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Rigidity for relative 0-cycles

FEDERICO BINDA AND AMALENDU KRISHNA

Abstract. We present a relation between the classical Chow group of relative 0-cycles on a regular scheme $\mathcal{X}$, projective and flat over an excellent Henselian discrete valuation ring, and the Levine-Weibel Chow group of 0-cycles on the special fiber. We show that these two Chow groups are isomorphic with finite coefficients under extra assumptions. This generalizes a result of Esnault, Kerz and Wittenberg.

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1. Introduction

Let A be an excellent Henselian discrete valuation ring with perfect residue field k of exponential characteristic $p \geq 1$. Let $\mathcal{X}$ be a regular scheme which is projective and flat over A. Let $X \subset \mathcal{X}$ be the reduced special fiber. If the map $\mathcal{X} \to \operatorname{Spec}(A)$ is an isomorphism, then Gabber's generalization of Suslin's rigidity theorem [16] says that the algebraic K-theory of $\mathcal{X}$ and X are isomorphic with coefficients prime to p. However, this rigidity theorem does not hold when the relative dimension of $\mathcal{X}$ over A is positive. One can then ask if it is possible to prove such an isomorphism for the higher Chow groups (which are the building block of K-theory in view of [13]) in certain bi-degrees. This is the context of the present work.

Let $\mathrm{CH}_1(\mathcal{X})$ denote the classical Chow group [15] of 1-dimensional cycles on $\mathcal{X}$. If $\mathcal{X}$ is smooth over A and k is finite or algebraically closed, Saito and Sato [36, Corollary 0.10] showed that there is a restriction map $\rho : \mathrm{CH}_1(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \to \mathrm{CH}_0(X) \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$ which is an isomorphism, whenever m is prime to the exponential characteristic of k.

As part of their proof of the above restriction isomorphism, Saito and Sato showed that the étale cycle class map for $\mathrm{CH}_1(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$ is an isomorphism more generally for every model $\mathcal{X} \to \operatorname{Spec}(A)$ with semi-stable reduction, *i.e.*,

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such that the reduced special fiber X has simple normal crossing (again, under the assumption that the residue field k is finite or algebraically closed). As an application of this, they proved that if K is a local field with finite residue field and Y is smooth and projective over K, then $\mathrm{CH}_0(Y) \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z}$ is finite, originally a conjecture due to Colliot-Thélène [8].

Inspired by an argument originally due to Bloch and discussed in [11, Appendix A], the result of Saito and Sato was revisited and generalized by Esnault, Kerz and Wittenberg in [10]. Under the assumption that the reduced special fiber X is a simple normal crossing divisor in $\mathcal{X}$, it was observed in [10] that it is possible to replace the classical Chow group (see [15]) $\mathrm{CH}_0(X)$ of the special fiber X with the Friedlander-Voevodsky [12] motivic cohomology $H^{2d}(X, \mathbb{Z}(d))$, where $d = \dim_k(X)$, and still prove the existence of an isomorphism

$$\rho : \mathrm{CH}_1(\mathcal{X}) \otimes_{\mathbb{Z}} \mathbb{Z}/m\mathbb{Z} \to H^{2d}(X, \mathbb{Z}/m\mathbb{Z}(d)), \tag{1.1}$$

provided that some extra assumptions on m or on the residue field are satisfied. This approach allowed Esnault, Kerz and Wittenberg to generalize the restriction isomorphism of Saito and Sato by allowing the field k to belong to a bigger class than just finite or algebraically closed fields, and the reduced special fiber to be a simple normal crossing divisor than just smooth. In fact, in the case of good reduction, they showed that ρ is an isomorphism for any perfect residue field k. Note that there is always a surjective map $H^{2d}(X, \mathbb{Z}(d)) \twoheadrightarrow \mathrm{CH}_0(X)$ for a simple normal crossings divisor $X \subset \mathcal{X}$. But this is not in general an isomorphism, even with finite coefficients.

In this paper, we show that if we further replace the $(2d, d)$ motivic cohomology group of the reduced special fiber X by its Levine-Weibel Chow group of 0-cycles [31], then the restriction isomorphism of Saito and Sato holds without any condition on X whenever the residue field is algebraically closed. More generally, we prove the following generalization of [10] for arbitrary perfect residue fields.

We let $\mathcal{Z}_1(\mathcal{X})$ denote the free Abelian group on the set of integral 1-dimensional closed subschemes of $\mathcal{X}$ and let $\mathcal{Z}_1^g(\mathcal{X})$ denote the subgroup of $\mathcal{Z}_1(\mathcal{X})$ generated by integral cycles which are flat over A and do not meet the singular locus of X. It follows from the moving lemma of Gabber, Liu and Lorenzini [17] that the composite map

$$\mathcal{Z}_1^g(\mathcal{X}) \hookrightarrow \mathcal{Z}_1(\mathcal{X}) \twoheadrightarrow \mathrm{CH}_1(\mathcal{X})$$

is surjective. For any reduced quasi-projective scheme Y over a field, let $\mathrm{CH}_0^{LW}(Y)$ denote the Levine-Weibel Chow group of 0-cycles on Y, first introduced in [31]. It is a quotient of the free Abelian group $\mathcal{Z}_0(Y \setminus Y_{\mathrm{sing}})$ of 0-cycles supported in the regular locus of Y (see 3.1 for a reminder of its definition). Let m be an integer prime to the exponential characteristic of k and let $\Lambda = \mathbb{Z}/m\mathbb{Z}$. For an Abelian group M, write $M_\Lambda = M \otimes_{\mathbb{Z}} \Lambda$. Then the following holds.

Theorem 1.1. *Let $\mathcal{X}$ be a regular scheme which is projective and flat over an excellent Henselian discrete valuation ring with perfect residue field. Let X denote the reduced special fiber of $\mathcal{X}$. Then there exists a commutative diagram*

$$
\begin{tikzcd}
\mathcal{Z}_1^g(\mathcal{X})_\Lambda \arrow[r, "\rho"] \arrow[d, two heads] & \mathcal{Z}_0(X \setminus X_{\mathrm{sing}})_\Lambda \arrow[d, two heads] \\
\mathrm{CH}_1(\mathcal{X})_\Lambda & \mathrm{CH}_0^{LW}(X)_\Lambda \arrow[l, "\gamma"']
\end{tikzcd}
\tag{1.2}
$$

such that γ is surjective.

Here, the vertical maps are the canonical projections, and $\widetilde{\rho}$ is the group homomorphism given by taking an 1-cycle in good position and intersecting it with the reduced special fiber X.

Let us explain how this theorem relates to the construction of [10]. Suppose that $\mathcal{X}$ has semi-stable reduction. Then by [10, Theorem 5.1] there exists a unique surjective homomorphism γ_{EKW} making the diagram

$$
\begin{tikzcd}
\mathcal{Z}_1^g(\mathcal{X})_\Lambda \arrow[r, "\rho"] \arrow[d, two heads] & \mathcal{Z}_0(X \setminus X_{\mathrm{sing}})_\Lambda \arrow[d, two heads] \\
\mathrm{CH}_1(\mathcal{X})_\Lambda & H^{2d}(X, \Lambda(d)) \arrow[l, "\gamma_{EKW}"']
\end{tikzcd}
$$

commutative. The group at the bottom right corner is the motivic cohomology group with Λ coefficients (as in (1.1)). Combining this with the cycle class map constructed in [5], we obtain then a commutative diagram of surjections

$$
\begin{tikzcd}
\mathrm{CH}_1(\mathcal{X})_\Lambda & \mathrm{CH}_0^{LW}(X)_\Lambda \arrow[l, two heads, "\gamma"'] \arrow[d, two heads, "\mathrm{cyc}_X^{\mathcal{M}}"] \\
& H^{2d}(X, \Lambda(d)) \arrow[ul, two heads, "\gamma_{EKW}"]
\end{tikzcd}
$$

so that we can interpret Theorem 1.1 as a lift to the Levine-Weibel Chow group of the inverse restriction map considered in [10]. Note that γ_{EKW} exists only in the semi-stable case, while the diagram (1.2) with γ exists without assumption on the special fiber.

One consequence of Theorem 1.1 is the following.

Corollary 1.2. *In the notation of Theorem* 1.1, *suppose that the map $\widetilde{\rho}\colon \mathcal{Z}_1^g(\mathcal{X})_\Lambda \to \mathcal{Z}_0(X \setminus X_{\mathrm{sing}})_\Lambda$ descends to a morphism between the Chow groups*

$$\rho\colon \mathrm{CH}_1(\mathcal{X})_\Lambda \to \mathrm{CH}_0^{LW}(X)_\Lambda. \tag{1.3}$$

Then ρ is an isomorphism. If moreover $\mathcal{X}$ has semi-stable reduction, then there is a commutative diagram of isomorphisms

$$\begin{tikzcd} \mathrm{CH}_1(\mathcal{X})_\Lambda \arrow[r, "\rho"] \arrow[dr] & \mathrm{CH}_0^{LW}(X)_\Lambda \arrow[d, two heads, "\mathrm{cyc}_X^{\mathcal{M}}"] \\ & H^{2d}(X, \Lambda(d)) \end{tikzcd} .$$

The diagonal arrow in the semi-stable case agrees with the map ρ of [10, Theorem 1.1]. We expect that the homomorphism ρ in (1.3) always exists, and the reason for such expectation is twofold. On one side, the Levine-Weibel Chow group is expected to be part of a satisfactory theory of cycles on singular varieties, closer to the K-theory of vector bundles than the cdh-motivic cohomology. The restriction homomorphism ρ should then be seen as a cycle-theoretic incarnation (in certain bi-degrees) of the restriction map on K-groups with Λ-coefficients

$$\iota^*\colon K_0(\mathcal{X};\Lambda) \to K_0(X;\Lambda)$$

induced by the inclusion $\iota\colon X \hookrightarrow \mathcal{X}$.

The relationship between the Levine-Weibel Chow group and the K_0 group has been object of investigation by many authors (we recall here [24,27–29,31,34,35] to name a few). It is known that the group $\mathrm{CH}_0^{LW}(X)$ can be used to detect invariants of "additive" type. For example, if X is an arbitrary reduced curve over a field k, we have

$$\mathrm{CH}_0^{LW}(X) \xrightarrow{\cong} \mathrm{Pic}(X) \cong H^1(X, \mathbb{G}_m)$$

generalizing the classical relationship between line bundles and Weil divisors, while

$$H^2(X, \mathbb{Z}(1)) \cong H^2(X_{\mathrm{sn}}, \mathbb{Z}(1)) \cong \mathrm{Pic}(X_{\mathrm{sn}})$$

where X_{sn} denotes the semi-normalization of X. This is reflecting the fact that the functor $X \mapsto \mathrm{Pic}(X)$ considered on $\mathbf{Sch}(k)$ rather than on $\mathbf{Sm}(k)$ is not $\mathbb{A}^1$-invariant, and thus can not be captured by an $\mathbb{A}^1$-invariant theory like Voevodsky's motivic cohomology.

On the other side, however, with torsion coefficients prime to the exponential characteristic of k, there are no additive invariants to detect, and the non-$\mathbb{A}^1$-invariant theory "collapses" to the classical one. This statement can be made precise in the context of the theory of motives with modulus, as recently developed by Kahn Saito and Yamazaki. See [20, Corollary 4.2.6 and Remark 4.2.7 b)] (using some results in [3]). We therefore conjecture that the cycle class map

$$\mathrm{cyc}_X^{\mathcal{M}}\colon \mathrm{CH}_0^{LW}(X)_\Lambda \to H^{2d}(X, \Lambda(d)) \tag{1.4}$$

is always an isomorphism with $\Lambda = \mathbb{Z}/m\mathbb{Z}$-coefficients. In a similar spirit, we expect that $\mathrm{cyc}_X^{\mathcal{M}}$ is an isomorphism with integral coefficients (if k admits resolution of singularities, or with $\mathbb{Z}[1/p]$-coefficients otherwise) if the the singularities of X are sufficiently mild, intuitively where additive phenomena do not occur. This is supported by the following result: if the residue field k is algebraically closed and $X \subset \mathcal{X}$ is a simple normal crossing divisor, it is shown in [5] that there is a canonical isomorphism

$$\mathrm{cyc}_X^{\mathcal{M}}\colon \mathrm{CH}_0^{LW}(X) \otimes_{\mathbb{Z}} [1/p] \xrightarrow{\cong} H^{2d}(X, \mathbb{Z}[1/p](d)), \tag{1.5}$$

which holds integrally if k admits resolution of singularities.

