Research Article

Alexander Lecke, Lorenzo Luperi Baglini* and Paolo Giordano

The classical theory of calculus of variations for generalized functions

DOI: 10.1515/anona-2017-0150
Received June 26, 2017; accepted July 29, 2017

Abstract: We present an extension of the classical theory of calculus of variations to generalized functions. The framework is the category of generalized smooth functions, which includes Schwartz distributions, while sharing many nonlinear properties with ordinary smooth functions. We prove full connections between extremals and Euler–Lagrange equations, classical necessary and sufficient conditions to have a minimizer, the necessary Legendre condition, Jacobi's theorem on conjugate points and Noether's theorem. We close with an application to low regularity Riemannian geometry.

Keywords: Calculus of variations, Schwartz distributions, generalized functions for nonlinear analysis, low regular Riemannian geometry

MSC 2010: 49-XX, 46F30, 53B20

1 Introduction and motivations

Singular problems in the calculus of variations have longly been studied both in mathematics and in relevant applications (see, e.g., [7, 18, 25, 43] and references therein). In this paper, we introduce an approach to variational problems involving singularities that allows the extension of the classical theory with very natural statements and proofs. We are interested in extremizing functionals which are either distributional themselves or whose set of extremals includes generalized functions. Clearly, distribution theory, being a linear theory, has certain difficulties when nonlinear problems are in play.

To overcome this type of problems, we are going to use the category of generalized smooth functions, see [12–15]. This theory seems to be a good candidate, since it is an extension of classical distribution theory, which allows the modeling of nonlinear singular problems, while at the same time sharing many nonlinear properties with ordinary smooth functions like the closure with respect to composition and several non-trivial classical theorems of calculus. One could describe generalized smooth functions as a methodological restoration of Cauchy–Dirac's original conception of generalized function, see [8, 24, 28]. In essence, the idea of Cauchy and Dirac (but also of Poisson, Kirchhoff, Helmholtz, Kelvin and Heaviside) was to view generalized functions as suitable types of smooth set-theoretical maps obtained from ordinary smooth maps depending on suitable infinitesimal or infinite parameters. For example, the density of a Cauchy–Lorentz distribution with an infinitesimal scale parameter was used by Cauchy to obtain classical properties, which nowadays are attributed to the Dirac delta function, cf. [24].

Alexander Lecke: Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria, e-mail: alexander.lecke@univie.ac.at
***Corresponding author: Lorenzo Luperi Baglini:** Wolfgang Pauli Institute, c/o Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria, e-mail: lorenzo.luperi.baglini@univie.ac.at. http://orcid.org/0000-0002-0559-0770
Paolo Giordano: Wolfgang Pauli Institute, c/o Fakultät für Mathematik, Universität Wien, Oskar-Morgenstern-Platz 1, 1090 Wien, Austria, e-mail: paolo.giordano@univie.ac.at. http://orcid.org/0000-0001-7653-1017

In the present work, the foundation of the calculus of variations is set for functionals defined by arbitrary generalized functions. This in particular applies to any Schwartz distribution and any Colombeau generalized function (see, e.g., [5, 6]), and hence justifies the title of the present paper.

For example, during the last years, the study of low regular Riemannian and Lorentzian geometry was intensified and made a huge amount of progress (cf. [26, 27, 30, 33, 40, 41]). It was shown that the exponential map is a bi-Lipschitz homeomorphism when metrics $g \in \mathcal{C}^{1,1}$ are considered [26, 34], or that Hawking's singularity theorem still holds when $g \in \mathcal{C}^{1,1}$, see [27]. However, the calculus of variations in the classical sense may cease to hold when metrics with $\mathcal{C}^{1,1}$ regularity, or below, are considered [19, 29]. This motivates the search for an alternative. In fact, if $p, q \in \mathbb{R}^d$ and $\Omega(p, q)$ denotes the set of all Lipschitz continuous curves connecting p and q, the natural question about what curves $\gamma \in \Omega(p, q)$ realize the minimal g-length leads to the corresponding geodesic equation, but the Jacobi equation is not rigorously defined. To be more precise, the Riemannian curvature tensor exists only as an L^∞_{loc} function on $\mathbb{R}^d$, and is evaluated along γ. However, the image $\mathrm{Im}(\gamma)$ of γ has Lebesgue-measure zero if $d > 1$. Thus, we cannot state the Jacobi equations properly.

In order to present a possible way out of the aforementioned problems, the singular metric g is embedded as a generalized smooth function. In this way, the embedding $\iota(g)$ has derivatives of all orders, valued in a suitable non-Archimedean ring ${}^\rho\widetilde{\mathbb{R}} \supseteq \mathbb{R}$ (i.e., a ring that contains infinitesimal and infinite numbers). Despite the total disconnectedness of the ground ring, the final class of smooth functions on ${}^\rho\widetilde{\mathbb{R}}$ behaves very closely to that of standard smooth functions; this is a typical step one can recognize in other topics such as analytic space theory [4, 37] and non-Archimedean analysis, see, e.g., [17] and references therein. We then apply our extended calculus of variations to the generalized Riemannian space $({}^\rho\widetilde{\mathbb{R}}^d, \iota(g))$, and sketch a way to translate the given problem into the language of generalized smooth functions, solve it there, and translate it back to the standard Riemannian space $(\mathbb{R}^d, g)$. Clearly, the process of embedding the singular metric g using $\iota(g)$ introduces infinitesimal differences. This is typical in a non-Archimedean setting, but the notion of *standard part* comes to help: if $x \in {}^\rho\widetilde{\mathbb{R}}$ is infinitely close to a standard real number s, i.e., $|x - s| \leq r$ for all $r \in \mathbb{R}_{>0}$, then the standard part of x is exactly s. We then show that (assuming that $(\mathbb{R}^d, g)$ is geodesically complete) the standard part of the minimal length in the sense of generalized smooth functions is the minimal length in the classical sense, and give a simple way to check if a given (classical) geodesic is a minimizer of the length functional or not. In this way, the framework of generalized smooth functions is presented as a method to solve standard problems rather than a proposal to switch into a new setting.

The structure of the present paper is as follows. We start with an introduction into the setting of generalized smooth functions and give basic notions concerning generalized smooth functions and their calculus that are needed for the calculus of variations (Section 2). The paper is self-contained in the sense that it contains all the statements required for the proofs of calculus of variations we are going to present. If proofs of preliminaries are omitted, we clearly give references to where they can be found. Therefore, to understand this paper, only a basic knowledge of distribution theory is needed.

In Section 3, we obtain some preliminary lemmas regarding the calculus of variations with generalized smooth functions. The first variation and the notion of critical point will be defined and studied in Section 4. We prove the fundamental lemma of calculus of variations and the full connection between critical points of a given functional and solutions of the corresponding Euler–Lagrange equation. In Section 5, we study the second variation and define the notion of local minimizer. We also extend to generalized functions classical necessary and sufficient conditions to have a minimizer, and we give a proof of the Legendre condition. In Section 6, we introduce the notion of Jacobi field and extend to generalized functions the definition of conjugate points, so as to prove the corresponding Jacobi theorem. In Section 7, we extend the classical Noether's theorem. We close with an application to $\mathcal{C}^{1,1}$ Riemannian geometry in Section 8.

Note that Konjik, Kunzinger and Oberguggenberger [25] already established the calculus of variations in the setting of Colombeau generalized functions, by using a comparable methodological approach. Indeed, generalized smooth functions are related to Colombeau generalized functions, and one could say that the former is a minimal extension of the latter so as to get more general domains for generalized functions, and hence the closure with respect to composition and a better behavior on unbounded sets. However, there are some conceptual advantages in our approach.

(i) Whereas generalized smooth functions are closed with respect to composition, Colombeau generalized functions are not. This forced the authors of [25] to consider only functionals defined using compactly supported Colombeau generalized functions, i.e., functions assuming only finite values, or tempered generalized functions.

(ii) The authors of [25] were forced to consider the so-called compactly supported points $c(\Omega)$ (i.e., finite points in $\Omega \subseteq \mathbb{R}^n$), where the setting of generalized smooth functions gives the possibility to consider more natural domains like the interval $[a, b] \subseteq {}^\rho\widetilde{\mathbb{R}}$. This leads us to extend in a natural way the statements of classical results of calculus of variations. Moreover, all our results still hold when we take as $a, b \in {}^\rho\widetilde{\mathbb{R}}$ two infinite numbers such that $a < b$, or as boundary points two unbounded points $p, q \in {}^\rho\widetilde{\mathbb{R}}^d$.

(iii) The theory of generalized smooth functions was developed to be very user friendly, in the sense that one can avoid cumbersome "ε-wise" proofs quite often, whereas the proofs in [25] frequently use this technique. Thus, one could say that some of the proofs based on generalized smooth functions are more "intrinsic" and close to the classical proofs in a standard smooth setting. This allows a smoother approach to this new framework.

(iv) The setting of generalized smooth functions depends on a fixed infinitesimal net $(\rho_\varepsilon)_{\varepsilon\in(0,1]} \downarrow 0$, whereas the Colombeau setting considers only $\rho_\varepsilon = \varepsilon$. This added degree of freedom allows to solve singular differential equations that are unsolvable in the classical Colombeau setting and to prove a more general Jacobi theorem on conjugate points.

(v) In [25], only the notion of global minimizer is defined, whereas we define the notion of local minimizer, as in [10], using a natural topology in space of generalized smooth curves.

(vi) We obtain more classical results like the Legendre condition, and the classical results about Jacobi fields and conjugate points.

(vii) The Colombeau generalized functions can be embedded into generalized smooth functions. Thus, our approach is a natural extension of [25].

2 Basic notions

The new ring of scalars

In this work, I denotes the interval $(0, 1] \subseteq \mathbb{R}$ and we will always use the variable ε for elements of I; we also denote ε-dependent nets $x \in \mathbb{R}^I$ simply by (x_ε). By $\mathbb{N}$, we denote the set of natural numbers, including zero.

We start by defining the new simple non-Archimedean ring of scalars that extends the real field $\mathbb{R}$. The entire theory is constructive to a high degree, e.g., no ultrafilter or non-standard method is used. For all the proofs in this section, see [13–15].

Definition 2.1. Let $\rho = (\rho_\varepsilon) \in \mathbb{R}^I$ be a net such that $\lim_{\varepsilon\to 0} \rho_\varepsilon = 0^+$.

(i) $\mathcal{I}(\rho) := \{(\rho_\varepsilon^{-a}) \mid a \in \mathbb{R}_{>0}\}$ is called the *asymptotic gauge* generated by ρ. The net ρ is called a *gauge*.

(ii) If $\mathcal{P}(\varepsilon)$ is a property of $\varepsilon \in I$, we use the notation $\forall^0\varepsilon : \mathcal{P}(\varepsilon)$ to denote $\exists\varepsilon_0 \in I, \forall\varepsilon \in (0, \varepsilon_0] : \mathcal{P}(\varepsilon)$. We can read $\forall^0\varepsilon$ as *for ε small*.

(iii) We say that a net $(x_\varepsilon) \in \mathbb{R}^I$ *is ρ-moderate* and write $(x_\varepsilon) \in \mathbb{R}_\rho$ if $\exists(J_\varepsilon) \in \mathcal{I}(\rho) : x_\varepsilon = O(J_\varepsilon)$ as $\varepsilon \to 0^+$.

(iv) Let $(x_\varepsilon), (y_\varepsilon) \in \mathbb{R}^I$. We say that $(x_\varepsilon) \sim_\rho (y_\varepsilon)$ if $\forall(J_\varepsilon) \in \mathcal{I}(\rho) : x_\varepsilon = y_\varepsilon + O(J_\varepsilon^{-1})$ as $\varepsilon \to 0^+$. This is a congruence relation on the ring $\mathbb{R}_\rho$ of moderate nets with respect to pointwise operations, and we can hence define

$$ {}^\rho\widetilde{\mathbb{R}} := \mathbb{R}_\rho / \sim_\rho, $$

which we call *Robinson–Colombeau ring of generalized numbers*, see [5, 6, 38]. We denote the equivalence class $x \in {}^\rho\widetilde{\mathbb{R}}$ simply by $x := [x_\varepsilon] := [(x_\varepsilon)]_\sim \in {}^\rho\widetilde{\mathbb{R}}$.

In the following, ρ will always denote a net as in Definition 2.1. The infinitesimal ρ can be chosen depending on the class of differential equations we need to solve for the generalized functions we are going to introduce, see [16]. For motivations concerning the naturality of ${}^\rho\widetilde{\mathbb{R}}$, see [14].

We can also define an order relation on $^\rho\widetilde{\mathbb{R}}$ by saying that $[x_\varepsilon] \le [y_\varepsilon]$ if there exists $(z_\varepsilon) \in \mathbb{R}^I$ such that $(z_\varepsilon) \sim_\rho 0$ (we then say that (z_ε) is *ρ-negligible*) and $x_\varepsilon \le y_\varepsilon + z_\varepsilon$ for ε small. Equivalently, we have that $x \le y$ if and only if there exist representatives $[x_\varepsilon] = x$ and $[y_\varepsilon] = y$ such that $x_\varepsilon \le y_\varepsilon$ for all ε. Clearly, $^\rho\widetilde{\mathbb{R}}$ is a partially ordered ring. The usual real numbers $r \in \mathbb{R}$ are embedded in $^\rho\widetilde{\mathbb{R}}$ by considering constant nets $[r] \in {}^\rho\widetilde{\mathbb{R}}$.

Even in the case where the order $\le$ is not total, we still have the possibility to define the infimum $\min([x_\varepsilon], [y_\varepsilon]) := [\min(x_\varepsilon, y_\varepsilon)]$, and analogously the supremum function $\max([x_\varepsilon], [y_\varepsilon]) := [\max(x_\varepsilon, y_\varepsilon)]$ and the absolute value $|[x_\varepsilon]| := [|x_\varepsilon|] \in {}^\rho\widetilde{\mathbb{R}}$. Note, e.g., that $x \le z$ and $-x \le z$ imply $|x| \le z$. In the following, we will also use the customary notation $^\rho\widetilde{\mathbb{R}}^*$ for the set of invertible generalized numbers. Our notations for intervals are $[a, b] := \{x \in {}^\rho\widetilde{\mathbb{R}} \mid a \le x \le b\}$ and $[a, b]_{\mathbb{R}} := [a, b] \cap \mathbb{R}$, and analogously for segments $[x, y] := \{x + r \cdot (y - x) \mid r \in [0, 1]\} \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $[x, y]_{\mathbb{R}^n} = [x, y] \cap \mathbb{R}^n$. Finally, we write $x \approx y$ to denote that $|x - y|$ is an infinitesimal number, i.e., $|x - y| \le r$ for all $r \in \mathbb{R}_{>0}$. This is equivalent to $\lim_{\varepsilon \to 0^+} |x_\varepsilon - y_\varepsilon| = 0$ for all representatives $x = [x_\varepsilon]$ and $y = [y_\varepsilon]$.

Topologies on $^\rho\widetilde{\mathbb{R}}^n$

On the $^\rho\widetilde{\mathbb{R}}$-module $^\rho\widetilde{\mathbb{R}}^n$, we can consider the natural extension of the Euclidean norm, i.e., $|[x_\varepsilon]| := [|x_\varepsilon|] \in {}^\rho\widetilde{\mathbb{R}}$, where $[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n$. Even if this generalized norm takes values in $^\rho\widetilde{\mathbb{R}}$, it shares several properties with usual norms, like the triangular inequality or the property $|y \cdot x| = |y| \cdot |x|$. It is therefore natural to consider on $^\rho\widetilde{\mathbb{R}}^n$ topologies generated by balls defined by this generalized norm and a set of radii $\mathfrak{R}$.

Definition 2.2. Let $\mathfrak{R} \in \{{}^\rho\widetilde{\mathbb{R}}^*_{\ge 0}, \mathbb{R}_{>0}\}$, $c \in {}^\rho\widetilde{\mathbb{R}}^n$ and $x, y \in {}^\rho\widetilde{\mathbb{R}}$.
(i) We write $x <_{\mathfrak{R}} y$ if $\exists r \in \mathfrak{R} : r \le y - x$.
(ii) $B^{\mathfrak{R}}_r(c) := \{x \in {}^\rho\widetilde{\mathbb{R}}^n \mid |x - c| <_{\mathfrak{R}} r\}$ for each $r \in \mathfrak{R}$.
(iii) For each $r \in \mathbb{R}_{>0}$, $B^{\mathrm{E}}_r(c) := \{x \in \mathbb{R}^n \mid |x - c| < r\}$ denotes an ordinary Euclidean ball in $\mathbb{R}^n$.

The relation $<_{\mathfrak{R}}$ has better topological properties compared to the usual strict order relation $a \le b$ and $a \ne b$ (that we will *never* use) because for $\mathfrak{R} \in \{{}^\rho\widetilde{\mathbb{R}}^*_{\ge 0}, \mathbb{R}_{>0}\}$ the set of balls $\{B^{\mathfrak{R}}_r(c) \mid r \in \mathfrak{R}, c \in {}^\rho\widetilde{\mathbb{R}}^n\}$ is a base for a topology on $^\rho\widetilde{\mathbb{R}}^n$. The topology generated in the case $\mathfrak{R} = {}^\rho\widetilde{\mathbb{R}}^*_{\ge 0}$ is called *sharp topology*, whereas the one with the set of radii $\mathfrak{R} = \mathbb{R}_{>0}$ is called *Fermat topology*. We will call *sharply open set* any open set in the sharp topology, and *large open set* any open set in the Fermat topology; clearly, the latter is coarser than the former. The existence of infinitesimal neighborhoods implies that the sharp topology induces the discrete topology on $\mathbb{R}$. This is a necessary result when one has to deal with continuous generalized functions which have infinite derivatives. In fact, if $f'(x_0)$ is infinite, we have $f(x) \approx f(x_0)$ only for $x \approx x_0$, see [11, 12]. With an innocuous abuse of language, we write $x < y$ instead of $x <_{{}^\rho\widetilde{\mathbb{R}}^*_{\ge 0}} y$, and $x <_{\mathbb{R}} y$ instead of $x <_{\mathbb{R}_{>0}} y$. For example, $^\rho\widetilde{\mathbb{R}}^*_{\ge 0} = {}^\rho\widetilde{\mathbb{R}}_{>0}$. We will simply write $B_r(c)$ to denote an open ball in the sharp topology and $B^{\mathrm{F}}_r(c)$ for an open ball in the Fermat topology. Also open intervals are defined using the relation $<$, i.e., $(a, b) := \{x \in {}^\rho\widetilde{\mathbb{R}} \mid a < x < b\}$.

The following result is useful to deal with positive and invertible generalized numbers (cf. [35]).

Lemma 2.3. *Let $x \in {}^\rho\widetilde{\mathbb{R}}$. Then the following are equivalent:*
(i) *x is invertible and $x \ge 0$, i.e., $x > 0$.*
(ii) *For each representative $(x_\varepsilon) \in \mathbb{R}_\rho$ of x, we have $\forall^0\varepsilon : x_\varepsilon > 0$.*
(iii) *For each representative $(x_\varepsilon) \in \mathbb{R}_\rho$ of x, we have $\exists m \in \mathbb{N}, \forall^0\varepsilon : x_\varepsilon > \rho^m_\varepsilon$*

We will also need the following result.

Lemma 2.4. *Let $a, b \in {}^\rho\widetilde{\mathbb{R}}$ such that $a < b$. Then the interior* $\mathrm{int}([a, b])$ *in the sharp topology is dense in $[a, b]$.*

Internal and strongly internal sets

A natural way to obtain sharply open, closed and bounded sets in $^\rho\widetilde{\mathbb{R}}^n$ is by using a net (A_ε) of subsets $A_\varepsilon \subseteq \mathbb{R}^n$. We have two ways of extending the membership relation $x_\varepsilon \in A_\varepsilon$ to generalized points $[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$.

Definition 2.5. Let (A_ε) be a net of subsets of $\mathbb{R}^n$.

(i) $[A_\varepsilon] := \{[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n \mid \forall^0\varepsilon : x_\varepsilon \in A_\varepsilon\}$ is called the *internal set* generated by the net (A_ε). See [36] for an introduction and an in-depth study of this notion in the case $\rho_\varepsilon = \varepsilon$.

(ii) Let (x_ε) be a net of points of $\mathbb{R}^n$. We say that $x_\varepsilon \in_\varepsilon A_\varepsilon$, and we read it as (x_ε) *strongly belongs to* (A_ε), if $\forall^0\varepsilon : x_\varepsilon \in A_\varepsilon$, and if $(x'_\varepsilon) \sim_\rho (x_\varepsilon)$, then also $x'_\varepsilon \in A_\varepsilon$ for ε small. Also, we set $\langle A_\varepsilon\rangle := \{[x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n \mid x_\varepsilon \in_\varepsilon A_\varepsilon\}$, and we call it the *strongly internal set* generated by the net (A_ε).

