ASYMPTOTIC GAUGES: GENERALIZATION OF COLOMBEAU TYPE ALGEBRAS

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ABSTRACT. We use the general notion of set of indices to construct algebras of nonlinear generalized functions of Colombeau type. They are formally defined in the same way as the special Colombeau algebra, but based on more general "growth condition" formalized by the notion of asymptotic gauge. This generalization includes the special, full and nonstandard analysis based Colombeau type algebras in a unique framework. We compare Colombeau algebras generated by asymptotic gauges with other analogous construction, and we study systematically their properties, with particular attention to the existence and definition of embeddings of distributions. We finally prove that, in our framework, for every linear homogeneous ODE with generalized coefficients there exists a minimal Colombeau algebra generated by asymptotic gauges in which the ODE can be uniquely solved. This marks a main difference with the Colombeau special algebra, where only linear homogeneous ODEs satisfying some restriction on the coefficients can be solved.

1. INTRODUCTION

Currently, a successful approach to modeling singularities as generalized functions in a nonlinear context is the theory of Colombeau-type algebras of generalized functions. The basic underlying idea is to regularize distributions (or even more singular quantities) through nets of smooth functions depending on a regularization parameter ε and then to quantify asymptotically the strength of singularities in terms of this parameter ε. In particular, these algebras contain the space of Schwartz distributions as a linear subspace and the algebra of smooth functions as a faithful subalgebra. We suppose the reader to have a certain familiarity with this topic and refer to [2, 3, 1, 4, 7, 6] for detailed information; as for terminology and notations we mainly follow [6].

Since the beginning of this theory, it was natural to generalize Colombeau construction replacing the family $(\varepsilon^n)_{\varepsilon\in(0,1],n\in\mathbb{N}}$ with different "scales". So, we have the notions of asymptotic scales (see e.g. [10, 11]), $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras (see e.g. [8, 14] and references therein), and sequence spaces with exponent weights ([9]).

In realizing this generalization, one can ask problems like:

- When do we obtain an algebra?
- When can we embed Schwartz distributions using the common method of regularization by means of a mollifier?

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- How can we use this generalization to solve differential problems having a singular growth, i.e. growing more than polynomially with ε?

The present work inscribes in this research thread. It is hence natural to clarify the relationship between our approach and the cited articles, and to highlight what we obtain more with respect to them. In this introduction, we start this clarification, so that the reader can more easily understand the general picture.

Smooth functions: First of all, a general approach is frequently preferred, e.g. by considering a sheaf $\mathcal{E}$ of topological algebras, and suitable families of seminorms. On the contrary, in the present work we will only consider the usual sheaf of smooth functions $\mathcal{C}^\infty(\Omega)$, for Ω open in $\mathbb{R}^n$, and the usual family of seminorms $\sup_{x\in K} |\partial^\alpha f(x)|$, where $K \Subset \Omega$ and $\alpha \in \mathbb{N}^n$. In spite of our choice, as it is clearly stated in [8, pag. 394]: "Except in a few cases, the sheaf $\mathcal{E}$ is chosen to be the sheaf of smooth functions". In this way, we can focus on the conditions we need to impose to the more general family of scales. We can thus avoid to add further conditions which trivially hold in the (almost) unique case that most of the readers will consider. We hence left to the interested reader the natural generalization of the present work to a more abstract framework.

Set of indices: The results of this paper are proved for a generic set of indices, a new unifying structure introduced in [12]. This permits to include several Colombeau algebras in the same framework and notations: the special algebra $\mathcal{G}^{\mathrm{s}}$, the full algebra $\mathcal{G}^{\mathrm{e}}$ and the nonstandard analysis based algebra of asymptotic functions $\hat{\mathcal{G}}$. Even if in the present article we are going to develop only the cases $\mathcal{G}^{\mathrm{s}}$, $\mathcal{G}^{\mathrm{e}}$ and $\hat{\mathcal{G}}$, we are strongly convinced that with minor modifications (see [12]), the results we are going to present can also be applied to the diffeomorphism invariant algebras $\mathcal{G}^{\mathrm{d}}$, $\mathcal{G}^2$ and $\hat{\mathcal{G}}$ of [6].

Both asymptotic scales and $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras apply only to the special case. Sequence spaces with exponent weights applies to $\mathcal{G}^{\mathrm{s}}$, $\mathcal{G}^{\mathrm{e}}$, $\mathcal{G}^{\mathrm{d}}$ but not to $\hat{\mathcal{G}}$.

Logical structure: One of the key features in using a set of indices is that for all the algebras $\mathcal{G}^{\mathrm{s}}$, $\mathcal{G}^{\mathrm{e}}$, $\hat{\mathcal{G}}$, $\mathcal{G}^{\mathrm{d}}$, $\mathcal{G}^2$ and $\hat{\mathcal{G}}$ of [6], we have the same logical structure of the simple Colombeau algebra, i.e. $\forall K\, \forall\alpha\, \exists N$ and $\forall K\, \forall\alpha\, \forall m$, followed by a suitable big-O asymptotic relation. Both $\mathcal{G}^{\mathrm{e}}$ and $\mathcal{G}^{\mathrm{d}}$ can be seen as sequence spaces with exponent weights, but continuing to use the usual more involved logical structure (hidden by the use of infinite intersections and unions). On the other hand, this permits to [9] to underscore the relationship between sequence spaces with exponent weights and Maddox sequence spaces.

Scales as primitive data: Like in [10, 9], we take the choice of the "scale" as one of the primitive data. This approach is methodologically a little different from $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras, where the scale is hidden in the pair (A, I) of the ring A of moderate and the ideal I of negligible scalars, but where the ring A usually contains much more than only the scales. For example, when A is polynomially overgenerated, it contains both infinitesimals and infinite nets. Moreover, polynomially overgenerated rings do not permit to obtain an algebra closed with respect to exponential (see [8]). This represents a limitation e.g. in solving even linear ODE with generalized constant coefficients. On the contrary, we prove in Thm. 51 that every linear ODE with constant (generalized) coefficients whose scale is of type $\mathcal{B}$ has a unique solution whose scale is of type $e^{\mathcal{B}}$ (see Def. 46). Moreover, the

Colombeau-like algebra defined starting from the scale $e^{\mathcal{B}}$ is the smallest among this type of algebras where every ODE of this type has a solution (see Thm. 56).

Embedding of distributions: In Thm. 44, we characterize which scales permit to embed distributions using a mollifier and respecting the product of smooth functions. Our results also clarify when, in other approaches, embeddings of distributions are possible and when they are not. About this problem, see also [11, pag. 1-2] and [9].

Generality: Considering [14, 9], it is already known that the $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras approach is the most general one. In section 3.0.1, we prove that when we consider the usual sheaf of smooth functions, and we consider only the special algebra case, the $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras approach is equivalent to our approach.

1.1. **Set of indices.** In this section, we recall notations and notions from [12] that we will use in the present work. For all the proofs, we refer to [12]. In the naturals $\mathbb{N}$ we always include zero.

If $\varphi \in \mathcal{D}(\mathbb{R}^n)$, $r \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}^n$, we use the notations $r \odot \varphi$ for the function $x \in \mathbb{R}^n \mapsto \frac{1}{r^n} \cdot \varphi\left(\frac{x}{r}\right) \in \mathbb{R}$ and $x \oplus \varphi$ for the function $y \in \mathbb{R}^n \mapsto \varphi(y - x) \in \mathbb{R}$. These new notations permit to highlight that $\odot$ is a free action of the multiplicative group $(\mathbb{R}_{>0}, \cdot, 1)$ on $\mathcal{D}(\mathbb{R}^n)$ and $\oplus$ is a free action of the additive group $(\mathbb{R}_{>0}, +, 0)$ on $\mathcal{D}(\mathbb{R}^n)$. We also have the distributive property $r \odot (x \oplus \varphi) = rx \oplus r \odot \varphi$.

Definition 1. We say that $\mathbb{I} = (I, \leqslant, \mathcal{I})$ is a *set of indices* if the following conditions hold:

(i) $(I, \leqslant)$ is a pre-ordered set, i.e. it is a non empty set I with a reflexive and transitive relation $\leqslant$.
(ii) $\mathcal{I}$ is a set of subsets of I such that $\varnothing \notin \mathcal{I}$ and $I \in \mathcal{I}$.
(iii) $\forall A, B \in \mathcal{I}\, \exists C \in \mathcal{I} : \ C \subseteq A \cap B$.

For all $e \in I$, set $(\varnothing, e] := \{\varepsilon \in I \mid \varepsilon \leqslant e\}$. As usual, we say $\varepsilon < e$ if $\varepsilon \leqslant e$ and $\varepsilon \neq e$. Using these notations, we state the last condition in the definition of set of indices:

(iv) If $e \leqslant a \in A \in \mathcal{I}$, the set $A_{\leqslant e} := (\varnothing, e] \cap A$ is downward directed by $<$, i.e., it is non empty and

$$\forall b, c \in A_{\leqslant e}\, \exists d \in A_{\leqslant e} : \ d < b\ ,\ d < c. \tag{1.1}$$

Henceforward, functions of the type $f : I \longrightarrow \mathbb{R}$ will also be called *nets*, and for their evaluation we will both use the notations f_ε or $f(\varepsilon)$, in case the subscript notation is too cumbersome. When the domain I is clear, we use also the notation $f = (f_\varepsilon)$ for the whole net. Analogous notations will be used for nets of smooth functions $u = (u_\varepsilon) \in \mathcal{C}(\Omega)^I$.

Example 2.

(i) Conditions (ii) and (iii) can be summarized saying that $\mathcal{I}$ is a filter base on I which contains I.
(ii) The simplest example of set of indices is given by $I^{\mathrm{s}} := (0, 1] \subseteq \mathbb{R}$, the relation $\leqslant$ is the usual order relation on $\mathbb{R}$, and $\mathcal{I}^{\mathrm{s}} := \{(0, \varepsilon_0] \mid \varepsilon_0 \in I\}$. We denote by $\mathbb{I}^{\mathrm{s}} := (I^{\mathrm{s}}, \leqslant, \mathcal{I}^{\mathrm{s}})$ this set of indices which, of course, is that used for the special algebra $\mathcal{G}^{\mathrm{s}}$.
(iii) In the context of [18], we set $\hat{I} := \mathcal{D}_0 = \mathcal{D}(\mathbb{R}^d)$. The pre-order relation is defined by $\varphi \leqslant \psi$ iff $\underline{\varphi} \leqslant \underline{\psi}$, where $\underline{\varphi} := \operatorname{diam}(\operatorname{supp}(\varphi))$ if $\varphi \neq 0$ and $\underline{\varphi} := 1$

otherwise. $\hat{\mathcal{I}}$ is the free ultrafilter on $\mathcal{D}_0$ used in [18]. Then $\hat{\mathbb{I}} := (\hat{I}, \leqslant, \hat{\mathcal{I}})$ is a set of indices.

(iv) With the usual notations of [6] for the full algebra $\mathcal{G}^{\mathrm{e}}$, we define $I^{\mathrm{e}} := \mathcal{A}_0$, $\mathcal{I}^{\mathrm{e}} := \{\mathcal{A}_q \mid q \in \mathbb{N}\}$, and for ε, $e \in I^{\mathrm{e}}$, we define $\varepsilon \leqslant e$ iff there exists $r \in \mathbb{R}_{>0}$ such that $r \leqslant 1$ and $\varepsilon = r \odot e$. Then $\mathbb{I}^{\mathrm{e}} := (I^{\mathrm{e}}, \leqslant, \mathcal{I}^{\mathrm{e}})$ is a set of indices.

As we mentioned in the introduction, in the present work we will actually consider only these examples of set of indices.

In each set of indices, we can define two notions of big-O that formally behave in the usual way. Since each set of the form $A_{\leqslant e} = (\varnothing, e] \cap A$ is downward directed, the first big-O is the usual one:

Definition 3. Let $\mathbb{I} = (I, \leqslant, \mathcal{I})$ be a set of indices. Let $a \in A \in \mathcal{I}$ and (x_ε), $(y_\varepsilon) \in \mathbb{R}^I$ be two nets of real numbers defined in I. We write

$$x_\varepsilon = O_{a,A}(y_\varepsilon) \text{ as } \varepsilon \in \mathbb{I} \tag{1.2}$$

if

$$\exists H \in \mathbb{R}_{>0}\, \exists \varepsilon_0 \in A_{\leqslant a}\, \forall \varepsilon \in A_{\leqslant \varepsilon_0} : \ |x_\varepsilon| \leqslant H \cdot |y_\varepsilon|. \tag{1.3}$$

Definition 4. Let $\mathbb{I} = (I, \leqslant \mathcal{I})$ be a set of indices. Let $\mathcal{J} \subseteq \mathcal{I}$ be a non empty subset of $\mathcal{I}$ such that

$$\forall A, B \in \mathcal{J}\, \exists C \in \mathcal{J} : \ C \subseteq A \cap B. \tag{1.4}$$

Finally, let (x_ε), $(y_\varepsilon) \in \mathbb{R}^I$ be nets of real numbers. Then we say

$$x_\varepsilon = O_{\mathcal{J}}(y_\varepsilon) \text{ as } \varepsilon \in \mathbb{I}$$

if

$$\exists A \in \mathcal{J}\, \forall a \in A : \ x_\varepsilon = O_{a,A}(y_\varepsilon).$$

We simply write $x_\varepsilon = O(y_\varepsilon)$ (as $\varepsilon \in \mathbb{I}$) when $\mathcal{J} = \mathcal{I}$, i.e. to denote $x_\varepsilon = O_{\mathcal{I}}(y_\varepsilon)$.

The simplification consequent to the use of the second notion of big-O is due to the following theorem, which states that also the second big-O formally behaves as expected:

Theorem 5. *Under the assumptions of Def. 4, the following properties of $O_{\mathcal{J}}$, as $\varepsilon \in \mathbb{I}$, hold:*

(i) $x_\varepsilon = O_{\mathcal{J}}(x_\varepsilon)$;
(ii) *if* $x_\varepsilon = O_{\mathcal{J}}(y_\varepsilon)$ *and* $y_\varepsilon = O_{\mathcal{J}}(z_\varepsilon)$ *then* $x_\varepsilon = O_{\mathcal{J}}(z_\varepsilon)$;
(iii) $O_{\mathcal{J}}(x_\varepsilon) \cdot O_{\mathcal{J}}(y_\varepsilon) = O_{\mathcal{J}}(x_\varepsilon \cdot y_\varepsilon)$;
(iv) $O_{\mathcal{J}}(x_\varepsilon) + O_{\mathcal{J}}(y_\varepsilon) = O_{\mathcal{J}}\left(|x_\varepsilon| + |y_\varepsilon|\right)$;
(v) $x_\varepsilon \cdot O_{\mathcal{J}}(y_\varepsilon) = O_{\mathcal{J}}(x_\varepsilon \cdot y_\varepsilon)$;
(vi) $O_{\mathcal{J}}(x_\varepsilon) + O_{\mathcal{J}}(x_\varepsilon) = O_{\mathcal{J}}(x_\varepsilon)$;
(vii) *if* $x_\varepsilon, y_\varepsilon \geqslant 0$ *for all* $\varepsilon \in I$, *then* $x_\varepsilon + O_{\mathcal{J}}(y_\varepsilon) = O_{\mathcal{J}}(x_\varepsilon + y_\varepsilon)$;
(viii) $\forall k \in \mathbb{R}_{\neq 0} : \ O_{\mathcal{J}}(k \cdot x_\varepsilon) = O_{\mathcal{J}}(x_\varepsilon)$;
(ix) $\forall k \in \mathbb{R} : \ k \cdot O_{\mathcal{J}}(x_\varepsilon) = O_{\mathcal{J}}(x_\varepsilon)$.

An analogue of Thm. 5 holds also for the first notion of big-O, i.e. for the relation $x_\varepsilon = O_{a,A}(y_\varepsilon)$ as $\varepsilon \in \mathbb{I}$.

The unifying properties of these notions are explained in the following results:

Corollary 6. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $(u_\varepsilon) \in \mathcal{C}^\infty(\Omega, \mathbb{R})$ be a net of smooth functions. We use the notations of [6] for moderate and negligible nets related to the special algebra $\mathcal{G}^{\mathrm{s}}(\Omega)$, and the notations of [18] for similar notions related to the algebra $\hat{\mathcal{G}}(\Omega)$ of asymptotic functions. Moreover, we recall that $\underline{\varepsilon} := \min\{\mathrm{diam}\left(\mathrm{supp}(\varepsilon)\right), 1\}$, where $\varepsilon \in \mathcal{D}(\mathbb{R}^d)$. Then*

(i) $(u_\varepsilon) \in \mathcal{E}^{\mathrm{s}}_M(\Omega)$ *if and only if*

$$\forall K \Subset \Omega\, \forall \alpha \in \mathbb{N}^n\, \exists N \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N}) \text{ as } \varepsilon \in \mathbb{I}^{\mathrm{s}};$$

(ii) $(u_\varepsilon) \in \mathcal{N}^{\mathrm{s}}(\Omega)$ *if and only if*

$$\forall K \Subset \Omega\, \forall \alpha \in \mathbb{N}^n\, \forall m \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{m}) \text{ as } \varepsilon \in \mathbb{I}^{\mathrm{s}};$$

(iii) $(u_\varepsilon) \in \mathcal{M}\left(\mathcal{E}(\Omega)^{\mathcal{D}_0}\right)$ *if and only if*

$$\forall K \Subset \Omega\, \forall \alpha \in \mathbb{N}^n\, \exists N \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\underline{\varepsilon}^{-N}) \text{ as } \varepsilon \in \hat{\mathbb{I}};$$

(iv) $(u_\varepsilon) \in \mathcal{N}\left(\mathcal{E}(\Omega)^{\mathcal{D}_0}\right)$ *if and only if*

$$\forall K \Subset \Omega\, \forall \alpha \in \mathbb{N}^n\, \forall m \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(\underline{\varepsilon}^{m}) \text{ as } \varepsilon \in \hat{\mathbb{I}}.$$

To arrive at a similar unifying result for the full algebra, we need the following

Definition 7. Let $\Omega \subseteq \mathbb{R}^n$ be an open set.

(i) If $\varepsilon \in \mathcal{A}_0$, then $\Omega_\varepsilon := \Omega \cap \{x \in \mathbb{R}^n \mid \mathrm{supp}(\varepsilon) \subseteq \Omega - x\}$.
(ii) $\mathcal{P}^{\mathrm{e}}(\Omega) := \prod_{\varepsilon \in I^{\mathrm{e}}} \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R})$.
(iii) If $g : X \longrightarrow Z^Y$ is a map, then $g^\vee : (x, y) \in X \times Y \mapsto g(x)(y) \in Z$.

