A GENERAL FRAMEWORK FOR NOETHERIAN WELL ORDERED POLYNOMIAL REDUCTIONS

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ABSTRACT. Polynomial reduction is one of the main tools in computational algebra with innumerable applications in many areas, both pure and applied. Since many years both the theory and an efficient design of the related algorithm have been solidly established. This paper presents a definition of the generic concept of polynomial reduction structure, studies its features and highlights the aspects needed in order to grant and to efficiently test the main properties (Noetherianity, confluence, ideal membership). The most significant aspect of this analysis is a negative reappraisal of the role of the notion of term ordering which is usually considered a central and crucial tool in the theory. In fact, as it was already established in the computer science context in relation with termination of algorithms, most of the properties can be obtained simply considering a well founded order, while the classical requirement that it be preserved by multiplication is irrelevant. The last part of the paper shows how the polynomial basis concepts present in literature are interpreted in our language and their properties are consequences of the general results established in the first part of the paper.

1. INTRODUCTION

Buchberger reduction was introduced in 1899 by Gordan [56] as a technical tool in his proof of Hilbert's *Basissatz* [62] but, at that time, at least the PDE community was aware of the concepts of generic initial ideal introduced in 1896 by Delassus [37] and of S-polynomials introduced in 1910 by Riquier [99]. This knowledge was summarized by Janet in [64].

When such theory was independently rediscovered by Buchberger [23, 24, 27] under the name of Groebner basis, the Pandora box was opened: Buchberger's Theory and Algorithm introduced for polynomial rings over a field [23, 24, 27] was extended to polynomial ring over the integers [68], over Euclidean domains [69], over each ring on which ideal membership is testable and syzygies are computable [112], over domains [91] and PIRs [84], to non-commutative rings which satisfy Poincaré-Birkhoff-Witt Theorem [12], Lie algebras [6, 7], solvable polynomial rings [70, 71], skew polynomial rings [110, 45, 46, 47], multivariate Ore extensions [92, 93, 30, 33], other algebras which satisfy Poincaré-Birkhoff-Witt Theorem [4, 74, 75], semigroup rings [101, 79, 80], function rings [97, 98], non-commutative free algebras [12], all effective rings [89, 34], reduction rings [106, 107, 108, 109, 81], involutive bases [94, 113, 95, 50, 53, 54, 51, 52, 55, 104, 105], marked bases [17, 16, 36].

Except [17, 16, 36] and Gordan, all these results make a strong and non-necessary requirement in order to grant termination of the reduction procedures; in fact they

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imposed a semigroup ordering on the set of the monomials, i.e. an ordering that preserves multiplication by variables, while a Noetherian well founded ordering can be sufficient.

It is true that the results reported by Janet assume that the Noetherian ordering preserves multiplication by variables, but their motivations are completely different:

- for the researchers developing techniques for solving PDE's multiplication by variable was just an algebraic notation for derivation; and in each calculus course, derivation of a formula by any single variable is naturally performed by scanning it;
- Hilbert's proof of the *Nullstellensatz* is done by inductively performing euclidean division in the univariate polynomial ring $K[x_1, \ldots, x_{n-1}][x_n]$; while Hilbert does not even make reference to an ordering of the monomials, it is obvious that a reformulation of Hilbert's reduction in terms of Buchberger's reduction requires the deglex ordering induced by $x_1 < \ldots < x_n$.

While the assumption of having a term ordering is obviously justified for historical reasons in the results reported by Janet, we cannot imagine any valid reason for maintaining such an irrelevant assumption in the research started from the introduction of Groebner bases theory.

Actually, this assumption hinders the study of Hilbert scheme; it is well-known [11] that deformations of the Groebner basis of an ideal I in the polynomial ring $\mathcal{P}$ are a flat family and can thus be applied for studying geometrical deformations of the scheme $\mathcal{X}$ defined by I. However such families of deformations in general cover only locally closed subschemes of Hilbert scheme and are not sufficient to study neighbourhood of deformations of $\mathcal{X}$, *id est* opens of Hilbert scheme; such opens can be obtained instead by considering [20] those ideals I' of $\mathcal{P}$ which share with I a fixed monomial basis of the quotient $\mathcal{P}/\mathsf{I}$. In order to determine the family of all such ideals I' of $\mathcal{P}$, term ordering free bases of polynomial ideals were introduced, under the label of *marked bases* in [17, 16, 36].

Following Riquier and Janet, given a finite set $\mathcal{F}$ of polynomials, they allow to multiply each $f \in \mathcal{F}$ only by a restricted set of variables (*multiplicative variables* in Janet formulation) or, in general, by an order ideal τ_f of terms (*multiplier set*). It is then sufficient for them to restrict the requirement of preserving leading terms to such subsets of multipliers for obtaining well founded orders which are not semigroup ones but however grant a Noetherian reduction. Clearly, this does not contradict Reeves-Sturmfels Theorem for the elementary reason that the aforementioned theorem requires the application of the whole set of terms as multipliers.

The aim of this paper is to study the main properties of the consequent Noetherian reduction (and its differences with Buchberger reduction); we cover Noetherianity, weak Noetherianity, confluency, canonical forms; moreover we import in our setting results available within the theories of Groebner bases and of involutiveness as Buchberger's and Möller's Criteria and Janet-Schreyer approach for computing resolutions.

Fixed the notation (Section 2) and introduced the definition and related notions of *reduction structure* (Section 3), we discuss (Section 4) *marked sets* and the associated rewriting rule $\to$, focusing on its main properties, Noetherianity, weak Noetherianity, their relation with the orderedness of the related reduction structure (Sections 5, 6) and with Reeves-Sturmfels Theorem (Theorem 5.10), the structure of the related Groebner representation (Proposition 6.2 of Section 6), confluency (Section 7), criteria for marked bases (Section 8) and for avoiding useless reductions (Section 9).

The notion of marked set and its main properties have been introduced and deeply studied in a long series of papers [36, 20, 17, 16, 32] devoted to a deep analysis of Hilbert schemes (Section 10).

Next we discuss stably ordered reduction structures (Section 11). Finally (Sections 12, 13) we cover the most important types of known polynomial bases coherent with a term ordering reformulating them in our language.

2. Notations.

Consider the polynomial ring

$$\mathcal{P} := A[x_1, ..., x_n] = \bigoplus_{d\in\mathbb{N}} \mathcal{P}_d$$

in n variables and coefficients in the base field A. For every set $V \subset \mathcal{P}$ we denote by $\langle V\rangle$ the A-vector space generated by V.

When an order on the variables comes into play, we consider $x_1 < x_2 < ... < x_n$.

The *set of terms* in the variables $x_1, ..., x_n$ is

$$\mathcal{T} := \{x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n},\ (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n\}.$$

For every polynomial $f \in \mathcal{P}$, $\deg(f)$ is its usual degree and $deg_i(f) = deg_z(f)$ is its degree with respect to the variable $x_i = z$.

Given a term $x^\alpha \in \mathcal{T}$, we denote $|\alpha| := \deg(x^\alpha)$ and set

$$\max(x^\alpha) = \max\{x_i \ :\ x_i \mid x^\alpha\},\ \min(x^\alpha) = \min\{x_i \ :\ x_i \mid x^\alpha\}$$

the maximal and the minimal variable appearing in x^α with nonzero exponent.

If $\{x_{j_1}, ..., x_{j_r}\} \subset \{x_1, ..., x_n\}$, we define

$$\mathcal{T}[x_{j_1}, ..., x_{j_r}] := \{x_{j_1}^{\alpha_{j_1}} \cdots x_{j_r}^{\alpha_{j_r}},\ (\alpha_{j_1}, ..., \alpha_{j_r}) \in \mathbb{N}^r\}.$$

For each $p \in \mathbb{N}$, and for all $V \subseteq \mathcal{P}$,

$$V_p := \{f \in V \ :\ f \text{ homogeneous and } \deg(f) = p\};$$

in particular:

$$\mathcal{T}_p := \{x^\alpha \in \mathcal{T} \,:\, \deg(x^\alpha) = p\}.$$

We also denote $\mathcal{T}_{\geq p} := \{x^\alpha \in \mathcal{T} \,:\, \deg(x^\alpha) \geq p\}$.

Once a well founded order $<$ is fixed in $\mathcal{T}$ then each $f \in \mathcal{P}$ has a unique representation as an ordered linear combination of terms $t \in \mathcal{T}$ with coefficients in A:

$$f = \sum_{i=1}^{s} c(f, t_i)t_i : c(f, t_i) \in A \setminus \{0\}, t_i \in \mathcal{T}, t_1 > \cdots > t_s.$$

The *support* of f is the set

$$\mathrm{Supp}(f) := \{t : c(f, t) \neq 0\} = \{t_1, \ldots, t_s\};$$

we further denote $\mathbf{T}(f) := t_1$ the *maximal term* of f, $\mathrm{lc}(f) := c(f, t_1)$ its *leading coefficient* and $\mathbf{M}(f) := c(f, t_1)t_1$ its *maximal monomial*.

