

THETA-REGULARITY OF CURVES AND BRILL–NOETHER LOCI

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ABSTRACT. We provide a bound on the Θ -regularity of an arbitrary reduced and irreducible curve embedded in a polarized abelian variety in terms of its degree and codimension. This is an “abelian” version of Gruson–Lazarsfeld–Peskine’s bound on the Castelnuovo–Mumford regularity of a non-degenerate curve embedded in a projective space. As an application, we provide a Castelnuovo type bound for the genus of a curve in a (non necessarily principally) polarized abelian variety. Finally, we bound the Θ -regularity of a class of higher dimensional subvarieties in Jacobian varieties, *i.e.* the Brill–Noether loci associated to a Petri general curve, extending earlier work of Pareschi–Popa.

1. INTRODUCTION

Given a reduced and irreducible non-degenerate curve $C \subset \mathbf{P}^r$ ($r \geq 3$) of degree d contained in the projective r -space, natural problems that have been considered by Castelnuovo first (*cf.* [Sz, §3.2 Theorem 1]), and by Gruson–Lazarsfeld–Peskine later ([GLP]), is to see for which integers $n = n(r, d)$ the following properties regarding the geometry of C hold:

(PA_n) The line bundle $\mathcal{O}_C(n)$ is non-special;

(PB_n) The homomorphism $H^0(\mathbf{P}^r, \mathcal{O}_{\mathbf{P}^r}(n)) \rightarrow H^0(C, \mathcal{O}_C(n))$ is surjective, *i.e.* hypersurfaces of degree n trace out a complete linear system on C ;

(PC_n) C is cut out in \mathbf{P}^r by hypersurfaces of degree n .

In particular, in [Sz] and [GLP] the following effective bounds have been proved:

$$\begin{aligned} (PA_n) \quad & \text{holds for all } n \geq \left\lfloor \frac{d-1}{r-1} \right\rfloor, \\ (PB_n) \quad & \text{holds for all } n \geq d-r+1, \\ (PC_n) \quad & \text{holds for all } n \geq d-r+2. \end{aligned}$$

The goal of this paper is to study analogous properties to (PA), (PB), and (PC) for integral curves C embedded in a polarized abelian variety (A, Θ) . These properties are listed below and take into account the fact that abelian varieties have a “continuous” Picard variety. Hence, as shown for instance by work of [GL], [PP1] and [PP3], it is natural and definitely more fruitful to consider the entire family $\{\mathcal{O}_A(\Theta) \otimes \alpha\}_{\alpha \in \text{Pic}^0(A)}$ of deformations rather than the polarization $\mathcal{O}_A(\Theta)$ itself.

- (AA_n) The cohomology groups $H^1(C, \mathcal{O}_C(n\Theta) \otimes \alpha|_C)$ vanish for all $\alpha \in \text{Pic}^0(A)$;
- (AA_n^h) There exists a closed algebraic subset $V \subset \text{Pic}^0(A)$ such that $\text{codim } V \geq h$ and $H^1(C, \mathcal{O}_C(n\Theta) \otimes \alpha|_C) = 0$ for all $\alpha \notin V$;
- (AB_n) The homomorphisms $H^0(A, \mathcal{O}_A(n\Theta) \otimes \alpha) \rightarrow H^0(C, \mathcal{O}_C(n\Theta) \otimes \alpha|_C)$ are surjective for all $\alpha \in \text{Pic}^0(A)$;
- (AB_n^h) There exists a closed algebraic subset $V \subset \text{Pic}^0(A)$ such that $\text{codim } V \geq h$ and the homomorphisms $H^0(A, \mathcal{O}_A(n\Theta) \otimes \alpha) \rightarrow H^0(C, \mathcal{O}_C(n\Theta) \otimes \alpha|_C)$ are surjective for all $\alpha \notin V$;
- (AC_n^{lin}) C is cut out by divisors linearly equivalent to $n\Theta$;
- (AC_n^{alg}) C is cut out by divisors algebraically equivalent to $n\Theta$.

As in the case of curves embedded in projective spaces where Castelnuovo–Mumford regularity is shown to govern properties (PA), (PB), and (PC), so in the case of curves in abelian varieties properties (AA), \dots , (AC) are controlled by a parallel notion of regularity called Θ -regularity. This notion was introduced by Pareschi–Popa in [PP1] to study generation properties of coherent sheaves on abelian varieties and satisfies analogous properties to Castelnuovo–Mumford regular sheaves on projective spaces (*cf.* Theorem 2.2 (iv) and [Laz2, Theorem 1.8.3]). More concretely, we say that a closed subvariety $X \subset A$ of a polarized abelian variety (A, Θ) is n - Θ -regular if

$$\text{codim}_{\text{Pic}^0(A)} V^i(\mathcal{I}_X((n-1)\Theta)) > i \quad \text{for all } i > 0$$

where

$$V^i(\mathcal{I}_X((n-1)\Theta)) := \{\alpha \in \text{Pic}^0(A) \mid H^i(A, \mathcal{I}_X((n-1)\Theta) \otimes \alpha) \neq 0\}$$

are the non-vanishing loci and \mathcal{I}_X is the ideal sheaf of X . To have a more global picture regarding computations of Θ -regularity, we mention that this has been only computed for two classes of curves and a few higher-dimensional varieties. For instance, Abel–Jacobi curves, Abel–Prym curves, Brill–Noether loci of type W_d on a smooth curve, and Fano surfaces of lines associated to smooth cubic threefolds are 3- Θ -regular (however there are quite deep differences in these examples as both Abel–Jacobi curves and the W_d ’s satisfy the stronger *strongly* 3- Θ -regularity; see Definition 2.1, [PP1, Theorem 4.1], [CMLV, Theorem A], [Ho, Theorem 1.2]).¹ Moreover, a result of Pareschi–Popa, extending to the “abelian” case a theorem of Bertram–Ein–Lazarsfeld, bounds the Θ -regularity of a smooth subvariety in terms of the degrees of its defining equations ([PP1, Theorem 6.5]).

The main result of this paper is a bound on the Θ -regularity of a reduced and irreducible curve on an abelian variety in terms of its codimension and degree.

¹A further property that relates the Brill–Noether loci W_d to Fano surfaces of lines is that they are both examples of subvarieties representing a minimal cohomology class $\frac{[\Theta]^d}{d!} \in H^{2d}(A, \mathbf{Z})$. Moreover, it is conjectured that these varieties should be the only ones that represent a minimal class and furthermore that strongly 3- Θ -regularity should provide a cohomological characterization for them ([PP2, Conjecture 2.2]).

Theorem 1.1. *If (A, Θ) is a polarized abelian variety of dimension $r > 1$ defined over an algebraically closed field and $C \subset A$ is a reduced and irreducible curve of degree $d := (\Theta \cdot C) = \deg \mathcal{O}_C(\Theta)$, then C is $(4d + 2r + 1)$ - Θ -regular. Moreover, if in addition Θ is globally generated, then C is $(d + r + 1)$ - Θ -regular.*

As a consequence, we deduce effective versions of properties $(AA), \dots, (AC^{\text{alg}})$.

Corollary 1.2. *With notation as in Theorem 1.1, the curve C satisfies properties*

$$(AA_n^3), \quad (AA_{n+1}), \quad (AB_n^2), \quad (AB_{n+1}), \quad (AC_{n+1}^{\text{lin}}), \quad \text{and} \quad (AC_n^{\text{alg}})$$

for all $n \geq 4d + 2r$. Moreover, if Θ is globally generated, then we can take $n \geq d + r$.

Theorem 1.1 was inspired by the sharp bound $(d - r + 2)$ obtained in [GLP, Theorem 1.1] for the Castelnuovo–Mumford regularity of a reduced and irreducible non-degenerate curve $C \subset \mathbf{P}^r$ of degree d . We do not expect the bounds of Theorem 1.1 to be sharp as in the projective case (for instance we do not assume any non-degeneracy’s type condition), but they do confirm the linearity of the Θ -regularity of curves and, moreover, to the knowledge of the authors, they are the first bounds in this direction.

As an application of Corollary 1.2, we deduce a Castelnuovo genus type bound of a reduced and irreducible curve in a polarized abelian variety. Previous results in this direction were given in [De, Theorem 5.1], and furthermore in [PP2, Theorem B] for principally polarized abelian varieties (see Remark 4.2). Our bound improves the above ones in a few cases. For instance, in the case of abelian surfaces whose Néron–Severi group is of rank one and generated by a globally generated line bundle, or abelian threefolds of the same type and with $h_{\min}^0 \geq 6$ (see (1) for the definition of h_{\min}^0). We refer to Theorem 4.1 for the most general version of the following result.

