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ESSAYS ON EQUILIBRIUM SELECTION AND GAME THEORY

by

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and advised by

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Introduction

The problem of equilibrium selection has been a relevant and fundamental topic in game theory since the own definition of the Nash equilibrium concept. However, despite its importance, this topic does not appear any longer as a top priority in the game-theoretic research agenda; at least, considering the absence of relevant contributions in the last two decades. In its present state, the existent literature consists of an eclectic collection of methods, each one dependent on a particular methodology, or tailored merely to a specific class of problems. Hence, it becomes almost impossible to organize those approaches under the umbrella of one or two research programs.

The current state is partially the product of the initial approach to the equilibrium multiplicity issue; that was the equilibrium refinement research program. Its purpose was the refinement of the equilibrium concept, seeking in the process to produce more reasonable predictions, and to reduce the number of possible solutions in a game. However, the result was a sequence of competing refinements and the absence of a consensual and unanimous concept. Even in extensive-form games, where the sequential equilibrium concept is possibly the most sensible candidate, we can find several variations that depend on the conditions for the consistency of beliefs, none of which being entirely satisfactory. The first approach aiming at the development of rational criteria for the selection of an equilibrium as the solution of a game emerges in the late 1980s, seen as the culmination of the refinement program and led by two of its main contributors, John Harsanyi, and Reinhard Selten. The subsequent interest resulted in some new methods and theories; however, none became utterly dominant. The slowdown that followed was also the product of an emerging paradigm, under which the information provided by the multiplicity of solutions is perhaps more useful than the prediction/prescription of a single one.

However, in my opinion, game theory has reached a stage that justifies renewed attention over this topic. In the one hand, the information provided by the multiplicity of solutions does not prevent the selection of a single solution. On the other hand, the dissemination of game theoretical tools in applied research raises the necessity for sensible predictions. Those predictions do not have to correspond to an exact behavior code; instead, they may just have a prescriptive orientation, matching the limitations that real agents face. The absence of such criteria limits the impact of game-theoretical tools and raises questions as to the usefulness of the theory itself. The product of behavioral and experimental economics research also has increased the criticism on game theory, based on the argument that a theory that does not make consistent and recurrent predictions cannot be effectively useful in social sciences.

In this dissertation, I treat the selection issue with a prescriptive orientation. Therefore, I propose an equilibrium selection method to static games with complete information and an extension of it to dynamic games with asymmetric information.

In the first paper - equilibrium selection in static games - I define criteria of risk and payoff dominance, which I combine into a single measure. That measure - the premium of an equilibrium - represents the risk of an equilibrium to a player, given his perception about risk, and the expected payoff. Such measure helps to rationalize the available experimental evidence by adjusting the importance of each dominance criterion to the selection of an equilibrium according to the characteristics of the game, namely, the distribution of the payoffs across the game outcomes. The solution of a game is an equilibrium that minimizes the premium to the player, that is, which minimizes the risk to a player given their perception of risk and their expected payoff, conditional on the same being true for every opponent. I provide a brief axiomatic characterization of that measure, and show that the solution set is nonempty and that almost all games have a unique solution; therefore, the set of games with multiple solutions have null Lebesgue measure.

In the second paper, I extend the method to dynamic games with asymmetric information. Considering the sequential nature of decision making, I show that a solution of an extensive-form game does not necessarily coincide with the solution of its reduced normal-form. I apply the method to the most basic version of Spence signaling game with just two types of worker. I obtain that the solution of such a game depends on the firms' prior concerning the players' types. In both dynamic and static games, I show that the method's solutions respect certain invariance properties.

I am then able to identify several directions of research. In one direction, we have the application of the selection method to specific problems in which the multiplicity of solutions play a critical role. Among such problems, I highlight bank-runs, and climate change negotiations. Additionally, I am interested in expanding the selection analysis of Spence game through the inclusion of additional types of worker. Another direction follows a different path, and focus on the identification and further characterization of the epistemic conditions of this selection method, and the comparison with the conditions in other selection methods and/or equilibrium refinement concepts.

Chapter 1

Equilibrium selection in static games with complete information

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Abstract

The multiplicity of equilibria in non-cooperative games as long been a fundamental issue in game theory. In certain cases, that situation may contribute or even improve the comprehension of a game. However, in general, we believe that a game-theoretic analysis benefits from the existence of rational criteria for choosing a particular sensible equilibrium as the game solution.

This topic has been the focus of two research programs; One program concerns the refinement of the equilibrium concept, while the other one focuses on the criteria for the selection of a solution. We take into consideration some of the issues that have conditioned the success of these programs, namely in the former one, and propose a method of equilibrium selection on static games with complete information. We propose criteria of risk and payoff dominance, which we combine into a single measure, that we call the premium of an equilibrium. We define a solution of a game as an admissible Nash equilibrium that conditionally minimizes the premium to the players, that is, which minimizes the risk to a player, given his perception about the risk of the equilibrium and its payoff, conditional on the same being true for every opponent. We discuss the conditions for the uniqueness of a solution and show that in a certain class of games, *almost all* have a single solution.

Keywords: equilibrium selection, risk-dominance, payoff-dominance, Nash equilibrium, equilibrium premium, game solution.

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1 Introduction

The multiplicity of equilibria in non-cooperative games has long been a fundamental issue in game theory. In certain cases, that situation may contribute to the analysis of different behaviors and likely outcomes, and improve the understanding of the strategic problem. That is the case, for example, in bank runs (Diamond & Dybvig, 1983), climate negotiations (DeCanio & Fremstad, 2013), natural resource disputes (Madani, 2010), and geopolitical tensions and conflicts (Kuran, 1989). In those and other cases, the multiplicity of solutions is a plausible scenario; one case in point is the decision of which side of the road to drive on, where the optimal choice depends on the country where the player is. However, we believe that the analysis of a game, in general, benefits from the existence of rational criteria for choosing a particular equilibrium as the game solution. The absence of criteria that recurrently select a single sensible equilibrium not only limits the impact of game-theoretic methods but also raises some questions about the predictive power of the theory itself.

This issue has been the focus of two research programs. One program concerns the refinement of the equilibrium concept, while the other one focuses on the criteria for the selection of a solution. We start by addressing the former one, which has also been the most prominent. In static games, we can divide it into two classes.[1]

One class contains equilibrium refinements which are based on the notion of outcome stability against small perturbations of the game payoffs. Among these refinements, we highlight the notions of essential equilibrium (Wen-Tsun & Jia-He, 1962) and regular equilibrium (Harsanyi, 1973). The former concept selects the equilibria which have an equilibrium nearby in every neighboring game[2]. The latter one represents a game outcome at which the Jacobian of a certain continuous and differentiable mapping associated with the game is non-singular.

The second class of refinements includes concepts based on the same stability principle, but against small perturbations on equilibrium strategies. There are many concepts to choose from, although many of them merely follow from the weaknesses of previous refinements; hence, we

[1]Because this work is not built directly on any of the following concepts we omit any in-depth discussion about their relations and distinctions. Moreover, because we restrict the scope of this work to games in strategic-form, we leave unmentioned any refinements to extensive-form games.

[2]A game in which each player's payoffs are in the neighborhood of those in the original game, which corresponds to a point in $\mathbb{R}^n$ with the ℓ_∞ norm.

just highlight three concepts. The first one is the strategic-form version of the perfect equilibrium (Selten, 1975). This refinement selects those strategic profiles which are optimal even when a player has a small probability of making a mistake when choosing which action to play. These equilibria are the limit points of sequences of equilibria from games with small perturbations on the equilibrium strategies. The second refinement concept is the proper equilibrium (Myerson, 1978). It directly refines the previous one, since it requires the small perturbations on the equilibrium strategies to match the best-reply characteristics of the actions available to each player. The last refinement we mention is the persistent equilibrium (Kalai & Samet, 1984). This concept can be understood as a generalization of a strict equilibrium, aiming at reflecting the idea that stable equilibria should be robust against small trembles in beliefs. An equilibrium is persistent if it belongs to a persistent retract. A retract is a Cartesian product of nonempty, closed, and convex subsets of players' actions, which include the best-replies against beliefs that are concentrated on it. Such a retract is absorbing if small perturbations of strategic profiles in the retract have best-replies in that same set. A retract is persistent if it is absorbing and minimal.

These refinements, independently of the class to which they belong, reduce the number of solutions in many games, but not consistently down to a single one, or even to a subset of reasonable size. These concepts are penalized by a variety of issues, such as nonexistence (essential equilibrium), selection of unreasonable equilibria (common to all), and non-invariance to the introduction of strictly dominated actions (perfect and proper equilibria) or duplicate actions[3] (persistent equilibrium). The fact that new refinements are commonly constructed on specific weaknesses of existing ones penalizes even further this research program, contributing to what appears to be an endless cycle of competing refinements instead of a convergent path aiming at an encompassing concept.

Set-valued theories, such as rationalizability (Bernheim, 1984; Pearce, 1984) and strategic stability (Kohlberg & Mertens, 1986), have emerged partly following the limitations of the refinement program. However, these theories, in some games, retrieve a large number of solutions. That not only penalizes their predictive power but also prevents them from being consistent alternatives to equilibrium refinements.

This context on the refinement program justifies the existence of an alternative program which focuses instead on the development of rational criteria for the selection of one particular equilibrium.

[3] See the definition of duplicate actions on Section 2.2.

We organize it into three branches.

The first branch includes theories and methods which view the theory of equilibrium selection as the culmination of the equilibrium refinement program. Among those are the selection theories from Harsanyi and Selten (1988) and Harsanyi (1995). The former one is better known for the introduction of criteria of risk and payoff dominance than for its selection method. It can be applied to both cooperative and non-cooperative games; however, the process itself is long and complicated, as it depends on a collection of procedures that decompose and reduce the game structure down to a stage in which it selects a unique solution through the use of dominance criteria, a notion of strategic distance, and the tracing procedure. However, the choice of using the tracing procedure instead of any other mechanism of internal deliberation is rather arbitrary. Precedence is given to payoff over risk dominance, even though the available empirical evidence does not corroborate such a relation (see Section 4), while risk dominance comparisons follow from possible intransitive bilateral relations. The alternative method by Harsanyi (1995) attempts to correct some of these issues but does so by introducing new ones. The most notorious one concerns the exclusion of payoff dominance from the set of selection criteria, a choice that, once again, does not find the necessary support on the available empirical evidence. This decision makes the selection procedure dependent only on the strength of an incentive measure based on risk dominance. Another problem concerns the arbitrary decision to choose as the solution of a game when two or more equilibria are selected, and if pre-play communication is allowed, a correlated equilibrium with a uniform distribution over those equilibrium outcomes. If pre-play communication is not permitted, then the solution of the game is left undefined.

The limitations of those theories, as well as their complexity and inability to provide recurrent sensible predictions, justify the existence of alternative approaches to the selection problem; we divide them into two branches of research. The first one includes contributions based on higher-order beliefs, such as the theory of global games (Carlsson & Van Damme, 1993) and the concept of p-dominance (Morris & Shin, 1995) as perhaps the most notorious examples. The second branch includes theories that rely in evolutionary methods as a mechanism of coordination on certain equilibrium outcomes, as in Ellison (1993), Young (1993), Kandori et al. (1993), Matsui and Matsuyama (1995), and Binmore and Samuelson (1997, 1999). The issue with the methods in these two branches does not necessarily concern their predictions, but rather the complexity of some procedures, the difficulty to elicit some parameters, and the limited number of games to which they can be effectively applied.

Hence, the limited success of this research program can be, in part, attributed to the absence of consensus regarding the assumptions, criteria, and methods. Consequently, we are left with an eclectic collection of approaches and methods, none of them being entirely successful.

In this work, we try to overcome some of these issues, while following certain principles, to provide a sensible alternative to Harsanyi and Selten (1988) and Harsanyi (1995) methods. In the first place, we consider that our method's predictions do not have to satisfy exogenous criteria necessarily. Contrarily to Harsanyi and Selten (1988), which aim at satisfying the Nash product property, we consider that any prediction should be weighted only according to the method's own merits. Moreover, we aim at proposing a method suitable for a broad application but which, at the same time, possesses a reasonable level of complexity. Finally, we take into account the available empirical evidence on the definition of the mechanism of selection; on that basis, we address the issue concerning the trade-off between the criteria of risk and payoff dominance, which according to experimental data, depends on the distribution of the payoffs across the game outcomes.

Given those principles, we propose an equilibrium selection method to static games with complete information and a finite set of Nash equilibria. That latter assumption is made for simplification, but we believe to be possible to successfully extend the method, under certain conditions, to games with an infinite number of equilibria. The selection of an equilibrium depends upon a collection of assumptions, which we introduce in Section 2; these assumptions aim at controlling for the existence of strategically irrelevant elements in the game, guaranteeing the consistent behavior of the method, and further characterizing a player's rational behavior. In Section 3, we propose criteria of risk and payoff dominance, and combine them in Section 4 into a single measure. We call that measure the premium of an equilibrium; it represents the risk of an equilibrium to a player, given his perception about the risk of an equilibrium and its payoff. This measure helps rationalizing the available empirical evidence by adjusting the importance of each dominance criterion to the selection of an equilibrium considering the characteristics of the game. We proceed with the characterization of the process of equilibrium selection, defining it as a problem of minimization of the premium to the players, and whose set of minimizers are the solutions of the game. We provide an additional axiomatic characterization of a preference relation based on the premium of an equilibrium, and discuss the conditions for the uniqueness of a solution, showing that in a certain class of games, *almost all* have a single solution. In Section 5, we summarize the main conclusions, and point directions for future research.

2 Preliminaries

2.1 The game

Consider a finite static game G with complete information in strategic-form, which has the following elements:

- a finite set $N = \{1, \ldots, i, \ldots, n\}$ of players;
- each player $i \in N$ has a nonempty finite set A_i of pure actions that he can conceivably choose from, and a set of mixed actions $\Delta(A_i)$, such that each mixed action α_i is a probability distribution over pure actions $a_i \in A_i$;
- the vectors of the players' pure and mixed actions belong to the sets $A := \prod_{i \in N} A_i$ and $\Delta(A) := \prod_{i \in N} \Delta(A_i)$. With some abuse of notation, those vectors without player i's action belong to the sets $A_{-i} := \prod_{j \neq i} A_j$ and $\Delta(A_{-i}) := \prod_{j \neq i} \Delta(A_j)$;
- a payoff function $u_i : A \to \mathbb{R}$ for each $i \in N$ which corresponds to a von Neumann and Morgenstern utility function.

We identify G with a strategic-form $\langle N, (A_i, u_i)_{i \in N} \rangle$, and denote as $E(G)$ its nonempty set of Nash equilibria, which is finite by our assumption.

2.2 The assumptions

The process of equilibrium selection depends on a set of assumptions that aim at excluding elements which are strategically irrelevant (Assumptions 1-3), guaranteeing consistent and reasonable results (Assumption 4), and further characterizing the behavior of a rational player (Assumption 5). We start with a trivial assumption concerning the players' rationality.

Assumption 1.

Each $i \in N$ is rational; therefore, no player chooses with positive probability a strictly dominated action[4]*, which is common knowledge among the players. Hence, we simplify G through the iterated elimination of strictly dominated actions whenever possible.*

Intuitively, this assumption is sensible and it is commonly made in the literature; it has though an indirect effect on the set of admissible equilibria of G, whose relevance on the process of equilibrium selection is highlighted in Assumption 5.

To understand that effect, notice that a Nash equilibrium is admissible if no player chooses with positive probability a weakly dominated action[5]. Hence, take $i \in N$, and some strictly dominated pure action $a_i \in A_i$, while assuming that such action is not necessarily chosen with null probability. Some non-admissible Nash equilibrium $\alpha \in E(G)$ may not be considered as such if a_i is the only reason for which the equilibrium action $\alpha_j := \text{proj}_j\alpha$ is not dominated for some $j \neq i$, i.e., α_j unique best-reply is against a_i[6]. The game G_0 in Figure 1 illustrates this issue.

Colin

		L	C	R
Rowena	U	1, 1	0, 0	-4, -4
	M	0, 0	0, 0	0, -3
	D	-4, -4	-3, 0	-3, -3

Figure 1: Game G_0.

G_0 has two admissible equilibria in pure actions, $\alpha^1 = (\text{U, L})$ and $\alpha^2 = (\text{M, C})$; but if we eliminate "D" and "R", which are strictly dominated by "M" and "C", then α^2 is no longer admissible.

We now proceed with two assumptions regarding the existence of duplicate elements in the game. The first one concerns the existence of duplicate actions, while the second one the existence of duplicate players.

[4]An action $a_i \in A_i$ is *strictly dominated* for $i \in N$ if there is some $a_i' \neq a_i$ such that, for every $a_{-i} \in A_{-i}$, $u_i(a_i, a_{-i}) < u_i(a_i', a_{-i})$.

[5]An action $a_i \in A_i$ is *weakly dominated* for $i \in N$ if there is some $a_i' \neq a_i$ such that, for every $a_{-i} \in A_{-i}$, $u_i(a_i, a_{-i}) \leq u_i(a_i', a_{-i})$, and for $\hat{a}_{-i} \in A_{-i}$, $u_i(a_i, \hat{a}_{-i}) < u_i(a_i', \hat{a}_{-i})$. We shall use the term dominance to denote weak dominance

[6]Consider $\text{proj}_i x$ the projection function of the i^{th} coordinate of an n-dimensional vector x, such that $i \leq n$.

Assumption 2.

In G, we substitute strategically equivalent actions by a single representative action.

Two actions $\alpha_i, \alpha_i' \in \Delta(A_i)$ are strategically equivalent, which we write $\alpha_i \sim_i^a \alpha_i'$, if for every $\alpha_{-i} \in \Delta(A_{-i})$ and each $j \in N$, $u_j(\alpha_i, \alpha_{-i}) = u_j(\alpha_i', \alpha_{-i})$. We substitute them by a representative element $\bar{\alpha}_i \in \Delta(A_i)$ of an equivalence class $[\bar{\alpha}_i] = \{\alpha_i \in \Delta(A_i) \,|\, \alpha_i \sim_i^a \bar{\alpha}_i\}$ that is part of the quotient set of $\Delta(A_i)$. If some $a_i \in A_i$ and $\alpha_i \in \Delta(A_i)$ belong to a equivalence class $[\bar{\alpha}_i]$, and $a_i \in \text{supp}(\alpha_i)$, then α_i is not eligible to represent $[\bar{\alpha}_i]$[7].

Assumption 3.

In G, we substitute duplicate players by a single representative player.

Players i and j are duplicates of one another if:

- for $k, \ell = i, j$ and $k \neq \ell$, if $\alpha_k \in \Delta(A_k)$, $\alpha_\ell, \alpha_\ell' \in \Delta(A_\ell)$, and $\alpha_{-k-\ell} \in \prod_{h \in N \setminus \{i,j\}} \Delta(A_h)$, $u_k(\alpha_k, \alpha_\ell, \alpha_{-k-\ell}) = u_k(\alpha_k, \alpha_\ell', \alpha_{-k-\ell})$;
- a permutation of players i and j in G produces an isomorphic game[8].

In the first point, the payoffs of a duplicate player are independent of the actions of another duplicate player. The second point implies the strategic invariance of the game to permutations between these players. In this case, players i and j belong to an equivalence class, and we substitute them by a unique representative player.

The next assumption aims at guaranteeing that the players' implement a selection mechanism that produces sensible and consistent results.

Assumption 4.

The utility function $u_i : X \to \mathbb{R}$, with $A \subseteq X$, of every $i \in N$ intersects in a point $x^o \in X$, in which $u_i(x^o) = 0$. Moreover, for every $i \in N$, $u_i(a) \geq 0$ for each $a \in A$. Hence, the players have interval-scale measurable payoffs with a common origin.

We characterize $u_i : X \to \mathbb{R}$ with respect to its general domain X, which is a compact subset of the Euclidean space $\mathbb{R}^n$, and includes the set A, which is the sub-domain of interest in G. In this case,

[7] The $\text{supp}(\alpha_i)$ denotes the support of the probability measure α_i over A_i for $i \in N$.

[8] We define the concept of isomorphism between games in strategic-form in Definition 8.

each player's payoff function $u_i(\cdot)$ in the domain A is invariant to positive affine transformations $\mathfrak{f} : \mathbb{R} \to \mathbb{R}$, where $\mathfrak{f}_i(u_i(a)) = c_i u_i(a) + c_i'$, for scalars $c_i \in \mathbb{R}^+$, $c_i' \in \mathbb{R}$, and $a \in A$. The existence of an element $x^o \mapsto u_i(x^o) = 0$ for each $i \in N$ implies that interval-scale measurability results from adding a common natural origin to each $u_i(\cdot)$, together with the idea that utilities are measurable under a cardinal scale. This assumption comes without loss of generality since each player i has a payoff function $u_i(\cdot)$ that by definition is a von Neumann-Morgenstern utility function. Hence, the common intersection at x^o and the non-negativity condition can follow from a positive affine transformation of the utility function. We show in Proposition 2 that such transformation does not affect the selection method.

We conclude this collection of assumptions by focusing at a player's behavior during the selection of an equilibrium. Denote as $\mathscr{A}_i$ the set of admissible pure actions of player i, and as $\Delta(\mathscr{A}_i)$ the set of admissible mixtures. Let $\mathcal{E}$ be the set of admissible Nash equilibria in G, and $\mathcal{A}_i = \Delta(\mathscr{A}_i) \cap \text{proj}_i \mathcal{E}$ the set of admissible actions that belong to an equilibrium vector in $\mathcal{E}$, where $\mathcal{A} := \prod_{i \in N} \mathcal{A}_i$. Given these elements, we introduce an auxiliary structure to G.

Definition 1 (Auxiliary game Γ)**.**
The strategic-form $\Gamma = \langle N, \mathcal{A}, u \rangle$ is an auxiliary game of G, where each $i \in N$ only chooses with positive probability actions in $\mathcal{A}_i$; hence, player i does not randomize.

The set of Nash equilibria in Γ is $E(\Gamma)$, such that $\mathcal{E} \subseteq E(\Gamma)$. However, because the solution of G has to be an equilibrium in it, only the equilibria in $\mathcal{E}$ are eligible for selection. We now complete the set of assumptions.