For each $f, g \in \mathcal{P}$ such that $\mathrm{lc}(f) = 1 = \mathrm{lc}(g)$, the *S-polynomial of f and g* [23, 24][88, II, Definition 25.1.2.] is the polynomial

$$S(g,f) := \frac{\mathrm{lcm}(\mathbf{T}(f),\mathbf{T}(g))}{\mathbf{T}(f)}f - \frac{\mathrm{lcm}(\mathbf{T}(f),\mathbf{T}(g))}{\mathbf{T}(g)}g.$$

For an ordered set $F = \{f_1, ..., f_s\} \subset \mathcal{P}$ we denote $Syz(F)$ its *syzygy module*

$$Syz(F) = \{(g_1, ..., g_s) \in \mathcal{P}^s,\ \sum_{i=1}^{s} g_i f_i = 0\},$$

of all the *syzygies of F*.

If I is either a monomial ideal or a *semigroup ideal*[1] we denote by $\mathsf{N}(I)$ the *order ideal* (or: *normal set*)[2] $\mathsf{N}(I) := \mathcal{T} \setminus I$.

3. Introducing Reduction Structures

Definition 3.1. A *reduction structure* (RS for short) $\mathcal{J}$ in $\mathcal{T}$ is a 3-tuple

$$(M, \lambda := \{\lambda_\alpha\,,\ x^\alpha \in M\}, \tau := \{\tau_\alpha\,,\ x^\alpha \in M\})$$

that satisfies the following conditions

- M is a *finite* set of terms; we denote by J the semigroup ideal generated by M;
- for all $x^\alpha \in M$, $\tau_\alpha \subseteq \mathcal{T}$ is an order ideal, called *multiplicative set of x^α*, s.t. $\bigcup_{x^\alpha \in M} \mathrm{cone}(x^\alpha) = J$, where $\mathrm{cone}(x^\alpha) := \{x^{\alpha+\eta} \mid x^\eta \in \tau_\alpha\}$ is the *cone* of x^α;
- for all $x^\alpha \in M$, λ_α is a finite subset of $\mathcal{T} \setminus \mathrm{cone}(x^\alpha)$ that we call *tail set* of x^α.

Remark 3.2. In Definition 3.1, we suppose that λ_α is a finite subset of $\mathcal{T} \setminus \mathrm{cone}(x^\alpha)$, so we suppose to have *finite tails*. It is possible to reformulate the definition of reduction structure so that infinite tails are allowed, but it is out of the scope of this paper.

Lemma 3.3. *Let $\mathcal{J}$ be an RS. Then, there is at least a term $x^\alpha \in M$ s.t. $\tau_\alpha = \mathcal{T}$. In particular it holds $\mathcal{T} = \bigcup_{x^\alpha \in M} \tau_\alpha$.*

Proof. Suppose that the assertion is false and, for each $x^{\alpha_i} \in M$, choose a term x^{η_i} not belonging to τ_{α_i}. We denote x^β the product of the terms $x^{\alpha_i+\eta_i}$, $x^{\alpha_i} \in M$. By definition of RS there is a term in M, let it be x^{α_1}, whose cone contains x^β, so $x^{\beta-\alpha_1}$ is a multiple of x^{η_1} and belongs to τ_{α_1}. Since τ_{α_1} is an order ideal, it contains also x^{η_1}, leading to a contradiction. □

Definition 3.4. We will call *substructure* of $\mathcal{J} = (M, \lambda, \tau)$ each RS of the form $\mathcal{J}' = (M, \lambda, \tau')$ s.t. for each $x^\alpha \in M$ it holds $\tau'_\alpha \subseteq \tau_\alpha$. In this case we will write $\mathcal{J}' \subseteq \mathcal{J}$.

Reduction Structures of the following type will be important in the whole paper

Definition 3.5. A Reduction Structure $\mathcal{J}$ is:

- *homogeneous* if $\forall x^\alpha \in M$ it holds $\lambda_\alpha \subset \mathcal{T}_{|\alpha|}$,
- with *reduced tails* if $\forall x^\alpha \in M$ it holds $\lambda_\alpha \subseteq \mathsf{N}(J)$,
- *coherent with a term ordering* $\prec$ if $\forall x^\alpha \in M$ and $\forall x^\gamma \in \lambda_\alpha$ it holds $x^\alpha \succ x^\gamma$,
- with *maximal cones* if $\forall x^\alpha \in M$ it holds $\tau_\alpha = \mathcal{T}$,

[1] i.e. $\forall x^\eta \in \mathcal{T}$, $x^\gamma \in I \Rightarrow x^{\eta+\gamma} \in I$.

[2] Observe that a set of terms $\mathsf{N} \subset \mathcal{T}$ is an order ideal if and only if the complementary set $I := \mathcal{T} \setminus \mathsf{N}$ is a semigroup ideal.

- with *disjoint cones* if $\forall x^\alpha, x^{\alpha'} \in M$, $x^\alpha \neq x^{\alpha'}$, it holds $\mathrm{cone}(x^\alpha) \cap \mathrm{cone}(x^{\alpha'}) = \emptyset$,
- with *multiplicative variables* if for each $x^\alpha \in M$ exists

$$\mu_\alpha \subseteq \{x_1, \dots, x_n\} \text{ s.t. } \tau_\alpha = \mathcal{T}[\mu_\alpha].$$

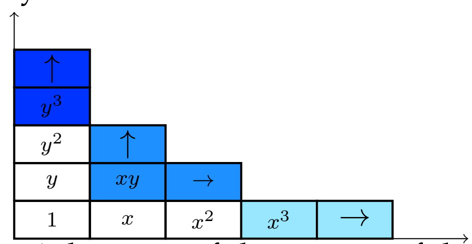
More generally, we will call x_i *multiplicative variable* for $x^\alpha \in M$ if $x_i \tau_\alpha \subset \tau_\alpha$.

As we will see in details in Section 12, there are RSs that give the natural framework in which we find Groebner bases and their properties. They are built as follows: M is any finite set of terms; for each term $x^\alpha \in M$, τ_α is the whole $\mathcal{T}$ and λ_α is the sets of terms lower than x^α w.r.t. a fixed term ordering. In the terminology just introduced, these RSs are coherent with a term ordering, have multiplicative variables and maximal cones.

On the other hand, our definition also includes strange RSs that cannot be included neither in a standard Groebner framework nor in any other type of polynomial bases that (in our knowledge) are already present in literature.

Example 3.6. In $A[x, y]$ let us consider the RS $\mathcal{J}$ given by

- $M = \{x^3, xy, y^3\}$;
- $\lambda_{x^3} = \lambda_{y^3} = \{x^2y, xy^2, x^2, xy, y^2, x, y, 1\}$, $\lambda_{xy} = \{x, y, 1\}$
- $\tau_{x^3} = \mathcal{T}[x]$, $\tau_{xy} = \mathcal{T}[x, y]$, $\tau_{y^3} = \mathcal{T}[y]$.



This RS is not coherent with a term ordering; however it has two of the most useful features that we can expect by a polynomial rewriting rule and that we will discuss in the following sections: Noetherianity and confluence.

4. Marked sets and rewriting rules

Definition 4.1 ([96]). A *marked polynomial* is a polynomial $f \in \mathcal{P}$ together with a fixed term $\mathrm{Ht}(f)$, its *marked term* that appears in f with coefficient 1_A.

We use RSs in order to investigate when and how marked polynomials can be efficiently applied as rewriting rules (for theoretical results on polynomial rewriting rules see [38, 29, 21]).

Definition 4.2. Given an RS $\mathcal{J} = (M, \lambda, \tau)$, consider for each $x^\alpha \in M$ a monic marked polynomial $f_\alpha \in \mathcal{P}$ s.t. $\mathrm{Ht}(f_\alpha) = x^\alpha$ and $\mathrm{Supp}(f_\alpha - x^\alpha) \subset \lambda_\alpha$. We call *marked term* of f_α such term x^α and *tail* of f_α the difference $f_\alpha - x^\alpha$.

The set $\mathcal{F} = \{f_\alpha\}_{x^\alpha \in M}$ of polynomials in $\mathcal{P}$ is called *marked set* on $\mathcal{J}$; note that M is indeed the set of the marked terms of $\mathcal{F}$.

We denote by $\tau\mathcal{F}$ the set $\tau\mathcal{F} := \{x^\eta f_\alpha : x^\eta \in \tau_\alpha\}$, by $\langle \tau\mathcal{F} \rangle$ the A-vector space generated by $\tau\mathcal{F}$ and by $(\mathcal{F})$ the ideal of $\mathcal{P}$ generated by $\mathcal{F}$.

A key notion in all the theory is the following

Definition 4.3. We say that a marked set $\mathcal{F}$ over an RS $\mathcal{J}$ is a *marked basis* on $\mathcal{J}$ if $\mathsf{N}(J)$ is a free set of generators for $A[x_1, \dots, x_n]/(\mathcal{F})$ as A-vector space, i.e. if it holds

$$(\mathcal{F}) \oplus \langle \mathsf{N}(J) \rangle = \mathcal{P}.$$

We can associate to a marked set $\mathcal{F}$ on $\mathcal{J}$ a reduction procedure $\to^+_{\mathcal{F}\mathcal{J}}$.

For $g, h \in \mathcal{P}$, it holds $g \to_{\mathcal{F}\mathcal{J}} h$ iff there are a term $x^\gamma \in \mathsf{Supp}(g)$, and an element $x^\alpha \in M$ s.t. $x^\gamma = x^{\alpha+\eta} \in \mathsf{cone}(x^\alpha)$ and $h = g - cx^\eta f_\alpha$, where $c = c(g, x^\gamma) \in A$ is the coefficient of x^γ in g.

We denote $\to^+_{\mathcal{F}\mathcal{J}}$ its transitive closure.

