of all functions from X to Y;
- $\mathfrak{F}(\mathbb{R}) = \mathfrak{F}(\mathbb{R}, \mathbb{R})$;
- $\mathcal{C}(\mathbb{R})$ denotes the set of continuous $f : \mathbb{R} \to \mathbb{R}$;
- $\mathcal{C}_0(\mathbb{R})$ denotes the set of functions in $\mathcal{C}(\mathbb{R})$ having compact support;
- $\mathcal{C}^k(\mathbb{R})$ denotes the set of functions in $\mathcal{C}(\mathbb{R})$ which have continuous derivatives up to the order k;
- $\mathcal{C}_0^k(\mathbb{R})$ denotes the set of functions in $\mathcal{C}^k(\mathbb{R})$ having compact support;
- $\mathcal{D}(\mathbb{R})$ denotes the set of the infinitely differentiable functions with compact support; $\mathcal{D}'(\mathbb{R})$ denotes the topological dual of $\mathcal{D}(\mathbb{R})$, namely the set of distributions on $\mathbb{R}$;
- if $\mathbb{K}$ is a linearly ordered field and $a, b \in \mathbb{K}$, then
  - $[a, b]_{\mathbb{K}} = \{x \in \mathbb{K} : a \leq x \leq b\}$;
  - $(a, b)_{\mathbb{K}} = \{x \in \mathbb{K} : a < x < b\}$;
- an element k of an ordered field $\mathbb{K}$ is infinite if $|k| > n$ for every natural number n;
- an ordered field $\mathbb{K}$ is non-archimedean if it contains infinite elements;
- a field $\mathbb{K}$ is superreal if it properly contains the field $\mathbb{R}$.

2 The main result

In this section we state the main result of the paper, which will be proved in section 3.2.

Theorem 1 *There exists an algebra* $(\mathfrak{A}, +, \cdot, \mathrm{D})$ *that satisfies the following properties:*

- *($\mathfrak{A}$-0) (**Algebraic structure**)* $\mathfrak{A} \subseteq \mathfrak{F}(\Sigma, \mathbb{K})$ *where* $\mathbb{K}$ *is a non-archimedean field and* Σ *is a set such that*

$$\mathbb{R} \subset \Sigma \subset \mathbb{K};$$

  $\mathfrak{A}$ *is an algebra equipped with the pointwise operations:*

$$(u+v)(x) = u(x) + v(x); \quad (u \cdot v)(x) = u(x) \cdot v(x).$$

- *($\mathfrak{A}$-1) (**Embedding of distributions**) There is a linear embedding*

$$\Phi : \mathcal{D}'(\mathbb{R}) \to \mathfrak{A}$$

  and a bilinear form $\langle \cdot, \cdot \rangle : \mathfrak{A} \times \mathfrak{A} \to \mathbb{K}$ *such that,* $\forall T \in \mathcal{D}'(\mathbb{R})$, $\forall \varphi \in \mathcal{D}(\mathbb{R})$,

$$T[\varphi] = \langle \Phi(T), \Phi(T_\varphi) \rangle .$$

- *($\mathfrak{A}$-2) (**Extension of the derivative**) There is a linear operator* $\mathrm{D} : \mathfrak{A} \to \mathfrak{A}$ *such that the diagram*

$$\begin{array}{ccc} \mathcal{D}'(\mathbb{R}) & \xrightarrow{\partial} & \mathcal{D}'(\mathbb{R}) \\ \downarrow \Phi & & \downarrow \Phi \\ \mathfrak{A} & \xrightarrow{\mathrm{D}} & \mathfrak{A} \end{array} \tag{3}$$

  commutes, where ∂ *is the usual distributional derivative.*

- *($\mathfrak{A}$-3) (**Extension of the product**) The restriction of* $\cdot$ *to* $\mathcal{C}^1(\mathbb{R})$ *agrees with the pointwise product namely, if* $f, g \in \mathcal{C}^1(\mathbb{R})$, *then*

$$\Phi(T_{fg}) = \Phi(T_f) \cdot \Phi(T_g) .$$

- *($\mathfrak{A}$-4) (**Weak Leibnitz rule**) For every* $u, v \in \mathfrak{A}$ *the following holds:*

$$\langle \mathrm{D}u, v \rangle + \langle u, \mathrm{D}v \rangle = [uv]_{-\beta}^{+\beta} ,$$

  where $\beta = \max(\Sigma)$, $-\beta = \min(\Sigma)$.

- *($\mathfrak{A}$-5) (**Locality of the extension**) If the support of a distribution* T *is included in* $[a, b] \subset \mathbb{R}$ *then, for every* $x \in \Sigma \setminus [a, b]_{\mathbb{K}}$, *we have*

$$\Phi(T)(x) = 0.$$

Let us observe that, since $\beta \in \mathbb{K} \setminus \mathbb{R}$, β is an infinite number in $\mathbb{K}$ and that every algebra given by Theorem 1 satisfies the requests (i), (ii), (iii)$_1$, (iv)' outlined in the introduction; moreover, as an immediate consequence of Theorem 1, the operator D and the scalar product $\langle \cdot, \cdot \rangle$ have properties similar to the duality of distributions. In the following corollary we identify every function $f \in \mathcal{C}_0^1(\mathbb{R})$ with its counterpart in $\mathfrak{A}$, namely with $\Phi(T_f)$.