Assumption 5.
In G, each $i \in N$ uses a selection mechanism $\mathfrak{s}_i : \Gamma \rightrightarrows \mathcal{E}$, such that, $\mathfrak{s}_i(\Gamma)$ is the set of equilibria selected by player i.[9]

Each $i \in N$ uses a selection mechanism $\mathfrak{s}_i(\cdot)$ to select one or more equilibria in $\mathcal{E}$. A comment is yet necessary as to the restriction to the set $\mathcal{A}$ during the selection process. As a first step, it is sensible to start by considering the set $\mathscr{A}_i$ instead of the entire set A_i for every $i \in N$. That assumption is made, for instance, in Luce and Raiffa (1957) characterization of decision-making under uncertainty,

[9]As a simplification, when we say that $i \in N$ chooses or selects $\alpha \in \mathcal{E}$, we mean that player i chooses an equilibrium action $\alpha_i := \text{proj}_i \alpha$, considering that every $j \neq i$ chooses an equilibrium action $\alpha_j := \text{proj}_j \alpha$.

and in Kohlberg and Mertens (1986) definition of the concept of strategic stability. But since we aim at making predictions that can be feasibly implementable by real people in real problems of strategic interaction, we do not consider sensible to select an equilibrium in G using the entire set $\Delta(\mathscr{A}_i)$; we take instead $\mathcal{A}$ a more reasonable domain of selection. The intuition is the following. Take $\alpha \in \mathcal{E}$, and consider a deviation by player i from the equilibrium action $\alpha_i := \text{proj}_i\alpha$ to some $\alpha_i' \neq \alpha_i$. Assume that α_i' belongs to some other equilibrium vector. In this case, player i considers a deviation to α_i' with positive probability if such action is a best-reply to the best-reply action α_j' by every $j \neq i$ to his initial deviation from α_i. Hence, we aim at reflecting the players' preference for equilibrium stability (i.e., that the outcome of the game is a Nash equilibrium), even when a player deviates from some equilibrium outcome.

3 Selection criteria

In this section, we propose and characterize criteria of risk and payoff dominance which are at the core of the method of equilibrium selection. We accompany their characterization of examples based on game G_1 in Figure 2. In Section 4, we provide some additional examples that highlight other features of the selection method.

		Colin	
		L	R
Rowena	U	8, 5	4, 1
	D	5, 2	7, 4

Figure 2: The game G_1.

G_1 is a generic game with complete information and the following admissible Nash equilibria: two equilibria in pure actions, $\hat{a} = (\text{U, L})$ and $\tilde{a} = (\text{D, R})$; and one equilibrium in mixed actions, $\alpha = \big(1/3[\text{U}] + 2/3[\text{D}], 1/2[\text{L}] + 1/2[\text{R}]\big)$.

3.1 Risk Dominance

We propose a criterion of risk-dominance that possesses some differences in comparison to Harsanyi and Selten (1988) version. For that reason, we believe to be necessary to comment on the motivation to follow a different route. In essence, both criteria share a similar intuition; the risk of an equilibrium depends on the losses following the players' unilaterally deviation from an equilibrium action. In Harsanyi and Selten (1988), that leads to a risk-dominance criterion based on pairwise comparisons. However, in games larger than 2×2 that relation is not necessarily transitive. To overcome that issue, they define, for such games, the risk criterion using the tracing procedure; that is a method of internal deliberation that transforms the players' prior beliefs into equilibrium choices. In that case, the method is significantly more complex. Moreover, the outcome of the tracing procedure depends on the prior chosen. The authors obtain that prior following the principle of insufficient reason[10]; but that option is not sensible in many cases. The entire definition of the criterion is also not objective, since there is no apparent reason for preferring the tracing procedure over any other method of internal deliberation. In comparison, our risk-dominance criterion allows for transitive comparisons between the equilibria; it possesses a reasonable level of complexity, and its arguments are elicitable, which should allow the empirical validation of its predictions, even on games larger than 2×2.

In the characterization of both dominance criteria, write an admissible Nash equilibrium $\alpha \in \mathcal{E}$, henceforth just equilibrium, as $\alpha = (\alpha_i, \alpha_{-i})$. Let $\alpha_{-i} = (\alpha_j)_{j \neq i}$, where $\alpha_j = \text{proj}_j\, \alpha$ for every $j \neq i$. We thus begin the characterization of the criterion of risk-dominance with the definition of the concept of best-deviation from an equilibrium action α_i by player i.

Definition 2 (Best-deviation)**.**
The best-deviation by player i from an equilibrium action α_i in $\alpha \in \mathcal{E}$ is an action $\alpha_i^1 \in \mathcal{A}_i \backslash \{\alpha_i\}$, such that, for every $\alpha_i' \in \mathcal{A}_i \backslash \{\alpha_i, \alpha_i^1\}$,

$$u_i(\alpha_i, \alpha_{-i}) \geq u_i(\alpha_i^1, \alpha_{-i}) \geq u_i(\alpha_i', \alpha_{-i}).$$

[10]The principle stating that in ignorance about the likelihood of some collection of events, each event is equally likely to take place.

This best-deviation induces a relative utility loss $\lambda_\alpha(\alpha_i^1)$, *where*

$$\lambda_\alpha(\alpha_i^1) = \frac{u_i(\alpha_i, \alpha_{-i}) - u_i(\alpha_i^1, \alpha_{-i})}{u_i(\alpha_i, \alpha_{-i}) - u_i(x^o)}. \tag{1}$$

A best-deviation α_i^1 from an equilibrium action α_i represents for player i his second best-reply against the equilibrium actions α_{-i} of every $j \neq i$. The relative utility loss $\lambda_i(\cdot)$ is computed with reference to $u_i(x^o)$ to guarantee the invariance of the ratio to an isomorphism of G (see Definition 8). That property, as we discuss later, ensures that the selection method can adequately reflect potential symmetries between elements of G. We now generalize the concept of best-deviation.

Definition 3 (Best-deviation generalization)**.**
Let $|\mathcal{A}_i \backslash \{\alpha_i\}| = m_i$ *and* $m_i > 1$; *the* 2^{nd} *best-deviation from an equilibrium action* α_i *in* $\alpha \in \mathcal{E}$ *is an action* $\alpha_i^2 \in \mathcal{A}_i \backslash \{\alpha_i, \alpha_i^1\}$, *such that, for every* $\alpha_i' \in \mathcal{A}_i \backslash \{\alpha_i, \alpha_i^1, \alpha_i^2\}$,

$$u_i(\alpha_i^2, \alpha_{-i}) \geq u_i(\alpha_i', \alpha_{-i}),$$

and such deviation induces a relative utility loss $\lambda_\alpha(\alpha_i^2)$, *where*

$$\lambda_\alpha(\alpha_i^2) = \frac{u_i(\alpha_i, \alpha_{-i}) - u_i(\alpha_i^2, \alpha_{-i})}{u_i(\alpha_i, \alpha_{-i}) - u_i(x^o)}. \tag{2}$$

By induction, the ℓ^{th} *best-deviation by player* i *from the equilibrium action* α_i *is an action* $\alpha_i^\ell \in \mathcal{A}_i \backslash \{\alpha_i\} \cup \{\alpha_i^k\}_{k=1}^{\ell-1}$, *such that, for every* $\alpha_i' \in \mathcal{A}_i \backslash \{\alpha_i\} \cup \{\alpha_i^k\}_{k=1}^{\ell}$,

$$u_i(\alpha_i^\ell, \alpha_{-i}) \geq u_i(\alpha_i', \alpha_{-i}),$$

and such deviation induces a relative utility loss $\lambda_\alpha(\alpha_i^\ell)$, *where*

$$\lambda_\alpha(\alpha_i^\ell) = \frac{u_i(\alpha_i, \alpha_{-i}) - u_i(\alpha_i^\ell, \alpha_{-i})}{u_i(\alpha_i, \alpha_{-i}) - u_i(x^o)}. \tag{3}$$

If $\mathcal{A}_i \backslash \{\alpha_i\} \cup \{\alpha_i^k\}_{k=1}^{\ell} = \emptyset$, *then* $\ell = m_i$, *and player* i *has no more available deviations.*

We now illustrate the previous concepts in the following example.

Example 1.

Consider the auxiliary game Γ_1 of G_1 in Figure 3.

		Colin		
		L	R	1/2[L] + 1/2[R]
	U	8, 5	4, 1	6, 3
Rowena	D	5, 2	7, 4	6, 3
	1/3[U] + 2/3[D]	6, 3	6, 3	6, 3

Figure 3: The auxiliary game Γ_1 of G_1.

In Table 1, we identify Rowena and Colin's best-deviations and relative utility losses from each equilibrium. Write an equilibrium as $\alpha = (\alpha_r, \alpha_c)$, where α_r stands for Rowena's action, and α_r for Colin's action.

	Rowena				**Colin**			
	α_r^1	$\lambda_{(\cdot)}(\alpha_r^1)$	α_r^2	$\lambda_{(\cdot)}(\alpha_r^2)$	α_c^1	$\lambda_{(\cdot)}(\alpha_c^1)$	α_c^2	$\lambda_{(\cdot)}(\alpha_c^2)$
$\hat{a}$	α_r	.25	D	.375	α_c	.4	R	.8
$\tilde{a}$	α_r	.143	U	.429	α_c	.25	L	.5
α	U	0	D	0	L	0	R	0

Table 1: Best-deviations and relative utility losses.

We read those results as follows. At the pure equilibrium $\hat{a}$, Rowena's best deviation is to $\alpha_r = 1/3[U] + 2/3[D]$, which implies a relative loss of .25 of her expected utility in equilibrium. Notice that in the mixed equilibrium α, both players are indifferent between a deviation to any of their admissible pure actions, since none leads to a loss in the expected utility. ▶

Denote now as $\delta_i(\alpha) = \{\alpha_i^k\}_{k=1}^{m_i}$ the set of all possible deviations by player i from $\alpha \in \mathcal{E}$. Write the collection of all deviations from $\alpha \in \mathcal{E}$ by every $i \in N$ as,

$$\delta(\alpha) = \bigcup_{i \in N} \delta_i(\alpha).$$

With some abuse of notation, denote that collection without player i's best deviations as,

$$\delta_{-i}(\alpha) = \bigcup_{j \neq i} \delta_j(\alpha).$$

Let $\mathfrak{P}\big(\delta_{-i}(\alpha)\big)$ be the collection of all possible combinations of the deviations in $\delta_{-i}(\alpha)$, which satisfies the following conditions:

a) for every set $\bar{\delta}(\alpha) \in \mathfrak{P}(\delta_{-i}(\alpha))$, let $|\bar{\delta}(\alpha) \cap \delta_i(\alpha)| \in \{0, 1\}$ and $0 < |\bar{\delta}(\alpha)| \leq n$, such that $\delta_{-i}(\alpha) \notin \mathfrak{P}\big(\delta_{-i}(\alpha)\big)$;

b) $\mathfrak{P}\big(\delta_{-i}(\alpha)\big) \subset \mathscr{P}\big(\delta_{-i}(\alpha)\big)$, where $\mathscr{P}\big(\delta_{-i}(\alpha)\big)$ is the power set of $\delta_{-i}(\alpha)$.

Condition a) implies that each combination of deviations $\bar{\delta}(\alpha)$ from $\alpha \in \mathcal{E}$ is nonempty and includes, at the most, one deviation per player. In the case that some player deviates for several times from an equilibrium, each of those deviations belong to a different vector of deviations. Condition b) means that $\mathfrak{P}\big(\delta_{-i}(\alpha)\big)$ has a strict inclusion in the power set of $\delta_{-i}(\alpha)$. We then define a function,

$$r_i : \mathfrak{P}\big(\delta_{-i}(\alpha)\big) \to [0, 1], \tag{4}$$

such that, for any $\bar{\delta}(\alpha) \in \mathfrak{P}\big(\delta_{-i}(\alpha)\big)$,

$$\bar{\delta}(\alpha) \mapsto r_i\big(\bar{\delta}(\alpha)\big) = \prod_{j \neq i} \lambda_\alpha(\alpha_j^k), \quad \forall \alpha_j^k \in \bar{\delta}(\alpha), \tag{5}$$

where $\lambda_\alpha(\alpha_j^k)$ is the relative utility loss of a deviation $\alpha_j^k \in \bar{\delta}(\alpha)$ by $j \neq i$ from $\alpha \in \mathcal{E}$.

For some $\bar{\delta}(\alpha) \in \mathfrak{P}\big(\delta_{-i}(\alpha)\big)$ and $\alpha \in \mathcal{E}$, when $r_i\big(\bar{\delta}(\alpha)\big)$ is close to zero, the likelihood of $\bar{\delta}(\alpha)$ taking place in G is high, and so is the risk of the equilibrium α to player i. Hence, the combination of deviations in $\mathfrak{P}\big(\delta_{-i}(\alpha)\big)$ with the lowest $r_i(\cdot)$ sets the highest risk of the equilibrium α to player i; we call it the risk of degree 1.

Definition 4 (Risk of degree 1)**.**

The risk of degree 1 *of an equilibrium to* $i \in N$ *is a function* $\rho_i^1 : \mathcal{E} \to [0,1]$*, such that, at* $\alpha \in \mathcal{E}$*,*

$$\rho_i^1(\alpha) = \begin{cases} 1 - r_i\big(\bar{\delta}(\alpha)\big), & \textit{for } r_i\big(\bar{\delta}(\alpha)\big) \leq r_i\big(\bar{\delta}'(\alpha)\big) \textit{ and } \forall \bar{\delta}'(\alpha) \in \mathfrak{P}\big(\delta_{-i}(\alpha)\big) \\ 0, & \textit{if } \mathfrak{P}\big(\delta_{-i}(\alpha)\big) = \emptyset. \end{cases} \tag{6}$$

A set $\bar{\delta}(\alpha) \in \mathfrak{P}\big(\delta_{-i}(\alpha)\big)$ containing the deviations of bigger number of players sets, in general, lower degrees of risk; hence, the risk that these deviations induce to player i is small. The intuition is that simultaneous unilateral deviations of multiple players are, in principle, less likely than, for example, the unilateral deviation of a single one. However, it is trivial to verify that a degree of risk is not necessarily monotonic in the number of simultaneous unilateral deviations. We now generalize the concept of degree of risk.

Definition 5 (Risk of degree k)**.**

Let $\bar{m} = \max_{i \in N} |\mathfrak{P}\big(\delta_{-i}(\alpha)\big)|$*; the risk of degree* k *of an equilibrium to* $i \in N$ *is a function* $\rho_i^k : \mathcal{E} \to [0,1]$*, with* $k \leq \bar{m}$*, such that, at* $\alpha \in \mathcal{E}$*,*

$$\rho_i^k(\alpha) = \begin{cases} 1 - r_i\big(\bar{\delta}''(\alpha)\big), & \textit{for } r_i\big(\bar{\delta}''(\alpha)\big) \leq r_i\big(\bar{\delta}'(\alpha)\big) \textit{ and } \forall \bar{\delta}'(\alpha) \in \mathfrak{P}\big(\delta_{-i}(\alpha)\big) \backslash \big\{\bar{\delta}^\ell(\alpha)\big\}_{\ell=1}^{k-1} \\ 0, & \textit{if } \mathfrak{P}\big(\delta_{-i}(\alpha)\big) \backslash \big\{\bar{\delta}^\ell(\alpha)\big\}_{\ell=1}^{k-1} = \emptyset. \end{cases} \tag{7}$$

Every $i \in N$ associates $\bar{m}$ degrees of risk to each $\alpha \in \mathcal{E}$, albeit the dimension of each $\mathcal{A}_i$ is not the same. In that way, we keep the coherence between the number of degrees of risk of each equilibrium to every player. We now aggregate the degrees of risk of $\alpha \in \mathcal{E}$ to player i into a unique measure, which is simultaneously the cornerstone for the criterion of risk-dominance.

Definition 6 (Risk)**.**

The risk of an equilibrium for $i \in N$ *is a function* $\rho_i : \mathcal{E} \to \mathbb{R}$*, such that, at* $\alpha \in \mathcal{E}$*,*

$$\rho_i(\alpha) = \sum_{j=1}^{\bar{m}} \eta_i^j(\alpha) \rho_i^j(\alpha). \tag{8}$$

Each weight $\eta_i^j(\alpha) \in [0,1]$ *is an element of a finite monotonic weakly decreasing sequence* $\eta_i(\alpha) =$

$\{\eta_i^k(\alpha)\}_{k=1}^{\bar{m}}$, such that, for $j \neq k$,

$$\eta_i^j(\alpha) = 1 - \sum_{k \in \{1,\dots,\bar{m}\} \setminus j} \eta_i^k(\alpha), \text{ and } \eta_i^k(\alpha) = \frac{\rho_i^k(\alpha)}{\rho_i^j(\alpha)} \eta_i^j(\alpha), \tag{9}$$

and

$$\sum_{\ell=1}^{\bar{m}} \eta_i^\ell(\alpha) = 1. \tag{10}$$

The function $\rho_i(\cdot)$ unifies the degrees of risk of an equilibrium α to player i. Because each degree of risk is not necessarily equally relevant, its contribution for the risk has a weight; the collection of such weights represents a vector of dimension $\bar{m}$. Using $\rho_i(\cdot)$, we can now compare, at the player level, the equilibria in $\mathcal{E}$.

Definition 7 (Risk-dominance)**.**

Each $i \in N$ has a binary relation $\succsim_i^\rho$, which we call a risk-dominance relation, over $\mathcal{E}$, such that, for any $\alpha', \alpha'' \in \mathcal{E}$, α' risk dominates α'', which we write $\alpha' \succ_i^\rho \alpha''$, if $\rho_i(\alpha') \leq \rho_i(\alpha'')$.

We conclude the following about the risk-dominance relation $\succsim_i^\rho$.

Proposition 1.

$\succsim_i^\rho$ is a preference relation.

In the next example, we compare the admissible equilibria of G_1 using $\succsim_i^\rho$.

Example 2.

We depict in Table 2 the degrees of risk and weights of each equilibrium of G_1 to each player.

	Rowena				**Colin**			
	$\rho_r^1(\cdot)$	$\rho_r^2(\cdot)$	$\eta_r^1(\cdot)$	$\eta_r^2(\cdot)$	$\rho_c^1(\cdot)$	$\rho_c^2(\cdot)$	$\eta_c^1(\cdot)$	$\eta_c^2(\cdot)$
$\hat{a}$	0.6	0.2	0.75	0.25	0.75	0.625	0.545	0.454
$\tilde{a}$	0.75	0.5	0.6	0.4	0.857	0.571	0.6	0.4
α	1	1	0.5	0.5	1	1	0.5	0.5

Table 2: Degrees of risk and respective weights of each admissible equilibrium in G_1.

Considering Definition 6: for Rowena, we have $\rho_r(\hat{a}) = 0.5$, $\rho_r(\tilde{a}) = 0.65$, and $\rho_r(\alpha) = 1$; and for Colin, we have $\rho_c(\hat{a}) \approx 0.693$, $\rho_c(\tilde{a}) \approx 0.743$, and $\rho_c(\alpha) = 1$. Hence, Rowena's risk-dominance preferences are $\hat{a} \succsim_r^\rho \tilde{a} \succsim_r^\rho \alpha$, and Colin's are $\hat{a} \succsim_c^\rho \tilde{a} \succsim_c^\rho \alpha$. ▶

We conclude the characterization of $\succsim_i^\rho$ by addressing the effect on it of symmetries between some elements of G; therefore, we check for its invariance to an isomorphism of the strategic-form of G. We start with the definition of that concept following the definition in Harsanyi and Selten (1988).

Definition 8 (Isomorphism of the strategic-form)**.**
An isomorphism between two strategic-form games $G = \langle N, (A_i, u_i)_{i \in N} \rangle$ *and* $\tilde{G} = \langle \tilde{N}, (\tilde{A}_i, \tilde{u}_i)_{i \in \tilde{N}} \rangle$ *is a bijection* $T : G \to \tilde{G}$ *which satisfy the following conditions:*

1. *There exists a bijection* $t : N \to \tilde{N}$*;*
2. *There exists a bijection* $\mathfrak{t}_i : A_i \to \tilde{A}_{t(i)}$ *for every* $i \in N$*;*
3. $u_{t(i)}\big(\mathfrak{t}(a)\big) = \theta_i u_i(a) + \vartheta_i$*, for every* $i \in N$ *and* $a \in A$*, where* $\theta_i > 0$ *and* $\vartheta_i \in \mathbb{R}$*.*

Conditions 1 and 2 imply the mapping of player i's actions in A_i in G onto the actions in $\mathfrak{t}(A_{t(i)})$) of player $t(i)$ in $\tilde{G}$. In condition 3, the payoff $u_{t(i)}(\mathfrak{t}(a))$ of any $\mathfrak{t}(a) \in \mathfrak{t}(A) := \prod_{t(i) \in \tilde{N}} \mathfrak{t}(A_{t(i)})$ to a player $t(i)$ in $\tilde{G}$ is a positive affine transformation of $u_i(a)$ of any $a \in A$ to $i \in N$ in G; hence, the preferences of the latter player are preserved. When $\theta_i = 1$ and $\vartheta_i = 0$, the isomorphism T just renames the players in N and actions in A_i for each $i \in N$. The following result states the desired invariance property of $\succsim_i^\rho$.

Proposition 2.
Let T *be an isomorphism between* $G = \langle N, (A_i, u_i)_{i \in N} \rangle$ *and* $\tilde{G} = \langle \tilde{N}, (\tilde{A}_i, \tilde{u}_i)_{i \in \tilde{N}} \rangle$*. For any* $\alpha, \alpha' \in \mathcal{E}$*, let* $\alpha \succsim_i^\rho \alpha'$ *in* G*. Hence,* $\mathfrak{t}(\alpha) \succsim_{t(i)}^\rho \mathfrak{t}(\alpha')$ *in* $\tilde{G}$*.*

3.2 Payoff Dominance

We define the criterion of payoff dominance considering the weight of an equilibrium payoff to player i on the sum of all the equilibrium payoffs to him in G.

Definition 9 (Weight of an equilibrium payoff)**.**
The weight of the payoff of an equilibrium to $i \in N$ *is a function* $w_i : \mathcal{E} \to \mathbb{R}$*, such that, at* $\alpha \in \mathcal{E}$*,*

$$w_i(\alpha) = \frac{u_i(\alpha) - u_i(x^o)}{\sum_{\alpha' \in \mathcal{E}} \big(u_i(\alpha') - u_i(x^o)\big) + 1}. \tag{11}$$

We can now compare the equilibria in $\mathcal{E}$ from a payoff perspective at the player level, while using the same unit of measure as in risk-dominance. As in the previous dominance criterion, we define a binary relation over $\mathcal{E}$.