We also remark that if $g \in \mathcal{P}$ and $\mathsf{Supp}(g) \in \mathsf{N}(J)$, then there is no $h \in \mathcal{P}, h \neq g$, such that $g \to^+_{\mathcal{F}\mathcal{J}} h$; if this happen, we say that g is *reduced* w.r.t. J or is a *J-remainder*[3]. More precisely:

Definition 4.4. If $g \to^+_{\mathcal{F}\mathcal{J}} l$ and l is a J-remainder (i.e. $\mathsf{Supp}(l) \subset \mathsf{N}(J)$), we call l a *reduced form* (or J-remainder) of g. If such a polynomial l exists, we say that g *has a complete reduction w.r.t. $\mathcal{F}\mathcal{J}$*.

When we want to stress that l is a J-remainder of g we will write

$$g \to^+_{\mathcal{F}\mathcal{J}} l \downarrow .$$

Definition 4.5. A *rewriting rule* is a couple $(\mathcal{F}, \to^+_{\mathcal{F}\mathcal{J}})$, where $\mathcal{F}$ is a marked set over an RS $\mathcal{J}$ and $\to^+_{\mathcal{F}\mathcal{J}}$ is the binary relation defined above.

Notice that there could be several terms in $\mathsf{Supp}(g) \cap J$ and that each of them can belong to several cones. Therefore, the reduction performed on a general polynomial g by a rewriting rule is in general far for being unique nor, in principle, is unique its output, unless [29, 21] it is both Noetherian and (locally) confluent.

Remark 4.6. If $\mathcal{F} = \{f_\alpha,\ x^\alpha \in M\}$ is a marked set over $\mathcal{J} = (M, \lambda, \tau)$, then it is also marked over every RS $\mathcal{J}' = (M', \lambda', \tau')$ such that $M' = M$ and $\lambda'_\alpha \supseteq \mathsf{Supp}(f_\alpha - x^\alpha)$ for every $x^\alpha \in M$. From a different point of view, note that $\mathcal{F}$ is marked also on every substructure $\mathcal{J}'$ of $\mathcal{J}$.

Some notion related to $\mathcal{F}$ depends on which RS we are considering, while others do not. For instance, it is obvious from the definition that the notion of marked basis does not depend on the RS. On the other hand, a same set of marked polynomials $\mathcal{F}$ related to several RSs gives rise to essentially different rewriting rules depending on the set of multiplicative terms.

An interesting example of RS on which $\mathcal{F}$ is marked and also the rewriting rule is not modified, is $\widetilde{\mathcal{J}} := (M, \{\lambda_\alpha \cap \mathsf{Supp}(f_\alpha)\}, \tau)$. The terms of each λ_α not appearing in f_α are irrelevant in the reduction steps involving f_α. Moreover, they are irrelevant for the steps not involving f_α. Anyway, notice that the set of marked sets over $\mathcal{J}'$ is a proper subset of the analogous over $\mathcal{J}$.

In principle the theory (but not the practice!) of RSs can cover Hironaka Theory and reduction of series in the setting of [86, 87] and [88, II.Hironaka Theorem 24.6.16]. This could be done by substituting the Noetherianity assumption with *inflimitedness* [88, II.Definition 24.5.2][88, IV.Definition 50.3.3]. Anyway, we restrict our paper to the polynomial setting; this is the reason we have defined reduction structures only with finite tails (see remark 3.2)

5. Noetherianity I: well founded orders

In this section we discuss some relations between the different types of RSs we have introduced in relation with the Noetherianity of the rewriting rules.

[3] Recall that J denotes the semigroup ideal generated by M.

Definition 5.1. We say that $\to^+_{\mathcal{F}\mathcal{J}}$ is *Noetherian* if there is no infinite reduction chain

$$g_1 \to^+_{\mathcal{F}\mathcal{J}} g_2 \to^+_{\mathcal{F}\mathcal{J}} g_3 \to^+_{\mathcal{F}\mathcal{J}} \cdots .$$

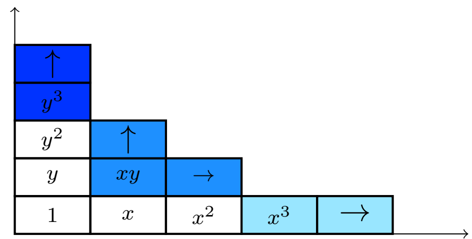
We call $\mathcal{J}$ *Noetherian* if for each marked set $\mathcal{F}$ on $\mathcal{J}$, the rewriting rule $\to^+_{\mathcal{F}\mathcal{J}}$ is Noetherian. The RS $\mathcal{J}$ is *weakly Noetherian* if it has a Noetherian substructure.

If $\mathcal{J}$ is weakly Noetherian, each polynomial g has a complete reduction $g \to^+_{\mathcal{F}\mathcal{J}} l \downarrow$, though there could be also infinite sequences of base steps of reduction starting on g.

Here an example of an RS that is not weakly Noetherian (nor a fortiori Noetherian).

Example 5.2. Let us consider the RS $\mathcal{J}$ given by

- $M = \{xy, x^3, y^3\}$;
- $\tau_{xy} = \mathcal{T}[x,y]$, $\tau_{x^3} = \mathcal{T}[x]$, $\tau_{y^3} = \mathcal{T}[y]$;
- $\lambda_{xy} = \{x^2, y^2\}$, $\lambda_{x^3} = \lambda_{y^3} = \emptyset$.



As the cones are disjoint, $\mathcal{J}$ has no proper substructure, hence it is sufficient to show that $\mathcal{J}$ itself is not Noetherian.

Let us consider the marked set $\mathcal{F} = \{f_{xy} = xy - x^2 - y^2, f_{x^3} = x^3, f_{y^3} = y^3\}$ over $\mathcal{J}$.

We obtain an infinite reduction chain as follows:

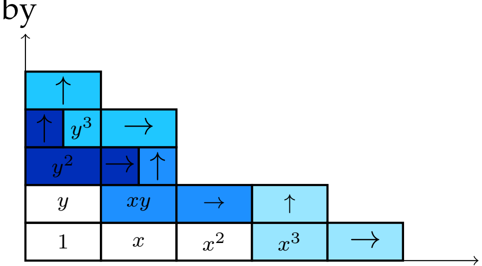
$$x^2y \longrightarrow_{\mathcal{F}\mathcal{J}} x^2y - xf_{xy} = x^3 + xy^2 \longrightarrow_{\mathcal{F}\mathcal{J}} x^3 + xy^2 - f_{x^3} = xy^2 \longrightarrow_{\mathcal{F}\mathcal{J}} xy^2 - yf_{xy} =$$
$$x^2y + y^3 \longrightarrow_{\mathcal{F}\mathcal{J}} x^2y + y^3 - f_{y^3} = x^2y \longrightarrow_{\mathcal{F}\mathcal{J}} x^2y - xf_{xy} \ldots .$$

Note that at each step of every possible sequence of reductions of x^2y we find a polynomial of the type $x^2y + nx^3 + my^3$ or $xy^2 + nx^3 + my^3$ with $n, m \in \mathbb{Z}$, and none of them is a J-remainder.

Here an example of a weakly Noetherian RS that is not Noetherian.

Example 5.3. Let us consider the RS $\mathcal{J}$ given by

- $M = \{xy, y^2, x^3, y^3\}$;
- $\tau_{xy} = \tau_{y^2} = \tau_{x^3} = \tau_{y^3} = \mathcal{T}[x,y]$;
- $\lambda_{xy} = \{x^2, y^2\}$, $\lambda_{y^2} = \lambda_{x^3} = \lambda_{y^3} = \emptyset$.



Every marked set over $\mathcal{J}$ has the shape

$$\mathcal{F}_{a,b} = \{f_{xy} = xy - ax^2 - by^2, f_{y^2} = y^2, f_{x^3} = x^3, f_{y^3} = y^3\},\ a, b \in A.$$

The RS $\mathcal{J}$ is not Noetherian since reducing x^2y with respect to the marked set $\mathcal{F}_{1,1}$ we may obtain the same infinite sequence of steps described in Example 5.2.

However, in this case for every polynomial there are also reductions leading to a J-remainder, since $\mathcal{J}$ has for instance the Noetherian substructure $\mathcal{J}'$ given by $\tau'_{xy} = \{1, x\}$ and $\tau'_{y^2} = \tau'_{x^3} = \tau'_{y^3} = \mathcal{T}[x,y]$.

In fact, for every $\mathcal{F}_{a,b}$ the reduction procedure $\to^+_{\mathcal{F}_{a,b}\mathcal{J}'}$ returns after the only possible first step of reduction the J-remainder of xy (it is $ax^2 + by^2$) and of every monomial

v that is multiple of either y^2 or x^3 (it is 0). Moreover, the only possible sequences of reduction of x^2y are

$$x^2y \to_{\mathcal{F}_{a,b}\mathcal{J}'} ax^3 + bxy^2 \to_{\mathcal{F}_{a,b}\mathcal{J}'} ax^3 \to_{\mathcal{F}_{a,b}\mathcal{J}'} 0 \downarrow$$
$$x^2y \to_{\mathcal{F}_{a,b}\mathcal{J}'} ax^3 + bxy^2 \to_{\mathcal{F}_{a,b}\mathcal{J}'} bxy^2 \to_{\mathcal{F}_{a,b}\mathcal{J}'} 0 \downarrow .$$

Lemma 5.4. *Let $\mathcal{J}$ be an RS. Then*

(i) if $\mathcal{J}$ is Noetherian, then it is also weakly Noetherian;
(ii) if $\mathcal{J}$ has disjoint cones, then also the converse of (i) holds true;
(iii) if $\mathcal{J}' \subseteq \mathcal{J}$ and $\mathcal{J}$ is Noetherian, then also $\mathcal{J}'$ is Noetherian;
(iv) if $\mathcal{J}' \subseteq \mathcal{J}$ and $\mathcal{J}'$ is weakly Noetherian, then $\mathcal{J}$ is weakly Noetherian.