(iii) We say that the internal set $K = [A_\varepsilon]$ is *sharply bounded* if there exists $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that $K \subseteq B_r(0)$. Analogously, a net (A_ε) is *sharply bounded* if the internal set $[A_\varepsilon]$ is sharply bounded.

Therefore, $x \in [A_\varepsilon]$ if there exists a representative $[x_\varepsilon] = x$ such that $x_\varepsilon \in A_\varepsilon$ for ε small, whereas this membership is independent from the chosen representative in the case of strongly internal sets. Note explicitly that an internal set generated by a constant net $A_\varepsilon = A \subseteq \mathbb{R}^n$ is simply denoted by $[A]$.

The following theorem shows that internal and strongly internal sets have dual topological properties:

Theorem 2.6. *For $\varepsilon \in I$, let $A_\varepsilon \subseteq \mathbb{R}^n$ and let $x_\varepsilon \in \mathbb{R}^n$. Then the following hold:*

(i) $[x_\varepsilon] \in [A_\varepsilon]$ *if and only if* $\forall q \in \mathbb{R}_{>0}, \forall^0\varepsilon : d(x_\varepsilon, A_\varepsilon) \le \rho_\varepsilon^q$. *Thus,* $[x_\varepsilon] \in [A_\varepsilon]$ *if and only if* $[d(x_\varepsilon, A_\varepsilon)] = 0 \in {}^\rho\widetilde{\mathbb{R}}$.

(ii) $[x_\varepsilon] \in \langle A_\varepsilon\rangle$ *if and only if* $\exists q \in \mathbb{R}_{>0}, \forall^0\varepsilon : d(x_\varepsilon, A^c_\varepsilon) > \rho_\varepsilon^q$, *where* $A^c_\varepsilon := \mathbb{R}^n \setminus A_\varepsilon$. *Hence, if* $(d(x_\varepsilon, A^c_\varepsilon)) \in \mathbb{R}_\rho$, *then* $[x_\varepsilon] \in \langle A_\varepsilon\rangle$ *if and only if* $[d(x_\varepsilon, A^c_\varepsilon)] > 0$.

(iii) $[A_\varepsilon]$ *is sharply closed and* $\langle A_\varepsilon\rangle$ *is sharply open.*

(iv) $[A_\varepsilon] = [\mathrm{cl}(A_\varepsilon)]$, *where* $\mathrm{cl}(S)$ *is the closure of* $S \subseteq \mathbb{R}^n$. *On the other hand,* $\langle A_\varepsilon\rangle = \langle \mathrm{int}(A_\varepsilon)\rangle$, *where* $\mathrm{int}(S)$ *is the interior of* $S \subseteq \mathbb{R}^n$.

Generalized smooth functions and their calculus

Using the ring ${}^\rho\widetilde{\mathbb{R}}$, it is easy to consider a Gaussian with an infinitesimal standard deviation. If we denote this probability density by $f(x, \sigma)$, and if we set $\sigma = [\sigma_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, where $\sigma \approx 0$, we obtain the net of smooth functions $(f(-, \sigma_\varepsilon))_{\varepsilon\in I}$. This is the basic idea we are going to develop in the following.

Definition 2.7. Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ be arbitrary subsets of generalized points. We say that $f : X \to Y$ is a *generalized smooth function* if there exists a net $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ defining f in the sense that $X \subseteq \langle\Omega_\varepsilon\rangle$, $f([x_\varepsilon]) = [f_\varepsilon(x_\varepsilon)] \in Y$ and $(\partial^\alpha f_\varepsilon(x_\varepsilon)) \in \mathbb{R}^d_\rho$ for all $x = [x_\varepsilon] \in X$ and all $\alpha \in \mathbb{N}^n$. The space of generalized smooth functions (GSF) from X to Y is denoted by ${}^\rho\mathcal{GC}^\infty(X, Y)$.

Let us note explicitly that this definition states minimal logical conditions to obtain a set-theoretical map from X into Y, defined by a net of smooth functions. In particular, the following theorem states that the equality $f([x_\varepsilon]) = [f_\varepsilon(x_\varepsilon)]$ is meaningful, i.e., that we have independence from the representatives for all derivatives $[x_\varepsilon] \in X \mapsto [\partial^\alpha f_\varepsilon(x_\varepsilon)] \in {}^\rho\widetilde{\mathbb{R}}^d$, $\alpha \in \mathbb{N}^n$.

Theorem 2.8. *Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ be arbitrary subsets of generalized points. Let $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$ be a net of smooth functions that defines a generalized smooth map of the type $X \to Y$. Then the following hold:*

(i) $\forall\alpha \in \mathbb{N}^n, \forall(x_\varepsilon), (x'_\varepsilon) \in \mathbb{R}^n_\rho : [x_\varepsilon] = [x'_\varepsilon] \in X \Rightarrow (\partial^\alpha u_\varepsilon(x_\varepsilon)) \sim_\rho (\partial^\alpha u_\varepsilon(x'_\varepsilon))$.

(ii) $\forall[x_\varepsilon] \in X, \forall\alpha \in \mathbb{N}^n, \exists q \in \mathbb{R}_{>0}, \forall^0\varepsilon : \sup_{y\in B^{\mathrm{E}}_{\varepsilon^q}(x_\varepsilon)} |\partial^\alpha u_\varepsilon(y)| \le \varepsilon^{-q}$.

(iii) *For all $\alpha \in \mathbb{N}^n$, the GSF $g : [x_\varepsilon] \in X \mapsto [\partial^\alpha f_\varepsilon(x_\varepsilon)] \in \widetilde{\mathbb{R}}^d$ is locally Lipschitz in the sharp topology, i.e., each $x \in X$ possesses a sharp neighborhood U such that $|g(x) - g(y)| \le L|x - y|$ for all $x, y \in U$ and some $L \in {}^\rho\widetilde{\mathbb{R}}$.*

(iv) *Each $f \in {}^\rho\mathcal{GC}^\infty(X, Y)$ is continuous with respect to the sharp topologies induced on X, Y.*

(v) *Assume that the GSF f is locally Lipschitz in the Fermat topology and that its Lipschitz constants are always finite, i.e., $L \in \mathbb{R}$. Then f is continuous in the Fermat topology.*

(vi) *$f : X \to Y$ is a GSF if and only if there exists a net $v_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^d)$ defining a generalized smooth map of type $X \to Y$ such that $f = [v_\varepsilon(-)]|_X$.*

(vii) *Subsets $S \subseteq {}^\rho\widetilde{\mathbb{R}}^s$ with the trace of the sharp topology, and generalized smooth maps as arrows form a subcategory of the category of topological spaces. We will call this category the* category of GSF, *and denote it by ${}^\rho\mathcal{GC}^\infty$.*

The differential calculus for GSF can be introduced by showing existence and uniqueness of another GSF serving as incremental ratio.

Theorem 2.9 (Fermat–Reyes theorem for GSF). *Let $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ be a sharply open set, let $v = [v_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n$, and let $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth map generated by the net of smooth functions $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R})$. Then the following hold:*

(i) *There exists a sharp neighborhood T of $U \times \{0\}$ and a generalized smooth map $r \in {}^\rho\mathcal{GC}^\infty(T, {}^\rho\widetilde{\mathbb{R}})$, called the* generalized incremental ratio *of f* along *v, such that $f(x + hv) = f(x) + h \cdot r(x, h)$ for all $(x, h) \in T$.*

(ii) *Any two generalized incremental ratios coincide on a sharp neighborhood of $U \times \{0\}$.*

(iii) *We have $r(x, 0) = [\frac{\partial f_\varepsilon}{\partial v_\varepsilon}(x_\varepsilon)]$ for every $x \in U$ and we can thus define $Df(x) \cdot v := \frac{\partial f}{\partial v}(x) := r(x, 0)$, so that $\frac{\partial f}{\partial v} \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$.*

If U is a large open set, then an analogous statement holds by replacing sharp neighborhoods by large neighborhoods.

Note that this result permits the consideration of the partial derivative of f with respect to an arbitrary generalized vector $v \in {}^\rho\widetilde{\mathbb{R}}^n$ which can be, e.g., infinitesimal or infinite. Using this result, we can also define subsequent differentials $D^j f(x)$ as j-multilinear maps, and we set

$$D^j f(x) \cdot h^j := D^j f(x)(\underbrace{h, \dots, h}_{j}).$$

The set of all the j-multilinear maps $({}^\rho\widetilde{\mathbb{R}}^n)^j \to {}^\rho\widetilde{\mathbb{R}}^d$ over the ring ${}^\rho\widetilde{\mathbb{R}}$ will be denoted by $L^j({}^\rho\widetilde{\mathbb{R}}^n, {}^\rho\widetilde{\mathbb{R}}^d)$. For $A = [A_\varepsilon(-)] \in L^j({}^\rho\widetilde{\mathbb{R}}^n, {}^\rho\widetilde{\mathbb{R}}^d)$, we set $|A| := [|A_\varepsilon|]$, the generalized number defined by the operator norms of the multilinear maps $A_\varepsilon \in L^j(\mathbb{R}^n, \mathbb{R}^d)$.

The following result follows from the analogous properties for the nets of smooth functions defining f and g.

Theorem 2.10. *Let $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ be an open subset in the sharp topology, and let $v \in {}^\rho\widetilde{\mathbb{R}}^n$ and $f, g \colon U \to {}^\rho\widetilde{\mathbb{R}}$ be generalized smooth maps. Then the following hold:*

(i) $\frac{\partial(f+g)}{\partial v} = \frac{\partial f}{\partial v} + \frac{\partial g}{\partial v}$.

(ii) $\frac{\partial(r \cdot f)}{\partial v} = r \cdot \frac{\partial f}{\partial v}$ *for all $r \in {}^\rho\widetilde{\mathbb{R}}$.*

(iii) $\frac{\partial(f \cdot g)}{\partial v} = \frac{\partial f}{\partial v} \cdot g + f \cdot \frac{\partial g}{\partial v}$.

(iv) *For each $x \in U$, the map $\mathrm{d}f(x) \cdot v := \frac{\partial f}{\partial v}(x) \in {}^\rho\widetilde{\mathbb{R}}$ is ${}^\rho\widetilde{\mathbb{R}}$-linear in $v \in {}^\rho\widetilde{\mathbb{R}}^n$.*

(v) *Let $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $V \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ be open subsets in the sharp topology, and let $g \in {}^\rho\mathcal{GC}^\infty(V, U)$ and $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ be generalized smooth maps. Then, for all $x \in V$ and all $v \in {}^\rho\widetilde{\mathbb{R}}^d$, we have $\frac{\partial(f \circ g)}{\partial v}(x) = \mathrm{d}f(g(x)) \cdot \frac{\partial g}{\partial v}(x)$.*

We also have a generalization of the Taylor formula.

Theorem 2.11. *Let $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function defined in the sharply open set $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$. Let $a, x \in {}^\rho\widetilde{\mathbb{R}}^n$ be such that the line segment $[a, x]$ belongs to U. Then, for all $n \in \mathbb{N}$, we have*

$$\exists \xi \in [a, x] : f(x) = \sum_{j=0}^{n} \frac{D^j f(a)}{j!} \cdot (x - a)^j + \frac{D^{n+1} f(\xi)}{(n+1)!} \cdot (x - a)^{n+1}. \tag{2.1}$$

If we further assume that all the n components $(x - a)_k \in {}^\rho\widetilde{\mathbb{R}}$ of $x - a \in {}^\rho\widetilde{\mathbb{R}}^n$ are invertible, then there exists $\rho \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, $\rho \le |x - a|$, such that

$$\forall k \in B_\rho(0), \exists \xi \in [a - k, a + k] : f(a + k) = \sum_{j=0}^{n} \frac{D^j f(a)}{j!} \cdot k^j + \frac{D^{n+1} f(\xi)}{(n+1)!} \cdot k^{n+1}, \tag{2.2}$$

$$\frac{D^{n+1} f(\xi)}{(n+1)!} \cdot k^{n+1} \approx 0. \tag{2.3}$$

Formula (2.1) corresponds to a direct generalization of Taylor formulas for ordinary smooth functions with Lagrange remainder. On the other hand, in (2.2) and (2.3), the possibility that the differential $D^{n+1} f$ may be infinite at some point is considered, and the Taylor formulas are stated so as to have infinitesimal remainder.

The following local inverse function theorem will be used in the proof of Jacobi's theorem (see [13] for a proof).

Theorem 2.12. *Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ and $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}}^n)$, and suppose that for some x_0 in the sharp interior of X, $Df(x_0)$ is invertible in $L({}^\rho\widetilde{\mathbb{R}}^n, {}^\rho\widetilde{\mathbb{R}}^n)$. Then there exists a sharp neighborhood $U \subseteq X$ of x_0 and a sharp neighborhood V of $f(x_0)$ such that $f : U \to V$ is invertible and $f^{-1} \in {}^\rho\mathcal{GC}^\infty(V, U)$.*

We can define right and left derivatives as, e.g., $f'(a) := f'_+(a) := \lim_{t\to a, a<t} f'(t)$, which always exist if $f \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$. The one-dimensional integral calculus of GSF is based on the following.

Theorem 2.13. *Let $f \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}})$ be a generalized smooth function defined in the interval $[a, b] \subseteq {}^\rho\widetilde{\mathbb{R}}$, where $a < b$. Let $c \in [a, b]$. Then there exists one and only one generalized smooth function $F \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}})$ such that $F(c) = 0$ and $F'(x) = f(x)$ for all $x \in [a, b]$. Moreover, if f is defined by the net $f_\varepsilon \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ and $c = [c_\varepsilon]$, then $F(x) = [\int_{c_\varepsilon}^{x_\varepsilon} f_\varepsilon(s)\,\mathrm{d}s]$ for all $x = [x_\varepsilon] \in [a, b]$.*

Definition 2.14. Under the assumptions of Theorem 2.13, we denote by $\int_c^{(-)} f := \int_c^{(-)} f(s)\,\mathrm{d}s \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}})$ the unique generalized smooth function such that

$$\int_c^c f = 0 \qquad \text{and} \qquad \Big(\int_u^{(-)} f\Big)'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_u^x f(s)\,\mathrm{d}s = f(x) \quad \text{for all } x \in [a, b].$$

All the classical rules of integral calculus hold in this setting:

Theorem 2.15. *Let $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ and $g \in {}^\rho\mathcal{GC}^\infty(V, {}^\rho\widetilde{\mathbb{R}})$ be generalized smooth functions defined on sharply open domains in ${}^\rho\widetilde{\mathbb{R}}$. Let $a, b \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$, and $c, d \in [a, b] \subseteq U \cap V$. Then*
(i) $\int_c^d (f + g) = \int_c^d f + \int_c^d g$,
(ii) $\int_c^d \lambda f = \lambda \int_c^d f$ *for all* $\lambda \in {}^\rho\widetilde{\mathbb{R}}$,
(iii) $\int_c^d f = \int_c^e f + \int_e^d f$ *for all* $e \in [a, b]$,
(iv) $\int_c^d f = -\int_d^c f$,
(v) $\int_c^d f' = f(d) - f(c)$,
(vi) $\int_c^d f' \cdot g = [f \cdot g]_c^d - \int_c^d f \cdot g$.

Theorem 2.16. *Let $f \in {}^\rho\mathcal{GC}^\infty(U, {}^\rho\widetilde{\mathbb{R}})$ and $\phi \in {}^\rho\mathcal{GC}^\infty(V, U)$ be generalized smooth functions defined on sharply open domains in ${}^\rho\widetilde{\mathbb{R}}$. Let $a, b \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$, such that $[a, b] \subseteq V$, $\phi(a) < \phi(b)$ and $[\phi(a), \phi(b)] \subseteq U$. Finally, assume that $\phi([a, b]) \subseteq [\phi(a), \phi(b)]$. Then*

$$\int_{\phi(a)}^{\phi(b)} f(t)\,\mathrm{d}t = \int_a^b f[\phi(s)] \cdot \phi'(s)\,\mathrm{d}s.$$

Embedding of Schwartz distributions and Colombeau functions

We finally recall two results that give a certain flexibility in constructing embeddings of Schwartz distributions. Note that both the infinitesimal ρ and the embedding of Schwartz distributions have to be chosen depending on the problem we aim to solve. A trivial example in this direction is the ODE $y' = y/\mathrm{d}\varepsilon$, which cannot be solved for $\rho = (\varepsilon)$, but it has a solution for $\rho = (e^{-1/\varepsilon})$. As another simple example, if we need the property $H(0) = 1/2$, where H is the Heaviside function, then we have to choose the embedding of distributions accordingly. See also [16, 32] for further details.

If $\phi \in \mathcal{D}(\mathbb{R}^n)$, $r \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}^n$, we use the notations $r \odot \phi$ for the function $x \in \mathbb{R}^n \mapsto \frac{1}{r^n} \cdot \phi(\frac{x}{r}) \in \mathbb{R}$ and $x \oplus \phi$ for the function $y \in \mathbb{R}^n \mapsto \phi(y - x) \in \mathbb{R}$. These notations permit to highlight that $\odot$ is a free action of the multiplicative group $(\mathbb{R}_{>0}, \cdot, 1)$ on $\mathcal{D}(\mathbb{R}^n)$ and $\oplus$ is a free action of the additive group $(\mathbb{R}_{>0}, +, 0)$ on $\mathcal{D}(\mathbb{R}^n)$. We also have the distributive property $r \odot (x \oplus \phi) = rx \oplus r \odot \phi$.

Lemma 2.17. *Let $b \in {}^\rho\widetilde{\mathbb{R}}$ be a net such that $\lim_{\varepsilon\to 0^+} b_\varepsilon = +\infty$. Let $d \in (0, 1)$. There exists a net $(\psi_\varepsilon)_{\varepsilon\in I}$ of $\mathcal{D}(\mathbb{R}^n)$ with the following properties:*

(i) $\operatorname{supp}(\psi_\varepsilon) \subseteq B_1(0)$ *for all* $\varepsilon \in I$.
(ii) $\int \psi_\varepsilon = 1$ *for all* $\varepsilon \in I$.
(iii) $\forall \alpha \in \mathbb{N}^n, \exists p \in \mathbb{N} : \sup_{x\in\mathbb{R}^n} |\partial^\alpha \psi_\varepsilon(x)| = O(b_\varepsilon^p)$ *as* $\varepsilon \to 0^+$.
(iv) $\forall j \in \mathbb{N}, \forall^0 \varepsilon : 1 \le |\alpha| \le j \Rightarrow \int x^\alpha \cdot \psi_\varepsilon(x)\,\mathrm{d}x = 0$.
(v) $\forall \eta \in \mathbb{R}_{>0}, \forall^0 \varepsilon : \int |\psi_\varepsilon| \le 1 + \eta$.
(vi) *If* $n = 1$, *then the net* $(\psi_\varepsilon)_{\varepsilon\in I}$ *can be chosen so that* $\int_{-\infty}^0 \psi_\varepsilon = d$.
In particular, $\psi_\varepsilon^b := b_\varepsilon^{-1} \odot \psi_\varepsilon$ *satisfies* (ii)–(v).

It is worth noting that the condition (iv) of null moments is well known in the study of convergence of numerical solutions of singular differential equations, see, e.g., [9, 21, 42] and references therein.

Concerning the embeddings of Schwartz distributions, we have the following result, where

$$c(\Omega) := \{[x_\varepsilon] \in [\Omega] \mid \exists K \Subset \Omega, \forall^0 \varepsilon : x_\varepsilon \in K\}$$

is called the set of *compactly supported points in* $\Omega \subseteq \mathbb{R}^n$.

Theorem 2.18. *Under the assumptions of Lemma 2.17, let* $\Omega \subseteq \mathbb{R}^n$ *be an open set and let* (ψ_ε^b) *be the net defined in Lemma 2.17. Then the mapping*

$$\iota_\Omega^b : T \in \mathcal{E}'(\Omega) \mapsto [(T * \psi_\varepsilon^b)(-)] \in {}^\rho\mathcal{GC}^\infty(c(\Omega), {}^\rho\widetilde{\mathbb{R}})$$

uniquely extends to a sheaf morphism of real vector spaces

$$\iota^b : \mathcal{D}' \to {}^\rho\mathcal{GC}^\infty(c((-)), {}^\rho\widetilde{\mathbb{R}}),$$

and satisfies the following properties:
(i) *If* $b \ge \mathrm{d}\rho^{-a}$ *for some* $a \in \mathbb{R}_{>0}$, *then* $\iota^b|_{\mathcal{C}^\infty(-)} : \mathcal{C}^\infty(-) \to {}^\rho\mathcal{GC}^\infty(c((-)), {}^\rho\widetilde{\mathbb{R}})$ *is a sheaf morphism of algebras.*
(ii) *If* $T \in \mathcal{E}'(\Omega)$, *then* $\operatorname{supp}(T) = \operatorname{supp}(\iota_\Omega^b(T))$.
(iii) $\lim_{\varepsilon\to 0^+} \int_\Omega \iota_\Omega^b(T)_\varepsilon \cdot \phi = \langle T, \phi\rangle$ *for all* $\phi \in \mathcal{D}(\Omega)$ *and all* $T \in \mathcal{D}'(\Omega)$.
(iv) ι^b *commutes with partial derivatives, i.e.,* $\partial^\alpha(\iota_\Omega^b(T)) = \iota_\Omega^b(\partial^\alpha T)$ *for each* $T \in \mathcal{D}'(\Omega)$ *and* $\alpha \in \mathbb{N}$.