We can say that elements of $\mathcal{P}^{\mathrm{e}}(\Omega)$ are I^{e}-indexed nets (u_ε) such that $u_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R})$. In [13] it is proved that $\mathcal{P}^{\mathrm{e}}(\Omega)$ is isomorphic (as diffeological space, and hence also as set) to the usual space $\mathcal{E}^{\mathrm{e}}(\Omega)$ (see [6]).

Theorem 8. *Let $u = (u_\varepsilon) \in \mathcal{P}^{\mathrm{e}}(\Omega)$, then*

(i) $u^\vee \in \mathcal{E}^{\mathrm{e}}_M(\Omega)$ *if and only if*

$$\forall K \Subset \Omega\, \forall \alpha \in \mathbb{N}^n\, \exists N \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O\left(\underline{\varepsilon}^{-N}\right) \text{ as } \varepsilon \in \mathbb{I}^{\mathrm{e}};$$

(ii) $u^\vee \in \mathcal{N}^{\mathrm{e}}(\Omega)$ *if and only if*

$$\forall K \Subset \Omega\, \forall \alpha \in \mathbb{N}^n\, \forall m \in \mathbb{N} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O\left(\underline{\varepsilon}^{m}\right) \text{ as } \varepsilon \in \mathbb{I}^{\mathrm{e}}.$$

The same unifying and simple formulation can be used for the diffeomorphism invariant algebra $\mathcal{G}^{\mathrm{d}}$, with only one difference: in the definition of moderate net we use a big-O relation of the type $O_{\mathcal{J}}$ for a suitable $\mathcal{J} \subset \mathcal{I}$ (see [12]). For this reason, we are strongly convinced that the following results can be generalized also to $\mathcal{G}^{\mathrm{d}}$.

2. Asymptotic Gauges

In this section, we are going to introduce some notions for a set of indices which permit to define what an asymptotic gauge is.

2.1. **"For ε sufficiently small" in a set of indices.** We start by introducing a useful notation which corresponds, in the set of indices for the special algebra $\mathbb{I}^{\mathrm{s}}$, to the usual "for ε sufficiently small".

Definition 9. Let $\mathbb{I} = (I, \leqslant, \mathcal{I})$ be a set of indices. Let $a \in A \in \mathcal{I}$ and $\mathcal{P}(-)$ be a property, then we say

$$\forall^{\mathbb{I}} \varepsilon \in A_{\leqslant a} : \ \mathcal{P}(\varepsilon),$$

and we read it *for ε sufficiently small in $A_{\leqslant a}$ the property $\mathcal{P}(\varepsilon)$ holds*, if

$$\exists e \leqslant a \, \forall \varepsilon \in A_{\leqslant e} : \ \mathcal{P}(\varepsilon). \tag{2.1}$$

Moreover, we say that

$$\forall^{\mathbb{I}} \varepsilon : \ \mathcal{P}(\varepsilon),$$

and we read it *for ε sufficiently small in $\mathbb{I}$ the property $\mathcal{P}(\varepsilon)$ holds, if* $\exists A \in \mathcal{I} \, \forall a \in A \, \forall^{\mathbb{I}} \varepsilon \in A_{\leqslant a} : \ \mathcal{P}(\varepsilon)$.

Example 10.

(i) By condition (iv) of Def. 1 of set of indices, it follows that $A_{\leqslant e} \neq \varnothing$. Therefore, (2.1) is equivalent to

$$\exists e \in A_{\leqslant a} \, \forall \varepsilon \in A_{\leqslant e} : \ \mathcal{P}(\varepsilon).$$

Analogously, we can reformulate similar properties we will see below.

(ii) We have $x_\varepsilon = O_{a,A}(y_\varepsilon)$ if and only if $\exists H \in \mathbb{R}_{>0} \, \forall^{\mathbb{I}} \varepsilon \in A_{\leqslant a} : \ |x_\varepsilon| \leqslant H \cdot |y_\varepsilon|$.

(iii) In the set of indices $\mathbb{I}^{\mathrm{s}}$, the following properties are equivalent:
  (a) $\exists \varepsilon_0 \in I^{\mathrm{s}} \, \forall \varepsilon \in (0, \varepsilon_0] : \ \mathcal{P}(\varepsilon)$;
  (b) $\exists A \in \mathcal{I}^{\mathrm{s}} \, \exists a \in A \, \forall^{\mathbb{I}^{\mathrm{s}}} \varepsilon \in A_{\leqslant a} : \ \mathcal{P}(\varepsilon)$;
  (c) $\exists A \in \mathcal{I}^{\mathrm{s}} \, \forall a \in A \, \forall^{\mathbb{I}^{\mathrm{s}}} \varepsilon \in A_{\leqslant a} : \ \mathcal{P}(\varepsilon)$;
  (d) $\forall A \in \mathcal{I}^{\mathrm{s}} \, \exists a \in A \, \forall^{\mathbb{I}^{\mathrm{s}}} \varepsilon \in A_{\leqslant a} : \ \mathcal{P}(\varepsilon)$.

(iv) In the set of indices $\hat{\mathbb{I}}$, we recall that a property $\mathcal{P}(\varepsilon)$ is said to hold *almost everywhere* iff $\{\varepsilon \in \mathcal{D}_0 \mid \mathcal{P}(\varepsilon)\} \in \hat{\mathcal{I}}$ (see [18]). Using this language, the following properties are equivalent:
  (a) $\mathcal{P}(\varepsilon)$ holds almost everywhere;
  (b) $\exists A \in \hat{\mathcal{I}} \, \exists a \in A \, \forall^{\hat{\mathbb{I}}} \varepsilon \in A_{\leqslant a} : \ \mathcal{P}(\varepsilon)$;
  (c) $\exists A \in \hat{\mathcal{I}} \, \forall a \in A \, \forall^{\hat{\mathbb{I}}} \varepsilon \in A_{\leqslant a} : \ \mathcal{P}(\varepsilon)$.

(v) In the set of indices $\mathbb{I}^{\mathrm{e}}$, assume that $\varphi \in \mathcal{A}_q$, then the following properties are equivalent
  (a) $\forall^{\mathbb{I}^{\mathrm{e}}} \varepsilon \in (\mathcal{A}_q)_{\leqslant \varphi} : \ \mathcal{P}(\varepsilon)$;
  (b) $\exists r \in (0, 1] \, \forall s \in (0, r] : \ \mathcal{P}(s \odot \varphi)$.

2.2. **Order relation in a set of indices.** All the scales $(\varepsilon^{-n})_{\varepsilon,n}$ of the special algebra are *positive functions*. Of course, if we change the function $\varepsilon \mapsto \varepsilon^{-n}$, only for $\varepsilon > \varepsilon_0$, so that it is not globally positive anymore, this will not change anything in the definition of $\mathcal{G}^{\mathrm{s}}(\Omega)$. In this section, we are going to define this order relation for functions of the type $I \longrightarrow \mathbb{R}$.

Definition 11. Let $\mathbb{I} = (I, \leqslant, \mathcal{I})$ be a set of indices, and $i, j : I \longrightarrow \mathbb{R}$ be maps. Then we say $i >_{\mathbb{I}} j$ if

$$\forall^{\mathbb{I}} \varepsilon : \ i_\varepsilon > j_\varepsilon.$$

Following the intuitive interpretation given in [12], we can say that $i >_{\mathbb{I}} j$ if we can find an accuracy class $A \in \mathcal{I}$ such that for each measuring instrument $a \in A$ in that class, we have $i_\varepsilon > j_\varepsilon$ for $\varepsilon \in A_{\leqslant a}$ sufficiently small.

Theorem 12. *Let $\mathbb{I} = (I, \leqslant, \mathcal{I})$ be a set of indices, and $i, j, k, z : I \longrightarrow \mathbb{R}$ be maps. Then we have:*

(i) $i \not>_{\mathbb{I}} i$.
(ii) *If $i >_{\mathbb{I}} j >_{\mathbb{I}} k$, then $i >_{\mathbb{I}} k$.*
(iii) *If $i >_{\mathbb{I}} 0$ and $k >_{\mathbb{I}} j$, then $i \cdot k >_{\mathbb{I}} i \cdot j$.*
(iv) *If $i >_{\mathbb{I}} j$ and $k >_{\mathbb{I}} z$, then $i + k >_{\mathbb{I}} j + z$.*
(v) *If $\exists A \in \mathcal{I}\, \forall a \in A\, \forall^{\mathbb{I}} \varepsilon \in A_{\leqslant a} : \ i_\varepsilon \leqslant j_\varepsilon$, then $i_\varepsilon = O(j_\varepsilon)$ as $\varepsilon \in \mathbb{I}$.*

Proof. (i): By contradiction, assume that $i >_{\mathbb{I}} i$, i.e.

$$\exists A \in \mathcal{I}\, \forall a \in A\, \forall^{\mathbb{I}} \varepsilon \in A_{\leqslant a} : \ i_\varepsilon > i_\varepsilon. \tag{2.2}$$

But $\exists a \in A$ since $\varnothing \notin \mathcal{I}$. This, (2.2) and (i) of Example 10 yield that for some $e \in A_{\leqslant a}$ we have $i_\varepsilon > i_\varepsilon$ for all $\varepsilon \in A_{\leqslant e}$. This yields the contradiction $i_e > i_e$.
(ii): We can write the assumptions of this claim as

$$\exists A \in \mathcal{I}\, \forall a \in A\, \forall^{\mathbb{I}} \varepsilon \in A_{\leqslant a} : \ i_\varepsilon > j_\varepsilon \tag{2.3}$$

$$\exists B \in \mathcal{I}\, \forall b \in B\, \forall^{\mathbb{I}} \varepsilon \in B_{\leqslant b} : \ j_\varepsilon > k_\varepsilon. \tag{2.4}$$

By (iii) we get the existence of $C \in \mathcal{I}$ which is contained in $A \cap B$, i.e. where both (2.3) and (2.4) hold. Fix a generic $c \in C$. Both the relations $i_\varepsilon > j_\varepsilon$ and $j_\varepsilon > k_\varepsilon$ hold for ε sufficiently small in $A_{\leqslant c}$ and $B_{\leqslant c}$ respectively. Hence

$$\exists e' \leqslant c\, \forall \varepsilon \in A_{\leqslant e'} : \ i_\varepsilon > j_\varepsilon \tag{2.5}$$

$$\exists e'' \leqslant c\, \forall \varepsilon \in A_{\leqslant e''} : \ j_\varepsilon > k_\varepsilon. \tag{2.6}$$

But $(\varnothing, c]$ is downward directed so that there exists $e \leqslant c$ such that $e < e'$ and $e < e''$. For each $\varepsilon \in C_{\leqslant e}$, from (2.5) and (2.6) we hence get the conclusion $i_\varepsilon > j_\varepsilon > k_\varepsilon$.
Properties (iii) and (iv) can be proved analogously. Property (v) also follows directly from the definitions. □

Example 13.

(i) In the set of indices $\mathbb{I}^{\mathrm{s}}$, we have $i >_{\mathbb{I}^{\mathrm{e}}} j$ if and only if $i_\varepsilon > j_\varepsilon$ for ε sufficiently small.
(ii) In the set of indices $\mathbb{I}^{\mathrm{e}}$ we have $i >_{\mathbb{I}^{\mathrm{e}}} j$ if and only if there exists $q \in \mathbb{N}$ such that for each $\varphi \in \mathcal{A}_q$ we have $i(\varepsilon \odot \varphi) > j(\varepsilon \odot \varphi)$ for $\varepsilon \in (0, 1]$ sufficiently small.
(iii) In the set of indices $\hat{\mathbb{I}}$ we have $i >_{\hat{\mathbb{I}}} j$ if and only if $i_\varepsilon > j_\varepsilon$ almost everywhere.

2.3. **Limits in a set of indices.** All the scales $(\varepsilon^{-n})_{\varepsilon,n}$ of the special algebra have limit $+\infty$ for $\varepsilon \to 0^+$. In this section, we want to define the notion of limit in a set of indices for functions of the type $I \longrightarrow \mathbb{R}$. This notion can be easily generalized to generic $f : I \longrightarrow T$, where T is a topological space.

Definition 14. Let $\mathbb{I} = (I, \leqslant, \mathcal{I})$ be a set of indices, $f : I \longrightarrow \mathbb{R}$ a map, and $l \in \mathbb{R} \cup \{+\infty, -\infty, \infty\}$. Then we say that *$l$ is the limit of f in* $\mathbb{I}$ if

$$\exists A \in \mathcal{I}\, \forall a \in A : \ l = \lim_{\varepsilon \leqslant a} f|_A(\varepsilon), \tag{2.7}$$

where the limit (2.7) is taken in the downward directed set $(\varnothing, a]$.

Remark.

(i) Writing $\lim_{\varepsilon\leqslant a} f|_A(\varepsilon)$ we mean (in case l is finite)
$$\forall r \in \mathbb{R}_{>0}\, \exists a_0 \leqslant a\, \forall \varepsilon \in A :\ \varepsilon \leqslant a_0 \quad \Longrightarrow \quad |l - f_\varepsilon| < r, \tag{2.8}$$
that is
$$\forall r \in \mathbb{R}_{>0}\, \forall^{\mathbb{I}} \varepsilon \in A_{\leqslant a} :\ |l - f_\varepsilon| < r. \tag{2.9}$$
From this, the following properties easily follow
- If $l = \lim_{\varepsilon\leqslant a} f|_A(\varepsilon)$ and $B \subseteq A$, $B \in \mathcal{I}$, then $l = \lim_{\varepsilon\leqslant a} f|_B(\varepsilon)$.
- There exists at most one l verifying (2.8).

(ii) Let us assume that l_1 and l_2 are both limits of f in $\mathbb{I}$. So, for some A, $B \in \mathcal{I}$ we have $l_1 = \lim_{\varepsilon\leqslant a} f|_A(\varepsilon)$ for all $e \in A$, and $l_2 = \lim_{\varepsilon\leqslant e} f|_B(\varepsilon)$ for all $e \in B$. We can always find $C \in \mathcal{I}$ and $e \in C$ such that $C \subseteq A \cap B$, so that $l_1 = \lim_{\varepsilon\leqslant e} f|_C(\varepsilon) = l_2$. Therefore, if this limit exists, it is unique and we can use the notation
$$l = \lim_{\mathbb{I}} f = \lim_{\varepsilon\in\mathbb{I}} f_\varepsilon.$$

Example 15.

(i) In the set of indices $\mathbb{I}^{\mathrm{s}}$, we have $l = \lim_{\mathbb{I}^{\mathrm{s}}} f$ if and only if $l = \lim_{\varepsilon\to 0^+} f_\varepsilon$.

(ii) In the set of indices $\mathbb{I}^{\mathrm{e}}$, we have $l = \lim_{\mathbb{I}^{\mathrm{e}}} f$ if and only if there exists $q \in \mathbb{N}$ such that for each $\varphi \in \mathcal{A}_q$ we have $l = \lim_{\varepsilon\to 0^+} f(\varepsilon \odot \varphi)$.

(iii) In the set of indices $\hat{\mathbb{I}}$, assume that $\hat{\mathcal{I}}$ is a P-point ([5]), and denote by ${}^*\mathbb{R}$ the hyperreals constructed as the ultrapower $\mathbb{R}^I/\hat{\mathcal{I}}$. Then we have
  (a) $\exists \lim_{\hat{\mathbb{I}}} f = l \in \mathbb{R}$ if and only if f is finite and l is the standard part of $[f]_{\hat{\mathcal{I}}} \in {}^*\mathbb{R}$.
  (b) $\exists \lim_{\hat{\mathbb{I}}} f = +\infty$ if and only if $f > 0$ and $[f]_{\hat{\mathcal{I}}}$ is infinite.
  (c) $\exists \lim_{\hat{\mathbb{I}}} f = -\infty$ if and only if $f < 0$ and $[f]_{\hat{\mathcal{I}}}$ is infinite.
  (d) $\exists \lim_{\hat{\mathbb{I}}} f = \infty$ if and only if $[|f|]_{\hat{\mathcal{I}}}$ is infinite.

An expected result is the following

Theorem 16. *Let $\mathbb{I} = (I, \leqslant, \mathcal{I})$ be a set of indices and f, g, $h : I \longrightarrow \mathbb{R}$ be maps such that the limits $\lim_{\mathbb{I}} f$, $\lim_{\mathbb{I}} g$ exists and are finite. Then*

(i) $\exists \lim_{\mathbb{I}}(f + g) = \lim_{\mathbb{I}} f + \lim_{\mathbb{I}} g$.

(ii) $\exists \lim_{\mathbb{I}}(f \cdot g) = \lim_{\mathbb{I}} f \cdot \lim_{\mathbb{I}} g$.

(iii) $\forall r \in \mathbb{R} :\ \exists \lim_{\mathbb{I}} r = r$.

(iv) *If* $\lim_{\mathbb{I}} g \neq 0$, *then* $\exists \lim_{\mathbb{I}} \frac{f}{g} = \frac{\lim_{\mathbb{I}} f}{\lim_{\mathbb{I}} g}$.

(v) *If* $f <_{\mathbb{I}} h <_{\mathbb{I}} g$ *and* $\lim_{\mathbb{I}} f = \lim_{\mathbb{I}} g =: l$, *then* $\exists \lim_{\mathbb{I}} h = l$.

(vi) *If* $\exists \lim_{\mathbb{I}} f > 0$, *then* $f >_{\mathbb{I}} 0$.

The proof is a direct consequence of our definition of limit and, as usual, property (iii) of Def. 1.

2.4. **Asymptotic gauges.** Asymptotic gauges represent our definition of scale for Colombeau like algebras. The idea is, essentially, to ask the *asymptotic* closure of $\mathcal{B} \subseteq \mathbb{R}^I$ with respect to algebraic operations.

Definition 17. Let $\mathbb{I} = (I, \leqslant, \mathcal{I})$ be a set of indices. All the big-O in this definition have to be meant as $O_{\mathbb{I}}$ (see Def. 4). Then we say that $\mathcal{B}$ is an *asymptotic gauge on* $\mathbb{I}$ (briefly: AG on $\mathbb{I}$) if

(i) $\mathcal{B} \subseteq \mathbb{R}^I$;

(ii) $\exists i \in \mathcal{B} :\ \lim_{\mathbb{I}} i = \infty$;

(iii) $\forall i, j \in \mathcal{B}\, \exists p \in \mathcal{B} :\ i \cdot j = O(p)$;

(iv) $\forall i \in \mathcal{B}\, \forall r \in \mathbb{R}\, \exists \sigma \in \mathcal{B}: \ r \cdot i = O(\sigma)$;
(v) $\forall i, j \in \mathcal{B}\, \exists s \in \mathcal{B}: \ s >_{\mathbb{I}} 0\ ,\ |i| + |j| = O(s)$.

Moreover, we say that:

- $\mathcal{B} >_{\mathbb{I}} 0$, and we read it as $\mathcal{B}$ *is positive*, if $i >_{\mathbb{I}} 0$ for each $i \in \mathcal{B}$;
- $\mathcal{B}$ *is totally ordered* if for all $i, j \in \mathcal{B}$ either $i = O(j)$ or $j = O(i)$;
- $\mathcal{B}_{>0} := \{i \in \mathcal{B} \mid i >_{\mathbb{I}} 0\}$.