Proof. All these properties are trivial consequences of the definitions. We only observe for (ii) that an RS with disjoint cones has no proper substructures. □

In order to find some effective way to check the Noetherianity of an RS, we now exploit arguments and results concerning the termination of algorithms based on rewriting rules, that have been developed mainly in the computer science context. They state a closed relation between the Noetherianity and the presence of a suitable well founded order.

We recall that an order < on a set W is called *well founded* if each nonempty subset of W contains minimal elements.

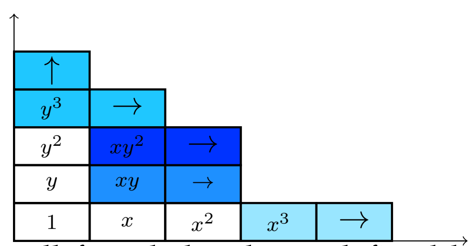
Definition 5.5. We say that an RS $\mathcal{J}$ is *ordered* if there is a well founded order > on $\mathcal{T}$ s.t.

$$\forall\, x^\alpha \in M,\ x^\gamma \in \lambda_\alpha,\ x^\eta \in \tau_\alpha \text{ it holds } x^{\alpha+\eta} > x^{\gamma+\eta}.$$

All the RSs coherent with a term ordering $\prec$ are obviously ordered. However there are ordered RSs that are not coherent with a term ordering; an easy example is the following.

Example 5.6. Let us consider the RS $\mathcal{J}$ given by

- $M := \{x^3, xy, xy^2, y^3\}$;
- $\tau_{x^3} = \tau_{xy} = \tau_{xy^2} := \mathcal{T}[x]$, $\tau_{y^3} := \mathcal{T}[x,y]$;
- $\lambda_{x^3} := \lambda_{xy^2} := \lambda_{y^3} := \emptyset$, $\lambda_{xy} := \{x^2, y^2\}$.



We prove $\mathcal{J}$ to be ordered, by considering the well founded order $<$ defined by $m_1 > m_2$ if and only if $m_1 = x^a y$ and either $m_2 = x^{a+1}$ or $m_2 = x^{a-1}y^2$ for some positive integer a. Of course there is no term ordering $\succ$ such that both $xy \succ x^2$ and $xy \succ y^2$.

Example 5.7. The RS of Example 3.6 is ordered by the well founded order < that we obtain refining the one given by the degree in the following way

$$\forall \mathbf{u}, \mathbf{v} \in \mathcal{T}[x,y]_d : \quad \mathbf{u} < \mathbf{v} \Longleftrightarrow \mathbf{u} \notin \{x^d, y^d\},\ \mathbf{v} \in \{x^d, y^d\}.$$

We would like to connect this definition of ordered RS to the rewriting rules on it. To this aim, we adapt to our situation a more general construction presented by Dershowitz and Manna in [38] and extend any order < on the set of monomials $\mathcal{T}$ to an order $\ll$ on the set of polynomials $\mathcal{P}$, by setting for every pair $f, g \in \mathcal{P}$, $f \gg g$ if and only if

$$\mathrm{Supp}(f) \neq \mathrm{Supp}(g) \text{ and } \forall m \in \mathrm{Supp}(g) \setminus \mathrm{Supp}(f)\, \exists m' \in \mathrm{Supp}(f) \text{ s.t. } m' > m.$$

Theorem 5.8. [38] *$(\mathcal{T}, <)$ is well founded if and only if $(\mathcal{P}, \ll)$ is.*

It is quite obvious that for every marked set $\mathcal{F}$ on an RS $\mathcal{J}$ ordered by $<$, $f \to_{\mathcal{F}\mathcal{J}} g$ implies $f \gg g$. We can then reformulate in our framework a well-know result by Manna and Ness concerning the termination of programs ([78],[38]).

Theorem 5.9. *Let $\mathcal{J}$ be an RS. Then*

$$\mathcal{J} \text{ is ordered } \iff \mathcal{J} \text{ is Noetherian}.$$

Due to the above result, **in the following we consider Noetherian and ordered as synonyms for what concerns the RSs.** Therefore, to every Noetherian RS we associate a well founded ordering $<$ on $\mathcal{T}$ and its extension $\ll$ on $\mathcal{P}$.

We conclude this section with a reformulation of a well known result by Reeves and Sturmfels in our language.

Theorem 5.10 (Reeves-Sturmfels, [96])**.** *Let $\mathcal{J} = (M, \lambda, \tau)$ be a RS with maximal cones. Then*

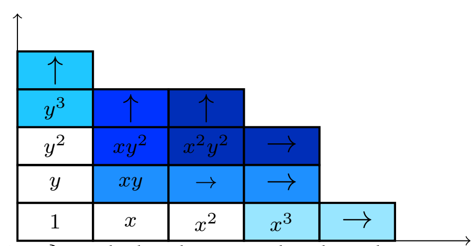
$$\mathcal{J} \text{ is noetherian} \iff \mathcal{J} \text{ is coherent with a term order}.$$

In the following example we present an RS which is *not* coherent with a term ordering while Noetherian.

By any similar RS, we can obtain examples of weakly Noetherian RSs with maximal cones, though non-coherent with a term ordering. Indeed, if $\mathcal{J}' = (M', \lambda', \tau')$ is a Noetherian, then $\mathcal{J} = (M = M', \lambda = \lambda', \{\tau_\alpha = \mathcal{T}\})$, of which $\mathcal{J}'$ is a substructure, is weakly Noetherian and has maximal cones.

Example 5.11. In $A[x, y]$ we consider

- $M = \{xy, x^3, y^3, xy^2, x^2y^2\}$;
- $\tau_{xy} = \tau_{x^3} = \mathcal{T}[x]$, $\tau_{y^3} = \tau_{xy^2} = \mathcal{T}[y]$, $\tau_{x^2y^2} = \mathcal{T}[x, y]$;
- $\lambda_{xy} = \{x^2, y^2\}$, $\lambda_{x^3} = \lambda_{y^3} = \lambda_{xy^2} = \lambda_{x^2y^2} = \emptyset$.



Let us consider the marked set $\mathcal{F} = \{f_{xy}, f_{x^3}, f_{y^3}, f_{xy^2}\}$; while the marked polynomials $f_{x^3}, f_{y^3}, f_{xy^2}$ are necessarily monomials, for xy we have to fix a polynomial with the shape $xy - ax^2 - by^2, a, b \in A$; the reduction we are discussing assume $a \neq 0 \neq b$ but is a trivialtask to check that our claim apply also when either $a = 0$ and/or $b = 0$. The RS $\mathcal{J} = (M, \lambda, \tau)$ is trivially *non*-coherent with a term ordering but is Noetherian.

In fact:

- if $v \in \mathrm{cone}(x^3) \cup \mathrm{cone}(y^3) \cup \mathrm{cone}(xy^2) \cup \mathrm{cone}(x^2y^2)$, trivially $v \to 0 \downarrow$;
- $xy \to_{\mathcal{F}\mathcal{J}} ax^2 + by^2 \downarrow \in \langle \mathsf{N}(J) \rangle$
- $x^2y = x(xy) \to_{\mathcal{F}\mathcal{J}} x(ax^2 + by^2) = ax^3 + bxy^2 \to^+_{\mathcal{F}\mathcal{J}} 0 \downarrow$
- $x^{i+3}y = x^{i+2}(xy) \to_{\mathcal{F}\mathcal{J}} x^{i+2}(ax^2 + by^2) = ax^{i+1} \cdot x^3 + by \cdot x^i \cdot x^2y^2 \to^+_{\mathcal{F}\mathcal{J}} 0 \downarrow, i \geq 0.$

Note that the example does not contradict Reeves-Sturmfels Theorem for the simple reason that the cones are *not* maximal.

6. Noetherianity II: lower representations of polynomials

In this section we relate the reduction of a polynomial g by a given marked set $\mathcal{F}$ and its linear/polynomial representation in terms of $\tau\mathcal{F}$. We recall that for a given marked set $\mathcal{F}$ over an RS $\mathcal{J} = (M, \lambda, \tau)$, we denote by $\tau\mathcal{F}$ the set of polynomials $x^\gamma f_\alpha$ with $f_\alpha \in \mathcal{F}$ and $x^\gamma \in \tau_\alpha$, and by $\langle \tau\mathcal{F} \rangle$ the A-vector space generated by $\tau\mathcal{F}$. Moreover, J denotes the semigroup ideal generated by M and $\mathsf{N}(J)$ the order ideal $\mathcal{T} \setminus J$.

Definition 6.1. Let $\mathcal{F}$ be a marked set over an RS $\mathcal{J}$ and let g be any polynomial in $\langle \tau\mathcal{F} \rangle$.

If $g = \sum_{i=1}^r c_i x^{\eta_i} f_{\alpha_i}$ with $c_i \in A$ and $x^{\eta_i} f_{\alpha_i}$ distinct elements of $\tau\mathcal{F}$ ($\alpha_i \neq \alpha_j$ for $i, j \in \{1, ..., r\}$, $i \neq j$), we say that the writing $\sum_{i=1}^r c_i x^{\eta_i} f_{\alpha_i}$ is a *representation of g by $\tau\mathcal{F}$*.