Corollary 2 $\forall u \in \mathfrak{A}$, $\forall f \in \mathcal{C}_0^1(\mathbb{R})$, $\langle \mathrm{D}u, f \rangle = -\langle u, \partial f \rangle$.

Proof. By ($\mathfrak{A}$-4) we have $\langle \mathrm{D}u, f \rangle + \langle u, \mathrm{D}f \rangle = [uf]_{-\beta}^{+\beta}$, and by ($\mathfrak{A}$-5) we get that $[uf]_{-\beta}^{\beta} = 0$. Moreover, by ($\mathfrak{A}$-2) it follows that $\mathrm{D}f = \Phi(\partial T_f) = \partial f$ (with respect to our identification), hence we can conclude. □

3 Construction of the Algebra

3.1 The Ultrafunctions

Throughout this section we assume that the reader has a basic knowledge of nonstandard analysis (for a general reference on the subject, see e.g. [11]). We work in a (at least) $(2^{\mathfrak{c}})^+$-saturated extension of the real numbers[1] (where $\mathfrak{c}$ stands for the cardinality of continuum), and we take as standard universe the superstructure $V(\mathbb{R})$ on $\mathbb{R}$. We recall that, given a set A in $V(\mathbb{R})$, A^σ is the set

$$A^\sigma = \{a^* \mid a \in A\}.$$

We let Λ denote a hyperfinite set in $\mathfrak{F}(\mathbb{R},\mathbb{R})^*$ with $\mathfrak{F}(\mathbb{R},\mathbb{R})^\sigma \subseteq \Lambda$. We let

$$\widetilde{\mathcal{C}^1}(\mathbb{R}) = Span\{\mathcal{C}^1(\mathbb{R})^* \cap \Lambda\}.$$

Let us observe that, by definition, $\widetilde{\mathcal{C}^1}(\mathbb{R})$ is an internal vector space of hyperfinite dimension and $\mathcal{C}^1(\mathbb{R})^\sigma \subseteq \widetilde{\mathcal{C}^1}(\mathbb{R})$.

Definition 3 *Let β be a positive infinite number. We call ultrafunctions the elements of the space V_Λ, where*

$$V_\Lambda = \{u_{\restriction [-\beta,\beta]} \mid u \in \widetilde{\mathcal{C}^1}(\mathbb{R})\}.$$

Remark 4 *In our previous works ([2], [3]) we called $\widetilde{\mathcal{C}^1}(\mathbb{R})$ the space of ultrafunctions generated by $\mathcal{C}^1(\mathbb{R})$ (which was constructed in a different, but equivalent, way). In this paper we slightly changed our definition of "ultrafunction".*

From now on, with some abuse of notation, we will say that a function φ is in V_Λ meaning that the restriction $\varphi_{\restriction [-\beta,\beta]} \in V_\Lambda$. Similarly, when we say that $f^* \in V_\Lambda$ we mean that $f^*_{\restriction [-\beta,\beta]} \in V_\Lambda$.

On the space V_Λ we can define a notion of derivative by duality as follows:

[1] We recall that, given a cardinal number k, a nonstandard model has the k^+-saturation property if for every family $\mathfrak{F}$ of internal sets with the finite intersection property and with $|\mathfrak{F}| \leq k$ the intersection $\bigcap_{A \in \mathfrak{F}} A$ is not empty.

Definition 5 *For every ultrafunction $u \in V_\Lambda$, the derivative Du of u is the unique ultrafunction such that, for every $v \in V_\Lambda$,*

$$\int_{-\beta}^{\beta} Du(x)v(x)dx = \int_{-\beta}^{\beta} \partial^* u(x)v(x)dx,$$

where $\int_{-\beta}^{\beta}$ denotes the extension of the Lebesgue integral to $\mathbb{R}^*$ with limits $-\beta, \beta$.

Let $P_{V_\Lambda} : \mathcal{C}^0(\mathbb{R})^* \to V_\Lambda$ be the orthogonal projection w.r.t. the L^2 scalar product defined on $[-\beta, \beta]$, namely for every $f \in \mathcal{C}^0(\mathbb{R})^*$ $P_{V_\Lambda} f$ is the unique ultrafunction such that, for every ultrafunction u, we have

$$\int_{-\beta}^{\beta} f(x)u(x)dx = \int_{-\beta}^{\beta} P_{V_\Lambda} f(x)u(x)dx.$$

Then D can be equivalently expressed by composition as follows:

$$D = P_{V_\Lambda} \circ \partial^*.$$

An immediate consequence of the definition is that, if $f \in \mathcal{C}^2(\mathbb{R})$, then

$$Df^* = (\partial f)^*.$$

In fact, if $f \in \mathcal{C}^2(\mathbb{R})$ then $\partial f \in \mathcal{C}^1(\mathbb{R})$ and, since $\mathcal{C}^1(\mathbb{R})^\sigma \subseteq \widetilde{\mathcal{C}^1}(\mathbb{R})$, we have $(\partial f)^* = P_{V_\Lambda}(\partial f)^* = Df^*$.