Definition 10 (Payoff-dominance)**.**
Each $i \in N$ has a binary relation $\succsim_i^u$, which we call a payoff-dominance relation, over $\mathcal{E}$, such that, for any $\alpha', \alpha'' \in \mathcal{E}$, α' payoff dominates α'', which we write $\alpha' \succsim_i^u \alpha''$, if $w_i(\alpha') \geq w_i(\alpha'')$.

The payoff-dominance criterion $\succsim_i^u$ is a preference relation. Such follows from $\succsim_i^u$ being induced by a measure $w_i(\cdot)$, whose range just contains utility factors of player i on the unit interval. In the next example, we compare once again the admissible equilibria of G_1, this time using the criterion of payoff-dominance.

Example 3.
We depict in Table 3 the equilibrium payoff weights of the equilibria of G_1.

	$\hat{a}$	$\tilde{a}$	α
w_r	0.381	0.333	0.286
w_c	0.417	0.333	0.25

Table 3: Equilibrium payoff weights in the game G_1.

According to the criterion of payoff-dominance, Rowena's preferences are $\hat{a} \succ_r^u \tilde{a} \succ_r^u \alpha$, and Colin's are $\hat{a} \succ_c^u \tilde{a} \succ_c^u \alpha$. ▶

Before introducing to the equilibrium selection mechanism, we make one remark.

Remark 1.
Consider an admissible pure equilibrium a and mixed equilibrium α. Suppose that, for every $i \in N$, $w_i(a) = w_i(\alpha)$, $\rho_i(a) = \rho_i(\alpha)$, and $\rho_i^k(a) = \rho_i^k(\alpha)$ for every $k \leq \bar{m}$; additionally, let $supp(\alpha_i) \cap a_i \neq \emptyset$, where $a_i = proj_i a$. In that case, only a is eligible for selection since not only is action a a limiting strategy of α, but because the players are indifferent between each equilibrium with respect to the selection criteria, these outcomes are, in some sense, duplicates of one another; hence, the presence of both bias the results by duplicating the best-deviations from the admissible equilibria. In that

case, both a and α belong to an equivalence class $[\bar{a}] = \{\alpha \in \mathcal{E} | w_i(\bar{a}) = w_i(\alpha), \rho_i(\bar{a}) = \rho_i(\alpha), \rho_i^k(\bar{a}) = \rho_i^k(\alpha), \forall k \leq \bar{m}, \forall i \in N\}$, which is represented by a; hence, only the latter equilibrium is eligible for selection, since for every $a_i := proj_i a$, $a_i \in supp(\alpha_i)$, where $\alpha_i := proj_i \alpha$, that is, α depends on the equilibrium actions in a, but not the contrary.

4 Equilibrium selection

4.1 Selection mechanism

One question that arises from Harsanyi and Selten (1988) method, and which is neglected afterwards in Harsanyi (1995), concerns the proper manner, if there is any, to trade-off the criteria of risk and payoff dominance. Several empirical works provide some insight on this issue. In general, they support the idea that a player's choice between playing one of multiple possible Nash equilibria takes into account both the criteria of risk and payoff dominance. However, their relative importance in the selection process depends on the characteristics of the game, namely, the distribution of the payoffs across the different game outcomes. We summarize the trade-off mechanism as follows.

On the one hand, the criterion of payoff-dominance is the more relevant one when comparing two equilibria, in the sense that the equilibrium which players prefer is payoff-dominant, whenever the difference between the risk of those equilibria falls below some threshold, and the difference between their respective payoffs is high enough. Such idea is supported by Friedman (1996), who shows that the payoff-dominant equilibrium is played more often as the difference between the equilibrium payoffs increase, and their risk is kept fixed. It is also backed by Battalio, Samuelson, and Van Huyck (2001), that on experiments using stag-hunt games obtain that as the difference between the risk of two equilibria decreases, the play of the Pareto dominant one becomes more frequent, and by Schmidt et al. (2003) and Février and Linnemer (2006), the former in experiments with stag-hunt games, and the latter through the analysis of the TV contest "the weakest link".

On the other hand, the conditions under which the criterion of risk-dominance is the more relevant one in such comparison, as one would expect, are the opposite of those that we describe above. We notice that the experimental evidence by Van Huyck, Battalio, and Beil (1990, 1991), Cooper et al. (1990), and Straub (1995) point to the fact that players, even if not entirely behaviorally oriented to the selection of the risk-dominant equilibrium in some game, follow the recommendations of that criterion more often than those prescribed by the criterion of payoff-dominance. One possible interpretation of those results is that the necessary differences between the risk and the payoff of the equilibria under comparison, which would allow the selection of the payoff dominant outcome, are rather strict and frequently not met.

We conclude, from the empirical evidence, that there is a trade-off between the dominance criteria on a player's decision to play a specific equilibrium action. Contrarily to Harsanyi (1995), we should not assume that risk-dominance is the only criterion taken into consideration by a player when comparing multiple equilibria; not even that such criterion always determines the solution of a game.

Hence, we characterize player i's process of evaluation of an equilibrium as a combination of its risk and the weight of its payoff, together with his perception about its risk. The player's perception of the risk brings some behavioral content to the selection analysis, and arises from the non-linear trade-off between the dominance criteria on the experiments above. It represents either the incorrect computation of the risk following, for instance, a sporadic mistake, or a bias or a over/underestimation of the true risk of an equilibrium. Given the possible large dimension of the sets of actions and equilibria, and the operations to perform, we find this approach sensible.

Therefore, we combine the risk and payoff characteristics of an equilibrium into a single measure, which we call the premium of an equilibrium.

Definition 11 (Premium)**.**

The premium of an equilibrium to player i is a function $\pi_i : \mathcal{E} \to \mathbb{R}$, such that, at $\alpha \in \mathcal{E}$ we have,

$$\pi_i(\alpha) = \frac{1}{1 + w_i(\alpha)} f_i\big(1 + \rho_i(\alpha)\big). \tag{12}$$

The function $f_i : [0, \infty) \to [1, \infty)$ is increasing, continuous, and either:

(i) convex, with a slope greater or equal than one in its entire domain;

(ii) concave, with a slope less or equal than one in its entire domain;

(iii) linear, with a slope equal to one in its entire domain,

and it represents the player's perception of the risk of an equilibrium.

The premium $\pi_i(\alpha)$ of $\alpha \in \mathcal{E}$ to player i represents the risk $\rho_i(\alpha)$ from playing an equilibrium action $\alpha_i := \text{proj}_i\alpha$, given his perception $f_i(\cdot)$, and the weight of its equilibrium payoff $w_i(\alpha)$. We assume that player i's function $f_i(\cdot)$ is the same in the evaluation of each $\alpha \in \mathcal{E}$. However, it may be reasonable to assume that, in some occasions, the player have different functions describing his perception about the risk of different subsets of equilibria. We stick with the first assumption for simplification and to minimize the notation, since it does not affect any of the results. Hence, let $f(\cdot) = (f_i(\cdot))_{i\in N}$ be the profile of functions describing the players' perception of the risk of an equilibrium, which is common to every equilibrium in $\mathcal{E}$.

The premium $\pi_i(\cdot)$ helps rationalizing some of the available empirical literature. To understand that, take some $i \in N$, and $\alpha, \alpha' \in \mathcal{E}$, such that $\rho_i(\alpha) \leq \rho_i(\alpha')$ with $|\rho_i(\alpha) - \rho_i(\alpha')| \leq \phi$, and $w_i(\alpha) \geq w_i(\alpha')$ where $|w_i(\alpha) - w_i(\alpha')| \geq \psi$. For some $\phi \leq \bar{\phi}$ and $\psi \geq \bar{\psi}$, the criterion of payoff-dominance is the more relevant one, and we select α. Otherwise, it is the risk-dominance criterion the main one, and we select α'.

The function $f_i(\cdot)$, which describes player i's perception of the risk of an equilibrium, has one of three general forms, each characterizing a different type of perception:

(i) It is *convex with a slope greater or equal than one in its entire domain* when player i's perception overweight the risk $\rho_i(\alpha)$ of $\alpha \in \mathscr{E}$ in its premium $\pi_i(\alpha)$. We say that player i aggravates his perception of the risk if such is then represented by a convex function $f_i'(\cdot)$, in which $Df_i'(\cdot) > Df_i(\cdot)$ in the entire domain $[0, +\infty)$, where D is a differential operator, such that, $\pi_{i,f}(\alpha) \leq \pi_{i,f'}(\alpha)$ for every $\alpha \in \mathcal{E}$, where $\pi_{i,f}(\alpha), \pi_{i,f'}(\alpha)$ are the premia of α when player i's perception is given by $f_i(\cdot)$ and $f_i'(\cdot)$;

(ii) It is *concave with a slope less or equal than one in its entire domain* when player i's perception underweight the risk $\rho_i(\alpha)$ of $\alpha \in \mathscr{E}$ in its premium $\pi_i(\alpha)$. We say that player i improves his perception of the risk of an equilibrium if such is then represented by a concave function $f_i'(\cdot)$, with $Df_i'(\cdot) < Df_i(\cdot)$ in the domain $[0, +\infty)$, such that $\pi_{i,f}(\alpha) \geq \pi_{i,f'}(\alpha)$ for every $\alpha \in \mathcal{E}$, where $\pi_{i,f}(\alpha), \pi_{i,f'}(\alpha)$ are the premia of α when player i's perception is given by $f_i(\cdot)$ and $f_i'(\cdot)$;

(iii) It is *linear with constant slope equal to one in its entire domain* when a player has a neutral perception of the risk of an equilibrium.

We define the process of equilibrium selection by each $i \in N$ as a minimization problem of the premium $\pi_i(\cdot)$ from choosing an equilibrium action $\alpha_i := \text{proj}_i\alpha$ belonging to $\alpha \in \mathcal{E}$ when every $j \neq i$ plays an equilibrium action $\alpha_j := \text{proj}_j\alpha$, such that,

$$\begin{aligned}
\min_{\alpha \in \mathcal{E}} \quad & \pi_i(\alpha) \\
\text{s.t.} \quad & \tau_i(\pi_i(\alpha')) + \sum_{j \neq i} \tau_j(\pi_j(\alpha')) \geq \tau_i(\pi_i(\alpha)) + \sum_{j \neq i} \tau_j(\pi_j(\alpha)), \quad \forall \alpha' \in \mathcal{E}, \\
& \pi_i(\alpha) \geq 0, \quad \forall \alpha \in \mathcal{E}.
\end{aligned} \tag{13}$$

Each $i \in N$ chooses $\alpha \in \mathcal{E}$ with the minimum premium $\pi_i(\alpha)$ among every equilibrium in $\mathcal{E}$; hence, if every $j \neq i$ plays that equilibrium, any deviation by i from α to some $\alpha' \in \mathcal{E}$ cannot improve, and possibly worsens, the premium to at least one $j \neq i$, i.e., $\pi_j(\alpha) \leq \pi_j(\alpha')$. If that latter relation holds in inequality, then α' is not a self-enforcing selection; player j has, in that case, an incentive to deviate from α' by playing $\alpha_j := \text{proj}_j\alpha$, hence enforcing the selection of α. Therefore, the solutions of the problem for player i are the equilibria in which him and the remaining $n-1$ players coordinate on, as he cannot enforce the coordination in one other equilibrium without penalizing the premium to some $j \neq i$.

In this problem, $\tau_i : [0, y) \to [0, \infty)$ is a concave function, where $\tau_i(x) = 0$ in $x \in \{0, y\}$, with $y \in (0, \infty)$, and $D\tau_i(x) = 0$ in some unique point $x^* \in (0, y)$. This function represents a *no dictator condition* in the selection problem. A dictator is some $j \in N$ who can determine the selection of his preferred (with respect to the premium) equilibrium $\alpha' \in \mathcal{E}$ over some $\alpha \neq \alpha'$, independently of the preferences of every other $i \neq j$. In the absence of $\tau_i(\cdot)$, player j would have a set of functions F_j describing his perception about the risk of an equilibrium, under which,

$$\pi_j(\alpha') - \pi_j(\alpha) > \sum_{i \neq j} \pi_i(\alpha') - \pi_i(\alpha). \tag{14}$$

The domain of $\tau_i(\cdot)$ is semi-open to guarantee that the contribution of the premium $\pi_i(\cdot)$ to player i in the aggregate premium $\pi(\cdot)$ is strictly positive for any $\pi_i(\cdot) > 0$.

Because each $i \in N$ faces the minimization problem in Equation (13), there are n minimization problems of the premium of an equilibrium. However, we can then just focus on the common restriction of each of those problems; hence, we can write the selection problem from a global perspective as,

$$\begin{aligned} \min_{\alpha \in \mathcal{E}} \quad & \sum_{i \in N} \tau_i(\pi_i(\alpha)). \\ \text{s.t.} \quad & \tau_i(\pi_i(\alpha)) \geq 0, \quad \forall \alpha \in \mathcal{E}, \forall i \in N. \end{aligned} \tag{15}$$

We can verify that the solution of this problem simultaneously solves each of the n individual minimization problems as defined in Equation (13); therefore, we can just analyze the selection problem under this setup. For simplification, we write

$$\pi(\alpha) = \sum_{i \in N} \tau_i(\pi_i(\alpha)), \tag{16}$$

and call it the aggregate premium of $\alpha \in \mathcal{E}$. Thus, we have that the solutions to the problem in Equation (15) are the equilibria we select as the solutions of G.

Definition 12 (Solution)**.**

The $\arg\min_{\alpha \in \mathcal{E}} \pi(\alpha) = \mathfrak{s}$ *are the solutions of* G*, where* $\mathfrak{s} = \prod_{i \in N} \mathfrak{s}_i$*. We call* $\alpha \in \mathfrak{s}$ *a solution of* G*.*

It is trivial to prove that $\mathfrak{s}$ is a nonempty set; hence, there is always at least one solution on G. We can now use $\pi(\cdot)$ to order $\mathcal{E}$.

Definition 13 (Global preferences)**.**

There is a binary relation $\succsim^{\pi}$ *over* $\mathcal{E}$*, such that, for any* $\alpha', \alpha'' \in \mathcal{E}$*,* α' *is globally preferred to* α''*, which we write* $\alpha' \succsim^{\pi} \alpha''$*, if* $\pi(\alpha') \leq \pi(\alpha'')$*.*

Consequently, we obtain the following result regarding $\succsim^{\pi}$.

Proposition 3.

The relation $\succsim^{\pi}$ *is a total order over* $\mathcal{E}$*.*

The proof is trivial. Since the risk $\rho_i(\cdot)$ of an equilibrium in $\mathcal{E}$ to each $i \in N$ and the weight $w_i(\cdot)$ of its payoff induce preference relations, then $\succsim^{\pi}$ is at least a partial order. Because the set $\mathcal{E}$ is finite, and every equilibrium induces a well-defined aggregate premium $\pi(\cdot)$, every admissible equilibrium is comparable, consequently satisfying the trichotomy law. We add the following remark.

Remark 2.

If there is only one $i \in N$ *in* G *for which* $|\mathcal{A}_i| > 1$*, then this player decides the solution of the game based on the equilibria that maximizes his expected payoff* $u_i(\cdot)$*.*

We complete the characterization of the selection process with two examples. In the first one, we just conclude the process of equilibrium selection on G_1.

Example 4.

In Table 4, we identify the premium of the equilibria of G_1. For simplification, we assume that Rowena and Colin have a neutral perception of the risk of an equilibrium, such that $f_i(\cdot) = (\cdot)$, where $i = r$ stands for Rowena, and $i = c$ for Colin.

	$\hat{a}$	$\tilde{a}$	α
π_r	1.086	1.238	1.555
π_c	1.195	1.307	1.6
π	2.281	2.544	3.155

Table 4: The premium of every equilibrium in G_1.

The global preferences over the equilibria of G_1 are $\hat{a} \succ^{\pi} \tilde{a} \succ^{\pi} \alpha$. Hence, $\hat{a}$ is its unique solution. ▶

In the second example, we analyze three versions of the stag-hunt game, similar to those used in the experiments of Battalio et al. (2001). We show that the trade-off between the criteria of risk and payoff dominance depends on the characteristics of the game, namely, the distribution of the payoffs across the game outcomes; hence, it supports the claim that our method contributes to the rationalization of some of the available empirical evidence on equilibrium selection.

Example 5.

Consider three versions of a stag-hunt game in Figure 4. Games G_3 and G_4 (center and right-hand side, respectively) are variations of the game G_2 (left-hand side) regarding the distribution of the payoffs; the best-reply correspondence is, nonetheless, the same in all three games.

		Colin	
		S	H
Rowena	S	70 70	55 1
	H	1 55	20 20

	Colin	
	S	H
S	70 70	55 1
H	1 55	10 10

	Colin	
	S	H
S	70 70	55 8
H	8 55	20 20

Figure 4: Original game G_2 (left-hand side), first variation G_3 (middle), and second variation G_4 (right-hand side).

These games have the following admissible Nash equilibria: two equilibria in pure actions, $a = (S, S)$ and $a' = (H, H)$; and one equilibrium in mixed actions, which is $\alpha = \big(19/34[S]{+}15/34[H], 19/34[S]{+}15/34[H]\big)$ in G_2, $\alpha = \big(3/8[S] + 5/8[H], 3/8[S] + 5/8[H]\big)$ in G_3, and $\alpha = \big(4/9[S] + 5/9[H], 4/9[S] + 5/9[H]\big)$ in G_4. Because these games are symmetric, when both players have the same perception of the risk of an equilibrium, we can focus on the selection problem of one player. For simplicity, we assume that this is the case, and set $f_i(\cdot) = (\cdot)^2$ for Rowena ($i = r$) and Colin ($i = c$).

Game G_2: The equilibrium a' risk dominates the other two equilibria since $\rho_r(a') \approx 0.429$, while $\rho_r(a) \approx 0.850$ and $\rho_r(\alpha) = 1$; however, the equilibrium a is payoff dominant. In this case, risk-dominance determines the equilibrium chosen, as it is corroborated by the premia, since $\pi_r(a) \approx 2.222$, $\pi_r(a') \approx 1.768$, and $\pi_r(\alpha) \approx 3.064$. Hence, a' is the solution of G_2.

Game G_3: This is a variation of G_2, where the payoff of (H, H) is reduced from 20 down to 10. In this case, $\rho_r(a')$ increases to approximately .589, and the difference between the weight of the equilibrium payoffs is larger. Hence, payoff-dominance now determines the equilibrium chosen, as reflected in the premia, since $\pi_r(a) \approx 2.019$ and $\pi_r(a') \approx 2.308$. Thus, a is the solution of G_3.

Game G_4: This is another variation of G_2, where the risk of both pure equilibria increase, though in a larger proportion on a'. The equilibrium a is the solution since $\pi_r(a) \approx 2.164$ and $\pi_r(a') \approx 2.252$. However, the necessary variation in the risk of a' to allow this result is larger than in G_3, since the change in the premium of the equilibria in G_3 is also due to a change in the weight of their payoffs. ▶

We conclude this subsection with an additional but brief axiomatic characterization of $\succsim^\pi$. We call it additional since $\succsim^\pi$ takes into account the assumptions made in Section 2; in this case, the addition of the axioms below intends to highlight some properties of $\succsim^\pi$.

Let $\mathfrak{G}$ be the collection of all static games with complete information and a finite set of admissible Nash equilibria. For any $G \in \mathfrak{G}$ with a set of equilibria $\mathscr{E}$, consider a generic binary relation $\succsim \subseteq \mathscr{E} \times \mathscr{E}$.

Axiom 1.

If T is an isomorphism between $G, G' \in \mathfrak{G}$, for $\alpha, \alpha' \in \mathscr{E}$, if $\alpha \succsim \alpha'$ in G, then $T(\alpha) \succsim T(\alpha')$ in G'.

This axiom just extends the analysis of the effects of a game isomorphism beyond the sphere of risk-dominance. As for the next axiom, let $a \in \mathscr{E}$ be a pure equilibrium; take $G' \in \mathfrak{G}$, where $G' = \langle N, \mathcal{A}, u' \rangle$, such that, for every $\alpha \in \mathscr{E} \backslash \{a\}$, $u'_i(\alpha) = u_i(\alpha)$, and $u'_i(a) = u_i(a) + b_i$, where $b_i \geq 0$ for every $i \in N$, and $b_j > 0$ for at least one $j \in N$. In this case, we say that G' is generated from G by the strengthening of $a \in \mathscr{E}$. If $\mathfrak{s}$ is the solution of G and $\mathfrak{s}'$ the solution of G', and if $\mathfrak{s}' \subseteq \mathfrak{s}$, we say that $\mathfrak{s}$ is payoff monotonous.

Axiom 2.

If G' is a game generated from G by the strengthening of $a \in \mathscr{E}$, and for $\alpha \in \mathscr{E} \backslash \{a\}$, $a \succsim \alpha$ in G, then $a \succ \alpha$ in G'.

This is an appealing axiom since it is reasonable to assume that an equilibrium should not be less attractive to the players if its payoff is not worse to any of them, and it is strictly better for at least one of them. We can accompany this axiom with an extension by considering that G' can be a game generated from G by weakening $a \in \mathcal{A}$, while keeping the same best-reply correspondence, such that, for every $\alpha \in \mathcal{A} \backslash \{a\}$, $u'_i(\alpha) = u_i(\alpha)$, and $u'_i(a) = u_i(a) + b_i$, where $b_i \leq 0$ for every $i \in N$, and $b_j > 0$ for at least one $j \in N$.

Axiom 2′.

Let $\alpha', \alpha'' \in \mathscr{E}$; if G' is a game generated from G by the weakening of $a \in \mathcal{A}$, where $a_i = proj_i a$ and $a_j = proj_j a$ for every $j \neq i$, such that, $a_j \in supp(\alpha'_j)$, for $\alpha'_j = proj_j \alpha'$, but $a_i \notin supp(\alpha'_i)$, if $\alpha' \succsim \alpha''$ in G, then $\alpha' \succ \alpha''$ in G'.

It means that if at least one player has more to lose following a deviation from an equilibrium action, while the remaining players suffer the same losses as before, then it is sensible to assume that an equilibrium is not less appealing than before. We conclude this collection of axioms with the following one.

Axiom 3.

Let $\alpha, \alpha', \alpha'' \in \mathscr{E}$; for every $i \in N$, if $\alpha \succsim_i^\rho \alpha' \succsim_i^\rho \alpha''$ and $\alpha' \succsim_i^u \alpha \succsim_i^u \alpha''$, then $\alpha'' \notin \mathfrak{s}$ in G.

This axiom just implies that players' have a strict preference for the minimization of the risk and the maximization of their payoff when they coordinate in one or more equilibria. Considering these axioms, we obtain the following result.