Of course, any solid subalgebra (see [14, Def. 9]) of $\mathbb{R}^I$ containing at least an infinite net is a trivial asymptotic gauge. To include this case, we only asked an existence in (ii) of the previous definition.

The name *gauge* gives the idea that using the infinities of $\mathcal{B}$, we are going to define moderate nets for our Colombeau-like algebras. To define negligible nets, we can use b^{-1} for all $b \in \mathcal{B}$ or another asymptotic gauge $\mathcal{Z}$ "at least as strong as $\mathcal{B}$".

The first example corresponds, of course, to the special algebra and so it starts from the set of indices $\mathbb{I}^{\mathrm{s}}$ and it is defined as $\mathcal{B}^{\mathrm{s}} := \{(\varepsilon^{-a}) \mid a \in \mathbb{R}_{>0}\}$. This AG is positive and totally ordered.

Property (v) of Def. 17 is equivalent to ask for the asymptotic closure of $\mathcal{B}$ with respect to sum and to absolute value, as it is stated in the following

Lemma 18. *Let $\mathcal{B} \subseteq \mathbb{R}^I$, then property (v) of Def. 17 is equivalent to*

(i) $\forall i, j \in \mathcal{B}\, \exists s \in \mathcal{B}: \ i + j = O(s)$;
(ii) $\forall i \in \mathcal{B}\, \exists a \in \mathcal{B}: \ a >_{\mathbb{I}} 0\ ,\ i = O(a)$.

Proof. That Def. 17.(v) is sufficient follows from $i + j \leqslant |i| + |j|$, $i \leqslant |i| + |i|$ and Thm. 12.(v). The condition is also necessary: if i, $j \in \mathcal{B}$, then from (ii) we get a, $b \in \mathcal{B}_{>0}$ such that $i = O(a)$ and $j = O(b)$; from (i) we have the existence of $s \in \mathcal{B}$ such that $a + b = O(s)$. Once again from (ii), we can assume $s >_{\mathbb{I}} 0$. Thus $|i| + |j| = O(|a| + |b|) = O(a + b) = O(s)$. □

A general way to obtain an asymptotic gauge on a generic set of indices $\mathbb{I}$ is to find a map $\rho : I \longrightarrow (0,1]$ such that $\lim_{\mathbb{I}} \rho = 0$ and to take the composition of ρ with the nets of an asymptotic gauge $\mathcal{B}$ on $\mathbb{I}^{\mathrm{s}}$, i.e.

$$\mathcal{B} \circ \rho := \{i \circ \rho \mid i \in \mathcal{B}\}\,.$$

For example, for the set of indices $\mathbb{I}^{\mathrm{e}}$ we can consider

$$\rho(\varphi) := \min\left\{\mathrm{diam}\left(\mathrm{supp}\varphi\right), 1\right\} \quad \forall \varphi \in I^{\mathrm{e}} = \mathcal{A}_0. \tag{2.10}$$

An analogue function ρ can be defined for the set of indices $\hat{\mathbb{I}}$ and in both cases they have limit zero. Several of the definitions we have introduced so far are motivated by the wish to obtain the following

Theorem 19. *Let $\mathbb{I} = (I, \leqslant, \mathcal{I})$ be a set of indices, and $\rho : I \longrightarrow (0,1]$ be a map such that $\lim_{\mathbb{I}} \rho = 0$. Let $\mathcal{B}$ be an asymptotic gauge on $\mathbb{I}^{\mathrm{s}}$, then*

(i) *$\mathcal{B} \circ \rho$ is an asymptotic gauge on $\mathbb{I}$.*
(ii) *If $\mathcal{B} >_{\mathbb{I}^{\mathrm{s}}} 0$ then $\mathcal{B} \circ \rho >_{\mathbb{I}} 0$.*
(iii) *If $\mathcal{B}$ is totally ordered, then also $\mathcal{B} \circ \rho$ is totally ordered.*

Proof. Property (i) of Def. 17 is clear. To prove (ii) of Def. 17, assume that $i \in \mathcal{B}$ is such that $\lim_{r \to 0^+} i_r = \infty$. Since $\lim_{\mathbb{I}} \rho = 0$, we get

$$\exists A \in \mathcal{I}\, \forall a \in A: \ 0 = \lim_{\varepsilon \leqslant a} \rho|_A(\varepsilon). \tag{2.11}$$

For each $R \in \mathbb{R}_{>0}$ there exists $\delta \in \mathbb{R}_{>0}$ such that $|i_r| > R$ for $r \in (0, \delta]$. But for each $a \in A$, 2.11 yields

$$\exists \varepsilon_0 \leqslant e \, \forall \varepsilon \in A : \ \varepsilon \leqslant \varepsilon_0 \quad \Longrightarrow \quad |\rho_\varepsilon| < \delta.$$

Therefore, $|i(\rho_\varepsilon)| > r$ for the same $\varepsilon \in A_{\leqslant \varepsilon_0}$. This proves that $\lim_{\mathbb{I}} i(\rho_\varepsilon) = \infty$.

Take $i, j \in \mathcal{B}$. We want to prove that both $|i \circ \rho| + |j \circ \rho|$ and $(i \circ \rho) \cdot (j \circ \rho)$ are asymptotically in $\mathcal{B} \circ \rho$. From the analogous property of the AG $\mathcal{B}$, we get the existence of $\sigma, \pi \in \mathcal{B}$ such that $|i| + |j| = O_{\mathbb{I}^{\mathrm{s}}}(\sigma)$ and $i \cdot j = O_{\mathbb{I}^{\mathrm{s}}}(\pi)$, that is $||i_r| + |j_r|| \leqslant H \cdot |\sigma_r|$ and $|i_r \cdot j_r| \leqslant K \cdot |\pi_r|$ for some $H, K \in \mathbb{R}_{>0}$ and for each $r \in (0, r_0]$. It suffices to take A as in 2.11 and $a \in A$ to have $|\rho_\varepsilon| < r_0$ for each $\varepsilon \in A_{\leqslant \varepsilon_1}$, for a suitable $\varepsilon_1 \leqslant a$. For all these $\varepsilon \in A_{\leqslant \varepsilon_1}$ we hence get $||i(\rho_\varepsilon)| + |j(\rho_\varepsilon)|| \leqslant H \cdot |\sigma(\rho_\varepsilon)|$ and $|i(\rho_\varepsilon) \cdot j(\rho_\varepsilon)| \leqslant K \cdot |\pi(\rho_\varepsilon)|$, which is our conclusion. Analogously, we can prove the asymptotic closure with respect to the product by scalars.

To prove (ii), assume that $i \in \mathcal{B} >_{\mathbb{I}^{\mathrm{s}}} 0$, so that $i_r > 0$ for $r \in (0, r_0]$. From (2.11) for each $a \in A$ we have that $|\rho_\varepsilon| < r_0$ for all $\varepsilon \in A_{\leqslant \varepsilon_1}$ and for some $\varepsilon_1 \leqslant a$. Therefore $i(\rho_\varepsilon) > 0$ for the same ε. This proves that $i(\rho_\varepsilon) > 0 \ \forall^{\mathbb{I}} \varepsilon \in A_{\leqslant a}$, and hence also that $i \circ \rho >_{\mathbb{I}} 0$.

Finally, to prove (iii), take $i, j \in \mathcal{B}$ such that $i_r = O_{\mathbb{I}^{\mathrm{s}}}(j_r)$ as $r \in \mathbb{I}^{\mathrm{s}}$. We want to prove that $i(\rho_\varepsilon) = O_{\mathbb{I}}(j(\rho_\varepsilon))$ as $\varepsilon \in \mathbb{I}$. Assume that $|i_r| \leqslant H \cdot |j_r|$ for $r \in (0, r_0]$. The limit relation (2.11) yields the existence of $A \in \mathcal{I}$ such that for each $a \in A$ we have

$$\exists \varepsilon_0 \leqslant a \, \forall \varepsilon \in A_{\leqslant \varepsilon_0} : \ |\rho_\varepsilon| < r_0,$$

and thus $|i(\rho_\varepsilon)| \leqslant H \cdot |j(\rho_\varepsilon)|$ for the same ε. By (iv) of Def. 1 of set of indices, we get $A_{\leqslant \varepsilon_0} \neq \varnothing$ which yields

$$\exists \varepsilon_1 \in A_{\leqslant a} \, \forall \varepsilon \in A_{\leqslant \varepsilon_1} : \ |i(\rho_\varepsilon)| \leqslant H \cdot |j(\rho_\varepsilon)|,$$

which proves that $i(\rho_\varepsilon) = O_{a,A}(j(\rho_\varepsilon))$, that is our conclusion. □

Example 20.

(i) Let ρ be the function defined in (2.10), then $\{(\rho_\varepsilon^{-a}) \mid a \in \mathbb{R}_{>0}\}$ is a totally ordered asymptotic gauge of positive functions on $\mathbb{I}^{\mathrm{e}}$. In the same way, we can proceed for $\hat{\mathbb{I}}$.

(ii) Define $\exp(x) := e^x$ for $x \in \mathbb{R}$, and $\exp^k := \exp \circ \overset{k}{\ldots} \circ \exp$, then $\mathcal{B}^{\exp}_{\mathrm{fin}} := \left\{\exp^k\left(\frac{1}{\varepsilon}\right) \mid k \in \mathbb{N}_{\neq 0}\right\}$ and $\mathcal{B}^{\exp}_{\infty} := \left\{\left(\exp^{[\varepsilon^{-1}]}(\varepsilon^{-1})^a\right) \mid a \in \mathbb{R}_{>0}\right\}$, where $[x]$ is the integer part of $x \in \mathbb{R}$, are totally ordered asymptotic gauges of positive functions.

(iii) Assuming that $\hat{\mathcal{I}}$ is a P-point on $\hat{I} = \mathcal{D}(\mathbb{R}^d)$, then the condition $\rho : \hat{I} \longrightarrow \mathbb{R}_{>0}$ and $\lim_{\hat{\mathbb{I}}} \rho = 0$ are equivalent to say that $[\rho]_{\hat{\mathcal{I}}} \in {}^*\mathbb{R} = \mathbb{R}^{\hat{I}}/\hat{\mathcal{I}}$ is infinitesimal. Therefore, Thm. 19 gives that $\hat{\mathcal{B}}_\rho := \{\rho^{-a} \mid a \in \mathbb{R}_{>0}\}$, $\hat{\mathcal{B}}^{\exp}_{\mathrm{fin}} := \{\exp^k(\rho^{-1}) \mid k \in \mathbb{N}_{\neq 0}\}$ and $\hat{\mathcal{B}}^{\exp}_{\infty} := \left\{\exp^{[\rho^{-1}]}(\rho^{-1})^a \mid a \in \mathbb{R}_{>0}\right\}$ are totally ordered asymptotic gauges of positive functions on $\hat{\mathbb{I}}$.

Definition 21. Let $\mathcal{B}$ be an AG on the set of indices $\mathbb{I} = (I, \leqslant, \mathcal{I})$. The set of *moderate nets generated by* $\mathcal{B}$ is

$$\mathbb{R}_M(\mathcal{B}) := \left\{x \in \mathbb{R}^I \mid \exists b \in \mathcal{B} : \ x_\varepsilon = O(b_\varepsilon) \text{ as } \varepsilon \in \mathbb{I}\right\}.$$

It is immediate to see that $\mathcal{B} \subseteq \mathbb{R}_M(\mathcal{B})$. Let us introduce also the following definition:

Definition 22. We say that an AG $\mathcal{B}$ is an *asymptotically closed ring* if

$$\mathcal{B} = \mathbb{R}_M(\mathcal{B}). \tag{2.12}$$

The following holds:

Theorem 23. *If $\mathcal{B}$ is an AG then $\mathbb{R}_M(\mathcal{B})$ is the minimal (with respect to inclusion) asymptotically closed solid ring containing $\mathcal{B}$.*

Proof. From Def. 17.(ii) and Thm. 12 we get $0 \in \mathbb{R}_M(\mathcal{B})$. If $x_\varepsilon = O(b_\varepsilon)$ and $y_\varepsilon = O(c_\varepsilon)$ for $b, c \in \mathcal{B}$, then $x_\varepsilon + y_\varepsilon = O(|b_\varepsilon| + |c_\varepsilon|)$. But $|b_\varepsilon| + |c_\varepsilon| = O(d_\varepsilon)$ for some $d \in \mathcal{B}_{>0}$ by Def. 17.(v). Therefore, $x_\varepsilon + y_\varepsilon = O(d_\varepsilon)$. Analogously, we can prove the closure of $\mathbb{R}_M(\mathcal{B})$ with respect to the product. It is immediate to see that $\mathbb{R}_M(\mathbb{R}_M(\mathcal{B})) = \mathbb{R}_M(\mathcal{B})$. Let us prove the minimality: let R be an asymptotically closed ring containing $\mathcal{B}$. Let $x \in \mathbb{R}_M(\mathcal{B})$, and let $a \in \mathcal{B}_{>0}$ be such that $x = O(a)$. Then, since $a \in \mathcal{B} \subseteq R$, we get $x \in \mathbb{R}_M(R)$. But $R = \mathbb{R}_M(R)$ because R is asymptotically closed, and we get that $x \in R$. So $\mathbb{R}_M(\mathcal{B}) \subseteq R$. The definition of $\mathbb{R}_M(\mathcal{B})$ directly gives that it is also asymptotically solid. □

Definition 24. Given two asymptotic gauges $\mathcal{B}_1, \mathcal{B}_2$ we say that $\mathcal{B}_1$ and $\mathcal{B}_2$ are *equivalent* if $\mathbb{R}_M(\mathcal{B}_1) = \mathbb{R}_M(\mathcal{B}_2)$.

Example 25.
(i) Every AG $\mathcal{B}$ is equivalent to $\mathbb{R}_M(\mathcal{B})$.
(ii) The asymptotic gauges $\mathcal{B}_1 = \{(\varepsilon^{-a}) \mid a \in \mathbb{R}\}$, $\mathcal{B}_2 = \{(\varepsilon^{-2a}) \mid a \in \mathbb{R}\}$ and $\mathcal{B}_3 = \{(\varepsilon^{-n}) \mid n \in \mathbb{N}\}$ on $I = (0, 1]$ are all equivalent.

3. Colombeau algebras generated by two asymptotic gauges

As we have already stated, every asymptotic gauge formalizes a notion of "growth conditions". For example, $\mathcal{B}^s$ (see Sec. 2.4) formalizes the idea of polynomial growth. We can hence use an asymptotic gauge $\mathcal{B}$ to define moderate nets and the reciprocals of nets taken from another asymptotic gauge $\mathcal{Z}$ to define negligible nets. From this point of view, it is natural to introduce the following definition:

Definition 26. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, let $\mathcal{B}, \mathcal{Z}$ be AG on a set of indices $\mathbb{I} = (I, \leqslant, \mathcal{I})$ and let $\mathcal{A}$ be a subalgebra of $\mathcal{C}^\infty(\Omega)^I$. The *set of $\mathcal{B}$-moderate nets in $\mathcal{A}$* is

$$\mathcal{E}_M(\mathcal{B}, \Omega, \mathcal{A}) := \{u \in \mathcal{A} \mid \forall K \Subset \Omega \,\forall \alpha \in \mathbb{N}^n \; \exists b \in \mathcal{B} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(b_\varepsilon) \text{ as } \varepsilon \in \mathbb{I}\}.$$

The *set of $\mathcal{Z}$-negligible nets in $\mathcal{A}$* is

$$\mathcal{N}(\mathcal{Z}, \Omega, \mathcal{A}) := \{u \in \mathcal{A} \mid \forall K \Subset \Omega \,\forall \alpha \in \mathbb{N}^n \; \forall z \in \mathcal{Z}_{>0} : \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = O(z_\varepsilon^{-1}) \text{ as } \varepsilon \in \mathbb{I}\}. \tag{3.1}$$

Moreover we set

$$\mathcal{E}_M(\mathcal{B}, \Omega) := \mathcal{E}_M(\mathcal{B}, \Omega, \mathcal{C}^\infty(\Omega)^I)$$

and

$$\mathcal{N}(\mathcal{Z},\Omega):=\mathcal{N}(\mathcal{Z},\Omega,\mathcal{C}^\infty(\Omega)^I).$$

Remark.

(i) If $z\in\mathcal{Z}_{>0}$ then $\forall^{\mathbb{I}}\varepsilon\in A_{\leqslant a}:\ z_\varepsilon>0$ for some $a\in A\in\mathcal{I}$, so that we can consider z_ε^{-1}. It is implicit in (3.1) that we are considering only these ε.

(ii) $\mathcal{E}_M(\mathcal{B},\Omega)\cap\mathbb{R}^I=\mathbb{R}_M(\mathcal{B})$ (by identifying a constant function $\Omega\longrightarrow\mathbb{R}$ with its value).

(iii) $\mathcal{E}_M(\mathcal{B},\Omega,\mathcal{A})=\mathcal{E}_M(\mathcal{B},\Omega)\cap\mathcal{A}$.

(iv) $\mathcal{N}(\mathcal{Z},\Omega,\mathcal{A}):=\mathcal{N}(\mathcal{Z},\Omega)\cap\mathcal{A}$.

We want to find conditions that ensure that the quotient $\mathcal{E}_M(\mathcal{B},\Omega)/\mathcal{N}(\mathcal{Z},\Omega)$ is an algebra. When this happens, we will use the following definition:

Definition 27. Let $\mathcal{B},\mathcal{Z}$ be AG and let $\mathcal{A}$ be a subalgebra of $\mathcal{C}^\infty(\Omega)^I$. The *Colombeau AG algebra generated by $\mathcal{B}$ and $\mathcal{Z}$ on $\mathcal{A}$* is the quotient

$$\mathcal{G}(\mathcal{B},\mathcal{Z},\mathcal{A}):=\mathcal{E}_M(\mathcal{B},\Omega,\mathcal{A})/\mathcal{N}(\mathcal{Z},\Omega,\mathcal{A}).$$

We also set $\mathcal{G}(\mathcal{B},\mathcal{Z},\mathcal{C}^\infty(\Omega)^I)=\mathcal{G}(\mathcal{B},\mathcal{Z},\Omega)$.

In the following, we will only consider the case $\mathcal{A}=\mathcal{C}^\infty(\Omega)^I$ and real valued nets of smooth functions. Nevertheless, all the results that we prove in this section can be easily generalized to the case of a generic subalgebra $\mathcal{A}\subseteq\mathcal{C}^\infty(\Omega)^I$ and to complex valued nets.

Let us observe that $\mathcal{G}^s(\Omega)=\mathcal{G}(\mathcal{B}^s,\mathcal{B}^s,\Omega)$. A known result is that, having fixed $\mathcal{Z}=\mathcal{B}^s$, $\mathcal{B}=\mathcal{Z}$ is the maximal choice such that $\mathcal{G}(\mathcal{B},\mathcal{Z},\Omega)$ is an algebra. We will prove that a similar property holds in our general setting.