Theorem 1.3. *Let C be a reduced and irreducible curve of geometric genus g embedded in a polarized abelian variety (A, Θ) of dimension $r > 1$. Denote by \tilde{C} the normalization of C and by $f : \tilde{C} \rightarrow A$ the induced morphism. Moreover define the non-negative integer*

$$(1) \quad h_{\min}^0 := \min\{h^0(\tilde{C}, f^*(\mathcal{O}_A(\Theta) \otimes \gamma)) \mid \gamma \in \text{Pic}^0(A)\}.$$

Then the following bound holds

$$g \leq (d - h_{\min}^0 + 1)(4d + 2r + 1).$$

Moreover, if in addition Θ is globally generated, then the bound

$$g \leq (d - h_{\min}^0 + 1)(d + r + 1)$$

holds and $h_{\min}^0 \geq 2$ ($h_{\min}^0 \geq 3$ if Θ is very ample).

In the second part of the paper (see §5), we provide a first answer to a question raised by Pareschi–Popa [PP3, Question 7.21] asking for the least positive integer n such that the Brill–Noether loci

$$W_d^r(C) = \{L \in \text{Pic}^d(C) \mid h^0(C, L) \geq r + 1\}$$

attached to a smooth curve C are n - Θ -regular (upon the choice of a point on C we can think of these spaces as embedded in the Jacobian $J(C)$). We bound this integer in the case C is Petri

general. As a byproduct we obtain a much better bound for the Θ -regularity of Brill–Noether curves which in particular only depends on the codimension.

Theorem 1.4. *Let C be a Petri general curve of genus $g \geq 3$ and let d, r be positive integers such that $\text{codim } W_d^r(C) = (r+1)(g-d+r) \geq 1$. Then $W_d^r(C)$ is*

$$\left(2 + 4 \left\lfloor \sqrt{\text{codim } W_d^r(C)} \right\rfloor \right)\text{-}\Theta\text{-regular.}$$

We refer to Theorem 5.3 and Corollary 5.4 for more efficient statements. Moreover, we mention that Theorem 1.4 extends previous work carried out by Pareschi–Popa on the Θ -regularity of Brill–Noether loci of type W_{g-1}^1 ([PP3, Proposition 7.22]). Finally, in Corollary 5.6, we spell out effective bounds to properties (AA), \dots , (AC) for the case of Brill–Noether curves.

The strategy of the proof of Theorem 1.1 goes as follows. Similarly to the projective case, the key idea is to construct a resolution (which is exact away an algebraic subset of dimension one) of the ideal sheaf \mathcal{I}_C by locally free sheaves such that the Θ -regularity of its terms is easy to compute. In order to do this, we use regularity theory of sheaves on abelian varieties as developed by Pareschi–Popa in [PP1] and [PP3]. More precisely, we notice that a resolution for \mathcal{I}_C is determined by the study of generation properties of sheaves of type

$$(2) \quad q_*(p^*Q \otimes \mathcal{I}_\Gamma) \otimes \mathcal{O}_A(\Theta)$$

where p and q are the projections from $\tilde{C} \times A$ onto the first and second factor respectively, \tilde{C} is the normalization of C , and Γ is the graph of the morphism $f : \tilde{C} \rightarrow A$. One of the technical results we prove, which boils down to checking the surjectivity of certain multiplication maps on global sections, is that the sheaves in (2) are *continuously globally generated* anytime $Q = B \otimes f^*\mathcal{O}_A(\Theta)$ and B is a general line bundle of degree $g(\tilde{C})$ on \tilde{C} . In other words, there exist a positive integer N and line bundles $\alpha_1, \dots, \alpha_N \in \text{Pic}^0(A)$ such that the sum of twisted evaluation maps

$$\bigoplus_{i=1}^N H^0(A, q_*(p^*Q \otimes \mathcal{I}_\Gamma) \otimes \mathcal{O}_A(\Theta) \otimes \alpha_i) \otimes \alpha_i^{-1} \xrightarrow{\psi} q_*(p^*Q \otimes \mathcal{I}_\Gamma) \otimes \mathcal{O}_A(\Theta)$$

is surjective (Proposition 3.3 and Lemma 3.4). Therefore, by considering an Eagon–Northcott complex associated to the composition of $\psi \otimes \mathcal{O}_A(-\Theta)$ with the inclusion $q_*(p^*Q \otimes \mathcal{I}_\Gamma) \subset H^0(\tilde{C}, Q) \otimes \mathcal{O}_A$, one obtains a resolution of \mathcal{I}_C which is exact away from C and whose terms are sums of line bundles of type $\mathcal{O}_A(-i\Theta) \otimes \beta_i$ with $\beta_i \in \text{Pic}^0(A)$ (Proposition 3.2).

On the other hand, for the proof of Theorem 1.4 we exploit the description of Brill–Noether loci as degeneracy loci of a morphism of vector bundles $\varphi : E \rightarrow F$ where E is a Picard bundle and F is a sum of topologically trivial line bundles. It is a general fact that if a degeneracy locus has the expected dimension, then its ideal sheaf admits a resolution whose terms are tensor products of Schur powers of the two bundles in play (*cf.* for instance [Lay]). Therefore, in our situation, the Θ -regularity of $W_d^r(C)$ is computed as soon as one computes the Θ -regularity of Picard bundles. This is the goal of Proposition 5.1 which in particular extends previous calculations of Pareschi–Popa on this subject ([PP3, Proposition 7.15 and Remark 7.17]).

Problem 1.5. It would be interesting to give a uniform bound for the Θ -regularity of an arbitrary subvariety of an abelian variety in terms of its dimension and degree in order to study higher-dimensional analogue of properties $(AA), \dots, (AC)$. In the projective setting the problem of bounding the Castelnuovo–Mumford regularity of subvarieties of projective spaces goes under the name of Eisenbud–Goto’s Conjecture ([EG] and [Laz2, Conjecture 1.8.47]). This has been solved in the case of smooth surfaces and for certain threefolds by means of generic projections (*cf.* [Laz1], [?], and [Ran]).

Notation. If \mathcal{F} is a coherent sheaf on a variety X and D a Cartier divisor, we denote by $\mathcal{F}(D)$ the tensor product $\mathcal{F} \otimes \mathcal{O}_X(D)$ of \mathcal{O}_X -modules.

2. PRELIMINARIES: REGULARITY OF SHEAVES ON ABELIAN VARIETIES

Let (A, Θ) be a polarized abelian variety defined over an algebraically closed field and let \mathcal{F} be a coherent sheaf on A . For each integer $i \geq 0$ we define the algebraic closed sets

$$V^i(\mathcal{F}) := \{\alpha \in \text{Pic}^0(A) \mid h^i(A, \mathcal{F} \otimes \alpha) > 0\},$$

usually called *cohomological support loci* or *non-vanishing loci*. We recall some terminology from [PP1].

Definition 2.1. Let \mathcal{F} be a coherent sheaf on a polarized abelian variety (A, Θ) .

- (i). \mathcal{F} satisfies *GV* (*Generic Vanishing*) if $\text{codim}_{\text{Pic}^0(A)} V^i(\mathcal{F}) \geq i$ for all $i > 0$.
- (ii). \mathcal{F} is *M-regular* (*Mukai regular*) if $\text{codim}_{\text{Pic}^0(A)} V^i(\mathcal{F}) > i$ for all $i > 0$.
- (iii). \mathcal{F} satisfies *I.T.0* (*Index Theorem with index 0*) if $V^i(\mathcal{F}) = \emptyset$ for all $i > 0$.
- (iv). \mathcal{F} is *n- Θ -regular* if $\mathcal{F}((n-1)\Theta)$ is *M-regular*.
- (v). \mathcal{F} is *strongly n- Θ -regular* if $\mathcal{F}((n-1)\Theta)$ satisfies *I.T.0*.
- (vi). \mathcal{F} is *continuously globally generated* if there exists a positive integer N such that for general line bundles $\alpha_1, \dots, \alpha_N \in \text{Pic}^0(A)$ the sum of twisted evaluation maps

$$\bigoplus_{i=1}^N H^0(A, \mathcal{F} \otimes \alpha_i) \otimes \alpha_i^\vee \longrightarrow \mathcal{F}$$

is surjective.

We also recall some properties of *M-regular* and Θ -regular sheaves which we will need later on.

Theorem 2.2. *Let (A, Θ) be a polarized abelian variety and \mathcal{F} a coherent sheaf on A .*

- (i). *If \mathcal{F} is M-regular, then \mathcal{F} is continuously globally generated.*

- (ii). If \mathcal{F} is continuously globally generated, then $\mathcal{F}(m\Theta)$ is globally generated for all $m \geq 1$.
- (iii). If \mathcal{F} is M -regular, then $\mathcal{F}(m\Theta)$ satisfies I.T.0 for all $m \geq 1$.
- (iv). If \mathcal{F} is 0 - Θ -regular, then \mathcal{F} is globally generated and m - Θ -regular for all $m \geq 1$. Moreover, the multiplication maps
- $$H^0(A, \mathcal{F}(\Theta)) \otimes H^0(A, \mathcal{O}_A(k\Theta)) \longrightarrow H^0(A, \mathcal{F}((k+1)\Theta))$$
- are surjective for all $k \geq 2$.