Concerning the embedding of Colombeau generalized functions, we recall that the special Colombeau algebra on Ω is defined as the quotient $\mathcal{G}^s(\Omega) := \mathcal{E}_M(\Omega)/\mathcal{N}^s(\Omega)$ of *moderate nets* over *negligible nets*, where the former is

$$\mathcal{E}_M(\Omega) := \Big\{(u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I \;\Big|\; \forall K \Subset \Omega, \forall \alpha \in \mathbb{N}^n, \exists N \in \mathbb{N} : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^{-N})\Big\}$$

and the latter is

$$\mathcal{N}^s(\Omega) := \Big\{(u_\varepsilon) \in \mathcal{C}^\infty(\Omega)^I \;\Big|\; \forall K \Subset \Omega, \forall \alpha \in \mathbb{N}^n, \forall m \in \mathbb{N} : \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = O(\varepsilon^m)\Big\}.$$

Using $\rho = (\varepsilon)$, we have the following compatibility result.

Theorem 2.19. *A Colombeau generalized function* $u = (u_\varepsilon) + \mathcal{N}^s(\Omega)^d \in \mathcal{G}^s(\Omega)^d$ *defines a generalized smooth map* $u : [x_\varepsilon] \in c(\Omega) \to [u_\varepsilon(x_\varepsilon)] \in \widetilde{\mathbb{R}}^d$, *which is locally Lipschitz on the same neighborhood of the Fermat topology for all derivatives. This assignment provides a bijection of* $\mathcal{G}^s(\Omega)^d$ *onto* ${}^\rho\mathcal{GC}^\infty(c(\Omega), {}^\rho\widetilde{\mathbb{R}}^d)$ *for every open set* $\Omega \subseteq \mathbb{R}^n$.

2.1 Extreme value theorem and functionally compact sets

For GSF, suitable generalizations of many classical theorems of differential and integral calculus hold such as the intermediate value theorem, mean value theorems, a sheaf property for the Fermat topology, local and global inverse function theorems, the Banach fixed point theorem and a corresponding Picard–Lindelöf theorem, see [13–15, 31].

Even though the intervals $[a, b] \subseteq \widetilde{\mathbb{R}}$, $a, b \in \mathbb{R}$, are neither compact in the sharp nor in the Fermat topology (see [15, Theorem 25]), analogously to the case of smooth functions, a GSF satisfies an extreme value theorem on such sets. In fact, we have the following theorem.

Theorem 2.20. *Let $f \in \mathcal{GC}^\infty(X, \widetilde{\mathbb{R}})$ be a generalized smooth function defined on the subset X of $\widetilde{\mathbb{R}}^n$. Let $\emptyset \neq K = [K_\varepsilon] \subseteq X$ be an internal set generated by a sharply bounded net (K_ε) of compact sets $K_\varepsilon \Subset \mathbb{R}^n$. Then*

$$\exists m, M \in K, \forall x \in K : f(m) \leq f(x) \leq f(M). \tag{2.4}$$

We shall use the assumptions on K and (K_ε) given in this theorem to introduce a notion of "compact subset" which behaves better than the usual classical notion of compactness in the sharp topology.

Definition 2.21. A subset K of $\widetilde{\mathbb{R}}^n$ is called *functionally compact*, denoted by $K \Subset_{\mathrm{f}} \widetilde{\mathbb{R}}^n$, if there exists a net (K_ε) such that
(i) $K = [K_\varepsilon] \subseteq \widetilde{\mathbb{R}}^n$,
(ii) (K_ε) is sharply bounded,
(iii) $\forall \varepsilon \in I : K_\varepsilon \Subset \mathbb{R}^n$.
If, in addition, $K \subseteq U \subseteq \widetilde{\mathbb{R}}^n$, then we write $K \Subset_{\mathrm{f}} U$. Finally, we write $[K_\varepsilon] \Subset_{\mathrm{f}} U$ if (ii), (iii) and $[K_\varepsilon] \subseteq U$ hold.

We motivate the name *functionally compact subset* by noting that on this type of subsets, GSF have properties very similar to those that ordinary smooth functions have on standard compact sets.

Remark 2.22. (i) By [36, Proposition 2.3], any internal set $K = [K_\varepsilon]$ is closed in the sharp topology. In particular, the open interval $(0, 1) \subseteq \widetilde{\mathbb{R}}$ is not functionally compact since it is not closed.
(ii) If $H \Subset \mathbb{R}^n$ is a non-empty ordinary compact set, then $[H]$ is functionally compact. In particular, we have that $[0, 1] = [[0, 1]_{\mathbb{R}}]$ is functionally compact.
(iii) For the empty set, we have $\emptyset = \widetilde{\emptyset} \Subset_{\mathrm{f}} \widetilde{\mathbb{R}}$.
(iv) $\widetilde{\mathbb{R}}^n$ is not functionally compact since it is not sharply bounded.
(v) The set of compactly supported points $\widetilde{\mathbb{R}}_c$ is not functionally compact because the GSF $f(x) = x$ does not satisfy the conclusion (2.4) of Proposition 2.20.

In the present paper, we need the following properties of functionally compact sets.

Theorem 2.23. *Let $K \subseteq X \subseteq \widetilde{\mathbb{R}}^n$, $f \in \mathcal{GC}^\infty(X, \widetilde{\mathbb{R}}^d)$. Then $K \Subset_{\mathrm{f}} \widetilde{\mathbb{R}}^n$ implies $f(K) \Subset_{\mathrm{f}} \widetilde{\mathbb{R}}^d$.*

As a corollary of this theorem and Remark 2.22 (ii), we get the following.

Corollary 2.24. *If $a, b \in \widetilde{\mathbb{R}}$ and $a \leq b$, then $[a, b] \Subset_{\mathrm{f}} \widetilde{\mathbb{R}}$.*

Let us note that $a, b \in \widetilde{\mathbb{R}}$ can also be infinite, e.g., $a = [-\varepsilon^{-N}]$, $b = [\varepsilon^{-M}]$ or $a = [\varepsilon^{-N}]$, $b = [\varepsilon^{-M}]$ with $M > N$. Finally, in the following result we consider the product of functionally compact sets.

Theorem 2.25. *Let $K \Subset_{\mathrm{f}} \widetilde{\mathbb{R}}^n$ and $H \Subset_{\mathrm{f}} \widetilde{\mathbb{R}}^d$, then $K \times H \Subset_{\mathrm{f}} \widetilde{\mathbb{R}}^{n+d}$. In particular, if $a_i \leq b_i$ for $i = 1, \dots, n$, then $\prod_{i=1}^n [a_i, b_i] \Subset_{\mathrm{f}} \widetilde{\mathbb{R}}^n$.*

A theory of compactly supported GSF has been developed in [13], and it closely resembles the classical theory of LF-spaces of compactly supported smooth functions. It establishes that for suitable functionally compact subsets, the corresponding space of compactly supported GSF contains extensions of all Colombeau generalized functions, and hence also of all Schwartz distributions.

3 Preliminary results for calculus of variations with GSF

In this section, we study extremal values of generalized functions at sharply interior points of intervals $[a, b] \subseteq {}^\rho\widetilde{\mathbb{R}}$. As in the classical calculus of variations, this will provide the basis for proving necessary and sufficient conditions for general variational problems. Since the new ring of scalars ${}^\rho\widetilde{\mathbb{R}}$ has zero divisors and is not totally ordered, the following extension requires a more refined analysis than in the classical case.

The following lemma shows that we can interchange integration and differentiation while working with generalized functions.

Lemma 3.1. *Let $a, b, c, d \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$ and $c < d$. Let also $f \in {}^\rho\mathcal{GC}^\infty(X, Y)$ and assume that $[a, b]\times[c, d] \subseteq X \subseteq {}^\rho\widetilde{\mathbb{R}}^2$ and $Y \subseteq {}^\rho\widetilde{\mathbb{R}}^d$. Then, for all $s \in [c, d]$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}s}\int_a^b f(\tau, s)\,\mathrm{d}\tau = \int_a^b \frac{\partial}{\partial s} f(\tau, s)\,\mathrm{d}\tau. \tag{3.1}$$

Proof. We first note that $f(\cdot\,, s) \in {}^\rho\mathcal{GC}^\infty([a, b], Y)$, by the closure of GSF with respect to composition. Therefore, $\frac{\partial}{\partial s} f(\cdot\,, s) \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$, and the right-hand side of (3.1) is well defined as an integral of a GSF. In order to show that also the left-hand side of (3.1) is well defined, we need to prove that also $\sigma \in [c, d] \mapsto \int_a^b f(\tau, \sigma)\,\mathrm{d}\tau \in {}^\rho\widetilde{\mathbb{R}}^d$ is a GSF. Let f be defined by the net $f_\varepsilon \in \mathcal{C}^\infty(\Omega_\varepsilon, \mathbb{R}^d)$, with $X \subseteq \langle\Omega_\varepsilon\rangle$, and let $[\sigma_\varepsilon] \in [c, d]$. Then $[a, b] \times \{[\sigma_\varepsilon]\} \Subset_{\mathrm{f}} {}^\rho\widetilde{\mathbb{R}}^2$ and the extreme value Theorem 2.20 applied to $\frac{\partial^n f}{\partial\sigma^n}$ yields the existence of $N \in \mathbb{R}_{>0}$ such that

$$\left|\frac{\mathrm{d}^n}{\mathrm{d}\sigma^n}\int_{a_\varepsilon}^{b_\varepsilon} f_\varepsilon(\tau, \sigma_\varepsilon)\,\mathrm{d}\tau\right| \le \int_{a_\varepsilon}^{b_\varepsilon}\left|\frac{\partial^n}{\partial\sigma^n} f_\varepsilon(\tau, \sigma_\varepsilon)\right|\mathrm{d}\tau \le \rho_\varepsilon^{-N}\cdot(b_\varepsilon - a_\varepsilon).$$

This proves that also the left-hand side of (3.1) is well defined as a derivative of a GSF. From the classical derivation under the integral sign, the Fermat–Reyes Theorem 2.9, and Theorem 2.13 about definite integrals of GSF, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}s}\int_a^b f(\tau, s)\,\mathrm{d}\tau &= \frac{\mathrm{d}}{\mathrm{d}s}\int_a^b [f_\varepsilon(\tau, s)]\,\mathrm{d}\tau \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\left[\int_{a_\varepsilon}^{b_\varepsilon} f_\varepsilon(\tau, s)\,\mathrm{d}\tau\right] \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}s}\int_{a_\varepsilon}^{b_\varepsilon} f_\varepsilon(\tau, s)\,\mathrm{d}\tau\right] \\ &= \int_a^b \left[\frac{\partial}{\partial s} f_\varepsilon(\tau, s)\right]\mathrm{d}\tau \\ &= \int_a^b \frac{\partial}{\partial s} f(\tau, s)\,\mathrm{d}\tau. \qquad \square\end{aligned}$$

The next result will be used frequently.

Lemma 3.2. *Let $(D, \ge)$ be a directed set and let $f : D \longrightarrow {}^\rho\widetilde{\mathbb{R}}$ be a set-theoretical map such that $f(d) \ge 0$ for all $d \in D$, and $\exists \lim_{d\in D} f(d) \in {}^\rho\widetilde{\mathbb{R}}$ in the sharp topology. Then $\lim_{d\in D} f(d) \ge 0$.*

Proof. Note that the internal set $[0, +\infty) = [[0, +\infty)_{\mathbb{R}}]$ is sharply closed by (iii) of Theorem 2.6. □

Remark 3.3. (i) If $x \in {}^\rho\widetilde{\mathbb{R}}$, then $x \ge 0$ if and only if $\exists A \in \mathbb{R}_{>0}, \forall a \in \mathbb{R}_{>A} : x \ge -\mathrm{d}\rho^a$. Indeed, it suffices to let $a \to +\infty$ in $f(a) = x + \mathrm{d}\rho^a$.

(ii) Assume that $x, y \in {}^\rho\widetilde{\mathbb{R}}^n$ and

$$\exists s_0 \in {}^\rho\widetilde{\mathbb{R}}_{>0}, \forall s \in {}^\rho\widetilde{\mathbb{R}}_{>0} : s \le s_0 \implies |x| \le s|y|.$$

Then taking $s \to 0$ in $f(s) = s|y| - |x|$, we get $x = 0$.

Definition 3.4. We call $x = (x_1, \dots, x_d) \in {}^\rho\widetilde{\mathbb{R}}^d$ *componentwise invertible* if and only if for all $k \in \{1, \dots, d\}$, we have that $x_k \in {}^\rho\widetilde{\mathbb{R}}$ is invertible.

Lemma 3.5. *Let $f \in {}^\rho\mathcal{GC}^\infty(U, Y)$, where $Y \subseteq {}^\rho\widetilde{\mathbb{R}}$, and $U \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ is a sharply open subset. Then $f \ge 0$ if and only if $f(x) \ge 0$ for all componentwise invertible $x \in U$.*

Proof. By Lemma 2.3, it follows that for $V \subseteq {}^\rho\widetilde{\mathbb{R}}$, the set of invertible points in V, i.e., $V \cap {}^\rho\widetilde{\mathbb{R}}^* \subseteq V$ is dense in V (with respect to the sharp topology). This implies that $U \cap ({}^\rho\widetilde{\mathbb{R}}^d)^* \subseteq U$ is dense. By Theorem 2.8 (iv), f is sharply continuous, so Lemma 3.2 yields $f(x) \geq 0$. The other direction is obvious. □

Analogously to the classical case, we say that $x_0 \in X$ is a local minimum of $f \in {}^\rho\mathcal{GC}^\infty(X)$ if there exists a sharply open neighborhood (in the trace topology) $Y \subseteq X$ of x_0 such that $f(x_0) \leq f(y)$ for all $y \in Y$. A local maximum is defined accordingly. We will write $f(x_0) = \min!$, which is a short hand notation to denote that x_0 is a (local) minimum of f.

Lemma 3.6. *Let $X \subseteq {}^\rho\widetilde{\mathbb{R}}$ and $f \in {}^\rho\mathcal{GC}^\infty(X, {}^\rho\widetilde{\mathbb{R}})$. If $x_0 \in X$ is a sharply interior local minimum of f, then $f'(x_0) = 0$.*

Proof. Without loss of generality, we can assume $x_0 = 0$, because of the closure of GSF with respect to composition. Let $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ be such that $B_{2r}(0) =: U \subseteq X$ and $f(0) = \min!$ over U. Take any $x \in {}^\rho\widetilde{\mathbb{R}}$ such that $0 < |x| < r$, so that $[-|x|, |x|] \subseteq U$. Thus, if $x > 0$, by Taylor's Theorem 2.11, there exists $\xi \in [0, x]$ such that

$$f(x) = f(0) + f'(0) \cdot x + \frac{f''(\xi)}{2} \cdot x^2.$$

Set $K := [B_{r_\varepsilon}(0)] \Subset_f B_{2r}(0) \subseteq U$ and $M := \max_{x \in K} |f''(x)| \in {}^\rho\widetilde{\mathbb{R}}_{\geq 0}$. Due to the fact that $f(0)$ is minimal, we have

$$f'(0) \cdot x + \frac{f''(\xi)}{2} \cdot x^2 = f(x) - f(0) \geq 0.$$

Thus, $-f'(0) \cdot x \leq \frac{M}{2} x^2$ and $-f'(0) \leq \frac{M}{2}|x|$, since $x > 0$. Analogously, if we take $x < 0$, we get $f'(0) \leq -\frac{M}{2} x = \frac{M}{2}|x|$. Therefore, $|f'(0)| \leq \frac{M}{2}|x|$, and the conclusion follows by Remark 3.3 (ii). □

As a consequence of Lemma 2.4 and Theorem 2.8 (iv), we have the following lemma.

Lemma 3.7. *Let $a, b \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$, and let $f \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$ be such that $f(x) = 0$ for all sharply interior points $x \in [a, b]$. Then $f = 0$ on $[a, b]$.*

Now, we are able to prove the "second-derivative-test" for GSF.

Lemma 3.8. *Let $a, b \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$, and let $f \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}})$ be such that $f(x_0) = \min!$ for some sharply interior $x_0 \in [a, b]$. Then $f''(x_0) \geq 0$. Vice versa, if $f'(x_0) = 0$ and $f''(x_0) > 0$, then $f(x_0) = \min!$.*

Proof. As above, we can assume that $x_0 = 0$. Let $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ be such that $B_{2r}(0) =: U \subseteq X$ and $f(0) = \min!$ over U. Take any $x \in {}^\rho\widetilde{\mathbb{R}}$ such that $0 < x < r$, so that $[0, x] \subseteq U$, and set $K := [B_{r_\varepsilon}(0)] \Subset_f B_{2r}(0) \subseteq U$ and $M := \max_{x \in K} |f'''(x)| \in {}^\rho\widetilde{\mathbb{R}}_{\geq 0}$. By Taylor's Theorem 2.11, for some $\xi \in [0, x]$, we obtain

$$f(x) = f(0) + f'(0)x + \frac{1}{2} f''(0)x^2 + \frac{1}{6} f'''(\xi)x^3.$$

By assumption, for all $a \in \mathbb{R}_{>0}$, we have

$$0 \leq f(x) - f(0) + \mathrm{d}\rho^a.$$

By Lemma 3.6, we know that $f'(0) = 0$. Thus, for all $a \in \mathbb{R}_{>0}$, we obtain

$$f(x) - f(0) = \frac{1}{2} f''(0)x^2 + \frac{1}{6} f'''(\xi)x^3 \geq -\mathrm{d}\rho^a.$$

Therefore, also $\frac{1}{2} f''(0)x^2 + \frac{1}{6} Mx^3 \geq -\mathrm{d}\rho^a$. In this inequality we can set $x = \mathrm{d}\rho^{a/3}$, assuming that $a > A$ and $\mathrm{d}\rho^A < r$. We get $f''(0) \geq -(2 + \frac{M}{3})\mathrm{d}\rho^{a/3}$, and the conclusion follows from Lemma 3.2 as $a \to +\infty$.

Now assume that $f'(0) = 0$ and $f''(0) > 0$, so that $f''(0) > \mathrm{d}\rho^a$ for some $a \in \mathbb{R}_{>0}$, by Lemma 2.3. Since $f'(0) = 0$, for all $x \in B_r(0)$, Taylor's formula gives

$$f(x) - f(0) = \frac{1}{2} f''(0)x^2 + \frac{1}{6} f'''(\xi_x)x^3,$$

where $\xi_x \in [0, x]$. Therefore, $f(x) - f(0) > x^2(\frac{1}{2}\mathrm{d}\rho^a + \frac{1}{6} f'''(\xi_x)x)$. Now

$$\left| \frac{1}{6} f'''(\xi_x)x \right| \leq \frac{1}{6} M|x| \to 0 \quad \text{as } x \to 0.$$

Thus,

$$\exists s \in {}^{\rho}\widetilde{\mathbb{R}}_{>0} : s < r,\ \forall x \in B_s(0) : -\frac{1}{4}\mathrm{d}\rho^a < \frac{1}{6}f'''(\xi_x)x < \frac{1}{4}\mathrm{d}\rho^a.$$

We can hence write $f(x) - f(0) > x^2(\frac{1}{2}\mathrm{d}\rho^a - \frac{1}{4}\mathrm{d}\rho^a) = x^2\frac{1}{4}\mathrm{d}\rho^a \geq 0$ for all $x \in B_s(0)$, which proves that $x = 0$ is a local minimum. □

For the generalization of Lemmas 3.6 and 3.8 to the multivariate case, one can proceed as above, using the ideas of [25]. Note, however, that we do not need this generalization in the present work.

4 First variation and critical points

In this section, we define the first variation of a functional and prove that some classical results have their counterparts in this generalized setting, for example, the fundamental lemma (Lemma 4.4) or the connection between critical points and the Euler–Lagrange equations (Theorem 4.5).