For our aims, the most important property of D is the following:

Theorem 6 *For every $u, v \in V_\Lambda$ we have*

$$\int_{-\beta}^{\beta} Du(x)v(x)dx = -\int_{-\beta}^{\beta} u(x)Dv(x)dx + [uv]_{-\beta}^{\beta}.$$

Proof. Let us compute $\int_{-\beta}^{\beta} Du(x)v(x)dx$:

$$\begin{aligned} \int_{-\beta}^{\beta} Du(x)v(x)dx &= \int_{-\beta}^{\beta} \partial u(x)v(x)dx = \\ -\int_{-\beta}^{\beta} u(x)\partial v(x)dx + [uv]_{-\beta}^{\beta} &= -\int_{-\beta}^{\beta} u(x)Dv(x)dx + [uv]_{-\beta}^{\beta}. \end{aligned}$$ □

This derivative will play a central role in the construction of the algebra $\mathfrak{A}$. One of its important properties is presented in the following:

Proposition 7 *For every $k \in \mathbb{N}^*$, for every $u \in V_\Lambda$, for every $\varphi \in \mathcal{D}(\mathbb{R})$ we have the following:*

$$\int_{-\beta}^{\beta} D^k u(x) \cdot \varphi^*(x)dx = (-1)^k \int_{-\beta}^{\beta} u(x)\partial^k \varphi^*(x)dx.$$

Proof. By internal induction on k: if $k=0$ there is nothing to prove. Let us suppose the statement true for k. Then

$$\begin{aligned}\int_{-\beta}^{\beta} D^{k+1}(u(x))\varphi^{*}(x)dx &= \int_{-\beta}^{\beta} D(D^{k}(u(x)))\varphi^{*}(x)dx = \\ &-\int_{-\beta}^{\beta} D^{k}(u(x))D\varphi^{*}(x)dx + \left[D^{k}u\cdot\varphi^{*}\right]_{-\beta}^{\beta}.\end{aligned}$$

Since $\varphi\in\mathcal{D}(\mathbb{R})$ we have $\left[D^{k}u\cdot\varphi^{*}\right]_{-\beta}^{\beta}=0$. Moreover $D\varphi^{*}=\partial\varphi^{*}\in\mathcal{D}(\mathbb{R})$. So by inductive hypothesis we have

$$\begin{aligned}-\int_{-\beta}^{\beta} D^{k}(u(x))D\varphi^{*}(x)dx &= -\int_{-\beta}^{\beta} D^{k}(u(x))\partial\varphi^{*}(x)dx = \\ &(-1)^{k+1}\int_{-\beta}^{\beta} u(x)\partial^{k+1}\varphi^{*}(x)dx,\end{aligned}$$

and the thesis is proved. □

As stated in Theorem 1, we want the algebra $\mathfrak{A}$ to be a subalgebra of $\mathfrak{F}(\Sigma,\mathbb{K})$, where $\Sigma\subseteq\mathbb{K}$ and $\mathbb{K}$ is a non-archimedean field. We fix $\mathbb{K}=\mathbb{R}^{*}$, and to choose Σ we use the notion of "independent set of points" (which has been introduced in [3]):

Definition 8 *Given a number $q\in\Omega^{*}$, we denote by $\boldsymbol{\delta}_{q}(x)$ an ultrafunction in V_{Λ} such that*

$$\forall v\in V_{\Lambda},\ \int_{-\beta}^{\beta} v(x)\boldsymbol{\delta}_{q}(x)dx=v(q). \tag{4}$$

$\boldsymbol{\delta}_{q}(x)$ is called Delta (or Dirac) ultrafunction centered in q.
A Delta-basis $\{\boldsymbol{\delta}_{a}(x)\}_{a\in\Sigma}$ ($\Sigma\subset[-\beta,\beta]$) is a basis for V_{Λ} whose elements are Delta ultrafunctions. Its dual basis $\{\sigma_{a}(x)\}_{a\in\Sigma}$ is called Sigma-basis. The set $\Sigma\subset[-\beta,\beta]$ is called set of independent points.

As we proved in [3], Theorem 19, for every $q\in[-\beta,\beta]$ there exists a unique Delta ultrafunction centered in q. Let us also note that, by saying that $\{\sigma_{a}(x)\}_{a\in\Sigma}$ is the dual basis of $\{\boldsymbol{\delta}_{a}(x)\}_{a\in\Sigma}$, we commit an abuse of language: in fact, in general, given a basis $\{e_{j}\}_{j=1}^{n}$ in a finite dimensional vector space V, the dual basis of $\{e_{j}\}_{j=1}^{n}$ is the basis $\{e_{j}'\}_{j=1}^{n}$ of the dual space V' defined, for every $1\leq j,k\leq n$, by the following relation:

$$e_{j}'\left[e_{k}\right]=\delta_{jk}.$$

When V has a scalar product $(\cdot\mid\cdot)$ there exists a base $g_{1},...,g_{n}$ of the space V such that, for every $1\leq j,k\leq n$, we have

$$(g_{j}\mid e_{k})=\delta_{jk}.$$

So $\left\{e_j'\right\}_{j=1}^n$ and $\{g_j\}_{j=1}^n$ can be identified, and $\{g_j\}_{j=1}^n$ will be called the dual basis of $\left\{e_j\right\}_{j=1}^n$.