Proposition 4.

$\succsim^{\pi}$ *satisfies Axioms 1,2,2′, and 3 in any* $G \in \mathfrak{G}$.

Naturally, $\succsim^{\pi}$ is unique up to a positive affine transformation of $\pi(\cdot)$ or to a positive monotonic transformation of either $\rho_i(\cdot)$ or $w_i(\cdot)$, or both, in $\pi_i(\cdot)$ for every $i \in N$. It is also clear that $\succsim^{\pi}$ would satisfy the axioms above if for every $i \in N$, $\pi_i(\cdot)$ would be the result of the addition of $\rho_i(\cdot)$ with $w_i(\cdot)$ instead of a ratio. However, we consider the latter form more sensible, since it is more sensitive to changes in each of those measures, and because it has a straight intuitive interpretation; it represents to a player the risk of an equilibrium weighted by its payoff. In the case of the addition, an interpretation is not immediate or even clear.

4.2 Uniqueness of a solution

We now discuss some of the conditions for the existence of a unique solution in G. We start by providing an example of a well-known game with multiple solutions.

Example 6.

Consider the game of the battle of the sexes in Figure 5. This game has two equilibria in pure actions, $a = (B, B)$ and $a' = (S, S)$, and one equilibrium in mixed actions, $\alpha = \big(2/3[B]+1/3[S]; 1/3[B]+2/3[S]\big)$. We assume that players have the same perception of the risk of an equilibrium, such that $f_i(\cdot) = (\cdot)^2$ for both Rowena ($i = r$) and Colin's ($i = c$).

		Colin	
		B	S
Rowena	B	3, 2	1, 1
	S	1, 1	2, 3

Figure 5: The battle of the sexes.

We obtain that $\pi(a) = \pi(a') \approx 3.835$, and $\pi(\alpha) = 6.4$. Hence, this game has two solutions. This result, albeit not a unique solution, is intuitively sensible due to the exclusion of an equilibrium that both players dislike concerning its risk and payoff characteristics. ▶

Considering this example, we can anticipate that, under certain conditions, we may obtain a unique solution in the battle of the sexes. To verify that, consider an alternative version of Example 6.

Example 7.

Consider again the game of the battle of the sexes in Figure 5. Assume that Rowena's perception of the risk of an equilibrium is given by $f_r(\cdot) = (\cdot)^2$, but that Colin's perception is now represented by a linear function $f_c(\cdot) = (\cdot)$. In this case, $\pi(a) \approx 2.855$, and $\pi(a') \approx 3.345$. Hence, although the game is symmetric, by inducing asymmetry in the players' perception of the risk of an equilibrium, we break down the multiplicity of solutions, and select a as the unique solution of the game. ▶

Therefore, the uniqueness or multiplicity of solutions in G depends, on the last instance, on the profile $f = (f_i(\cdot))$ that describes the players' perception of the risk of the equilibria in $\mathcal{E}$.

Denote as F_i the set of all functions $f_i(\cdot)$ that can describe player i's perception of the risk of an equilibrium; without any additional restriction, let $F_i = F_j$ for every $i, j \in N$. Take $F = \prod_{i \in N} F_i$ as the space of such functions. We then introduce the following concept.

Definition 14 (f-variation of G)**.**
Consider a triple $(G, f, \zeta(f))$, where $f \in F$ is a profile $f = (f_i(\cdot))_{i \in N}$ and $\zeta : F \rightrightarrows \mathcal{E}$ is a correspondence, such that, $\zeta(f) = \mathfrak{s}$ is the set of solutions of G under the profile f. We call this triple an f-variation of G.

Among all possible f-variations of G, we focus on one specific class, which we call divisible f-variations.

Definition 15 (Divisible f-variations of G)**.**
Fix the game G; let $\mathscr{G}$ be a collection of its f-variations $(G, \cdot, \cdot)$. We say that $\mathscr{G}$ is a collection with divisible f-variations if for any $\alpha, \alpha' \in \mathcal{E}$, each $i \in N$, and every $\alpha'' \in \mathcal{E} \backslash \{\alpha, \alpha'\}$, we do not have $\rho_i(\alpha) = \rho_i(\alpha') \leq \rho_i(\alpha'')$, and $w_i(\alpha) = w_i(\alpha') \geq w_i(\alpha'')$.

In this class of games, we obtain the following result concerning the uniqueness of a solution.

Proposition 5.
Let $\mathscr{G}$ be a collection with divisible f-variations; in "almost all" f-variations $(G, f, \zeta(f))$ of G we have $|\zeta(f)| = 1$, i.e., we have a unique solution.

We provide the proof of this proposition in the Appendix A.2.. We show that the set of f-variations $(G, f, \zeta(f))$ of G in which $|\zeta(f)| \geq 2$ has a null Lesbegue measure. Therefore, games with multiple solutions are exceptional in this class.

5 Conclusion

The equilibrium selection method we propose in this work shares some of the intuition underlying the method by Harsanyi and Selten (1988), but addresses some important issues on that same work, that were not solved in Harsanyi (1995). The main one, in particular, concerns the importance of risk and payoff dominance in the decision to play a given equilibrium action, and the trade-off between those criteria on such decision.

The selection method depends on a collection of five assumptions, although only two of them can be empirically falsified. Those assumptions (Assumption 1 and Assumption 5) are, however, the more relevant ones for the selection process. We propose criteria of risk and payoff dominance as alternatives to the criteria in Harsanyi and Selten (1988), although the difference in the latter one is mainly in terms of representation. In comparison to Harsanyi and Selten (1988), our risk-dominance criterion allows for pairwise transitive comparisons between the equilibria that are eligible for selection, and incorporates a reasonable level of complexity, even in games with a substantial number of actions, which should permit its empirical validation.

One of the main questions that we address, and which has been left open in the theoretical literature on equilibrium selection, concerns how to combine and trade-off the criteria of risk and payoff dominance. For that purpose, we take into account the available empirical literature, and propose a measure that combines these dominance criteria, and rationalizes some of the experimental results. The subsequent measure, that we call the premium of an equilibrium, represents the risk for a player from choosing to play some equilibrium action, given his perception of the risk, and the payoff in equilibrium. Using this measure, we define the equilibrium selection as a problem of minimization of the premium for all the players from playing some action in a particular equilibrium, whose set of minimizers are the solutions of the game. We show that a preference relation which has the premium as the underlying measure satisfies a collection of sensible axioms. We also show that the possible multiplicity of solutions, which is an issue in the majority of the methods available in the literature, is not a critical matter in our method. Under certain conditions, which are quite common and sensible in static games, we prove that *almost all* games have a unique solution.

We can identify at least two paths as future directions of research. The first one, which is perhaps the more evident, concerns the extension of this selection method to other types of game, such as, dynamic games with asymmetries of information. A second one regards the identification of the epistemic conditions of this selection method, and its comparison with the conditions in other selection methods, and equilibrium refinements.

Appendix

Appendix A.1.

Proof of Proposition 1.

Concerning completeness, take $\alpha, \alpha' \in \mathcal{E}$. We either have $\rho_i(\alpha) \geq \rho_i(\alpha')$ or $\rho_i(\alpha) \leq \rho_i(\alpha')$, since $\rho_i(\alpha)$ and $\rho_i(\alpha')$ are real numbers. Therefore, $\alpha \succsim_i^\rho \alpha'$ or $\alpha' \succsim_i^\rho \alpha$, which implies that $\succsim_i^\rho$ is complete. Regarding transitivity, take $\alpha, \alpha', \alpha'' \in \mathcal{E}$. If $\alpha \succsim_i^\rho \alpha'$ and $\alpha' \succsim_i^\rho \alpha''$, then $\rho_i(\alpha) \geq \rho_i(\alpha')$ and $\rho_i(\alpha') \geq \rho_i(\alpha'')$. Since $\geq$ is transitive on $\mathbb{R}$, we have $\alpha \succsim_i^\rho \alpha''$, and $\succsim_i^\rho$ is transitive. $\square$

Proof of Proposition 2.

We check for the effect on $\succsim_i^\rho$ of the conditions satisfied by an isomorphism T. Considering the first two conditions, there are bijections t and $\mathfrak{t} = (\mathfrak{t}_i)_{i \in N}$ which rename the players in N and for each $i \in N$ the actions in A_i. These bijections have no strategic impact on G; hence, they preserve the game's best-reply correspondence, and the deviations from the equilibria by any $i \in N$. Regarding the third condition, since $\alpha \succsim_i^\rho \alpha'$ for $\alpha, \alpha' \in \mathcal{E}$ in G, by definition, $\rho_i(\alpha) \leq \rho_i(\alpha')$. Hence, we can write the risk of the equilibrium $\mathfrak{t}(\alpha)$ for player $t(i)$ in $\tilde{G}$ as,

$$\rho_{t(i)}\Big(\mathfrak{t}(\alpha)\Big) = \sum_{k=1}^{\bar{m}} \eta_{t(i)}^k\Big(\mathfrak{t}(\alpha)\Big)\rho_{t(i)}^k\Big(\mathfrak{t}(\alpha)\Big). \tag{17}$$

Expanding it, we get

$$\sum_{k=1}^{\bar{m}} \eta_{t(i)}^{k}\Big(\mathfrak{t}(\alpha)\Big)\rho_{t(i)}^{k}\Big(\mathfrak{t}(\alpha)\Big) = \sum_{k=1}^{\bar{m}} \eta_{t(i)}^{k}\Big(\mathfrak{t}(\alpha)\Big)\left[1-\prod_{t(j)} \frac{u_{t(j)}\Big(\mathfrak{t}_j(\alpha_j), \mathfrak{t}_{-j}(\alpha_{-j})\Big) - u_{t(j)}\Big(\mathfrak{t}_j(\alpha_j^q), \mathfrak{t}_{-j}(\alpha_{-j})\Big)}{u_{t(j)}\Big(\mathfrak{t}_j(\alpha_j), \mathfrak{t}_{-j}(\alpha_{-j})\Big) - u_{t(j)}(x^o)}\right],$$

which means that,

$$\rho_{t(i)}\Big(\tau(\alpha)\Big) = \sum_{k=1}^{\bar{m}} \eta_i^k(\alpha)\left[1 - \prod_j \frac{(\theta_j u_j(\alpha_j, \alpha_{-j}) + \beta_j) - (\theta_j u_j(\alpha_j^q, \alpha_{-j}) + \beta_j)}{(\theta_j u_j(\alpha_j, \alpha_{-j}) + \beta_j) - (\theta_j u_j(x^o) + \beta_j)}\right] = \rho_i(\alpha), \tag{18}$$

where $\alpha_j^q \in \mathcal{A}_j$ is the q^{th} best-deviation of $j \neq i$ in G. The risk of the equilibrium $\mathfrak{t}(\alpha^{'})$ for $t(i) \in t(N)$ follows likewise from the risk of α' in G. Hence, we conclude that $\succsim_i^\rho$ is invariant to a game isomorphism. $\square$

Proof of Proposition 4.

We start with Axiom 1. Since for every $i \in N$, $\succsim_i^\rho$ and $\succsim_i^u$ are invariant to an isomorphism T of G, given that $\pi(\cdot)$ is the aggregation of $pi_i(\cdot)$, each resulting from the combination of the measures underlying those preference relations, then we conclude that $\succsim^\pi$ is also invariant to T. Regarding Axiom 2, if we strength $a \in \mathscr{E}$, then for at least some $i \in N$, $w_i'(a) > w_i(a)$, and for every $j \neq i$, $\rho_j(a) > \rho_j'(a)$, where $w_i'(\cdot)$ and $\rho_j(\cdot)$ are the payoff weight and risk functions in G'. Then if $a \succsim^\pi \alpha$ in G, for $\alpha \in \mathscr{E}$, it just follows that $a \succ^\pi \alpha$ in G'. In the extension of this axiom, when we weaken $a \in \mathcal{A}$, and such profile respects the conditions of Axiom 2′, then $\rho_j(\alpha') > \rho_j'(\alpha')$; assuming that the payoff weights are intact, we have that if $\alpha' \succsim^\pi \alpha''$ in G, then $\alpha' \succ^\pi \alpha''$ in G'. Regarding Axiom 3, if for every $i \in N$, $w_i(\alpha) \geq w_i(\alpha'')$ and $w_i(\alpha') \geq w_i(\alpha'')$, while $\rho_i(\alpha) \geq \rho_i(\alpha'')$ and $\rho_i(\alpha') \geq \rho_i(\alpha'')$, it is trivial that $\pi(\alpha) \leq \pi(\alpha'')$ and $\pi(\alpha') \leq \pi(\alpha'')$, guaranteeing that an equilibrium which is risk and payoff dominated is not part of the solution set of G.

Appendix A.2.

We construct the proof of Proposition 5 through some lemmata. We start by identifying the vector space. Fix the game G; we identify $\mathscr{G}$ with $\mathbb{R}^{n|\mathcal{E}|}$ by considering a f-variation $(G, f, \zeta(f)) \in \mathscr{G}$, where $f = (f_i(\cdot))_{i \in N}$. Denote as $p_{i,f} := (\pi_{i,f}(\alpha))_{\alpha \in \mathcal{E}}$ the vector of premia of every $\alpha \in \mathcal{E}$ under $f_i(\cdot)$ for each $i \in N$. We can view $p_{i,f}$ as an element of $\mathbb{R}^{\mathcal{E}}$, and $p_f = (p_{1,f}, \ldots, p_{n,f})$ as a vector in $\mathbb{R}^{n|\mathcal{E}|}$. Additionally, consider two subsets $\mathscr{G}^1$ and $\mathscr{G}^2$ of $\mathscr{G}$; the former includes every f-variation $(G, f, \zeta(f))$ in which $|\zeta(f)| = 1$, and the latter, every f-variation $(G, f', \zeta(f'))$ with $f' \neq f$, in which $|\zeta(f')| \geq 2$.

Lemma 1.

If $\mathscr{G}$ is not a collection with divisible f-variations $(G, f, \zeta(f))$, for every $f \in F$, we have $\zeta(f) \geq 2$.

Proof. Take an f-variation $(G, f, \zeta(f)) \in \mathscr{G}$ in which $|\zeta(f)| \geq 2$. Assume that there is an $i \in N$ for which $\rho_i(\alpha) \neq \rho_i(\tilde{\alpha})$, where $\alpha, \tilde{\alpha} \in \zeta(f)$. Denote as $f_i(\cdot) = \text{proj}_i f$ the function describing i's risk perception. Consider an alternative function $f'_i(\cdot) \neq f_i(\cdot)$, such that $Df'_i(\cdot) \neq Df_i(\cdot)$. Under the profile f, $\pi_f(\alpha) = \pi_f(\tilde{\alpha})$. If instead, we take $f' = (f'_i(\cdot), f'_{-i}(\cdot))$, in which $f'_{-i} = (f_j(\cdot))_{j \neq i}$ for every $f_j(\cdot) =$ $\text{proj}_j f$ and each $j \neq i$, then

$$|\pi_{i,f}(\alpha) - \pi_{i,f'}(\alpha)| \neq |\pi_{i,f}(\tilde{\alpha}) - \pi_{i,f'}(\tilde{\alpha})|. \tag{19}$$

Hence, $\pi_{f'}(\alpha) \neq \pi_{f'}(\tilde{\alpha})$; it means that there is an $(G, f', \zeta(f')) \in \mathscr{G}$ in which $|\zeta(f')| = 1$, such that, either $\alpha \in \zeta(f')$ or $\tilde{\alpha} \in \zeta(f')$, providing the necessary contradiction. □

Lemma 2.

If $\mathscr{G}$ is a collection with divisible f-variations, then $\mathscr{G}^1$ is infinite and uncountable.

Proof. Consider an f-variation $(G, f, \zeta(f)) \in \mathscr{G}$ where $\zeta(f) = \{\alpha\}$, and let $f = (f_i(\cdot), f_{-i}(\cdot))$. Take some $i \in N$ for which $\rho_i(\alpha) \neq \rho_i(\alpha')$ in every $\alpha' \in \mathcal{E} \backslash \{\alpha\}$. If we substitute $f_i(\cdot)$ by a positive linear transformation $f'_i(\cdot) = cf_i(\cdot)$, where $c \in \mathbb{R}^+$, in a certain interval of c, we have

$$\pi_{i,f'}(\alpha) - \pi_{i,f'}(\alpha') \geq \pi_{i,f}(\alpha) - \pi_{i,f}(\alpha'). \tag{20}$$

Considering that $\tau_i(\cdot)$ is a concave function, there is an interval $[a, b]$, for $a, b \in \mathbb{R}$ and $b > a$, where for every $c \in [a, b]$, the positive linear transformation $f'_i(\cdot) = cf_i(\cdot)$ satisfies Equation (20). Since $\tau_i(\cdot)$ has a semi-open domain, it trivially follows that the interval between a and b is not necessarily closed. Since some open interval $[a, b) \subset \mathbb{R}$ is uncountable, then also is the set of positive linear transformations $f'_i(\cdot) = cf_i(\cdot)$ respecting condition Equation (20), which completes the proof. □

Lemma 3.

If $\mathscr{G}$ is a collection with divisible f-variations, then $\mathscr{G}^2$ is countable.

Proof. We first show that $\mathscr{G}^2$ can be empty. Fix G; for each $i \in N$, let $\rho_i(\alpha) < \rho_i(\alpha')$ and $w_i(\alpha) > w_i(\alpha')$ for some $\alpha \in \mathcal{E}$ and every $\alpha' \in \mathcal{E}\backslash\{\alpha\}$. Independently of $f \in F$, $\pi(\alpha) < \pi(\alpha')$; thus in every f-variation $(G, f, \zeta(f)) \in \mathscr{G}$, $\zeta(f) = \{\alpha\}$. If $\mathscr{G}^2 \neq \emptyset$, take an f-variation $(G, f, \zeta(f)) \in \mathscr{G}$ where, for simplicity, $\zeta(f) = \{\alpha, \alpha'\}$, although a higher cardinality does not change the proof. Consider a profile $f' \in F$ in some $(G, f', \zeta(f')) \in \mathscr{G}$, where $f_{-i}(\cdot) = f'_{-i}(\cdot)$ with $f_{-i}(\cdot) = (f_j(\cdot))_{j\neq i}$ and $f_j(\cdot) = \text{proj}_j f$, and $f_i(\cdot) \neq f'_i(\cdot)$, in which $f_i(\cdot) = \text{proj}_i f$ and $f'_i(\cdot) = \text{proj}_i f'$.

Define a function $\mathfrak{c} : \mathscr{G}^2 \to \mathbb{N}$ for which $\mathfrak{c}(G, f, \zeta(f)) = \mathfrak{c}(G, f', \zeta(f')) = m$, for some $m \in \mathbb{N}$, if $f' = (f'_i(\cdot), f_{-i}(\cdot))$ and $\zeta(f') = \zeta(f)$. We can define an equivalence class $[\mathscr{G}^2_m] \subseteq \mathscr{G}^2$, in which $[\mathscr{G}^2_m] = \{(G, f, \zeta(f)) \in \mathscr{G}^2 | \mathfrak{c}(G, f, \zeta(f)) = m\}$. We can show that $\#[\mathscr{G}^2_m] \leq n$. Assume that $\tau_i(\cdot)$ is a linear function, such that, to each $i \in N$, the weight of the premium $\pi_i(\cdot)$ is the same in $\pi(\cdot)$. If $(G, f, \zeta(f))$ is an f-variation where $\zeta(f) = \{\alpha, \alpha'\}$, then for $f' \in F$,

$$\pi_{i,f}(\alpha) - \pi_{i,f'}(\alpha) \neq \pi_{i,f}(\alpha') - \pi_{i,f'}(\alpha'). \tag{21}$$

Then, $\pi_{f'}(\alpha) \neq \pi_{f'}(\alpha')$. If now $\tau_i(\cdot)$ is a concave function with a single maximum, then there is, at the most, a unique pair $f_i, f'_i \in F_i$, for which

$$\tau_i(\pi_{i,f}(\alpha)) - \tau_i(\pi_{i,f'}(\alpha)) = \tau_i(\pi_{i,f}(\alpha')) - \tau_i(\pi_{i,f'}(\alpha')). \tag{22}$$

Since that holds for every $i \in N$, $\#[\mathscr{G}^2_m]$ is bounded by n. It is then trivial to check that each f-variation $(G, f, \zeta(f))$ in which $|\zeta(f)| \geq 2$ belongs to an equivalence class as $[\mathscr{G}^2_m]$. □

Proof of Proposition 5.

Considering Lemma 1, if $\mathscr{G}$ is a collection with divisible f-variations, there is some $(G, f, \zeta(f)) \in \mathscr{G}$ in which $|\zeta(f)| = 1$; thus, $\mathscr{G}^1 \neq \emptyset$. Denote as $\mu(\cdot)$ the Lebesgue measure on $\mathbb{R}^{n|\mathcal{E}|}$. We have not identified the dimension of $\mathscr{G}^2$. However, by Lemma 3, we know that every f-variation $(G, f, \zeta(f)) \in \mathscr{G}^2$ belongs to some equivalence class $[\mathscr{G}^2_m] \subset \mathscr{G}^2$, where $m \in \mathbb{N}$, which has a finite number of elements. Because $\mathscr{G}^2$ is countable following Lemma 3, then $\mu(\mathscr{G}^2) = 0$; hence, "almost all" f-variations $(G, f, \zeta(f)) \in \mathscr{G}$ have $|\zeta(f)| = 1$. □

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Chapter 2

Equilibrium selection in dynamic games with asymmetric information

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Abstract

In this work, we extend the author's equilibrium selection method, initially applied to static games with complete information, to dynamic games with asymmetric information. We adjust the method to games in extensive-form by reformulating its assumptions, and some features of the selection mechanism. We define the solution of a dynamic game as a sequential equilibrium that conditionally minimizes the risk to the players, given their perception of the risk and its payoff. We show that a solution is invariant to different forms of isomorphism of the extensive-form; however, due to the sequential nature of decision-making, we show that a solution of the extensive-form does not necessarily coincide with the solution from its reduced normal-form. We conclude the paper with an application of this method to the most simple version of Spence signaling game with just two types of worker.

Keywords: equilibrium selection, sequential equilibrium, dynamic game, asymmetric information, Spence signaling game.