Definition 4.1. If $a, b \in {}^{\rho}\widetilde{\mathbb{R}}$ and $a < b$, we define

$${}^{\rho}\mathcal{GC}_0^{\infty}(a, b) := \{\eta \in {}^{\rho}\mathcal{GC}^{\infty}({}^{\rho}\widetilde{\mathbb{R}}, {}^{\rho}\widetilde{\mathbb{R}}^d) : \eta(a) = 0 = \eta(b)\}.$$

When the use of the points a, b is clear from the context, we adopt the simplified notation ${}^{\rho}\mathcal{GC}_0^{\infty}$. We also note here that ${}^{\rho}\mathcal{GC}_0^{\infty}(a, b)$ is an ${}^{\rho}\widetilde{\mathbb{R}}$-module.

One of the positive features of the use of GSF for the calculus of variations is their closure with respect to composition. For this reason, the next definition of functional is formally equal to the classical one, though it can be applied to arbitrary generalized functions F and u.

Theorem 4.2. *Let $a, b \in {}^{\rho}\widetilde{\mathbb{R}}$ with $a < b$. Let $u \in {}^{\rho}\mathcal{GC}^{\infty}([a, b], {}^{\rho}\widetilde{\mathbb{R}}^d)$ and $F \in {}^{\rho}\mathcal{GC}^{\infty}([a, b] \times {}^{\rho}\widetilde{\mathbb{R}}^d \times {}^{\rho}\widetilde{\mathbb{R}}^d, {}^{\rho}\widetilde{\mathbb{R}})$, and define*

$$I(u) := \int_a^b F(t, u, \dot{u})\,\mathrm{d}t. \tag{4.1}$$

Let also $\eta \in {}^{\rho}\mathcal{GC}_0^{\infty}$. Then

$$\delta I(u; \eta) := \frac{\mathrm{d}}{\mathrm{d}s}I(u + s\eta)\Big|_{s=0} = \int_a^b \eta\Big(F_u(t, u, \dot{u}) - \frac{\mathrm{d}}{\mathrm{d}t}F_{\dot{u}}(t, u, \dot{u})\Big)\,\mathrm{d}t.$$

Proof. Using Theorems 2.10 and 2.15, and Lemma 3.1, we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}s}I(u + s\eta)\Big|_{s=0} &= \frac{\mathrm{d}}{\mathrm{d}s}\int_a^b F(t, u + s\eta, \dot{u} + s\dot{\eta})\,\mathrm{d}t\Big|_{s=0} \\
&= \int_a^b \frac{\partial}{\partial s}F(t, u + s\eta, \dot{u} + s\dot{\eta})\Big|_{s=0}\,\mathrm{d}t \\
&= \int_a^b \eta F_u(t, u, \dot{u}) + \dot{\eta}F_{\dot{u}}(t, u, \dot{u})\,\mathrm{d}t \\
&= [\eta F_{\dot{u}}(t, u, \dot{u})]_a^b + \int_a^b \eta\Big(F_u(t, u, \dot{u}) - \frac{\mathrm{d}}{\mathrm{d}t}F_{\dot{u}}(t, u, \dot{u})\Big)\,\mathrm{d}t \\
&= \int_a^b \eta\Big(F_u(t, u, \dot{u}) - \frac{\mathrm{d}}{\mathrm{d}t}F_{\dot{u}}(t, u, \dot{u})\Big)\,\mathrm{d}t.
\end{aligned}$$

□

We call $\delta I(u;\eta)$ the *first variation* of I. In addition, we call $u \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}}^d)$ a *critical point* of I if $\delta I(u;\eta) = 0$ for all $\eta \in {}^\rho\mathcal{GC}^\infty_0$.

To prove the fundamental lemma of the calculus of variations, Lemma 4.4, we first show that every GSF can be approximated using generalized strict delta nets.

Lemma 4.3. *Let $a, b \in {}^\rho\widetilde{\mathbb{R}}$ be such that $a < b$ and let $f \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}})$. Let $x \in [a,b]$ and $R \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ be such that $B_R(x) \subseteq [a,b]$. Assume that $G_t \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}})$ has the following properties:*
(i) $\int_{-R}^{R} G_t = 1$ *for $t \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ small.*
(ii) *For t small, $(G_t)_{t\in{}^\rho\widetilde{\mathbb{R}}_{>0}}$ is zero outside every ball $B_\delta(0)$, $0 < \delta < R$, i.e.,*

$$\forall \delta \in {}^\rho\widetilde{\mathbb{R}}_{>0}, \exists \rho \in {}^\rho\widetilde{\mathbb{R}}_{>0}, \forall t \in B_\rho(0) \cap {}^\rho\widetilde{\mathbb{R}}_{>0}, \forall y \in [-R, -\delta] \cup [\delta, R] : G_t(y) = 0.$$

(iii) $\exists M \in {}^\rho\widetilde{\mathbb{R}}_{>0}, \exists \rho \in {}^\rho\widetilde{\mathbb{R}}, \forall t \in B_\rho(0) : \int_{-R}^{R} |G_t(y)|\,dy \le M.$
Then

$$\lim_{t\to 0^+} \int_{-R}^{R} f(x-y)G_t(y)\,dy = f(x).$$

Proof. We only have to generalize the classical proof concerning limits of convolutions with strict delta nets. We first note that

$$\int_{-R}^{R} f(x-y)G_t(y)\,dy = \int_{x-R}^{x+R} f(y)G_t(x-y)\,dy,$$

and so these integrals exist because $(x-R, x+R) = B_R(x) \subseteq [a,b]$. Using (i), for t small, say for $0 < t < S \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, we get

$$\left|\int_{-R}^{R} f(x-y)G_t(y)\,dy - f(x)\right| = \left|\int_{-R}^{R} [f(x-y) - f(x)]G_t(y)\,dy\right| \le \int_{-R}^{R} |f(x-y) - f(x)| \cdot |G_t(y)|\,dy.$$

For each $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, the sharp continuity of f at x yields $|f(x-y) - f(x)| < r$ for all y such that $|y| < \delta \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, and we can take $\delta < R$. By (ii), for $0 < |t| < \min(\rho, S)$, we have

$$\left|\int_{-R}^{R} f(x-y)G_t(y)\,dy - f(x)\right| \le r \int_{-\delta}^{+\delta} |G_t(y)|\,dy. \tag{4.2}$$

The right-hand side of (4.2) can be taken arbitrarily small in ${}^\rho\widetilde{\mathbb{R}}_{>0}$, because of (iii), the fact that $[-\delta, \delta] \Subset_f {}^\rho\widetilde{\mathbb{R}}$ and the application of the extreme value Theorem 2.20 to the GSF G_t. □

Lemma 4.4 (Fundamental lemma of the calculus of variations). *Let $a, b \in {}^\rho\widetilde{\mathbb{R}}$ be such that $a < b$, and let $f \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}})$. If*

$$\int_a^b f(t)\eta(t)\,dt = 0 \quad \text{for all } \eta \in {}^\rho\mathcal{GC}^\infty_0, \tag{4.3}$$

then $f = 0$.

Proof. Let $x \in [a,b]$. Because of Theorem 2.8 (iv) and Lemma 2.4, without loss of generality, we can assume that x is a sharply interior point, so that $B_R(x) \subseteq [a,b]$ for some $R \in {}^\rho\widetilde{\mathbb{R}}_{>0}$. Let $\phi \in \mathcal{D}_{[-1,1]}(\mathbb{R})$ be such that $\int \phi = 1$. Set $G_{t,\varepsilon}(x) := \frac{1}{t_\varepsilon}\phi(\frac{x}{t_\varepsilon})$, where $x \in \mathbb{R}$ and $t \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, and $G_t(x) := [G_{t,\varepsilon}(x_\varepsilon)]$ for all $x \in {}^\rho\widetilde{\mathbb{R}}$. Then, for t sufficiently small, we have $G_t(x - \cdot) \in {}^\rho\mathcal{GC}^\infty_0$ and (4.3) yields $\int_a^b f(y)G_t(x-y)\,dy = 0$. For t small, we both have that $G_t(x - \cdot) = 0$ on $[a, x-R] \cup [x+R, b]$ and the assumptions of Lemma 4.3 hold. Therefore,

$$0 = \int_a^b f(y)G_t(x-y)\,dy = \int_{x-R}^{x+R} f(y)G_t(x-y)\,dy = \int_{-R}^{R} f(x-y)G_t(y)\,dy,$$

and hence Lemma 4.3 yields $f(x) = 0$. □

Thus, we obtain the following theorem.

Theorem 4.5. *Let $a, b \in {}^\rho\widetilde{\mathbb{R}}$ be such that $a < b$, and let $u \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$. Then u solves the Euler–Lagrange equations*

$$F_u - \frac{\mathrm{d}}{\mathrm{d}t} F_{\dot{u}} = 0, \tag{4.4}$$

for I given by (4.1), if and only if $\delta I(u; \eta) = 0$ for all $\eta \in {}^\rho\mathcal{GC}_0^\infty$, i.e., if and only if u is a critical point of I.

5 Second variation and minimizers

As in the classical case (see, e.g., [10]), thanks to the extreme value Theorem 2.20 and the property of the interval $[a, b]$ of being functionally compact, we can naturally define a topology on the space ${}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$.

Definition 5.1. Let $a, b \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$. Let $m \in \mathbb{N}$ and $v \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$. Then

$$\|v\|_m := \max_{\substack{n \le m \\ 1 \le i \le d}} \max\left(\left|\frac{d^n}{dt^n} v^i(M_{ni})\right|, \left|\frac{d^n}{dt^n} v^i(m_{ni})\right|\right) \in {}^\rho\widetilde{\mathbb{R}},$$

where $M_{ni}, m_{ni} \in [a, b]$ satisfy

$$\frac{d^n}{dt^n} v^i(m_{ni}) \le \frac{d^n}{dt^n} v^i(t) \le \frac{d^n}{dt^n} v^i(M_{ni}) \quad \text{for all } t \in [a, b].$$

The following result permits the calculation of the (generalized) norm $\|v\|_m$ using any net (v_ε) that defines v.

Lemma 5.2. *Under the assumptions of Definition 5.1, let $a = [a_\varepsilon]$ and $b = [b_\varepsilon]$ be such that $a_\varepsilon < b_\varepsilon$ for all ε. Then the following hold:*

(i) *If the net (v_ε) defines v, then*

$$\|v\|_m = \left[\max_{\substack{n \le m \\ 1 \le i \le d}} \max_{t \in [a_\varepsilon, b_\varepsilon]} \left|\frac{d^n}{dt^n} v_\varepsilon^i(t)\right|\right].$$

(ii) $\|v\|_m \ge 0$.

(iii) $\|v\|_m = 0$ *if and only if* $v = 0$.

(iv) $\|c \cdot v\|_m = |c| \cdot \|v\|_m$ *for all* $c \in {}^\rho\widetilde{\mathbb{R}}$.

(v) *For all $u \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$, we have $\|u + v\|_m \le \|u\|_m + \|v\|_m$ and $\|u \cdot v\|_m \le c_m \cdot \|u\|_m \cdot \|v\|_m$ for some $c_m \in {}^\rho\widetilde{\mathbb{R}}_{>0}$.*

Proof. By the standard extreme value theorem applied ε-wise, we get the existence of $\bar{m}_{ni\varepsilon}, \bar{M}_{ni\varepsilon} \in [a_\varepsilon, b_\varepsilon]$ such that

$$\frac{d^n}{dt^n} v_\varepsilon^i(\bar{m}_{ni\varepsilon}) \le \frac{d^n}{dt^n} v_\varepsilon^i(t) \le \frac{d^n}{dt^n} v_\varepsilon^i(\bar{M}_{ni\varepsilon}) \quad \text{for all } t \in [a_\varepsilon, b_\varepsilon].$$

Hence,

$$\left|\frac{d^n}{dt^n} v_\varepsilon^i(t)\right| \le \max\left(\left|\frac{d^n}{dt^n} v_\varepsilon^i(\bar{m}_{ni\varepsilon})\right|, \left|\frac{d^n}{dt^n} v_\varepsilon^i(\bar{M}_{ni\varepsilon})\right|\right).$$

Thus,

$$\max_{\substack{n \le m \\ 1 \le i \le d}} \max_{t \in [a_\varepsilon, b_\varepsilon]} \left|\frac{d^n}{dt^n} v_\varepsilon^i(t)\right| \le \max_{\substack{n \le m \\ 1 \le i \le d}} \max\left(\left|\frac{d^n}{dt^n} v_\varepsilon^i(\bar{m}_{ni\varepsilon})\right|, \left|\frac{d^n}{dt^n} v_\varepsilon^i(\bar{M}_{ni\varepsilon})\right|\right).$$

But $\bar{m}_{ni\varepsilon}, \bar{M}_{ni\varepsilon} \in [a_\varepsilon, b_\varepsilon]$, so

$$\begin{aligned}\left[\max_{\substack{n \le m \\ 1 \le i \le d}} \max_{t \in [a_\varepsilon, b_\varepsilon]} \left|\frac{d^n}{dt^n} v_\varepsilon^i(t)\right|\right] &= \left[\max_{\substack{n \le m \\ 1 \le i \le d}} \max\left(\left|\frac{d^n}{dt^n} v_\varepsilon^i(\bar{m}_{ni\varepsilon})\right|, \left|\frac{d^n}{dt^n} v_\varepsilon^i(\bar{M}_{ni\varepsilon})\right|\right)\right] \\ &= \max_{\substack{n \le m \\ 1 \le i \le d}} \max\left(\left|\frac{d^n}{dt^n} v^i(\bar{m}_{ni})\right|, \left|\frac{d^n}{dt^n} v^i(\bar{M}_{ni})\right|\right).\end{aligned}$$

This proves both that $\|v\|_m$ is well defined, i.e., it does not depend on the particular choice of points m_{ni}, M_{ni} as in Definition 5.1, and claim (i). The remaining properties (ii)–(v) follows directly from (i) and the usual properties of standard $\mathcal{C}^m$-norms. □

Using these ${}^\rho\widetilde{\mathbb{R}}$-valued norms, we can naturally define a topology on the space ${}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}}^d)$.

Definition 5.3. Let $a, b \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$, $m \in \mathbb{N}$, $u \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}}^d)$ and $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$.
(i) We set $B^m_r(u) := \{v \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}}^d) \mid \|v - u\|_m < r\}$.
(ii) If $U \subseteq {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}}^d)$, then we say that U is a *sharply open set* if

$$\forall u \in U, \exists m \in \mathbb{N}, \exists r \in {}^\rho\widetilde{\mathbb{R}}_{>0} : B^m_r(u) \subseteq U.$$

As in [15, Theorem 2], one can easily prove that sharply open sets form a topology on ${}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}}^d)$. Using this topology, we can assess when a curve is a minimizer of the functional I. Note explicitly that there are no restrictions on the generalized numbers $a, b \in {}^\rho\widetilde{\mathbb{R}}$, $a < b$, e.g., they can also both be infinite.

Definition 5.4. Let $a, b \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$, and $u \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}}^d)$.
(i) For all $p, q \in {}^\rho\widetilde{\mathbb{R}}^d$, we set

$${}^\rho\mathcal{GC}^\infty_{\mathrm{bd}}(p,q) := \{v \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}}^d) \mid v(a) = p, v(b) = q\}.$$

Note that ${}^\rho\mathcal{GC}^\infty_{\mathrm{bd}}(0,0) = {}^\rho\mathcal{GC}^\infty_0$. The subscript "bd" stands here for "boundary values".
(ii) We say that u is a *local minimizer of I in ${}^\rho\mathcal{GC}^\infty_{\mathrm{bd}}(p,q)$* if $u \in {}^\rho\mathcal{GC}^\infty_{\mathrm{bd}}(p,q)$ and

$$\exists r \in {}^\rho\widetilde{\mathbb{R}}_{>0}, \exists m \in \mathbb{N}, \forall v \in B^m_r(u) \cap {}^\rho\mathcal{GC}^\infty_{\mathrm{bd}}(p,q) : I(v) \geq I(u). \tag{5.1}$$

(iii) We define the *second variation* of I in the direction $\eta \in {}^\rho\mathcal{GC}^\infty_0$ as

$$\delta^2 I(u;\eta) := \left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_0 I(u + s\eta).$$

Note also explicitly that the points $p, q \in {}^\rho\widetilde{\mathbb{R}}^d$ can have infinite norm, e.g., $|p_\varepsilon| \to +\infty$ as $\varepsilon \to 0$. By using the standard Einstein's summation conventions, we calculate

$$\begin{aligned}
\delta^2 I(u;\eta) &= \left.\frac{\mathrm{d}^2}{\mathrm{d}s^2}\right|_0 \int_a^b F(t, u + s\eta, \dot{u} + s\dot{\eta})\,\mathrm{d}t \\
&= \int_a^b \left.\frac{\partial^2}{\partial s^2}\right|_0 F(t, u + s\eta, \dot{u} + s\dot{\eta})\,\mathrm{d}t \\
&= \int_a^b F_{u^i u^j}(t,u,\dot{u})\eta^i\eta^j + 2F_{u^i\dot{u}^j}(t,u,\dot{u})\eta^i\dot{\eta}^j + F_{\dot{u}^i\dot{u}^j}(t,u,\dot{u})\dot{\eta}^i\dot{\eta}^j\,\mathrm{d}t,
\end{aligned}$$

which we abbreviate as

$$\delta^2 I(u;\eta) = \int_a^b F_{uu}(t,u,\dot{u})\eta\eta + 2F_{u\dot{u}}(t,u,\dot{u})\eta\dot{\eta} + F_{\dot{u}\dot{u}}(t,u,\dot{u})\dot{\eta}\dot{\eta}\,\mathrm{d}t.$$

The following results establish classical necessary and sufficient conditions to decide if a function u is a minimizer for the given functional (4.1).

Theorem 5.5. *Let $a, b \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$, $F \in {}^\rho\mathcal{GC}^\infty([a,b] \times {}^\rho\widetilde{\mathbb{R}}^d \times {}^\rho\widetilde{\mathbb{R}}^d, {}^\rho\widetilde{\mathbb{R}})$, $p, q \in {}^\rho\widetilde{\mathbb{R}}^d$, and let u be a local minimizer of I in ${}^\rho\mathcal{GC}^\infty_{\mathrm{bd}}(p,q)$. Then*
(i) *$\delta I(u;\eta) = 0$ for all $\eta \in {}^\rho\mathcal{GC}^\infty_0$,*
(ii) *$\delta^2 I(u;\eta) \geq 0$ for all $\eta \in {}^\rho\mathcal{GC}^\infty_0$.*

Proof. Let $r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ be such that (5.1) holds. Since $\eta \in {}^{\rho}\mathcal{GC}_0^{\infty}$, the map $s \in {}^{\rho}\widetilde{\mathbb{R}} \mapsto u + s\eta \in {}^{\rho}\mathcal{GC}^{\infty}_{\mathrm{bd}}(p,q)$ is well defined and continuous with respect to the trace of the sharp topology in its codomain. Therefore, we can find $\bar{r} \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ such that $u + s\eta \in B_r^m(u) \cap {}^{\rho}\mathcal{GC}^{\infty}_{\mathrm{bd}}(p,q)$ for all $s \in B_{\bar{r}}(0)$. We hence have $I(u+s\eta) \geq I(u)$. This shows that the GSF $s \in B_{\bar{r}}(0) \mapsto I(u+s\eta) \in {}^{\rho}\widetilde{\mathbb{R}}$ has a local minimum at $s = 0$. Now, by employing Lemmas 3.6 and 3.8, the claims are proven. □

Theorem 5.6. *Let $a, b \in {}^{\rho}\widetilde{\mathbb{R}}$, with $a < b$, and $p, q \in {}^{\rho}\widetilde{\mathbb{R}}^d$. Let $u \in {}^{\rho}\mathcal{GC}^{\infty}_{\mathrm{bd}}(p,q)$ be such that*
(i) $\delta I(u;\eta) = 0$ *for all* $\eta \in {}^{\rho}\mathcal{GC}_0^{\infty}$,
(ii) $\delta^2 I(v;\eta) \geq 0$ *for all* $\eta \in {}^{\rho}\mathcal{GC}_0^{\infty}$ *and all* $v \in B_r^m(u) \cap \mathcal{GC}^{\infty}_{\mathrm{bd}}(p,q)$, *where* $r \in {}^{\rho}\widetilde{\mathbb{R}}_{>0}$ *and* $m \in \mathbb{N}$.
Then u is a local minimizer of the functional I in ${}^{\rho}\mathcal{GC}^{\infty}_{\mathrm{bd}}(p,q)$.

Moreover, if $\delta^2 I(v;\eta) > 0$ *for all* $\eta \in {}^{\rho}\mathcal{GC}_0^{\infty}$ *such that* $\|\eta\|_m > 0$ *and all* $v \in B_{2r}^m(u) \cap \mathcal{GC}^{\infty}_{\mathrm{bd}}(p,q)$, *then* $I(v) > I(u)$ *for all* $v \in B_r^m(u) \cap \mathcal{GC}^{\infty}_{\mathrm{bd}}(p,q)$ *such that* $\|v-u\|_m > 0$.