Proof. For (i) see [PP1, Proposition 2.13]; for (ii) see [PP1, Proposition 2.12]; for (iii) see [PP1, Proposition 2.9]; for (iv) see [PP1, Theorem 6.3]. \square

3. REGULARITY OF CURVES IN ABELIAN VARIETIES

Throughout the whole section we fix the following notation. Let A be an abelian variety of dimension $r > 1$ defined over an algebraically closed field and let Θ be an ample and globally generated divisor on A . Let $\iota : C \hookrightarrow A$ be a reduced and irreducible curve of geometric genus g with normalization $\nu : \tilde{C} \rightarrow C$ and set

$$f := \iota \circ \nu : \tilde{C} \longrightarrow A \quad \text{and} \quad d := \deg(f^* \mathcal{O}_A(\Theta)) = (\Theta \cdot C).$$

Finally, we denote by p and q the projections from $\tilde{C} \times A$ onto the first and second factor respectively, and by Γ the graph of f .

In order to compute the Θ -regularity of \mathcal{I}_C , we will show that it is enough to check the continuous global generation of sheaves of type $q_*(p^*Q \otimes \mathcal{I}_\Gamma) \otimes \mathcal{O}_A(\Theta)$ where Q is a globally generated line bundle on \tilde{C} (see Proposition 3.2). Before doing so, we present a simple lemma along the lines of [GLP, Lemma 1.6] computing cohomological support loci of sheaves admitting resolutions which are exact at most away an algebraic closed subset of dimension one.

Lemma 3.1. *Let*

$$\mathcal{E} : \quad \dots \longrightarrow \mathcal{E}_2 \xrightarrow{d_2} \mathcal{E}_1 \xrightarrow{d_1} \mathcal{E}_0 \xrightarrow{d_0} \mathcal{J} \longrightarrow 0$$

be a finite complex of coherent sheaves on a smooth projective irregular variety Y such that d_0 is surjective. If \mathcal{E} is exact away from an algebraic closed subset of dimension at most one, then there are inclusions

$$V^i(\mathcal{J}) \subset V^i(\mathcal{E}_0) \cup V^{i+1}(\mathcal{E}_1) \cup \dots \cup V^{\dim Y}(\mathcal{E}_{\dim Y - i}) \quad \text{for any } i \geq 1.$$

Proof. For any $j \geq 0$ we set

$$\mathcal{K}_j := \text{Ker } d_j, \quad \mathcal{I}_j := \text{Im } d_j, \quad \text{and} \quad \mathcal{H}_j := \mathcal{K}_j / \mathcal{I}_{j+1}.$$

We note that since $\dim \text{Supp } \mathcal{H}_j \leq 1$ we must have

$$V^i(\mathcal{H}_j) = \emptyset \quad \text{for all } j \geq 0 \quad \text{and} \quad i > 1.$$

We prove the lemma by chopping \mathcal{E} into short exact sequences $0 \rightarrow \mathcal{K}_j \rightarrow \mathcal{E}_j \rightarrow \mathcal{I}_j \rightarrow 0$ from which we get inclusions

$$(3) \quad V^i(\mathcal{I}_j) \subset V^i(\mathcal{E}_j) \cup V^{i+1}(\mathcal{K}_j) \text{ for all } j \geq 0 \text{ and } i \geq 1.$$

Moreover, there are exact sequences $0 \rightarrow \mathcal{I}_{j+1} \rightarrow \mathcal{K}_j \rightarrow \mathcal{H}_j \rightarrow 0$ giving the further inclusions

$$(4) \quad V^{i+1}(\mathcal{K}_j) \subset V^{i+1}(\mathcal{I}_{j+1}) \cup V^{i+1}(\mathcal{H}_j) = V^{i+1}(\mathcal{I}_{j+1}) \text{ for all } j \geq 0 \text{ and } i \geq 1.$$

Therefore, by putting together (3) and (4), we obtain inclusions for all $i \geq 1$

$$V^i(\mathcal{J}) \subset V^i(\mathcal{E}_0) \cup V^{i+1}(\mathcal{E}_1) \cup \dots \cup V^{\dim Y}(\mathcal{E}_{\dim Y-i}) \cup V^{\dim Y+1}(\mathcal{K}_{\dim Y-i}),$$

which conclude the proof as $V^{\dim Y+1}(\mathcal{K}_{\dim Y-i}) = \emptyset$. \square

The following proposition can be considered as an analogous of [GLP, Proposition 1.2] in the case of abelian varieties.

Proposition 3.2. *Let Q be a globally generated line bundle on \tilde{C} . If the sheaf $q_*(p^*Q \otimes \mathcal{I}_\Gamma) \otimes \mathcal{O}_A(\Theta)$ is continuously globally generated, then the ideal sheaf \mathcal{I}_C is $(h^0(\tilde{C}, Q) + r)$ - Θ -regular.*

Proof. From the short exact sequence defining Γ

$$(5) \quad 0 \longrightarrow \mathcal{I}_\Gamma \longrightarrow \mathcal{O}_{\tilde{C} \times A} \longrightarrow \mathcal{O}_\Gamma = (\text{id}_{\tilde{C}}, f)_* \mathcal{O}_{\tilde{C}} \longrightarrow 0$$

we deduce the following sequence

$$(6) \quad 0 \longrightarrow q_*(p^*Q \otimes \mathcal{I}_\Gamma) \longrightarrow H^0(\tilde{C}, Q) \otimes \mathcal{O}_A \xrightarrow{ev} f_*Q \longrightarrow R^1q_*(p^*Q \otimes \mathcal{I}_\Gamma)$$

where, since Q is globally generated, the evaluation map ev may fail to be surjective at most at the singular points of C . We denote by \mathcal{G} the image of ev . Moreover, by hypotheses there exist an integer $N > 0$ and line bundles $\alpha_1, \dots, \alpha_N \in \text{Pic}^0(A)$ such that the sum of twisted evaluation maps

$$(7) \quad \bigoplus_{i=1}^N H^0(A, q_*(p^*Q \otimes \mathcal{I}_\Gamma) \otimes \mathcal{O}_A(\Theta) \otimes \alpha_i) \otimes \alpha_i^{-1} \longrightarrow q_*(p^*Q \otimes \mathcal{I}_\Gamma) \otimes \mathcal{O}_A(\Theta)$$

is surjective. For simplicity we define

$$W_i := H^0(A, q_*(p^*Q \otimes \mathcal{I}_\Gamma) \otimes \mathcal{O}_A(\Theta) \otimes \alpha_i) \quad \text{for } i = 1, \dots, N.$$

Therefore, we obtain a presentation of $\mathcal{G} = \text{Im}(ev)$:

$$(8) \quad \bigoplus_{i=1}^N W_i \otimes \alpha_i^{-1} \otimes \mathcal{O}_A(-\Theta) \xrightarrow{\varphi} H^0(\tilde{C}, Q) \otimes \mathcal{O}_A \longrightarrow \mathcal{G} \longrightarrow 0$$

by putting together (6) and (7). Moreover set

$$E := \bigoplus_{i=1}^N W_i \otimes \alpha_i^{-1} \otimes \mathcal{O}_A(-\Theta), \quad F := H^0(\tilde{C}, Q) \otimes \mathcal{O}_A, \quad e := \text{rank } E, \quad a := \dim H^0(\tilde{C}, Q)$$

and denote by $\mathcal{J} \subset \mathcal{O}_A$ the 0-th Fitting ideal of \mathcal{G} . Since Q is globally generated, \mathcal{G} is supported on C . Moreover, as C is reduced, there is an inclusion of sheaves $\mathcal{J} \subset \mathcal{I}_C$ such that the quotient $\mathcal{I}_C/\mathcal{J}$ may fail to be trivial at most at the (finitely many) singular points of C (see [GLP, p. 496] for a similar argument). As \mathcal{J} coincides with the image of $\wedge^a \varphi$, there is an Eagon–Northcott complex associated to $\varphi : E \rightarrow F$ of the form (see [Laz2, Theorem B.2.2], or

[GLP, §0]):

$$0 \longrightarrow \bigwedge^e E \otimes \mathrm{Sym}^{e-a} F^\vee \longrightarrow \dots \longrightarrow \bigwedge^{a+1} E \otimes F^\vee \longrightarrow \bigwedge^a E \longrightarrow \mathcal{J} \longrightarrow 0$$

which is exact away from $\mathrm{Supp}(\mathrm{coker} \varphi) = C$. We note that, for all $i \geq 0$, the terms $E_i := \bigwedge^{a+i} E \otimes \mathrm{Sym}^{e-a-i} F^\vee$ of the resolution can be written as

$$E_i \simeq \bigoplus_t \mathcal{O}_A((-a-i)\Theta) \otimes \beta_{it} \quad \text{for some } \beta_{it} \in \mathrm{Pic}^0(A).$$

Moreover, as $\mathcal{I}_C/\mathcal{J}$ is supported at finitely many points, the sheaf \mathcal{I}_C is $(a+r)$ - Θ -regular as soon as \mathcal{J} is $(a+r)$ - Θ -regular. But an application of Lemma 3.1 to the sequence (3) provides inclusions of cohomological support loci for all $i > 0$

$$V^i(\mathcal{J}((a+r-1)\Theta)) \subset \bigcup_{k=i}^r V^k(E_{k-i}((a+r-1)\Theta)) = \bigcup_{k,t} V^k(\mathcal{O}_A((r-k+i-1)\Theta) \otimes \beta_{kt})$$

from which we see that

$$\begin{aligned} \dim V^1(\mathcal{J}((a+r-1)\Theta)) &= 0 \quad \text{and} \\ V^i(\mathcal{J}((a+r-1)\Theta)) &= \emptyset \quad \text{for } i > 1. \end{aligned}$$

So in particular \mathcal{J} is $(a+r)$ - Θ -regular and the proposition is proved. \square

In the next proposition we will give sufficient conditions for the hypotheses of Proposition 3.2 to hold (recall that M -regular sheaves are continuously globally generated by Theorem 2.2 (i)).