JEL classification: C72

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1 Introduction

In Silva (2019) we propose an equilibrium selection method to static games with complete information. In that work, we define the solution of a static game as an admissible Nash equilibrium that minimizes the risk to a player, given his perception about the risk and its payoff, conditional on the same being true for every opponent. We show that in a particular class of games, "almost all" have a unique solution; the set of games with multiple solutions has a null Lebesgue measure, thus being negligible. In this work, we extend the method in its more immediate and reasonable direction, which is to dynamic games with asymmetric information. In the first part of the work, we discuss and characterize the main features of such extension. In the second part, we further adjust the method to signaling games with a finite number of player types and apply it to the most simple version of Spence signaling game (Spence, 1973) with just two types.

As in static games with complete information, there is a large literature in equilibrium selection in dynamic games; however, in the former type of games, it is more challenging to organize the literature into distinct research programs. The only general methods of selection on this type of games are from Harsanyi and Selten (1988), Harsanyi (1995a) that we have already discussed in Silva (2019), and Harsanyi (1995b), which is a direct generalization of Harsanyi's method to games with incomplete information. We roughly divide into two groups. One group includes equilibrium refinement concepts. In comparison to static games, one concept, the sequential equilibrium, in one of its many forms (depending on the requirements on belief formation and consistency), is seen as the most sensible option. The other group includes methods and criteria with some degree of symbiosis with a sequential equilibrium, i.e., they take that refinement as the starting point of the selection process. As in Harsanyi and Selten (1988), the methods and criteria in the latter group represent the culmination point of the equilibrium refinement program. We shall briefly address the main theories in each group.

Concerning the equilibrium refinements to dynamic games, since many of them are just variations of older ones, we just highlight three concepts. We start with the concept of subgame perfection (Selten, 1975). The intuition underlying it is that a player's optimal strategy should prescribe a best-reply in every subgame that the play of the game reaches with positive probability.

In perfect information games, and with some exceptions in non-generic games [1], or games without non-trivial subgames[2], it reduces the set of Nash equilibria down to a singleton. With imperfect information, that result is no longer always possible, as subgame perfection may select multiple Nash equilibria, among which, some may not be sensible. However, these issues are common to the following refinements.

The first one is the extensive-form version of the perfect equilibrium (Selten, 1975). As in its strategic-form version, it selects equilibrium actions that are optimal, at each information set, even when a player has a small probability of deviating to another action. Perfect equilibria are limit points of sequences of equilibrium outcomes from a collection of games in which players have small but positive, probabilities of deviating from their equilibrium strategies. However, this refinement demands excessively from the players' rationality. To understand that claim, we have first to introduce the concept of sequential equilibrium (Kreps & Wilson, 1982b). This refinement shares the intuition of a subgame perfect equilibrium since it prescribes a player to choose strategies that are optimal in every information set that the play of the game can reach with positive probability, i.e., strategies that are sequentially rational. However, sequential rationality should consider the player's consistent beliefs about the play of the game reaching each of the nodes in his information sets, and how on the game will proceed afterward; we provide a definition of consistency in Section 2. In comparison to perfectness, a sequential equilibrium requires players' to be sequentially rational at the limit of a sequence of perturbed games, while in the former concept that is required in every perturbed game; however, in generic extensive-form games this difference almost disappears, as the sets of perfect and sequential equilibria nearly coincide.

Regarding the theories of equilibrium selection in dynamic games, they further refine the sequential equilibrium predictions. We start with two popular selection criteria, which aim at excluding unreasonable sequential equilibria of a game, and at providing behavioral foundations to Kohlberg and Mertens (1986) notion of strategic stability. These criteria are valid in signaling games, which although less complex than general extensive-form games play an important role in economics; for instance, in the analysis of labor markets, entry and deterrence problems, or IPO's. The first one is the intuitive criterion (Cho & Kreps, 1987). It follows the principle that in a sensible sequential

[1] A game in which a player has two or more best replies to some particular vector of opponents' strategies that are payoff equivalent.

[2] An extensive-form game without another subgame than the game itself.

equilibrium we do not expect some types of sender to choose specific strategies. Hence, a sequential equilibrium satisfies the intuitive criterion if no type of sender can profitably deviate to an off-the-equilibrium strategy. This criterion selects a single sequential equilibrium in signaling games with two types of sender, such as a two types version of the Spence (1973) signaling model. However, it cannot guarantee the same result as the number of types increase. In that case, if more than one type of sender has a profitable deviation, the beliefs of the receiver associate the same deviation likelihood to each of the sender's types. The second one is the divinity criterion (Banks & Sobel, 1987). It minimizes some of the issues in the intuitive criterion, as it provides more sensible predictions in signaling games with a larger number of types, and assumes that the beliefs of a receiver can associate distinct likelihoods of deviation to different types. However, it is incapable of selecting a single sequential equilibrium in many signaling games.

An alternative to those criteria explores the effects of reputation on the selection of an equilibrium. In signaling games with a single long-run player (an incumbent) and multiple short-run players (entrants), both Kreps and Wilson (1982a) and Milgrom and Roberts (1982) challenge the sequential equilibrium prediction in which every entrant enters, and the incumbent accommodates the entry. They conclude it is sufficient to assume the existence of a certain probability of the incumbent fighting every time the entrant enters. Extensions can be found in Fudenberg and Levine (1989), which covers a broader class of games, and introduces a single patient long-run player[3], and Aumann and Sorin (1989), which considers multiple long-run players. However, works considering the effects of reputation provide firm conclusions only on a restrict number of games[4] and do not necessarily guarantee the selection of a single solution.

Another approach, common in the analysis of multilateral vertical contracting problems, considers the effect of different types' of beliefs.[5] The selection of an equilibrium depends on the beliefs that a downstream firm has about the off-the-equilibrium path behavior in a game. Those beliefs are of one of three types. The first type, which is simultaneously the most common assumption in this literature, are called "passive" beliefs (Rey & Tirole, 2007). In this case, a downstream firm

[3]The patient long-run player discounts his payoffs with a factor near one, and the short-term players with a factor near zero. In the chain-store game, it implies that the incumbent has a stronger incentive to fight.

[4]For example, in dynamic games with only long-run players, reputation effects do not provide reasonable predictions.

[5]These are contracting problems with an upstream firm, usually a monopolist, who sells a product to possible several downstream firms. Some examples are the relationship between a manufacturer and several distributors, and between a franchisor and his franchisees.

who receives an off-the-equilibrium offer does not update its beliefs about the offers made to the other firms. The second type are called "symmetric" beliefs (Pagnozzi & Piccolo, 2012). Therefore, conditional on observing a deviation, a downstream firm believes that all the other downstream firms have also received off-the-equilibrium offers. The last type are known as "wary" beliefs (Rey & Vergé, 2004), in which case, a downstream firm believes that an upstream firm only deviates if it is to an optimal strategy.

The number of selection methods in dynamic games, although not residual, is smaller than in static games. A couple of reasons may justify this scenario. The first one is the lower level of complexity of games in strategic-form. The second one concerns Kohlberg and Mertens (1986) arguments, under which methods to strategic-form games can be applied to extensive-form games if represented in their reduced normal-form. However, we follow the branch of the literature that considers that the information of a dynamic game cannot be entirely encompassed in its strategic-form representation. Therefore, we see the necessity for a selection method that is directly applied and adjusted to extensive-form games.

We organize the paper as follows. In Section 2, we introduce the assumptions underlying the process of equilibrium selection. In Section 3, we define risk and payoff dominance criteria, and characterize the selection mechanism. These sections are mainly a straight adaptation of Silva (2019). We proceed to show that a solution is invariant to symmetries between elements of the game; however, a solution in the extensive-form representation of a game does not necessarily coincide with the solution of its reduced normal-form. In Section 4, we apply the selection method to the most simple version of Spence (1973) signaling game with only two types of worker; we show that a solution depends on the prior probability of a worker being of a highly productive type, and compare it with the results in Cho and Kreps (1987) and in the application of Harsanyi and Selten (1988) made by van Damme and Güth (1991). In Section 5, we summarize the main conclusions and provide some directions for future research.

2 Preliminaries

2.1 The game

Consider a finite dynamic game G with imperfect information in extensive-form and a set of players $N = \{1, \ldots, i, \ldots, n\}$.[6]

2.1.1 Game tree

Let A be a finite set of actions, and X a set of nodes. Each $x \in X$ is a finite sequence of actions $a^k \in A$, such that, $x = (a^1, \ldots a^k)$, and $a^1, \ldots, a^k \in A$. We define a partial order $\succ$ over X, such that, x follows x', which we write $x \succ x'$, if $x = (x', a^1, \ldots, a^k)$ and $k \geq 1$. When $k = 1$, we say that x immediately follows x', and write $x \succeq x'$.

Let $W \subset X$ be the set of initial nodes. The game starts at one node $x_0 \in W$ controlled by Nature, who moves first and just once according to a family of probability distributions $p = (p^{x_0})_{x_0 \in W}$, such that, p^{x_0} is the realization probability of the nodes immediately following $x_0 \in W$. We assume that $p^{x_0}(x) > 0$ for every $x \succ x_0$, and that p is common knowledge among the players. The game ends at one or more terminal nodes in a set $Z := \{x \in X \,|\, (x, a) \notin X, \forall a \in A\}$.

We call $x \in X \backslash (W \cup Z)$ a decision node; the nodes that immediately follow it belong to the set $C(x) = \{x' \in X \backslash (W \cup Z) \,|\, x' \succeq x\}$, while the the feasible actions at x belong to $A(x) := \{a \in A \,|\, (x, a) \in X\}$. The player who moves at $x \in X \backslash (W \cup Z)$, which we call the active player at x, is identified by $\iota : X \backslash (W \cup Z) \to N$. The set $X_i := \{x \in X \backslash (W \cup Z) \,|\, \iota(x) = i\}$ includes the decision nodes in which player $i \in N$ is active, and the set $A_i := \{a \in A \,|\, (x, a) \in X, \iota(x) = i\}$ the available actions at each $x \in X_i$.

We divide the nodes in X_i of player i into information sets. We say that two decision nodes $x, x' \in X \backslash (W \cup Z)$ belong to the same information set of player i if $\iota(x) = \iota(x')$ and $A(x) = A(x')$. We denote as $h_i(x)$ the information set containing $x \in X$, while the collection $H_i = \{h_i(x) \,|\, \iota(x) = i, \text{ for } x \in X \backslash (W \cup Z)\}$ includes all information sets of player i, and $H := \bigcup_{i \in N} H_i$ all the information sets in G. We also assume that G has perfect recall, so we can establish an unambiguously partial order over the information sets in H according to the partial order $\succ$.

[6] We consider that a game G is dynamic if the problem involves sequential decision-making. In that case, there is a unique extensive-form representation of the problem, which we preserve throughout the selection analysis.

Definition 1 (Perfect recall)**.**

We write xR^px' *for any* $x, x' \in X\backslash(W \cup Z)$ *if* $h(x) = h(x')$ *implies that:*

- *there are* $\hat{x}, \tilde{x} \in X\backslash(W \cup Z)$*, such that,* $\iota(x) = \iota(\hat{x})$*, with* $\tilde{x} \in C(\hat{x})$ *and* $x \succ \tilde{x}$*;*
- *there are* $\hat{x}', \tilde{x}' \in X\backslash(W \cup Z)$*, for which* $h(\hat{x}') = h(\hat{x})$*, with* $\tilde{x}' \in C(\hat{x}')$*, and where* $x' \succ \tilde{x}'$ *and* $A(\tilde{x}) = A(\tilde{x}')$*.*

We write $xR^p = \{x' \in X\backslash(W \cup Z) \,|\, xR^px'\}$*;* G *has perfect recall if* $xR^p = h(x)$ *for each* $x \in X\backslash(W \cup Z)$*.*

The payoff function $u_i : Z \to \mathbb{R}$ is a von Neumann and Morgenstern utility function. Therefore, we identify G with an extensive-form $\langle X, \succ, N, W, X, H, A, u, p \rangle$.

2.1.2 Strategies

A pure strategy of player i is a function $s_i : H_i \to A$ where $s_i\big(h_i(x)\big) \in A(x)$ for any $x \in X\backslash(W \cup Z)$, with $\iota(x) = i$. We denote as S_i the set of pure strategies of player i. A mixed strategy σ_i is a probability distribution over S_i; instead, a behavior strategy β_i is a probability distribution over the actions available to player i at each $h_i \in H_i$[7]. Denote as $A(h_i)$ the set of actions available at $h_i \in H_i$, such that, $\beta_i : A(h_i) \to [0, 1]$ for every $h_i \in H_i$, in which $\sum_{a_i \in A(h_i)} \beta_i(h_i; a_i) = 1$. We call a behavioral strategy β_i completely mixed if $\beta_i(h_i; a_i) > 0$ for every $a_i \in A_i(h_i)$ and $h_i \in H_i$; we call it pure if $\beta_i(h_i; a_i) = 1$ for some $a_i \in A_i(h_i)$ and every $h_i \in H_i$, denoting it as $\beta_i^p \in B_i$. Each $i \in N$ has a set B_i of behavioral strategies, and a set B_i^p of pure behavioral strategies. We denote β as a generic vector of behavioral strategies, such that, $\beta \in B := \prod_{i \in N} B_i$, and β_{-i} a vector in $B_{-i} := \prod_{j \neq i} B_j$.

2.1.3 Equilibrium

A system of beliefs is a function $\mu : X\backslash(W \cup Z) \to [0, 1]$ where $\sum_{x \in h(x)} \mu(x) = 1$ for each $h \in H$. Therefore, a belief $\mu(x)$ is the probability that player $\iota(x)$ assigns to x if $h(x)$ is reached with positive probability.

We call a pair (β, μ) an assessment. The expected payoff of (β, μ) to player i if the game reaches $h_i \in H_i$ with positive probability, and if all the players follow β, given that the play of the game is

[7]We denote an information set of player $i \in N$ as h_i whenever it does not lead to any ambiguity.

at x, is

$$v_i(\beta|h_i, \mu) = \sum_{x \in h_i} \mu(x) u_i(\beta|x). \tag{1}$$

An assessment (β, μ) may have a couple of desirable properties.

Definition 2 (Consistency)**.**
An assessment (β, μ) is consistent if there is a sequence of completely mixed behavioral strategies β^n converging to β, such that, a sequence of systems of beliefs μ^n, each one derived from the Bayes rule, converges to μ.

Definition 3 (Sequential rationality)**.**
An assessment (β, μ) is sequentially rational if for every $i \in N$, $h_i \in H_i$, and $\beta_i' \in B_i$, we have for $\beta_{-i} \in B_{-i}$,

$$v_i(\beta|h_i, \mu) \geq v_i(\beta_i', \beta_{-i}|h_i, \mu).$$

Thus, we define a sequential equilibrium following the definition in Kreps and Wilson (1982b).

Definition 4 (Sequential equilibrium)**.**
An assessment (β, μ) is a sequential equilibrium if it is consistent and sequentially rational.

Denote as $E(G)$ the set of Nash equilibria, and as $\mathcal{S}(G)$ set of sequential equilibria; the latter one may include an infinite number of assessments. In Section 3, we discuss under which conditions we allow the set $\mathcal{S}(G)$ to be infinite, and which procedure do we follow in that case.

2.2 The assumptions

As in static games, the process of equilibrium selection depends on a collection of assumptions. These assumptions aim at eliminating strategically irrelevant elements from the game, guaranteeing that the method is well-behaved, and characterizing a player's behavior further. We start with two assumptions that are almost identical to assumptions 1 and 4 in Silva (2019).

Assumption 1.
Each $i \in N$ is rational; therefore, no player chooses with positive probability a strictly dominated strategy, which is common knowledge among the players. Hence, we simplify G through an iterated process of elimination of strictly dominated behavioral strategies whenever possible.

Assumption 2.

The utility function $u_i : X \to \mathbb{R}$*, with* $Z \subseteq X$*, of every* $i \in N$ *intersects in a point* $x^o \in X$*, such that* $u_i(x^o) = 0$*. Moreover, for every* $i \in N$*,* $u_i(z) \geq 0$ *in each* $z \in Z$*. Hence, the players have interval-scale measurable payoffs with a common origin.*

The next assumption adapts assumption 2 in Silva (2019), controlling for the existence of duplicate behavioral strategies.

Assumption 3.

In G*, we substitute strategically equivalent behavioral strategies by a single representative strategy.*

Two behavioral strategies $\beta_i, \hat{\beta}_i \in B_i$ are strategically equivalent, which we write $\beta_i \approx_i \hat{\beta}_i$, if:

- $u_j(\beta_i, \beta_{-i}) = u_j(\hat{\beta}_i, \beta_{-i})$ for every $\beta_{-i} \in B_{-i}$ and $j \in N$;
- for any $a_i \in \text{supp}(\beta_i)$ and $\hat{a}_i \in \text{supp}(\hat{\beta}_i)$, $h_j(x_i, a_i) = h_j(x_i, \hat{a}_i)$ for any $j \in N$ and $h_j \in H_j$, where $x_i \in h_i$ and $(x_i, a_i), (x_i, \hat{a}_i) \in X \backslash Z$.

We substitute these strategies by a representative element $\mathsf{b}_i \in B_i$ of an equivalence class $[\mathsf{b}_i] = \{\beta_i \in B_i \,|\, \beta_i \sim_i^b \mathsf{b}_i\}$. If we assume that $\mathsf{b}_i = \beta_i$ for some $\beta_i \in [\mathsf{b}_i]$, then every $\hat{a}_i \in \text{supp}(\hat{\beta}_i)$ in any $\hat{\beta}_i \in [\mathsf{b}_i] \backslash \{\beta_i\}$ is played with null probability in every behavioral strategy both in G and every neighboring game[8] unless $\hat{a}_i \in \text{supp}(\hat{\beta}_i) \cap \text{supp}(\beta_i)$ and $\beta_i \notin [\mathsf{b}_i]$. Additionally, if $\beta_i^p \in B_i^p$ and $\beta_i \in B_i$ belong to $[\mathsf{b}_i]$, and $\text{supp}(\beta_i^p) \subset \text{supp}(\beta_i)$, then β_i cannot be the representative of $[\mathsf{b}_i]$.

This assumption has an important effect on the set of sequential equilibria. To understand it, consider the game G_1 in Figure 1.

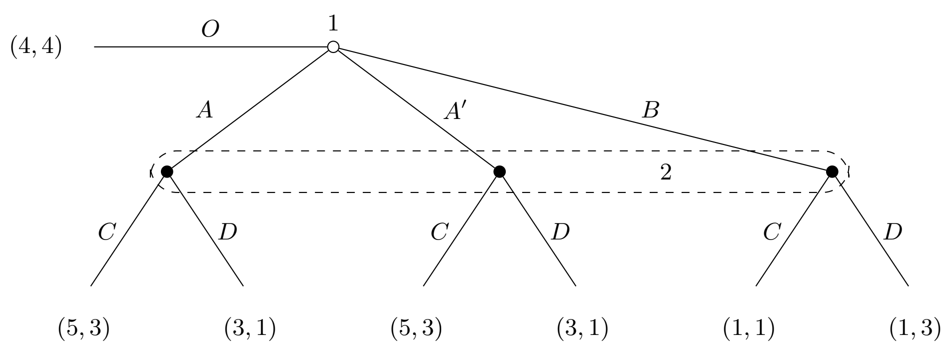

Figure 1: Game G_1.

[8] A game where a player's payoffs are in the neighborhood of those in G, which is a point in $\mathbb{R}^n$ with the ℓ_∞ norm.

In G_1 there is one sequential equilibria where:

- player 1 plays $\beta_1(\emptyset; O, A, A', B) = (1, 0, 0, 0)$;
- player 2 best-replies with $\beta_2(\{A, A', B\}; C, D) = (1-p, p)$ for $p \geq 1/6$, which is sequentially rational given a system of beliefs $\mu(\{A, A', B\}; A, A', B) = (1/3, 1/3, 1/3)$.

However, $\beta_1(\emptyset; A)$ and $\beta_1(\emptyset; A')$ are strategically equivalent. If we assume that $\beta_1(\emptyset; A') = 0$, then player 1 chooses A' with null probability. Hence, the previous sequential equilibrium no longer holds, and in this case:

- if player 1 plays $\beta_1(\emptyset; O, A, A', B) = (1, 0, 0, 0)$;
- it is sequentially rational for player 2 to choose $\beta_2(\{A, B\}; C, D) = (1-q, q)$ for $q \geq 1/2$, given that $\mu(\{A, B\}; A, B) = (1/2, 1/2)$.

The next assumption adapts another assumption from Silva (2019), this time concerning the existence of duplicate players.

Assumption 4.

In G, we apply the coalescing of duplicate players whenever possible.

Players i and j are duplicates of one another if:

- for $k, \ell = i$, and $k \neq \ell$, with $\beta_k \in B_k$ and $\beta_\ell, \beta'_\ell \in B_\ell$, and $\beta_{-k-\ell} \in \prod_{h \in N \setminus \{i,j\}} B_h$, $u_k(\beta_k, \beta_\ell, \beta_{-k-\ell}) = u_k(\beta_k, \beta'_\ell, \beta_{-k-\ell})$;
- a permutation of the players i and j in G generates an isomorphic game[9].

In this case, players i and j belong to an equivalence class N_d, which is represented by $\bar{i}$. However, the coalescing of these players only takes place if we satisfy an additional condition:

(a) Let $X_{\bar{i}} = X_i \cup X_j$, $H_{\bar{i}} = H_i \cup H_j$, and $A_{\bar{i}} = A_i \cup A_j$ if and only if for $x \in h_i(\hat{x})$ and $x' \in h_j(\tilde{x})$, we have $xR^p = h_i(\hat{x})$ and $x'R^p = h_j(\tilde{x})$.

If this condition is not satisfied, players i and j are no longer duplicates of one another; therefore, we preserve the perfect recall in G. That is not the case, for example, in a similar transformation by Hoshi and Isaac (2010).

[9]We provide two definitions of isomorphism of the extensive-form in Definitions 19 and 20.

Before making a final assumption, we introduce some additional elements. We start by classifying the nodes in G according with their position in the tree.

Definition 5 (Levels).