In view of the above discussion, the existence of the map ρ in (1.3) is therefore coherent with the expectation of [10] in the semi-stable reduction case, as explained in [10, 1].

We are in the situation of the Corollary if we put some extra assumption.

Theorem 1.3. *Let $\mathcal{X}$ be as in Theorem 1.1, and assume moreover that A has equal characteristic. Then the map $\widetilde{\rho}$ in (1.2) descends to a morphism between the Chow groups in the following cases*

(1) *If X has only isolated singularities and k is finite;*
(2) *If $\dim(X) = 2$, (with no further assumptions on the singularities of X).*

In both cases, the map $\tilde{\rho}$ induces an isomorphism

$$\rho\colon \mathrm{CH}_1(\mathcal{X})_\Lambda \xrightarrow{\cong} \mathrm{CH}_0^{LW}(X)_\Lambda,$$

and both groups are finite if k is finite.

If A has equal characteristic, then the Gersten conjecture for Milnor K-theory holds, thanks to [22], and the existence of the map ρ can be deduced from the validity of the Bloch-Quillen formula for singular varieties. See Section 5.3 for details (and for a comment about the assumption on the singularities of X in the case of relative dimension 2).

Remark 1.4. If the residue field k is algebraically closed, the cycle class map to étale cohomology $\mathrm{cyc}_X^{\text{ét}}\colon \mathrm{CH}_0^{LW}(X)_\Lambda \to H^{2d}_{\text{ét}}(X, \Lambda(d))$ is an isomorphism (see 5.1). This gives in particular that the map $\tilde{\rho}$ of (1.2) descends to a morphism between the Chow groups $\rho\colon \mathrm{CH}_1(\mathcal{X})_\Lambda \xrightarrow{\cong} \mathrm{CH}_0^{LW}(X)_\Lambda$, and so $\mathrm{CH}_1(\mathcal{X})_\Lambda \xrightarrow{\cong} H^{2d}_{\text{ét}}(\mathcal{X}, \Lambda(d))$ by proper base change. Note however that this isomorphism between $\mathrm{CH}_1(\mathcal{X})_\Lambda$ and the étale cohomology group was already obtained by Bloch [11, Theorem A.1], and we do not get more information in the algebraically closed field case.

We end the introduction with a brief outline of this text. The proofs of our main theorems are inspired by the ideas of Esnault, Kerz and Wittenberg [10]. The new insight is the introduction of the Levine-Weibel Chow group and its modified version from [4] in the picture and to show how this leads to the above generalizations, using the moving lemmas of Gabber, Liu and Lorenzini [17], some ideas from the Bertini theorems of Jannsen and Saito [36] and a construction of cycle class maps to étale cohomology and to the Nisnevich cohomology of Milnor K-sheaves. These cycle class maps play an important role in the calculation of $\mathrm{CH}_0^{LW}(X)$ with torsion coefficients.

In Section 2, we discuss some forms of Bertini theorems over a base and in Section 3, we prove our result for relative curves. We finish the proof of Theorem 1.1 in Section 4. In Section 5, we construct the cycle class maps for the Levine-Weibel Chow group and prove Theorem 1.3.

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2. Bertini type theorems over a base

In this section, we discuss some of the technical lemmas which we need in order to prove Theorem 1.1 when $\dim(\mathcal{X})$ is at least two. As some of these results are of independent interest and also used elsewhere, we state them separately. We fix the following general framework.

2.1. Setting

Let S be the spectrum of a discrete valuation ring A with field of fractions K. Let η be the generic point of S and s its closed point. Write k for the residue field of A, which is assumed to be perfect. We let $\mathfrak{M} = (\pi)$ denote the maximal ideal of A. Throughout this text, we fix a regular scheme $\mathcal{X}$ which is flat and projective over S. We let $\phi : \mathcal{X} \to S$ be the structure morphism and let $d \geq 0$ denote the relative dimension of $\mathcal{X}$ over S. Write $\mathcal{X}_s = \mathcal{X} \times_A k := \mathcal{X} \times_S \mathrm{Spec}\,(k)$ for the special fiber of ϕ and $X = (\mathcal{X}_s)_{\mathrm{red}} \hookrightarrow \mathcal{X}_s$ for the reduced special fiber. Given any scheme Y, we write $Y_{\mathrm{sing}} \subsetneq Y$ for the singular locus of Y_{red}. In this section, we shall assume k to be infinite.

Definition 2.1. A hyperplane $H \subset \mathbb{P}^N_S$ of the projective space $\mathbb{P}^N_S$ over S is a closed subscheme of $\mathbb{P}^N_S$ corresponding to an S-rational point of the dual $(\mathbb{P}^N_S)^\vee := \mathrm{Gr}_S(N-1, N)$.

By definition, an S-point of $\mathrm{Gr}_S(N-1, N)$ corresponds to (an isomorphism class of) a surjection $q \colon \mathcal{O}_S^{\oplus N+1} \to \mathcal{Q}$, where $\mathcal{Q}$ is locally free (hence free, since S is the spectrum of a DVR) of rank N. Fixing a basis $\{e_0, \ldots, e_N\}$ of $\mathcal{O}_S^{\oplus N+1}$, we can write the kernel of q as $\sum_{i=0}^{N}\langle a_i\rangle e_i \subset \mathcal{O}_S^{N+1}$ for elements $a_i \in A$, not all in $\mathfrak{M}$. Here, $\langle a\rangle$ is the submodule of $\mathcal{O}_S$ generated by $a \in A$. If $X_0, \dots, X_n$ are the homogeneous coordinate functions on $\mathbb{P}^N_S$, then the hyperplane H corresponding to q is the zero locus of the linear polynomial $q(x) = \sum_{i=0}^{N} a_i X_i$.

The same equation defines the hyperplane $H_\eta \subset \mathbb{P}^N_K$, the generic fiber of H. We denote by H_s the hyperplane in $\mathbb{P}^N_k$ defined by the reduction of $q(x) \mod \pi$. In order to show the existence of good hyperplanes of $\mathbb{P}^N_S$, we will frequently use the following simple but crucial remark, due to Jannsen and Saito.

Lemma 2.2. ([18, Theorem 0.1]) *Let P be a projective S-scheme and let* $\mathrm{sp}\colon P(K) \to P(k)$ *be the specialization map, given by $x \mapsto \overline{\{x\}} \cap P_s$. Let $V_1 \subset P_\eta$ and $V_2 \subset P_s$ be two open dense subsets of P_η and P_s, respectively. Assume that* sp *is surjective, P has irreducible fibers and P_s is a rational variety over k. Then the set*

$$U := V_1(K) \cap \mathrm{sp}^{-1}(V_2(k))$$

is non-empty.

Proof. This is extracted from the middle of the proof of [18, Theorem 0.1]. Before we give the proof, we note that if $x \in P(K)$, then the map $\overline{\{x\}} \to S$ must be an isomorphism and hence $\overline{\{x\}} \cap P_s$ is a unique closed point. In particular, the map $\mathrm{sp}\colon P(K) \to P(k)$ is well-defined.

Let $Z_1 = P_\eta \setminus V_1$ and $Z_2 = P_s \setminus V_2$ be the (reduced) closed complements of V_1 and V_2, respectively. Write $\overline{Z_1}$ for the closure of Z_1 in P. One clearly has that $Z_1(K) \subset \mathrm{sp}^{-1}((\overline{Z_1} \cap P_s)(k))$, so that the interesting set U contains $\mathrm{sp}^{-1}((V_2 \setminus (\overline{Z_1} \cap P_s))(k))$. Since sp is surjective by assumption, it's enough to observe that $(V_2 \setminus (\overline{Z_1} \cap P_s))(k)$ is non-empty. Now, we are given that V_2 is a dense open subset and $(\overline{Z_1} \cap P_s)$ is a proper closed subset of the irreducible scheme P_s. It follows that $V_2 \setminus (\overline{Z_1} \cap P_s)$ is open and dense in P_s. Since k is infinite and P_s is rational over k, one knows that $V_2 \setminus (\overline{Z_1} \cap P_s)(k)$ is dense in P_s. This finishes the proof. $\square$

If we take $P = (\mathbb{P}^N_S)^\vee$, the three conditions of the Lemma are satisfied. Since any hyperplane $H \subset \mathbb{P}^S_N$ is completely determined by its generic fiber H_η (as $(\mathbb{P}^N_S)^\vee(S) = (\mathbb{P}^N_K)^\vee(K)$), we see that the 'good' hyperplanes over S are parameterized by subsets of the form $V(K) \cap \mathrm{sp}^{-1}(U(k))$, for good open subsets V of $(\mathbb{P}^N_K)^\vee$ and U of $(\mathbb{P}^N_k)^\vee$, representing the prescribed behavior of the generic fiber and of the special fiber of H. We call a hyperplane H corresponding to a K-rational point of a set of the form $V(K) \cap \mathrm{sp}^{-1}(U(k))$ *general*. Our first application is the following proposition.

Proposition 2.3. *Let $\mathcal{X} \subset \mathbb{P}^N_S$ be as in Subsection* 2.1 *such that $d \geq 2$. Then a general hyperplane $H \subset \mathbb{P}^N_S$ intersects $\mathcal{X}$ transversely,* i.e., *the fiber product $\mathcal{X} \times_{\mathbb{P}^N_S} H$ is regular and flat S-scheme. If the generic fiber $\mathcal{X}_\eta$ of $\mathcal{X}$ is smooth over K, then H_η is smooth as well.*

Proof. We first note that since $\mathcal{X} \to S$ is flat and both $\mathcal{X}$ and S are regular, it follows that $X = (\mathcal{X}_s)_{\mathrm{red}}$ is equi-dimensional of dimension d. We begin by claiming that there exists an open subset U of $(\mathbb{P}^N_k)^\vee$ with the dense subset $U(k)$ of k-rational points such that the following hold. Let H be the hyperplane of $\mathbb{P}^N_k$ lying in $U(k)$. Then H does not contain any component of X, and if h denotes the image in $\mathcal{O}_{X,x}$ of a local equation for H at a closed point $x \in X$, either h is a unit or $h \in \mathfrak{m}_{X,x} \setminus \mathfrak{m}^2_{X,x}$.

It is clear that there exists a dense open subset U' of $(\mathbb{P}^N_k)^\vee$ such that no hyperplane corresponding to a k-rational point of U' contains any irreducible component of X. So we only need to find an open subset U of $(\mathbb{P}^N_k)^\vee$ with the dense subset $U(k)$ such that if H is the hyperplane of $\mathbb{P}^N_k$ corresponding to a point of $U(k)$ and if h denotes the image in $\mathcal{O}_{X,x}$ of a local equation for H at a closed point $x \in X$, either h is a unit or $h \in \mathfrak{m}_{X,x} \setminus \mathfrak{m}^2_{X,x}$.

To prove this latter claim, we first assume that $k = \overline{k}$ is separably (hence algebraically, since k is perfect) closed. Let W be the incidence variety $W \subset X \times (\mathbb{P}^N_k)^\vee$ consisting of points (x, H) such that either H contains a component of X or H does not contain any component of X but for any local equation h of H at x, one has $h \in \mathfrak{m}^2_{X,x} \subset \mathcal{O}_{X,x}$. We need to estimate the dimension of W.