In our case the scalar product that we consider is the extension of the L^2 scalar product to V_Λ, namely the scalar product such that, for every $u, v \in V_\Lambda$, we have

$$(u, v) = \int_{-\beta}^{\beta} u(x)v(x)dx.$$

So a Sigma-basis is characterized by the fact that, $\forall a, b \in \Sigma$,

$$\int_{-\beta}^{\beta} \delta_a(x)\sigma_b(x)dx = \delta_{ab}. \tag{5}$$

The existence of a Delta-basis (and, consequently, of a Sigma-basis) is an immediate consequence of the following fact:

Remark 9 *The set $\{\boldsymbol{\delta}_a(x)|a \in [-\beta, \beta]\}$ generates all V_Λ. In fact, let G be the vector space generated by the set $\{\boldsymbol{\delta}_a(x) \mid a \in [-\beta, \beta]\}$ and let us suppose that G is properly included in V_Λ. Then the orthogonal $G^\perp$ of G in V_Λ contains a function $f \neq 0$. But, since $f \in G^\perp$, for every $a \in [-\beta, \beta]$ we have*

$$f(a) = \int_{-\beta}^{\beta} f(x)\boldsymbol{\delta}_a(x)dx = 0,$$

so $f_{\upharpoonright_{[-\beta,\beta]}} = 0$ and this is absurd. Thus the set $\{\boldsymbol{\delta}_a(x) \mid a \in [-\beta, \beta]\}$ generates V_Λ, hence it contains a basis.

Finally, let us recall the properties of a Sigma basis that we will use (see [3], Theorem 22 for a proof of these results):

Theorem 10 *A Sigma-basis $\{\sigma_q(x)\}_{q\in\Sigma}$ satisfies the following properties:*

1. *if $u \in V_\Lambda$ then*
$$u(x) = \sum_{q\in\Sigma} u(q)\sigma_q(x);$$

2. *if two ultrafunctions u and v coincide on a set of independent points then they are equal;*

3. *if Σ is a set of independent points and $a, b \in \Sigma$ then $\sigma_a(b) = \delta_{ab}$.*

For our aims, we need to fix an independent set Σ that extends $\mathbb{R} \cup \{-\beta, \beta\}$. This is possible, as the following Theorem shows:

Theorem 11 *There exists an independent set $\Sigma \subseteq [-\beta, \beta]$ such that*

$$\mathbb{R} \cup \{-\beta, \beta\} \subseteq \Sigma.$$

Proof. Given $a \in \mathbb{R}$ let

$$\Sigma_a = \{\Sigma \subseteq [-\beta, \beta] \mid \Sigma \text{ is an independent set and } a, -\beta, \beta \in \Sigma\}.$$

Each set Σ_a is internal so, if we prove that the family $\{\Sigma_a\}_{a \in \mathbb{R}}$ has the finite intersection property, we can conclude by $\mathfrak{c}^+$-saturation (which holds, since we have chosen to work in a $(2^{\mathfrak{c}})^+$ −saturated model).

Let $a_1, ..., a_n$ be distinct real numbers. To prove that $\Sigma_{a_1} \cap ... \cap \Sigma_{a_n} \neq \emptyset$ it is sufficient to show that the functions $\boldsymbol{\delta}_{a_1}, ..., \boldsymbol{\delta}_{a_n}, \boldsymbol{\delta}_{-\beta}, \boldsymbol{\delta}_{\beta}$ are linearly independent (by duality, this fact entails that $\sigma_{a_1}, ..., \sigma_{a_n}, \sigma_{-\beta}, \sigma_{\beta}$ are linearly independent, and hence we have our thesis). We want to prove this fact.

First of all, $\boldsymbol{\delta}_{-\beta}$ and $\boldsymbol{\delta}_{\beta}$ are linearly independent, otherwise we would find an hyperreal number ξ such that $\boldsymbol{\delta}_{\beta} = \xi \boldsymbol{\delta}_{-\beta}$, so $u(\beta) = \xi u(-\beta)$ for every ultrafunction u, and this is clearly false. For the general case let us suppose, by contrast, that

$$\boldsymbol{\delta}_{a_1}(x) = \sum_{i=2}^{n} c_i \boldsymbol{\delta}_{a_i}(x) + d_1 \boldsymbol{\delta}_{-\beta}(x) + d_2 \boldsymbol{\delta}_{\beta}(x).$$

Let $f \in \mathcal{C}_0^1(\mathbb{R})$ be such that $f(a_1) \neq 0$ while $f(a_i) = 0$ for every $i = 2, ..., n$. Since $f \in \mathcal{C}_0^1(\mathbb{R})$ we have

$$\int_{-\beta}^{\beta} f^*(x) \boldsymbol{\delta}_{\beta}(x) dx = \int_{-\beta}^{\beta} f^*(x) \boldsymbol{\delta}_{-\beta}(x) dx = 0.$$

Then

$$\begin{aligned} 0 \quad &\neq \quad f^*(a_1) = \int_{-\beta}^{\beta} f^*(x) \boldsymbol{\delta}_{a_1}(x) = \\ \int_{-\beta}^{\beta} f^*(x) \sum_{i=2}^{n} c_i \boldsymbol{\delta}_{a_i}(x) dx \quad &= \quad \sum_{i=2}^{n} c_i \int_{-\beta}^{\beta} f^*(x) \boldsymbol{\delta}_{a_i}(x) dx = 0, \end{aligned}$$

which is absurd. □

In the next section we will use an indipendent set of point Σ to define the notion of restricted ultrafunction. The algebra that we are searching for will be precisely an algebra of restricted ultrafunctions.