In G, we divide the nodes of X in levels, such that:

- *Level 1 nodes: every $x \in X$ for which $x \succ x_0$ and $x_0 \in W$. Let $X_1 \subset X$ be the set of those nodes, and $X_{i,1} \subseteq X_1$ the set of player i's nodes;*
- *Level 2 nodes: every $x \in X \backslash X_1$ for which $x' \succ x$ and $x \in X_1$. Let $X_2 \subset X$ be the set of those nodes, and $X_{i,2} \subseteq X_2$ the set of player i's nodes;*
- *Level k nodes: every $x \in X \backslash \bigcup_{\ell=1}^{k-1} X_\ell$ for which $x'' \succ x$ and $x \in X_{k-1}$. Let $X_k \subset X$ be the set of those nodes, and $X_{i,k} \subseteq X_k$ the set of player i's nodes;*

Let $\mathscr{B}_i$ be player i's set of admissible behavioral strategies. Take $\mathcal{E}$ as the set of admissible Nash equilibria β belonging to a sequential equilibrium assessment (β, μ), i.e., for each $\beta \in \mathcal{E}$, $(\beta, \mu) \in \mathcal{S}(G)$. The set $\mathcal{B}_i = \mathscr{B}_i \cap \text{proj}_i\mathcal{E}$ includes admissible behavioral strategies $\beta_i := \text{proj}_i\beta$ that belong to some admissible equilibrium $\beta \in \mathcal{E}$, such that, (β, μ) is a sequential equilibrium. Finally, let $\mathscr{S}$ be the set of sequential equilibria (β, μ) in which $\beta \in \mathcal{E}$. We then introduce the concept of auxiliary game.

Definition 6 (Auxiliary game $\Gamma(1)$).

The tuple $\Gamma = \langle G, \mathcal{B}(1), \mathscr{S} \rangle$ is an auxiliary game of G, in which $\mathcal{B}(1) = \mathcal{B}$, and $\mathcal{B} = \prod_{i \in N} \mathcal{B}_i$, such that each $i \in N$ only chooses behavioral strategies in $\mathcal{B}_i$.

The auxiliary game $\Gamma(1)$ is an auxiliary structure of G that restricts the strategies that each $i \in N$ can conceivably choose, and consequently the sequential equilibria eligible for selection.

We define additional auxiliary games of G, where players' decisions depend on certain collections of nodes. Therefore, write a behavioral strategy $\beta_i = (\beta_{i,1}, \beta_{i,2}, \ldots, \beta_{i,k}, \ldots, \beta_{i,K})$, where $\beta_{i,k}$ is the probability distribution over the actions $A_{i,k}(h_{i,k})$ available at $h_{i,k} \in H_{i,k}$, such that, $H_{i,k} \supset X_{i,k}$, and $X_{i,k}$ is the set of nodes of player i at level k of the game tree of G. If the set $X_{i,k} = \emptyset$, then $\beta_{i,k}$ is empty.

Definition 7 (Auxiliary game $\Gamma(k)$)**.**

The tuple $\Gamma(k) = \langle G, \mathcal{B}(k), \mathscr{S} \rangle$ *with* $k > 1$ *is an auxiliary game* $\Gamma(k)$ *of* G*, where* $\mathcal{B}_i(k) \subset \mathcal{B}_i$*. Write a behavioral strategy* $\beta_i(k) \in \mathcal{B}_i(k)$ *as* $\beta_i(k) = (\beta_{i,1}, \ldots, \beta_{i,k-1}, \beta_{i,k}, \ldots, \beta_{i,K})$*, where* $(\beta_{i,1}, \ldots, \beta_{i,k-1})$ *are fixed, and* $(\beta_{i,k}, \ldots, \beta_{i,K})$ *are player* i*'s probability distributions over actions available at the nodes in* $\bigcup_{\ell=k}^{K} X_{i,\ell}$*. For simplicity, we write* $X_i(k) = \bigcup_{\ell=k}^{K} X_{i,\ell}$*, denote as* $A_i(k)$ *the set of available actions at those nodes, and as* $H_i(k)$ *the collection of information sets.*

In Definition 7, a fixed $(\beta_{i,1}, \ldots, \beta_{i,k-1})$ means that player i has already chosen those distributions as part of the behavioral strategy $\beta_i(k)$. Let $\Gamma = \{\Gamma(k)\}_{k=1}^{K}$ be the collection of all auxiliary games of G, which we call its full auxiliary structure. We then make one final assumption.

Assumption 5.

Fix N *and* $\mathcal{B}(1), \ldots, \mathcal{B}(K)$*; let* $\mathcal{G}$ *be a collection of all auxiliary structures* Γ *of* G *with a set of players* N *and spaces of behavioral strategies* $\mathcal{B}(1), \ldots, \mathcal{B}(K)$*. Then each* $i \in N$ *uses a selection mechanism* $\mathfrak{s}_i : \mathcal{G} \rightrightarrows \mathscr{S}$*, such that,* $\mathfrak{s}_i(\Gamma)$ *includes the sequential equilibria selected by player* i *in* G*.*[10]

The intuition for Assumption 5 is that each $i \in N$ do not only conduct a process of selection ex-ante, which corresponds to the application of the selection mechanism to the auxiliary game $\Gamma(1)$; he also conducts such selection process in several interim stages, each one corresponding to each of several auxiliary games in $\Gamma \backslash \Gamma(1)$.

3 Criteria and selection

In this section, we characterize the process of equilibrium selection. We start by defining the criteria of risk and payoff dominance given the sequential nature of decision making, and the method's assumptions. Afterwards we describe the process of selection of a solution. We omit, whenever that information is available in Silva (2019), the intuition and details of some elements and the proofs of some results.

[10] As in static games, when we say that $i \in N$ chooses or selects a sequential equilibrium $(\beta, \mu) \in \mathscr{S}$, we mean that player i chooses an equilibrium behavioral strategy $\beta_i := \mathrm{proj}_i \beta$ considering that every $j \neq i$ choose equilibrium behavioral strategies $\beta_j := \mathrm{proj}_j \beta$.

3.1 Dominance criteria

Write a sequential equilibrium $(\beta,\mu) \in \mathscr{S}$ as $\big(\beta_i, \beta_{-i}; \mu\big)$, where $\beta_{-i} := (\beta_j)_{j\neq i}$, and each $\beta_j := \mathrm{proj}_j\beta$, and $\beta_i := \mathrm{proj}_i\beta$. Assume, for now, that $\mathscr{S}$ is finite. In each $\Gamma(k) \in \Gamma$, write as $u_i(\beta|\mu, x)$ the expected payoff of (β,μ) for player i, given that all the players' follow β and the play of the game is at $x \in X_k$. Denote as $\mathcal{H}_i(k) = \{h_i \in H_i(k)|\mu_\beta(x) > 0, x \in h_i\}$ the information sets that the play of the game reaches with positive probability when the players' beliefs are given by μ and the strategies are in β. In the auxiliary game $\Gamma(1)$, we just write $u_i(\beta|\mu)$ since the play of the game is at $x_0 \in W$; that auxiliary game corresponds to the ex-ante version of G. Considering the auxiliary games in Γ, we introduce the concept of best-deviation in each of them.

Definition 8 (Best-deviation)**.**
In an auxiliary game $\Gamma(k)$, given that the play of the game is at $x \in X_k$, the best-deviation by player i from an equilibrium strategy β_i in $(\beta_i, \beta_{-i}; \mu) \in \mathscr{S}$ is a strategy $\beta_i^1(k) \in \mathcal{B}_i(k)\backslash\{\beta_i\}$, such that, for every $\beta_i'(k) \in \mathcal{B}_i(k)\backslash\{\beta_i, \beta_i^1(k)\}$,

$$v_i(\beta_i, \beta_{-i}|\mu, h_i) \geq v_i(\beta_i^1(k), \beta_{-i}|\mu, h_i) \geq v_i(\beta_i'(k), \beta_{-i}|\mu, h_i), \quad \forall h_i \in \mathcal{H}_i(k).$$

This best-deviation induces a relative utility loss $\lambda_{\beta,\mu}(\beta_i^1(k))$, where

$$\lambda_{\beta,\mu}(\beta_i^1(k)) = \frac{u_i(\beta_i, \beta_{-i}|\mu, x) - u_i(\beta_i^1(k), \beta_{-i}|\mu, x)}{u_i(\beta_i, \beta_{-i}|\mu, x) - u_i(x^o)}.$$

We now generalize the concept of best-deviation considering that $\mathcal{B}_i(k)$ in each $\Gamma(k) \in \Gamma$ does not have necessarily just two behavioral strategies.

Definition 9 (Best-deviation generalization)**.**
In an auxiliary game $\Gamma(k)$, given that the play of the game is at $x \in X_k$, let $|\mathcal{B}_i(k)\backslash\{\beta_i\}| = m_i^k$; the ℓ^{th} best-deviation by player i from an equilibrium strategy β_i in $(\beta_i, \beta_{-i}), \mu) \in \mathscr{S}$ is a strategy $\beta_i^\ell(k) \in \mathcal{B}_i(k)\backslash\{\beta_i\} \cup \{\beta_i^h(k)\}_{h=1}^{\ell-1}$, such that, for every strategy $\beta_i'(k) \in \mathcal{B}_i(k)\backslash\{\beta_i\} \cup \{\beta_i^h(k)\}_{h=1}^{\ell}$,

$$v_i(\beta_i^\ell(k), \beta_{-i}|\mu, h_i) \geq v_i(\beta_i'(k), \beta_{-i}|\mu, h_i), \quad \forall h_i \in \mathcal{H}_i.$$

This best-deviation induces a relative utility loss $\lambda_{\beta,\mu}(\beta_i^\ell(k))$, *where*

$$\lambda_{\beta,\mu}(\beta_i^\ell(k)) = \frac{u_i(\beta_i, \beta_{-i}|\mu, x) - u_i(\beta_i^\ell(k), \beta_{-i}|\mu, x)}{u_i(\beta_i, \beta_{-i}|\mu, x) - u_i(x^o)}.$$

If $\mathcal{B}_i(k)\backslash\{\beta_i\} \cup \{\beta_i^h(k)\}_{h=1}^{\ell} = \emptyset$, *then* $\ell = m_i^k$, *and player* i *has no more possible deviations.*

Denote as $\delta_i^k(\beta,\mu) = \{\beta_i^\ell(k)\}_{\ell=1}^{m_i^k}$ the set of all possible deviations by player i from $(\beta,\mu) \in \mathscr{S}$ in $\Gamma(k) \in \Gamma$. The collection of all possible deviations from that sequential equilibrium by every $i \in N$ in $\Gamma(k)$ is,

$$\delta^k(\beta,\mu) = \bigcup_{i \in N} \delta_i^k(\beta,\mu).$$

Let the collection without player i's deviations be,

$$\delta_{-i}^k(\beta,\mu) = \bigcup_{j \neq i} \delta_j^k(\beta,\mu).$$

Consider a collection $\mathfrak{P}\big(\delta_{-i}^k(\beta,\mu)\big)$ of all possible combinations of deviations in $\delta_{-i}^k(\beta,\mu)$ in $\Gamma(k)$, which satisfies the following conditions:

a) for every set $\bar{\delta}^k(\beta,\mu) \in \mathfrak{P}\big(\delta_{-i}^k(\beta,\mu)\big)$, let $|\bar{\delta}^k(\beta,\mu) \cap \delta_i^k(\beta,\mu)| = \{0,1\}$, and $0 < |\bar{\delta}^k(\beta,\mu)| \leq n$, such that $\delta_{-i}^k(\beta,\mu) \notin \mathfrak{P}\big(\delta_{-i}^k(\beta,\mu)\big)$;

b) $\mathfrak{P}\big(\delta_{-i}^k(\beta,\mu)\big) \subset \mathscr{P}\big(\delta_{-i}^k\big(\beta,\mu\big)\big)$, in which $\mathscr{P}\big(\delta_{-i}\big(\beta,\mu\big)\big)$ is the power set of $\delta_{-i}^k(\beta,\mu)$.

We define a function,

$$r_i : \mathfrak{P}\big(\delta_{-i}^k(\beta,\mu)\big) \to [0,1],$$

such that, for any $\bar{\delta}^k(\beta,\mu) \in \mathfrak{P}\big(\delta_{-i}^k(\beta,\mu)\big)$ we have,

$$\bar{\delta}^k(\beta,\mu) \mapsto r_i\big(\bar{\delta}^k((\beta,\mu))\big) = \prod_{j \neq i} \lambda_{\beta,\mu}(\beta_j^\ell(k)), \quad \forall \beta_j^\ell(k) \in \bar{\delta}^k(\beta,\mu).$$

Using the function $r_i(\cdot)$, we can identify in each $\Gamma(k) \in \Gamma$ the degrees of risk of every $(\beta,\mu) \in \mathscr{S}$ to player i.

Definition 10 (Degrees of risk)**.**

The risk of degree 1 *of a sequential equilibrium to* $i \in N$ *in* $\Gamma(k)$ *is a function* $\rho^1_{i,k} : \mathscr{S} \to [0,1]$*, such that on* $(\beta,\mu) \in \mathscr{S}$*,*

$$\rho^1_{i,k}(\beta,\mu) = \begin{cases} 1 - r_i\big(\bar{\delta}^k(\beta,\mu)\big), & \textit{for } r_i\big(\bar{\delta}^k(\beta,\mu)\big) \le r_i\big(\bar{\delta}^{k\prime}(\beta,\mu)\big), \forall \bar{\delta}^{k\prime}(\beta,\mu) \in \mathfrak{P}\big(\delta^k_{-i}(\beta,\mu)\big), \\ 0, & \textit{if } \mathfrak{P}\big(\delta^k_{-i}(\beta,\mu)\big) = \emptyset. \end{cases}$$

Let $\bar{m}^k = \max_{i \in N} |\mathfrak{P}\big(\delta^k_{-i}(\beta,\mu)\big)|$*; the risk of degree* ℓ *of a sequential equilibrium to* $i \in N$ *is a function* $\rho^\ell_{i,k} : \mathscr{S} \to [0,1]$*, such that,* $\ell \le \bar{m}^k$*, and on* $(\beta,\mu) \in \mathscr{S}$*,*

$$\rho^\ell_{i,k}(\beta,\mu) = \begin{cases} 1 - r_i\big(\bar{\delta}^k(\beta,\mu)\big), & r_i\big(\bar{\delta}^k(\beta,\mu)\big) \le r_i\big(\bar{\delta}^{k\prime}(\beta,\mu)\big), \forall \bar{\delta}^{k\prime}(\beta,\mu) \in \mathfrak{P}\big(\delta^k_{-i}(\beta,\mu)\big) \backslash \big\{\bar{\delta}^{k^h}(\beta,\mu)\big\}_{h=1}^{\ell-1}, \\ 0, & \textit{if } \mathfrak{P}\big(\delta^k_{-i}(\beta,\mu)\big) \backslash \big\{\bar{\delta}^{k^h}(\beta,\mu)\big\}_{h=1}^{\ell-1} = \emptyset. \end{cases}$$

We unify the degrees of risk of $(\beta,\mu) \in \mathscr{S}$ to player i in $\Gamma(k) \in \Gamma$ into a unique measure.

Definition 11 (Risk)**.**

The risk of a sequential equilibrium to $i \in N$ *in* $\Gamma(k)$ *is a function* $\rho_{i,k} : \mathscr{S} \to \mathbb{R}$*, such that on* $(\beta,\mu) \in \mathscr{S}$*,*

$$\rho_{i,k}(\beta,\mu) = \sum_{j=1}^{\bar{m}^k} \eta^j_{i,k}(\beta,\mu)\rho^j_{i,k}(\beta,\mu).$$

Each weight $\eta^j_{i,k}(\beta,\mu) \in [0,1]$ *is an element of a finite monotonic weakly decreasing sequence of weights* $\eta_{i,k}(\beta,\mu) = \{\eta^\ell_{i,k}(\beta,\mu)\}_{\ell=1}^{\bar{m}^k}$*, such that, for* $j \neq h$*,*

$$\eta^j_{i,k}(\beta,\mu) = 1 - \sum_{h \in \{1,\ldots,\bar{m}^k\} \backslash j}^{\bar{m}^k} \eta^h_{i,k}(\beta,\mu), \ \textit{and } \eta^h_{i,k}(\beta,\mu) = \frac{\rho^h_{i,k}(\beta,\mu)}{\rho^j_{i,k}(\beta,\mu)} \eta^j_{i,k}(\beta,\mu),$$

and,

$$\sum_{j=1}^{\bar{m}^k} \eta^j_{i,k}(\beta,\mu) = 1.$$

We assume that the risk of $(\beta,\mu) \in \mathscr{S}$ is fully characterized by deviations of each player i from their equilibrium strategies both on and off-the-equilibrium path. We could have chosen a different route, looking instead to deviations in the beliefs (e.g., a mistake or tremble). However, that approach is more complex, given the necessary restrictions on the mistakes that a player can eventually make. Since there would exist a bijection between the strategies that are sequentially rational to those incorrect beliefs, and those that are best-deviations, we opted for the latter approach. We can now define the criterion of risk-dominance on $\Gamma(k)$.

Definition 12 (Risk-dominance)**.**
In $\Gamma(k)$, each $i \in N$ has a binary relation $\succsim^{\rho}_{i,k}$, a risk-dominance relation, over $\mathscr{S}$, such that for any $(\beta^{'}, \mu^{'}), (\beta^{''}, \mu^{''}) \in \mathscr{S}$, $(\beta^{'}, \mu^{'})$ risk dominates $(\beta^{''}, \mu^{''})$, which we write $(\beta^{'}, \mu^{'}) \succ^{\rho}_{i,k} (\beta^{''}, \mu^{''})$, if $\rho_{i,k}(\beta^{'}) \leq \rho_{i,k}(\beta^{''})$.

The risk-dominance relation $\succsim^{\rho}_{i,k}$ is a preference relation according to Proposition 1 in Silva (2019). As for the criterion of payoff dominance, we adapt the definition in Silva (2019) to a different class of equilibria.

Definition 13 (Weight of an equilibrium payoff)**.**
In $\Gamma(k)$, given that the play of the game is at $x \in X_k$, the weight of the payoff of a sequential equilibrium to player i is a function $w_{i,k} : \mathscr{S} \to \mathbb{R}$, such that, on $(\beta, \mu) \in \mathscr{S}$ we have,

$$w_{i,k}(\beta, \mu) = \frac{u_i(\beta|\mu, x) - u_i(x^o)}{\sum\limits_{(\beta', \mu') \in \mathscr{S}} \big(u_i(\beta'|\mu', x) - u_i(x^o)\big) + 1}.$$

We define the criterion of payoff-dominance as follows.

Definition 14 (Payoff-dominance)**.**
In $\Gamma(k)$, each $i \in N$ has a binary relation $\succsim^{u}_{i,k}$ over $\mathscr{S}$, such that for any $(\beta, \mu), (\beta', \mu') \in \mathscr{S}$, (β, μ) payoff dominates (β', μ'), which we write $(\beta, \mu) \succsim^{u}_{i,k} (\beta', \mu')$, if $w_{i,k}(\beta, \mu) \geq w_{i,k}(\beta', \mu')$.

Remark 1.
Take $(\beta, \mu), (\beta', \mu') \in \mathscr{S}$, and let (β, μ) be risk and payoff dominant over (β', μ') for each $i \in N$ in every $\Gamma(k) \in \Gamma$; hence, (β', μ') is not eligible for selection, and the set $\mathcal{B}_i = \mathscr{B}_i \cap proj_i \mathcal{E} \backslash \beta'$ if $\beta \neq \beta'$ for every $i \in N$. Since (β', μ') is not the solution of G, deviations by the players to equilibrium strategies part of β' do not matter in the process of equilibrium selection (in Silva (2019), we provide the intuition for the set $\mathcal{B}_i$, which completes this remark).

Remark 2.

We do not restrict the dimension of the set $\mathscr{S}$ *in* G*. However, we only apply the selection mechanism when* $\mathscr{S}$ *has an infinite number of sequential equilibria if only a finite number of them are not risk and payoff dominated for every player* $i \in N$*. We do not make this remark in Silva (2019), but it is trivial to verify that it is also valid in static games.*

3.2 Equilibrium selection

The process of equilibrium selection in G is not exactly the same as in Silva (2019). The difference is that, due to the sequential nature of decision-making, we consider several stages of evaluation of an equilibrium, i.e., ex-ante and interim stages. Nonetheless, the intuition for the selection of a solution in static and dynamic games remains similar. We start the characterization of the selection process of a sequential equilibrium in $\Gamma(k)$ with the definition of the concept of premium.

Definition 15 (Premium in $\Gamma(k)$)**.**

The premium $\pi_{i,k}$ *of a sequential equilibrium in* $\Gamma(k)$ *to player* i *is a function* $\pi_{i,k} : \mathscr{S} \to \mathbb{R}$*, such that, at* $(\beta, \mu) \in \mathscr{S}$ *we have,*

$$\pi_{i,k}(\beta, \mu) = \frac{1}{1 + w_{i,k}(\beta, \mu)} f_i\Big(1 + \rho_{i,k}(\beta, \mu)\Big).$$

The function $f_i : [0, \infty) \to [1, \infty)$*, which is the same in every* $\Gamma(k) \in \Gamma$*, describes player* i*'s perception of the risk of a sequential equilibrium, being increasing, continuous, and either: (i) convex, with a slope greater or equal than one in its entire domain; (ii) concave, with a slope less or equal than one in its entire domain; (iii) linear, with a slope of exactly one in its entire domain.*

In each $\Gamma(k) \in \Gamma$, every $(\beta, \mu) \in \mathscr{S}$ has a particular premium $\pi_{i,k}(\beta, \mu)$ for player i. We can now properly construct the premium of a sequential equilibrium in G.

Definition 16 (Premium in G)**.**

Consider the collection $\{\pi_{i,k}(\beta, \mu)\}_{k=1}^{K}$ *of the premia of* $(\beta, \mu) \in \mathscr{S}$ *to player* i *in each* $\Gamma(k) \in \Gamma$*. Its premium in* G *is,*

$$\pi_i(\beta, \mu) = \sum_{k=1}^{K} \nu_{i,k}(\pi_{i,k}(\beta, \mu))\pi_{i,k}(\beta, \mu).$$

Each weight $\nu_{i,k}(\pi_{i,k}(\beta,\mu)) \in [0,1]$ is an element of a sequence $\nu_i(\pi_i(\beta_\mu)) = \{\nu_{i,k}(\pi_{i,k}(\beta,\mu))\}_{k=1}^{K}$, and is computed as the weights $\eta_{i,k}(\cdot)$ in Definition 11, by substituting each $\eta_{i,k}^{j}(\cdot)$ by $\nu_{i,k}(\cdot)$, the risk function $\rho_{i,k}^{j}(\cdot)$ by the premium $\pi_{i,k}(\cdot)$, and the size of the sequence $\bar{m}_i^k$ by K.