If, moreover, $\mathcal{J}$ is Noetherian with well founded ordering $<$ and x^δ is any term, we say that a representation $g = \sum_{i=1}^r c_i x^{\eta_i} f_{\alpha_i}$ by $\tau\mathcal{F}$ is a *x^δ-lower representation* ($x^\delta - \mathrm{LRep}$ for short) and, respectively, a *x^δ-strictly lower representation* ($x^\delta - \mathrm{SLRep}$ for short) if, for every $i = 1, \dots, r$, it holds $x^{\eta_i + \alpha_i} \leq x^\delta$ and respectively $x^{\eta_i + \alpha_i} < x^\delta$.

We observe that, as an obvious consequence of the definition of reduction procedure, if $g \to^+_{\mathcal{F}\mathcal{J}} h$, then $g - h$ has a representation by $\tau\mathcal{F}$ given by the steps of the reduction (summing up the coefficients of each element of $\tau\mathcal{F}$ used more than once during the reduction).

Proposition 6.2. *Let $\mathcal{F}$ be a marked set over a weakly Noetherian RS $\mathcal{J}$ and let $g \in \mathcal{P}$.*

i) There exists a reduced form l of g obtained by $\mathcal{F}$ and $g - l$ has a representation by $\tau\mathcal{F}$.
ii) If $\mathcal{J}$ has disjoint cones, then there is only one polynomial l (the canonical form *of g) with Supp$(l) \subset \mathsf{N}(J)$ and $g - l \in \langle \tau\mathcal{F} \rangle$; moreover, there is a unique representation of $g - l$ by $\tau\mathcal{F}$.*
iii) If $\mathcal{J}$ is Noetherian (with well founded order $<$) and, for a reduced polynomial l, $g - l$ has a representation $\sum_{i=1}^r c_i x^{\gamma_i} f_{\alpha_i}$ by $\tau\mathcal{F}$ with all distinct heads $x^{\gamma_i + \alpha_i}$, then $g \to^+_{\mathcal{F}\mathcal{J}} l \downarrow$ and, for each i, $x^{\gamma_i + \alpha_i} \leq x^\delta$ for some $x^\delta \in$ Supp(g).
Vice versa, from $g \to^+_{\mathcal{F}\mathcal{J}} l \downarrow$ one deduces that $g - l$ has a representation $\sum_{i=1}^r c_i x^{\gamma_i} f_{\alpha_i}$ by $\tau\mathcal{F}$ with all distinct heads s.t. for each i it holds $x^{\gamma_i + \alpha_i} \leq x^\delta$ for some $x^\delta \in$ Supp(g).
iv) In the same hypotheses and setting of iii), if g is a term x^δ, then $x^\delta - l$ has a $x^\delta - \mathrm{LRep}$ by $\tau\mathcal{F}$.

Proof. i) follows from the definition of $\to^+_{\mathcal{F}\mathcal{J}}$ and the weak Noetherianity of $\mathcal{J}$.

In order to prove ii) we observe that $\mathcal{J}$ is in fact Noetherian, since an RS with disjoint cones has no proper substructures.

Consider two reduced polynomials l, l' such that $g - l, g - l' \in \langle \tau\mathcal{F} \rangle$ and take some representations $g - l = \sum_{i=1}^r c_i x^{\gamma_i} f_{\alpha_i}$ and $g - l' = \sum_{i=1}^r d_i x^{\gamma_i} f_{\alpha_i}$ in $\tau\mathcal{F}$; we may suppose that the indices of the two summations are the same, possibly adding some zeroes.

We have then $l - l' = \sum_{i=1}^r (d_i - c_i) x^{\gamma_i} f_{\alpha_i}$ and we deduce that $c_i = d_i$ for $i = 1, \dots, r$. If, in fact, this were not true, we could choose a maximal element in the set $\{x^{\gamma_i + \alpha_i}, i = 1, \dots, r, c_i - d_i \neq 0\}$: suppose it is $x^{\gamma_1 + \alpha_1}$. Then $x^{\gamma_1 + \alpha_1}$ appears in the support of $\sum_{i=1}^r (d_i - c_i) x^{\gamma_i} f_{\alpha_i}$: indeed this term is different from $x^{\gamma_i + \alpha_i}$ for $i = 2, \dots, r$, since by hypothesis $\mathcal{J}$ has disjoint cones, and it does not appear in the support of $x^{\gamma_i} f_{\alpha_i} - x^{\gamma_i + \alpha_i}$ for some $i = 1, \dots r$, by maximality. We get then a contradiction since the support of $l - l'$ is contained in $\mathsf{N}(J)$. Then $c_i = d_i$ and $l = l'$.

In order to prove iii), we proceed by induction on the number r of the summands. If $r = 1$ then $g = l + c_1 x^{\gamma_1} f_{\alpha_1}$, and $x^{\gamma_1+\alpha_1}$ necessarily appears in Supp(g), since it cannot coincide neither with a term in the support of l nor with a term of $x^{\gamma_1} f_{\alpha_1} - x^{\gamma_1+\alpha_1}$. We can get l from g via a base reduction step on the term $x^{\gamma_1+\alpha_1}$ using f_{α_1}.
Setting $x^\delta := x^{\gamma_1+\alpha_1}$, we trivially have $x^{\gamma_1+\alpha_1} \leq x^\delta \in$ Supp(g). Moreover, each term x^β in the support of $c_1 x^{\gamma_1} f_{\alpha_1}$ satisfies $x^\beta \leq x^\delta$ since each term $x^\gamma \in$ Supp$(f_{\alpha_1} \setminus \{x^{\alpha_1}\})$ satisfies $x^{\gamma_1+\gamma} < x^{\gamma_1+\alpha_1} \leq x^\delta$.
Suppose by inductive hypothesis that the assertion is true in the case in which we have $r-1$ summands. We can suppose that $x^{\gamma_r+\alpha_r}$ is maximal in the set $\{x^{\gamma_i+\alpha_i}, i = 1, ..., r\}$ and so it is also maximal in $\{x^\epsilon \mid x^\epsilon \in \text{Supp}(x^{\gamma_i} f_{\alpha_i}), i = 1, \ldots, r\}$. Then $x^{\gamma_r+\alpha_r}$ appears in the support of $\sum_{i=1}^r c_i x^{\gamma_i} f_{\alpha_i}$ and so also in the support of g (remember that Supp$(l) \subset \mathsf{N}(J)$). We execute the first reduction step on g choosing exactly that term and setting $g' = g - c_r x^{\gamma_r} f_{\alpha_r}$.
Setting $x^\delta := x^{\gamma_r+\alpha_r}$, we trivially have, for each i, $x^{\gamma_i+\alpha_i} \leq x^{\gamma_r+\alpha_r} = x^\delta \in$ Supp(g). Thus we obtain $g' - l = \sum_{i=1}^{r-1} c_i x^{\gamma_i} f_{\alpha_i}$ and we conclude by inductive hypothesis.
The converse statement immediately follows from the fact that $\mathcal{J}$ is ordered and also from the hypothesis.

iv) is a consequence of iii) and of Definition 6.1. □

Corollary 6.3. *Let $\mathcal{F}$ be a marked set over a weakly Noetherian RS $\mathcal{J}$. Then*

$$\langle \tau\mathcal{F} \rangle + \langle \mathsf{N}(J) \rangle = \mathcal{P}.$$

If, moreover, $\mathcal{J}$ has disjoint cones, then

$$\langle \tau\mathcal{F} \rangle \oplus \langle \mathsf{N}(J) \rangle = \mathcal{P}.$$

In particular, take $x^\eta \in \mathcal{T}$, $g, l \in \mathcal{P}$ s.t. Supp$(l) \subseteq \mathsf{N}(J)$ and $x^\gamma \leq x^\eta$, for every $x^\gamma \in$ Supp(g). Then

$$g - l \in \langle \tau\mathcal{F} \rangle \iff g \to^+_{\mathcal{F}\mathcal{J}} l \downarrow \iff g - l \text{ has a } x^\eta - LRep \text{ by } \tau\mathcal{F}.$$

Proof. The first assertion comes from Proposition 6.2. Indeed, for all $g \in \mathcal{P}$, from $g = \sum_{i=1}^r c_i x^{\gamma_i} f_{\alpha_i} + l$ with $x^{\gamma_i} f_{\alpha_i} \in \tau\mathcal{F}$ and Supp$(l) \subset \mathsf{N}(J)$ we deduce $g \in \langle \tau\mathcal{F} \rangle + \langle \mathsf{N}(J) \rangle$. So $\langle \tau\mathcal{F} \rangle + \langle \mathsf{N}(J) \rangle \supseteq \mathcal{P}$. The other implication is obvious.

For the second assertion it is then sufficient to prove that $\langle \tau\mathcal{F} \rangle \cap \langle \mathsf{N}(J) \rangle = 0$ and this comes from Proposition 6.2 *ii)*.