Let $V = H^0(\mathbb{P}^N_k, \mathcal{O}_{\mathbb{P}^N_k}(1))$ be the $(N+1)$-dimensional k-vector space of linear forms, with basis $\{X_0, X_1, \dots, X_n\}$. Let $x \in X$ be a closed point. Up to a change of coordinates, we can assume that the hyperplane cut out by X_0 does not pass through x. We then get an isomorphism $V \xrightarrow{\simeq} \mathcal{O}_{\mathbb{P}^N_k,x}/\mathfrak{m}^2_{\mathbb{P}^N_k,x}$, sending X_0 to 1. By composition, we have a surjection

$$\phi_x \colon V \twoheadrightarrow \mathcal{O}_{X,x}/\mathfrak{m}^2_{X,x}$$

and the kernel of ϕ_x is the k-vector space $V_x = \{H \in (\mathbb{P}^N_k)^\vee(k) \mid x \in H \text{ and } h \in \mathfrak{m}^2_{X,x}\}$. Moreover, V_x consists precisely of the hyperplanes which are bad at x. Notice now that we have an exact sequence of k-vector spaces

$$0 \to \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}^2_{X,x} \to \mathcal{O}_{X,x}/\mathfrak{m}_{X,x} = k \to 0.$$

In particular, we get $\dim_k(\mathcal{O}_{X,x}/\mathfrak{m}^2_{X,x}) \geq 1 + \dim(\mathcal{O}_{X,x}) = 1 + d$. Thus $\dim_k(V_x) \leq (N+1) - (d+1) = N - d$.

If W_x denotes the fiber at x of W along the first projection $p_1 \colon W \to X \times (\mathbb{P}^N_k)^\vee \to X$, then we have $W_x = \mathbb{P}(V_x)$ and this implies from the previous estimate that $\dim_k(W_x) \leq N - d - 1$. Since the projection p_1 is surjective, X is equi-dimensional of dimension d, and for each $x \in X$, the fiber W_x is a projective space of dimension at most $N - d - 1$, we deduce that W has dimension at most $(N - d - 1) + d = N - 1$. Since X is proper over k, the second projection map $p_2 \colon W \to (\mathbb{P}^N_k)^\vee$ is closed, hence the image is a proper closed subset of dimension at most $N-1$. We conclude that $U := (\mathbb{P}^N_k)^\vee \setminus p_2(W)$ is open and dense in $(\mathbb{P}^N_k)^\vee$.

Suppose now that k is an arbitrary infinite perfect field and let $\overline{k}$ be an algebraic closure of k. Let $X_{\overline{k}}$ denote the base change of X to $\overline{k}$ and let $U \subset (\mathbb{P}^N_{\overline{k}})^\vee$ be the dense open subset of good hyperplanes over $\overline{k}$ obtained as above. Since k is infinite and $(\mathbb{P}^N_k)^\vee$ is rational, we know that the set of closed points in $(\mathbb{P}^N_{\overline{k}})^\vee$ which are defined over k is dense in U. Let $H \in U(k)$ be any such point. Let $x \in X$ be any closed point and let h denote the local equation of H in $\mathcal{O}_{X,x}$. Suppose that h is not a unit in $\mathcal{O}_{X,x}$ so that $h \in \mathfrak{m}_{X,x}$.

We know that $\pi^{-1}(x)$ is a finite set of closed points $\{x_1, \ldots, x_r\}$, where $\pi : X_{\overline{k}} \to X$ is the projection. Moreover, $H_{\overline{k}}$ has the property that its local equation h lies in $\mathfrak{m}_{X_{\overline{k}},x_i} \setminus \mathfrak{m}^2_{X_{\overline{k}},x_i}$ for each i. It follows that h must lie in $\mathfrak{m}_{X,x} \setminus \mathfrak{m}^2_{X,x}$. In other words, there is an open subset $U \subset (\mathbb{P}^N_k)^\vee$ with the dense subset $U(k)$ such that every member of $U(k)$ satisfies the desired property. This proves the claim.

We now let $\mathrm{sp}\colon (\mathbb{P}^N_K)^\vee(K) \to (\mathbb{P}^N_k)^\vee(k)$ be the specialization map, and let H be any hyperplane corresponding to a K-rational point of $\mathrm{sp}^{-1}(U(k))$ (note that this set is non-empty). Since this is a point in a projective space, say of coordinates $(a_0 : a_1 : \ldots : a_N)$, we can assume that not all the a_i's are divisible by π. In particular, H is not vertical, *i.e.*, it is not contained in the special fiber $\mathbb{P}^N_k$. Hence it is automatically flat over S.

Let $x \in X$ be a closed point and let h be the image in $\mathcal{O}_{\mathcal{X},x}$ of a local equation defining $\mathcal{X} \cdot H = \mathcal{X} \times_S H$ in a neighborhood of x. If h is a unit in $\mathcal{O}_{\mathcal{X},x}$, then $x \notin \mathcal{X} \cdot H$ and there is nothing to say. Assume then that $h \in \mathfrak{m}_{\mathcal{X},x} \subset \mathcal{O}_{\mathcal{X},x}$ and write $\overline{h}$ for the image of h in $\mathcal{O}_{X,x}$. By construction, $\overline{h}$ is a local equation for $X \cdot H_s$ and hence $\overline{h} \in \mathfrak{m}_{X,x} \setminus \mathfrak{m}^2_{X,x}$ by our choice of U. But this forces $h \in \mathfrak{m}_{\mathcal{X},x} \setminus \mathfrak{m}^2_{\mathcal{X},x}$ as well. Since $\mathcal{O}_{\mathcal{X},x}$ is regular by assumption, this implies that $\mathcal{O}_{\mathcal{X},x}/(h) = \mathcal{O}_{\mathcal{X}\cdot H,x}$ is a regular local ring. We have thus shown that every closed point $x \in (\mathcal{X} \cdot H)_s$ has an open neighborhood in $\mathcal{X} \cdot H$ where $\mathcal{X} \cdot H$ is regular. Since $\mathcal{X}$ is proper over S, these neighborhoods form a cover of $\mathcal{X} \cdot H$, proving that $\mathcal{X} \cdot H$ is regular, as required.

For the last assertion, suppose that $\mathcal{X}_\eta$ is smooth over K. In this case, the classical theorem of Bertini (see, for example, [19, 6.11]) asserts that there exists a dense Zariski open set $V \subset (\mathbb{P}^N_K)^\vee$ parametrizing hyperplanes H_η of $\mathbb{P}^N_K$ such that the intersection $\mathcal{X}_\eta \cdot H_\eta$ is smooth. It is then enough to take $H \in V(K) \cap \mathrm{sp}^{-1}(U(k))$, which is non-empty by Lemma 2.2, to get a general hyperplane of $\mathbb{P}^N_S$ which satisfies all the required conditions. □

Remark 2.4. The proof of Proposition 2.3 gives in fact a bit more. In the setting of this proposition, we can consider the following situation. Let $(P)_s$ be any property which is generically satisfied by a hyperplane section of X in $\mathbb{P}^N_k$. An example of such property could be 'being Cohen-Macaulay' if X is Cohen-Macaulay, or 'being irreducible' if X is irreducible (see [19]). Here, generically means that the property is satisfied by each hyperplane in a open dense subset V_P of $(\mathbb{P}^N_k)^\vee$. The set $U \cap V_P$ for the open set U constructed above is then open and dense in $(\mathbb{P}^N_k)^\vee$. Thus, any hyperplane H of $\mathbb{P}^N_S$ which corresponds to a K-rational point of $\mathrm{sp}^{-1}((U \cap V_P)(k))$ will intersect $\mathcal{X}$ transversely, and its special fiber will moreover satisfy the property (P).

We will now show that, under some extra conditions, there is a weak version of the Theorem of Altman and Kleiman [1] on hypersurface sections containing a subscheme. The proof of this fact uses a combination of ideas from Bloch's appendix to [11] and from [36, Theorem 4.2].

Proposition 2.5. *Let $\mathcal{X} \subset \mathbb{P}^N_S$ be as in Subsection 2.1 such that $d \geq 2$. Let $Z \subset \mathcal{X}$ be a regular, integral, flat relative 0-cycle over S. Let $\mathcal{O}_{\mathcal{X}}(1)$ be the restriction of the line bundle $\mathcal{O}_{\mathbb{P}^N_S}(1)$ to $\mathcal{X}$, and let $\mathcal{I} \subset \mathcal{O}_{\mathbb{P}^N_S}$ be the ideal sheaf of Z in $\mathbb{P}^N_S$. Assume that $Z \cap X$ is supported on one closed point $x \in X$. Then, for all integers $n \gg 0$ and a general section $\sigma \in H^0(\mathbb{P}^N_S, \mathcal{I}(n))$, the hypersurface $H = (\sigma)$ defined by σ has the following properties:*

(1) *$\mathcal{X} \cdot H$ is regular, flat and projective over S;*
(2) *$H \supset Z$.*

Proof. Let $W = Z \times_S X$ be the scheme-theoretic intersection of Z with the reduced special fiber. We start by noting that the embedding dimension $e_x(W) := \dim_{k(x)} \mathfrak{m}_{W,x}/\mathfrak{m}^2_{W,x}$ is at most 1. Indeed, $W \subset Z$ and Z is regular, finite and flat over S by assumption. Hence $e_x(W) \leq e_x(Z) = \dim(Z) = 1$. As a consequence, if we let $I_{W,x}$ denote the ideal of W in $\mathcal{O}_{X,x}$, we see that $I_{W,x}/(I_{W,x} \cap \mathfrak{m}^2_{X,x}) \neq 0$. In fact, suppose that $I_{W,x} \subset \mathfrak{m}^2_{X,x}$. Then $\mathfrak{m}_{X,x}/(\mathfrak{m}^2_{X,x} + I_{W,x}) = \mathfrak{m}_{X,x}/\mathfrak{m}^2_{X,x}$ has dimension $d \geq 2$. But $\mathfrak{m}_{X,x}/(\mathfrak{m}^2_{X,x} + I_{W,x}) = \mathfrak{m}_{W,x}/\mathfrak{m}^2_{W,x}$ has dimension at most one as shown above. This leads to a contradiction.

Let $\overline{\mathcal{I}}$ be the ideal sheaf of W in $\mathbb{P}^N_k$ and let $n \gg 0$ be any integer such that $\overline{\mathcal{I}}(n)$ is generated by the global sections $V = H^0(\mathbb{P}^N_k, \overline{\mathcal{I}}(n)) \subset H^0(\mathbb{P}^N_k, \mathcal{O}(n))$. We now claim that there exists a non-empty open subset U in the space $\mathbb{P}(V)$ such that for any $\sigma \in U(k)$, the image σ_x of σ in $\mathcal{O}_{X,x}$ (for a closed point $x \in X$) is either a unit or an element of $\mathfrak{m}_{X,x} \setminus \mathfrak{m}^2_{X,x}$.

Since the linear system associated to a basis of V has base locus W, this condition is satisfied for σ in $U' \subset V$ for each $y \neq x$ thanks to the proof of Proposition 2.3, with U' open and non-empty. For $n \gg 0$, there is clearly another non-empty open $U'' \subset V$ such that for $\sigma \in U''$, the restriction of σ has non-zero image in $I_{W,x}/(I_{W,x} \cap \mathfrak{m}^2_{X,x})$ (which is itself non-zero by the argument above). Let $U = U' \cap U''$.