The premium of $(\beta,\mu) \in \mathscr{S}$ to player i is the weighted sum of the premia in every $\Gamma(k) \in \Gamma$. Hence, we define the process of equilibrium selection of player i as a problem of minimization of that premium, such that,

$$\begin{aligned} \min_{(\beta,\mu)\in\mathscr{S}} \quad & \pi_i(\beta,\mu) \\ \text{s.t.} \quad & \tau_i(\pi_i(\beta,\mu)) + \sum_{j\neq i}\tau_j(\pi_j(\beta,\mu)) \leq \tau_i(\pi_i(\beta',\mu')) + \sum_{j\neq i}\tau_j(\pi_j(\beta',\mu')), \forall(\beta',\mu') \in \mathscr{S} \\ & \pi_i(\beta,\mu), \quad \forall(\beta,\mu) \in \mathscr{S}. \end{aligned} \tag{2}$$

The function $\tau_i : [0,y) \to [0,\infty)$ is concave with $\tau_i(x) = 0$ in $x \in \{0,y\}$, $y \in (0,\infty)$, and $D\tau_i(x) = 0$ in a unique point $x^* \in (0,y)$, where D is a differential operator. For the reasons in Silva (2019), we can just focus on the global selection problem,

$$\begin{aligned} \min_{(\beta,\mu)\in\mathscr{S}} \quad & \sum_{i\in N}\tau_i(\pi_i(\beta,\mu)) \\ \text{s.t.} \quad & \tau_i(\pi_i(\beta,\mu)), \quad \forall(\beta,\mu) \in \mathscr{S}, \forall i \in N. \end{aligned} \tag{3}$$

We write $\pi(\beta,\mu) = \sum_{i\in N}\tau_i(\pi_i(\beta,\mu))$, and call it the aggregate premium of $(\beta,\mu) \in \mathscr{S}$. The solutions of the problem in Equation (3) are consequently the solutions of G.

Definition 17 (Solution)**.**
Let $\arg\min_{(\beta,\mu)\in\mathscr{S}} \pi(\beta,\mu) = \mathfrak{s}$; then $\mathfrak{s} = \prod_{i\in N}\mathfrak{s}_i$ is the set of the solutions of G.

The set $\mathfrak{s}$ is trivially nonempty. One alternative to this process of selection would be the definition of minimization problem in Equation (3) in each $\Gamma(k)$. If we would write the solution of each $\Gamma(k)$ as $\mathfrak{s}_k$, we could define the solution of G as the set $\mathfrak{s} = \bigcap_{k=1}^{K}\mathfrak{s}_k$. The caveat is that $\mathfrak{s}$ may be empty; the reason is the dependency of the selection procedure on the payoff cardinality. Therefore, we opt for a a procedure whose solution minimizes the weighted average of the risk of a sequential equilibrium in the different auxiliary games, thus reflecting the different stages of interim reasoning. Considering $\pi(\cdot)$, we now define a binary relation $\succsim^{\pi}$ over $\mathscr{S}$.

Definition 18 (Global preferences)**.**

There is a binary relation $\succsim^{\pi}$ *over* $\mathscr{S}$*, such that, for any* $(\beta,\mu),(\beta',\mu') \in \mathscr{S}$*,* (β,μ) *is globally preferred to* (β',μ')*, which we write* $(\beta,\mu) \succsim^{\pi} (\beta',\mu')$*, if* $\pi(\beta,\mu) \leq \pi(\beta',\mu')$*.*

The binary relation $\succsim^{\pi}$ is a total order over $\mathscr{S}$ (see Proposition 3 in Silva (2019)). The conditions for the uniqueness of a solution in G are analogous to those in static games, and the same holds as to the class in which almost all games have a unique solution (see Proposition 4 in Silva (2019)).

3.3 Structure and Symmetry

In this section, we check if the existence of symmetries between elements of the game has any effect on the process of equilibrium selection; we determine that by checking for the invariance of a solution to isomorphic transformations of the extensive-form. In strategic-form games, the definition of an isomorphism follows from Harsanyi and Selten (1988), which is unique and unambiguous; however, in extensive-form games that is not the case. In particular, the literature distinguishes between two main concepts. The first one is the concept of strong isomorphism (Peleg, Rosenmüller, & Sudhölter, 1999), which corresponds to a bijective mapping between the sets of nodes of two extensive-form games, while preserving the order of the moves.

Definition 19 (Strong isomorphism)**.**

A strong isomorphism between games $G = \langle X, \succ, N, W, (X_i, H_i, A_i, u_i)_{i\in N}, p\rangle$ *and* $\tilde{G} = \langle \tilde{X}, \succ, \tilde{N}, \tilde{W}, (\tilde{X}_i, \tilde{H}_i, \tilde{A}_i, \tilde{u}_i)_{i\in N}, \tilde{p}\rangle$ *is a bijection* $\phi : G \to \tilde{G}$ *which satisfies the following properties:*

(A) For every $x, x' \in X$*, if* $x' \succ x$*, then* $\phi(x') \succ \phi(x)$*;*

(B) There exists a bijection $t : N \to \tilde{N}$*;*

(C) $\phi(X_i) = \tilde{X}_{t(i)}$ *and* $\phi(H_i) = \tilde{H}_{t(i)}$ *for every* $i \in N$*;*

(D) $\phi(W) = \tilde{W}$ *and* $p^{x_0}(x) = p^{\phi(x_0)}\big(\phi(x)\big)$ *for every* $x_0 \in W$ *and* $x \succeq x_0$*;*

(E) If $C(h_i) = \bigcup_{x\in h_i} C(x)$*, then* $C\big(\phi(h_i)\big) = \bigcup_{\phi(x)\in\phi(h_i)} C\big(\phi(x)\big)$*. Hence, there is a bijection* $\mathfrak{t} : A \to \tilde{A}$*, such that* $\mathfrak{t}(A_i(h_i)) = \tilde{A}_{t(i)}\big(\phi(h)_{t(i)}\big)$ *and* $\mathfrak{t}(A_i(x)) = \tilde{A}_{t(i)}\big(\phi(x)\big)$ *for every* $i \in N$ *and* $h \in H$*;*

(F) $u_{t(i)}\big(\phi(z)\big) = \theta_i u_i(z) + \vartheta_i$*, for* $z \in Z$*, in which* $\theta_i > 0$ *and* $\vartheta_i \in \mathbb{R}$ *are constants.*

Condition (A) implies that the trees of two isomorphic games are isomorphic. Condition (B) renames the players, and conditions (C) and (D) imply that the set of nodes and the information sets of each player are mapped onto those of a single player in $\tilde{G}$, and that the initial nodes controlled by Nature are independent of that transformation if they have the same probability distribution on both G and $\tilde{G}$. By condition (E), the actions are mapped jointly with the nodes. In the last condition, since $\phi(Z) = \tilde{Z}$, the preferences of a player in G are mapped onto the preferences of a single player in $\tilde{G}$, whose payoffs are a positive affine transformation of those in G. We provide in Figure 2 an example of a strong isomorphism.

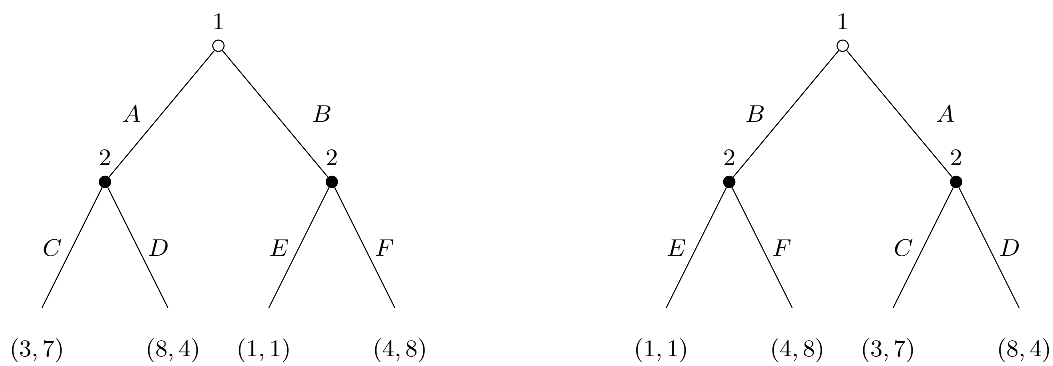

Figure 2: Game G (left), and game $\tilde{G}$ (right) following a strong isomorphism.

Because this isomorphism preserves the order of the moves, we cannot successfully apply this transformation to every dynamic game; it does not preserve strategic symmetries when common to both the strategic and the extensive-form representations. Hence, it is incompatible with the existence of multiple extensive-form representations of a strategic-form game. To overcome this issue, Peleg, Rosenmüller, and Sudhölter (2000) propose an alternative concept, which they apply to the canonical extensive-form of a game. In this representation, a chance mechanism selects, without the players' awareness, one among multiple possible extensive-form representations of a game, thus connecting a player's sets of nodes and information sets of each representation. This concept guarantees the representation of symmetries, but increases the complexity of the game representation, which is less intuitive, and of the isomorphic transformation. A sensible alternative to it is the concept of weak isomorphism (Casajus, 2001). It is based on Selten (1983) notion of symmetry in two-player extensive-form games, corresponding to a bijective mapping between actions instead of nodes. In comparison to a strong isomorphisms, it just keeps the the strategical content of the order of moves, to what he calls its essential extent; consequently, that allows to preserve the equilibria of the game. This transformation is especially sensible if we do not regard the sequence of moves as being a real one.

Denote a path as $\jmath(x) = \{x, x' \in X \,|\, x' \succ x, x' \in X \backslash W\}$, which contains x and all preceding nodes with exception of the initial one. Take $a(\jmath(z)) = \{a \in A \,|\, a \in A(x), x \in \jmath(z)\}$ as the set of the actions that lead to z, and whose path is $\jmath(z)$.

Definition 20 (Weak isomorphism)**.**

A weak isomorphism between two games $G = \langle X, \succ, N, W, (X_i, H_i, A_i, u_i)_{i \in N}, p \rangle$ *and* $\tilde{G} = \langle \tilde{X}, \succ, \tilde{N}, \tilde{W}, (\tilde{X}_i, \tilde{H}_i, \tilde{A}_i, \tilde{u}_i)_{i \in N}, \tilde{p} \rangle$ *is a bijection* $\varphi : G \to \tilde{G}$ *satisfying the following properties:*

(a) There are bijections $t^1 : N \to \tilde{N}$*,* $t^2 : H \to \tilde{H}$*,* $t^3 : W \to \tilde{W}$*, and* $t^4 : Z \to \tilde{Z}$*;*

(b) $\varphi(A(h)) = \tilde{A}(t^2(h))$*,* $\varphi(A_i) = \tilde{A}_{t^1(i)}$*,* $\varphi(A(x_0)) = \tilde{A}(t^3(x_0))$*, for every* $i \in N$*,* $h \in H$*,* $x_0 \in W$*;*

(c) If $p^{x_0}(x) = p(a)$*, then* $p(a) = p(\varphi(a))$*, where* $a \in A(x_0)$*;*

(d) $\varphi(a(\jmath(z))) = \tilde{a}(\jmath(t^4(z)))$ *for every* $z \in Z$*;*

(e) $u_{t(i)}(t^4(z)) = \theta_i u_i(z) + \vartheta_i$*, for* $z \in Z$*, in which* $\theta_i > 0$ *and* $\vartheta_i \in \mathbb{R}$ *are constants.*

Comparing both isomorphisms, condition (e) is equivalent to condition (F), while conditions (b), and (c) correspond to conditions (C) and (D). The bijections in condition (a) are equivalent to condition (B) and to the bijection ϕ, while condition (E) is related to the bijection φ. The difference between the isomorphisms lies in conditions (A) and (d). The latter condition requires plays given by a set of certain actions in G to map to plays given by specific actions in $\tilde{G}$; thus, it requires a path to contain the same actions leading to z, but not necessarily in the same order, as condition (A) requires. In Figure 3 we give an example of a weak isomorphism.

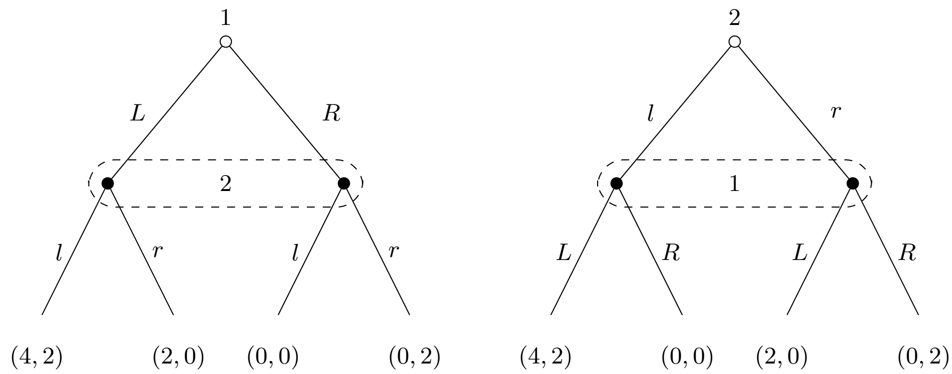

Figure 3: The games G (left), and $\tilde{G}$ (right) following a weak isomorphism.

According to Casajus (2001, Corollaries 3.4.1 and 3.4.2), both a weak and a strong isomorphism are equivalent under certain game path conditions. In a later work, Casajus (2006) further relaxes the latter concept, which he calls a super weak isomorphism. In that case, the isomorphism is independent of the distribution over the initial nodes, and of the assignment of payoffs to terminal nodes. He shows that such isomorphism preserves the set of sequential equilibria, and it is equivalent to an isomorphism of the agent-normal form for a larger number of games than a weak isomorphism.

In general dynamic games, the sequence of moves may have some importance. However, given the complexity of the analysis of games in canonical form, for simplicity, we pair the analysis of the effect of a strong and a weak isomorphism on the mechanism of selection. We start with the effect of the former type of isomorphism on $\succsim^{\rho}_{i,k}$ in $\Gamma(k)$.

Proposition 1.

$\succsim^{\rho}_{i,k}$ *in every* $\Gamma(k) \in \Gamma$ *is invariant to a strong isomorphism* ϕ*.*

As for the effect of a weak isomorphism on $\succsim^{\rho}_{i}$, we start by noticing that the order of moves is not preserved, in the sense of Definition 19. However, according to Casajus (2001, theorem 3.3.7.), this isomorphism still preserves the set of sequential equilibria. In that case, we say that the isomorphism preserves the order of moves to its essential extent. We notice that this approach inverts the standard paradigm in equilibrium analysis, since an equilibrium validates the concept, and not the other way around.

Proposition 2.

$\succsim^{\rho}_{i,k}$ *in every* $\Gamma(k) \in \Gamma$ *is invariant to a weak isomorphism* φ*.*

Considering the invariance properties of $\succsim^{\rho}_{i,k}$ in each $\Gamma(k)$, we obtain the following corollary.

Corollary 1.

$\succ^{\pi}$ is invariant to a strong isomorphism ϕ and weak isomorphism φ.

We have shown in Proposition 1 and 2 that $\succsim^{\rho}_{i,k}$ is invariant to both types of isomorphism in every $\Gamma(k)$. We have not done the same with respect to $\succsim^{w}_{i,k}$. However, such property follows from $\succsim^{w}_{i,k}$ being induced by a utility factor in each $\Gamma(k)$. Hence, also $\pi(\cdot)$ is invariant to those isomorphisms, since such relation is a combination of the risk and the equilibrium payoff weight in each auxiliary game.

3.4 Strategic equivalence

We conclude this section with a discussion on the sensitivity of $\succ^{\pi}$ and $\mathfrak{s}$ to elementary operations of the extensive-form of a game. We aim at verifying if the solutions in the extensive-form and strategic-form representations of a game necessarily coincide.

The elementary operations - inflate-deflate, addition of a superfluous node, coalescing of information sets, and interchange of moves - proposed by Thompson (1952), when applied in the correct order, transform the extensive-form game into its reduced normal-form. For that reason, according to Kohlberg and Mertens (1986), the latter type of representation is sufficient for the analysis of a game. However, that conclusion is arguable. On the one hand, the set of sequential equilibria is not invariant to some elementary operations. On the other hand, two of these operations - inflation-deflation and the addition of a superfluous decision node - may eliminate perfect recall. For the latter reason, we focus on the elementary operations proposed by Elmes and Reny (1994) - interchange of decision nodes, coalescing of information sets, and addition of superfluous decision nodes - which preserve perfect recall[11]. We properly characterize each of these operations in Appendix A.2

We obtain that $\succ^{\pi}$ and $\mathfrak{s}$ are not independent of a transformation of the extensive-form representation of a game into its reduced normal-form because they are sensitive to one particular elementary operation.

[11] There are other approaches in the strategic equivalence literature in extensive-form games, which are still relevant, such as Bonanno (1992) transformation between games in set-theoretical and extensive forms, and Hoshi and Isaac (2010) extension of Thompson (1952) transformations to games with unawareness.

Proposition 3.

$\succ^{\pi}$ is not invariant to the coalescing of information sets.

We do not prove it here, but we could show that $\succ^{\pi}$ is invariant to the other two elementary operations. Regarding the interchange of decision nodes, the invariance follows from $\succ^{\pi}$ being invariant to a weak isomorphism. Concerning the addition of superfluous decision nodes, the invariance of $\succ^{\pi}$ follows from such operation not changing the number of actions, behavioral strategies, and game tree levels.

4 The Spence signaling game

We now apply the selection method to the most basic version of Spence (1973) job market signaling game, in which there are only two types of worker (high and low productive types). We have chosen an application to this model for several reasons. On the one hand, it is a game with economic meaning, which compensates for more abstract games, as the ones we provide in Silva (2019). On the other hand, the structure of the game is simple, and we do not need to make many modifications on the method to general extensive-form games. Its mathematical structure is common to other economic signaling models, which would allow to extrapolate some arguments and conclusions to those models.

4.1 Notation and equilibria

In a general extensive-form game, players select one or more sequential equilibria as the solutions of the game. In a signaling game, however, a player's strategy and payoff, hence the respective deviations and losses in utility, depend on the type representing him in the game. We assume, as in Harsanyi (1995b), that the various types of each $i \in N$ in a signaling game are those who actually select a solution; hence, the strategies and the payoffs of a player, and naturally his deviations, are the strategies and payoffs of his types.

The most simple version of Spence's signaling game has one worker, which is one of possible $\theta = \{H, L\}$ types. The player's type determines his productivity, such that $H > L > 0$; we assume that L is high enough for every game payoff to be nonnegative. A chance move determines the player's type, such that, with prior probability p he is of type H, and with probability $1 - p$ he is of type L. The worker moves first by choosing a level of education $e \in [0, \infty)$. Two risk-neutral firms, upon observing the education choice e, simultaneously offer a wage $r \in [0, \infty)$ in a Bertrand competition regime. If both firms offer the same wage, the worker decides which offer to take according to the result of a fair coin toss. The payoff of a firm who does not attract the worker is zero; it is $\theta - r$ if the firm attracts a worker of type θ. A type θ payoff is $r - (e/\theta)$, where (e/θ) is the disutility of education. In equilibrium, the firms bid the expected productivity of the worker, such that,

$$\mathbb{E}(\theta|e) = \sum_{\theta=\{H,L\}} \mu(\theta|e)\theta, \quad e \in [0, \infty),$$

where $\mu(\cdot|e)$ is the firm's belief about the worker's type upon observing his level of education choice.

This game has three categories of sequential equilibrium. The first one contains separating equilibria, where different types of worker choose different levels of education, and each firm, upon correctly identifying the type, proposes wages matching the types' productivity. The level of education supporting a separating equilibrium is $e^* \in [e_1, e_2]$, where $e_1 > 0$ is the lowest level of education satisfying $H - (e^*/L) \leq L$, and $e_2 > e_1$ the highest level of education satisfying $H - (e^*/H) \geq L$. Each firm believes that an education level $e < e^*$ signals a type L, and $e \geq e^*$ a type H. Type L chooses an education level $e_0 = 0$, and obtains a payoff L; type H chooses an education level e^*, and his payoff is $H - (e^*/H)$. The incentive constraints which avoid a deviation by any of the types are for type L,

$$H - \frac{e^*}{L} \leq L,$$

and for type H,

$$H - \frac{e^*}{H} \geq L.$$

In the second category we include pooling equilibria. In this case, both types L and H pool in a level of education e^p. Since the firms cannot identify their types, they offer a wage that corresponds to the worker's prior expected productivity $pH + (1-p)L$. Several systems of beliefs support this equilibrium; for instance, one system implies that the firms consider any education level $e \neq e^p$ as signal of a type L. The worker's types do not have an incentive to deviate, in this case, if

$$pH + (1-p)L - \frac{e^p}{\theta} \geq L.$$

There are multiple pooling equilibria in which types pool in $e^p \in [0, b]$ with $b \geq 0$, since a firm's beliefs are not restricted off-the-equilibrium path.

The third category includes the hybrid equilibria, in which one or both types randomize between pooling or separating levels of education. An example of a hybrid equilibrium has type L separating, and type H randomizing. Hence, the former type chooses an education level $e = 0$, while the latter one randomizes between $e = 0$ with probability q, and an education level e^* with probability $1 - q$. The firm's posterior probability that the worker's type is H, given $e = 0$, by the Bayes rule, is

$$\frac{qp}{qp + (1-p)}.$$

The firm's offer a wage,

$$r = \frac{qp}{qp + (1-p)}H + \frac{1-p}{qp + (1-p)}L.$$

Type H is indifferent between education levels $e = 0$ and e^* if

$$\frac{qp}{qp + (1-p)}H + \frac{1-p}{qp + (1-p)}L - \frac{e^*(1-q)}{H} = H - \frac{e^*}{H}. \tag{4}$$

One system of beliefs that supports this equilibrium has the firms believing that any level of education $e' < e^*$ signals a type L, while a level of education $e'' > e^*$ leads to the update of the posterior probabilities of the worker being type L or H.

4.2 Solutions

Denote a separating equilibrium as $(e_0, e^*; r^s)$ where $e^* \in [e_1, e_2]$, a pooling equilibrium as $(e^p, e^p; r^p)$ with $e^p \in [0, b]$, and a hybrid equilibrium as $(\beta_L(e), \beta_H(e); r^h)$[12].

Because there is an infinite number of equilibria, we select a solution by seeking for an equilibrium which is dominant over all others for both types of worker according to the criterion of risk and payoff dominance. To simplify the analysis, due to the heavy notation of the model, we assume that the productivity of the types is $H = 2$ and $L = 1$. In van Damme and Güth (1991) application of Harsanyi and Selten (1988) selection model to this same version of the Spence signaling game, they assume instead that $H = 1$ and $L = 0$. In comparison, we apply an affine transformation of

[12]For simplification, we do not identify the system of beliefs supporting the equilibrium assessment.

these productivity levels; otherwise the utility functions are not continuous everywhere. Hence, we obtain that $e_1 = 1$, and $e_2 = 2$. The intuition for this setup is that type H is fully productive, while L is completely unproductive. Following the application of the selection mechanism, we obtain the following result.