Now we prove the last assertion. If $g - l \in \langle \tau\mathcal{F} \rangle$, by 6.2 *ii)*, then $g - l$ has a unique representation $\sum_{i=1}^r c_i x^{\gamma_i} f_{\alpha_i}$ by $\tau\mathcal{F}$; as $\mathcal{J}$ has disjoint cones, the heads $x^{\gamma_i+\alpha_i}$ are distinct. By 6.2 *iii)* we obtain $g \to^+_{\mathcal{F}\mathcal{J}} l \downarrow$ and $x^{\gamma_i+\alpha_i} \leq x^\delta$ for some $x^\delta \in$ Supp(g); then for every x^ϵ in the support of $x^{\gamma_i} f_{\alpha_i}$ it holds $x^\epsilon \leq x^\eta$, namely $\sum_{i=1}^r c_i x^{\gamma_i} f_{\alpha_i}$ is a $x^\eta -$ LRep.

The other implications are obvious. □

Corollary 6.4. *Let $\mathcal{J} = (M, \lambda, \tau)$ be a Noetherian RS. Then there is a Noetherian RS $\mathcal{J}^{Red} = (M, \lambda^{Red}, \tau)$ with reduced tails, such that for every marked set $\mathcal{F}$ over $\mathcal{J}$ there is a $\mathcal{J}^{Red}$-marked set $\mathcal{F}^{Red}$ that satisfies $\langle \tau\mathcal{F} \rangle = \langle \tau\mathcal{F}^{Red} \rangle$. Moreover, $\mathcal{F}$ is a marked basis iff $\mathcal{F}^{Red}$ is. If $\mathcal{J}$ is also confluent, then $\mathcal{F}^{Red}$ is unique and for all $g \in \mathcal{P}$*

$$g \to^+_{\mathcal{F}\mathcal{J}} l \downarrow \iff g \to^+_{\mathcal{F}^{Red}\mathcal{J}^{Red}} l \downarrow .$$

Proof. Assume that $\mathcal{J}$ is ordered by $<$ (Theorem 5.9). For every $x^\alpha \in M$ we choose as λ^{Red}_α the support of any polynomial ℓ_α such that $x^\alpha - f_\alpha \to^+_{\mathcal{F}\mathcal{J}'} \ell_\alpha$ and set $\mathcal{F}^{Red} = \{f^{Red}_\alpha := x^\alpha - \ell_\alpha \mid x^\alpha \in M\}$.

Let us assume that $\langle \tau\mathcal{F} \rangle \neq \langle \tau\mathcal{F}^{Red} \rangle$ and consider a minimal element $x^{\eta+\alpha} \in J$ such that $x^\eta \in \tau_\alpha$ and either $x^\eta f_\alpha \notin \langle \tau\mathcal{F}^{Red} \rangle$ or $x^\eta f^{Red}_\alpha \notin \langle \tau\mathcal{F} \rangle$. Therefore, if $x^\delta \in \tau_\beta$ and $x^{\delta+\beta} < x^{\eta+\alpha}$, then both $x^\delta f_\beta \in \langle \tau\mathcal{F}^{Red} \rangle$ and $x^\delta f^{Red}_\beta \in \langle \tau\mathcal{F} \rangle$ hold.

By Proposition 6.2 (ii), the difference $x^\eta f^{Red}_\alpha - x^\eta f_\alpha$ has a $x^{\eta+\alpha}$-SLR $\sum c_i x^{\eta_i} f_{\alpha_i}$ in $\tau\mathcal{F}$; by the assumption every $x^{\eta_i} f_{\alpha_i}$ also belongs to $\langle \tau\mathcal{F}^{Red} \rangle$. Then we get a contradiction, since $x^\eta f^{Red}_\alpha = x^\eta f_\alpha + \sum c_i x^{\eta_i} f_{\alpha_i} \in \langle \tau\mathcal{F} \rangle$ and $x^\eta f_\alpha = x^\eta f^{Red}_\alpha - \sum c_i x^{\eta_i} f_{\alpha_i} \in \langle \tau\mathcal{F}^{Red} \rangle$.

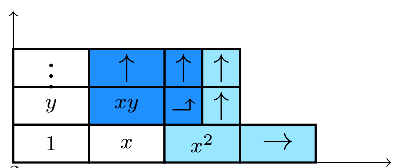
As a consequence $(\mathcal{F}) = (\mathcal{F}^{Red})$, so that $\mathcal{F}$ is a basis iff $\mathcal{F}^{Red}$ is by Corollary 6.3.

The other assertions are direct consequence of the above construction. □

In what follows, we will use the second assertion of Proposition 6.2 *(iii)*. Indeed, if one wants to use induction in proofs, it will be useful to consider the fact that not only a certain polynomial g is in $\langle \tau\mathcal{F} \rangle$, but also that g can be written as a linear combination of elements in $\tau\mathcal{F}$ whose heads satisfy the property underlined in *(iii)*.

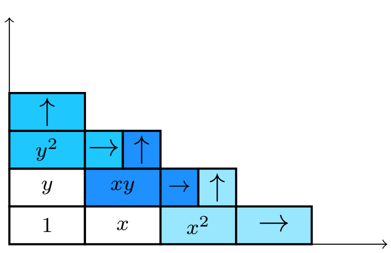
The following two examples show that the hypotheses of the various points of Proposition 6.2 are necessary. Point ii) does not necessarily hold if $\mathcal{J}$ has non-disjoint cones. Moreover, the conditions $g - l \in \langle \tau\mathcal{F} \rangle$ and $\mathrm{Supp}(l) \subset \mathsf{N}(J)$ do not necessarily imply that $g \to^+_{\mathcal{F}\mathcal{J}} l$.

Example 6.5. In $A[x, y]$, let
$\mathcal{J} = (M = \{x^2, xy\}, \{\lambda_{x^2} = \lambda_{xy} = \{x\}\}, \{\tau_{x^2} = \mathcal{T}, \tau_{xy} = \{y^k, xy^k,\ k \in \mathbb{N}\}\})$; notice that $\mathcal{J}$ is Noetherian, since it is coherent with any degree compatible term ordering, and it has cones which are not disjoint ($x^2y = x^2 \cdot y = xy \cdot x$).



Let moreover $\mathcal{F} = \{f_{x^2} = x^2 - x, f_{xy} = xy\}$ and $g = x^2y - xy$. For each reduction $g \to^+_{\mathcal{F}\mathcal{J}} l \downarrow$ we have $l = 0$, but g has two representations of the form of Proposition 6.2 ii): $g = yf_{x^2} = xf_{xy} - f_{xy}$.

Example 6.6. Consider the RS $\mathcal{J} = (M = \{x^2, xy, y^2\}, \{\lambda_{x^2} = \lambda_{y^2} = \lambda_{xy} = \{1\}\}, \{\tau_{x^2} = \tau_{y^2} = \tau_{xy} = \mathcal{T}\})$ and the marked set $\mathcal{F} = \{x^2 - 1, xy, y^2\}$ in $A[x, y]$.



For $g = y^3$ and $l = y$ we have $g - l = yf_{x^2} - xf_{xy} + yf_{y^2} \in \langle \tau\mathcal{F} \rangle$ and $\mathrm{Supp}(l) \subset \mathsf{N}(J)$, but g has only one possible complete reduction $g \to_{\mathcal{F}\mathcal{J}} 0 \downarrow$ by means of f_{y^2}; therefore, $g \to^+_{\mathcal{F}\mathcal{J}} l \downarrow$ does not hold. Notice that $g = y^3 = yf_{y^2} \in \langle \tau\mathcal{F} \rangle$, whereas $l = -yf_{x^2} + xf_{xy} \in \langle \tau\mathcal{F} \rangle \cap \langle \mathsf{N}(J) \rangle$ is exactly the S-polynomial $S(f_{x^2}, f_{xy})$ (see Remark 7.8).

7. Confluence and ideal membership

The reduction procedure on a polynomial f with respect to a given marked set $\mathcal{F}$ over an RS $\mathcal{J} = (M, \lambda, \tau)$ in general is not unique.

For instance, we start the reduction choosing a monomial u in $\mathrm{Supp}(f) \cap J$ (there could be several) and a term m in M such that $u \in \mathrm{cone}(m)$ (there could be several). If $\mathcal{J}$ is Noetherian, after a finite number of steps we obtain a reduced form l. It is natural to ask whether l could be independent of the choices we performed, namely under which conditions the procedure is confluent.

Definition 7.1. Let $\mathcal{F}$ be a marked set over a weakly Noetherian RS $\mathcal{J}$. If for each polynomial g there is one and only one l s.t. $g \to^+_{\mathcal{F}\mathcal{J}} l \downarrow$, then we call $\to^+_{\mathcal{F}\mathcal{J}}$ *confluent*.

We call $\mathcal{J}$ *confluent* if for each marked set $\mathcal{F}$ over $\mathcal{J}$, the reduction procedure $\to^+_{\mathcal{F}\mathcal{J}}$ is confluent.

The most significant case of confluent RS is the one presented in the following

Remark 7.2. If $\mathcal{J} = (M, \lambda, \tau)$ is a weakly Noetherian RS with disjoint cones, then it is Noetherian and confluent.

Since Noetherianity follows by Lemma 5.4, we need to show that each marked set $\mathcal{F}$ over $\mathcal{J}$ is confluent. If there are $g \in \mathcal{P}$, $l, l' \in \mathcal{P}$, $\mathrm{Supp}(l) \subset \mathsf{N}(J)$, $\mathrm{Supp}(l') \subset \mathsf{N}(J)$, s.t. $g \to^+_{\mathcal{F}\mathcal{J}} l \downarrow$ and $g \to^+_{\mathcal{F}\mathcal{J}} l' \downarrow$, then by Corollary 6.3, we would have $l' - l = (g-l)-(g-l') \in \langle \tau\mathcal{F} \rangle$, hence $l' - l = 0$.