Proposition 3.3. *If B is a line bundle on \tilde{C} such that f_*B is M -regular on A , then $q_*(p^*(B \otimes f^*\mathcal{O}_A(\Theta))) \otimes \mathcal{I}_\Gamma \otimes \mathcal{O}_A(\Theta)$ is also M -regular.*

Proof. We continue to denote by Γ the graph of the morphism f , and by p and q the projections from $\tilde{C} \times A$ onto the first and second factor respectively. We define

$$\mathcal{H} := q_*(\mathcal{I}_\Gamma \otimes p^*(B \otimes f^*\mathcal{O}_A(\Theta)))$$

so that it is enough to check $\mathrm{codim}_{\mathrm{Pic}^0(A)} V^i(\mathcal{H} \otimes \mathcal{O}_A(\Theta)) > i$ for all $i > 0$.

By Theorem 2.2 the sheaf $f_*B \otimes \mathcal{O}_A(m\Theta)$ is globally generated and satisfies *I.T.0* for all $m \geq 1$. Therefore, from the sequence (5) and the isomorphism

$$(9) \quad H^0(\tilde{C}, B \otimes f^*\mathcal{O}_A(\Theta)) \simeq H^0(A, f_*B \otimes \mathcal{O}_A(\Theta))$$

obtained with the use of projection formula, we obtain exact sequences

$$(10) \quad 0 \rightarrow \mathcal{H} \otimes \alpha(\Theta) \rightarrow H^0(\tilde{C}, B \otimes f^*\mathcal{O}_A(\Theta)) \otimes \alpha(\Theta) \rightarrow f_*B \otimes \alpha(2\Theta) \rightarrow 0$$

for any α in $\mathrm{Pic}^0(A)$. As both $\mathcal{O}_A(\Theta)$ and $f_*B \otimes \mathcal{O}_A(2\Theta)$ satisfy *I.T.0*, we deduce

$$V^i(\mathcal{H} \otimes \mathcal{O}_A(\Theta)) = \emptyset \quad \text{for all } i \geq 2$$

so that we are left with the study of the dimension of $V^1(\mathcal{H} \otimes \mathcal{O}_A(\Theta))$. To this end we denote by

$$(11) \quad m_\alpha : H^0(A, f_*B \otimes \mathcal{O}_A(\Theta)) \otimes H^0(A, \mathcal{O}_A(\Theta) \otimes \alpha) \rightarrow H^0(A, f_*B \otimes \mathcal{O}_A(2\Theta) \otimes \alpha)$$

the multiplication map induced by (10) and (9), and furthermore we point out the identification

$$(12) \quad V^1(\mathcal{H} \otimes \mathcal{O}_A(\Theta)) = \{\alpha \in \text{Pic}^0(A) \mid m_\alpha \text{ is not surjective}\}.$$

In order to conclude the proof, we claim that it is enough to prove the following inclusion of algebraic sets

$$(13) \quad (-1_{\text{Pic}^0(A)})(V^1(\mathcal{H} \otimes \mathcal{O}_A(\Theta))) \subset V^1(f_*B)$$

where $(-1_{\text{Pic}^0(A)}) : \text{Pic}^0(A) \rightarrow \text{Pic}^0(A)$ is the isomorphism mapping α to α^{-1} . In fact, this inclusion would immediately imply

$$\dim V^1(\mathcal{H} \otimes \mathcal{O}_A(\Theta)) \leq \dim V^1(f_*B) \leq r - 2$$

as f_*B is M -regular by hypotheses.

Now we show the inclusion (13). Since for any $\alpha \in \text{Pic}^0(A)$ the sheaf $\mathcal{O}_A(\Theta) \otimes \alpha$ is globally generated, we get exact sequences

$$(14) \quad 0 \longrightarrow M_{\mathcal{O}_A(\Theta) \otimes \alpha} \longrightarrow H^0(A, \mathcal{O}_A(\Theta) \otimes \alpha) \otimes \mathcal{O}_A \longrightarrow \mathcal{O}_A(\Theta) \otimes \alpha \longrightarrow 0$$

where $M_{\mathcal{O}_A(\Theta) \otimes \alpha}$ is the kernel bundle of the evaluation map of $\mathcal{O}_A(\Theta) \otimes \alpha$. Furthermore, by tensorizing (14) by $\mathcal{O}_A(\Theta) \otimes \nu_*B$, we get further exact sequences

$$0 \longrightarrow \iota^*(M_{\mathcal{O}_A(\Theta) \otimes \alpha} \otimes \mathcal{O}_A(\Theta)) \otimes \nu_*B \longrightarrow H^0(A, \mathcal{O}_A(\Theta) \otimes \alpha) \otimes \iota^*\mathcal{O}_A(\Theta) \otimes \nu_*B \xrightarrow{\text{ev}_\alpha} \iota^*(\mathcal{O}_A(2\Theta) \otimes \alpha) \otimes \nu_*B \longrightarrow 0.$$

We note now that the map induced by ev_α on global sections coincides with the multiplication map m_α introduced in (11). Hence, by using the identification (12), and from the fact that

$$(15) \quad H^1(C, \iota^*\mathcal{O}_A(\Theta) \otimes \nu_*B) \simeq H^1(A, f_*B \otimes \mathcal{O}_A(\Theta)) = 0$$

(here we used projection formula and the fact that $f_*B \otimes \mathcal{O}_A(\Theta)$ satisfies *IT.0*), we see that

$$(16) \quad \alpha \in V^1(\mathcal{H} \otimes \mathcal{O}_A(\Theta)) \implies H^1(C, \iota^*(M_{\mathcal{O}_A(\Theta) \otimes \alpha} \otimes \mathcal{O}_A(\Theta)) \otimes \nu_*B) \neq 0.$$

Now pick an arbitrary element $\alpha \in V^1(\mathcal{H} \otimes \mathcal{O}_A(\Theta))$ and set

$$W := \text{Im}(H^0(A, \mathcal{O}_A(\Theta) \otimes \alpha) \longrightarrow H^0(C, \iota^*(\mathcal{O}_A(\Theta) \otimes \alpha))).$$

We note that W generates the sheaf $\iota^*(\mathcal{O}_A(\Theta) \otimes \alpha)$. Therefore, by Castelnuovo's pencil trick, we can pick two general sections s_1 and s_2 in W such that they generate $\iota^*(\mathcal{O}_A(\Theta) \otimes \alpha)$. We have then a commutative diagram:

$$\begin{array}{ccccccc} 0 \rightarrow \iota^*(\mathcal{O}_A(-\Theta) \otimes \alpha^{-1}) & \longrightarrow & \mathcal{O}_C \oplus \mathcal{O}_C & \xrightarrow{(s_1, s_2)} & \iota^*(\mathcal{O}_A(\Theta) \otimes \alpha) & \rightarrow & 0 \\ & & \downarrow & & \parallel & & \\ 0 \longrightarrow \iota^*M_{\mathcal{O}_A(\Theta) \otimes \alpha} & \longrightarrow & H^0(A, \mathcal{O}_A(\Theta) \otimes \alpha) \otimes \mathcal{O}_C & \rightarrow & \iota^*(\mathcal{O}_A(\Theta) \otimes \alpha) & \rightarrow & 0. \end{array}$$

By defining

$$\tilde{V} := H^0(A, \mathcal{O}_A(\Theta) \otimes \alpha) / \langle s_1, s_2 \rangle,$$

the Snake Lemma produces an exact sequence

$$0 \longrightarrow \iota^*(\mathcal{O}_A(-\Theta) \otimes \alpha^{-1}) \longrightarrow \iota^*M_{\mathcal{O}_A(\Theta) \otimes \alpha} \longrightarrow \tilde{V} \otimes \mathcal{O}_C \longrightarrow 0,$$

and hence the exact sequence

$$0 \longrightarrow \iota^*\alpha^{-1} \otimes \nu_*B \longrightarrow \iota^*(M_{\mathcal{O}_A(\Theta) \otimes \alpha} \otimes \mathcal{O}_A(\Theta)) \otimes \nu_*B \longrightarrow \tilde{V} \otimes \iota^*\mathcal{O}_A(\Theta) \otimes \nu_*B \longrightarrow 0.$$

Now by (15) and (16) we deduce

$$H^1(C, \iota^* \alpha^{-1} \otimes \nu_* B) \simeq H^1(A, f_* B \otimes \alpha^{-1}) \neq 0$$

(once again we used projection formula) from which we finally get $\alpha^{-1} \in V^1(f_* B)$. Since the line bundle $\alpha \in V^1(\mathcal{H} \otimes \mathcal{O}_A(\Theta))$ was arbitrary, this concludes the proof of the inclusion (13), and hence of the proposition. □

Before proceeding with the proof of Theorem 1.1, we check when the hypotheses of Proposition 3.3 hold. In other words we establish which line bundles B on the smooth curve \tilde{C} make the sheaf $f_* B$ M -regular on A .