3.2 The Algebra of Restricted Ultrafunctions

Let us fix an independent set of points Σ with $\mathbb{R} \cup \{-\beta, \beta\} \subseteq \Sigma$. By point (1) in Proposition 10 it follows that every ultrafunction u depends only on the values it attains on an independent set of points; therefore, if $\mathfrak{I}(\Sigma, \mathbb{R}^*)$ is the family of internal functions $u : \Sigma \to \mathbb{R}^*$, then the operator of restriction $\Psi : V_\Lambda \to \mathfrak{I}(\Sigma, \mathbb{R}^*)$ given by

$$\Psi[f] := f_{\upharpoonright \Sigma}$$

is an isomorphism. The set $\mathfrak{I}(\Sigma, \mathbb{R}^*)$ will be denoted by $V(\Sigma)$.

Definition 12 *The elements of $V(\Sigma)$ will be called restricted ultrafunctions.*

In order to simplify the notation, if u is a restricted ultrafunction we will write

$$\widetilde{u} := \Psi^{-1}[u].$$

Namely, if $\{\sigma_a(x)\}_{a\in\Sigma}$ is the Sigma-basis of V_Σ associated to the independent set of points Σ, then

$$\widetilde{u} = \sum_{a\in\Sigma} u(a)\sigma_a(x).$$

The restricted ultrafunctions present the advantage that they form an algebra with respect to the pointwise sum and product:

$$(f+g)(x) = f(x) + g(x);\ (f\cdot g)(x) = f(x)\cdot g(x).$$

Moreover every restricted ultrafunction can be written as follows

$$u(x) = \sum_{a\in\Sigma} u(a)\delta_{ax},$$

where $\delta_{ax} : \Sigma \to \{0,1\}$ is the usual Kronecker delta.

The spaces V_Λ and $V(\Sigma)$ are isomorphic with respect to many operations (e.g., with respect to the operations of sum and multiplication by a constant) but not to all. This can be seen if we observe that, when endowed with the pointwise multiplication, $V(\Sigma)$ is an algebra while V_Λ is not. In particular, if u and v are restricted ultrafunctions, $\widetilde{u}\cdot\widetilde{v}$ is not in general an extended ultrafunction, namely $\widetilde{u}\cdot\widetilde{v} \notin V_\Lambda$ and

$$\widetilde{u}\cdot\widetilde{v} \neq \widetilde{u\cdot v}.$$

In any case, $\widetilde{u}\cdot\widetilde{v}$ and $\widetilde{u\cdot v}$ coincide on the points of Σ.

A nice feature of $V(\Sigma)$ is that it contains an extension of every function $f \in \mathfrak{F}(\mathbb{R})$:

Definition 13 *Given a function $f \in \mathfrak{F}(\mathbb{R})$, its hyperfinite extension (denoted by f°) is the restricted ultrafunction*

$$f^\circ(x) = \sum_{a\in\Sigma} f^*(a)\delta_{ax}.$$

We observe that, by definition, given any function $f \in \mathfrak{F}(\mathbb{R})$ we have

$$\widetilde{f^\circ}(x) = \sum_{a\in\Sigma} f^*(a)\sigma_a(x).$$

So, in general, $\widetilde{f^\circ}(x) \neq f^*(x)$, even if for every $f \in \mathcal{C}^1(\mathbb{R})$ we have $\widetilde{f^\circ}(x) = f^*(x)$ (equivalently, for every $f \in \mathcal{C}^1(\mathbb{R})$ we have $f^\circ = \Psi(f^*)$).

We now introduce a scalar product on $V(\Sigma)$ that will play a central role in what follows:

Definition 14 *We denote by* $\langle \cdot, \cdot \rangle : V(\Sigma) \to \mathbb{R}^*$ *the scalar product such that, for every* $u, v \in V(\Sigma)$, *we have*

$$\langle u, v \rangle = \int_{-\beta}^{\beta} \widetilde{u}(x) \cdot \widetilde{v}(x) dx.$$

Notice that, in general, $\langle u, v \rangle \neq \int_{-\beta}^{\beta} \widetilde{u \cdot v}(x)\ dx$; in fact

$$\int_{-\beta}^{\beta} \widetilde{uv}(x)\ dx = \sum_{a \in \Sigma} u(a)v(a)\eta_a,$$

while

$$\langle u, v \rangle = \sum_{a,b \in \Sigma} u(a)v(b)\eta_{ab},$$

where, for every $a, b \in \Sigma$, we set

$$\eta_a = \int_{-\beta}^{\beta} \sigma_a(x)dx; \quad \eta_{ab} = \int_{-\beta}^{\beta} \sigma_a(x)\sigma_b(x)dx.$$

Nevertheless, given any $f, g \in \mathcal{C}^1(\mathbb{R})$, we have

$$\langle f^\circ, g^\circ \rangle = \int_{-\beta}^{+\beta} f^*(x)g^*(x)dx$$

so, in particular, if $f, g \in \mathcal{C}_0^1(\mathbb{R})$ then

$$\langle f^\circ, g^\circ \rangle = \int f(x)g(x)dx.$$

We use this scalar product to define the derivative D: $V(\Sigma) \to V(\Sigma)$ by duality:

Definition 15 *The derivative of a restricted ultrafunction* u *(denoted by* $\mathrm{D}u$*) is the unique restricted ultrafunction such that,* $\forall \varphi \in V(\Sigma)$, *we have*

$$\langle \mathrm{D}u, \varphi \rangle = \int_{-\beta}^{\beta} \partial^* \widetilde{u}(x)\widetilde{\varphi}(x)dx.$$

Let us observe that, since $\int_{-\beta}^{\beta} \partial^* \widetilde{u}(x)\widetilde{\varphi}(x)dx = \int_{-\beta}^{\beta} D\widetilde{u}(x)\widetilde{\varphi}(x)dx$, then

$$\widetilde{\mathrm{D}u} = D\widetilde{u}.$$

So we can equivalently define D as follows:

$$\mathrm{D} = \Psi \circ D \circ \Psi^{-1} = \Psi \circ P_{V_\Lambda} \circ \partial^* \circ \Psi^{-1}.$$

In particular $\mathrm{D}f^\circ = (\partial f)^\circ$ whenever $f \in \mathcal{C}^2(\mathbb{R})$.

By combining Theorem 6 with the definitions of the scalar product $\langle \cdot, \cdot \rangle$ and of the operator D we obtain the following result:

Theorem 16 *For every $u, v \in V(\Sigma)$ we have*

$$\langle \mathrm{D}u(x), v(x) \rangle = -\langle u(x), \mathrm{D}v(x) \rangle + [uv]_{-\beta}^{\beta}.$$

Proof. Let us compute $\langle \mathrm{D}u(x), v(x) \rangle$:

$$\begin{aligned}
\langle \mathrm{D}u(x), v(x) \rangle &= \int_{-\beta}^{\beta} \widetilde{\mathrm{D}u(x)}\widetilde{v(x)}dx = \int_{-\beta}^{\beta} D\widetilde{u(x)}\widetilde{v(x)}dx = \\
-\int_{-\beta}^{\beta} \widetilde{u(x)}D\widetilde{v(x)}dx + [\widetilde{u}\widetilde{v}]_{-\beta}^{\beta} &= -\int_{-\beta}^{\beta} \widetilde{u(x)}\widetilde{\mathrm{D}v(x)}dx + [\widetilde{u}\widetilde{v}]_{-\beta}^{\beta} \\
&= -\langle u(x), \mathrm{D}v(x) \rangle + [uv]_{-\beta}^{\beta}. \qquad \square
\end{aligned}$$

Now we want to define a (in some sense canonical) embedding of distributions

$$\Phi : \mathcal{D}'(\mathbb{R}) \to V(\Sigma).$$

A known representation theorem for distributions states that for every distribution $T \in \mathcal{D}'(\mathbb{R})$ and for every compact set $[a, b] \subseteq \mathbb{R}$ there exist $f \in \mathcal{C}^1(\mathbb{R})$ and $k \in \mathbb{N}$ such that $T_{\upharpoonright [a,b]} = \partial^k f$. By transfer we deduce that, in particular, for every distribution $T \in \mathcal{D}'(\mathbb{R})$ there exists $\varphi \in \mathcal{C}^1(\mathbb{R})^*$ and $k \in \mathbb{N}^*$ such that $T^*_{\upharpoonright [-\beta,\beta]} = \partial^k \varphi$. So the following definition makes sense:

Definition 17 *Given $T \in \mathcal{D}'(\mathbb{R})$ we define*

$$d_T = \min\{k \in \mathbb{N}^* \mid \exists \varphi \in \mathcal{C}^1(\mathbb{R})^* \textit{such that } T^*_{\upharpoonright [-\beta,\beta]} = \partial^k \varphi\}$$

and

$$R_T = \{\varphi \in \mathcal{C}^1(\mathbb{R})^* | T^*_{\upharpoonright [-\beta,\beta]} = \partial^{d_T} \varphi\}.$$

It is known that if the weak derivative of a continuous function is zero on an interval $[a, b]$ then the continuous function is constant on $[a, b]$. It is easy to generalize this result and to prove the following Lemma:

Lemma 18 *Let $f \in \mathcal{C}^0(\mathbb{R})$, $k \in \mathbb{N}$. If $\partial^k f_{\upharpoonright [a,b]} = 0$ then there exists a polynomial $P(x)$, with $\deg(P(x)) < k$, such that $f(x)_{\upharpoonright [a,b]} = P(x)$.*

This standard result allows us to prove the following

Lemma 19 *If $\varphi_1, \varphi_2 \in R_T$ then then there exists a polynomial $P(x) \in \mathcal{C}^1(\mathbb{R})^*$, with $\deg(P(x)) < d_T$, such that $(\varphi_1 - \varphi_2)_{\upharpoonright [-\beta,\beta]} = P(x)$.*

Proof. By construction, $\varphi_1 - \varphi_2 \in \mathcal{C}^1(\mathbb{R})^*$ and the d_T-th weak derivative of $\varphi_1 - \varphi_2$ is zero on $[-\beta, \beta]$. We apply the nonstandard version of Lemma 18 obtained by transfer and we deduce the thesis. $\square$

Let us also observe that $d_{\partial T} = d_T + 1$ and that by Lemma 19 it follows that

$$R_{\partial T} = \{\varphi_T + r x^{d_T} \mid \varphi_T \in R_T, r \in \mathbb{R}^*\}.$$

Theorem 20 *There exists a hyperfinite set H such that $R_T \subseteq Span(H)$ for every $T \in \mathcal{D}'(\mathbb{R})$.*