Proof. For any $v \in B_r^m(u) \cap \mathcal{GC}^{\infty}_{\mathrm{bd}}(p,q)$, we set $\psi(s) := I(u + s(v-u)) \in {}^{\rho}\widetilde{\mathbb{R}}$ for all $s \in B_1(0)$, so that we have $u + s(v-u) \in B_r^m(u)$. Since $(v-u)(a) = 0 = (v-u)(b)$, we have $v - u \in {}^{\rho}\mathcal{GC}_0^{\infty}$, and properties (i), (ii) yield $\psi'(0) = \delta I(u; v-u) = 0$ and $\psi''(s) = \delta^2 I(u + s(v-u); v-u) \geq 0$ for all $s \in B_1(0)$. We claim that $s = 0$ is a minimum of ψ. In fact, for all $s \in B_1(0)$, by Taylor's Theorem 2.11, we have

$$\psi(s) = \psi(0) + s\psi'(0) + \frac{s^2}{2}\psi''(\xi)$$

for some $\xi \in [0,s]$. But $\psi'(0) = 0$, and hence $\psi(s) - \psi(0) = \frac{s^2}{2}\psi''(\xi) \geq 0$. Finally, Lemma 3.2 yields

$$\lim_{s\to 1^-} \psi(s) = I(v) \geq \psi(0) = I(u),$$

which is our conclusion. Note explicitly that if $\delta^2 I(v;\eta) = 0$ for all $\eta \in {}^{\rho}\mathcal{GC}_0^{\infty}$ and all $v \in B_r^m(u) \cap \mathcal{GC}^{\infty}_{\mathrm{bd}}(p,q)$, then $\psi''(\xi) = 0$ and hence $I(v) = I(u)$.

Now, assume that $\delta^2 I(v;\eta) > 0$ for all $\eta \in {}^{\rho}\mathcal{GC}_0^{\infty}$ such that $\|\eta\|_m > 0$ and all $v \in B_{2r}^m(u) \cap \mathcal{GC}^{\infty}_{\mathrm{bd}}(p,q)$, and take $v \in B_r^m(u) \cap \mathcal{GC}^{\infty}_{\mathrm{bd}}(p,q)$ such that $\|v-u\|_m > 0$. As above, set $\psi(s) := I(u + s(v-u)) \in {}^{\rho}\widetilde{\mathbb{R}}$ for all $s \in B_{3/2}(0)$, so that $u + s(v-u) \in B_{2r}^m(u)$. We have $\psi'(0) = 0$ and $\psi''(s) = \delta^2 I(u + s(v-u); v-u) > 0$ for all $s \in B_{3/2}(0)$ because $\|v-u\|_m > 0$. Using Taylor's theorem, we get $\psi(1) = \psi(0) + \frac{1}{2}\psi''(\xi)$ for some $\xi \in [0,1]$. Therefore, $\psi(1) - \psi(0) = I(v) - I(u) = \frac{1}{2}\psi''(\xi) > 0$. □

Lemma 5.7. *Let $(a_k)_{k\in\mathbb{N}}$, $(b_k)_{k\in\mathbb{N}}$ and $(c_k)_{k\in\mathbb{N}}$ be sequences in ${}^{\rho}\widetilde{\mathbb{R}}_{>0}$. Assume that both $(a_k)_k, (b_k)_k \to 0$ and $\frac{c_k}{a_k+b_k} \to 1$ in the sharp topology as $k \to +\infty$. Let $f \in {}^{\rho}\mathcal{GC}^{\infty}([a_1,b_1], {}^{\rho}\widetilde{\mathbb{R}})$. Finally, let $a_k < t < b_k$ for all $k \in \mathbb{N}$. Then*

$$f(t) = \lim_{k\to\infty} \frac{1}{c_k} \int_{t-a_k}^{t+b_k} f(s)\,\mathrm{d}s.$$

Proof. We can apply the integral mean value theorem for each ε and each defining net (f_ε) of f to get the existence of $\tau_k \in [t-a_k, t+b_k]$ such that

$$f(\tau_k) = \frac{1}{b_k + a_k} \int_{t-a_k}^{t+b_k} f(s)\,\mathrm{d}s = \frac{c_k}{b_k + a_k} \frac{1}{c_k} \int_{t-a_k}^{t+b_k} f(s)\,\mathrm{d}s.$$

Now, we take the limit for $k \to \infty$, and the claim follows by assumption and Theorem 2.8 (iv), i.e., by the sharp continuity of f. □

We now derive the so-called necessary Legendre condition.

Theorem 5.8. *Let $a, b \in {}^{\rho}\widetilde{\mathbb{R}}$, with $a < b$, and let $u \in {}^{\rho}\mathcal{GC}^{\infty}([a,b], {}^{\rho}\widetilde{\mathbb{R}}^d)$ be a minimizer of the functional I. Then $F_{\dot{u}\dot{u}}(t, u(t), \dot{u}(t))$ is positive semi-definite for all $t \in [a,b]$, i.e.,*

$$F_{\dot{u}^i\dot{u}^j}(t, u(t), \dot{u}(t))\lambda^i\lambda^j \geq 0 \quad \text{for all } \lambda = (\lambda^1, \dots, \lambda^d) \in {}^{\rho}\widetilde{\mathbb{R}}^d. \tag{5.2}$$

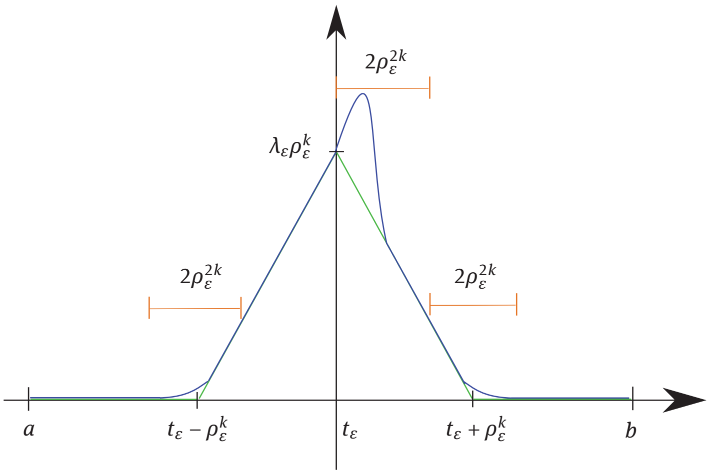

Figure 1. This figure illustrates the function ϑ_ε we are considering (blue). The dotted green triangle symbolizes the function which is used in the classical proofs of the Legendre necessary condition (cf. [23, Theorem 1.3.2]).

Proof. Let $\lambda = [\lambda_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^d$ and let $k, h \in \mathbb{N}$ be arbitrary. Let $t = [t_\varepsilon] \in [a, b]$. We can assume that t is a sharply interior point, because otherwise we can use sharp continuity of the left-hand side of (5.2) and Lemma 3.2. We can also assume that λ is componentwise invertible because of Lemma 3.5. We want to mimic the classical proof of [23, Theorem 1.3.2], but considering a "regularized" version of the triangular function used there (see Figure 1). In particular: (1) The smoothed triangle must have an infinitesimal height which is proportional to λ, and we will take ρ_ε^k as this infinitesimal. (2) In the proof we need that the derivative at t is equal to λ, and this justifies the drawing of the peak in Figure 1. (3) To regularize the singular points of the triangular function, we need a smaller infinitesimal, and we can take, e.g., ρ_ε^{2k}. So, consider a net of smooth functions ϑ_ε on $[a_\varepsilon, b_\varepsilon]$ such that the following properties hold:
(i) $\vartheta_\varepsilon(x) = 0$ for $x \le t_\varepsilon - \rho_\varepsilon^k - \rho_\varepsilon^{2k}$.
(ii) $\vartheta_\varepsilon(x) = 0$ for $x \ge t_\varepsilon + \rho_\varepsilon^k + \rho_\varepsilon^{2k}$.
(iii) $\vartheta_\varepsilon(x) = \lambda(x - t_\varepsilon) + \rho_\varepsilon^k\lambda$ for $x \in [t_\varepsilon - \rho_\varepsilon^k + \rho_\varepsilon^{2k}, t_\varepsilon]$.
(iv) $\vartheta_\varepsilon(x) = -\lambda(x - t_\varepsilon) + \rho_\varepsilon^k\lambda$ for $x \in [t_\varepsilon + \rho_\varepsilon^{2k}, t_\varepsilon + \rho_\varepsilon^k - \rho_\varepsilon^{2k}]$.
(v) $|\vartheta_\varepsilon(x)| \le \rho_\varepsilon^k \cdot |\lambda| + 2\rho_\varepsilon^{2k}|\lambda|$.
(vi) $|\dot\vartheta_\varepsilon(x)| \le 2|\lambda|$ for all x.

The net (ϑ_ε) defines a GSF $\vartheta := [\vartheta_\varepsilon(-)] \in {}^\rho\mathcal{G}\mathcal{C}_0^\infty$ because t is a sharply interior point. Setting for simplicity $a_k := \mathrm{d}\rho^k + \mathrm{d}\rho^{2k}$, by assumption, we have

$$0 \le \delta^2 I(u, \vartheta) = \int_{t-a_k}^{t+a_k} F_{uu}(t, u, \dot u)\vartheta\vartheta + 2F_{\dot u u}(t, u, \dot u)\dot\vartheta\vartheta + F_{\dot u\dot u}(t, u, \dot u)\dot\vartheta\dot\vartheta \,\mathrm{d}t. \tag{5.3}$$

Now, setting $M := \max_{[a,b]}|F_{uu}(t, u, \dot u)|$ and $N := \max_{[a,b]}|F_{u\dot u}(t, u, \dot u)|$, by (v), we have

$$\left|\int_{t-a_k}^{t+a_k} F_{uu}(t, u, \dot u)\vartheta\vartheta \,\mathrm{d}t\right| \le M \cdot |\vartheta(t)|^2 \cdot 2a_k = O(\mathrm{d}\rho^{3k}),$$

where we used the evident notation $G_k = O(\mathrm{d}\rho^k)$ to denote that there exists some $A \in {}^\rho\widetilde{\mathbb{R}}_{>0}$ such that $G_k \le A \cdot \mathrm{d}\rho^k$ for all $k \in \mathbb{N}$. Using (v) and (vi), we analogously have

$$\left|\int_{t-a_k}^{t+a_k} F_{\dot u u}(t, u, \dot u)\dot\vartheta\vartheta \,\mathrm{d}t\right| \le 4N \cdot |\vartheta(t)| \cdot a_k \cdot |\lambda| = O(\mathrm{d}\rho^{2k}).$$

Note that there always exists $C \in {}^\rho\widetilde{\mathbb{R}}$ such that $|\lambda| \le C\,\mathrm{d}\rho^k$. Therefore,

$$\lim_{k\to+\infty} \frac{1}{2\mathrm{d}\rho^k} \int_{t-a_k}^{t+a_k} F_{uu}(t,u,\dot{u})\vartheta\vartheta + 2F_{\dot{u}u}(t,u,\dot{u})\dot{\vartheta}\vartheta\,\mathrm{d}t = 0. \tag{5.4}$$

Using Lemmas 5.7 and 3.2, (5.4) and (5.3), we obtain

$$F_{\dot{u}\dot{u}}(t,u(t)\dot{u}(t))\dot{\vartheta}(t)\dot{\vartheta}(t) = \lim_{k\to+\infty} \frac{1}{2\mathrm{d}\rho^k} \int_{t-a_k}^{t+a_k} F_{\dot{u}\dot{u}}(t,u,\dot{u})\dot{\vartheta}\dot{\vartheta}\,\mathrm{d}t \ge 0.$$

But (iii) yields $\dot{\vartheta}(t) = \lambda$, and this concludes the proof. □

6 Jacobi fields

As in the classical case, Theorem 5.5 (ii) motivates us to define the *accessory* integral

$$Q(\eta) := \int_a^b \psi(t,\eta,\dot{\eta})\,\mathrm{d}t \quad \text{for all } \eta \in {}^\rho\mathcal{GC}_0^\infty,$$

where

$$\psi(t,l,v) := F_{uu}(t,u,\dot{u})ll + 2F_{u\dot{u}}(t,u,\dot{u})lv + F_{\dot{u}\dot{u}}(t,u,\dot{u})vv \tag{6.1}$$

for all $t \in [a,b]$ and $(l,v) \in {}^\rho\widetilde{\mathbb{R}}^d \times {}^\rho\widetilde{\mathbb{R}}^d$. Note that if u minimizes I, then

$$Q(\eta) \ge 0 \quad \text{for all } \eta \in {}^\rho\mathcal{GC}_0^\infty.$$

As usual, we note that $\eta = 0$ is a minimizer of the functional Q, and we are interested to know if there are others. In order to solve this problem, we consider the *Euler–Lagrange equations* for Q, which are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_{\dot{\eta}}(t,\eta,\dot{\eta}) = \psi_\eta(t,\eta,\dot{\eta}).$$

In other words,

$$\frac{\mathrm{d}}{\mathrm{d}t}\{F_{\dot{u}\dot{u}}(t,u,\dot{u})\dot{\eta} + F_{u\dot{u}}(t,u,\dot{u})\eta\} = F_{u\dot{u}}(t,u,\dot{u})\dot{\eta} + F_{uu}(t,u,\dot{u})\eta. \tag{6.2}$$

Since u is given, (6.2) is an ${}^\rho\widetilde{\mathbb{R}}$-linear system of second order equations in the unknown GSF η and with time dependent coefficients in ${}^\rho\widetilde{\mathbb{R}}$. We call (6.2) the *Jacobi equations I with respect to u*. As in the classical setting, we introduce the following definition.

Definition 6.1. A solution $\eta \in {}^\rho\mathcal{GC}_0^\infty$ of the Jacobi equations (6.2) is called a *Jacobi field along u*.

The following result confirms that the intuitive interpretation of a Jacobi field as the tangent space of a smooth family of solutions of the Euler–Lagrange equation still holds in this generalized setting.

Lemma 6.2. *Let $u \in {}^\rho\mathcal{GC}^\infty([-\delta,\delta]\times[a,b], {}^\rho\widetilde{\mathbb{R}}^d)$, where $\delta \in {}^\rho\widetilde{\mathbb{R}}_{>0}$. We write $u_s := u(s,-)$ for all $s \in [-\delta,\delta]$. Assume that each u_s satisfies the Euler–Lagrange equations* (4.4)*:*

$$\frac{\mathrm{d}}{\mathrm{d}t}F_{\dot{u}}(t,u_s,\dot{u}_s) = F_u(t,u_s,\dot{u}_s) \quad \textit{for all } s \in [-\delta,\delta].$$

Then

$$\eta(t) := \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 u_s(t) \quad \textit{for all } t \in [a,b]$$

is a Jacobi field along u.

Proof. A straight forward calculation gives

$$\begin{aligned} 0 &= \left.\frac{\mathrm{d}}{\mathrm{d}s}\right|_0 \left(\frac{\mathrm{d}}{\mathrm{d}t}F_{\dot{u}}(t,u_s,\dot{u}_s) - F_u(t,u_s,\dot{u}_s)\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}(F_{\dot{u}\dot{u}}(t,u,\dot{u})\dot{\eta} + F_{u\dot{u}}(t,u,\dot{u})\eta) - F_{u\dot{u}}(t,u,\dot{u})\dot{\eta} - F_{uu}(t,u,\dot{u})\eta. \end{aligned}$$

□

6.1 Conjugate points and Jacobi's theorem

The classical key result concerning Jacobi fields relates conjugate points and minimizers. The main aim of the present section is to derive this theorem in our generalized framework, by extending the ideas of the proof of [23, Theorem 1.3.4].

A crucial notion is hence that of piecewise GSF.

Definition 6.3. We call *piecewise GSF* an n-tuple $(f_1, \dots, f_n)$ with the following properties:
(i) For all $i = 1, \dots, n$ there exist $a_i, a_{i+1} \in {}^\rho\widetilde{\mathbb{R}}$ such that $a_i < a_{i+1}$ and $f_i \in {}^\rho\mathcal{GC}^\infty([a_i, a_{i+1}], {}^\rho\widetilde{\mathbb{R}}^d)$. Note that $[a, b] = [a', b']$ implies $a = a'$ and $b = b'$ because the relation $\le$ is antisymmetric. Therefore, the points a_i, a_{i+1} are uniquely determined by the set-theoretical function f_i.
(ii) For all $i = 1, \dots, n$, we have $f_i(a_{i+1}) = f_{i+1}(a_{i+1})$.
Every pointwise GSF $(f_1, \dots, f_n)$ defines a set-theoretical function:
(iii) For all $t \in \bigcup_{i=1}^n [a_i, a_{i+1}]$, we set $(f_1, \dots, f_n)(t) := f_i(t)$ if $t \in [a_i, a_{i+1}]$.
We also use the arrow notation $(f_1, \dots, f_n) \colon \bigcup_{i=1}^n [a_i, a_{i+1}] \to {}^\rho\widetilde{\mathbb{R}}^d$ to say that both (i) and (ii) hold.

Remark 6.4. (i) Clearly, $t \in [a_i, a_{i+1}] \cap [a_{i+1}, a_{i+2}]$ implies $t = a_{i+1}$, so that condition (ii) yields that the evaluation (iii) is well defined.
(ii) Since the order relation $\le$ is not a total one, we do not have that $[a_i, a_{i+1}] \cup [a_{i+1}, a_{i+2}] = [a_i, a_{i+2}]$.
(iii) If $v \colon [a_1, a_2] \cup [a_2, a_3] \to {}^\rho\widetilde{\mathbb{R}}^d$ is a set-theoretical function originating from a piecewise GSF (f_1, f_2), then neither the GSF f_i nor the points a_i are uniquely determined by v. For this reason, we prefer to stress our notations with symbols like $(f_1, f_2)(t) \in {}^\rho\widetilde{\mathbb{R}}^d$.
(iv) Every GSF $f \in {}^\rho\mathcal{GC}^\infty([a_1, a_2], {}^\rho\widetilde{\mathbb{R}}^d)$ can be seen as a particular case of a piecewise GSF.
(v) If $(g_1, \dots, g_n), (f_1, \dots, f_n) \colon \bigcup_{i=1}^n [a_i, a_{i+1}] \to {}^\rho\widetilde{\mathbb{R}}^d$ and $r \in {}^\rho\widetilde{\mathbb{R}}$, then also $(g_1, \dots, g_n) + (f_1, \dots, f_n) := (g_1 + f_1, \dots, g_n + f_n)$ and $r \cdot (f_1, \dots, f_n) := (r \cdot f_1, \dots, r \cdot f_n)$ are piecewise GSF, and we hence have a structure of an ${}^\rho\widetilde{\mathbb{R}}$-module.
(vi) If $(f_1, \dots, f_n) \colon \bigcup_{i=1}^n [a_i, a_{i+1}] \to {}^\rho\widetilde{\mathbb{R}}^d$ and $F \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}^d, {}^\rho\widetilde{\mathbb{R}}^n)$, then we can define the composition

$$F \circ (f_1, \dots, f_n) := (F \circ f_1, \dots, F \circ f_n) \colon \bigcup_{i=1}^n [a_i, a_{i+1}] \to {}^\rho\widetilde{\mathbb{R}}^n.$$

Piecewise GSF inherit from their defining components a well-behaved differential and integral calculus. The former is even more general and taken from [1].

Definition 6.5. Let $x = [x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}$. Then we set
(i) $v(x) := \sup\{b \in \mathbb{R} \mid |x_\varepsilon| = O(\rho_\varepsilon^b)\} \in \mathbb{R} \cup \{+\infty\}$,
(ii) $|x|_e := e^{-v(x)} \in \mathbb{R}_{\ge 0}$,
(iii) $\mathrm{d}\rho(x) := \mathrm{d}\rho^{-\log|x|_e} \in {}^\rho\widetilde{\mathbb{R}}_{>0}$.

It is worth noting that $|-|_e \colon {}^\rho\widetilde{\mathbb{R}} \to \mathbb{R}_{\ge 0}$ induces an ultrametric on ${}^\rho\widetilde{\mathbb{R}}$ that generates exactly the sharp topology, see, e.g., [2, 11] and references therein. However, we will not use this ultrametric structure in the present paper, and we only introduced it to get an invertible infinitesimal $\mathrm{d}\rho(x)$ that goes to zero with x. It is in fact easy to show that

$$\lim_{x \to 0} \frac{x}{\mathrm{d}\rho(x)} = 1$$

in the sharp topology. The following definition is based on [1, Definition 2.2].