Lemma 3.4. *Let D be a smooth and irreducible curve of genus $g \geq 2$, and let $f : D \rightarrow A$ be a morphism to an abelian variety A of dimension r . If B is a general line bundle on D of degree $b \geq g - 2$, then $\dim V^1(f_* B) \leq g + r - b - 2$. In particular, the sheaf $f_* B$ is M -regular (resp. satisfies I.T.0) on A if B is a general line bundle of degree $b \geq g$ (resp. $b \geq g + r - 1$).*

Proof. Without loss of generality, we can assume that f is non-constant since in this case $V^1(f_* B) = \emptyset$ for any $B \in \text{Pic}(D)$. We note that the algebraic set $V^1(B)$ is irreducible as Serre duality yields an isomorphism $V^1(B) \simeq W_{2g-2-b}(D)$ (here $W_{2g-2-b}(D)$ is the Brill–Noether locus parameterizing line bundles on D of degree $2g - 2 - b$ with at least one non-zero global section). Moreover, the algebraic group $\text{Pic}^0(D)$ acts on itself via translations: for any $\gamma \in \text{Pic}^0(D)$ we write $\gamma V^1(B)$ for the image of $V^1(B)$ under the action of γ . Then, by Kleiman’s Transversality Theorem [Kl, Theorem 2], there exists an open dense subset $V \subset \text{Pic}^0(D)$ such that for all $\gamma \in V$ the fiber product $\gamma V^1(B) \times_{\text{Pic}^0(D)} \text{Pic}^0(A)$ is either empty or of dimension

$$\dim V^1(B) + \dim \text{Pic}^0(A) - \dim \text{Pic}^0(D) = \dim V^1(B) + r - g.$$

As $\gamma V^1(B) \simeq V^1(B \otimes \gamma^{-1})$ for any $\gamma \in V$, we obtain closed immersions $V^1(f_*(B \otimes \gamma^{-1})) \hookrightarrow V^1(B \otimes \gamma^{-1}) \times_{\text{Pic}^0(D)} \text{Pic}^0(A)$ by the universal property of the fiber product and the projection formula. Hence, for any $\gamma \in V$, we have

$$\begin{aligned} \dim V^1(f_*(B \otimes \gamma^{-1})) &\leq \dim (\gamma V^1(B) \times_{\text{Pic}^0(D)} \text{Pic}^0(A)) \\ &= \dim V^1(B) + r - g \\ &= \dim W_{2g-2-b}(D) + r - g \\ &= r + g - b - 2. \end{aligned}$$

□

At this point Theorem 1.1 follows by putting together the previous lemmas and propositions.

Proof of Theorem 1.1. We focus first on the case Θ is globally generated. By Lemma 3.4 we can pick a line bundle B on \tilde{C} of degree $g = g(\tilde{C})$ such that $f_* B$ is M -regular on A . Moreover, we can take B sufficiently general so that $B \otimes f^* \mathcal{O}_A(\Theta)$ is globally generated. Hence by Proposition 3.3 the sheaf $q_*(p^*(B \otimes f^* \mathcal{O}_A(\Theta)) \otimes \mathcal{I}_\Gamma) \otimes \mathcal{O}_A(\Theta)$ is M -regular, and thus continuously globally

generated by Theorem 2.2 (i). Therefore, we can apply Proposition 3.2 with $Q = B \otimes f^* \mathcal{O}_A(\Theta)$ which tells us that \mathcal{I}_C is $(h^0(\tilde{C}, Q) + r)$ - Θ -regular. But by Riemann–Roch’s formula

$$h^0(\tilde{C}, B \otimes f^* \mathcal{O}_A(\Theta)) = d + 1$$

as $h^1(\tilde{C}, B \otimes f^* \mathcal{O}_A(\Theta)) = 0$ by (15).

If Θ is not globally generated, then by Lefschetz’s theorem 2Θ is ([Mum, p. 57-58]). Therefore, by the previous argument, $\mathcal{I}_C((r + d')(2\Theta))$ is M -regular where $d' = \deg(f^* \mathcal{O}_A(2\Theta)) = 2d$, and hence $\mathcal{I}_C((4d + 2r)\Theta)$ is M -regular. This is equivalent to saying that \mathcal{I}_C is $(4d + 2r + 1)$ - Θ -regular. \square

Proof of Corollary 1.2. The statements relative to properties (AA) follow from Theorem 1.1 and the fact that, for any integer $n \geq 2$, the ideal sheaf of a curve \mathcal{I}_C is n - Θ -regular if and only if properties (AA_{n-1}^3) and (AB_{n-1}^2) hold. Moreover, in this case $\mathcal{I}_C(n\Theta)$ satisfies *I.T.0* by Theorem 2.2 (ii), and therefore property (AA_n) hold.

The statements relative to property (AC^{lin}) follow again from Theorem 1.1 since if \mathcal{I}_C is n - Θ -regular, then $\mathcal{I}_C(n\Theta)$ is 0- Θ -regular, and hence globally generated (Theorem 2.2 (iv)). Finally, the statement relative to property (AC^{alg}) is an application of the fact that M -regular sheaves are continuously globally generated (Theorem 2.2 (i)). \square

4. GENUS BOUNDS

The following theorem proves and extends Theorem 1.3.

Theorem 4.1. *Let (A, Θ) be a polarized abelian variety of dimension $r > 1$ defined over an algebraically closed field and let $C \subset A$ be a reduced and irreducible curve of degree $d = (\Theta \cdot C)$ with normalization \tilde{C} . Denote by $f : \tilde{C} \rightarrow A$ the induced morphism and set*

$$\begin{aligned} h_{\min}^0 &:= \min \left\{ h^0 \left(\tilde{C}, f^* (\mathcal{O}_A(\Theta) \otimes \gamma) \right) \mid \gamma \in \text{Pic}^0(A) \right\}, \\ h_{\max}^0 &:= \max \left\{ h^0 \left(\tilde{C}, f^* (\mathcal{O}_A(\Theta) \otimes \gamma) \right) \mid \gamma \in \text{Pic}^0(A) \right\}. \end{aligned}$$

Then the following bound holds for the geometric genus $g = g(\tilde{C})$ of C :

$$g \leq \min \left\{ (d - h_{\min}^0 + 1)(4d + 2r + 1), (d - h_{\max}^0 + 1)(4d + 2r + 2) \right\}.$$

Moreover, if in addition Θ is globally generated, then the following bound holds:

$$g \leq \min \left\{ (d - h_{\min}^0 + 1)(d + r + 1), (d - h_{\max}^0 + 1)(d + r + 2) \right\},$$

and moreover $h_{\max}^0 \geq h_{\min}^0 \geq 2$ (≥ 3 if Θ is very ample).

Proof. We prove the theorem in the case Θ is globally generated by stressing out the parts that should be changed accordingly in case Θ is not globally generated.

The fact that $h_{\min}^0 \geq 2$ easily follows since f is a finite morphism and $\mathcal{O}_A(\Theta) \otimes \gamma$ is globally generated and ample for all γ . Similarly, the bound $h_{\min}^0 \geq 3$ follows as the restriction of a very ample line bundle is still very ample and there are no rational curves in an abelian variety.