Proof. By saturation, the intersection

$$\bigcap_{T \in D'} [d_T, +\infty)$$

is nonempty. Let $\alpha \in \bigcap_{T \in D'} [d_T, +\infty)$. For every $T \in \mathcal{D}'(\mathbb{R})$ let $\varphi_T \in R_T$ and let

$$H_T = \{H \subseteq \mathcal{C}^1(\mathbb{R})^* \mid H \text{ is hyperfinite and } 1, x, ..., x^\alpha, \varphi_T \in H\}.$$

For every $T \in \mathcal{D}'(\mathbb{R})$ the set H_T is nonempty; moreover, the family $\{H_T\}_{T \in D'}$ has the finite intersection property since, for every $H_1 \in H_{T_1}, H_2 \in H_{T_2}$, we have $H_1 \cup H_2 \in H_{T_1} \cap H_{T_2}$. Then by saturation we have

$$\bigcap_{T \in D'} H_T \neq \emptyset.$$

Let $H \in \bigcap_{T \in D'} H_T$. For every $T \in \mathcal{D}'(\mathbb{R})$ we have that $1, x, ..., x^{d_T}, \varphi_T \in H$ hence, by Lemma 19, we conclude that $R_T \in Span(H)$. $\square$

Now let H and α be given as in Theorem 20. Let

$$\widetilde{H} = \{P^s(h) \mid 0 \leq s \leq \alpha \text{ and } h \in H\},$$

where P denotes the operator that maps a function in $\mathcal{C}^1(\mathbb{R})^*$ to (one of) its primitive with respect to ∂. From now on we consider $\Lambda \subseteq \mathbb{C}^1(\mathbb{R})^*$ to be a hyperfinite set with

$$\widetilde{H} \cup \mathfrak{F}(\mathbb{R}, \mathbb{R})^\sigma \subseteq \Lambda$$

and we construct our model by mean of this hyperfinite set Λ. Let us note that, as a consequence of this choice, we have that

$$\forall T \in \mathcal{D}'(\mathbb{R}) \; \forall \varphi_T \in R_T \; \forall s \in \mathbb{N}^* \cap [0, \alpha] \; P^s(\varphi_T) \text{ is an ultrafunction.}$$

In particular every polynomial $P(x)$ with $\deg(P(x)) \leq \alpha$ is an ultrafunction.

Lemma 21 *Let $P(x) \in V_\Lambda$ be a polynomial and let $\deg(P(x)) < \alpha$. Then $\mathrm{D}^{k+1}(\Psi(P(x))) = 0$.*

Proof. Since $\deg(P) < \alpha$, $P(x), \partial P(x), ..., \partial^{k+1} P(x)$ are ultrafunctions, so we deduce that $D^i P(x) = \partial^i P(x)$ for every $0 \leq i \leq k+1$. Then $D^{k+1} P(x) = \partial^{k+1} P(x) = 0$, and we obtain the thesis by recalling that $\mathrm{D}^{k+1}(\Psi(P(x))) = \Psi(D^{k+1} P(x))$. $\square$

Definition 22 *We denote by $\Phi : \mathcal{D}'(\mathbb{R}) \to V(\Sigma)$ the function such that for every $T \in \mathcal{D}'(\mathbb{R})$*

$$\Phi(T) = \mathrm{D}^{d_T}(\Psi(\varphi_T)), \tag{6}$$

where $\varphi_T \in R_T$.

Lemma 21 entails that Φ is well defined since it does not depend on the particular choice of $\varphi_T \in R_T$. The function Φ has a few important properties:

Theorem 23 *We have the following properties:*

1. *if* $f \in \mathcal{C}^1(\mathbb{R})$ *then* $\Phi(T_f) = f^\circ$;
2. $\forall T \in \mathcal{D}'(\mathbb{R}),\ \forall \varphi \in \mathcal{D}(\mathbb{R}), T(\varphi) = \langle \Phi(T), \varphi^\circ \rangle$;
3. *the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{D}'(\mathbb{R}) & \xrightarrow{\partial} & \mathcal{D}'(\mathbb{R}) \\ \downarrow \Phi & & \downarrow \Phi \\ V(\Sigma) & \xrightarrow{\mathrm{D}} & V(\Sigma) \end{array}$$

   where ∂ *is the usual distributional derivative;*

4. *the restriction of* $\cdot$ *to* $\mathcal{C}^1(\mathbb{R})$ *agrees with the pointwise product, namely if* $f, g \in \mathcal{C}^1(\mathbb{R})$ *then*

$$\Phi(T_{fg}) = \Phi(T_f) \cdot \Phi(T_g).$$

Proof. 1) If $f \in \mathcal{C}^1(\mathbb{R})$ then $T_f = f$. So $d_f = 0$ and $R_T = \{f^*\}$ hence, by definition, $\Phi(T_f) = \Psi(f^*) = f^\circ$.