Definition 6.6. Let $T \subseteq {}^\rho\widetilde{\mathbb{R}}$ and let $f \colon T \to {}^\rho\widetilde{\mathbb{R}}^d$ be an arbitrary set-theoretical function. Let $t_0 \in T$ be a sharply interior point of T. Then we say that f *is differentiable at* t_0 if

$$\exists m \in {}^\rho\widetilde{\mathbb{R}}^d : \lim_{h \to 0} \frac{f(t + h) - f(t_0) - m \cdot h}{\mathrm{d}\rho(h)} = 0.$$

In this case, using Landau little-oh notation, we can hence write

$$f(t + h) = f(t_0) + m \cdot h + o(\mathrm{d}\rho(h)) \quad \text{as } h \to 0. \tag{6.3}$$

As in the classical case, (6.3) implies the uniqueness of $m \in {}^\rho\widetilde{\mathbb{R}}^d$, so that we can define $f'(t_0) := \dot{f}(t_0) := m$, and the usual elementary rules of differential calculus. By the Fermat–Reyes theorem, this definition of derivative generalizes that given for GSF.

In particular, this notion of derivative applies to the set-theoretical function induced by a piecewise GSF $(f_1, \dots, f_n)$. We therefore have that $(f_1, \dots, f_n)(-)$ is differentiable at each $a_i < t < a_{i+1}$, and that $(f_1, \dots, f_n)'(t) = f_i'(t)$, but clearly there is no guarantee that $(f_1, \dots, f_n)(-)$ is also differentiable at each point a_i.

The notion of definite integral is naturally introduced in the following definition.

Definition 6.7. Let $(f_1, \dots, f_n)\colon \bigcup_{i=1}^n [a_i, a_{i+1}] \to {}^\rho\widetilde{\mathbb{R}}^d$ be a piecewise GSF. Then

$$\int_{a_1}^{a_{n+1}} (f_1, \dots, f_n)(t)\,\mathrm{d}t := \sum_{i=1}^{n} \int_{a_i}^{a_{i+1}} f_i(t)\,\mathrm{d}t.$$

Since our main aim in using piecewise GSF is to prove Jacobi's theorem, we do not need to prove that the usual elementary rules of integration hold, since we will always reduce to integrals of GSF.

Having a notion of derivative and of definite integral also for piecewise GSF allows to study functionals of the form

$$v := (f_1, \dots, f_n),\ a_1 = a,\ a_n = b \ \Rightarrow\ I(v) := \int_a^b F(t, v(t), \dot{v}(t))\,\mathrm{d}t \in {}^\rho\widetilde{\mathbb{R}}. \tag{6.4}$$

This leads to the following natural definition: we say that a piecewise GSF v is a piecewise GSF (global) minimizer if $I(v) \le I(\tilde{v})$ for all $\tilde{v} \in {}^\rho\mathcal{GC}_0^\infty$. For the proof of Jacobi's theorem, we will only need this particular notion of global minimizer. Note explicitly that in (6.4), we only need the existence of right and left derivatives of GSF, because of Definition 6.7, and of Definition 2.14 of a definite integral of a GSF.

Classically, several proofs of Jacobi's theorem use both some form of implicit function theorem and of uniqueness of solution for linear ODE.

Theorem 6.8 (Implicit function theorem). *Let $U \subseteq {}^\rho\widetilde{\mathbb{R}}^n$, $V \subseteq {}^\rho\widetilde{\mathbb{R}}^d$ be sharply open sets. Let $F \in {}^\rho\mathcal{GC}^\infty(U \times V, {}^\rho\widetilde{\mathbb{R}}^d)$ and $(x_0, y_0) \in U \times V$. If $\partial_2 F(x_0, y_0)$ is invertible in $L({}^\rho\widetilde{\mathbb{R}}^d, {}^\rho\widetilde{\mathbb{R}}^d)$, then there exists a sharply open neighborhood $U_1 \times V_1 \subseteq U \times V$ of (x_0, y_0) such that*

$$\forall x \in U_1, \exists! y_x \in V_1 : F(x, y_x) = F(x_0, y_0).$$

Moreover, the function $f(x) := y_x$ for all $x \in U_1$ is a GSF $f \in {}^\rho\mathcal{GC}^\infty(U_1, V_1)$ and satisfies

$$Df(x) = -(\partial_2 F(x, f(x)))^{-1} \circ \partial_1 F(x, f(x)).$$

Proof. The usual deduction of the implicit function theorem from the inverse function theorem in Banach spaces can be easily adapted by using Theorem 2.12 and noting that $\det[\partial_2 F(-,-)]$ is a GSF such that $|\det[\partial_2 F(x_0, y_0)]| \in {}^\rho\widetilde{\mathbb{R}}_{>0}$. □

In the next theorem, the dependence of the entire theory on the initial infinitesimal net $\rho = (\rho_\varepsilon) \downarrow 0$ plays an essential role. Indirectly, the same important role will reverberate in the final Jacobi's theorem.

Theorem 6.9 (Solution of first order linear ODE). *Let $A \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^{d\times d})$, where $a, b \in {}^\rho\widetilde{\mathbb{R}}$, $a < b$, and let $t_0 \in [a, b]$ and $y_0 \in {}^\rho\widetilde{\mathbb{R}}^d$. Assume that*

$$\left| \int_{t_0}^{t} A(s)\,\mathrm{d}s \right| \le -C \cdot \log \mathrm{d}\rho \quad \text{for all } t \in [a, b], \tag{6.5}$$

where $C \in \mathbb{R}_{>0}$. Then there exists one and only one $y \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$ such that

$$\begin{cases} y'(t) = A(t) \cdot y(t) & \text{if } t \in [a, b], \\ y(t_0) = y_0. \end{cases} \tag{6.6}$$

Moreover, this y is given by $y(t) = \exp(\int_{t_0}^t A(s)\,\mathrm{d}s) \cdot y_0$ for all $t \in [a, b]$.

Proof. We first note that

$$\exp\left(\int_{t_0}^{t} A(s)\,\mathrm{d}s\right) = \left[\exp\left(\int_{t_{0\varepsilon}}^{t_\varepsilon} A_\varepsilon(s)\,\mathrm{d}s\right)\right],$$

where $t = [t_\varepsilon]$, $t_0 = [t_{0\varepsilon}]$ and $A(s) = [A_\varepsilon(s_\varepsilon)] \in {}^\rho\widetilde{\mathbb{R}}^{d\times d}$. This exponential matrix in ${}^\rho\widetilde{\mathbb{R}}^{d\times d}$ is a GSF because for all $t \in [a, b]$, we have

$$\exp\left(\int_{t_0}^{t} A(s)\,\mathrm{d}s\right) \le e^{-C\log \mathrm{d}\rho} \le \mathrm{d}\rho^{-C}.$$

Therefore, all values of $y(t) = \exp(\int_{t_0}^{t} A(s)\,\mathrm{d}s)\cdot y_0$ are ρ-moderate. Analogously, one can prove that also $y^{(k)}(t)$ are moderate for all $k \in \mathbb{N}$ and $t \in [a, b]$. Considering that derivatives can be calculated ε-wise, we have that this GSF y satisfies (6.6), and this proves the existence part.

To show uniqueness, we can proceed as in the smooth case. Assume that $z \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$ satisfies (6.6), and set $h(t) := \exp(-\int_{t_0}^{t} A(s)\,\mathrm{d}s)$ for all $t \in [a, b]$. Since $h' = -A\cdot h$, we have

$$(hz)' = h'z + hz' = -Ahz + hAz = -Ahz + Ahz = 0.$$

From the uniqueness of primitives of GSF and Theorem 2.13, we have that $h\cdot z = h(t_0)\cdot z(t_0) = y_0$. Therefore, $z = h^{-1}\cdot y_0$. □

If $\alpha, \beta \in {}^\rho\widetilde{\mathbb{R}}$, we write $\alpha = O_{\mathbb{R}}(\beta)$ to denote that there exists $C \in \mathbb{R}_{>0}$ such that $|\alpha| \le C\cdot|\beta|$. Therefore, assumption (6.5) can be written as $\int_{t_0}^{t} A(s)\,\mathrm{d}s = O_{\mathbb{R}}(\log \mathrm{d}\rho)$. Note that this assumption is weaker, in general, than

$$(b-a)\cdot \max_{t\in[a,b]} |A(t)| = O_{\mathbb{R}}(\log \mathrm{d}\rho).$$

The following result is the key regularity property that is needed to prove Jacobi's theorem.

Lemma 6.10. *Let $a, a', b \in {}^\rho\widetilde{\mathbb{R}}$, with $a < a' < b$, and let $K \in {}^\rho\mathcal{GC}^\infty([a, b]\times{}^\rho\widetilde{\mathbb{R}}^d\times{}^\rho\widetilde{\mathbb{R}}^d, {}^\rho\widetilde{\mathbb{R}})$. Let, in addition, $v = (\eta, \beta)\colon [a, a']\cup[a', b] \to {}^\rho\widetilde{\mathbb{R}}^d$ be a piecewise GSF which satisfies the Euler–Lagrange equation*

$$K_u(t, v(t), \dot v(t)) - \frac{\mathrm{d}}{\mathrm{d}t}K_{\dot u}(t, v(t), \dot v(t)) = 0 \quad \text{for all } t \in [a, a')\cup(a', b]. \tag{6.7}$$

Finally, assume that $\det(K_{\dot u_i\dot u_j}(a', \eta(a'), \dot\eta(a'))_{i,j=i,\dots,d}) \in {}^\rho\widetilde{\mathbb{R}}$ *is invertible. Then*

$$\lim_{\substack{t\to a'\\ t<a'}} \dot v(t) = \lim_{\substack{t\to a'\\ a'<t}} \dot v(t) = \dot\eta(a').$$

In particular, if $\beta \equiv 0|_{[a',b]}$, then $\dot\eta(a') = 0$.

Proof. Set $\Phi(t, l, v, q) := K_{\dot u}(t, l, v) - q$ for all $t \in [a, b]$ and all $l, v, q \in {}^\rho\widetilde{\mathbb{R}}^d$. For simplicity, set

$$(t_0, l_0, v_0, q_0) := (a', \eta(a'), \dot\eta(a'), K_{\dot u}(a', \eta(a'), \dot\eta(a'))).$$

Our assumption on the invertibility of $K_{\dot u\dot u}(a', \eta(a'), \dot\eta(a')) = \partial_v\Phi(t_0, l_0, v_0, q_0)$ makes it possible to apply the implicit function Theorem 6.8 to conclude that there exists a neighborhood $T\times L\times V\times Q$ of (t_0, l_0, v_0, q_0) such that

$$\forall (t, l, q) \in T\times L\times Q, \exists! v \in V : \Phi(t, l, v, q) = \Phi(t_0, l_0, v_0, q_0). \tag{6.8}$$

But we have $\Phi(t_0, l_0, v_0, q_0) = K_{\dot u}(a', \eta(a'), \dot\eta(a')) - q_0 = 0$. Moreover, the unique function ϕ defined by $\Phi(t, l, \phi(t, l, q), q) = 0$ for all $(t, l, q) \in T\times L\times Q$ is a GSF that belongs to ${}^\rho\mathcal{GC}^\infty(T\times L\times Q, V)$. Now, for all $t \in [a, a')\cup(a', b]$, we have

$$\Phi(t, v(t), \dot v(t), K_{\dot u}(t, v(t), \dot v(t))) = K_{\dot u}(t, v(t), \dot v(t)) - K_{\dot u}(t, v(t), \dot v(t)) = 0.$$

Therefore, the uniqueness in (6.8) yields

$$\dot v(t) = \phi(t, v(t), K_{\dot u}(t, v(t), \dot v(t))) \quad \text{for all } t \in [a, a')\cup(a', b].$$

We now integrate the Euler–Lagrange equation (6.7) on $[a, t]$, obtaining

$$K_{\dot{u}}(t, v(t), \dot{v}(t)) = \int_a^t K_u(s, v(s), \dot{v}(s))\,\mathrm{d}s + K_{\dot{u}}(a, \eta(a), \dot{\eta}(a)) \quad \text{for all } t \in [a, a') \cup (a', b].$$

This entails that we can write

$$\dot{v}(t) = \phi\Big(t, v(t), \int_a^t K_u(s, v(s), \dot{v}(s))\,\mathrm{d}s + K_{\dot{u}}(a, \eta(a), \dot{\eta}(a))\Big) \quad \text{for all } t \in [a, a') \cup (a', b]. \tag{6.9}$$

But the function $t \in [a, a') \cup (a', b] \mapsto \int_a^t K_u(s, v(s), \dot{v}(s))\,\mathrm{d}s \in {}^\rho\widetilde{\mathbb{R}}^d$ has equal limits on the left and on the right of a' because on $[a, a')$ and on $(a', b]$ it is a GSF. In fact, for $t < a'$, we have

$$\left|\int_a^t K_u(s, v(s), \dot{v}(s))\,\mathrm{d}s - \int_a^{a'} K_u(s, v(s), \dot{v}(s))\,\mathrm{d}s\right| \le \max_{t\in[a,a']} |K_u(s, \eta(s), \dot{\eta}(s))| \cdot |t - a'|,$$

and this goes to 0 as $t \to a'$, $t < a'$. Analogously, we can proceed for $t > a'$ using β. Therefore,

$$\lim_{\substack{t\to a'\\ t<a'}} \int_a^t K_u(s, v(s), \dot{v}(s))\,\mathrm{d}s = \lim_{\substack{t\to a'\\ t>a'}} \int_a^t K_u(s, v(s), \dot{v}(s))\,\mathrm{d}s.$$

Applying this equality in (6.9), we get $\lim_{t\to a', t<a'} \dot{v}(t) = \dot{\eta}(a') = \lim_{t\to a', a'<t} \dot{v}(t)$, as claimed. Finally, if $\beta \equiv 0|_{[a',b]}$, then $\lim_{t\to a', a'<t} \dot{v}(t) = 0$. □

In the following definition and below, we use the complete notation ${}^\rho\mathcal{GC}_0^\infty(a, a')$ (see Definition 4.1).

Definition 6.11. Let $a, a', b \in {}^\rho\widetilde{\mathbb{R}}$, where $a < a' < b$. We call a' *conjugate to* a with respect to the variational problem (4.1) if there exists a non-identically vanishing Jacobi field $\eta \in {}^\rho\mathcal{GC}_0^\infty(a, a')$ along $u|_{[a,a']}$ such that $\eta(a) = 0 = \eta(a')$, where ψ is given by (6.1).

Jacobi's theorem shows that we cannot have minimizers if there are interior points conjugate to a. In order to prove it in the present generalized context, we finally need the following lemma.

Lemma 6.12. *Let $u \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}}^d)$ and $a' \in (a, b)$. Let $\eta \in {}^\rho\mathcal{GC}_0^\infty(a, a')$ be a Jacobi field along $u|_{[a,a']}$, with $\eta(a) = 0 = \eta(a')$. Then*

$$\int_a^{a'} \psi(t, \eta, \dot{\eta})\,\mathrm{d}t = 0.$$

Proof. Since ψ is ${}^\rho\widetilde{\mathbb{R}}$-homogeneous of second order in $(\eta, \dot{\eta})$, we have

$$2\psi(t, \eta, \dot{\eta}) = \psi_\eta(t, \eta, \dot{\eta})\eta + \psi_{\dot{\eta}}(t, \eta, \dot{\eta})\dot{\eta}.$$

Thus, by integration by parts, we calculate

$$2\int_a^{a'} \psi(t, \eta, \dot{\eta})\,\mathrm{d}t = \int_a^{a'} \eta\psi_\eta(t, \eta, \dot{\eta}) + \dot{\eta}\psi_{\dot{\eta}}(t, \eta, \dot{\eta})\,\mathrm{d}t = \int_a^{a'} \eta\Big(\psi_\eta(t, \eta, \dot{\eta}) - \frac{\mathrm{d}}{\mathrm{d}t}\psi_{\dot{\eta}}(t, \eta, \dot{\eta})\Big)\,\mathrm{d}t = 0,$$

where we used the fact that η is a Jacobi field. □

After these preparations, we can finally prove Jacobi's theorem.

Theorem 6.13 (Jacobi). *Let $a, b \in {}^\rho\widetilde{\mathbb{R}}$, with $a < b$, $F \in {}^\rho\mathcal{GC}^\infty([a, b] \times {}^\rho\widetilde{\mathbb{R}}^d \times {}^\rho\widetilde{\mathbb{R}}^d, {}^\rho\widetilde{\mathbb{R}})$ and $u \in {}^\rho\mathcal{GC}^\infty([a, b], {}^\rho\widetilde{\mathbb{R}})$. Assume that the following hold:*
(i) *$a' \in (a, b)$ is conjugate to a.*
(ii) *$\det F_{\dot{u}\dot{u}}(t, u(t), \dot{u}(t)) \in {}^\rho\widetilde{\mathbb{R}}$ is invertible for all $t \in [a, b]$.*

(iii) *For all* $t \in [a, a']$,

$$\int_{a'}^{t} F_{\dot u\dot u}^{-1}(s, u(s), \dot u(s)) \cdot \left[\frac{\mathrm{d}}{\mathrm{d}s} F_{u\dot u}(s, u(s), \dot u(s)) - F_{uu}(s, u(s), \dot u(s))\right] \mathrm{d}s = O_{\mathbb{R}}(\log \mathrm{d}\rho),$$

$$\int_{a'}^{t} F_{\dot u\dot u}^{-1}(s, u(s), \dot u(s)) \cdot \frac{\mathrm{d}}{\mathrm{d}s} F_{\dot u\dot u}(s, u(s), \dot u(s))\, \mathrm{d}s = O_{\mathbb{R}}(\log \mathrm{d}\rho).$$

Then u *cannot be a local minimizer of* I. *Therefore, for any* $r \in {}^\rho\widetilde{\mathbb{R}}_{>0}$, *there exists* $v \in {}^\rho\mathcal{GC}^\infty_{\mathrm{bd}}(u(a), u(b))$ *and* $m \in \mathbb{N}$ *such that* $\|v - u\|_m < r$ *but* $I(u) \not\leq I(v)$.

Proof. By contradiction, assume that u is a local minimizer, and let $\eta \in {}^\rho\mathcal{GC}^\infty_0(a, a')$ be a Jacobi field along $u|_{[a,a']}$ such that the conditions from Definition 6.11 hold for η. We want to prove that $\eta \equiv 0$. Define $v := (\eta, 0|_{[a',b]})$, which is a piecewise GSF since $\eta(a') = 0$. Since also $\eta(a) = 0$, Lemma 6.12 and the homogeneity of ψ yield

$$Q(v) = \int_a^b \psi(t, v(t), \dot v(t))\, \mathrm{d}t = \int_a^{a'} \psi(t, \eta(t), \dot\eta(t))\, \mathrm{d}t + \int_{a'}^{b} \psi(t, 0, 0)\, \mathrm{d}t = 0.$$

Thus, Theorem 5.5 (necessary condition for u being a minimizer) gives $Q(\tilde v) \geq 0 = Q(v)$ for all $\tilde v \in {}^\rho\mathcal{GC}^\infty_0(a, b)$. Therefore, v is a minimizer of the functional Q. Since v is only a piecewise GSF, we cannot directly apply Theorem 4.5 (Euler–Lagrange equations). But, for all $\phi \in {}^\rho\mathcal{GC}^\infty_0(a, b)$ and all $s \in {}^\rho\widetilde{\mathbb{R}}$, we have

$$Q(v + s\phi) = \int_a^b \psi(t, v + s\phi, \dot v + s\dot\phi)\, \mathrm{d}t = \int_a^{a'} \psi(t, \eta + s\phi, \dot\eta + s\dot\phi)\, \mathrm{d}t + \int_{a'}^{b} \psi(t, s\phi, s\dot\phi)\, \mathrm{d}t. \tag{6.10}$$

This shows that $s \in {}^\rho\widetilde{\mathbb{R}} \mapsto Q(v + s\phi) \in {}^\rho\widetilde{\mathbb{R}}$ is a GSF, and hence $s = 0$ is a minimum for this function. By Lemma 3.6 and (6.10), we get

$$\begin{aligned}
\delta Q(v, \phi) = 0 &= \frac{\mathrm{d}}{\mathrm{d}s} Q(v + s\phi)\Big|_0 \\
&= \int_a^{a'} \left(\psi_\eta(t, \eta, \dot\eta) - \frac{\mathrm{d}}{\mathrm{d}t}\psi_{\dot\eta}(t, \eta, \dot\eta)\right)\phi\, \mathrm{d}t + \int_{a'}^{b} (\phi\psi_\eta(t, 0, 0) + \dot\phi\psi_{\dot\eta}(t, 0, 0))\, \mathrm{d}t \\
&= \int_a^{a'} \left(\psi_\eta(t, \eta, \dot\eta) - \frac{\mathrm{d}}{\mathrm{d}t}\psi_{\dot\eta}(t, \eta, \dot\eta)\right)\phi\, \mathrm{d}t.
\end{aligned}$$

By the fundamental Lemma 4.4, this implies that η satisfies the Euler–Lagrange equations for ψ in the interval $[a, a')$. Therefore, v satisfies the same equations in $[a, a') \cup (a', b]$. Moreover, $\psi_{\dot\eta\dot\eta}(a', \eta(a'), \dot\eta(a')) = F_{\dot u\dot u}(a', u(a'), \dot u(a'))$ is invertible by assumption (ii). Thus, all the hypotheses of the regularity Lemma 6.10 hold, and we derive that $\dot\eta(a') = 0$.