If $n \gg 0$, the map $a\colon H^0(\mathbb{P}^N_S, \mathcal{I}(n)) \to H^0(\mathbb{P}^N_k, \overline{\mathcal{I}}(n))$ is surjective. Then any $\sigma \in H^0(\mathbb{P}^N_S, \mathcal{I}(n))$ such that $a(\sigma) \in U$ will satisfy the conditions of the proposition. Indeed, it is clear by our choice that $H = (\sigma)$ contains Z, while the regularity of $\mathcal{X} \cdot H$ is proved exactly as in Proposition 2.3. □

Remark 2.6. The reader can easily see that when A is Henselian (which is the case for the rest of this text), the assumption in Proposition 2.5 that $Z \cap X$ be supported on one closed point $x \in X$, is redundant.

3. Lifting of zero-cycles

In this section we shall recall the definitions of the Chow groups which are used in the statements of the main results. We shall then show how the 0-cycles on the special fiber can be lifted to good 1-cycles on $\mathcal{X}$. Using this lifting, we shall give a proof of the base case of Theorem 1.1, namely, the case of relative curves. This case will be used in the next section to prove the general case of Theorem 1.1. We keep the notation of Subsection 2.1. Throughout this section, we shall assume that the base ring A is excellent and Henselian, with perfect residue field k which is not necessarily infinite.

3.1. The Chow groups of the model and the special fiber

Let $\mathcal{Z}_1(\mathcal{X})$ be the free Abelian group on the set of integral 1-dimensional cycles in $\mathcal{X}$. Let $\mathcal{R}_1(\mathcal{X})$ be the subgroup of $\mathcal{Z}_1(\mathcal{X})$ generated by the cycles which are rationally equivalent to zero (see, for example, [17, Section 1] or [15, Chapter 20]). Let $\mathrm{CH}_1(\mathcal{X}) = \mathcal{Z}_1(\mathcal{X})/\mathcal{R}_1(\mathcal{X})$ be the Chow group of 1-cycles on $\mathcal{X}$ modulo rational equivalence.

We call an integral cycle $Z \in \mathcal{Z}_1(\mathcal{X})$ *good* if it is flat over S and $Z \cap X_{\mathrm{sing}} = \emptyset$. We let $\mathcal{Z}_1^g(\mathcal{X}) \subset \mathcal{Z}_1(\mathcal{X})$ be the free Abelian group on the set of good cycles. In a similar spirit, we write $\mathcal{Z}_1^{vg}(\mathcal{X}) \subset \mathcal{Z}_1^g(\mathcal{X})$ for the free Abelian group on the set of integral flat 1-cycles Z which are good in the above sense and are regular as schemes. We call these cycles *very good* on $\mathcal{X}$.

As $\mathcal{X}$ is projective over S, it is an FA-scheme in the sense of [17, 2.2(1)]. Therefore, the moving Lemma of Gabber, Liu and Lorenzini [17, Theorem 2.3] tells us that the canonical map

$$\frac{\mathcal{Z}_1^g(\mathcal{X})}{\mathcal{Z}_1^g(\mathcal{X}) \cap \mathcal{R}_1(\mathcal{X})} \to \mathrm{CH}_1(\mathcal{X}) \tag{3.1}$$

is an isomorphism. In other words, every cycle $\alpha \in \mathrm{CH}_1(\mathcal{X})$ has a representative $\alpha = \sum_{i=1}^n n_i[Z_i]$ with each Z_i a good integral cycle. This will play a crucial role in the proofs of our main results.

We now recall the definition of Levine-Weibel Chow group of 0-cycles on X from [31] and its modified version from [4]. Let X_{reg} denote the disjoint union of the smooth loci of the d-dimensional irreducible components of X. A regular (or smooth) closed point of X will mean a closed point lying in X_{reg}. Let $Y \subsetneq X$ be a closed subset not containing any d-dimensional component of X such that $X_{\mathrm{sing}} \subseteq Y$. Let $\mathcal{Z}_0(X, Y)$ be the free Abelian group on closed points of $X \setminus Y$. We shall often write $\mathcal{Z}_0(X, X_{\mathrm{sing}})$ as $\mathcal{Z}_0(X)$.

Definition 3.1. Let C be a reduced scheme which is of pure dimension one over k. We shall say that a pair (C, Z) is *a good curve relative to* X if there exists a finite morphism $\nu\colon C \to X$ and a closed proper subscheme $Z \subsetneq C$ such that the following hold.

(1) No component of C is contained in Z;
(2) $\nu^{-1}(X_{\text{sing}}) \cup C_{\text{sing}} \subseteq Z$;
(3) ν is local complete intersection at every point $x \in C$ such that $\nu(x) \in X_{\text{sing}}$.

Let (C, Z) be a good curve relative to X and let $\{\eta_1, \cdots, \eta_r\}$ be the set of generic points of C. Let $\mathcal{O}_{C,Z}$ denote the semilocal ring of C at $S = Z \cup \{\eta_1, \cdots, \eta_r\}$. Let $k(C)$ denote the ring of total quotients of C and write $\mathcal{O}_{C,Z}^\times$ for the group of units in $\mathcal{O}_{C,Z}$. Notice that $\mathcal{O}_{C,Z}$ coincides with $k(C)$ if $|Z| = \emptyset$. As C is Cohen-Macaulay, $\mathcal{O}_{C,Z}^\times$ is the subgroup of group of units in the ring of total quotients $k(C)^\times$ consisting of those $f \in \mathcal{O}_{C,x}$ which are regular and invertible for every $x \in Z$ (see [10, Section 1] for further details).

Given any $f \in \mathcal{O}_{C,Z}^\times \hookrightarrow k(C)^\times$, we denote by $\operatorname{div}_C(f)$ (or $\operatorname{div}(f)$ in short) the divisor of zeros and poles of f on C, which is defined as follows. If $C_1, \ldots, C_r$ are the irreducible components of C, and f_i is the factor of f in $k(C_i)$, we set $\operatorname{div}(f)$ to be the 0-cycle $\sum_{i=1}^r \operatorname{div}(f_i)$, where $\operatorname{div}(f_i)$ is the usual divisor of a rational function on an integral curve in the sense of [15]. As f is an invertible regular function on C along Z, $\operatorname{div}(f) \in \mathcal{Z}_0(C, Z)$.

By definition, given any good curve (C, Z) relative to X, we have a push-forward map $\mathcal{Z}_0(C, Z) \xrightarrow{\nu_*} \mathcal{Z}_0(X)$. We shall write $\mathcal{R}_0(C, Z, X)$ for the subgroup of $\mathcal{Z}_0(X)$ generated by the set $\{\nu_*(\operatorname{div}(f)) | f \in \mathcal{O}_{C,Z}^\times\}$. Let $\mathcal{R}_0(X)$ denote the subgroup of $\mathcal{Z}_0(X)$ generated by the image of the map $\mathcal{R}_0(C, Z, X) \to \mathcal{Z}_0(X)$, where (C, Z) runs through all good curves relative to X. We let $\mathrm{CH}_0^{BK}(X) = \frac{\mathcal{Z}_0(X)}{\mathcal{R}_0(X)}$.

If we let $\mathcal{R}_0^{LW}(X)$ denote the subgroup of $\mathcal{Z}_0(X)$ generated by the divisors of rational functions on good curves as above, where we further assume that the map $\nu : C \to X$ is a closed immersion, then the resulting quotient group $\mathcal{Z}_0(X)/\mathcal{R}_0^{LW}(X)$ is denoted by $\mathrm{CH}_0^{LW}(X)$. Such curves on X are called the *Cartier curves*. There is a canonical surjection $\mathrm{CH}_0^{LW}(X) \twoheadrightarrow \mathrm{CH}_0^{BK}(X)$. The Chow group $\mathrm{CH}_0^{LW}(X)$ was discovered by Levine and Weibel [31] in an attempt to describe the Grothendieck group of a singular scheme in terms of algebraic cycles. The modified version $\mathrm{CH}_0^{BK}(X)$ was introduced in [4].

We remark here that the definition of $\mathrm{CH}_0^{LW}(X)$ given above is mildly different from the one given in [31] because we do not allow non-reduced Cartier curves. However, it does agree with the definition of [31] if k is infinite by [28, Lemmas 1.3, 1.4]. Note that over finite fields the situation is unclear (but see [6] for the case of surfaces), since the standard norm trick to reduce to the case of infinite fields for comparison does not work for the Levine-Weibel Chow group. The situation is substantially better if one uses its variant [4] instead.

3.2. Lifting 0-cycles on the special fiber to 1-cycles on $\mathcal{X}$

From the above definitions of $\mathrm{CH}_1(\mathcal{X})$ and $\mathrm{CH}_0^{LW}(X)$, it is not clear if the 1-cycles on $\mathcal{X}$ always restrict to admissible 0-cycles on X, nor if the restriction (whenever defined) preserves the rational equivalence. This question will be addressed in the next section. Here, we solve the reverse problem, namely, we show that the Levine-Weibel 0-cycles on X can be lifted to good 1-cycles on $\mathcal{X}$, following the idea of [10]. Using this lifting, we shall prove Theorem 1.1. We fix an integer m prime to the exponential characteristic of k and let $\Lambda = \mathbb{Z}/m\mathbb{Z}$. For an Abelian group M, we let $M_\Lambda = M \otimes_{\mathbb{Z}} \Lambda$.

Let $[Z] \in \mathcal{Z}_1^g(\mathcal{X})$ be an integral good 1-cycle. Intersecting $[Z]$ with the reduced special fiber X gives rise to a 0-cycle $[Z \cap X]$, which is supported in the regular locus of X. Here, $[Z \cap X]$ is the 0-cycle in $\mathcal{Z}_0(X)$ associated to the (possibly non-reduced) 0-dimensional scheme-theoretic intersection $Z \cap X$. This gives rise to the *restriction* homomorphism on the cycle group

$$\widetilde{\rho} : \mathcal{Z}_1^g(\mathcal{X}) \to \mathcal{Z}_0(X, X_{\mathrm{sing}}), \quad [Z] \mapsto [Z \cap X]. \tag{3.2}$$

To prove Theorem 1.1, we begin by recalling the following result. The proof is classical, and in this form is essentially taken from [10]. We review the proof in order to fix our notation.

Proposition 3.2. ([10, Section 4]) *Given a regular closed point $x \in X$, there exists an integral* 1*-cycle $Z_x \subset \mathcal{X}$ which is regular, finite and flat over S such that $Z_x \times_S X = \{x\}$ scheme-theoretically. In particular, the restriction map $\widetilde{\rho}$ of* (3.2) *is surjective.*

Proof. Let $x \in X_{\mathrm{reg}}$ be a closed point and let $\mathcal{O}_{\mathcal{X},x}$ be the local ring of $\mathcal{X}$ at x. Since $\mathcal{X}$ is regular, $\mathcal{O}_{\mathcal{X},x}$ is a regular local ring. In particular, it is a unique factorization domain. There is then a prime element $\sigma \in \mathfrak{m}_{\mathcal{X},x} \setminus \mathfrak{m}^2_{\mathcal{X},x}$ and an integer $n > 0$ such that $\sigma^n = \pi c$, where $\pi \in \mathcal{O}_{\mathcal{X},x}$ is the uniformizer of A and c is a unit. Indeed, π can not be a product of distinct prime elements in $\mathcal{O}_{\mathcal{X},x}$, since $\mathcal{O}_{\mathcal{X},x} \otimes_A (A/(\pi)) = \mathcal{O}_{\mathcal{X}_s,x}$ has a unique minimal prime (its reduction, $\mathcal{O}_{X,x}$ is a regular local ring). We can now complete σ to a regular sequence $(\sigma, a_1, \ldots, a_d)$ generating the maximal ideal $\mathfrak{m}_{\mathcal{X},x}$ such that the images $(\overline{a}_1, \ldots, \overline{a}_d)$ in $\mathcal{O}_{X,x} = \mathcal{O}_{\mathcal{X},x}/(\sigma)$ form a regular sequence, generating the maximal ideal $\mathfrak{m}_{X,x}$.