Example 7.3. The set of all marked sets over the RS

$$(M, \lambda := \{\lambda_\alpha\,,\ x^\alpha \in M\}, \tau := \{\tau_\alpha\,,\ x^\alpha \in M\})$$

with $\lambda_\alpha = \emptyset$ for all $x^\alpha \in M$, consists of the single set $\{f_\alpha = x^\alpha : x^\alpha \in M\}$ namely with the monomial set M it self. Therefore $\mathcal{J}$ is obviously Noetherian and confluent.

If however we assume (as in Buchberger's Theory) $\tau_\alpha = \mathcal{T}$ for all $x^\alpha \in M$, then the cones of two different monomials in M are not disjoint!

Of course, an RS $\mathcal{J} = (M, \lambda, \tau)$ coherent with a term ordering and with maximal cones is both Noetherian and with non-disjoint cones (unless $\#M = 1$). In this "natural" setting confluency is related with ideal membership. On the other side, Janet (followed by all research in involutiveness) introduced, in the reduction step related with *membership test*, the restriction to disjoint cones thus trivially guaranteeing confluence. The counterpart, clearly, is that one has to transfer to a different procedure the task of granting that the A-vector space $\langle \tau\mathcal{F} \rangle$ generated by the set $\tau\mathcal{F}$ of all polynomials $x^\gamma f_\alpha$ with $f_\alpha \in \mathcal{F}$ and $x^\gamma \in \tau_\alpha$ which, in principle is just a sub-vector-space of the ideal $(\mathcal{F})$ generated by $\mathcal{F}$, really coincides with it. Janet approach was, originally via Riquier's *completion*, later, in connection with Cartan test, with complete linear reduction of sufficiently many vector-spaces $\mathcal{F}_d$.

Let $\mathcal{J}$ be a weakly Noetherian RS. Even if the cones in $\mathcal{J}$ are not disjoint, we can "simulate" this property in the following way.

Let $\tilde{\tau} = \{\tilde{\tau}_\alpha, x^\alpha \in M\}$ be s.t. each $\tilde{\tau}_\alpha$ is a subset of τ_α; in what follows we will denote by $\to^+_{\tilde{\tau}\mathcal{F}\mathcal{J}}$ the reduction process obtained by using only polynomials of $\tilde{\tau}\mathcal{F} := \{x^\eta f_\alpha \,|\, f_\alpha \in \mathcal{F},\ x^\eta \in \tilde{\tau}_\alpha\}$.

Lemma 7.4. *Let $\mathcal{J} = (M, \lambda, \tau)$ be a weakly Noetherian RS. Then, there is a list of sets of terms $\overline{\tau} = \{\overline{\tau}_\alpha\}_{x^\alpha \in M}$ with $\overline{\tau}_\alpha \subseteq \tau_\alpha$ s.t.*

- *$\forall x^\alpha, x^{\alpha'} \in M$, $x^\alpha \neq x^{\alpha'}$, one has $x^\alpha \overline{\tau}_\alpha \cap x^{\alpha'} \overline{\tau}_{\alpha'} = \emptyset$*
- *$\bigcup_{x^\alpha \in M} x^\alpha \overline{\tau}_\alpha = J$*
- *for each marked set $\mathcal{F}$ on $\mathcal{J}$, the reduction process $\to^+_{\overline{\tau}\mathcal{F}\mathcal{J}}$ is Noetherian.*

Proof. By hypothesis there is a Noetherian substructure $\mathcal{J}' = (M, \lambda, \tau')$ of $\mathcal{J}$, so $\tau'_\alpha \subseteq \tau_\alpha$ and $J = \bigcup_{x^\alpha \in M} x^\alpha {\tau'}_\alpha$.

We can construct the required subsets $\overline{\tau}_\alpha$ of τ'_α as follows: for each $x^\beta \in J$ we choose randomly one and only one monomial $x^\alpha \in M$ s.t. $x^\beta = x^{\gamma+\alpha}$ with $x^\gamma \in \tau'_\alpha$ and insert x^γ in $\overline{\tau}_\alpha$.

Of course this is all one needs to find subsets $\overline{\tau}_\alpha \subseteq \tau_\alpha$ and grant that the first two conditions are satisfied; moreover, Noetherianity of $\mathcal{J}'$ grants Noetherianity of $\to^+_{\overline{\tau}\mathcal{F}\mathcal{J}}$. $\square$

By a restriction to disjoint cones we can now reinforce point iii) of Proposition 6.2.

Proposition 7.5. *Let $\mathcal{F}$ be a marked set over a weakly Noetherian RS $\mathcal{J}$, $\mathcal{J}'$ be a Noetherian substructure (with order $<$) and $\overline{\tau}$ be as in Lemma 7.4.*

Then, for all $g \in \mathcal{P}$, there exists a unique J-remainder l s.t. $g \to^+_{\overline{\tau}\mathcal{F}\mathcal{J}} l$. Moreover, $g-l$ has a representation $\sum_j c_j x^{\gamma_j} f_{\alpha_j}$ by $\overline{\tau}\mathcal{F}$ with all distinct heads and $x^{\gamma_j+\alpha_j} < x^\delta$ for some $x^\delta \in Supp(g)$, and $l=0$ if and only if $g \in \langle \overline{\tau}\mathcal{F}\rangle$. Therefore

$$\langle \overline{\tau}\mathcal{F}\rangle \oplus \langle \mathsf{N}(J)\rangle = \mathcal{P}.$$

Proof. For every polynomial $g \in \mathcal{P}$, the J-remainder l exists and is unique by Lemma 7.4. Notice that the elements of $\overline{\tau}\mathcal{F}$ have all distinct heads; moreover $\to^+_{\overline{\tau}\mathcal{F}\mathcal{J}}$ is Noetherian since $\mathcal{J}'$ is Noetherian. We conclude by Corollary 6.3. $\square$

We can now characterize confluency of marked sets over weakly Noetherian RSs

Theorem 7.6. *Let $\mathcal{F}$ be a marked set over a weakly Noetherian RS $\mathcal{J}$ and let $\mathcal{J}'$ and $\overline{\tau}$ be as in Lemma 7.4. The following statements are equivalent:*

i) $\to^+_{\mathcal{F}\mathcal{J}}$ *is confluent.*
ii) $\langle \tau\mathcal{F}\rangle \oplus \langle \mathsf{N}(J)\rangle = \mathcal{P}$.
iii) $\langle \tau\mathcal{F}\rangle \cap \langle \mathsf{N}(J)\rangle = 0$.
iv) $\langle \tau\mathcal{F}\rangle = \langle \tau'\mathcal{F}\rangle = \langle \overline{\tau}\mathcal{F}\rangle$.
v) for each $x^\eta f_\alpha \in \tau\mathcal{F} \setminus \overline{\tau}\mathcal{F}$ *it holds* $x^\eta f_\alpha \to^+_{\overline{\tau}\mathcal{F}\mathcal{J}} 0\downarrow$.
vi) for each $x^\eta f_\alpha \in \tau\mathcal{F}$, *for each reduction* $x^\eta f_\alpha \to^+_{\mathcal{F}\mathcal{J}'} l\downarrow$ *it holds* $l=0$.
vii) for each $x^\eta f_\alpha \in \tau\mathcal{F}$, $x^{\eta'} f_{\alpha'} \in \tau'\mathcal{F}$ *with* $x^{\eta+\alpha} = x^{\eta'+\alpha'}$ *it holds* $x^\eta f_\alpha - x^{\eta'} f_{\alpha'} \to^+_{\mathcal{F}\mathcal{J}'} 0\downarrow$.
viii) for each $x^\eta f_\alpha \in \tau\mathcal{F}$, *for each reduction* $x^\eta f_\alpha \to^+_{\mathcal{F}\mathcal{J}} l\downarrow$ *it holds* $l=0$.
ix) for each $x^\eta f_\alpha, x^{\eta'} f_{\alpha'} \in \tau\mathcal{F}$ *with* $x^{\eta+\alpha} = x^{\eta'+\alpha'}$, *for each reduction* $x^\eta f_\alpha - x^{\eta'} f_{\alpha'} \to^+_{\mathcal{F}\mathcal{J}} l\downarrow$ *it holds* $l=0$.

Proof. ii) $\Leftrightarrow$ iii): the assertion trivially follows from Corollary 6.3.

iii) $\Rightarrow$ i): notice that if $g \to^+_{\mathcal{F}\mathcal{J}} l\downarrow$ and $g \to^+_{\mathcal{F}\mathcal{J}} l'\downarrow$, then the difference $l-l'$ belongs to $\langle \tau\mathcal{F}\rangle \cap \langle \mathsf{N}(J)\rangle$, so $l-l'=0$.

ii) $\Leftrightarrow$ iv) $\Leftrightarrow$ v) : follow from Proposition 7.5 and from the fact that by construction $\tau\mathcal{F} \supseteq \tau'\mathcal{F} \supseteq \overline{\tau}\mathcal{F}$.

viii) $\Rightarrow$ vi) $\Rightarrow$ v) are trivial, since the reductions by $\to^+_{\overline{\tau}\mathcal{F}\mathcal{J}}$ are particular cases of the ones by $\to^+_{\mathcal{F}\mathcal{J}'}$, that are particular cases of the ones by $\to^+_{\mathcal{F}\mathcal{J}}$. Notice that $\mathcal{F}$ is weakly Noetherian, so each polynomial has at least a total reduction.

i) $\Rightarrow$ viii) is again obvious; indeed every polynomial $x^\eta f_\alpha \in \tau\mathcal{F}$ has at least the complete reduction $x^\eta f_\alpha \to_{\mathcal{F}\mathcal{J}} x^\eta f_\alpha - x^\eta f_\alpha = 0\downarrow$.