For all $t \in [a, b]$, we define

$$\begin{aligned}
\xi(t) &:= -F_{\dot u\dot u}^{-1} \cdot \left[\frac{\mathrm{d}}{\mathrm{d}t} F_{u\dot u}(t, u, \dot u) - F_{uu}(t, u, \dot u)\right], \\
\vartheta(t) &:= -F_{\dot u\dot u}^{-1} \cdot \frac{\mathrm{d}}{\mathrm{d}t} F_{\dot u\dot u}(t, u, \dot u),
\end{aligned}$$

so that we can re-write the Jacobi equations (6.2) for η on $[a, a']$ as a system of first order ODE:

$$\begin{cases}
\dot y := \begin{pmatrix} \dot\eta \\ \dot z \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \xi & \vartheta \end{pmatrix} \cdot \begin{pmatrix} \eta \\ z \end{pmatrix} =: A \cdot y \quad \text{for all } t \in [a, a'], \\
y(a') = \begin{pmatrix} \eta(a') \\ \dot\eta(a') \end{pmatrix} = 0.
\end{cases}$$

By assumptions (iii), we obtain $\int_{a'}^{t} A(t) = O_{\mathbb{R}}(\log \mathrm{d}\rho)$ for all $t \in [a, a']$, and we can hence apply Theorem 6.9, obtaining $y \equiv 0$, and thus $\eta \equiv 0$. □

Note that if one of the quantities in (iii) depends even only polynomially on ε, then we are forced to take, e.g., $\rho_\varepsilon = \varepsilon^{1/\varepsilon}$ to fulfill this assumption. This underlines the importance of the parameter ρ, making the entire theory dependent on the parameter ρ, in order to avoid unnecessary constraints on the scope of the functionals we look upon.

7 Noether's theorem

In this section, we state and prove Noether's theorem by following the lines of [3]. We first note that any $X \in {}^\rho\mathcal{GC}^\infty(J \times X, Y)$, where $J \subseteq {}^\rho\widetilde{\mathbb{R}}$, can also be considered as a family in GSF which smoothly depends on the parameter $s \in J$. In this case, we hence say that $(X_s)_{s\in J}$ *is a generalized smooth family in* ${}^\rho\mathcal{GC}^\infty(X, Y)$. In particular, we can reformulate in the language of GSF the classical definition of *one-parameter group of generalized diffeomorphisms of* X as follows:
(i) $(X_s)_{s\in{}^\rho\widetilde{\mathbb{R}}}$ is a generalized smooth family in ${}^\rho\mathcal{GC}^\infty(X, X)$.
(ii) For all $s \in {}^\rho\widetilde{\mathbb{R}}$, the map $X_s : X \to X$ is invertible, and $X_s^{-1} \in {}^\rho\mathcal{GC}^\infty(X, X)$.
(iii) $X_0(x) = x$ for all $x \in X$.
(iv) $X_s \circ X_t = X_{s+t}$ for all $s, t \in {}^\rho\widetilde{\mathbb{R}}$.
In our proofs, we will in fact only use properties (i) and (iii).

The proof of Noether's theorem is classically anticipated by the following time-independent version, which the general case is subsequently reduced to.

Theorem 7.1. *Let $K \in {}^\rho\mathcal{GC}^\infty(L \times V, {}^\rho\widetilde{\mathbb{R}})$, where $L, V \subseteq {}^\rho\widetilde{\mathbb{R}}^n$ are sharply open sets. Let $w \in {}^\rho\mathcal{GC}^\infty((a, b), L)$ be a solution of the Euler–Lagrange equation corresponding to K, i.e., for all $t \in (a, b)$,*

$$\dot{w}(t) \in V, \quad K_u(w(t), \dot{w}(t)) = \frac{\mathrm{d}}{\mathrm{d}t} K_{\dot{u}}(w(t), \dot{w}(t)). \tag{7.1}$$

Suppose that 0 is a sharply interior point of $J \subseteq {}^\rho\widetilde{\mathbb{R}}$ and $(X_s)_{s\in J}$ is a generalized smooth family in ${}^\rho\mathcal{GC}^\infty(L, L)$ such that for all $t \in (a, b)$,
(i) $\frac{\partial}{\partial t} X_s(w(t)) \in V$,
(ii) $X_0(w(t)) = w(t)$,
(iii) *K is invariant under $(X_s)_{s\in J}$ along w, i.e.,*

$$K(w(t), \dot{w}(t)) = K\Big(X_s(w(t)), \frac{\partial}{\partial t} X_s(w(t))\Big) \quad \textit{for all } s \in J. \tag{7.2}$$

Then, the quantity

$$K_{\dot{u}^j}(w(t), \dot{w}(t)) \frac{\partial}{\partial s}\Big|_{s=0} X_s^j(w(t)) \in {}^\rho\widetilde{\mathbb{R}}$$

is constant in $t \in (a, b)$.

Proof. We first note that both sides of (7.2) are in ${}^\rho\mathcal{GC}^\infty((a, b), {}^\rho\widetilde{\mathbb{R}})$. Let $\tau \in (a, b)$ be arbitrary but fixed. Since $s = 0 \in J$ is a sharply interior point, we can consider $\frac{\mathrm{d}}{\mathrm{d}s}|_{s=0}$. Then, using (7.2) and (ii), we obtain

$$0 = \frac{\partial}{\partial s}\Big|_{s=0} K\Big(X_s(w), \frac{\partial}{\partial t} X_s(w)\Big) = \int_a^\tau K_u(w, \dot{w}) \frac{\partial}{\partial s}\Big|_{s=0} X_s(w) + K_{\dot{u}}(w, \dot{w}) \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Big|_{s=0} X_s(w)\,\mathrm{d}t.$$

Since the Euler–Lagrange equations (7.1) for K are given by $K_u(w, \dot{w}) = \frac{\mathrm{d}}{\mathrm{d}t} K_{\dot{u}}(w, \dot{w})$, we have

$$0 = \frac{\mathrm{d}}{\mathrm{d}t}(K_{\dot{u}}(w, \dot{w})) \frac{\partial}{\partial s}\Big|_{s=0} X_s(w) + K_{\dot{u}}(w, \dot{w}) \frac{\partial}{\partial t}\frac{\partial}{\partial s}\Big|_{s=0} X_s(w) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(K_{\dot{u}}(w, \dot{w}) \frac{\partial}{\partial s}\Big|_{s=0} X_s(w)\Big),$$

which is our conclusion by the uniqueness part of Theorem 2.13. □

We are now able to prove Noether's theorem. For the convenience of the reader, in its statement and proof we use the variables t, T, l, L, v, V so as to recall *tempus*, *locus*, *velocitas*, respectively.

Theorem 7.2 (Noether). *Let $a, b \in {}^\rho\widetilde{\mathbb{R}}^d$, with $a < b$, and $F \in {}^\rho\mathcal{GC}^\infty([a,b] \times {}^\rho\widetilde{\mathbb{R}}^d \times {}^\rho\widetilde{\mathbb{R}}^d, {}^\rho\widetilde{\mathbb{R}})$. Let, in addition, $u \in {}^\rho\mathcal{GC}^\infty([a,b], {}^\rho\widetilde{\mathbb{R}}^d)$ be a solution of the Euler–Lagrange equation (4.4) corresponding to F. Suppose that 0 is a sharply interior point of $J \subseteq {}^\rho\widetilde{\mathbb{R}}$ and $(X_s)_{s\in J}$ is a generalized smooth family in ${}^\rho\mathcal{GC}^\infty((a,b) \times {}^\rho\widetilde{\mathbb{R}}^d, (a,b) \times {}^\rho\widetilde{\mathbb{R}}^d)$. We denote by $T_s(t,l) := X_s^1(t,l) \in (a,b)$ and $L_s(t,l) := X_s^2(t,l) \in {}^\rho\widetilde{\mathbb{R}}^d$, for all $(t,l) \in (a,b) \times {}^\rho\widetilde{\mathbb{R}}^d$, the two projections of X_s on (a,b) and ${}^\rho\widetilde{\mathbb{R}}^d$, respectively. We assume that for all $t \in (a,b)$,*

(i) $\frac{\partial}{\partial t} T_s(t, u(t)) \in {}^\rho\widetilde{\mathbb{R}}$ *is invertible,*

(ii) $T_0(t, u(t)) = t$ *and* $L_0(t, u(t)) = u(t)$,

(iii) $F(t, u(t), \dot u(t)) = F\left[T_s(t,u), L_s(t,u), \frac{\frac{\partial}{\partial t} L_s(t,u)}{\frac{\partial}{\partial t} T_s(t,u)}\right] \cdot \frac{\partial}{\partial t} T_s(t,u)$ *for all $s \in J$.*

Then, the quantity

$$F_{\dot u^j}(t,u(t),\dot u(t)) \frac{\partial}{\partial s}\bigg|_{s=0} L_s^j(t,u(t)) + \left[F(t,u(t),\dot u(t)) - F_{\dot u^k}(t,u(t),\dot u(t))\dot u^k(t)\right] \frac{\partial}{\partial s}\bigg|_{s=0} T_s(t,u(t)) \tag{7.3}$$

is constant in $t \in [a,b]$.

Proof. Since (7.3) is a GSF in $t \in [a,b]$, by sharp continuity it suffices to prove the claim for all $t \in (a,b)$. Set $L := (a,b) \times {}^\rho\widetilde{\mathbb{R}}^d$, $V := {}^\rho\widetilde{\mathbb{R}}^* \times {}^\rho\widetilde{\mathbb{R}}^d$ (we recall that ${}^\rho\widetilde{\mathbb{R}}^*$ denotes the set of all invertible generalized numbers in ${}^\rho\widetilde{\mathbb{R}}$). Define $K \in {}^\rho\mathcal{GC}^\infty(L \times V, {}^\rho\widetilde{\mathbb{R}})$ by

$$K(t,l;p,v) := F\Big(t, l, \frac{v}{p}\Big) \cdot p \quad \text{for all } (t,l) \in L \text{ and all } (p,v) \in V, \tag{7.4}$$

and $w \in {}^\rho\mathcal{GC}^\infty((a,b), L)$ by $w(t) := (t, u(t))$ for all $t \in (a,b)$. We note that $L, V \subseteq {}^\rho\widetilde{\mathbb{R}}^{d+1}$ are sharply open subsets and that $\dot w(t) = (1, \dot u(t)) \in V$. The notations for partial derivatives used in the present work result from the symbolic writing $K(u^1, \dots, u^{d+1}; \dot u^1, \dots, \dot u^{d+1})$, so that the variables used in (7.4) yield

$$K_{u^j}(t,l;p,v) = \begin{cases} K_t(t,l;p,v) = F_t(t,l,\frac{v}{p}) \cdot p & \text{if } j = 1, \\ K_{l^j}(t,l;p,v) = F_{u^{j-1}}(t,l,\frac{v}{p}) \cdot p & \text{if } j = 2, \dots, d+1, \end{cases} \tag{7.5}$$

and

$$K_{\dot w^j}(t,l;p,v) = \begin{cases} K_p(t,l;p,v) = F(t,l,\frac{v}{p}) - F_{\dot u^k}(t,l,\frac{v}{p})\frac{v^k}{p} & \text{if } j = 1, \\ K_{v^j}(t,l;p,v) = F_{\dot u^{j-1}}(t,l,\frac{v}{p}) & \text{if } j = 2, \dots, d+1. \end{cases} \tag{7.6}$$

From these, for all $t \in (a,b)$ and all $j = 2, \dots, d+1$, it follows that

$$K_{u^1}(w,\dot w) - \frac{\mathrm{d}}{\mathrm{d}t} K_{\dot u^1}(w,\dot w) = \left[\frac{\mathrm{d}}{\mathrm{d}t} F_{\dot u^k}(t,u,\dot u) - F_{u^k}(t,u,\dot u)\right] \cdot \dot u^k,$$
$$K_{u^j}(w,\dot w) - \frac{\mathrm{d}}{\mathrm{d}t} K_{\dot u^j}(w,\dot w) = F_{u^{j-1}}(t,u,\dot u) - \frac{\mathrm{d}}{\mathrm{d}t} F_{\dot u^{j-1}}(t,u,\dot u).$$

Therefore, since u satisfies the Euler–Lagrange equations for F, this entails that w is a solution of the analogous equations for K in (a,b). Now, (i) gives

$$\frac{\partial}{\partial t} X_s(w(t)) = \left(\frac{\partial}{\partial t} T_s(t,u(t)), \frac{\partial}{\partial t} L_s(t,u(t))\right) \in {}^\rho\widetilde{\mathbb{R}}^* \times {}^\rho\widetilde{\mathbb{R}}^d = V.$$

Moreover, (ii) gives $X_0(w(t)) = (T_0(t,u(t)), L_0(t,u(t))) = w(t)$. Finally,

$$K(w,\dot w) = F(t,u,\dot u),$$
$$K\Big(X_s(w), \frac{\partial}{\partial t} X_s(w)\Big) = F\left[T_s(t,u), L_s(t,u), \frac{\frac{\partial}{\partial t} L_s(t,u)}{\frac{\partial}{\partial t} T_s(t,u)}\right] \cdot \frac{\partial}{\partial t} T_s(t,u).$$

We can hence apply Theorem 7.1, and from (7.5) and (7.6), we get that

$$K_{\dot w^j}(w,\dot w) \frac{\partial}{\partial s}\bigg|_0 X_s^j(w) = F_{\dot u^j}(t,u(t),\dot u(t)) \frac{\partial}{\partial s}\bigg|_{s=0} L_s^j(t,u(t))$$
$$+ \left[F(t,u(t),\dot u(t)) - F_{\dot u^k}(t,u(t),\dot u(t))\dot u^k(t)\right] \frac{\partial}{\partial s}\bigg|_{s=0} T_s(t,u(t))$$

is constant in $t \in (a,b)$. □

8 Application to $\mathcal{C}^{1,1}$ Riemannian metric

In the following, we apply what we did so far to the problem of length minimizers in $(\mathbb{R}^d, g)$, where $g \in \mathcal{C}^{1,1}$ is a Riemannian metric. Furthermore, we assume that $(\mathbb{R}^d, g)$ is geodesically complete. Note that the seeming restriction of considering only $\mathbb{R}^d$ as our manifold weighs not so heavy. Indeed, the question of length minimizers can be considered to be a local one, since it is not guaranteed that global minimizers exist at all, whereas local minimizers always exist. Additionally, note that it was shown that it suffices to consider smooth manifolds (cf. [20, Theorem 2.9]) instead of $\mathcal{C}^k$ manifolds with $1 \le k < +\infty$. Therefore, there is no need to consider non-smooth charts.

In this section, we fix an embedding $(\iota_\Omega^b)_\Omega$, where $b \in {}^\rho\widetilde{\mathbb{R}}$ satisfies $b \ge \mathrm{d}\rho^{-a}$ for some $a \in \mathbb{R}_{>0}$, and where $\Omega \subseteq \mathbb{R}^d$ is an arbitrary open set, see Theorem 2.18. Actually, the embedding also depends on the dimension $d \in \mathbb{N}_{>0}$, but to avoid cumbersome notations, we denote embeddings always with the symbol ι.

By [27, Remark 2.6.2], it follows that we can always find a net of smooth functions (g_{ij}^ε) such that by setting $\tilde{g} := \iota(g) =: [g_{ij}^\varepsilon(-)] \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}^d \times {}^\rho\widetilde{\mathbb{R}}^d, {}^\rho\widetilde{\mathbb{R}})$, we have that g_{ij}^ε is a Riemannian metric for all ε. By Theorem 2.18 (iii), it follows that $g_{ij}^\varepsilon \to g_{ij}$ in $\mathcal{C}^0$ norm. Let Γ_{ij}^ε be the Christoffel symbols of g^ε, and set $\tilde{\Gamma}_{ij} := [\Gamma_{ij}^\varepsilon(-)] \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}^d, {}^\rho\widetilde{\mathbb{R}}^d)$. A curve $\gamma \in {}^\rho\mathcal{GC}^\infty(J, {}^\rho\widetilde{\mathbb{R}}^d)$, with J being a sharply open subset of ${}^\rho\widetilde{\mathbb{R}}$, is said to be a *geodesic* of $({}^\rho\widetilde{\mathbb{R}}^d, \tilde{g})$ if

$$\ddot{\gamma}(t) + \tilde{\Gamma}_{ij}(\gamma(t))\dot{\gamma}^i(t)\dot{\gamma}^j(t) = 0 \quad \text{for all } t \in J. \tag{8.1}$$

Definition 8.1. We say that $({}^\rho\widetilde{\mathbb{R}}^d, \tilde{g})$ is *geodesically complete* if every solution of the geodesic equation belongs to ${}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}}^d)$, i.e., if for all $p \in {}^\rho\widetilde{\mathbb{R}}^d$ and all $v \in {}^\rho\widetilde{\mathbb{R}}^d$, there exists a geodesic $\gamma \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}}^d)$ of $({}^\rho\widetilde{\mathbb{R}}^d, \tilde{g})$ such that $\gamma(0) = p$ and $\dot{\gamma}(0) = v$.

This definition includes also the possibility that the point p or the vector v could be infinite. By Theorem 2.19, it follows that if we consider only finite p and v, then any geodesic $\gamma \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}, {}^\rho\widetilde{\mathbb{R}}^d)$ induces a Colombeau generalized function $\gamma|_{\mathrm{c}(\mathbb{R})} \in \mathcal{G}^s(\mathbb{R})^d$. Therefore, the space $(\mathrm{c}(\mathbb{R}^d), \tilde{g}|_{\mathrm{c}(\mathbb{R}^d)\times\mathrm{c}(\mathbb{R}^d)})$ is geodesically complete in the sense of [39]. We recall that $\mathrm{c}(\Omega)$ is the set of compactly supported (i.e., finite) generalized points in Ω (see Theorem 2.18).

The definition of length of a (non-singular) curve needs the following.

Remark 8.2. We set

$$\sqrt{-} = (-)^{1/2} : x = [x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}_{>0} \mapsto [\sqrt{x_\varepsilon}] \in {}^\rho\widetilde{\mathbb{R}}_{>0}.$$

Lemma 2.3 readily implies that $\sqrt{-} \in {}^\rho\mathcal{GC}^\infty({}^\rho\widetilde{\mathbb{R}}_{>0}, {}^\rho\widetilde{\mathbb{R}}_{>0})$. Therefore, the square root is defined on every strictly positive infinitesimal, but it cannot be extended to ${}^\rho\widetilde{\mathbb{R}}_{\ge 0}$.

Definition 8.3. (i) Let $\tilde{p}, \tilde{q} \in {}^\rho\widetilde{\mathbb{R}}^d$, then

$${}^\rho\mathcal{GC}^\infty_{>0}(\tilde{p}, \tilde{q}) := \{\lambda \in {}^\rho\mathcal{GC}^\infty([0,1], {}^\rho\widetilde{\mathbb{R}}^d) \mid \lambda(0) = \tilde{p}, \lambda(1) = \tilde{q}, |\dot{\lambda}(t)| > 0 \; \forall t \in [0,1]\}.$$

Moreover, for $\lambda \in {}^\rho\mathcal{GC}^\infty_{>0}(\tilde{p}, \tilde{q})$, we set

$$L_{\tilde{g}}(\lambda) := \int_0^1 (\tilde{g}_{ij}(\alpha(t))\dot{\alpha}^i(t)\dot{\alpha}^j(t))^{1/2}\, \mathrm{d}t \in {}^\rho\widetilde{\mathbb{R}}.$$

(ii) Let $x = [x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}^n$. Then we set $\mathrm{st}(x) := \lim_{\varepsilon\to 0} x_\varepsilon \in \mathbb{R}^d$, if this limit exists. Note that $x \approx \mathrm{st}(x)$ in this case.

Note that (8.1) are the usual geodesic equations for the generalized metric $\tilde{g}$, whose derivation is completely analogous to that in the smooth case. Thus, they are the Euler–Lagrange equations of $L_{\tilde{g}}$.

We are interested only in global minimizers of the functional $L_{\tilde{g}}$, i.e., curves $\lambda_0 \in X(\tilde{p}, \tilde{q})$ such that $L_{\tilde{g}}(\lambda_0) \le L_{\tilde{g}}(\lambda)$ for all $\lambda \in {}^\rho\mathcal{GC}^\infty_{>0}(\tilde{p}, \tilde{q})$.