Let $\mathrm{Spec}\,(\mathcal{O}_{\mathcal{X},x}/(a_1, \ldots, a_d))$ be the closed subscheme of $\mathrm{Spec}\,(\mathcal{O}_{\mathcal{X},x})$ associated to the ideal $(a_1, \ldots, a_d)$. It is clearly integral, regular, local, 1-dimensional and flat over S. If we let $\widetilde{Z}_x$ denote its closure in $\mathcal{X}$, then $\widetilde{Z}_x$ is projective and dominant of relative dimension zero over A. In particular, it is finite and flat over S. We can therefore write $\widetilde{Z}_x = \mathrm{Spec}\,(B)$.

Since S is Henselian, the finite A-algebra B is totally split. Hence, there is a unique irreducible component Z_x of $\widetilde{Z}_x$ such that $x \in Z_x$. The scheme Z_x is then regular because its local ring at the unique closed point x agrees with $\mathcal{O}_{\mathcal{X},x}/(a_1, \ldots, a_d)$. Furthermore, Z_x is integral, finite and flat over S with $Z_x \times_S X = \{x\}$. $\square$

Note that thanks to the proposition above, we have in fact shown that the composite map

$$\mathcal{Z}_1^{vg}(\mathcal{X}) \hookrightarrow \mathcal{Z}_1^g(\mathcal{X}) \xrightarrow{\widetilde{\rho}} \mathcal{Z}_0(X, X_{\mathrm{sing}})$$

is surjective.

3.3. The case of relative dimension one

We continue with the assumption that A is Henselian and k is perfect (but not necessarily infinite). Suppose that $\dim_S(\mathcal{X}) = 1$ so that $\mathcal{X}$ is a family of projective *curves* over S. We shall now give the proof of Theorem 1.1 in this case.

Since X is reduced by construction, we have by [4, Lemma 3.12], the canonical isomorphisms $\mathrm{CH}_0^{LW}(X) \xrightarrow{\simeq} \mathrm{CH}_0^{BK}(X) \xrightarrow{\simeq} \mathrm{Pic}(X) \cong H^1_{\text{ét}}(X, \mathbb{G}_m)$. As a scheme, $\mathcal{X}$ is integral and purely two-dimensional so that we can identify $\mathrm{CH}_1(\mathcal{X})$ with $\mathrm{CH}^1(\mathcal{X})$. Since $\mathcal{X}$ is moreover separated, regular (hence locally factorial) and Noetherian, there are classical isomorphisms $\mathrm{CH}^1(\mathcal{X}) \xrightarrow{\simeq} \mathrm{Pic}(\mathcal{X}) \cong H^1_{\text{ét}}(\mathcal{X}, \mathbb{G}_m)$. Tensoring these groups with $\Lambda = \mathbb{Z}/m$, the Kummer sequence gives us injections

$$\mathrm{CH}^1(\mathcal{X})_\Lambda \xrightarrow{\cong} H^1_{\text{ét}}(\mathcal{X}, \mathbb{G}_m)_\Lambda \hookrightarrow H^2_{\text{ét}}(\mathcal{X}, \Lambda(1))$$

$$\mathrm{CH}_0^{LW}(X)_\Lambda \xrightarrow{\cong} H^1_{\text{ét}}(X, \mathbb{G}_m)_\Lambda \hookrightarrow H^2_{\text{ét}}(X, \Lambda(1)).$$

Using these injections, we get a diagram of solid arrows

$$\begin{tikzcd}
& \mathcal{Z}_1^g(\mathcal{X})_\Lambda \arrow[r, two heads, "\widetilde{\rho}"] \arrow[dl, "\alpha_{\mathcal{X}}"'] \arrow[d] & \mathcal{Z}_0(X, X_{\mathrm{sing}})_\Lambda \arrow[d] \arrow[dr, "\alpha_X"] \arrow[dl, dashed] & \\
\mathrm{CH}_1(\mathcal{X})_\Lambda \arrow[r, "\cong"] \arrow[dr, "cyc^{\text{ét}}_{\mathcal{X}}"'] & \mathrm{Pic}(\mathcal{X})_\Lambda \arrow[r, "\rho"] \arrow[d, hook] & \mathrm{Pic}(X)_\Lambda \arrow[d, hook] & \mathrm{CH}_0^{LW}(X)_\Lambda \arrow[l, "\cong"'] \arrow[dl, "cyc^{\text{ét}}_X"] \\
& H^2_{\text{ét}}(\mathcal{X}, \Lambda(1)) \arrow[r, "\cong"] & H^2_{\text{ét}}(X, \Lambda(1)). &
\end{tikzcd} \tag{3.3}$$

All horizontal arrows in the middle are induced by the restriction to the reduced special fiber. In particular, the two squares in the middle are commutative. The two triangles on the top left and top right can be easily seen to be commutative by recalling the construction of the isomorphism between the Picard group and the Chow group of codimension one cycles. The two triangles on the bottom left and bottom right commute by the definition of the cycle class maps to étale cohomology.

The bottom horizontal arrow (3.3) is an isomorphism by the rigidity theorem for étale cohomology (a consequence of the proper base change theorem, see [33, Chapter VI, Corollary 2.7]). The top horizontal arrow is surjective by Proposition 3.2. From the commutativity of (3.3), we immediately see that the canonical map $\alpha_{\mathcal{X}} : \mathcal{Z}_1^g(\mathcal{X})_\Lambda \twoheadrightarrow \mathrm{CH}_1(\mathcal{X})_\Lambda$ factors via $\widetilde{\rho}$. Equivalently, we have $\mathrm{Ker}(\widetilde{\rho}) \subseteq \mathrm{Ker}(\alpha_{\mathcal{X}})$. This gives the dashed arrow $\widetilde{\gamma} : \mathcal{Z}_0(X, X_{\mathrm{sing}})_\Lambda \to \mathrm{CH}_1(\mathcal{X})_\Lambda$, which is automatically surjective.

A second inspection of (3.3), using this time the fact that $\mathrm{CH}_1(\mathcal{X})_\Lambda \to H^2_{\text{ét}}(\mathcal{X}, \Lambda(1))$ is injective, shows similarly that $\mathrm{Ker}(\alpha_X) \subseteq \mathrm{Ker}(\widetilde{\gamma})$. Combining all this, we finally get a surjective group homomorphism γ fitting in the commutative diagram

$$\begin{tikzcd} \mathrm{CH}_1(\mathcal{X})_\Lambda \arrow[d, hook] & \mathrm{CH}_0^{LW}(X)_\Lambda \arrow[l, "\gamma"'] \arrow[d, hook] \\ H^2_{\text{ét}}(\mathcal{X}, \Lambda(1)) \arrow[r, "\cong"] & H^2_{\text{ét}}(X, \Lambda(1)). \end{tikzcd} \tag{3.4}$$

We also deduce from (3.4) that γ has to be injective as well. Since γ is clearly an inverse of the map $\widetilde{\rho}$ on the generators, we have then shown the following result which proves Theorem 1.1 and a general form of the part (1) of Theorem 1.3 for curves.

Proposition 3.3. *Let A be an excellent Henselian discrete valuation ring with perfect residue field. Let $\mathcal{X}$ be a regular scheme, flat and projective over S of relative dimension one. Then the restriction homomorphism $\widetilde{\rho}$ of* (3.2) *induces an isomorphism*

$$\rho\colon \mathrm{CH}_1(\mathcal{X})_\Lambda \xrightarrow{\cong} \mathrm{CH}_0^{LW}(X)_\Lambda.$$

4. Proof of Theorem 1.1

We shall now prove Theorem 1.1 using the Bertini theorems of Section 2 and the lifting proposition of Section 3. We assume A to be an excellent Henselian discrete valuation ring with perfect residue field k. The rest of the assumptions and notation are same as in Subsection 2.1.

4.1. Factorization of $\alpha_{\mathcal{X}}$ via $\widetilde{\rho}$

We begin by showing the first part of Theorem 1.1, *i.e.*, we show that the canonical surjection $\alpha_{\mathcal{X}} : \mathcal{Z}_1^g(\mathcal{X})_\Lambda \twoheadrightarrow \mathrm{CH}_1(\mathcal{X})_\Lambda$ factors through $\widetilde{\rho}$. This is a consequence of the following result, whose proof goes through the steps of [10, Proposition 4.1], using Proposition 2.5 instead of the Bertini Theorem of Jannsen-Saito proved in [36].

Proposition 4.1. ([10, Proposition 4.1]) *Let $Z \in \mathcal{Z}_1^g(\mathcal{X})$ be a good, integral 1-cycle and let $n[x] = [Z \cap X]$ for some $x \in X_{\mathrm{reg}}$ and $n > 0$. Then $\alpha_{\mathcal{X}}(Z - nZ_x) = 0$ in $\mathrm{CH}_1(\mathcal{X})_\Lambda$, where Z_x is as in Proposition* 3.2.

Proof. By the standard pro-ℓ-extension argument, we can assume that the residue field of A is infinite. The proof is now by induction on the relative dimension of $\mathcal{X}$ over S. The case $d = 0$ is trivial and the case $d = 1$ is provided by Proposition 3.3. We now assume that $d \geq 2$.

Assume first that Z is regular as well. The general case will be treated later, using a trick due to Bloch [11, Appendix A]). By an iterated application of Proposition 2.5, we can find

(1) a hypersurface section H of $\mathcal{X}$ which is regular, flat and projective over S such that $Z \subset H$, and

(2) a relative curve H' over S (*i.e.*, $\dim_S H' = 1$) which is regular, flat and projective over S and contains Z_x.

We can also assume that $Z'' := H' \cap H$ is regular as well, and that $H' \cap H \cap X$ consists only of the reduced point x. Note that we can do this since $x \in X$ is in the regular locus of X, so that we can choose H' and H which meets transversely there.

By our induction hypothesis, we have that $\alpha_H(Z - nZ'') = 0$ in $\mathrm{CH}_1(H)_\Lambda$. Moreover, it follows from Proposition 3.3 that $\alpha_{H'}(Z'' - Z_x) = 0$ in $\mathrm{CH}_1(H')_\Lambda = \mathrm{Pic}(H')_\Lambda$. In particular, we get $n\alpha_{H'}(Z'' - Z_x) = 0$. But then, we get

$$\alpha_{\mathcal{X}}(Z - nZ_x) = (\iota_H)_*(\alpha_H(Z - nZ'')) + (\iota_{H'})_*(\alpha_{H'}(nZ'' - nZ_x)) = 0$$

in $\mathrm{CH}_1(\mathcal{X})_\Lambda$, as required. Here, ι_H (respectively $\iota_{H'}$) is the inclusion $H \hookrightarrow \mathcal{X}$ (respectively $H' \hookrightarrow \mathcal{X}$).

Suppose now that Z is not necessarily regular. Following an idea of Bloch, we let Z^N be the normalization of Z. Since A is excellent and Z is finite over A (as it is a good 1-cycle), the map $Z^N \to Z$ is finite. In particular, there is a factorization

$$\begin{tikzcd} Z^N \arrow[r, hook] \arrow[d] & \mathbb{P}^M_{\mathcal{X}} \arrow[d, "q"] \\ Z \arrow[r, hook] & \mathcal{X}, \end{tikzcd}$$

where q is the canonical projection. We are then reduced to prove the statement in $\mathbb{P}^M_{\mathcal{X}}$ for Z^N and any regular lift of Z_x to $\mathbb{P}^M_{\mathcal{X}}$, chosen so that it contains $Z^N \cap \mathbb{P}^M_X$. Since Z^N is now regular, the claim follows from the previous case. □

An immediate consequence of Proposition 4.1 is the following.