Proposition 4.

The Spence signaling game has the following solutions:

- *if* $p > 1/2$, *the pooling equilibrium* $(e^p, e^p; r^p)$ *with* $e^p = 0$ *and firms' beliefs that any* $e \neq e^p$ *signals a type* L;
- *if* $p \leq 1/2$, *the separating equilibrium* $(e_0, e^*; r^s)$ *with* $e^* = e_1$.

We present the proof of Proposition 4 in the Appendix A.3.. This result only supports Cho and Kreps (1987) intuitive criterion prediction in the interval of the prior distribution $p \in [0, 1/2]$. The importance of this prior was also addressed in van Damme and Güth (1991). In that case, however, the predictions are different, since they support the pooling equilibrium $(e^p, e^p; r^p)$, in which $e^p = 0$, as the solution of the game if the prior probability $p \leq 1/2$, and the separating equilibrium $(e_0, e^*; r^s)$, in which $e^* = e_1$, if $p > 1/2$. They do not provide any economic intuition for the result; but they point that some of the sensibility of that result derives from the solution corresponding to Wilson (1977) E_2-equilibrium, that is, a sequential equilibrium that is the best outcome for type H[13]. However, it is not explicitly stated what is meant by being the best to type H. Nevertheless, our solution not only is the one that maximizes the expected payoff of type H in each of those intervals in the prior distribution, but it is also not possible to further improve neither his payoff nor the payoff of type L with the introduction of new contracts.

We can also provide some intuition for the result in Proposition 4. In both cases, $\mathcal{B}_L = \{\beta_L(0)\}$, where $\beta_L(0) = 1$, meaning that type L always chooses an education level of zero, and cannot enforce the selection of any particular equilibrium. Therefore, in both cases, type H selects his most preferred equilibrium. When $p > 1/2$, type H maximizes his expected payoff by participating of a pooling equilibrium, and when $p \leq 1/2$, the maximization of his payoff occurs in a separating

[13]We can describe E_2-equilibrium as a solution in which it is not possible to add any more contracts to the game, which are profitable, after all contracts that induce losses are eliminated from the set of contracts offered to the worker.

equilibrium. The idea, in the former case, is that when firms' have a high prior on a worker being high productive, type H's incentive to invest in education is low, as it becomes less important to signal his type. In the latter case, however, as the prior puts less mass on a type being highly productive, type H's maximization of the expected payoff depends on differentiation, which implies an investment on education to signal his type.

5 Conclusion

In this work, we extend Silva (2019) selection method from static to dynamic games, in which asymmetries of information are possible. The assumptions underlying the methods in both types of game are similar. The process of selection, however, differs, since in dynamic games we take into account the sequential nature of decision-making. Consequently, as we show Proposition 3, the solution of a game in extensive-form does not necessarily coincide with the solution in its respective reduced normal-form. The remaining properties, such as the invariance to the existence of symmetries between elements of the game, and the existence and uniqueness of a solution, also hold in this version of the method.

From the application of the method to Spence's signaling model we obtain a result partially concordant with Cho and Kreps (1987) intuitive criterion, when the prior probability of the worker being type H is $p \leq 1/2$; however, it contradicts the prediction from van Damme and Güth (1991) application of Harsanyi and Selten (1988). Our result mainly follows from type L not having any mean to enforce a particular equilibrium among those that are eligible for selection, since it is optimal for him to choose a null level of education. In that case, it is sensible to just focus on the choices of type H, which aim at maximizing his own expected payoff. That argument partially extends the equilibrium dominance criterion in Cho and Kreps (1987), justifying the match between our solution and the intuitive criterion for a given interval of the prior probability. With more than two types of worker, we conjecture that hybrid equilibria may play a more important role. We also believe that in that case, separation may also arise as a more likely solution, given the increase in importance of the criterion of risk-dominance. These are some questions that we intend to investigate in future research.

Appendix

Appendix A.1

Proof of Proposition 1.

We first show that the set $\delta_{i,k}(\beta,\mu)$ from $(\beta,\mu) \in \mathscr{S}$ by $i \in N$ is independent of a strong isomorphism. It then follows that $\rho_{i,k}(\cdot)$ also satisfies that property.

From condition (D), both W and the family of probability distributions p are independent of a strong isomorphism ϕ. By condition (C), the sets X_i and H_i of each $i \in N$ are mapped onto the sets of $t(i) \in t(N)$, which by condition (E) it also carries the feasible actions $A_i(h_i)$ at each $h_i \in H_i$. Condition (A) implies that the sequence of moves is the same. Hence, there is a bijection $g : \mathcal{B} \to \tilde{\mathcal{B}}$, such that $g(\mathcal{B}_i) = \tilde{\mathcal{B}}_{t(i)}$ for every $i \in N$. By condition (A), the order of moves leading to any $z \in Z$ is kept, and by condition (F) the payoff of such outcome to $t(i) \in t(N)$ is the same. Therefore, $\lambda(\cdot)$ from each deviation is the same. Therefore, we can propose another bijection $g' : \delta_k(\beta,\mu) \to \delta_k(\tilde{\beta},\tilde{\mu})$, such that, $g'(\delta_{i,k}(\beta,\mu)) = \delta_{t(i),k}(\tilde{\beta},\mu)$ for every $i \in N$, $(\beta,\mu) \in \mathscr{S}$, and $\Gamma(k) \in \Gamma$. Since we prove the invariance of $\rho_{i,k}(\cdot)$ to an affine transformation of the utility function in Silva (2019), we conclude that $\succsim^{\rho}_{i,k}$ is invariant to ϕ. $\square$

Proof of Proposition 2.

According to Casajus (2001, theorem 3.3.7.), a weak isomorphism φ preserves the set $\mathscr{S}$. We have seen that between a weak and strong isomorphism, only conditions (A) and (f) were not equivalent; those conditions concern the order of moves. In φ, considering player i's deviations $\delta_{i,k}(\beta,\mu)$ from $(\beta,\mu) \in \mathscr{S}$ in $\Gamma(k)$, we can propose a bijection $g'' : \delta_{i,k}(\beta,\mu) \to \delta_{t^1(i),k}(\beta,\mu)$, such that, the deviations and respective losses from a sequential equilibrium are independent of the order of the moves. Therefore, $\rho_{i,k}(\beta,\mu) = \rho_{t^1(i),k}(\tilde{\beta},\mu)$. Since $\rho_{i,k}(\cdot)$ is independent of an affine transformation of the payoff function in each $\Gamma(k) \in \Gamma$, then we conclude that $\succsim^{\rho}_{i,k}$ is independent of φ. $\square$

Proof of Proposition 3.

According to Kohlberg and Mertens (1986), the set $\mathscr{S}$ is not invariant to the coalescing of information sets. That would suffice to justify why $\succ^{\pi}$ is not invariant to that operation. However, we can show that the selection method, and not only its set of arguments, is affected by this operation. Assume that $\succ^{\pi}$ is invariant to the coalescing of information sets, given as a mapping $O^C : G \to G'$.

Denote as $\mathscr{S}_G$ and $\mathscr{S}_{G'}$ the sets of sequential equilibria in G and G', such that, $\mathscr{S}_G = \mathscr{S}_{G'}$, and $\mathscr{S}_G \subseteq \mathscr{S}$, and where $\mathscr{S}$ is the set of all the sequential equilibria of G which exist or not in G'. Let $\succ^{\pi}_G$ and $\succ^{\pi}_{G'}$ be total orders over $\mathscr{S}_G$ and $\mathscr{S}_{G'}$, which are equivalent, i.e., for every $(\beta, \mu), (\beta', \mu') \in \mathscr{S}_G$, if $(\beta, \mu) \succ^{\pi}_G (\beta', \mu')$, then $(\beta, \mu) \succ^{\pi}_{G'} (\beta', \mu')$. By definition, $\pi^G(\beta, \mu) \leq \pi^G(\beta', \mu')$ in G for every $(\beta, \mu), (\beta', \mu') \in \mathscr{S}_G$, which is equivalent to $\sum_{i \in N} \tau_i(\pi_i^G(\beta, \mu)) \leq \sum_{i \in N} \tau_i(\pi_i^G(\beta', \mu'))$. Write the premium of a sequential equilibrium to player i in G' as $\pi_i^{G'}(\cdot)$. Assume that G has K levels, and that G' just has $K-1$ levels; at some level k in G, we have $X_k = X_{i,k}$, and the nodes in $X_{i,k}$ are coalesced into $X_{i,k+1}$. Hence, for each $j \neq i$, we have $\pi_j^G(\beta, \mu) \neq \pi_j^{G'}(\beta, \mu)$, since the weights of the premium induced by (β, μ) in each auxiliary game of G and G' are $\nu_j^G(\pi_i(\beta_\mu)) \neq \nu_j^{G'}(\pi_i(\beta_\mu))$, and as long $\pi_{i,k}^G(\beta, \mu) \neq 0$. It is then not possible to guarantee that $\pi^G(\beta, \mu) \leq \pi^G(\beta', \mu')$ holds for every $(\beta, \mu), (\beta', \mu') \in \mathscr{S}_G$ in G', which concludes the proof. $\square$

Appendix A.2

An elementary operation is a transformation $O : G \to G'$ of the extensive-form representation of G, where $G = \langle X, \succ, N, W, H, A, u, p \rangle$ and $G' = \langle X', \succ, N', W', H', A', u', p' \rangle$. Let $F(x) = \{\tilde{x} \in X \,|\, \tilde{x} \succ x\}$ be the set of all successor nodes of x. Denote the continuation game of G at node x as G_x, which contains a set of nodes $F(x) \cup \{x\}$, while the remaining of its structure is adjusted to the set of nodes (i.e., information sets, actions, and payoffs). We start with the definition of the interchange of decision nodes.

Definition 21 (Interchange of decision nodes)**.**
The interchange of decision nodes is mapping $O^I : G \to G'$, where for nodes $x, x^1, x^2, x^3, x^4, x^5, x^6 \in X \backslash W \cup Z$:

(a) $C(x) = \{x^1, x^2\}$, $C(x^1) = \{x^3, x^4\}$, and $C(x^2) = \{x^5, x^6\}$; $h(x^1) = h(x^2)$; $A(x^3) = A(x^5)$ and $A(x^4) = A(x^6)$;

(b) For every $\hat{x} \in X \backslash \{x^1, x^2\}$, $C(\hat{x}) = C'(\hat{x})$, while $C'(x^1) = \big[C(x^1) \cup \{x^5\}\big] \backslash \{x^4\}$ and $C'(x^2) = \big[C(x^2) \cup \{x^4\}\big] \backslash \{x^5\}$;

(c) For every $\hat{x} \notin h(x) \cup h(x^1)$, we have $h'(\hat{x}) = h(\hat{x})$ and $\iota'(\hat{x}) = \iota(\hat{x})$. We have $h'(x) = \{h(x^1) \cup \{x\}\} \backslash \{x^1, x^2\}$ and $h'(x^1) = \{h(x) \cup \{x^1, x^2\}\} \backslash \{x\}$, while for every $\hat{x} \in h'(x)$ and $\tilde{x} \in h'(x^1)$, we have $\iota'(\hat{x}) = \iota(x^1)$ and $\iota'(\tilde{x}) = \iota(x)$;

(d) $A'(x^1) = A(x)$ *and* $A'(x) = A(x^2)$*;*

(e) $G = \langle X \backslash F(x) \cup \{x\}, \prec, N, \iota, W, H, A, p \rangle$ *and* $G' = \langle X' \backslash F'(x) \cup \{x\}, \prec, N', \iota', W', H',\ A', p' \rangle$ *are strongly isomorphic;*

(f) For every $\hat{x} \not\prec x^1$ *or* $\hat{x} \not\prec x^2$*,* $G_{\hat{x}}$ *and* $G'_{\hat{x}}$ *are strongly isomorphic.*

Point (a) describes the organization of the nodes in G, and point (b) how they relate with the nodes in G'. Point (c) implies that the information of a player does not change with O^I, and point (d) that the actions change according with a change in the nodes. Points (e) and (f) guarantee that the game remains the same in nodes not affected by O^I. In Figure 3, we have an example of this operation. The second elementary operation is the coalescing of information sets.

Definition 22 (Coalescing of information sets)**.**
Take $h(x^i) = \{x^1, \ldots, x^n\}$ *and* $h(\hat{x}^i) = \{\hat{x}^1, \ldots, \hat{x}^n\}$*; the coalescing of information sets is a mapping* $O^C : G \to G'$*, in which:*

(A) $\iota(x^i) = \iota(\hat{x}^i)$ *for* $i = 1, \ldots, n$*;*

(B) $\hat{x}^i \in C(x^i)$*;*

(C) G' *is strongly isomorphic to* $G = \langle X \backslash \{\hat{x}^1, \ldots, \hat{x}^n\}, \succ, N, \iota, H, A, u, p \rangle$*.*

Points (A) and (B) describe the organization of the nodes, and point (C) indicate that we merge two consecutive nodes into a single one. We depict in Figure 4 an example of this operation.

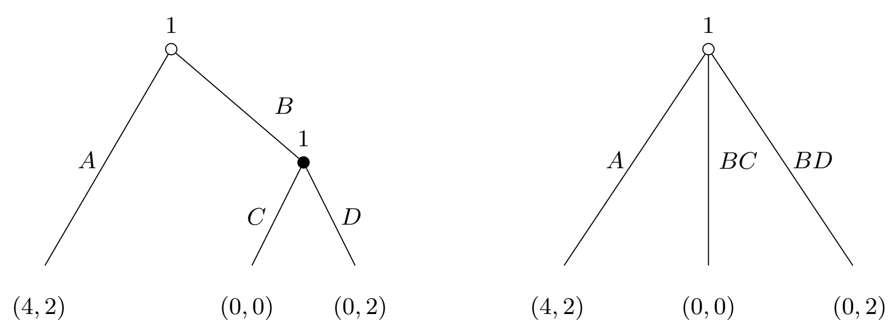

Figure 4: Game G (left) and game G' (right) following the coalescing of information sets.

The third elementary operation is the addition of superfluous decision nodes, which in the current form, and contrarily to Thompson (1952), preserves perfect recall.

Definition 23 (Addition of superfluous decision nodes)**.**

The addition of superfluous decision nodes $\{x^1, \dots, x^n\}$ *to* G *is a mapping* $O^A : G \to G'$*, such that, for nodes* $\{x^{1,1}, \dots, x^{1,m}, \dots, x^{n,1}, \dots, x^{n,m}\}$*, in which* $i, j = 1, \dots, n$ *and* $k, \ell = 1, \dots, m$*:*

1. *For every* $x' \in h(x^i)$*,* $x'R^p = h(x^i) = h(x^j)$*;*
2. $C(x^i) = \{x^{i,1}, \dots, x^{i,m}\}$*;*
3. $\Gamma_{x^{i,k}}$ *and* $\Gamma_{x^{i,\ell}}$ *are strongly isomorphic for every* $x^{i,k}, x^{i,\ell} \in C(x^i)$*;*
4. $A(x^{i,k}) = A(x^{j,k})$*;*
5. *If* $x \succ x^{i,k}$ *and* $\underline{x} \succ x^{i,\ell}$*, and we have* $h(x) = h(\underline{x})$*,* $\hat{x} \in C(x)$*,* $\underline{\hat{x}} \in C(\underline{x})$*, and* $\underline{\hat{x}} = \phi(\hat{x})$*, then* $A(\underline{\hat{x}}) = A(\hat{x})$*;*
6. *If* $\iota(x) \neq \iota(x^i)$*, and* $x \succ x^{i,k}$*, then* $h(\phi(x)) = h(x)$*; if* $\iota(x) = \iota(x^i)$*, and* $x \succ x^{i,k}$*,* $\hat{x} \succ x^{j,k}$*, then: (i) when* $\bar{x} \in h(x)$ *then* $\bar{x} \succ x^{\ell,k}$*, for* $\ell = 1, \dots, n$*; (ii) when* $\bar{x} \in h(\phi(x))$ *then* $\bar{x} \succ x^{\ell,q}$ *for* $\ell = 1, \dots, n$ *and* $q = 1, \dots, m$*; (iii)* $h(x) = h(\hat{x})$ *if and only if* $h(\phi(x)) = h(\phi(\hat{x}))$*;*
7. $G = \langle X_{x^i}, \prec, N, \iota, H, A_{x^i}, u, p \rangle$ *and* G' *are strongly isomorphic, such that* $X_{x^i} = [X \backslash \{x \mid x \succ x^i\}] \cup \{\hat{x} \mid \hat{x} \succ x^{i,k}\}$*, for* $i = 1, \dots, n$ *and* $k = 1, \dots, m$*, and* A_{x^i} *is the set of available actions at each* $x^i \in X_{x^i}$*.*

By point 1, we preserve perfect recall. From each successor of an added node following point 2, we obtain subgames that are mutually isomorphic (point 3), while ensuring that a player with additional nodes has the same available actions as before (point 4). Points 5 and 6 imply that the players who did not receive additional nodes have the same choices and information as before, and that those who do have additional nodes also have the same information as before. Point 7 implies that G is strongly isomorphic to various parts of G'. We provide an example of this transformation in Figure 5.

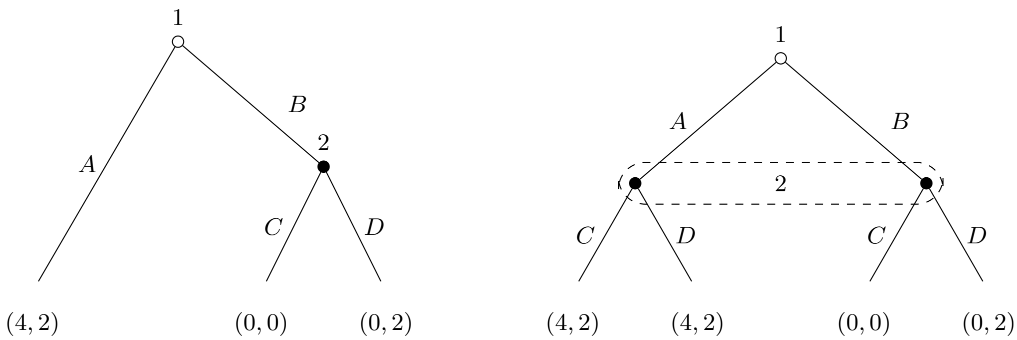

Figure 5: Games G (left) and game G' (right) following the addition of a superfluous decision node.

Appendix A.3

Lemma 1.

The equilibrium $(e_0, e_1; r^s)$ is the separating equilibrium with the lowest premium.

Proof. Consider any separating equilibrium where type H chooses $e' > e_1$. That equilibrium is payoff dominated for type H, since $H - (e_1/H) > H - (e'/H)$. The risk of each separating equilibrium follows from the deviations of type L; given that in each of those equilibria, type L plays e_0, any deviation induces the same payoff loss in each equilibrium. Hence, $\rho_H(\cdot)$ is the same in each equilibrium; we do not evaluate the risk for type L because this type cannot enforce a specific separating equilibrium, since in every one it chooses e_0. In this case, the criterion of payoff-dominance acts as a tie-breaker, which concludes the proof. □

Lemma 2.

The equilibrium $(e^p, e^p; r^p)$ in which $e^p = 0$ and the firm's system of beliefs are that any $e \neq e^p$ signals a type L is the pooling equilibrium with the lowest premium.

Proof. There are multiple pooling equilibria with $e^p \in [0, b]$, where the firms' belief system mainly determines the set of deviations $\delta_\theta(\cdot)$ of each type θ, and thus $\lambda_{(\cdot)}(\beta_\theta(\cdot))$. Concerning payoff-dominance, the pooling equilibrium with $e^p = 0$ is payoff-dominant for any type θ, since $pH + (1 - p)L < pH + (1 - p)L - (e/\theta)$ for any $e > 0$. The firms' system of beliefs that induce the largest losses to any type θ but which supports a pooling equilibrium with $e^p \in [0, b]$ considers that any deviation to $e \neq e^p$ signals a type L. In that case, the payoff from a deviation by type θ is $L - e/\theta$. In pooling equilibria with either $e^p = 0$ or $e^p = b$, for each type θ, $\lambda_{(0,0;r^p)}(\beta_\theta(e)) = \lambda_{(b,b;r^p)}(\beta_\theta(e))$

for every $e \neq e^p$, where $e \in \delta_\theta(0, 0; r^p)$ and $e \in \delta_\theta(b, b; r^p)$. Hence, payoff-dominance determines if we select the polling equilibrium with support on $e^p = 0$ or $e^p = b$, which concludes the proof. □

Lemma 3.

The hybrid equilibria of the game are not eligible for selection.

Proof. Extending the reasoning of Lemma 1 and Lemma 2 to hybrid equilibria, we exclude mixtures with support on separating and pooling levels of education not eligible for selection. We are left with hybrid equilibria in which type H randomizes between separating and pooling levels of education, and type L chooses an education level of zero. We describe such hybrid equilibrium in Section 4.1. Denote that semi-separating equilibrium as $(0, \beta_H(e); r^h)$. In that case, for both players, $(0, \beta_H(e); r^h) \sim^u_\theta (e_0, e_1; r^s)$. If the firms' beliefs imply that a deviation to $e < \sigma(e)$ signals a type L, then both types H and L have exactly the same relative payoff losses following an admissible deviation in $(0, \beta_H(e); r^h)$ and $(e_0, e_1; r^s)$. Hence, as we discuss in (Silva, 2019, Remark 1), the hybrid equilibrium $(0, \beta_H(e); r^h)$ is not eligible for selection. Since this reasoning holds on a semi-pooling equilibrium, we conclude the proof. □

Proof of Proposition 4.

Following Lemma 1, Lemma 2, and Lemma 3, the set of equilibria eligible to selection contains the separating equilibrium $(e_0, e_1; r^s)$, and the pooling equilibrium $(0, 0; r^p)$. When the prior probability p of the worker being type H is

$$p > \frac{H - L - (e_1/H)}{H - L} = \frac{1}{2}, \tag{5}$$

the pooling equilibrium is payoff-dominant for type H and L, and thus the solution of the game. Otherwise, it is the separating equilibrium, since only $|\mathcal{B}_H| > 1$, meaning that type H chooses, as the solution of the game, the equilibrium that maximizes his expected payoff. □

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