Lemma 8.4. *Let $p, q \in \mathbb{R}^d$ and $\tilde{p}, \tilde{q} \in {}^\rho\widetilde{\mathbb{R}}^d$ be such that $\mathrm{st}(\tilde{p}) = p$ and $\mathrm{st}(\tilde{q}) = q$. Let $\lambda = [\lambda_\varepsilon(-)] \in {}^\rho\mathcal{GC}^\infty_{>0}(\tilde{p}, \tilde{q})$ be such that there exists*

$$\bar{\lambda} \in \mathcal{C}^1_{>0}(p, q) := \{w \in \mathcal{C}^1([0,1]_\mathbb{R}, \mathbb{R}^d) \mid w(0) = p, w(1) = q, |\dot{w}(t)| > 0 \; \forall t \in [0,1]_\mathbb{R}\}$$

such that $\lambda_\varepsilon \to \bar{\lambda}$ in $\mathcal{C}^1$ as $\varepsilon \to 0$. Then $\mathrm{st}(L_{\tilde{g}}(\lambda)) = L_g(\bar{\lambda})$.

Proof. We calculate

$$\left|\int_0^1 (g^\varepsilon_{ij}(\lambda_\varepsilon)\dot\lambda^i_\varepsilon\dot\lambda^j_\varepsilon)^{1/2} - (g_{ij}(\bar\lambda)\dot{\bar\lambda}^i\dot{\bar\lambda}^j)^{1/2}\,\mathrm{d}t\right| = \left|\int_0^1 \frac{g^\varepsilon_{ij}(\lambda_\varepsilon)\dot\lambda^i_\varepsilon\dot\lambda^j_\varepsilon - g_{ij}(\bar\lambda)\dot{\bar\lambda}^i\dot{\bar\lambda}^j}{(g^\varepsilon_{ij}(\lambda_\varepsilon)\dot\lambda^i_\varepsilon\dot\lambda^j_\varepsilon)^{1/2} + (g_{ij}(\bar\lambda)\dot{\bar\lambda}^i\dot{\bar\lambda}^j)^{1/2}}\,\mathrm{d}t\right|.$$

By assumption, $(g^\varepsilon_{ij}(\lambda_\varepsilon)\dot\lambda^i_\varepsilon\dot\lambda^j_\varepsilon)^{1/2} \to (g_{ij}(\bar\lambda)\dot{\bar\lambda}^i\dot{\bar\lambda}^j)^{1/2}$, so that there exists $C \in \mathbb{R}_{>0}$ such that

$$\left|\int_0^1 (g^\varepsilon_{ij}(\lambda_\varepsilon)\dot\lambda^i_\varepsilon\dot\lambda^j_\varepsilon)^{1/2} - (g_{ij}(\bar\lambda)\dot{\bar\lambda}^i\dot{\bar\lambda}^j)^{1/2}\,\mathrm{d}t\right| \le C\int_0^1 \left|(g^\varepsilon_{ij}(\lambda_\varepsilon) - g_{ij}(\lambda_\varepsilon) + g_{ij}(\lambda_\varepsilon) - g_{ij}(\bar\lambda))\dot\lambda^i_\varepsilon\dot\lambda^j_\varepsilon + g_{ij}(\bar\lambda)(\dot\lambda^i_\varepsilon\dot\lambda^j_\varepsilon - \dot{\bar\lambda}^i\dot{\bar\lambda}^j)\right|\mathrm{d}t.$$

We hence obtain the claim by the triangle inequality and the convergence of λ_ε, $\dot\lambda_\varepsilon$ and g^ε_{ij} to $\bar\lambda$, $\dot{\bar\lambda}$ and g_{ij}, respectively. □

Now, we consider $p, q \in \mathbb{R}^d$, with $p \neq q$. Let

$$u \in \{u \in \mathcal{C}^{2,1}([0,1], \mathbb{R}^d) \mid u(0) = p, u(1) = q\}$$

be a solution of the geodesic equation

$$\begin{cases} \ddot u = -\Gamma_{ij}(u)\dot u^i\dot u^j, \\ p = u(0), \\ q = u(1). \end{cases} \tag{8.2}$$

Let $c_0 := \dot u(0)$. Obviously, u is also the unique solution of

$$\begin{cases} \ddot u = -\Gamma_{ij}(u)\dot u^i\dot u^j, \\ p = u(0), \\ c_0 = \dot u(0). \end{cases}$$

Using these initial conditions, for each fixed ε, we can solve the following problem:

$$\begin{cases} \ddot y = -\Gamma^\varepsilon_{ij}(y)\dot y^i\dot y^j, \\ p = y(0), \\ c_0 = \dot y(0) \end{cases} \tag{8.3}$$

for a unique $y_\varepsilon \in \mathcal{C}^\infty([-d_\varepsilon, d_\varepsilon]_\mathbb{R}, \mathbb{R}^d)$ and some $d_\varepsilon \in \mathbb{R}_{>0}$.

Lemma 8.5. *Let u and y_ε be as above. Then the following hold:*
(i) *For ε sufficiently small, the solution y_ε can be extended to a solution $y_\varepsilon \in \mathcal{C}^\infty([0,1]_\mathbb{R}, \mathbb{R}^d)$ of* (8.3) *such that $y_\varepsilon(1) = q$.*
(ii) *$y_\varepsilon \to u$ in $\mathcal{C}^2$.*
(iii) *The net (y_ε) defines a GSF, i.e., $y := [y_\varepsilon(-)] \in {}^\rho\mathcal{GC}^\infty_{>0}(p, q)$.*

Proof. (i)–(ii) For all i, j, we have that $\Gamma^\varepsilon_{ij} \to \Gamma_{ij}$ locally uniformly. Thus, we obtain these claims by (8.2) and by continuous dependence on parameters in ODE, see, e.g., [26, Lemma 2.3].

(iii) If $y := [y_\varepsilon(-)] \in {}^\rho\mathcal{GC}^\infty([0,1], {}^\rho\widetilde{\mathbb{R}}^d)$, then we have to show that for all ε, all the derivatives of y_ε are moderate. This is obviously true for y_ε, $\dot y_\varepsilon$ and $\ddot y_\varepsilon$. The claim follows now from the fact that

$$\frac{\mathrm{d}^{n+2}}{\mathrm{d}t^{n+2}} y_\varepsilon = -\frac{\mathrm{d}^n}{\mathrm{d}t^n}(\Gamma^{\varepsilon k}_{ij}(y)\dot y^i_\varepsilon\dot y^j_\varepsilon),$$

so that there exists a polynomial P such that

$$\frac{\mathrm{d}^n}{\mathrm{d}t^n}(\Gamma^{\varepsilon k}_{ij}(y)\dot y^i_\varepsilon\dot y^j_\varepsilon) = P\Big(y_\varepsilon, \frac{\mathrm{d}}{\mathrm{d}t}y_\varepsilon, \dots, \frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}}y_\varepsilon, \Gamma^{\varepsilon k}_{ij}, \mathrm{D}\Gamma^{\varepsilon k}_{ij}, \dots, \mathrm{D}^n\Gamma^{\varepsilon k}_{ij}\Big).$$

If $|\dot y(t)| > 0$ for all $t \in [0,1]$, then, by (ii), we have that $y_\varepsilon \to u$ in $\mathcal{C}^2$. Furthermore, $g_\varepsilon \to g$ in $\mathcal{C}^1$, by assumption, and we know that $g(\dot u, \dot u) = c > 0$ for some $c \in \mathbb{R}_{>0}$, since u is a g-geodesic (cf. [22, Lemma 1.4.5]). Therefore, we obtain that $g_\varepsilon(\dot y_\varepsilon, \dot y_\varepsilon) > \frac{c}{2} > 0$ for $\varepsilon > 0$ small enough. □

Finally, the standard part of the generalized length of y is the length of u.

Theorem 8.6. *Let u and y_ε be as above. We conclude (using Lemma 8.4) that* $\mathrm{st}(L_{\tilde{g}}(y)) = L_g(u)$.

Proposition 8.7. *Let $y = [y_\varepsilon(-)]$ be as above. In addition, assume that each y_ε is L_{g_ε}-minimizing. Then $L_{\tilde{g}}(y)$ is minimal.*

Proof. Let $\lambda = [\lambda_\varepsilon(-)] \in {}^\rho\mathcal{GC}^\infty_{>0}(p,q)$. We have that $L_{\tilde{g}}(\lambda) = [L_{g_\varepsilon}(\lambda_\varepsilon)]$ and that $L_{\tilde{g}}(y) = [L_{g_\varepsilon}(y_\varepsilon)]$. By assumption, for all ε, we have $L_{g_\varepsilon}(\lambda_\varepsilon) \ge L_{g_\varepsilon}(y_\varepsilon)$. Therefore, $L_{\tilde{g}}(\lambda) \ge L_{\tilde{g}}(y)$, as claimed. □

Corollary 8.8. *Let $\lambda \in {}^\rho\mathcal{GC}^\infty_{>0}(p,q)$ be a minimizer of $L_{\tilde{g}}$ and assume that for ε small, y_ε is L_{g_ε}-minimizing. Then $L_{\tilde{g}}(y) = L_{\tilde{g}}(\lambda)$.*

Corollary 8.8 gives us a way to answer the question if a certain classical geodesic between two given classical points p and q is a length-minimizer.

Furthermore, we are able to prove the following theorem, relating generalized minimizers to classical minimizers.

Theorem 8.9. *Let $p, q \in \mathbb{R}^d$ and let $\gamma \in {}^\rho\mathcal{GC}^\infty_{>0}(p,q)$ such that $L_{\tilde{g}}(\gamma)$ is minimal. Assume that* $\mathrm{st}(L_{\tilde{g}}(\gamma))$ *exists and that there exists $w \in \mathcal{C}^1_{>0}(p,q)$ such that $L_g(w) = \mathrm{st}(L_{\tilde{g}}(\gamma))$. Then w is g-minimizing and g-geodesic.*

Proof. Assume to the contrary that there exists a curve $\sigma \in C^2$ connecting p and q (without loss of generality, σ is a g-geodesic) such that $L_g(\sigma) < L_g(w)$. Now we construct (as done above) g_ε and σ_ε, and set $\tilde{\sigma} := [\sigma_\varepsilon]$. Then

$$\mathrm{st}(L_{\tilde{g}}(\tilde{\sigma})) = L_g(\sigma) < L_g(w) = \mathrm{st}(L_{\tilde{g}}(\gamma)).$$

But, by assumption, we have that $L_{\tilde{g}}(\gamma) \le L_{\tilde{g}}(\tilde{\sigma})$, which implies

$$\mathrm{st}(L_{\tilde{g}}(\gamma)) \le \mathrm{st}(L_{\tilde{g}}(\tilde{\sigma})) < \mathrm{st}(L_{\tilde{g}}(\gamma)).$$

This is a contradiction. □

9 Conclusions

We can summarize the present work as follows.

(i) The setting of GSF allows to treat Schwartz distributions more closely to classical smooth functions. The framework is so flexible and the extensions of classical results are so natural in many ways that one may treat it like smooth functions.

(ii) One key step of the theory is the change of the ring of scalars into a non-Archimedean one, and the use of the strict order relation $<$ to deal with topological properties. So, the use of $<$ and of ${}^\rho\widetilde{\mathbb{R}}$-valued norms allows a natural approach to topology, even of infinite dimensional spaces (cf. Definition 5.3). On the other hand, the use of a ring with zero divisors and a non-total order relation requires a more refined and careful analysis. However, as proved in the present work, very frequently classical proofs can be formally repeated in this context, but paying particular attention to using the relation $<$ and invertibility instead of being non-zero in $\mathbb{R}$, and avoiding the total order property.

(iii) Others crucial properties are the closure of GSF with respect to composition and the use of the gauge ρ, because they do not force to narrow the theory into particular cases.

(iv) The present extension of the classical theory of calculus of variations shows that the use of GSF is a powerful analytical technique. The final application shows how to use them as a method to address problems in an Archimedean setting based on the real field $\mathbb{R}$.

Concerning possible future developments, we note the following.

(v) A generalization of the whole construction to piecewise GSF seems possible.

(vi) A more elegant approach for the integration of piecewise GSF could use the existence of right and left limits of $(f_1, \dots, f_n)(-)$ and hyperfinite Riemann-like sums, i.e.,

$$\sum_{i=1}^{N} f(x_i')(x_i - x_{i-1}) := \left[\sum_{i=1}^{N_\varepsilon} f_\varepsilon(x_{i,\varepsilon}')(x_{i,\varepsilon} - x_{i-1,\varepsilon}) \right] \in {}^\rho\widetilde{\mathbb{R}}^d,$$

extended to $N \in \widetilde{\mathbb{N}} := \{[\mathrm{int}(x_\varepsilon)] \mid [x_\varepsilon] \in {}^\rho\widetilde{\mathbb{R}}\}$, where $\mathrm{int}(-)$ is the integer part function.

The present work could lay the foundations for further works concerning the possibility to extend other results of the calculus of variations in this generalized setting.

Acknowledgment: We would like to thank M. Kunzinger for helpful discussions and several suggestions that have led to considerable improvements.

Funding: Alexander Lecke has been supported by the uni:doc fellowship programme of the University of Vienna. Lorenzo Luperi Baglini has been supported by grant M1876-N35 of the Austrian Science Fund FWF. Paolo Giordano has been supported by grants P25311-N25 and P25116-N25 of the Austrian Science Fund FWF.

References

[1] J. Aragona, R. Fernandez and S. O. Juriaans, A discontinuous Colombeau differential calculus, *Monatsh. Math.* **144** (2005), no. 1, 13–29.
[2] J. Aragona and S. O. Juriaans, Some structural properties of the topological ring of Colombeau's generalized numbers, *Comm. Algebra* **29** (2001), no. 5, 2201–2230.
[3] A. Avez, *Differential Calculus*, John Wiley & Sons, Chichester, 1986.
[4] V. G. Berkovich, *Spectral Theory and Analytic Geometry over Non-Archimedean Fields*, Math. Surveys Monogr. 33, American Mathematical Society, Providence, 1990.
[5] J.-F. Colombeau, Multiplication of distributions, *Bull. Amer. Math. Soc. (N.S.)* **23** (1990), no. 2, 251–268.
[6] J.-F. Colombeau, *Multiplication of Distributions*, Lecture Notes in Math. 1532, Springer, Berlin, 1992.
[7] A. M. Davie, Singular minimisers in the calculus of variations in one dimension, *Arch. Ration. Mech. Anal.* **101** (1988), no. 2, 161–177.
[8] P. A. M. Dirac, The physical interpretation of quantum mechanics, *Proc. Roy. Soc. London. Ser. A.* **180** (1942), 1–40.
[9] B. Engquist, A.-K. Tornberg and R. Tsai, Discretization of Dirac delta functions in level set methods, *J. Comput. Phys.* **207** (2005), no. 1, 28–51.
[10] I. M. Gelfand and S. V. Fomin, *Calculus of Variations*, Prentice-Hall, Englewood Cliffs, 1963.
[11] P. Giordano and M. Kunzinger, New topologies on Colombeau generalized numbers and the Fermat–Reyes theorem, *J. Math. Anal. Appl.* **399** (2013), no. 1, 229–238.
[12] P. Giordano and M. Kunzinger, A convenient notion of compact sets for generalized functions, preprint (2014), https://arxiv.org/abs/1411.7292v1.
[13] P. Giordano and M. Kunzinger, Inverse function theorems for generalized smooth functions, in: *Generalized Functions and Fourier Analysis*, Oper. Theory Adv. Appl. 260, Birkhäuser, Cham (2017), 95–114.
[14] P. Giordano, M. Kunzinger and R. Steinbauer, A new approach to generalized functions for mathematical physics, preprint, http://www.mat.univie.ac.at/~giordap7/GenFunMaps.pdf.
[15] P. Giordano, M. Kunzinger and H. Vernaeve, Strongly internal sets and generalized smooth functions, *J. Math. Anal. Appl.* **422** (2015), no. 1, 56–71.
[16] P. Giordano and L. Luperi Baglini, Asymptotic gauges: Generalization of Colombeau type algebras, *Math. Nachr.* **289** (2016), no. 2–3, 247–274.
[17] H. Glöckner, A. Escassut and K. Shamseddine, *Advances in Non-Archimedean Analysis*, Contemp. Math. 665, American Mathematical Society, Providence, (2016).
[18] L. M. Graves, Discontinuous solutions in the calculus of variations, *Bull. Amer. Math. Soc.* **36** (1930), no. 12, 831–846.
[19] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time*, Cambridge Monogr. Math. Phys. 1, Cambridge University Press, London, 1973.
[20] M. W. Hirsch, *Differential Topology*, Graduate Texts in Math. 33, Springer, Berlin, 1976.
[21] B. Hosseini, N. Nigam and J. M. Stockie, On regularizations of the Dirac delta distribution, *J. Comput. Phys.* **305** (2016), 423–447.
[22] J. Jost, *Riemannian Geometry and Geometric Analysis*, 6th ed., Universitext, Springer, Heidelberg, 2011.

[23] J. Jost and X. Li-Jost, *Calculus of Variations*, Cambridge Stud. Adv. Math. 64, Cambridge University Press, Cambridge, 1998.
[24] M. G. Katz and D. Tall, A Cauchy–Dirac delta function, *Found. Sci.* **18** (2013), no. 1, 107–123.
[25] S. Konjik, M. Kunzinger and M. Oberguggenberger, Foundations of the calculus of variations in generalized function algebras, *Acta Appl. Math.* **103** (2008), no. 2, 169–199.
[26] M. Kunzinger, R. Steinbauer and M. Stojković, The exponential map of a $C^{1,1}$-metric, *Differential Geom. Appl.* **34** (2014), 14–24.
[27] M. Kunzinger, R. Steinbauer, M. Stojković and J. A. Vickers, A regularisation approach to causality theory for $C^{1,1}$-Lorentzian metrics, *Gen. Relativity Gravitation* **46** (2014), no. 8, Artcile ID 1738.
[28] D. Laugwitz, Definite values of infinite sums: Aspects of the foundations of infinitesimal analysis around 1820, *Arch. Hist. Exact Sci.* **39** (1989), no. 3, 195–245.
[29] A. Lecke, *Non-smooth Lorentzian geometry and causality theory*, Ph.D. thesis, Universität Wien, 2016.
[30] A. Lecke, R. Steinbauer and R. Švarc, The regularity of geodesics in impulsive *pp*-waves, *Gen. Relativity Gravitation* **46** (2014), no. 1, Article ID 1648.
[31] L. Luperi Baglini and P. Giordano, Fixed point iteration methods for arbitrary generalized ODE, preprint, http://www.mat.univie.ac.at/~giordap7/preprint_ODE.pdf.
[32] L. Luperi Baglini and P. Giordano, The category of Colombeau algebras, *Monatsh. Math.* **182** (2017), no. 3, 649–674.
[33] A. Lytchak and A. Yaman, On Hölder continuous Riemannian and Finsler metrics, *Trans. Amer. Math. Soc.* **358** (2006), no. 7, 2917–2926.
[34] E. Minguzzi, Convex neighborhoods for Lipschitz connections and sprays, *Monatsh. Math.* **177** (2015), no. 4, 569–625.
[35] M. Oberguggenberger, Generalized functions, nonlinear partial differential equations, and Lie groups, in: *Geometry, Analysis and Applications* (Varanasi 2000), World Scientific, River Edge (2001), 271–281.
[36] M. Oberguggenberger and H. Vernaeve, Internal sets and internal functions in Colombeau theory, *J. Math. Anal. Appl.* **341** (2008), no. 1, 649–659.
[37] S. Payne, Topology of nonarchimedean analytic spaces and relations to complex algebraic geometry, *Bull. Amer. Math. Soc. (N.S.)* **52** (2015), no. 2, 223–247.
[38] A. Robinson, Function theory on some nonarchimedean fields, *Amer. Math. Monthly* **80** (1973), no. 6, 87–109.
[39] C. Sämann and R. Steinbauer, Geodesic completeness of generalized space-times, in: *Pseudo-Differential Operators and Generalized Functions*, Oper. Theory Adv. Appl. 245, Birkhäuser/Springer, Cham (2015), 243–253.
[40] C. Sämann, R. Steinbauer, A. Lecke and J. Podolský, Geodesics in nonexpanding impulsive gravitational waves with Λ. Part I, *Classical Quantum Gravity* **33** (2016), no. 11, Article ID 115002.
[41] M. Stojković, *Causality theory for* $\mathcal{C}^{1,1}$*-metrics*, Ph.D. thesis, Universität Wien, 2015.
[42] A.-K. Tornberg and B. Engquist, Numerical approximations of singular source terms in differential equations, *J. Comput. Phys.* **200** (2004), no. 2, 462–488.
[43] C. Tuckey, *Nonstandard Methods in the Calculus of Variations*, Pitman Research Notes in Math. Ser. 297, Longman Scientific & Technical, Harlow, 1993.