By Corollary 1.2 C satisfies (AA_{d+r+1}^3) and therefore there exists a closed subset $V \subset \text{Pic}^0(A)$ such that $\text{codim } V \geq 3$ and the line bundles $\mathcal{O}_C((d+r+1)\Theta) \otimes \alpha|_C$ are non-special for all $\alpha \notin V$. (If Θ is not globally generated, we would need to work with the bundles $\mathcal{O}_C((4d+2r+1)\Theta) \otimes \alpha|_C$ instead.) Denoting by $\iota : C \hookrightarrow A$ the closed immersion and by $f : \tilde{C} \rightarrow A$ the composition of the normalization $\nu : \tilde{C} \rightarrow C$ with ι , we see that the line bundles $f^*(\mathcal{O}_A((d+r+1)\Theta) \otimes \alpha)$ are non-special too. This easily follows by tensorizing by $\iota^*(\mathcal{O}_A((d+r+1)\Theta) \otimes \alpha)$ the following short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \nu_*\mathcal{O}_{\tilde{C}} \longrightarrow Q \longrightarrow 0,$$

where Q is a sheaf supported at the singular points of C , and by noting that the induced long exact sequence in cohomology, together with projection formula, yields

$$H^1(\tilde{C}, f^*(\mathcal{O}_A((d+r+1)\Theta) \otimes \alpha)) \simeq H^1(C, \iota^*(\mathcal{O}_A((d+r+1)\Theta) \otimes \alpha) \otimes \nu_*\mathcal{O}_{\tilde{C}}) = 0.$$

This immediately gives

$$\chi(f^*\mathcal{O}_A((d+r+1)\Theta) \otimes \alpha) = h^0(\tilde{C}, f^*(\mathcal{O}_A((d+r+1)\Theta) \otimes \alpha)) \quad \text{for all } \alpha \notin V,$$

and moreover, by writing $\alpha = \gamma^{\otimes d+r+1}$ for some $\gamma \in \text{Pic}^0(A)$, we obtain that for all $\alpha \notin V$

$$\begin{aligned} h^0(\tilde{C}, f^*(\mathcal{O}_A((d+r+1)\Theta) \otimes \alpha)) &\geq \\ (d+r+1)h^0(\tilde{C}, f^*(\mathcal{O}_A(\Theta) \otimes \gamma)) - (d+r) &\geq (d+r+1)h_{\min}^0 - (d+r) \end{aligned}$$

where the first inequality is [Ha, Lemma 5.5]. On the other hand, Riemann–Roch’s formula gives

$$\chi(f^*(\mathcal{O}_A((d+r+1)\Theta) \otimes \alpha)) = d(d+r+1) - g + 1,$$

from which we obtain the bound $g \leq (d - h_{\min}^0 + 1)(d+r+1)$. (If Θ is not globally generated one obtains the bound $g \leq (d - h_{\min}^0 + 1)(4d+2r+1)$ with the same strategy.)

Now we prove the bound $g \leq (d - h_{\max}^0 + 1)(d+r+2)$. Exploiting the fact that C satisfies (AA_{d+r+2}) , this time the line bundles $\mathcal{O}_C((d+r+2)\Theta) \otimes \alpha|_C$ are non-special for all $\alpha \in \text{Pic}^0(A)$, and therefore so the $f^*(\mathcal{O}_A((d+r+2)\Theta) \otimes \alpha)$ are. Therefore, by writing $\alpha = \gamma^{\otimes d+r+2}$ for any $\alpha \in \text{Pic}^0(A)$ and by means of [Ha, Lemma 5.5], we deduce that for all $\gamma \in \text{Pic}^0(A)$:

$$\begin{aligned} d(d+r+2) - g + 1 &= \chi(f^*\mathcal{O}_A((d+r+2)\Theta) \otimes \alpha) \\ &= h^0(\tilde{C}, f^*(\mathcal{O}_A((d+r+2)\Theta) \otimes \alpha)) \\ &\geq (d+r+2)h^0(\tilde{C}, f^*\mathcal{O}_A(\Theta) \otimes \gamma) - (d+r+1). \end{aligned}$$

At this point the claimed bound follows at once. If Θ is not globally generated, then C satisfies $(AA_{4d+2r+2})$ and therefore the line bundles $\mathcal{O}_C((4d+2r+2)\Theta) \otimes \alpha|_C$ are non-special. Therefore one concludes as in the globally generated case. \square

Remark 4.2. The bounds in Theorem 4.1 improve, in some particular cases, a previous genus bound

$$g(C) < \frac{(2d-1)^2}{2(r-1)}$$

established by Debarre in [De, Theorem 5.1]. For instance, this is the case if A is an abelian surface and Θ is a globally generated line bundle on A such that $NS(A) = \mathbf{Z}[\Theta]$. Other examples are provided by polarized abelian threefold (A, Θ) with Θ globally generated, $NS(A) = \mathbf{Z}[\Theta]$,

and $h_{\min}^0 \geq 6$. Another genus bound which is sharp in the case of principally polarized surfaces can be found in [PP2, Theorem B].

5. REGULARITY OF BRILL–NOETHER LOCI

In this section we work over the field of the complex numbers. Let C be a smooth curve of genus $g \geq 3$ and denote by

$$W_d^r(C) = W_d^r := \{L \in \text{Pic}^d(C) \mid h^0(C, L) \geq r + 1\}$$

the Brill–Noether loci endowed with their usual scheme structure given by means of Fitting ideals. Up to tensorizing line bundles with a fixed line bundle of degree $-d$, we think of these spaces as subvarieties of the Jacobian $(J(C), \Theta)$ of C . We aim to bound the Θ -regularity of $W_d^r(C)$ for all r and d in case C is Petri general. Before doing so we compute the Θ -regularity of Picard bundles.

5.1. Regularity of Picard bundles. Let p and q be the projections from the product $C \times J(C)$ onto the first and second factor respectively. We identify $J(C)$ with its dual abelian variety via the principal polarization Θ and denote by $\iota : C \hookrightarrow J(C)$ an Abel–Jacobi map. We denote by

$$\Phi_{\mathcal{E}} : \mathbf{D}(C) \longrightarrow \mathbf{D}(J(C)), \quad \mathcal{F} \mapsto \mathbf{R}q_*(p^*(\mathcal{F}) \otimes \mathcal{E})$$

the Fourier–Mukai transform defined by the kernel $\mathcal{E} := (\iota \times \text{id}_{J(C)})^*P$ where P is a Poincaré bundle on $J(C) \times J(C)$.

An n -th Picard bundle E ($n \geq 2g - 1$) is a locally free sheaf of the form

$$q_*(p^*L \otimes \mathcal{E})$$

where L is a line bundle on C of degree n . It is straightforward to check that L satisfies *W.I.T.0* with respect to $\Phi_{\mathcal{E}}$, *i.e.* the cohomology sheaves $H^i(\Phi_{\mathcal{E}}(L))$ vanish for all $i \neq 0$, and that $E \simeq \widehat{\Phi_{\mathcal{E}}(L)} := H^0(\Phi_{\mathcal{E}}(L))$.

The regularity of Picard bundles has been studied in [PP3, Remark 7.17] where the authors give an effective bound on the degree n so that the product $E^{\otimes k} \otimes \mathcal{O}_{J(C)}(\Theta)$ ($1 \leq k \leq g - 1$) satisfies *I.T.0*. Here we notice that if one is only interested in weaker notions of regularity, such as *M*-regularity or *GV*, then it is possible to take smaller values of n . The proof of the following proposition follows the proof of [PP3, Proposition 7.15], the hints of [PP3, Remark 7.17], and also part of an argument of [PP1, Proposition 4.4].

Proposition 5.1. *Let C be a smooth curve of genus $g \geq 2$ and let E be an n -th Picard bundle. If $n \geq 4g - 5$ (resp. $n \geq 4g - 4$), then for any $1 \leq k \leq g - 1$ the bundles $E^{\otimes k}(\Theta)$ satisfy *GV* (resp. are *M*-regular).*

Proof. We only prove the *GV*ness of $E^{\otimes k}(\Theta)$ as the other statement is completely analogous. We can suppose that $E \simeq \widehat{\Phi_{\mathcal{E}}(L)}$ where L a line bundle of degree $n \geq 4g - 5$ on C . Denote by $\iota : C \hookrightarrow J(C)$ the Abel–Jacobi map $x \mapsto \mathcal{O}_C(x - x_0)$ where $x_0 \in C$ is a point and let $\pi_k : C^k \rightarrow J(C)$ be the morphism $(y_1, \dots, y_k) \mapsto \iota(y_1) + \dots + \iota(y_k)$. First of all we will show

that

$$(17) \quad R^i \pi_{k*}(L^{\boxtimes k}) = 0 \quad \text{for all } i > 0.$$

Note that $\pi_k = u_k \circ \sigma_k$ where $\sigma_k : C^k \rightarrow C_k$ is the quotient map under the action of the k -symmetric group on C^k , C_k is the k -fold symmetric product of C , and $u_k : C_k \rightarrow J(C)$ is the Abel–Jacobi map $(x_1, \dots, x_k) \mapsto \mathcal{O}_C(x_1 + \dots + x_k - kx_0)$. According to [Laz2, Lemma 4.3.10] in order to show (17) it suffices to show the vanishings of

$$H^i(C^k, L^{\boxtimes k} \otimes \sigma_k^* u_k^* \mathcal{O}_{J(C)}(l\Theta)) \quad \text{for } i > 0 \quad \text{and } l \gg 0.$$

But these follow from Kodaira Vanishing Theorem since $L^{\boxtimes k}$ can be written as $\omega_{C^k} \otimes (L')^{\boxtimes k}$ where L' is a line bundle on C of positive degree and $\sigma_k^* u_k^* \mathcal{O}_{J(C)}(l\Theta)$ is a nef line bundle on C^k . Then, as $(L')^{\boxtimes k}$ is ample on C^k , the product $(L')^{\boxtimes k} \otimes \sigma_k^* u_k^* \mathcal{O}_{J(C)}(l\Theta)$ is ample for all $l \geq 0$, and hence the desired vanishings follow.