Lemma 28. *For every $\mathcal{B},\mathcal{Z}$ asymptotic gauges and $\Omega\subseteq\mathbb{R}^n$ the inclusion $\mathcal{N}(\mathcal{Z},\Omega)\subseteq\mathcal{E}_M(\mathcal{B},\Omega)$ holds.*

Proof. Let $b\in\mathcal{B}$ and $z\in\mathcal{Z}$ be infinite nets: $\lim_{\varepsilon\in\mathbb{I}}b_\varepsilon=\lim_{\varepsilon\in\mathbb{I}}z_\varepsilon=\infty$. Then $\lim_{\varepsilon\in\mathbb{I}}z_\varepsilon^{-1}=0$, so that, for some $a\in A\in\mathcal{I}$, $|z_\varepsilon^{-1}|<1\ \forall^{\mathbb{I}}\varepsilon\in A_{\leqslant a}$. Analogously, $1<|b_\varepsilon|\ \forall^{\mathbb{I}}\varepsilon\in B_{\leqslant b}$ for some $b\in B\in\mathcal{I}$. Therefore, $z_\varepsilon^{-1}=O(b_\varepsilon)$. Now, let $K\Subset\Omega$, $\alpha\in\mathbb{N}^n$ and $u\in\mathcal{N}(\mathcal{Z},\Omega)$. As

$$\sup_{x\in K}|\partial^\alpha u_\varepsilon(x)|=O(z_\varepsilon^{-1})$$

and $z_\varepsilon^{-1}=O(b_\varepsilon)$, we obtain that

$$\sup_{x\in K}|\partial^\alpha u_\varepsilon(x)|=O(b_\varepsilon),$$

so $(u_\varepsilon)\in\mathcal{E}_M(\mathcal{B},\Omega)$. □

Lemma 29. *For every $\mathcal{B},\mathcal{Z}$ AG and $\Omega\subseteq\mathbb{R}^n$ open set, both $\mathcal{E}_M(\mathcal{B},\Omega)$ and $\mathcal{N}(\mathcal{Z},\Omega)$ are rings.*

Proof. Let $u,v\in\mathcal{E}_M(\mathcal{B},\Omega)$. Let $K\Subset\Omega$, $\alpha\in\mathbb{N}^n$ and let $b,c\in\mathcal{B}_{>0}$ be such that

$$\sup_{x\in K}|\partial^\alpha u_\varepsilon(x)|=O(b_\varepsilon),\ \sup_{x\in K}|\partial^\alpha v_\varepsilon(x)|=O(c_\varepsilon).$$

Finally, let $d\in\mathcal{B}_{>0}$ be such that $|b_\varepsilon|+|c_\varepsilon|=O(d_\varepsilon)$. Then

$$\sup_{x\in K}|\partial^\alpha(u_\varepsilon+v_\varepsilon)(x)|=O(b_\varepsilon)+O(c_\varepsilon)=O(|b_\varepsilon|+|c_\varepsilon|)=O(d_\varepsilon),$$

so $u+v\in\mathcal{E}_M(\mathcal{B},\Omega)$. Similarly, we can proceed for the product.

Now let $u, v \in \mathcal{N}(\mathcal{Z}, \Omega)$. Let $K \Subset \Omega$, $\alpha \in \mathbb{N}^n$ and let $z \in \mathcal{Z}_{>0}$. Then

$$\sup_{x \in K} |\partial^\alpha (u_\varepsilon + v_\varepsilon)(x)| = O(z_\varepsilon^{-1}) + O(z_\varepsilon^{-1}) = O(z_\varepsilon^{-1}),$$

so $u + v \in \mathcal{N}(\mathcal{Z}, \Omega)$. Similarly, we can proceed for the product. □

To have that $\mathcal{G}(\mathcal{B}, \mathcal{Z}, \Omega)$ is an algebra, we need that the product of a moderate net by a negligible one is always negligible. This implies that if $b \in \mathcal{B}$ and $z \in \mathcal{Z}_{>0}$, then we can find a $w \in \mathcal{Z}_{>0}$, depending on b and sufficiently small, such that $w_\varepsilon^{-1} \cdot b_\varepsilon$ is bounded by z_ε^{-1}. This forces a relation between the AG $\mathcal{B}$ and $\mathcal{Z}$ which can be summarized by saying that the scale $\mathcal{Z}$ is stronger or equal to that of $\mathcal{B}$, as it is precisely stated in the following theorem.

Theorem 30. *Let $\mathcal{B}$, $\mathcal{Z}$ be AG and $\Omega \subseteq \mathbb{R}^n$ be an open set. Then the following properties are equivalent*

(i) $\mathbb{R}_M(\mathcal{B}) \subseteq \mathbb{R}_M(\mathcal{Z})$;

(ii) $\forall b \in \mathcal{B}\, \forall z \in \mathcal{Z}_{>0}\, \exists w \in \mathcal{Z}_{>0} :\ w_\varepsilon^{-1} \cdot b_\varepsilon = O(z_\varepsilon^{-1})$.

If these hold, then

(iii) $\mathcal{N}(\mathcal{Z}, \Omega)$ *is a multiplicative ideal in* $\mathcal{E}_M(\mathcal{B}, \Omega)$ *(so, in particular, the quotient* $\mathcal{G}(\mathcal{B}, \mathcal{Z}, \Omega)$ *is an algebra).*

Moreover, if

$$\forall b \in \mathcal{B}\, \forall z \in \mathcal{Z} :\ b_\varepsilon = O(z_\varepsilon) \text{ or } z_\varepsilon = O(b_\varepsilon) \tag{3.2}$$

then (iii) entails (i).

Proof. (i) $\Rightarrow$ (ii): Let $b \in \mathcal{B}$ and $z \in \mathcal{Z}_{>0}$, we have to prove that $z \cdot b \in \mathbb{R}_M(\mathcal{Z})$. But $b \in \mathcal{B} \subseteq \mathbb{R}_M(\mathcal{B}) \subseteq \mathbb{R}_M(\mathcal{Z})$ and $z \in \mathcal{Z}_{>0} \subseteq \mathbb{R}_M(\mathcal{Z})$. Since $\mathbb{R}_M(\mathcal{Z})$ is a ring, the conclusion follows.

(ii) $\Rightarrow$ (i): Take $x \in \mathbb{R}_M(\mathcal{B})$, so that $x_\varepsilon = O(b_\varepsilon)$ for some $b \in \mathcal{B}$. Take an infinite net $z \in \mathcal{Z}$: $\lim_{\varepsilon \in \mathbb{I}} z_\varepsilon = \infty$. Then $b_\varepsilon = O(z_\varepsilon \cdot b_\varepsilon)$ and hence $x_\varepsilon = O(z_\varepsilon \cdot b_\varepsilon)$. By (ii) we get $w \in \mathcal{Z}_{>0}$ such that $z_\varepsilon \cdot b_\varepsilon = O(w_\varepsilon)$, which implies $x \in \mathbb{R}_M(\mathcal{Z})$.

(ii) $\Rightarrow$ (iii): Let $u \in \mathcal{N}(\mathcal{Z}, \Omega)$ and $v \in \mathcal{E}_M(\mathcal{B}, \Omega)$, let $K \Subset \Omega$, $\alpha \in \mathbb{N}^n$. Since

$$\sup_{x \in K} |\partial^\alpha (u_\varepsilon \cdot v_\varepsilon)(x)| \leqslant \sum_{\substack{\beta \in \mathbb{N}^n \\ \beta \leqslant \alpha}} \binom{\alpha}{\beta} \sup_{x \in K} \left|\partial^\beta u_\varepsilon(x)\right| \cdot \sup_{x \in K} \left|\partial^{\alpha-\beta} v_\varepsilon(x)\right| \quad \forall \varepsilon \in I,$$

for each $\beta \leqslant \alpha$ we can find $b_\beta = (b_{\beta\varepsilon}) \in \mathcal{B}_{>0}$ such that

$$\sup_{x \in K} |\partial^{\alpha-\beta} v_\varepsilon(x)| = O(b_{\beta\varepsilon}).$$

Therefore, for all $w \in \mathcal{Z}_{>0}$ we obtain

$$\begin{aligned}\sup_{x \in K} |\partial^\alpha (u_\varepsilon \cdot v_\varepsilon)(x)| &= \sum_{\substack{\beta \in \mathbb{N}^n \\ \beta \leqslant \alpha}} \binom{\alpha}{\beta} O(w_\varepsilon^{-1}) O(b_{\beta\varepsilon}) = \\ &= O\left(w_\varepsilon^{-1}\right) \cdot O\left(\sum_{\substack{\beta \in \mathbb{N}^n \\ \beta \leqslant \alpha}} b_{\beta\varepsilon}\right).\end{aligned} \tag{3.3}$$

Let $b \in \mathcal{B}_{>0}$ such that $\sum_{\substack{\beta \in \mathbb{N}^n \\ \beta \leqslant \alpha}} b_{\beta\varepsilon} = O(b_\varepsilon)$ and $z \in \mathcal{Z}_{>0}$. By (ii) there exists $(w_\varepsilon) \in \mathcal{Z}_{>0}$ such that $w_\varepsilon^{-1} \cdot b_\varepsilon = O(z_\varepsilon^{-1})$, and hence $\sup_{x \in K} |\partial^\alpha (u_\varepsilon \cdot v_\varepsilon)(x)| = O(z_\varepsilon^{-1})$ by (3.3).

(iii) and (3.2) $\Rightarrow$ (i): Let us assume that (i) does not hold. Then there exists $x \in \mathbb{R}_M(\mathcal{B})\backslash\mathbb{R}_M(\mathcal{Z})$, i.e. $x_\varepsilon = O(b_\varepsilon)$ for some $b \in \mathcal{B}_{>0}$. Since $x \notin \mathbb{R}_M(\mathcal{Z})$, it is easy to see that $b \notin \mathbb{R}_M(\mathcal{Z})$. From $b \notin \mathbb{R}_M(\mathcal{Z})$, we get $b_\varepsilon \neq O(z_\varepsilon)$ for all $(z_\varepsilon) \in \mathcal{Z}$. Thus, (3.2) yields $z_\varepsilon = O(b_\varepsilon)$ for all $(z_\varepsilon) \in \mathcal{Z}$. Hence $b_\varepsilon^{-1} = O(z_\varepsilon^{-1})$ for all $(z_\varepsilon) \in \mathcal{Z}_{>0}$, so we have that

$$(b_\varepsilon) \in \mathcal{E}_M(\mathcal{B}, \Omega);$$
$$(b_\varepsilon)^{-1} \in \mathcal{N}(\mathcal{Z}, \Omega);$$
$$(b_\varepsilon) \cdot (b_\varepsilon^{-1}) \notin \mathcal{N}(\mathcal{Z}, \Omega),$$

so $\mathcal{N}(\mathcal{Z}, \Omega)$ is not a multiplicative ideal in $\mathcal{E}_M(\mathcal{B}, \Omega)$, which is absurd. □

Let us note that, in particular, when (3.2) holds, $\mathcal{G}(\mathcal{B}, \mathcal{Z}, \Omega)$ is an algebra if and only if $\mathbb{R}_M(\mathcal{B}) \subseteq \mathbb{R}_M(\mathcal{Z})$, so the maximal possible choice for $\mathcal{B}$ is to take $\mathcal{B}$ equivalent to $\mathcal{Z}$. This is the generalization to our context of the known result for $\mathcal{G}^s(\Omega)$.

We conclude this section by proving that equivalent asymptotic gauges give the same Colombeau AG algebras:

Theorem 31. *Let $\mathcal{B}, \mathcal{Z}$ be AG, and let $\Omega \subseteq \mathbb{R}^n$. The following conditions hold:*

(i) $\mathcal{E}_M(\mathcal{B}, \Omega) = \mathcal{E}_M(\mathbb{R}_M(\mathcal{B}), \Omega)$;

(ii) $\mathcal{N}(\mathcal{Z}, \Omega) = \mathcal{N}(\mathbb{R}_M(\mathcal{Z}), \Omega)$.

In particular, $\mathcal{G}(\mathcal{B}, \mathcal{Z}, \Omega) = \mathcal{G}(\mathbb{R}_M(\mathcal{B}), \mathbb{R}_M(\mathcal{Z}), \Omega)$.

Proof. The proofs follow from the definitions. □

A consequence of Theorem 31 and Proposition 23 is that the theory could be developed in terms of asymptotically closed rings. This is the point of view followed by [8, 14]. Nevertheless, we think that it is useful to consider the notion of asymptotic gauge because many growth conditions are more easily expressed in terms of asymptotic gauges than in terms of asymptotically closed rings: note e.g. that the assumption (3.2) is too restrictive if $\mathcal{B}$ is a ring.

Example 32. Let $\mathcal{B}$ be an AG on $\mathbb{I}^{\mathrm{s}}$, then Thm. 19 yields that

$$\mathcal{B}^{\mathrm{e}} := \left\{ \left(b_{\underline{\varepsilon}}\right)_{\varepsilon \in \mathcal{A}_0} \mid b \in \mathcal{B} \right\} \ , \ \hat{\mathcal{B}} := \left\{ \left(b_{\underline{\varepsilon}}\right)_{\varepsilon \in \mathcal{D}(\mathbb{R}^d)} \mid b \in \mathcal{B} \right\}$$

are AG on $\mathbb{I}^e$ and $\hat{\mathbb{I}}$ respectively. Therefore, if $\mathbb{R}_M(\mathcal{B}) \subseteq \mathbb{R}_M(\mathcal{Z})$ then $\mathcal{G}(\mathcal{B}^{\mathrm{e}}, \mathcal{Z}^{\mathrm{e}}, \Omega)$ and $\mathcal{G}(\hat{\mathcal{B}}, \hat{\mathcal{Z}}, \Omega)$ generalize the full Colombeau algebra $\mathcal{G}^{\mathrm{e}}(\Omega)$ and the algebra $\hat{\mathcal{G}}(\Omega)$ of asymptotic functions (see also Cor. 6 and Thm. 8).

3.0.1. *A comparison with asymptotic scales, $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ algebras and exponent weights.* To study the relations between AG and the generalizations of Colombeau algebras cited in the title, in this section we only consider the set of indices $\mathbb{I}^{\mathrm{s}}$ of the special algebra, the sheaf $\mathcal{C}^\infty$of ordinary smooth functions, and the usual family of norms

$$S := \left\{ \|\partial^\alpha \cdot \|_{L^\infty(K)} \mid K \Subset \Omega, \alpha \in \mathbb{N}^n \right\}.$$

Let $\mathcal{B}$, $\mathcal{Z}$ be AG, with $\mathbb{R}_M(\mathcal{B}) \subseteq \mathbb{R}_M(\mathcal{Z})$. We already know (Thm. 23) that $\mathbb{R}_M(\mathcal{B})$ is a solid ring. Set

$$J_{\mathcal{Z}} := \left\{ x \in \mathbb{R}^{I^{\mathrm{s}}} \mid \forall z \in \mathcal{Z}_{>0} : \ x_\varepsilon = O(z_\varepsilon^{-1}) \right\}. \tag{3.4}$$

If $b \in \mathbb{R}_M(\mathcal{B})$, $x \in J_{\mathcal{Z}}$ and $z \in \mathcal{Z}_{>0}$, then Thm. 30.(ii) yields $w_\varepsilon^{-1} \cdot b_\varepsilon = O(z_\varepsilon^{-1})$ for some $w \in \mathcal{Z}_{>0}$. By the definition (3.4), we have $x_\varepsilon = O(w_\varepsilon^{-1})$, so that $x_\varepsilon \cdot b_\varepsilon =$

$O(w_\varepsilon^{-1} \cdot b_\varepsilon) = O(z_\varepsilon^{-1})$, which proves that $x \cdot b \in J_{\mathcal{Z}}$, i.e. that $J_{\mathcal{Z}}$ is an ideal of $\mathbb{R}_M(\mathcal{B})$. Directly from the definitions, we get

$$\mathcal{A}_{\mathbb{R}_M(\mathcal{B})/J_{\mathcal{Z}},\mathcal{C}^\infty,S} = \mathcal{G}(\mathcal{B},\mathcal{Z},-), \tag{3.5}$$

where the left hand side is the $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebra associated to the ring $\mathbb{R}_M(\mathcal{B})/J_{\mathcal{Z}}$.

Vice versa, let us assume that A is a solid subring of $\mathbb{R}^{I^s}$ containing at least one infinite net

$$\exists a \in A : \lim_{\varepsilon\to 0} a_\varepsilon = \infty.$$

Then $\mathcal{B} := A$ is an AG. We can now consider the solid ideal associated to A, i.e.

$$I_A = \left\{x \in \mathbb{R}^{I^s} \mid \forall a \in A^* : x = O(a)\right\}.$$

It is easy to prove that $I_A = \left\{x \in \mathbb{R}^{I^s} \mid \forall a \in A^*_{>0} : x = O(a^{-1})\right\}$ so that we can set $\mathcal{Z} := I_A$, which is an AG. Directly from the definitions, we get

$$\mathcal{A}_{A/I_A,\mathcal{C}^\infty,S} = \mathcal{G}(\mathcal{B},\mathcal{Z},-). \tag{3.6}$$

This proves that, in the framework of the special algebra, $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras and Colombeau AG algebra are essentially equivalent.

In [14], the relations between $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras and asymptotic algebras generated by an asymptotic scales are already clarified, so that our previous (3.5), (3.6) would also give the relations with the latter. Anyway, let us assume that $a = (a_m)_{m\in\mathbb{Z}}$, $a_m \in \mathbb{R}^{I^s}_{>0}$, is an asymptotic scale:

$$\forall m \in \mathbb{Z} : a_{m+1} = o(a_m), a_{-m} = \frac{1}{a_m}, \exists M \in \mathbb{Z} : a_M = O(a_m^2).$$

Then, we can define the AG $\mathcal{B} := \mathcal{Z} := \left\{x \in \mathbb{R}^{I^s} \mid \exists m \in \mathbb{Z} : x_\varepsilon = O(a_{m\varepsilon})\right\}$, and this yields

$$\mathcal{A}_a(\mathcal{C}^\infty, S) = \mathcal{G}(\mathcal{B},\mathcal{Z},-).$$

Finally, let $r : \mathbb{N} \longrightarrow \mathbb{R}_{>0}$ be a sequence of weights (see [9]) and, instead of the set of indices $\mathbb{I}^s$, we consider the set of indices $\mathbb{I}^\infty := (\mathbb{N}, \leqslant, \mathcal{F})$, where $\leqslant$ is the usual order relation on $\mathbb{N}$ and $\mathcal{F}$ is the Fréchet filter on $\mathbb{N}$. Setting

$$\mathcal{B} := \mathcal{Z} := \{x \in \mathbb{R}^{\mathbb{N}} \mid \forall p \in S : \|x\|_{p,r} < \infty\}$$

we get an AG and

$$\mathcal{G}_{S,r} = \mathcal{G}(\mathcal{B},\mathcal{Z},-).$$

See [9] for more details and for the notations $\|-\|_{p,r}$ and $\mathcal{G}_{S,r}$.