Corollary 4.2. *The lifting of 0-cycles of Proposition* 3.2 *gives rise to a well-defined group homomorphism* $\widetilde{\gamma} : \mathcal{Z}_0(X, X_{\mathrm{sing}})_\Lambda \to \mathrm{CH}_1(\mathcal{X})_\Lambda$ *such that the diagram*

$$\begin{tikzcd} \mathcal{Z}_1(\mathcal{X})_\Lambda \arrow[r, "\widetilde{\rho}"] \arrow[dr, "\alpha_{\mathcal{X}}"'] & \mathcal{Z}_0(X, X_{\mathrm{sing}})_\Lambda \arrow[d, "\widetilde{\gamma}"] \\ & \mathrm{CH}_1(\mathcal{X})_\Lambda \end{tikzcd} \tag{4.1}$$

commutes.

Proof. Let $Z \in \mathcal{Z}_1^g(\mathcal{X})$ be a good, integral 1-cycle. Since A is Henselian and Z is finite over A, the intersection $Z \cap X$ must be supported on a (regular) closed point, say, $x \in X$. In particular, we must have $[Z \cap X] = n[x]$ for some integer $n > 0$. Now, it follows from Proposition 4.1 that

$$\alpha_{\mathcal{X}}([Z]) - \widetilde{\gamma} \circ \widetilde{\rho}([Z]) = \alpha_{\mathcal{X}}([Z] - n[Z_x]) = 0$$

and this proves the corollary. □

4.2. Factorization of $\widetilde{\gamma}$ through rational equivalence

Now that we have constructed the map $\widetilde{\gamma}$ at the level of the cycle groups, our next goal is to show that it factors through the cohomological Chow group $\mathrm{CH}_0^{LW}(X)_\Lambda$ of the reduced special fiber X. In fact, we shall show (probably) more in the sense that $\widetilde{\gamma}$ actually has a factorization

$$\widetilde{\gamma} : \mathcal{Z}_0(X, X_{\mathrm{sing}})_\Lambda \twoheadrightarrow \mathrm{CH}_0^{LW}(X)_\Lambda \twoheadrightarrow \mathrm{CH}_0^{BK}(X)_\Lambda \to \mathrm{CH}_1(\mathcal{X})_\Lambda. \tag{4.2}$$

As we will see below, apart from giving us a stronger statement, the approach of working with $\mathrm{CH}_0^{BK}(X)$ also allows us to simplify the Cartier curves that give relations in $\mathcal{R}_0^{LW}(X)$ which we want to kill in $\mathrm{CH}_1(\mathcal{X})_\Lambda$. It allows us to assume that the Cartier curves are regularly embedded in X. This is an essential requirement in our proof.

It is not known if the canonical map $\mathrm{CH}_0^{LW}(X) \twoheadrightarrow \mathrm{CH}_0^{BK}(X)$ is an isomorphism in general. We refer to [4, Theorem 3.17] for some positive results.

We shall closely follow the proof of [10, Theorem 5.1] (and we keep similar notation for the reader's convenience), with one simplification and one complication. The simplification is that using the Levine-Weibel Chow group (or, rather, its variant introduced in [4]), we don't have to deal with the "type-1" relations (see [10, Section 2.2]), arising from the relations in the Suslin homology group $H_0^S(X_{\mathrm{reg}})$. On the other hand, the complication is that without any assumption on the geometry of X, we have to consider arbitrary l.c.i. curves C (and not simply SNC subcurves in X as in *loc.cit.*). Note that these l.c.i. curves may not even be embedded inside X. In order to lift our complicated relations in X to the model $\mathcal{X}$, we shall use the argument of [17, Lemma 2.5].

We will need the following commutative algebra Lemma whose proof can be obtained from [32, Theorem 16.3].

Lemma 4.3. *Let R be a Noetherian local ring and let $I \subset R$ be an ideal generated by a regular sequence $a_1, \ldots, a_n$. Let $b_1, \ldots, b_n \in I$ be elements such that the image of $\{b_1, \ldots, b_n\}$ in I/I^2 is a basis over R/I. Then $b_1, \ldots, b_n$ is a regular sequence in R.*

Proposition 4.4. *The lifting map $\widetilde{\gamma} \colon \mathcal{Z}_0(X, X_{\mathrm{sing}})_\Lambda \to \mathrm{CH}_1(\mathcal{X})_\Lambda$ of Corollary 4.2 factors through $\mathrm{CH}_0^{BK}(X)_\Lambda$.*

Proof. Since the case of relative dimension one is already shown in Subsection 3.3, we shall assume that $d = \dim_S(\mathcal{X}) \geq 2$. We need to show that for any good curve $\nu \colon C \to X$ in the sense of Definition 3.1 and any rational function f on C which is regular along $\nu^{-1}(X_{\mathrm{sing}})$, we have $\widetilde{\gamma}(\nu_*(\mathrm{div}(f))) = 0$ in $\mathrm{CH}_1(\mathcal{X})_\Lambda$.

We will first show that this relation holds when the curve C is regularly embedded inside X (*i.e.*, when the morphism ν is a regular closed embedding). The general case will be handled by factoring ν as a regular closed embedding $C \hookrightarrow \mathbb{P}^N_X$ followed by the projection $\mathbb{P}^N_X \to X$, and using the fact that the Chow groups $\mathrm{CH}_1(\mathcal{X})$ and $\mathrm{CH}_0^{BK}(X)$ admit proper push-forward for smooth morphisms.

So, let $C \hookrightarrow X$ be such an embedded l.c.i. curve. Write C_∞ for the finite set of points $(C \cap X_{\text{sing}}) \cup \{\eta_1, \ldots, \eta_r\}$, where each η_i is a generic point of C and $C \cap X_{\text{sing}}$ denotes the set of closed points of the intersection of C with X_{sing}. Let $\mathcal{O}_{X,C_\infty}$ be the semi-local ring of X at C_∞ and let I_{C,C_∞} be the ideal of C in $\mathcal{O}_{X,C_\infty}$ so that $\mathcal{O}_{C,C_\infty} = \mathcal{O}_{X,C_\infty}/I_{C,C_\infty}$. By definition, C is regularly embedded at each point $x \in C \cap X_{\text{sing}}$, and it is regularly embedded at the generic points. Hence, as a module over $\mathcal{O}_{C,C_\infty}$, the conormal sheaf $I_{C,C_\infty}/I^2_{C,C_\infty}$ admits a free set of generators, given by the image in $I_{C,C_\infty}/I^2_{C,C_\infty}$ of a regular sequence $a_1, \ldots, a_{d-1}$ in $\mathcal{O}_{X,C_\infty}$.

We shall inductively modify the sequence $a_1, \ldots, a_{d-1}$ (without changing the induced basis of $I_{C,C_\infty}/I^2_{C,C_\infty}$) in order to construct a good lifting of C to the model $\mathcal{X}$, following the recipe of [17, Lemma 2.5]. First, we note that according to Definition 3.1, the curve C is not contained in X_{sing}. By a moving argument, we can also assume that C does not contain any component of X_{sing}. Indeed, the Cartier condition of C implies that it will contain a component of X_{sing} only if $\dim(X_{\text{sing}}) = 0$. On the other hand, in this latter case, we can use a moving argument to ensure that C does not hit X_{sing} (see [9, Lemma 1.3]). Thus, the ideal I_{C,C_∞} of $\mathcal{O}_{X,C_\infty}$ does not contain, and it is not contained in the localization of any minimal prime $\mathfrak{p}$ of X_{sing} in $\mathcal{O}_{X,C_\infty}$.

Up to possibly adding an element of I^2_{C,C_∞} to $a_1 \in I_{C,C_\infty} \subset \mathcal{O}_{X,C_\infty}$, we can now choose $\hat{a}_1 \in \mathcal{O}_{\mathcal{X},C_\infty}$, lifting a_1, with the property that $\hat{a}_1$ does not belong to any minimal prime of X_{sing} in $\mathcal{O}_{\mathcal{X},C_\infty}$. In other words, $V(\hat{a}_1)$ in Spec $(\mathcal{O}_{\mathcal{X},C_\infty})$ does not contain any irreducible component of X_{sing}. Moreover, each irreducible component of $V(\hat{a}_1)$ has codimension exactly one in $X_{\text{sing}} \times_{\mathcal{X}} \text{Spec}\,(\mathcal{O}_{\mathcal{X},C_\infty})$ with the reduced induced closed subscheme structure of X_{sing}. Note that thanks to Lemma 4.3, the modification by adding elements of I^2_{C,C_∞} gives another regular sequence defining I_{C,C_∞}.

We now fix $\hat{a}_1$ and a_1 chosen above, and proceed. Since locally $V(\hat{a}_1) \cap C = C$ in Spec $(\mathcal{O}_{X,C_\infty})$, the ideal I_{C,C_∞} is not contained in any minimal prime of $V(\hat{a}_1) \cap X_{\text{sing}}$. Thus, we can alter a_2 by an element of I^2_{C,C_∞} so that we can assume that a_2 in particular is not in any minimal prime of $V(\hat{a}_1) \cap X_{\text{sing}}$. We now lift a_2 to $\hat{a}_2 \in \mathcal{O}_{\mathcal{X},C_\infty}$ and look at $V(\hat{a}_1, \hat{a}_2)$ in Spec $(\mathcal{O}_{\mathcal{X},C_\infty})$. As before, it follows by our construction that each irreducible component of $V(\hat{a}_1, \hat{a}_2)$ has codimension exactly one in $X_{\text{sing}} \cap V(\hat{a}_1)$. We fix this $\hat{a}_2$ and the corresponding a_2. Again, $a_1, a_2, \ldots, a_{d-1}$ (with a_2 accordingly modified) form a regular sequence generating I_{C,C_∞}, thanks to Lemma 4.3.

In general, the choice of $\hat{a}_i$ depends on the previously chosen $\hat{a}_1, \ldots, \hat{a}_{i-1}$. It is chosen with the property that $\hat{a}_i$ is a unit at each generic point of $X_{\text{sing}} \cap V(\hat{a}_1, \ldots, \hat{a}_{i-1})$, and that $\hat{a}_i$ lifts $a_i \in I_{C,C_\infty}$. This can be achieved, up to elements of I^2_{C,C_∞}, since locally $V(\hat{a}_1, \ldots, \hat{a}_{i-1}) \cap C = C \not\supset V(\hat{a}_1, \ldots, \hat{a}_{i-1}) \cap X_{\text{sing}}$.

At the end of the process, we get $\hat{a}_1, \ldots, \hat{a}_{d-1} \in \mathcal{O}_{\mathcal{X},C_\infty}$ with the following properties:

(1) The sequence $\{\hat{a}_1, \ldots, \hat{a}_{d-1}\}$ restricts to a regular sequence $\{a_1, \ldots, a_{d-1}\}$

generating the ideal I_{C,C_∞} in $\mathcal{O}_{X,C_\infty}$. The images of $\{a_1, \dots, a_{d-1}\}$ in $I_{C,C_\infty}/I^2_{C,C_\infty}$ are the basis we started from;

(2) Let $V(\hat{a}_1,\dots,\hat{a}_{d-1}) \subset \mathrm{Spec}\,(\mathcal{O}_{\mathcal{X},C_\infty})$ be the closed subscheme of $\mathrm{Spec}\,(\mathcal{O}_{\mathcal{X},C_\infty})$ defined by the ideal $(\hat{a}_1, \dots, \hat{a}_{d-1})$. Then $V(\hat{a}_1, \dots, \hat{a}_{d-1})$ intersects X_{sing} in at most finitely many points (the intersection could be empty);

(3) Let $\hat{C}$ be the closure of $V(\hat{a}_1, \dots, \hat{a}_{d-1})$ in $\mathcal{X}$. Then $\hat{C}$ is flat over S and there exists an open neighborhood U of C_∞ in X such that $(\hat{C} \cap X) \cap U$ and $C \cap U$ coincide scheme-theoretically. In particular, if T denotes the (finite) set of closed points of $\hat{C} \cap X_{\mathrm{sing}}$ together with the generic points of $\hat{C} \cap X$, then we have an isomorphism $\mathcal{O}_{\hat{C}\cap X,T} \cong \mathcal{O}_{C,C_\infty} \times R$, with R an 1-dimensional semi-local ring.