We now denote by $\Phi_P : \mathbf{D}(J(C)) \rightarrow \mathbf{D}(J(C))$ the Fourier–Mukai transform induced by a Poincaré bundle P . We say that a complex \mathcal{F} satisfies *W.I.T.i* (with respect to Φ_P) if the cohomology sheaves $H^j(\Phi_P(\mathcal{F}))$ vanish for all $j \neq i$. In this case we denote by $\widehat{\mathcal{F}}$ the sheaf such that $\Phi_P(\mathcal{F}) \simeq \widehat{\mathcal{F}}[-i]$, *i.e.* $\widehat{\mathcal{F}} = H^i(\Phi_P(\mathcal{F}))$. It is easy to check that $\pi_{k*}(L^{\boxtimes k})$ satisfies *I.T.0*, and hence *W.I.T.0*. In fact, by using the Leray spectral sequence, the vanishings in (17), and Kodaira vanishing we obtain

$$H^i(J(C), \pi_{k*}(L^{\boxtimes k}) \otimes \alpha) \simeq H^i(C^k, L^{\boxtimes k} \otimes \pi_k^* \alpha) = 0 \quad \text{for all } \alpha \in \text{Pic}^0(J(C)), i > 0.$$

Moreover there are isomorphisms

$$\widehat{\pi_{k*}(L^{\boxtimes k})} \simeq (m_*((\iota_* L)^{\boxtimes k}))^\wedge \simeq \widehat{\iota_* L^{\otimes k}} = E^{\otimes k}$$

where for the second isomorphism we used the exchange formula of Pontrjagin product and tensor product [Muk, (3.7)] (see also [PP3, Lemma 7.16]). Therefore by Mukai’s inversion theorem [Muk, Theorem 2.2] the bundle $E^{\otimes k}$ satisfies *W.I.T.g* and furthermore

$$\widehat{E^{\otimes k}} \simeq (-1_{J(C)})^* \pi_{k*}(L^{\boxtimes k}).$$

Hence, for any $\alpha \in \text{Pic}^0(J(C))$, we have the following chain of isomorphisms:

$$\begin{aligned} H^i(J(C), E^{\otimes k} \otimes \mathcal{O}_{J(C)}(\Theta) \otimes \alpha) &\simeq \text{Ext}^i(\mathcal{O}_{J(C)}(-\Theta) \otimes \alpha^{-1}, E^{\otimes k}) \\ &\simeq \text{Ext}^i((\mathcal{O}_{J(C)}(-\Theta) \otimes \alpha^{-1})^\wedge, \widehat{E^{\otimes k}}) \\ &\simeq \text{Ext}^i((-1_{J(C)})^*(\mathcal{O}_{J(C)}(\Theta) \otimes \alpha), (-1_{J(C)})^* \pi_{k*}(L^{\boxtimes k})) \\ &\simeq H^i(J(C), \pi_{k*}(L^{\boxtimes k}) \otimes \mathcal{O}_{J(C)}(-\Theta) \otimes \alpha^{-1}) \end{aligned}$$

(the third isomorphism is consequence of [Muk, Theorem 3.13 (v)]).

We conclude that it is enough to prove the inequalities

$$(18) \quad \text{codim}_{\text{Pic}^0(J(C))} V^i(\pi_{k*}(L^{\boxtimes k}) \otimes \mathcal{O}_{J(C)}(-\Theta)) \geq i \quad \text{for all } i > 0.$$

On the other hand, since $R^i \pi_{k*}(L^{\boxtimes k}) = 0$ for all $i > 0$, the vanishing of

$$H^i(J(C), \pi_{k*}(L^{\boxtimes k}) \otimes \mathcal{O}_{J(C)}(-\Theta) \otimes \alpha)$$

where $\alpha \in \text{Pic}^0(J(C))$ is equivalent to the vanishing of

$$(19) \quad H^i(C^k, L^{\boxtimes k} \otimes \pi_k^*(\mathcal{O}_{J(C)}(-\Theta) \otimes \alpha)).$$

Furthermore, as $k \leq g - 1$ we recall from [Iz, Appendix A.1] the isomorphism:

$$\pi_k^* \mathcal{O}_{J(C)}(\Theta) \simeq (\omega_C \otimes A^{-1})^{\boxtimes k} \otimes \mathcal{O}_{C^k}(-\Delta)$$

where Δ is the union of all diagonal divisors $\Delta_{i,j} := \{(x_1, \dots, x_k) \mid x_i = x_j\} \subset C^k$ ($1 \leq i < j \leq k$) and A is a line bundle of degree $g - k - 1$ on C . Then the vanishing of the group in (19) is equivalent to the vanishing of

$$H^{k-i}(C^k, (L^{-1} \otimes A^{-1} \otimes \omega_C^{\otimes 2})^{\boxtimes k} \otimes \mathcal{O}_{C^k}(-\Delta) \otimes \sigma_k^* u_k^* \alpha^{-1})$$

where we used Serre duality and wrote $\pi_k = u_k \circ \sigma_k$.

At this point, as the bundle $(L^{-1} \otimes A^{-1} \otimes \omega_C^{\otimes 2})^{\boxtimes k} \otimes \mathcal{O}_{C^k}(-\Delta)$ is invariant under the action of the symmetric group, it can be written as $\sigma_k^* \mathcal{L}'_{k,E}$ where $\mathcal{L}'_{k,E}$ is a line bundle on C_k and E is a divisor such that $\mathcal{O}_C(E) \simeq L^{-1} \otimes A^{-1} \otimes \omega_C^{\otimes 2}$. Therefore it is enough to check the vanishings of

$$H^{k-i}(C_k, \mathcal{L}'_{k,E} \otimes u_k^* \alpha) \quad \text{for } i > 0$$

which, using again [Iz], are identified to

$$\text{Sym}^{k-i} H^1(C, L^{-1} \otimes A^{-1} \otimes \omega_C^{\otimes 2} \otimes \iota^* \alpha) \otimes \wedge^i H^0(C, L^{-1} \otimes A^{-1} \otimes \omega_C^{\otimes 2} \otimes \iota^* \alpha)$$

for $1 \leq i \leq k - 1$, and

$$\wedge^k H^0(C, L^{-1} \otimes A^{-1} \otimes \omega_C^{\otimes 2} \otimes \iota^* \alpha)$$

for $i = k$ (here $\iota^* : \text{Pic}^0(J(C)) \rightarrow \text{Pic}^0(C)$ is the pull-back isomorphism).

In conclusion, as $\deg(L^{-1} \otimes A^{-1} \otimes \omega_C^{\otimes 2}) = 3g - 3 - n + k$, we have proved that

$$V^i(\pi_{k*}(L^{\boxtimes k}) \otimes \mathcal{O}_{J(C)}(-\Theta)) \subset W_{3g-3-n+k}(C) \quad \text{for } i = 1, \dots, k$$

and

$$V^i(\pi_{k*}(L^{\boxtimes k}) \otimes \mathcal{O}_{J(C)}(-\Theta)) = \emptyset \quad \text{for } i > k.$$

Therefore

$$\text{codim}_{\text{Pic}^0(J(C))} V^i(\pi_{k*}(L^{\boxtimes k}) \otimes \mathcal{O}_{J(C)}(-\Theta)) \geq n + 3 - 2g - k \quad \text{for } i = 1, \dots, k$$

and one can easily check that conditions (18) are true, for all $1 \leq k \leq g - 1$, as soon as $n \geq 4g - 5$. \square

Remark 5.2. We recall that if $n \geq 4g - 3$ then $E^{\otimes k}(\Theta)$ satisfies *I.T.0* for all $1 \leq k \leq g - 1$ ([PP3, Remark 7.17]).

5.2. Regularity of Brill–Noether loci. We bound the Θ -regularity of the Brill–Noether loci W_d^r .

Theorem 5.3. *Let C be a Petri general curve of genus $g \geq 3$, and let $d > 0$ and $r \geq 0$ be integers such that $0 \leq \rho := g - (r + 1)(g - d + r) \leq g - 1$. Write*

$$(20) \quad g - \rho + \lfloor \sqrt{g - \rho} \rfloor (4g - r - 5) = k(g - 1) + q$$

for some integers $k \geq 0$ and $0 \leq q < g - 1$. Moreover, set $s = 1$ if $q > 0$, and $s = 0$ else. Then $\mathcal{I}_{W_d^r}((k + s)\Theta)$ is M -regular.