2) Let $T^*_{\upharpoonright_{[-\beta,\beta]}} = \partial^{d_T} f$. We compute $\langle \Phi(T), \varphi^\circ \rangle$:

$$\begin{aligned} \langle \Phi(T), \varphi^\circ \rangle &= \int_{-\beta}^{\beta} \widetilde{\Phi(T)} \cdot \widetilde{\varphi^\circ} dx = \\ \int_{-\beta}^{\beta} \widetilde{\mathrm{D}^{d_T}(\Psi(f))} \cdot \varphi^* dx &= \int_{-\beta}^{\beta} D^{d_T} f \cdot \varphi^* dx. \end{aligned}$$

Now by Proposition 7 it follows that

$$\int_{-\beta}^{\beta} D^{d_T} f \cdot \varphi^* dx = (-1)^{d_T} \int_{-\beta}^{\beta} f \cdot \partial^{d_T} \varphi^* dx.$$

So

$$\begin{aligned} \langle \Phi(T), \varphi^\circ \rangle &= \int_{-\beta}^{\beta} D^{d_T} f \cdot \varphi^* dx = \\ (-1)^{d_T} \int_{-\beta}^{\beta} f \cdot \partial^{d_T} \varphi^* dx &= T^*[\varphi^*] = \\ (T[\varphi])^* = T[\varphi]. \end{aligned}$$

3) Let $T \in \mathcal{D}'(\mathbb{R})$, $T^*_{\upharpoonright_{[-\beta,\beta]}} = \partial^{d_T} f$, $f \in R_T$. Let us compute $\mathrm{D}(\Phi(T))$:

$$\begin{aligned} \mathrm{D}(\Phi(T)) &= \mathrm{D}\left(\Psi(D^{d_T}(f))\right) = \\ \Psi(D^{d_T+1}(f)) &= \Phi(\partial T), \end{aligned}$$

since $d_{\partial T} = d_T + 1$ and $f \in R_{\partial T}$.

4) Since $f, g \in \mathcal{C}^1(\mathbb{R})$ then $\Phi(T_{fg}) = (fg)^\circ = f^\circ g^\circ = \Phi(T_f) \cdot \Phi(T_g)$. $\square$

Corollary 24 $\Phi : \mathcal{D}'(\mathbb{R}) \to V(\Sigma)$ *is an embedding of vector spaces.*

Proof. Φ is injective: if $T_1 \neq T_2$ are distributions then there is a test function $\varphi \in \mathcal{D}(\mathbb{R})$ such that $T_1[\varphi] \neq T_2[\varphi]$. In particular,

$$\langle \Phi(T_1), \varphi^\circ \rangle = T_1[\varphi] \neq T_2[\varphi] = \langle \Phi(T_2), \varphi^\circ \rangle ,$$

hence $\Phi(T_1) \neq \Phi(T_2)$.

Φ is a linear map: let $T^*_{1|_{[-\beta,\beta]}} = \partial^{d_{T_1}} f$, $T^*_{2|_{[-\beta,\beta]}} = \partial^{d_{T_2}} g$, f, g ultrafunctions. Let us suppose that $d_{T_1} = d_{T_2} + s$. Necessarily, $0 \leq s \leq \alpha$ since both $d_{T_1}, d_{T_2} \leq \alpha$. So

$$(T_1 + T_2)^*_{|_{[-\beta,\beta]}} = \partial^{d_{T_1}}(f + P^s(g)),$$

therefore

$$\begin{aligned} \Phi(T_1 + T_2) &= \mathrm{D}^{d_{T_1}}(\Psi(f + P^s(g))) = \\ &\quad \mathrm{D}^{d_{T_1}}(\Psi(f)) + \mathrm{D}^{d_{T_1}}(\Psi(P^s(g))). \end{aligned}$$

Now, by definition $\mathrm{D}^{d_{T_1}}(\Psi(f)) = \Phi(T_1)$. Moreover

$$\begin{aligned} \mathrm{D}^{d_{T_1}}(\Psi(P^s(g))) &= \mathrm{D}^{d_{T_2}+s}(\Psi(P^s(g))) = \\ \mathrm{D}^{d_{T_2}}(\mathrm{D}^s(\Psi(P^s(g)))) &= \mathrm{D}^{d_{T_2}}(\Psi(D^s(P^s(g)))). \end{aligned}$$

By our choice of Λ in the construction of the space of ultrafunctions we have that $g, P(g), ..., P^s(g)$ are ultrafunctions. So $D^s(P^s(g)) = \partial^s(P^s(g)) = g$, hence

$$\begin{aligned} \mathrm{D}^{d_{T_1}}(\Psi(P^s(g))) &= \mathrm{D}^{d_{T_2}}(\Psi(D^s(P^s(g)))) = \\ \mathrm{D}^{d_{T_2}}(\Psi(g)) &= \Phi(T_2). \end{aligned}$$

To prove that

$$\Phi(rT) = r\Phi(T)$$

it is sufficient to observe that $d_{rT} = d_T$ and $R_{rT} = rR_T$. This proves that Φ is linear. In particular if $r = 0$ we get that the image of the zero distribution is the zero ultrafunction, as expected. $\square$

We are now ready to prove Theorem 1:

Proof of Theorem 1: Let us pose $\mathfrak{A} = V(\Sigma)$, and let us consider Φ,D, $\langle \cdot, \cdot \rangle$ as introduced in this section. Then ($\mathfrak{A}$-0) follows by the definition of $\mathfrak{A}$; ($\mathfrak{A}$-1), ($\mathfrak{A}$-2) and ($\mathfrak{A}$-3) have been proved in Theorem 23 and Corollary 24; ($\mathfrak{A}$-4) has been proved in Theorem 16 and ($\mathfrak{A}$-5) follows immediatly by the definition of f°. $\square$

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