4. Embeddings of Distributions

In this section we let $\Omega \subseteq \mathbb{R}^n$ be a fixed open set and we let $\mathcal{B}, \mathcal{Z}$ be two fixed asymptotic gauges with $\mathbb{R}_M(\mathcal{B}) \subseteq \mathbb{R}_M(\mathbb{Z})$. We will define an embedding

$$i : \mathcal{D}'(\Omega) \to \mathcal{G}(\mathcal{B},\mathcal{Z},\Omega)$$

by slightly modifying the construction usually considered in $\mathcal{G}^s(\Omega)$. Our construction will follow the same approach used by [6]. We start by defining our mollificator.

Definition 33. Let $b \in \mathcal{B}_{>0}$ be infinite: $\lim_{\mathbb{I}} b = +\infty$; for simplicity, we can assume that $b_\varepsilon > 0$ for all $\varepsilon \in I$. Let $\rho \in \mathcal{S}(\mathbb{R}^n)$ be such that

(i) $\int \rho(x)\,\mathrm{d}x = 1$;
(ii) $\int \rho(x) x^k\,\mathrm{d}x = 0$ for every $k \geqslant 1$.

We set
$$\rho_\varepsilon := b_\varepsilon^{-1} \odot \rho,$$
so that $\rho_\varepsilon(x) = b_\varepsilon^n \cdot \rho(b_\varepsilon x)$ for all $x \in \mathbb{R}^n$.

Lemma 34. *Let $w \in \mathcal{E}'(\Omega)$ and $\varphi \in \mathcal{D}(\Omega)$, then*
$$\lim_{\varepsilon \in \mathbb{I}} \int_\Omega (w * \rho_\varepsilon) \cdot \varphi = \langle w, \varphi \rangle.$$

Proof. As usual, by the continuity of convolution, the conclusion is equivalent to
$$\lim_{\varepsilon \leqslant e} \int \rho_\varepsilon \cdot \varphi = \varphi(0), \tag{4.1}$$
where $e \in I$. In fact, this would prove that $(\rho_\varepsilon)_{\varepsilon \leqslant e} \to \delta$ in $\mathcal{D}'$ with respect to the directed set $(\varnothing, e]$. To prove (4.1), we consider
$$\begin{aligned}\left| \int \rho_\varepsilon(x)\varphi(x)\,\mathrm{d}x - \varphi(0) \right| &= \left| \int \rho(t) \left[\varphi\left(\frac{t}{b_\varepsilon}\right) - \varphi(0) \right] \mathrm{d}t \right| \leqslant \\ &\leqslant \int |\rho(t)| \left| \varphi\left(\frac{t}{b_\varepsilon}\right) - \varphi(0) \right| \mathrm{d}t.\end{aligned}$$
Since φ is compactly supported, we have
$$\forall r \in \mathbb{R}_{>0}\, \exists \varepsilon_0 \leqslant e\, \forall \varepsilon \leqslant \varepsilon_0\, \forall t \in \operatorname{supp}\varphi : \ \left| \varphi\left(\frac{t}{b_\varepsilon}\right) - \varphi(0) \right| < r,$$
so
$$\forall r \in \mathbb{R}_{>0}\, \exists \varepsilon_0 \leqslant e\, \forall \varepsilon \leqslant \varepsilon_0 : \ \left| \int \rho_\varepsilon(x)\varphi(x)\,\mathrm{d}x - \varphi(0) \right| \leqslant r \int |\rho|\,.$$
□

Theorem 35. *For every open $\Omega \subseteq \mathbb{R}^n$ the map*
$$\begin{aligned} i_0 : \mathcal{E}'(\Omega) &\to \mathcal{G}(\mathcal{B}, \mathcal{Z}, \Omega) \\ w &\mapsto [(w * \rho_\varepsilon)|_\Omega] \end{aligned} \tag{4.2}$$
is a linear embedding.

Proof. We have to prove that
(i) i_0 is linear;
(ii) $\forall w \in \mathcal{E}'(\Omega) : \ (w * \rho_\varepsilon)|_\Omega \in \mathcal{E}_M(\mathcal{B}, \Omega)$;
(iii) $\operatorname{Ker}(i_0) = \{0\}$.

That i_0 is linear follows immediately by the definition, since the convolution is a linear operator. Let us prove (ii). By the local structure theorem for distributions, it suffices to consider the case $w = \partial^\alpha f \in \mathcal{E}'(\Omega)$, with $f \in \mathcal{D}(\Omega)$ and $\alpha \in \mathbb{N}^n$. Let $x \in K \Subset \Omega$, then
$$\begin{aligned}(w * \rho_\varepsilon)(x) = f * \partial^\alpha \rho_\varepsilon(x) &= \int f(x-y) \partial^\alpha \rho_\epsilon(y)\,\mathrm{d}y = \\ &= \int f(x-y) b_\varepsilon^{n+|\alpha|} (\partial^\alpha \rho)(b_\varepsilon \cdot y)\,\mathrm{d}y = \\ &= b_\varepsilon^{|\alpha|} \int f\left(x - \frac{t}{b_\varepsilon}\right) \cdot \partial^\alpha \rho(t)\,\mathrm{d}t = O(b_\varepsilon^{|\alpha|}) = O(c_\varepsilon),\end{aligned}$$

for some $(c_\varepsilon) \in \mathcal{B}$ with $b_\varepsilon^{|\alpha|} = O(c_\varepsilon)$. The same argument applies to the derivative $\partial^\beta (f * \partial^\alpha \rho_\varepsilon) = f * \partial^{\alpha+\beta}\rho_\varepsilon$.

To prove (iii), let $w \in \mathcal{E}'(\Omega)$ be such that $[(w * \rho_\varepsilon)\,|_\Omega] = 0$ and let $\varphi \in \mathcal{D}(\Omega)$. Thus, setting $K := \operatorname{supp}(\varphi)$, we have

$$\forall z \in \mathcal{Z}_{>0} : \sup_{x \in K} |(w * \rho_\varepsilon)(x)| = O(z_\varepsilon^{-1})$$

and hence $\lim_{\varepsilon \in \mathbb{I}} \sup_{x \in K} |(w * \rho_\varepsilon)(x)| = 0$. From this and Lemma 34, we hence obtain

$$|\langle w, \varphi\rangle| = \lim_{\varepsilon \in \mathbb{I}} \left| \int_\Omega (w * \rho_\varepsilon)\varphi \right| \leqslant \lim_{\varepsilon \in \mathbb{I}} \sup_{x \in K} |(w * \rho_\varepsilon)(x)| \cdot \int_K |\varphi| = 0.$$

□

Let us note that the embedding (4.2) depends on the open set Ω. We will use the notation $i_{0\Omega}$ when we want to underline this dependence.

We denote by σ the constant embedding of $\mathcal{C}^\infty(\Omega)$ into $\mathcal{G}(\mathcal{B}, \mathcal{Z}, \Omega)$, namely $\sigma(f) = [f]$. We would like to prove the analogue of [6, Prop. 1.2.11]. As usual, the idea is to start with $f \in \mathcal{D}(\Omega)$ and to use Taylor's formula obtaining

$$\begin{aligned}
(f * \rho_\varepsilon - f)(x) &= \int (f(x-y) - f(x))\rho_\varepsilon(y)\,\mathrm{d}y = \\
&= \int \left(f\left(x - \frac{t}{b_\varepsilon}\right) - f(x) \right) \rho(t)\,\mathrm{d}t = \\
&= \int \sum_{0<|\alpha|<m} \frac{1}{\alpha!} \left(-\frac{t}{b_\varepsilon}\right)^\alpha \partial^\alpha f(x)\rho(t)\,\mathrm{d}t + \\
&\quad + \int \sum_{|\alpha|=m} \frac{1}{\alpha!} \left(-\frac{t}{b_\varepsilon}\right)^\alpha \partial^\alpha f\left(x - \Theta \frac{t}{b_\varepsilon}\right) \rho(t)\,\mathrm{d}t = \\
&= 0 + b_\varepsilon^{-m} \cdot \int \sum_{|\alpha|=m} \frac{1}{\alpha!} (-t)^\alpha\, \partial^\alpha f\left(x - \Theta \frac{t}{b_\varepsilon}\right) \rho(t)\,\mathrm{d}t = \\
&= O\left(b_\varepsilon^{-m}\right).
\end{aligned} \tag{4.3}$$

Therefore, to have $(f - (f * \rho_\varepsilon)|_\Omega) \in \mathcal{N}(\mathcal{Z}, \Omega)$ we need a further condition of the form

$$\forall z \in \mathcal{Z}_{>0}\, \exists m \in \mathbb{N} : \ b_\varepsilon^{-m} = O(z_\varepsilon^{-1}) \quad \text{i.e.} \quad z_\varepsilon = O(b_\varepsilon^m).$$

This implies $\mathcal{Z}_{>0} \subseteq \mathbb{R}_M(\mathrm{AG}(b)) \subseteq \mathbb{R}_M(\mathcal{B})$, where $\mathrm{AG}(b)$ is the AG

$$\mathrm{AG}(b) := \{b^m \mid m \in \mathbb{N}\}.$$

We have thus $\mathbb{R}_M(\mathcal{Z}) = \mathbb{R}_M(\mathcal{B}) = \mathbb{R}_M(\mathrm{AG}(b))$.

Definition 36. Let $\mathcal{B}$ be an AG. If $b \in \mathcal{B}$ is such that $\mathbb{R}_M(\mathrm{AG}(b)) \supseteq \mathbb{R}_M(\mathcal{B})$ then we will say that b is a *generator of* $\mathcal{B}$ and that $\mathcal{B}$ is a *principal AG.*

Let us note that in the previous definition we could equivalently substitute the condition $\mathbb{R}_M(\mathrm{AG}(b)) \supseteq \mathbb{R}_M(\mathcal{B})$ with $\mathbb{R}_M(\mathrm{AG}(b)) = \mathbb{R}_M(\mathcal{B})$. Moreover if $\mathcal{B}$ is principal AG then, if necessary, we can always find a positive generator of $\mathcal{B}$: in fact, if b is any generator of $\mathcal{B}$ and $c \in \mathcal{B}_{>0}$, $c \in O(|b|)$, then also c is a generator of $\mathcal{B}$.

Every AG of Ex. 20, other than $\mathcal{B}_{\mathrm{fin}}^{\mathrm{exp}}$ and $\hat{\mathcal{B}}_{\mathrm{fin}}^{\mathrm{exp}}$, is principal; for example, ε^{-1} is a generator of $\mathcal{B}^{\mathrm{s}}$. Moreover, a solid subalgebra of $\mathbb{R}^I$ (containing an infinite net) generally speaking is not a principal AG. In the latter case, the embedding of distributions using a mollifier is not possible (see Thm. 44).

By Theorem 31 we can also assume, without loss of generality, that $\mathcal{B} = \mathcal{Z}$, which is the subject of our next

Assumption: $\mathcal{Z} = \mathcal{B}$ is a principal AG. Moreover we assume that the mollifier (ρ_ε) is constructed with a fixed generator b.

Let us observe that a generator of an asymptotic gauge is necessarily an infinite element in $\mathbb{R}^I$, i.e. $\lim_{\varepsilon\in\mathbb{I}} b_\varepsilon = \infty$, and that every principal AG is totally ordered. As a first consequence of our assumption, we have

$$\mathcal{N}(\mathcal{Z},\Omega) = \mathcal{N}(\mathcal{B},\Omega) =$$
$$\left\{ u \in \mathcal{C}^\infty(\Omega)^I \mid \forall K \Subset \Omega \, \forall \alpha \in \mathbb{N}^n \, \forall m \in \mathbb{N} : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(b_\varepsilon^{-m}) \text{ as } \varepsilon \in \mathbb{I} \right\}.$$

Henceforward, we will thus use the simplified notation $\mathcal{G}(\mathcal{B},\Omega) := \mathcal{G}(\mathcal{B},\mathcal{B},\Omega)$.

Theorem 37. $i_0|_{\mathcal{D}(\Omega)} = \sigma$. *Consequently,* i_0 *is an injective homomorphism of algebras on* $\mathcal{D}(\Omega)$.

Proof. The second statement follows from the first like in [6]. The remaining part is proved in (4.3). □

The notions of support $\mathrm{supp}(u)$ of a generalized function $u \in \mathcal{G}(\mathcal{B},\Omega)$ and of restriction $u|_{\Omega'} \in \mathcal{G}(\mathcal{B},\Omega')$ can be defined exactly like in [6, pag. 12].

Theorem 38. *If* $w \in \mathcal{E}'(\Omega)$ *then* $\mathrm{supp}(w) = \mathrm{supp}(i_0(w))$.

Proof. Let us prove that $\mathrm{supp}(i_0(w)) \subseteq \mathrm{supp}(w)$. We have to prove that

$$i_0(w)|_{\mathrm{supp}(w)^c} = 0$$

in $\mathcal{G}(\mathcal{B},\mathrm{supp}(w)^c)$. Let $K \Subset \mathrm{supp}(w)^c$, let $\alpha \in \mathbb{N}^n$ be such that $w = \partial^\alpha f$, with $f \in \mathcal{D}(\mathbb{R}^n \backslash K)$. Then $i_0(w) = [(f * \partial^\alpha \rho_\varepsilon)|_\Omega]$ and

$$\begin{aligned}
(f * \partial^\alpha \rho_\varepsilon)(x) &= \int f(x-y)\partial^\alpha \rho_\varepsilon(y)\,\mathrm{d}y = \\
&= \int f(x-y)\cdot b_\varepsilon^{|\alpha|+n}\partial^\alpha \rho(b_\varepsilon y)\,\mathrm{d}y = \\
&= \int f\left(x-\frac{t}{b_\varepsilon}\right)\cdot b_\varepsilon^{|\alpha|}\partial\rho(t)\,\mathrm{d}t = \\
&= \int_{|t|<\sqrt{b_\varepsilon}} f\left(x-\frac{t}{b_\varepsilon}\right)\cdot b_\varepsilon^{|\alpha|}\partial\rho(t)\,\mathrm{d}t + \\
&\quad + \int_{|t|\geqslant\sqrt{b_\varepsilon}} f\left(x-\frac{t}{b_\varepsilon}\right)\cdot b_\varepsilon^{|\alpha|}\partial\rho(t)\,\mathrm{d}t.
\end{aligned}$$

Recall that $b_\varepsilon > 0$ for each $\varepsilon \in I$. Since $\mathrm{supp}(f) \cap K = \emptyset$, if $x \in K$ there exist $A \in \mathcal{I}$ such that for each $a \in A$

$$\forall^{\mathbb{I}}\varepsilon \in A_{\leqslant a}\,\forall t : \ |t| < \sqrt{b_\varepsilon} \quad \Longrightarrow \quad x - \frac{t}{b_\varepsilon} \notin \mathrm{supp}(f),$$

so the first integral is zero. For the same ε sufficiently small, the second integral can be estimated as follows:

$$\int_{|t|\geqslant\sqrt{b_\varepsilon}} f\left(x-\frac{t}{b_\varepsilon}\right)\cdot b_\varepsilon^{|\alpha|}\partial\rho(t)\,\mathrm{d}t \leqslant b_\varepsilon^{|\alpha|}\cdot\|f\|_\infty\cdot\int_{|t|\geqslant\sqrt{b_\varepsilon}} \left|\partial^{|\alpha|}\rho(t)\right|\,\mathrm{d}t.$$

Since $\rho \in \mathcal{S}(\mathbb{R}^n)$ for any $m \in \mathbb{N}$ there exists a constant $c_m > 0$ such that $\left|\partial^{|\alpha|}\rho(t)\right| \leqslant c_m(1+|t|)^{-2m-n-1}$. Thus $\left|\partial^{|\alpha|}\rho(t)\right| \leqslant c_m\left(\sqrt{b_\varepsilon}\right)^{-2m}(1+|t|)^{-n-1}$ for $|t| \geqslant \sqrt{b_\varepsilon}$, and

$$\int_{|t|\geqslant\sqrt{b_\varepsilon}} b_\varepsilon^{|\alpha|}\left|\partial^{|\alpha|}\rho(t)\right| \mathrm{d}t \leqslant c_m b_\varepsilon^{|\alpha|-m} \cdot \int_{|t|\geqslant\sqrt{b_\varepsilon}} (1+|t|)^{-n-1}\,\mathrm{d}t = \widetilde{c_m} \cdot b_\varepsilon^{|\alpha|-m}.$$

Since m is arbitrary we can treat the derivative of $i_0(w)$ in the same way, and this gives the desired estimates that show that $\mathrm{supp}(i_0(w)) \subseteq \mathrm{supp}(w)$.

Let us now prove that $\mathrm{supp}(w) \subseteq \mathrm{supp}(i_0(w))$. Let $x_0 \in \mathrm{supp}(w)$. For every $\eta > 0$ there exists $\varphi \neq 0$ in $\mathcal{D}(\mathbb{R})$ such that $\mathrm{supp}(\varphi) \subseteq B_\eta(x_0)$ and $|\langle w, \varphi\rangle| =: c > 0$. Since $\lim_{\varepsilon\in\mathbb{I}} w * \rho_\varepsilon = w$ in $\mathcal{D}'(\Omega)$ (Lemma 34), this implies that $|\langle w * \rho_\varepsilon, \varphi\rangle| > \frac{c}{2}$ for $\varepsilon \in \mathbb{I}$ small. But setting $K := \overline{B_\eta(x_0)}$ we have

$$0 < \frac{c}{2} < |\langle w * \rho_\varepsilon, \varphi\rangle| = \left|\int_K (w * \rho_\varepsilon)(x) \cdot \varphi(x)\,\mathrm{d}x\right| \leqslant \sup_{x\in K} |(w * \rho_\varepsilon)(x)| \cdot \int_K |\varphi|\,. \quad (4.4)$$

The equality $\left[(w * \rho_\varepsilon)|_{B_\eta(x_0)}\right] = 0$ in $\mathcal{G}(\mathcal{B}, B_\eta(x_0))$ would imply

$$\lim_{\varepsilon\in\mathbb{I}} \sup_{x\in K} |(w * \rho_\varepsilon)(x)| = 0,$$

which is impossible by (4.4), so $i_0(w)|_{B_\eta(x_0)} = \left[(w * \rho_\varepsilon)|_{B_\eta(x_0)}\right] \neq 0$ and therefore $x_0 \in \mathrm{supp}(i_0(w))$. □

To prove that i_0 can be extended to an embedding $i : \mathcal{D}'(\Omega) \to \mathcal{G}$ we can now use the following result, whose proof can be conducted exactly like in [6]:

Theorem 39.