Proof. The idea of the proof is to exploit the fact that ideal sheaves of degeneracy loci of the expected dimension admit an explicit resolution by locally free sheaves (*cf.* [Lay] and the references therein).

Let $m \geq 2g - d - 1$ be an integer. When C is Petri general, the loci W_d^r are $(m + d - g - r)$ -th degeneracy loci of a morphism of locally free sheaves

$$\gamma : E \longrightarrow F$$

where E is an $(m + d - g + 1)$ -th Picard bundle and F is a direct sum of m topologically trivial line bundles on C ([ACGH, §VII]). Therefore, from [Lay, Theorem on p. 747 and Remark 1.1], we get a resolution of $\mathcal{I}_{W_d^r}$

$$(21) \quad 0 \longrightarrow K^{g-\rho} \longrightarrow \dots \longrightarrow K^2 \longrightarrow K^1 \longrightarrow \mathcal{I}_{W_d^r} \longrightarrow 0$$

where the K^i 's are locally free sheaves described as follows.

Denote by \mathcal{P} the set of non-increasing sequences $I = (i_1, i_2, \dots)$ such that $i_j \geq i_{j+1}$ for any $j \geq 1$ and $i_j = 0$ for $j \gg 0$. Denote also by $|I| = \sum_j i_j$ the weight of I . The Young diagram of a partition I is the set of pairs (p, q) such that $p \leq i_q$. We denote by \bar{I} the adjoint partition to I determined by the involution $(p, q) \mapsto (q, p)$ acting on Young diagrams. The rank of a partition I is the largest integer l such that the pair (l, l) belongs to the corresponding Young diagram. If I is a partition of rank l , we define the k -augmented partition $I(k)$ of I by setting

$$I(k) = (i_1, i_2, \dots, i_l, \underbrace{l, \dots, l}_{k\text{-times}}, i_{l+1}, \dots).$$

We denote by S_I the Schur functor associated to a partition I . Finally, by setting $S_0(i) := \{I \in \mathcal{P} \mid |I| = i\}$, the above mentioned sheaves K^i are defined as:

$$K^i = \bigoplus_{I \in S_0(i)} S_{\bar{I}(m+d-g-r)} E \otimes S_{I(m+d-g-r)} F^*.$$

We now bound the regularity of the K^i 's. First of all, we note that $S_{\bar{I}(m+d-g-r)} E$ is a direct summand of the bundle $E^{\otimes (|\bar{I}(m+d-g-r)|)}$, and similarly so $S_{I(m+d-g-r)} F^*$ is a summand of $(F^*)^{\otimes (|I(m+d-g-r)|)}$. Moreover, by the Linearity Formula of [JPW], we have that $S_{I(m+d-g-r)} F^*$ is still a direct sum of topological trivial line bundles as F^* is so. Choosing the least m so that $\text{rk } E = m + d - g + 1 \geq 4g - 4$, *i.e.* $m = 5g - 5 - d$, by Proposition 5.1 we have that for any $1 \leq j \leq g - 1$ the sheaves $E^{\otimes j}(\Theta)$ are M -regular. Therefore, all we have to do, is to compute the integers $|\bar{I}(m + d - g - r)| = |\bar{I}(4g - 5 - r)|$ for any partitions I . But this is an easy calculation as

$$\max_{I \in S_0(i)} \{|\bar{I}(4g - 5 - r)|\} = |I| + \left\lfloor \sqrt{|I|} \right\rfloor (4g - 5 - r) = i + \left\lfloor \sqrt{i} \right\rfloor (4g - 5 - r)$$

since

$$\max_{I \in S_0(i)} \{\text{rank}(I)\} = \left\lfloor \sqrt{i} \right\rfloor.$$

Therefore, by letting k to be the integer as in (20), we have that for all $I \in S_0(i)$ and $i = 1, \dots, g - \rho$ the sheaves $(S_{\bar{I}(4g-5-r)} E)((k + s)\Theta)$ are M -regular. Moreover, for any index i , the tensor products $K^i((k + s)\Theta)$ are also M -regular as the tensor product of an M -regular

sheaf with a direct a sum of topologically trivial line bundles is still M -regular. Finally, we conclude that $\mathcal{I}_{W_d^r}((k+s)\Theta)$ is M -regular by chopping (21) into short exact sequences. \square

Corollary 5.4. *Assume the same hypotheses of Theorem 5.3 and in addition suppose that $r > 0$. Then the ideal sheaf $\mathcal{I}_{W_d^r}$ is $(2 + 4\lfloor\sqrt{\text{codim } W_d^r}\rfloor)$ - Θ -regular.*

Proof. We can compute the integers k and q of Theorem 5.3:

$$k = 4\lfloor\sqrt{\text{codim } W_d^r}\rfloor, \quad q = \text{codim } W_d^r - (r+1)\lfloor\sqrt{\text{codim } W_d^r}\rfloor \leq g-2$$

(if $\text{codim } W_d^r - (r+1)\lfloor\sqrt{\text{codim } W_d^r}\rfloor < 0$, then we can take an even smaller value for k). The corollary follows from the bound of Theorem 5.3 and the very definition of Θ -regularity. \square

Remark 5.5. With the same proof of Theorem 5.3 it is possible to give similar results about the other notions of regularity, namely GV and $I.T.0$. More precisely, if k and q are the quotient and the remainder of the division between $g-\rho + \lfloor\sqrt{g-\rho}\rfloor(4g-r-6)$ and $g-1$ respectively, and if $s = s(q)$ is defined as in Theorem 5.3, then $\mathcal{I}_{W_d^r}((k+s)\Theta)$ satisfies GV . On the other hand, if k and q are the quotient and remainder of the division between $g-\rho + \lfloor\sqrt{g-\rho}\rfloor(4g-r-4)$ and $g-1$ respectively, then $\mathcal{I}_{W_d^r}((k+s)\Theta)$ satisfies $I.T.0$.

5.3. Brill–Noether curves. Brill–Noether curves are Brill–Noether loci of dimension one. Brill–Noether curves of the form W_{a+2}^1 associated to a Petri general curve C of genus $g = 2a+1$ are of particular interest. In fact, Ortega proves that the Jacobian of a general curve of genus $2a+1$ is a Prym–Tyurin variety for the curve W_{a+2}^1 (see [Or]). Here we spell out for which integers n Brill–Noether curves satisfy the properties (AA_n) , (AB_n) , (AC_n^{lin}) , and (AC_n^{alg}) of the Introduction. The weaker conditions (AA_n^h) and (AB_n^h) may be worked out as well.

Corollary 5.6. *The sheaf*

$$\mathcal{I}_{W_{a+2}^1} \quad \text{is} \quad \begin{cases} (2 + 4\lfloor\sqrt{2a}\rfloor)\text{-}\Theta\text{-regular} & \text{if } a \geq 3 \\ 9\text{-}\Theta\text{-regular} & \text{if } a = 2. \end{cases}$$

Moreover, by setting $E_a := W_{a+2}^1$ and $e_a := 1 + 4\lfloor\sqrt{2a}\rfloor$, we have:

(i). *The groups $H^1(E_a, \mathcal{O}_{E_a}(n_a\Theta) \otimes \alpha|_{E_a})$ vanish for any $\alpha \in \text{Pic}^0(J(C))$ and $n_a \geq e_a$.*

(ii). *The restriction homomorphisms*

$$H^0(J(C), \mathcal{O}_{J(C)}(n_a\Theta) \otimes \alpha) \longrightarrow H^0(E_a, \mathcal{O}_{E_a}(n_a\Theta) \otimes \alpha|_{E_a})$$

are surjective for any $\alpha \in \text{Pic}^0(J(C))$ and $n_a \geq e_a$.

(iii). *The curve E_a is cut out by divisors linearly equivalent to $n_a\Theta$ for $n_2 \geq 9$ and $n_a \geq e_a + 1$ if $a \geq 3$.*

(iv). *The curve E_a is cut out by divisors algebraically equivalent to $n_a\Theta$ for $n_2 \geq 8$ and $n_a \geq e_a$ if $a \geq 3$.*

Proof. For the first statement we use Theorem 5.3 and note that in (20) we can take $k = 4\lfloor\sqrt{2a}\rfloor$ and $q = 2a - \lfloor\sqrt{2a}\rfloor$. The statements in (i) and (ii) follow as $\mathcal{I}_C(n_a\Theta)$ satisfies *I.T.0* for $n_a \geq e_a$ by Remark 5.5. The statements in (iii) and (iv) reduce to check the (continuous) global generation of $\mathcal{O}_{J(C)}(n_a\Theta)$ for which we apply Theorem 2.2 (i) and (ii). \square

Remark 5.7. The Θ -regularity bound of the previous corollary should be compared to the weaker bound

$$\left(2(2a+1) + 4\frac{(2a+1)!}{a!(a+1)!} + 1\right)\text{-}\Theta\text{-regular}$$

provided by Theorem 1.1.

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