Property (2) follows from the fact that, at each step, the generic points of $V(\hat{a}_1,\dots,\hat{a}_i)$ have height exactly one at each generic point of $X_{\mathrm{sing}} \cap V(\hat{a}_1, \dots, \hat{a}_{i-1})$. Property (3) is clearly a consequence of (1) and of the construction. It tells us in particular that we can harmlessly throw away any component of $\hat{C}$ which happens to be completely vertical (*i.e.*, the structure map to S factors through the closed point). This is because such a component has to be disjoint from C in a neighborhood of C_∞. Note that $\hat{C}$ can be taken with the reduced scheme structure, but it may not be integral even if C is.

It follows from (3) that the map on units $\mathcal{O}^\times_{\hat{C},T} \to \mathcal{O}^\times_{\hat{C}\cap X,T} \times R$ is surjective. We can therefore find an element $\hat{f}$ in the ring of total quotients of $\hat{C}$ (which is by (3) a product of fields) which is a regular and invertible function in a neighborhood of T and which restricts to $(f, 1)$ (where f was the given function on C). In particular, this implies that $\widetilde{\rho}(\mathrm{div}_{\hat{C}}(\hat{f})) = \mathrm{div}_C(f)$. By $\mathrm{div}_{\hat{C}}(\hat{f})$, here we mean the sum of the divisors on the irreducible components of $\hat{C}$ if $\hat{C}$ is not integral. Note that $\mathrm{div}_{\hat{C}}(\hat{f})$ is an element of $\mathcal{Z}^g_1(\mathcal{X})$ and that, we have $\mathrm{div}_{\hat{C}}(\hat{f}) = \widetilde{\gamma}(\mathrm{div}_C(f))$ by construction. Since we clearly have $\mathrm{div}_{\hat{C}}(\hat{f}) = 0$ in $\mathrm{CH}_1(\mathcal{X})_\Lambda$, this completes the proof of the proposition when $\nu : C \hookrightarrow X$ is a regular closed immersion.

We now prove the general case. So suppose we are given a good curve $\nu : C \to X$ and a rational function f on C as in the beginning of the proof of the proposition. By [4, Lemma 3.5], we can assume that the map $\nu : C \to X$ is a complete intersection morphism. Now, we can find a commutative diagram

$$\begin{tikzcd} C \arrow[r, hook, "\nu'"] \arrow[dr, "\nu"'] & \mathbb{P}^M_X \arrow[r] \arrow[d, "q"] & \mathbb{P}^M_{\mathcal{X}} \arrow[d, "q"] \arrow[dr] & \\ & X \arrow[r] & \mathcal{X} \arrow[r] & S \end{tikzcd} \tag{4.3}$$

for some $M \gg 0$ such that ν' is a regular closed embedding. Letting $\mathcal{Y} = \mathbb{P}^M_{\mathcal{X}}$ and $Y = \mathbb{P}^M_X$, this gives a diagram

$$\begin{tikzcd} \mathcal{Z}_0(Y, Y_{\mathrm{sing}})_\Lambda \arrow[r, "\widetilde{\gamma}_Y"] \arrow[d, "q_*"'] & \mathrm{CH}_1(\mathcal{Y})_\Lambda \arrow[d, "q_*"] \\ \mathcal{Z}_0(X, X_{\mathrm{sing}})_\Lambda \arrow[r, "\widetilde{\gamma}_X"] & \mathrm{CH}_1(\mathcal{X})_\Lambda. \end{tikzcd} \tag{4.4}$$

Note that the push-forward map q_* on the left is defined since q is smooth (see [4, Proposition 3.18]). It is easily seen from the construction of the cycle Z_x associated to a regular closed point of X in Proposition 3.2 that (4.4) commutes. We thus get

$$\widetilde{\gamma}_X \circ \nu_*(\mathrm{div}(f)) = \widetilde{\gamma}_X \circ q_*(\mathrm{div}_C(f)) = q_* \circ \widetilde{\gamma}_Y(\mathrm{div}_C(f)) = 0.$$

This finishes the proof of the proposition. □

Proof of Theorem 1.1. The construction of $\widetilde{\rho}$ is given in (3.2). The existence of the map γ such that (1.2) commutes, follows directly from Corollary 4.2 and Proposition 4.4, using the fact that the surjection $\mathcal{Z}_0(X, X_{\mathrm{sing}}) \twoheadrightarrow \mathrm{CH}_0^{BK}(X)$ factors as $\mathcal{Z}_0(X, X_{\mathrm{sing}}) \twoheadrightarrow \mathrm{CH}_0^{LW}(X) \twoheadrightarrow \mathrm{CH}_0^{BK}(X)$. The surjectivity of γ follows from Proposition 3.2. □

4.3 In the above notation, we have constructed a surjective group homomorphism

$$\gamma \colon \mathrm{CH}_0^{LW}(X)_\Lambda \twoheadrightarrow \mathrm{CH}_1(\mathcal{X})_\Lambda,$$

which is (by construction) an inverse on the level of generators of the naive restriction map

$$\widetilde{\rho} \colon \mathcal{Z}_1^g(\mathcal{X}) \to \mathcal{Z}_0(X, X_{\mathrm{sing}})$$

for any regular projective and flat scheme $\mathcal{X}$ over S without any assumption on the residue field (apart from it being perfect). This also does not depend on the geometry of the reduced special fiber X. In particular, we can summarize what we have shown as follows.

Corollary 4.5. *Let A be a Henselian discrete valuation ring with perfect residue field. Let $\mathcal{X}$ be a regular scheme which is projective and flat over A with reduced special fiber X. Suppose that the map $\widetilde{\rho} \colon \mathcal{Z}_1^g(\mathcal{X}) \to \mathcal{Z}_0(X, X_{\mathrm{sing}})$ descends to a morphism between the Chow groups*

$$\rho \colon \mathrm{CH}_1(\mathcal{X})_\Lambda \to \mathrm{CH}_0^{LW}(X)_\Lambda. \tag{4.5}$$

Then ρ is an isomorphism.

5. The restriction isomorphism

We shall prove Theorem 1.3 in this section. In other words, we shall show that the restriction homomorphism ρ of (4.5) does exist if additional assumption on the field k or on the DVR A hold.

5.1. The étale cycle class map

The cycle class map from the Chow groups to the étale cohomology is well known for smooth schemes. More generally, the étale realization of Voevodsky's motives tells us that there are such maps from the Friedlander-Voevodsky motivic cohomology of singular schemes to their étale cohomology. But this is not good enough for us since we do not work with the motivic cohomology. In this section, we give a construction of the cycle class map from the Levine-Weibel Chow group of a singular scheme to its étale cohomology using Gabber's Gysin maps [14].

We let k be a perfect field and let X be an equi-dimensional quasi-projective scheme of dimension d over k. Let m be an integer prime to the exponential characteristic of k and let $\Lambda = \mathbb{Z}/m\mathbb{Z}$.

Let $x \in X_{\mathrm{reg}}$ be a regular closed point of X. We have the sequence of maps

$$\mathbb{Z} \twoheadrightarrow \Lambda \xrightarrow{\cong} H^0_{\text{ét}}(k(x), \Lambda) \underset{\cong}{\xrightarrow{(1)}} H^{2d}_{\{x\},\text{ét}}(X, \Lambda(d)) \xrightarrow{(2)} H^{2d}_{\text{ét}}(X, \Lambda(d)). \tag{5.1}$$

The arrow labeled (1) is the Gysin map [16], using the fact that x is a regular closed point of X. This is an isomorphism by the purity and excision theorems in étale cohomology. The arrow labeled (2) is the natural 'forget support' map. Let δ_x denote the composite of all maps in (5.1). We let $cyc^{\text{ét}}_X(x) = \delta_x(1)$ and extend it linearly to define a group homomorphism

$$cyc^{\text{ét}}_X \colon \mathcal{Z}_0(X, X_{\mathrm{sing}}) \to H^{2d}_{\text{ét}}(X, \Lambda(d)). \tag{5.2}$$

We shall now show that this map factors through the modified Chow group $\mathrm{CH}^{BK}_0(X)$. It will then follow that it factors through $\mathrm{CH}^{LW}_0(X)$ as well. So let $\nu \colon (C, Z) \to X$ be a good curve as in Definition 3.1. As in the proof of Proposition 4.4, we can assume that ν is a local complete intersection morphism. In this case, Gabber's construction of push-forward map in étale cohomology [14] gives us a push-forward map

$$\nu_* \colon H^2_{\text{ét}}(C, \Lambda(1)) \to H^{2d}_{\text{ét}}(X, \Lambda(d))$$

and a diagram

$$\begin{tikzcd} \mathcal{Z}_0(C, Z) \arrow[r, "\nu_*"] \arrow[d, "cyc^{\text{ét}}_C"'] & \mathcal{Z}_0(X, X_{\mathrm{sing}}) \arrow[d, "cyc^{\text{ét}}_X"] \\ H^2_{\text{ét}}(C, \Lambda(1)) \arrow[r, "\nu_*"] & H^{2d}_{\text{ét}}(X, \Lambda(d)). \end{tikzcd} \tag{5.3}$$

If $x \in C \setminus Z$ is a closed point so that $\nu(x) \in X_{\mathrm{reg}}$, the functoriality of the Gysin maps implies that the composite $H^0_{\text{ét}}(k(x), \Lambda(0)) \to H^2_{\text{ét}}(C, \Lambda(1)) \xrightarrow{\nu_*} H^{2d}_{\text{ét}}(X, \Lambda(d))$ is the push-forward map associated to the finite complete intersection map $\mathrm{Spec}\,(k(x)) \to X$. Using this fact and the description (5.1) of the cycle class map on generators, it follows that (5.3) is commutative.

We can identify $\mathrm{CH}_0(C, Z)$ with $\mathrm{Pic}(C)$ according to [4, Lemma 4.12]. The Kummer sequence then shows that there is a commutative diagram

$$\begin{tikzcd}
\mathcal{Z}_0(C, Z) \arrow[rr] \arrow[dr, "cyc_C^{\text{ét}}"'] & & \mathrm{Pic}(C) \arrow[dl] \\
& H^2_{\text{ét}}(C, \Lambda(1)). &
\end{tikzcd}$$

This immediately shows that for any rational function f on $k(C)$ such that $\mathrm{div}_C(f) \in \mathcal{R}_0(C, Z)$, we have $cyc_C^{\text{ét}}(\mathrm{div}_C(f)) = 0$ in $H^2_{\text{ét}}(C, \Lambda(1))$. But then, the commutativity of (5.3) proves that $\nu_*(\mathrm{div}_C(f))$ goes to zero in $H^{2d}_{\text{ét}}(X, \Lambda(d))$. We have therefore shown that the map $cyc_X^{\text{ét}}$ in (5.2) descends to a cycle class map on the Chow group:

$$cyc_X^{\text{ét}} : \mathrm{CH}_0^{BK}(X) \to H^{2d}_{\text{ét}}(X, \Lambda(d)). \tag{5.4}$$

We shall denote its composite with the canonical surjection $\mathrm{CH}_0^{LW}(X) \twoheadrightarrow \mathrm{CH}_0^{BK}(X)$ also by $cyc_X^{\text{ét}}$.