(i) *$\mathcal{G}(\mathcal{B}, -) : \Omega \mapsto \mathcal{G}(\mathcal{B}, \Omega)$ is a sheaf of differential algebras on $\mathbb{R}^n$;*

(ii) *There is a unique sheaf morphism of real vector spaces $i : \mathcal{D}' \longrightarrow \mathcal{G}(\mathcal{B}, -)$ such that:*

- *(a) i extends the embedding $i_0 : \mathcal{E}' \longrightarrow \mathcal{G}(\mathcal{B}, -)$ defined in (4.2), i.e. such that $i_\Omega|_{\mathcal{E}'(\Omega)} = i_{0\Omega}$.*
- *(b) i commutes with partial derivatives, i.e. $\partial^\alpha(i_\Omega(T)) = i_\Omega(\partial^\alpha T)$ for each $T \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}$.*
- *(c) $i|_{\mathcal{C}^\infty(-)} : \mathcal{C}^\infty(-) \longrightarrow \mathcal{G}(\mathcal{B}, -)$ is a sheaf morphism of algebras.*

4.1. **Embedding with a strict δ-net.** A simpler way to embed distributions is by means of a strict δ-net rather than a model δ-net (i.e. a net obtained by scaling a single function ρ). This can be done for a generic set of indices and a principal AG simply by generalizing [17, Lem. A1, Cor. A2]:

Theorem 40. *Let $\mathbb{I}$ be a set of indices and $\mathcal{B}$ be a principal AG on $\mathbb{I}$ generated by $b \in \mathcal{B}_{>0}$. There exists a net $(\psi_\varepsilon)_{\varepsilon\in I}$ of $\mathcal{D}(\mathbb{R}^n)$ with the properties:*

(i) *$supp(\psi_\varepsilon) \subseteq B_1(0)$ for all $\varepsilon \in I$;*

(ii) *$\int \psi_\varepsilon = 1$ for all $\varepsilon \in I$;*

(iii) *$\forall \alpha \in \mathbb{N}^n\, \exists p \in \mathbb{N} :\ \sup_{x\in\mathbb{R}^n} |\partial^\alpha \psi_\varepsilon(x)| = O(b_\varepsilon^p)$ as $\varepsilon \in \mathbb{I}$;*

(iv) *$\forall j \in \mathbb{N}\, \forall^{\mathbb{I}}\varepsilon :\ 1 \leqslant |\alpha| \leqslant j \Rightarrow \int x^\alpha \cdot \psi_\varepsilon(x)\,\mathrm{d}x = 0$;*

(v) *$\forall \eta \in \mathbb{R}_{>0}\, \forall^{\mathbb{I}}\varepsilon :\ \int |\psi_\varepsilon| \leqslant 1 + \eta$.*

In particular

$$\rho_\varepsilon := b_\varepsilon^{-1} \odot \psi_\varepsilon \quad \forall \varepsilon \in I$$

satisfies (ii) - (v).

Proof. For $m \in \mathbb{N}$ and $\eta \in \mathbb{R}_{>0}$ define the sets

$$\mathcal{A}_m := \left\{ \varphi \in \mathcal{D}(\mathbb{R}^n) \mid \mathrm{supp}(\varphi) \subseteq B_1(0), \int \varphi = 1, \int x^\alpha \varphi(x)\,\mathrm{d}x = 0 \ 1 \leqslant |\alpha| \leqslant m \right\},$$

$$\mathcal{A}'_m(\eta) := \left\{ \varphi \in \mathcal{A}_m \mid \int |\varphi| \leqslant 1 + \eta \right\}.$$

In [17] it is proved that $\mathcal{A}_m \neq \varnothing \neq \mathcal{A}'_m(\eta)$. For each $m \in \mathbb{N}_{>0}$, we choose $\varphi_m \in \mathcal{A}'_m\left(\frac{1}{m}\right)$ and we set

$$M_m := \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leqslant m}} |\partial^\alpha \varphi_m(x)|,$$

$$\mathcal{A}_{m,\varepsilon} := \left\{ \varphi \in \mathcal{A}'_m\left(\frac{1}{m}\right) \mid \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leqslant m}} |\partial^\alpha \varphi(x)| \leqslant b_\varepsilon \right\} \quad \forall m \in \mathbb{N}_{>0}\, \forall \varepsilon \in I.$$

Therefore, $\varnothing \neq \mathcal{A}_{m+1,\varepsilon} \subseteq \mathcal{A}_{m,\varepsilon}$ and $\varphi_m \in \mathcal{A}_{m,\varepsilon}$ whenever $M_m \leqslant b_\varepsilon$. Since $\lim_{m \to +\infty} M_m = +\infty$ (see [17]), for each fixed $\varepsilon \in I$, we have $b_\varepsilon < M_{m+1}$ for m sufficiently big. We denote by m_ε the minimum $m \in \mathbb{N}$ such that $b_\varepsilon < M_{m+1}$, so that

$$M_{m_\varepsilon} \leqslant b_\varepsilon < M_{m_\varepsilon+1} \quad \forall \varepsilon \in I \tag{4.5}$$

and hence $\varphi_{m_\varepsilon} \in \mathcal{A}_{m_\varepsilon,\varepsilon}$ for all $\varepsilon \in I$. Define $\psi_\varepsilon := \varphi_{m_\varepsilon}$ for all $\varepsilon \in I$, so that $\psi_\varepsilon = \varphi_{m_\varepsilon} \in \mathcal{A}_{m_\varepsilon,\varepsilon} \subseteq \mathcal{A}_{m_\varepsilon}$, which proves (i), (ii). The remaining properties can be proved like in [17]. We have only to note that if $\alpha \in \mathbb{N}^n$, then $|\alpha| \leqslant b_\varepsilon$ for $\varepsilon \in \mathbb{I}$ sufficiently small because $\lim_{\varepsilon \in \mathbb{I}} b_\varepsilon = +\infty$. Therefore, (4.5) yields $|\alpha| \leqslant m_\varepsilon$ and hence $\mathcal{A}_{|\alpha|,\varepsilon} \supseteq \mathcal{A}_{m_\varepsilon,\varepsilon} \ni \psi_\varepsilon$. ☐

We finally have the following results, whose proof is just a mild variation of the previous proofs about the embedding with a model δ-net.

Corollary 41. *If (ρ_ε) is the net defined like in Thm. 40, then the mapping*

$$i_\Omega : T \in \mathcal{D}'(\Omega) \mapsto [T * \rho_\varepsilon] \in \mathcal{G}(\mathcal{B}, \Omega)$$

is a sheaf morphism of real vector spaces $i : \mathcal{D}' \longrightarrow \mathcal{G}(\mathcal{B}, -)$, and satisfies the following properties:

- *(i)* *i commutes with partial derivatives, i.e. $\partial^\alpha(i_\Omega(T)) = i_\Omega(\partial^\alpha T)$ for each $T \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}$;*
- *(ii)* *$i|_{\mathcal{C}^\infty(-)} : \mathcal{C}^\infty(-) \longrightarrow \mathcal{G}(\mathcal{B}, -)$ is a sheaf morphism of algebras;*
- *(iii)* *If $w \in \mathcal{E}'(\Omega)$ then $\mathrm{supp}(w) = \mathrm{supp}(i_\Omega(w))$;*
- *(iv)* *$i_\Omega(T) \approx T$ for each $T \in \mathcal{D}'(\Omega)$, i.e. $\lim_{\varepsilon \in \mathbb{I}} \int_\Omega (T * \rho_\varepsilon) \cdot \varphi = \langle T, \varphi \rangle$ for all $\varphi \in \mathcal{D}(\Omega)$.*

4.2. **Comparison of embeddings.** We close this section by facing a natural problem: let us define two embeddings i_b, i_c like (4.2) but using two different generators b, $c \in \mathcal{B}$:

$$i_b(w) := \left[w * (b_\varepsilon^{-1} \odot \rho) \right],$$
$$i_c(w) = \left[w * (c_\varepsilon^{-1} \odot \rho) \right].$$

It is well known that $i_b(T) \approx T \approx i_c(T)$, but when are they equal?

Theorem 42. *Let $b, c \in \mathcal{B}_{>0}$ be generators of the AG $\mathcal{B}$. Assume that $\rho(0) \neq 0$; then $i_b = i_c$ if and only if $[b_\varepsilon] = [c_\varepsilon]$ in $\mathcal{G}(\mathcal{B}, \mathbb{R})$, i.e. iff they generate the same $\mathcal{B}$-Colombeau generalized number.*

Proof. If $i_b = i_c$, then $i_b(\delta) = [b_\varepsilon^n \cdot \rho(b_\varepsilon \cdot -)] = i_c(\delta) = [c_\varepsilon^n \cdot \rho(c_\varepsilon \cdot -)]$. Setting $K = \{0\}$ in the definition of negligible net, we get

$$\forall m \in \mathbb{N} : \ |b_\varepsilon^n \rho(0) - c_\varepsilon^n \rho(0)| = O(b_\varepsilon^{-m}),$$

that is $[b_\varepsilon^n] = [c_\varepsilon^n]$. The conclusion follows by applying the smooth function $\sqrt[n]{-} \in \mathcal{C}^\infty(\mathbb{R}_{>0})$.

Vice versa, assume that $[b_\varepsilon] = [c_\varepsilon]$; we want to prove that

$$\left[w * \left(b_\varepsilon^{-1} \odot \rho - c_\varepsilon^{-1} \odot \rho\right)\right] = 0 \quad \forall w \in \mathcal{E}'(\Omega).$$

It suffices to prove that $\lim_{\varepsilon \in \mathbb{I}} \left(b_\varepsilon^{-1} \odot \rho - c_\varepsilon^{-1} \odot \rho\right) = 0$ in $\mathcal{D}'(\Omega)$. For each $\varphi \in \mathcal{D}(\Omega)$ we have

$$\int \left(b_\varepsilon^{-1} \odot \rho - c_\varepsilon^{-1} \odot \rho\right) \varphi = \int \rho(t) \cdot \left[\varphi\left(\frac{t}{b_\varepsilon}\right) - \varphi\left(\frac{t}{c_\varepsilon}\right)\right]. \tag{4.6}$$

The composition of the generalized functions $\varphi \in \mathcal{C}^\infty(\mathbb{R}^n) \subseteq \mathcal{G}(\mathcal{B}, \mathbb{R}^n)$ and

$$\left[x \mapsto \frac{x}{b_\varepsilon}\right] = \left[x \mapsto \frac{x}{c_\varepsilon}\right] \in \mathcal{G}(\mathcal{B}, \mathbb{R}^n)$$

is well defined since the latter is compactly supported. Therefore, for $K := \text{supp}(\varphi)$, $\lim_{\varepsilon \in \mathbb{I}} \sup_{t \in K} \left|\varphi\left(\frac{t}{b_\varepsilon}\right) - \varphi\left(\frac{t}{c_\varepsilon}\right)\right| = 0$. From this and (4.6) the conclusion follows. □

The assumption $\rho(0) \neq 0$ clearly holds if we define ρ as the inverse Fourier transform of a positive function identically equal to 1 in a neighborhood of 0.

For example, $i_{(\varepsilon^{-k})}$ and $i_{(\varepsilon^{-h})}$ permit to deal with different speeds at the origin of different models of the Heaviside function H. Finally, as we already said at the beginning of the present work, it could be interesting to apply these results about different embeddings also to the full algebra $\mathcal{G}^e(\mathcal{B}^e, \Omega)$, e.g. in case we need particular properties like $H(0) = 0$. This is only a first step in the study of the infinitesimal (and infinite) differences between two embeddings i_b and i_c. In our opinion, this study could be very useful in nonlinear modeling.

4.3. **Necessity of a principal AG to embed distributions with a mollifier.** The assumption that $\mathcal{B}$ is a principal AG is quite natural if one looks at (4.3). In this section we want to prove that this is indeed a necessary condition if we want to have a pair i_0, σ of embeddings (where i_0 is defined like in Thm. 35) which coincide on a suitable set. More precisely, to state the following result, we set

Definition 43. Let $\mathcal{B}$ be an AG, then $\mathcal{E}'_M(\mathcal{B}, \Omega) := \mathcal{E}_M(\mathcal{B}, \Omega) \cap \mathcal{D}(\Omega)^I$ denotes the set of moderate nets of compactly supported functions.

We also recall that if $(z_k)_k$ is a sequence of $A_{\leqslant a}$, then we say $(z_k)_k \to \varnothing$ in $A_{\leqslant a}$ if

$$\forall a_0 \in A_{\leqslant a} \, \exists K \in \mathbb{N} \, \forall k \in \mathbb{N}_{\geqslant K} : \ z_k < a_0.$$

The existence of such a sequence is always verified in all our examples of set of indices (see [12]).

Theorem 44. *Let $\mathcal{B}$, $\mathcal{Z}$ be AG on the set of indices $\mathbb{I}$. Assume that for each $a \in A \in \mathcal{I}$ there exists a sequence $(z_k)_k \to \varnothing$ in $A_{\leqslant a}$. Let b, ρ, ρ_ε as in Def. 33. Then the following are equivalent:*

(i) $\forall (f_\varepsilon) \in \mathcal{E}'_M(\mathrm{AG}(b),\mathbb{R})\, \forall x \in \mathbb{R}: \ (f_\varepsilon * \rho_\varepsilon)(x) = f_\varepsilon(x) + \mathcal{N}(\mathcal{Z},\mathbb{R})$;
(ii) b *is a generator of* $\mathcal{Z}$.

Proof. We prove (i) $\Rightarrow$ (ii) only for the case $n = 1$, even if slightly more general notations can be used to repeat this proof for a generic dimension. As in (4.3), we can use in (i) a Taylor formula of order $m \in \mathbb{N}$, with Peano remainder, at $x \in \Omega$, so that for each $f_\varepsilon \in \mathcal{D}(\Omega)$ we have a (unique) remainder $R_\varepsilon = R(m, f_\varepsilon, x) \in \mathcal{D}(\Omega)$ such that for each $\varepsilon \in I$

$$|(f_\varepsilon * \rho_\varepsilon)(x) - f_\varepsilon(x)| = b_\varepsilon^{-m} \cdot \int_{\mathbb{R}} R_\varepsilon\left(x - \frac{t}{b_\varepsilon}\right) \cdot \rho(t)\,\mathrm{d}t$$
$$R_\varepsilon(y) = o\left(|y-x|^m\right) \text{ as } y \to x. \tag{4.7}$$

We set $c_\varepsilon(m, f_\varepsilon, x) := \int_{\mathbb{R}} R_\varepsilon\left(x - \frac{t}{b_\varepsilon}\right) \cdot \rho(t)\,\mathrm{d}t$. Without lack of generality, we can assume that $\rho(0) > 0$; analogously, we can proceed if $\rho(x) \neq 0$ at another point $x \in \mathbb{R}$.

Now, for each $\varepsilon \in I$ and $m \in \mathbb{N}_{>0}$, we want to define a function $f_{\varepsilon m}$ to use in (4.7) such that:

- $f_{\varepsilon m}$ is equal to its Peano remainder of order m at $x = 0$. This permits to directly have $R_\varepsilon = f_{\varepsilon m}$ in (4.7) and in the definition of $c_\varepsilon(m, f_{\varepsilon m}, 0)$.
- $c_\varepsilon(m, f_{\varepsilon m}, 0) \geqslant L_m > 0$, where L_m doesn't depend on ε and is infinitesimal for $m \to +\infty$.

$f_{\varepsilon m}$ can be defined in infinite ways; in its definition we will always respect the following criteria:

(i) We firstly fix $p, q \in \mathbb{R}_{>0}$ such that

$$\rho(t) > 0 \quad \forall t \in (-p, p)$$
$$q < \min(p, 1).$$

(ii) $f_{\varepsilon m} \in \mathcal{E}_M(\mathrm{AG}(b),\mathbb{R})$ and $\mathrm{supp}(f_{\varepsilon m}) \subseteq \left[-\frac{p}{b_\varepsilon}, \frac{p}{b_\varepsilon}\right]$.
(iii) $f_{\varepsilon m}(s) \geqslant 0$ for each s.
(iv) $f_{\varepsilon m}(s) = s^{2m} \cdot b_\varepsilon^{2m}$ for each $s \in \left(-\frac{q}{b_\varepsilon}, \frac{q}{b_\varepsilon}\right)$.

For example, we can take $\psi \in \mathcal{D}(\mathbb{R})$ such that $\mathrm{supp}(\psi) \subseteq [-p, p]$, $\psi \geqslant 0$, $\psi(s) = 1$ for $s \in [-q, q]$ and set $f_{\varepsilon m}(s) = s^{2m} \cdot b_\varepsilon^{2m} \cdot \psi(b_\varepsilon \cdot s)$; let us note that $\lim_{\varepsilon \in \mathbb{I}} [f_{\varepsilon m}] = s^{2m} \cdot b^{2m-n} \cdot \delta(s)$ in $\mathcal{G}(\mathrm{AG}(b),\mathbb{R})$ in the sharp topology, so that the limit of this net is a nonlinear generalized Colombeau function.
We have $f_{\varepsilon m}(s) = o(s^{m+1})$ as $s \to 0$ by (iv), so $f_{\varepsilon m}$ equals its Taylor remainder of order $m > 0$. Moreover $t \in (-p, p)$ iff $-\frac{t}{b_\varepsilon} \in \left(-\frac{p}{b_\varepsilon}, \frac{p}{b_\varepsilon}\right)$, so that

$$c_\varepsilon(m, f_{\varepsilon m}, 0) = \int_{-p}^{p} f_{\varepsilon m}\left(-\frac{t}{b_\varepsilon}\right) \cdot \rho(t)\,\mathrm{d}t \geqslant \int_{-q}^{q} t^{2m} \cdot \rho(t)\,\mathrm{d}t =: L_m > 0, \tag{4.8}$$

where we have used (iv), (iii) and (ii). Since $q < 1$, $t^{2m} \cdot \rho(t) < \rho(t)$ for every $t \in [-q, q]$. So, by dominated convergence, $\lim_{m\to+\infty} L_m = 0$. Now, (4.7) yields

$$|(f_{\varepsilon m} * \rho_\varepsilon)(0) - f_{\varepsilon m}(0)| = b_\varepsilon^{-m} \cdot c_\varepsilon(m, f_{\varepsilon m}, 0) \quad \forall \varepsilon, m,$$

so that, considering a generic $z \in \mathcal{Z}_{>0}$, assumption (i) gives

$$\forall m \in \mathbb{N}_{>0}\, \exists A_m \in \mathcal{I}\, \forall a \in A_m: \ b_\varepsilon^{-m} \cdot c_\varepsilon(m, f_{\varepsilon m}, 0) = O_{a, A_m}(z_\varepsilon^{-1}). \tag{4.9}$$

We proceed by contradiction assuming that

$$\forall m \in \mathbb{N} : \ b_\varepsilon^{-m} \neq O(z_\varepsilon^{-1}).$$

Taking a generic $m \in \mathbb{N}_{>0}$, this means

$$\forall A \in \mathcal{I}\, \exists a \in A : \ b_\varepsilon^{-m} \neq O_{a,A}(z_\varepsilon^{-1}).$$

We apply this with $A = A_m$ obtaining

$$\exists a_m \in A_m : \ b_\varepsilon^{-m} \neq O_{a_m,A_m}(z_\varepsilon^{-1}).$$

By Thm. 15 of [12], we obtain that for each $H \in \mathbb{R}_{>0}$ there exists a sequence $(\varepsilon_k)_{k\in\mathbb{N}} : \mathbb{N} \longrightarrow (A_m)_{\leqslant a_m}$ (depending on m and H) such that:

$$(\varepsilon_k)_k \to \varnothing \text{ in } (A_m)_{\leqslant a_m}$$
$$b_{\varepsilon_k}^{-m} > H \cdot z_{\varepsilon_k}^{-1} \quad \forall k \in \mathbb{N}. \tag{4.10}$$

We set $H := L_m^{-2} > 0$ in (4.10), obtaining a sequence $(\varepsilon_k)_k \to \varnothing$ in $(A_m)_{\leqslant a_m}$ (depending only on m) such that

$$b_{\varepsilon_k}^{-m} z_{\varepsilon_k} > L_m^{-2} \quad \forall k, m \in \mathbb{N}. \tag{4.11}$$

But from (4.9) with $a = a_m$ and (4.8), we get

$$\forall m \in \mathbb{N}_{>0}\, \exists T \in \mathbb{R}_{>0}\, \forall^{\mathbb{I}} \varepsilon \in (A_m)_{\leqslant a_m} : \ b_\varepsilon^{-m} \cdot z_\varepsilon \leqslant T \cdot c_\varepsilon(m, f_{\varepsilon m}, 0)^{-1} \leqslant T \cdot L_m^{-1}.$$

Applying this for $\varepsilon = \varepsilon_k$, with k sufficiently big, we get $b_{\varepsilon_k}^{-m} z_{\varepsilon_k} \leqslant T \cdot L_m^{-1}$ which, together with (4.11), yields $L_m^{-2} < T \cdot L_m^{-1}$ for each m. This is impossible since $L_m \to 0$ for $m \to +\infty$.

To prove (ii) $\Rightarrow$ (i), we can proceed as in (4.3) considering that $\partial^\alpha f_\varepsilon$ is bounded, on a fixed compact sets $K \Subset \mathbb{R}$, by a suitable power $b_\varepsilon^{-N_\alpha}$. Therefore, for $x \in \mathbb{R}$ fixed and ε sufficiently small, $x - \frac{t}{b_\varepsilon} \in \overline{B_1(x)} =: K$, and for each $m \in \mathbb{N}$, we can write

$$|(f_\varepsilon * \rho_\varepsilon)(x) - f_\varepsilon(x)| \leqslant b_\varepsilon^{-m} \cdot \sum_{|\alpha|=m} \sup_{x\in K} |\partial^\alpha f_\varepsilon(x)| \int \frac{|t|^\alpha}{\alpha!} |\rho(t)| \, \mathrm{d}t \leqslant$$
$$\leqslant b_\varepsilon^{-m} \sum_{|\alpha|=m} b_\varepsilon^{-N_\alpha} \int \frac{|t|^\alpha}{\alpha!} |\rho(t)| \, \mathrm{d}t \leqslant b_\varepsilon^{-m-N_m} \cdot C_m,$$

where $N_m := \max_{|\alpha|=m} N_\alpha$ and $C_m := \sum_{|\alpha|=m} \int \frac{|t|^\alpha}{\alpha!} |\rho(t)| \, \mathrm{d}t$. If $z \in \mathcal{Z}_{>0}$, by (ii) we get the existence of $k \in \mathbb{N}$ such that $b_\varepsilon^{-k} = O(z_\varepsilon^{-1})$. It suffices to take m sufficiently big so that $m + N_m > k$ so that for this fixed m and for ε small, $b_\varepsilon^{-m-N_m} \cdot C_m \leqslant b_\varepsilon^{-k}$. □

Let us note that condition (i) is stronger than the equality $i_0|_{\mathcal{D}(\Omega)} = \sigma$, which can be applied only to a single function $f \in \mathcal{D}(\Omega)$ instead of a whole net. Indeed, in the previous proof, we used this condition with the net $(f_{\varepsilon m})_\varepsilon$, which effectively depend on ε.
We can say that if b is a generator of $\mathcal{Z}$, then the equality $i_0(f) = \sigma(f)$ for $f \in \mathcal{D}(\Omega)$ can be extended to any net of compactly supported function which are AG(b)-moderated. Therefore, the only possibility to have an embedding using a mollifier but without using a generator is to avoid a natural property like (i), which is undesirable. We can summarize our results concerning the embedding of distributions by saying:

(i) The embedding of distributions by using a mollifier forces us to take only one principal AG: $\mathcal{B} = \mathcal{Z} = \mathrm{AG}(b)$.
(ii) If we are interested in using two different AG, $\mathcal{B} \neq \mathcal{Z}$, or a non principal AG, we have to consider a particular set of indices, e.g. the full one $\mathbb{I}^{\mathrm{e}}$, where an intrinsic embedding is possible. Of course, this is incompatible with particular properties like $H(0) = 0$.

5. Solving linear homogeneous ODE with generalized coefficients

Studying Colombeau theory, one senses a sort of delusion by seeing that these algebras, invented to find solutions of differential equations which are not solvable in $\mathcal{D}'$, are not able to find solutions of ODE of the simplest type. One way to bypass this problem is to assume ad hoc growing conditions of logarithmic type, i.e. to adapt the differential problem to the constraints of the theory (see e.g. [15, 16] for linear ODE). Another solution is to guess that this deficiency is due to the chosen polynomial growing condition, and that a generalization could be possible. This is one of the basic motivations to generalize Colombeau theory by defining notions like asymptotic scales, $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ algebras, exponent weights or AG. Here, the point of view is more similar to that used in algebra: given an equation we have to find the best space where it has a solution, i.e. we adapt the theory to the equation.

We start this section by defining the module of Colombeau generalized numbers where we will take the coefficients of our linear ODE.

Definition 45. Let $\mathcal{B}$, $\mathcal{Z}$ be AG on a set of indeces $\mathbb{I}$ such that $\mathbb{R}_M(\mathcal{B}) \subseteq \mathbb{R}_M(\mathcal{Z})$, and let $d \in \mathbb{N}_{>0}$, then
(i) $\Omega_M(\mathcal{B}) := \left\{(x_\varepsilon) \in \Omega^I \mid \exists b \in \mathcal{B} : \ x_\varepsilon = O(b_\varepsilon)\right\}$;
(ii) $(x_\varepsilon) \sim_{\mathcal{Z}} (y_\varepsilon)$ iff $\forall z \in \mathcal{Z}_{>0} : \ x_\varepsilon - y_\varepsilon = O(z_\varepsilon^{-1})$, where (x_ε), $(y_\varepsilon) \in \Omega_M(\mathcal{B})$;
(iii) $\widetilde{\Omega}(\mathcal{B}, \mathcal{Z}) := \Omega_M(\mathcal{B}) / \sim_{\mathcal{Z}}$;
(iv) $\widetilde{\mathbb{R}}^d(\mathcal{B}, \mathcal{Z}) := \mathbb{R}^d_M(\mathcal{B}) / \sim_{\mathcal{Z}}$.

Like in Thm. 30, we have that $\widetilde{\mathbb{R}}(\mathcal{B}, \mathcal{Z})$ is a ring. Moreover, $\widetilde{\mathbb{R}}(\mathcal{B}, \mathcal{Z})$ can be identified with a subring of $\widetilde{\mathbb{R}}(\mathcal{B}', \mathcal{Z}')$ if

$$\mathbb{R}_M(\mathcal{B}) \subseteq \mathbb{R}_M(\mathcal{B}') \subseteq \mathbb{R}_M(\mathcal{Z}') \subseteq \mathbb{R}_M(\mathcal{Z}). \tag{5.1}$$

A sufficient condition for these inclusions is $\mathcal{B} \subseteq \mathcal{B}' \subseteq \mathcal{Z}' \subseteq \mathcal{Z}$. The proof of Prop. 1.2.35 of [6] can be directly generalized to every set of indices, so that if Ω is connected and $u \in \mathcal{G}(\mathcal{B}, \mathcal{Z}, \Omega)$, then $Du = 0$ if and only if $u \in \widetilde{\mathbb{R}}(\mathcal{B}, \mathcal{Z})$.

As we mentioned above, if a differential equation $\dot{x}(t) = F(t, x(t))$ is well-defined in $\mathcal{G}(\mathcal{B}, \mathcal{Z}, \Omega)^n$, i.e. if $\Omega \subseteq \mathbb{R}^{1+n}$ and $F \in \mathcal{G}(\mathcal{B}, \mathcal{Z}, \Omega)^n$, then we will have to deal with moderate solutions bounded by terms of the form $e^b := \left(e^{b_\varepsilon}\right)$, for some $b \in \mathcal{B}$. It is therefore natural to set the following

Definition 46. Let $\mathcal{B}$ be an AG, then

$$e^{\mathcal{B}} := \left\{e^{H \cdot b} \mid H \in \mathbb{R}_{>0}, b \in \mathcal{B}\right\}$$

is called the *exponential of* $\mathcal{B}$.

The problem with $e^{\mathcal{B}}$ is that it is never a principal AG since it always contains (bounds of) e^{b^m}, whereas a single generator gives terms of the form e^{mb}.

Lemma. *Let $\mathcal{B}$ be an AG, then:*
(i) $e^{\mathcal{B}}$ *is a positive AG;*

(ii) $\mathbb{R}_M(\mathcal{B}) \subseteq \mathbb{R}_M\left(e^{\mathcal{B}}\right)$;

(iii) $\mathbb{R}_M(e^{\mathcal{B}}) = \mathbb{R}_M(e^{\mathbb{R}_M(\mathcal{B})})$;

(iv) *if* $\mathcal{B} = \mathrm{AG}(b)$ *then* $\mathbb{R}_M\left(e^{\mathcal{B}}\right) = \mathbb{R}_M(\{e^{b^k} \mid k \in \mathbb{N}\}) = \bigcup_{k\in\mathbb{N}} \mathbb{R}_M(\mathrm{AG}(e^{b^k})) \subseteq \mathbb{R}_M\left(\mathrm{AG}\left(e^{e^b}\right)\right)$;

(v) $e^{\mathcal{B}}$ *is not a principal AG.*

Proof. We only prove (v) since the other properties follow almost directly from the definitions. Assume that $e^{\mathcal{B}} = \mathrm{AG}\left(e^{H\cdot b}\right)$, where $H \in \mathbb{R}_{>0}$ and $b \in \mathcal{B}$. Then $b^2 = O(c)$, for some $c \in \mathcal{B}_{>0}$. Therefore, for $\varepsilon \in I$ sufficiently small we have

$$e^{b_\varepsilon^2} \leqslant e^{K\cdot c_\varepsilon} \tag{5.2}$$

for some $K \in \mathbb{R}_{>0}$. But $e^{Kc} \in e^{\mathcal{B}} = \mathrm{AG}\left(e^{Hb}\right)$ so $e^{Kc} = \left(e^{Hb}\right)^m = e^{mHb}$ for some $m \in \mathbb{N}$. From this and (5.2) we get $b_\varepsilon^2 \leqslant mHb_\varepsilon$ for ε small. This implies that b is bounded, so e^{Hb} is also bounded and it cannot generate e^i, where $i \in \mathcal{B}_{>0}$ is infinite. □

We want to consider linear ODE whose coefficients are, in some sense, "bounded by $\mathcal{B}$", but whose solutions are in $\mathcal{G}(\mathcal{B}', \mathcal{Z}', \mathbb{R})$, where $\mathbb{R}_M(\mathcal{B}') \supseteq \mathbb{R}_M\left(e^{\mathcal{B}}\right)$. We have to clarify this point, also because it is desirable to have some kind of preservation of old solutions: if x is a solution already in $\mathcal{G}(\mathcal{B}, \mathcal{Z}, \mathbb{R})$, e.g. because the coefficients have a growth of logarithmic type, then x must also be (in some sense) the unique solution in the new space $\mathcal{G}(\mathcal{B}', \mathcal{Z}', \mathbb{R})$. To have a relation between $\mathcal{G}(\mathcal{B}, \mathcal{Z}, \mathbb{R})$ and $\mathcal{G}(\mathcal{B}', \mathcal{Z}', \mathbb{R})$, a condition like (5.1) is too strong because if e.g. $\mathcal{B} = \mathcal{Z}$ are of polynomial type, then (5.1) implies that $\mathcal{B}'$ and $\mathcal{Z}'$ cannot be of exponential type. On the other hand, it is clear how to set the following

Definition 47. Let $\mathcal{B}$, $\mathcal{B}'$, $\mathcal{Z}'$ be AG such that $\mathbb{R}_M(\mathcal{B}) \subseteq \mathbb{R}_M(\mathcal{B}') \subseteq \mathbb{R}_M(\mathcal{Z}')$ and let $u \in \mathcal{G}(\mathcal{B}', \mathcal{Z}', \Omega)$. We say that u *is bounded by* $\mathcal{B}$ if

$$\exists (u_\varepsilon) \in \mathcal{E}_M(\mathcal{B}, \Omega) : \ u = [u_\varepsilon].$$

We also set

$$\mathcal{G}_{\mathcal{B}}(\mathcal{B}', \mathcal{Z}', \Omega) := \left\{u \in \mathcal{G}(\mathcal{B}', \mathcal{Z}, \Omega) \mid u \text{ is bounded by } \mathcal{B}\right\}.$$

Since element of $\widetilde{\mathbb{R}}(\mathcal{B}', \mathcal{Z}')$ can be identified with constant functions of $\mathcal{G}(\mathcal{B}', \mathcal{Z}', \mathbb{R})$, we have an analogous notion for elements of the ring $\widetilde{\mathbb{R}}(\mathcal{B}', \mathcal{Z}')$.

Like in Lem. 28, we can prove that u is bounded by $\mathcal{B}$ if and only if whenever we consider a representative $u = [u_\varepsilon]$, we have that $(u_\varepsilon) \in \mathcal{E}_M(\mathcal{B}, \Omega)$. In the statement of the next result, we use the point value of a Colombeau generalized function. We recall (see e.g. [12] and references therein) that this point value characterizes Colombeau generalized functions:

Definition 48. Let $\mathcal{B}$, $\mathcal{Z}$ be AG such that $\mathbb{R}_M(\mathcal{B}) \subseteq \mathbb{R}_M(\mathcal{Z})$, then

(i) $[x_\varepsilon] \in \widetilde{\Omega}_c(\mathcal{B}, \mathcal{Z})$ iff $[x_\varepsilon] \in \widetilde{\Omega}(\mathcal{B}, \mathcal{Z})$ and $\exists K \Subset \Omega \, \forall^{\mathbb{I}} \varepsilon : \ x_\varepsilon \in K$.

(ii) If $u = [u_\varepsilon] \in \mathcal{G}(\mathcal{B}, \mathcal{Z}, \Omega)$ and $x \in \widetilde{\Omega}_c(\mathcal{B}, \mathcal{Z})$, then $u(x) := [u_\varepsilon(x_\varepsilon)]$.

Lemma 49. *Let $\mathcal{B}$, $\mathcal{Z}$, $\mathcal{B}'$, $\mathcal{Z}'$ be AG such that*

$$\begin{array}{ccc} \mathbb{R}_M(\mathcal{B}) & \hookrightarrow & \mathbb{R}_M(\mathcal{Z}) \\ \downarrow & & \downarrow \\ \mathbb{R}_M(\mathcal{B}') & \hookrightarrow & \mathbb{R}_M(\mathcal{Z}') \end{array} \tag{5.3}$$

Then the following properties hold:

(i) $\mathcal{G}_{\mathcal{B}}(\mathcal{B}',\mathcal{Z}',\Omega)$ *is a differential subalgebra of* $\mathcal{G}(\mathcal{B}',\mathcal{Z}',\Omega)$.

(ii) *The map*

$$\bar{(-)} : [u_\varepsilon] \in \mathcal{G}_{\mathcal{B}}(\mathcal{B}',\mathcal{Z}',\Omega) \mapsto (u_\varepsilon) + \mathcal{N}(\mathcal{Z},\Omega) \in \mathcal{G}(\mathcal{B},\mathcal{Z},\Omega)$$

is a surjective morphism of differential algebras.

(iii) *Let* $J := (a,b) \subseteq \mathbb{R}$, $x \in \mathcal{G}_{\mathcal{B}}(\mathcal{B}',\mathcal{Z}',(a,b))$, $F \in \mathcal{G}_{\mathcal{B}}(\mathcal{B}',\mathcal{Z}',(a,b)\times\Omega)$ *be such that*

$$\forall t \in \widetilde{J}_c(\mathcal{B}',\mathcal{Z}') : \ t \text{ is bounded by } \mathcal{B} \Rightarrow \dot{x}(t) = F(t, x(t)) \tag{5.4}$$

holds in $\widetilde{\mathbb{R}}(\mathcal{B}',\mathcal{Z}')$. *Then*

$$\dot{\bar{x}}(t) = \bar{F}(t, \bar{x}(t)) \quad \forall t \in \widetilde{J}_c(\mathcal{B},\mathcal{Z}) \tag{5.5}$$

holds in $\widetilde{\mathbb{R}}(\mathcal{B},\mathcal{Z})$.

Proof. Property (i) follows by Lem. 29 and by the closure of $\mathcal{E}_M(\mathcal{B},\Omega)$ with respect to derivatives.

Property (ii) follows by the ε-pointwise definitions of all the operations. The counter-image of $(u_\varepsilon) + \mathcal{N}(\mathcal{Z},\Omega) \in \mathcal{G}(\mathcal{B},\mathcal{Z},\Omega)$ is $[u_\varepsilon]$, which is bounded by $\mathcal{B}$ since $(u_\varepsilon) \in \mathcal{E}_M(\mathcal{B},\Omega)$.