5.2. The case of algebraically closed fields

In the notation of Subsection 5.1, suppose moreover that k is separably (hence algebraically) closed and X is projective over k. Write $X = \cup_{i=1}^n X_i$, where the X_i's are the the irreducible components of X. In this case, we have a natural 'trace' map

$$\tau_X : H^{2d}_{\text{ét}}(X, \Lambda(d)) \xrightarrow{\cong} \oplus_{i=1}^n H^{2d}_{\text{ét}}(X_i, \Lambda(d)) \xrightarrow{\cong} \oplus_{i=1}^n \Lambda. \tag{5.5}$$

It follows by combining the exact sequence

$$H^{2d-1}_{\text{ét}}(X_{\text{sing}}, \Lambda(d)) \to H^{2d}_{c,\text{ét}}(X_{\text{reg}}, \Lambda(d)) \to H^{2d}_{\text{ét}}(X, \Lambda(d)) \to H^{2d}_{\text{ét}}(X_{\text{sing}}, \Lambda(d)),$$

[33, Chapter VI, Lemma 11.3] and the cohomological dimension bound $cd_\Lambda(X_{\text{sing}}) \le 2d - 2$ (as k is separably closed) that the map τ_X in (5.5) is an isomorphism.

Note further that for any regular closed point $x \in X_{\text{reg}}$, the composition

$$\Lambda \xrightarrow{\cong} H^0(k(x), \Lambda) \to H^{2d}_{\text{ét}}(X, \Lambda(d)) \xrightarrow{\cong} \oplus_{i=1}^n H^{2d}_{\text{ét}}(X_i, \Lambda(d)) \xrightarrow{\tau} \oplus_{i=1}^n \Lambda$$

sends $1 \in \Lambda$ to the element 1 in the direct summand of $\oplus_{i=1}^n \Lambda$ associated to the unique component of X containing x and to zero in all other summands.

Recall now from [9, Section 1] that there is a degree map $\deg\colon \mathrm{CH}_0^{BK}(X)_\Lambda \to \bigoplus_{i=1}^n \Lambda$. This is considered in *loc. cit.* for the Levine-Weibel Chow group, but the discussion there easily shows that it actually factors through the quotient $\mathrm{CH}_0^{BK}(X)$. This map is given by the sum of the degree maps for 0-cycles on the irreducible components of X. In particular, for any regular closed point $x \in X$, the degree of x is the element 1 in the direct summand of $\oplus_{i=1}^n \Lambda$ associated to the unique component of X containing x and is zero in all other summands.

Combining these two facts, we have a commutative diagram

$$\begin{tikzcd} \mathrm{CH}_0^{BK}(X)_\Lambda \arrow[rr, "\deg"] \arrow[dr, "cyc_X^{\text{ét}}"'] & & \oplus_{i=1}^n \Lambda \\ & H^{2d}_{\text{ét}}(X, \Lambda(d)). \arrow[ur, "\tau_X", "\cong"'] & \end{tikzcd} \tag{5.6}$$

In this setting, we have

Lemma 5.1. *The degree map induces an isomorphism* $\mathrm{CH}_0^{LW}(X)_\Lambda \xrightarrow{\cong} \oplus_{i=1}^n \Lambda$. *In particular, the étale cycle class map* $cyc_X^{ét} : \mathrm{CH}_0^{LW}(X)_\Lambda \to H^{2d}_{ét}(X, \Lambda(d))$ *is an isomorphism.*

Proof. The second statement is a consequence of the first by (5.6). Since the degree map is clearly surjective (as k is algebraically closed), it is enough to prove its injectivity. Since we are working with $\mathbb{Z}/m$-coefficients with $m \in k^\times$, it is in fact enough to prove that the subgroup $\mathrm{CH}_0(X)_{\deg=0}$ of 0-cycles of degree zero is m-divisible. But this well known as k is algebraically closed. Indeed, given any 0-cycle $\alpha \in \mathcal{Z}_0(X, X_{\mathrm{sing}})$ of degree zero, we can find a reduced Cartier curve $C \subset X$ which is regular along the support of α. This implies that α lies in the image of the push-forward map $\mathrm{Pic}^0(C) \to \mathrm{CH}_0^{LW}(X)$. It is therefore enough to know that $\mathrm{Pic}^0(C)$ is m-divisible. But this is elementary. □

It is a straightforward exercise to deduce from the previous Lemma the isomorphism discussed in Remark 1.4.

5.3. Results over non-algebraically closed fields

In this final section, we suppose that the Gersten conjecture for Milnor K-theory holds for schemes over A. Thanks to [22], this is the case if $k \subset A$, *i.e.*, if A is an equicharacteristic DVR.

Let $\Lambda = \mathbb{Z}/m\mathbb{Z}$, with m prime to p, and let $n \geq 0$ be a non-negative integer. Recall (see e.g., [10, 8.2]), that the n-th Milnor K-theory sheaf $\mathcal{K}^M_{n,\Lambda}$ with Λ-coefficients is defined as the (Zariski or Nisnevich) sheafification of the presheaf on affine schemes sending an A-algebra R to the quotient of $\Lambda \otimes_{\mathbb{Z}} T_n(R)$ by the two-sided ideal generated by elements of the form $a \otimes (1-a)$ with $a, 1-a \in R^\times$. Here $T_n(R)$ is the n-th tensor algebra of $R^\times$ (over $\mathbb{Z}$). Since in what follows we will only consider Λ-coefficients (unless explicitly mentioned), we drop it from the notation. Write $\mathcal{K}^M_{n,Y}$ for the restriction of $\mathcal{K}^M_{n,\Lambda}$ to the small (Zariski or Nisnevich) site of Y for any A-scheme Y. If the residue field of A is finite, we denote by the same symbol the sheaf of improved Milnor K-theory, with Λ coefficients, in the sense of Kerz [23].

Let $\mathcal{X}$ be again a regular scheme which is projective and flat over A, of relative dimension $d \geq 0$. One of the consequences of the Gersten conjecture is the so called Bloch formula, relating Milnor K-theory with the Chow groups. In particular, there is a canonical isomorphism

$$cyc^M_{\mathcal{X}} \colon \mathrm{CH}_1(\mathcal{X})_\Lambda \xrightarrow{\simeq} H^d(\mathcal{X}_{\mathrm{Nis}}, \mathcal{K}^M_{d,\mathcal{X}})$$

which is induced by the tautological "cycle class map"

$$cyc^M_{\mathcal{X}} \colon \mathcal{Z}_1(\mathcal{X}) = \bigoplus_{x \in \mathcal{X}_{(1)}} \mathbb{Z} \cong \bigoplus_{x \in \mathcal{X}_{(1)}} K^M_0(k(x)),$$

where the right hand side appears as the last term of the Gersten resolution for $\mathcal{K}^M_{d,\mathcal{X}}$.

Let now X denote as before the reduced special fiber of $\mathcal{X}$, and let $x \in X_{\mathrm{reg}}$ be a regular closed point of X. We have a sequence of maps

$$\Lambda \cong K^M_0(k(x)) \otimes_{\mathbb{Z}} \Lambda \underset{\cong}{\xrightarrow{(1)}} H^d_{\{x\}}(X_{\mathrm{Zar}}, \mathcal{K}^M_{d,X}) \xrightarrow{(2)} H^d(X_{\mathrm{Zar}}, \mathcal{K}^M_{d,X}) \xrightarrow{(3)} H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X})$$

where the isomorphism (1) follows from Kato's computation [21, Theorem 2] (using again the regularity of the point x), the map (2) is the canonical forget support map, and the map (3) is the change of topology from Zariski to the Nisnevich site.

Extending this map linearly, we get a cycle class map

$$cyc^M_X \colon \mathcal{Z}_0(X, X_{\mathrm{sing}}) \to H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X}).$$

Recall now the following result from [26]

Theorem 5.2. (26, Theorem 4.1) *The cycle class map cyc^M_X induces a surjective homomorphism*

$$cyc^M_X \colon \mathrm{CH}^{LW}_0(X) \twoheadrightarrow H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X}).$$

With this result at disposal, we can consider the following diagram.

$$\begin{tikzcd}
\mathcal{Z}_1(\mathcal{X})^g_\Lambda \arrow[r] \arrow[d, "\tilde{\rho}"] & \mathrm{CH}_1(\mathcal{X})_\Lambda \arrow[r, "cyc^M_{\mathcal{X}}", "\cong"'] \arrow[d, dashed, "\rho"] & H^d(\mathcal{X}_{\mathrm{Nis}}, \mathcal{K}^M_{d,\mathcal{X}}) \arrow[d, "\rho^M"] \\
\mathcal{Z}_0(X, X_{\mathrm{sing}})_\Lambda \arrow[r] & \mathrm{CH}^{LW}_0(X)_\Lambda \arrow[r, "cyc^M_X"] & H^d(X_{\mathrm{Nis}}, \mathcal{K}^M_{d,X})
\end{tikzcd} \tag{5.7}$$

Note that the outer rectangle in (5.7) commutes, since the left vertical map is the restriction on cycles (3.2), the right vertical map is the restriction homomorphism on Milnor K-theory and the composite horizontal maps are by definition the cycle class maps. We can therefore ask wether there exists a homomorphism making the left square commutative as well, *i.e.*, if the map $\tilde{\rho}$ descends to a morphism between the Chow groups.

This is clearly implied by the following Conjecture.

Conjecture 5.3 (Bloch-Quillen formula). The cycle class map cyc_X^M is an isomorphism.

If X is regular, this is a well-known fact. It was originally proved by Bloch in [7] for surfaces, and generalized by Kato [21] in higher dimension. If X is of dimension 1, it can be interpreted as the chain of isomorphisms (with integral coefficients)

$$\mathrm{CH}_0^{LW}(X) \xrightarrow{\cong} \mathrm{Pic}(X) \cong H^1(X, \mathcal{O}_X^\times)$$

where the cohomology is taken with respect to the Zariski or the Nisnevich topology.

For singular varieties of dimension ≥ 1, the status of this conjecture is summarized here.

Theorem 5.4. *Conjecture* 5.3 *is true in the following cases, with integral coefficients:*

(i) *X is a quasi-projective surface with isolated singularities, over any field k;*
(ii) *X is a quasi-projective surface with arbitrary singularities;*
(iii) *X is an affine surface over any perfect field;*
(iv) *X is projective and regular in codimension* 1*, over an algebraically closed field;*
(v) *X is quasi-projective with isolated singularities over a finite field.*

Item i) was first verified by Pedrini and Weibel in [34], and in the affine case by Levine and Weibel [31]. The case ii) is due to Levine [29] in the case of algebraically closed fields. A modification of Levine's argument can be used to extend the result to the case of an arbitrary (perfect) ground field, provided that one replaces the Levine-Weibel Chow group with its modified version introduced in [4]. This is done in [6].

The affine case iii) and the case of singularities in codimension at least 2 iv) are shown in [26, Theorem 1.1 and 1.2] (the arguments are independent from the arguments used in [6]), while case v) is [25, Theorem 1.6]. Older results in the affine case where obtained by Barbieri-Viale in [2]. We can now give another application of Theorem 1.1.

Corollary 5.5. *Let $\mathcal{X}$ and A be as above. Then the restriction homomorphism $\tilde{\rho}$ of* (3.2) *factors through the rational equivalence classes if k is finite and if X has only isolated singularities, or if* $\dim(X) = 2$ *(without restrictions on the type of singularities). In these cases, it induces an isomorphism*

$$\rho\colon \mathrm{CH}_1(\mathcal{X})_\Lambda \xrightarrow{\cong} \mathrm{CH}_0^{LW}(X)_\Lambda.$$

If k is finite, both groups are finite.

Proof. It is an immediate consequence of the commutative diagram (5.7), given Theorem 5.4. By Corollary 1.2, the induced map ρ is automatically an isomorphism. Finally, by [25, Theorem 1.2], the group $\mathrm{CH}_0^{LW}(X)_\Lambda$ is finite if the residue field k is finite. □

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Dipartimento di Matematica "Federigo Enriques"
Università degli Studi di Milano
Via Cesare Saldini 50
20133 Milano, Italia
federico.binda@unimi.it

School of Mathematics
Tata Institute of Fundamental Research
1 Homi Bhabha Road
Colaba, Mumbai, India
amal@math.tifr.res.in