Assumption (5.4) means that for each $t = [t_\varepsilon] \in \widetilde{J}_c(\mathcal{B}',\mathcal{Z}')$, if t is bounded by $\mathcal{B}$ then

$$[x_\varepsilon(t_\varepsilon)] \in \widetilde{\Omega}_c(\mathcal{B}',\mathcal{Z}') \quad \text{and} \quad [\dot{x}_\varepsilon(t_\varepsilon)] = [F_\varepsilon(t_\varepsilon, x_\varepsilon(t_\varepsilon))]. \tag{5.6}$$

For simplicity, we use the symbol $[\![u_\varepsilon]\!] := (u_\varepsilon) + \mathcal{N}(\mathcal{Z},\Omega)$ for the equivalence classes in $\mathcal{G}(\mathcal{B},\mathcal{Z},\Omega)$ (and hence also in $\widetilde{\mathbb{R}}(\mathcal{B},\mathcal{Z})$). If $t = [\![t_\varepsilon]\!] \in \widetilde{J}_c(\mathcal{B},\mathcal{Z})$, then $[t_\varepsilon] \in \widetilde{\mathbb{R}}(\mathcal{B}',\mathcal{Z}')$ is bounded by $\mathcal{B}$ and (5.6) yields $[\dot{x}_\varepsilon(t_\varepsilon)] = [F_\varepsilon(t_\varepsilon, x_\varepsilon(t_\varepsilon))]$. Both sides of this equality are bounded by $\mathcal{B}$, so that we can apply the morphism $\bar{(-)}$ obtaining $[\![\dot{x}_\varepsilon(t_\varepsilon)]\!] = [\![F_\varepsilon(t_\varepsilon, x_\varepsilon(t_\varepsilon))]\!]$. Moreover, $[\![x_\varepsilon(t_\varepsilon)]\!] \in \widetilde{\Omega}_c(\mathcal{B},\mathcal{Z})$ because $[x_\varepsilon]$ and $[t_\varepsilon]$ are both bounded by $\mathcal{B}$. We therefore have

$$\begin{aligned}
\dot{\bar{x}}(t) &= \frac{d}{dt}\left([\![x_\varepsilon]\!]\right)\left([\![t_\varepsilon]\!]\right) = [\![\dot{x}_\varepsilon]\!]([\![t_\varepsilon]\!]) = \\
&= [\![\dot{x}_\varepsilon(t_\varepsilon)]\!] = [\![F_\varepsilon(t_\varepsilon, x_\varepsilon(t_\varepsilon))]\!] = [\![F_\varepsilon]\!]\left([\![t_\varepsilon, x_\varepsilon(t_\varepsilon)]\!]\right) = \\
&= [\![F_\varepsilon]\!]\left([\![t_\varepsilon]\!], [\![x_\varepsilon]\!]([\![t_\varepsilon]\!])\right) = \\
&= \bar{F}\left(t, \bar{x}(t)\right)
\end{aligned}$$

□

Condition (iii) states that any ODE framed in $\mathcal{G}(\mathcal{B}',\mathcal{Z}',\Omega)$, but restricted to elements which are bounded by $\mathcal{B}$, corresponds, via the morphisms $\bar{(-)}$, to an ODE framed in $\mathcal{G}(\mathcal{B},\mathcal{Z},\Omega)$. This is our way to formalize that any bounded solution of an ODE of bounded type in the "bigger" algebra $\mathcal{G}(\mathcal{B}',\mathcal{Z}',\Omega)$ is also a solution in the "smaller" algebra $\mathcal{G}(\mathcal{B},\mathcal{Z},\Omega)$. The use of this order relation between algebras is formally introduced in the following

Definition 50. Let $\mathcal{B}$, $\mathcal{Z}$, $\mathcal{B}'$, $\mathcal{Z}'$ be AG, then we write $\mathcal{G}(\mathcal{B},\mathcal{Z},\Omega)\preceq\mathcal{G}(\mathcal{B}',\mathcal{Z}',\Omega)$, and we say that $\mathcal{G}(\mathcal{B},\mathcal{Z},\Omega)$ *is smaller than* $\mathcal{G}(\mathcal{B}',\mathcal{Z}',\Omega)$ if (5.3) holds.

The relation $\preceq$ is an order and, if $\mathcal{Z}=\mathcal{Z}'$, then $\mathcal{G}(\mathcal{B},\mathcal{Z},\Omega)\preceq\mathcal{G}(\mathcal{B}',\mathcal{Z},\Omega)$ if and only if $\mathcal{G}(\mathcal{B},\mathcal{Z},\Omega)\subseteq\mathcal{G}(\mathcal{B}',\mathcal{Z},\Omega)$ if and only if $\mathbb{R}_M(\mathcal{B})\subseteq\mathbb{R}_M(\mathcal{B}')$. In this case, the morphism $\bar{(-)}$ of Lem. 49 is also injective, and we have $\mathcal{G}_{\mathcal{B}}(\mathcal{B}',\mathcal{Z},\Omega)\simeq\mathcal{G}(\mathcal{B},\mathcal{Z},\Omega)$. We will use later the order relation $\preceq$.

In the following result, the main assumption is the inclusion $\mathbb{R}_M\left(e^{\mathcal{B}}\right)\subseteq\mathbb{R}_M(\mathcal{B}')$; in it we can therefore set $\mathcal{B}'=e^{\mathcal{B}}$ or $\mathcal{B}'=\mathrm{AG}\left(e^{e^b}\right)$ if we are interested to a principal AG.

Theorem 51. *Let $\mathcal{B}$, $\mathcal{B}'$, $\mathcal{Z}'$ be AG such that*

$$\mathbb{R}_M\left(e^{\mathcal{B}}\right)\subseteq\mathbb{R}_M(\mathcal{B}')\subseteq\mathbb{R}_M(\mathcal{Z}'). \tag{5.7}$$

Let $t_0\in\mathbb{R}$, $c\in\widetilde{\mathbb{R}}(\mathcal{B}',\mathcal{Z}')^d$ and $A\in\mathcal{M}_d(\widetilde{\mathbb{R}}(\mathcal{B}',\mathcal{Z}'))$ be a $d\times d$ matrix with entries in the ring $\widetilde{\mathbb{R}}(\mathcal{B}',\mathcal{Z}')$. Assume that both c and A are bounded by $\mathcal{B}$. Then the problem

$$\begin{cases} x'(t)+A\cdot x(t)=0\\ x(t_0)=c.\end{cases} \tag{5.8}$$

has a unique solution in $\mathcal{G}\left(\mathcal{B}',\mathcal{Z}',\mathbb{R}\right)^d$.

We split the proof of Theorem 51 in two parts: existence and uniqueness.

To prove exitence we will use the following

Lemma 52. *Let $d\in\mathbb{N}_{>0}$, let $A=(a_{ij})_{i,j\leqslant d}\in\mathcal{M}_d(\mathbb{R})$ and let $M=\max_{i,j}|a_{ij}|$. Then for every entry $x_{ij}(t)$ of the matrix e^{At}, we have*

$$|x_{ij}(t)|\leqslant M\cdot e^{dM|t|}.$$

Proof. For every $k\in\mathbb{N}$ let $A^k=:(a_{ijk})_{i,j\leqslant d}$ and let $M_k:=\max_{i,j}|a_{ijk}|$. We claim that, for every $k\geqslant 1$, $M_k\leqslant d^{k-1}M^k$. Let us prove this inequality by induction. If $k=1$ the conclusion is trivial. Let us assume that the claim is true for k. Let us suppose that $M_{k+1}=|a_{ij,k+1}|$. Then

$$M_{k+1}=\left|\sum_{r,s=1}^{d}a_{irk}\cdot a_{sj1}\right|\leqslant\sum_{r,s=1}^{d}|a_{irk}\cdot a_{sj1}|\leqslant\sum_{r,s=1}^{d}M_k\cdot M\leqslant d^kM^{k+1}.$$

The claim is proved. Therefore, since by definition $e^{At}=\sum_{k=0}^{\infty}\frac{A^kt^k}{k!}$, we have

$$|x_{ij}(t)|=\left|\sum_{k=0}^{\infty}\frac{a_{ijk}t^k}{k!}\right|\leqslant\sum_{k=0}^{\infty}\frac{|a_{ijk}||t|^k}{k!}\leqslant\sum_{k=0}^{\infty}\frac{d^kM^{k+1}|t|^k}{k!}=M\cdot e^{dM|t|},$$

hence the thesis is proved. □

Lemma 53 (Existence)**.** *Under the assumptions of Thm. 51, the problem (5.8) has a solution in $\mathcal{G}\left(\mathcal{B}',\mathcal{Z}',\mathbb{R}\right)^d$.*

Proof. Let $A=[A_\varepsilon]$ and $c=[c_\varepsilon]$. For every $\varepsilon\in I$ let $x_\varepsilon\in\mathcal{C}^\infty(\mathbb{R},\mathbb{R}^d)$ be the unique solution of the problem

$$\begin{cases} x'(t)+A_\varepsilon\cdot x(t)=0\\ x(t_0)=c_\varepsilon.\end{cases} \tag{5.9}$$

We claim that $[x_\varepsilon]$ is a solution of (5.8) in $\mathcal{G}\left(\mathcal{B}',\mathcal{Z}',\mathbb{R}\right)^d$. Since $\mathbb{R}_M\left(e^{\mathcal{B}}\right)\subseteq\mathbb{R}_M(\mathcal{B}')$, in order to prove that $[x_\varepsilon]$ is a solution in $\mathcal{G}\left(\mathcal{B}',\mathcal{Z}',\mathbb{R}\right)^d$ it is sufficient to show that $(x_\varepsilon)\in\mathcal{E}_M\left(e^{\mathcal{B}},\mathbb{R}\right)^d$. In fact, since also x'_ε verify the same equation (5.9), with initial $\mathcal{B}$-bounded value $x'_\varepsilon(t_0)=-A_\varepsilon\cdot c_\varepsilon$, we can proceed by proving moderateness of (x_ε) only. Without loss of generality, we suppose $t_0=0$. For every $\varepsilon\in I$ and $t\in\mathbb{R}$, we have

$$x_\varepsilon(t)=e^{-A_\varepsilon t}c_\varepsilon.$$

For every ε, set $A_\varepsilon=:(a_{ij\varepsilon})_{i,j\leqslant d}$ for the entries of the matrix $A_\varepsilon\in\mathcal{M}_d(\mathbb{R})$, $M_\varepsilon:=\max_{i,j}a_{ij\varepsilon}$, $x_\varepsilon(t)=:(x_{i\varepsilon}(t))_{i\leqslant d}$, and $c_\varepsilon=:(c_{i\varepsilon})_{i\leqslant d}$ for the components of the vectors $x_\varepsilon(t)$, $c_\varepsilon\in\mathbb{R}^d$. By lemma 52 we deduce that

$$|x_{i\varepsilon}(t)|\leqslant\sum_{j=1}^{d}M_\varepsilon\cdot e^{dM_\varepsilon|t|}c_{j\varepsilon}.$$

Since both $[M_\varepsilon]$ and $[c_\varepsilon]$ are bounded by $\mathcal{B}$, we have that $(\sum_{j=1}^{d}M_\varepsilon\cdot e^{dM_\varepsilon|t|}c_{j\varepsilon})\in\mathcal{E}_M\left(e^{\mathcal{B}},\mathbb{R}\right)$, therefore $(x_\varepsilon)\in\mathcal{E}_M\left(e^{\mathcal{B}},\mathbb{R}\right)^d$. □

Lemma 54 (Uniqueness)**.** *Under the assumptions of Thm. 51, if* $x\in\mathcal{G}\left(\mathcal{B}',\mathcal{Z}',\mathbb{R}\right)^d$ *is such that*

$$\begin{cases} x'(t)+A\cdot x(t)=0\\ x(t_0)=0\end{cases}\tag{5.10}$$

in $\mathcal{G}\left(\mathcal{B}',\mathcal{Z}',\mathbb{R}\right)$*, then* $x=0$.

Proof. Without loss of generality we can suppose that $t_0=0$. The generalized function $x\in\mathcal{G}\left(\mathcal{B}',\mathcal{Z}',\mathbb{R}\right)^d$ is a solution of (5.10) so there exist $(n_\varepsilon),(v_\varepsilon)\in\mathcal{N}(\mathcal{Z}',\mathbb{R})$ such that, for every $\varepsilon\in I$,

$$\begin{cases} x'_\varepsilon(t)+A_\varepsilon\cdot x_\varepsilon(t)=n_\varepsilon\\ x_\varepsilon(0)=v_\varepsilon.\end{cases}\tag{5.11}$$

The unique solution of (5.11) is $x_\varepsilon(t)=e^{-tA_\varepsilon}v_\varepsilon+\int_0^t e^{(s-t)A_\varepsilon}\cdot n_\varepsilon\,\mathrm{d}s$. So

$$\begin{aligned}|x_\varepsilon(t)|&=\left|e^{-tA_\varepsilon}v_\varepsilon+\int_0^t e^{(s-t)A_\varepsilon}\cdot n_\varepsilon\,\mathrm{d}s\right|\leqslant\\ &\leqslant e^{|t||A_\varepsilon|}|v_\varepsilon|+|t|e^{|t||A_\varepsilon|}|n_\varepsilon|,\end{aligned}$$

where we used the integral mean value theorem. If $K\Subset\mathbb{R}$, then

$$\sup_{t\in K}|x_\varepsilon(t)|\leqslant e^{R|A_\varepsilon|}|v_\varepsilon|+Re^{R|A_\varepsilon|}|n_\varepsilon|,$$

where $R:=\sup_{k\in K}|k|$. We have $\left(e^{R|A_\varepsilon|}|v_\varepsilon|+Re^{R|A_\varepsilon|}|n_\varepsilon|\right)\in\mathcal{N}(\mathcal{Z}',\mathbb{R})$ since $(v_\varepsilon),(n_\varepsilon)\in\mathcal{N}(\mathcal{Z}',\mathbb{R})$ and $\left(e^{R|A_\varepsilon|}\right),\left(Re^{R|A_\varepsilon|}\right)\in\mathbb{R}_M(e^{\mathcal{B}})\subseteq\mathbb{R}_M(\mathcal{B}')$ because $A=[A_\varepsilon]$ is bounded by $\mathcal{B}$. □

The results of Lemma 53 and by Lemma 54 provide a proof of Theorem 51.

Example 55. Let $\mathbb{I}=(I,\leqslant,\mathcal{I})$ be a set of indices, and $\rho:I\longrightarrow(0,1]$ be a map such that $\lim_{\mathbb{I}}\rho=0$. Let $\mathcal{B}^{\mathrm{s}}$ be the usual polynomial AG of the special Colombeau algebra, so that $\mathcal{B}^{\mathrm{s}}\circ\rho$ is an AG on $\mathbb{I}$ by Thm. 19. As we showed in Ex. 32, this framework generalizes the special, the full and the NSA based cases. The following problem:

$$\begin{cases} x'(t) + \left[\frac{1}{\varepsilon}\right] \cdot x(t) = 0 \\ x(0) = 1 \end{cases}$$

has not solution in $\mathcal{G}(\mathcal{B}^{\mathrm{s}} \circ \rho, \Omega)$, but it has a unique solution in $\mathcal{G}\left(e^{\mathcal{B}^{\mathrm{s}} \circ \rho}, \mathbb{R}\right) = \mathcal{G}\left(e^{\mathcal{B}^{\mathrm{s}} \circ \rho}, e^{\mathcal{B}^{\mathrm{s}} \circ \rho}, \mathbb{R}\right)$ and in $\mathcal{G}\left(\mathrm{AG}\left(e^{e^{\frac{1}{\varepsilon}}}\right) \circ \rho, \mathbb{R}\right) = \mathcal{G}\left(\mathrm{AG}\left(e^{e^{\frac{1}{\varepsilon}}}\right) \circ \rho, \mathrm{AG}\left(e^{e^{\frac{1}{\varepsilon}}}\right) \circ \rho, \mathbb{R}\right)$, namely

$$[x_\varepsilon(t)] = \left[e^{-\frac{1}{\varepsilon} t}\right],$$

where the equivalence class has to be meant differently in the two algebras. Let us note explicitly that we have applied Thm. 51 with $\mathcal{B} = \mathcal{B}^{\mathrm{s}} \circ \rho$ and $\mathcal{B}' = \mathcal{Z}' = e^{\mathcal{B}^{\mathrm{s}} \circ \rho}$ for the former algebra and $\mathcal{B}' = \mathcal{Z}' = \mathrm{AG}\left(e^{e^{\frac{1}{\varepsilon}}}\right)$ for the latter. Moreover, this problem has also a unique solution in the algebra $\mathcal{G}\left(\mathrm{AG}\left(e^{\frac{1}{\varepsilon}}\right) \circ \rho, \mathbb{R}\right)$. This shows one particular feature of our construction: if we want to have an algebra in which we can uniquely solve all the ODEs whose coefficients are bounded by a given AG $\mathcal{B}$ then (as we will show in Thm. 56) the minimal possible choice is $\mathcal{G}\left(e^{\mathcal{B}}, \mathbb{R}\right)$, whilst if we are interested only in a finite number of linear ODE with coefficients bounded by $\mathcal{B}$, then it is possible to find a solution to these ODE in the algebra $\mathcal{G}\left(\mathrm{AG}\left(e^{b}\right), \mathbb{R}\right)$, where $b \in \mathcal{B}_{>0}$ is any element such that $|c| = O(b)$ for every coefficient c that appears in the finite set of ODE.

We note that Theorem 51 can be reformulated in the following way:

Theorem 56. *Let $\mathcal{B}$ be an AG. Then $\mathcal{G}\left(e^{\mathcal{B}}, \mathbb{R}\right)$ is the smallest Colombeau algebra (with respect to the order relation $\preceq$ of Def. 50) in which every linear homogeneous ODE with coefficients in $\mathbb{R}_M(\mathcal{B})$ can be solved.*

Proof. By Theorem 51 we know that every linear homogeneous ODE with coefficients in $\mathbb{R}_M(\mathcal{B})$ can be solved in $\mathcal{G}\left(e^{\mathcal{B}}, e^{\mathcal{B}}, \mathbb{R}\right)$. Now let $\mathcal{B}'$ be an AG such that every linear homogeneous ODE with coefficients bounded by $\mathcal{B}$ can be solved in $\mathcal{G}\left(\mathcal{B}', \mathcal{B}', \mathbb{R}\right)$. In particular, for every $b \in \mathcal{B}$ we can solve the problem

$$\begin{cases} x'(t) + b \cdot x(t) = 0 \\ x(0) = 1. \end{cases}$$

As we showed in Lemma 53, the solution of this problem is $\left[e^{b_\varepsilon t}\right]$. This means that $(e^{b_\varepsilon t}) \in \mathcal{E}(\mathcal{B}', \mathbb{R})$ for every $b \in \mathcal{B}$. In particular this entails that $(e^{H \cdot b_\varepsilon}) \in \mathbb{R}_M(\mathcal{B}')$ for every $b \in \mathcal{B}$ and $H \in \mathbb{R}_{>0}$, so $e^{\mathcal{B}} \subseteq \mathbb{R}_M(\mathcal{B}')$ and $\mathbb{R}_M(e^{\mathcal{B}}) \subseteq \mathbb{R}_M(\mathcal{B}')$. Therefore, condition (5.3) holds for $\mathcal{Z} = e^{\mathcal{B}}$ and $\mathcal{Z}' = \mathcal{B}'$ so $\mathcal{G}\left(e^{\mathcal{B}}, e^{\mathcal{B}}, \mathbb{R}\right) \preceq \mathcal{G}\left(\mathcal{B}', \mathcal{B}', \mathbb{R}\right)$. □

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