As a consequence of what proved so far, the conditions i), ii), iii), iv), v), vi), viii) are equivalent.

iii) $\Rightarrow$ ix): it is sufficient to observe that in the hypotheses ix) the polynomial $x^\eta f_\alpha - x^{\eta'} f_{\alpha'}$ belongs to $\tau\mathcal{F}$ and l is in the intersection $\langle \tau\mathcal{F}\rangle \cap \langle \mathsf{N}(J)\rangle$.

ix) $\Rightarrow$ vii) directly follows by the same argument used to prove "viii) $\Rightarrow$ vi) $\Rightarrow$ v) ".

We finally prove vii) $\Rightarrow$ iv). By Proposition 6.2 i), condition vii) implies $\langle \tau\mathcal{F}\rangle \subseteq \langle \tau'\mathcal{F}\rangle$. Then, it is sufficient to prove that $\langle \tau'\mathcal{F}\rangle \subseteq \langle \overline{\tau}\mathcal{F}\rangle$, the opposite inclusions being obvious.

Assume by contradiction that the set $\tau'\mathcal{F} \setminus \langle \overline{\tau}\mathcal{F}\rangle$ is not empty and choose in it an element $x^\eta f_\alpha$ such that is minimal $x^{\eta+\alpha}$, w.r.t. the ordering $<$ associated to $\mathcal{J}'$. Let, moreover, $x^{\eta'} f_{\alpha'}$ the only element in $\overline{\tau}\mathcal{F}$ such $x^{\eta+\alpha} = x^{\eta'+\alpha'}$: we may apply vii) to these two elements (as $x^\eta f_\alpha \in \tau'\mathcal{F} \subseteq \tau\mathcal{F}$ and $x^{\eta'} f_{\alpha'} \in \overline{\tau}\mathcal{F} \subseteq \tau'\mathcal{F}$) finding a reduction $x^\eta f_\alpha - x^{\eta'} f_{\alpha'} \to^+_{\mathcal{F}\mathcal{J}'} 0\downarrow$.

We observe that every term $x^\gamma \in \mathrm{Supp}(x^\eta f_\alpha - x^{\eta'} f_{\alpha'})$ is either in $\mathrm{Supp}(x^\eta f_\alpha - x^{\eta+\alpha})$ or in $\mathrm{Supp}(x^{\eta'} f_{\alpha'} - x^{\eta'+\alpha'})$. In both cases, $x^\gamma < x^{\eta+\alpha} = x^{\eta'+\alpha'}$.

Then, by Corollary 6.3, the polynomial $x^\eta f_\alpha - x^{\eta'} f_{\alpha'}$ has a $x^{\eta+\alpha}$ – SLRep in $\tau'\mathcal{F}$ of the type $\sum_{i=1}^r c_i x^{\gamma_i} f_{\alpha_i}$. By the minimality of $x^{\eta+\alpha}$ we deduce that the summands $x^{\gamma_i} f_{\alpha_i}$ belong to $\langle \overline{\tau}\mathcal{F}\rangle$, hence also $x^\eta f_\alpha - x^{\eta'} f_{\alpha'}$ does. This is the wanted contradiction, as $x^\eta f_\alpha \notin \langle \overline{\tau}\mathcal{F}\rangle$ and $x^{\eta'} f_{\alpha'} \in \overline{\tau}\mathcal{F}$. $\square$

Now we assume that $\mathcal{J}$ is a weakly Noetherian RS and see which conditions have to be satisfied by a marked set $\mathcal{F}$ over $\mathcal{J}$ in order that the rewriting rule $\to^+_{\mathcal{F}\mathcal{J}}$ gives a criterion which is equivalent to the belonging to the ideal generated by $\mathcal{F}$, i.e. in order that

$$g \equiv g' \mod (\mathcal{F}) \Longleftrightarrow \forall g \to^+_{\mathcal{F}\mathcal{J}} l\downarrow \text{ and } \forall g' \to^+_{\mathcal{F}\mathcal{J}} l'\downarrow, \text{ it holds } l = l'$$

is satisfied.

We can observe that in order to apply the test implied by $\Leftarrow$ for deciding ideal equivalence (and in particular, ideal membership) we must require that $\mathcal{J}$ is weakly Noetherian; indeed, if there is a polynomial g without complete reductions, the reduction cannot allow us to establish whether g belongs to $(\mathcal{F})$[4]

The ideal membership can be reformulated through the notion of marked bases (Definition 4.3), which constitutes a central point for the whole theory.

Theorem 7.7. *Let $\mathcal{J} = (M, \lambda, \tau)$ be a weakly Noetherian RS and let $\mathcal{J}'$ and $\overline{\tau}$ be as in Lemma 7.4. Moreover, let $\mathcal{F}$ be a marked set over $\mathcal{J}$.*

If $\mathcal{F}$ is a marked basis, then $\to^+_{\mathcal{F}\mathcal{J}}$ is confluent.

On the other hand, if we suppose that $\to^+_{\mathcal{F}\mathcal{J}}$ is confluent, then $\mathcal{F}$ is a marked basis if and only if one of the following equivalent conditions holds:

i) $(\mathcal{F}) = \langle \overline{\tau}\mathcal{F}\rangle$,
ii) $(\mathcal{F}) = \langle \tau'\mathcal{F}\rangle$,
iii) $(\mathcal{F}) = \langle \tau\mathcal{F}\rangle$,
iv) for each $x^\alpha \in M$ and each $x^\gamma \notin \overline{\tau}_\alpha$ it holds $x^\gamma f_\alpha \to^+_{\overline{\tau}\mathcal{F}\mathcal{J}} 0\downarrow$,
v) for each $x^\alpha \in M$ and $x^\gamma \notin \tau'_\alpha$ it holds $x^\gamma f_\alpha \to^+_{\mathcal{F}\mathcal{J}'} 0\downarrow$,
vi) for each $x^\alpha \in M$ and $x^\gamma \notin \tau_\alpha$ it holds $x^\gamma f_\alpha \to^+_{\mathcal{F}\mathcal{J}} 0\downarrow$.

Proof. If $\mathcal{F}$ is a marked basis, we have $\langle \tau\mathcal{F}\rangle \cap \langle \mathsf{N}(J)\rangle \subseteq (\mathcal{F}) \cap \langle \mathsf{N}(J)\rangle = 0$; then $\to^+_{\mathcal{F}\mathcal{J}}$ is confluent by Theorem 7.6 iii) $\Rightarrow$ i).

Now, assume that $\to^+_{\mathcal{F}\mathcal{J}}$ is confluent.

The conditions *i)*, *ii)*, *iii)* are equivalent by Theorem 7.6 and the conditions *iv)*, *v)*, *vi)* are equivalent since the confluence of $\to^+_{\mathcal{F}\mathcal{J}}$ grants also the confluence of $\to^+_{\mathcal{F}\mathcal{J}'}$ and $\to^+_{\overline{\tau}\mathcal{F}\mathcal{J}}$. Notice that in each of the three conditions *iv)*, *v)*, *vi)* the restriction on x^γ could

[4]This is the "flaw" of Hironaka Theory (see [86]).

be omitted; indeed, if for instance $x^\gamma \in \tau_\alpha$, then by a single step of reduction on $x^{\gamma+\alpha}$ we obtain $x^\gamma f_\alpha \to_{\mathcal{F}\mathcal{J}} x^\gamma f_\alpha - x^\gamma f_\alpha = 0 \downarrow$. Finally, the equivalence between *i)* and *iv)* is consequence of Proposition 7.5 and of the above remark about x^γ.

We conclude observing that $\langle \overline{\tau}\mathcal{F} \rangle \subseteq (\mathcal{F})$, so by Proposition 7.5, $\mathcal{F}$ is a marked basis if and only if *i)* holds. □

Remark 7.8. We can reformulate the characterizations of confluence of Theorem 7.6 and of marked bases of Theorem 7.7 using the reduction w.r.t. polynomials of the form $x^\eta f_\alpha - x^{\eta'} f_{\alpha'}$ with $x^{\eta+\alpha} = x^{\eta'+\alpha'}$. Notice, anyway, that they are not only the S-polynomials $S(f_{\alpha'}, f_\alpha) := x^\eta f_\alpha - x^{\eta'} f_{\alpha'}$, with $x^{\eta+\alpha} = x^{\eta'+\alpha'} = \mathrm{lcm}(x^\alpha, x^{\alpha'})$, but a priori also all their (infinite) multiples.

We conjecture that for every weakly Noetherian RS there exists a finite set of controls using reductions that are sufficient to ensure that a marked set is a marked basis. In particular, we do not have neither a proof nor a counter-example to the reasonable conjecture that the set of S-polynomials could be sufficient to this purpose.

For this reason, for practical purposes, it is necessary to consider RSs with particular properties, allowing to execute those verifications with a known, finite (possibly small) set of reductions. We will examine two sufficiently general cases in Sections 8 and 11; in both of them the set of controls corresponds to the set of S-polynomials or to a subset of it.

8. Maximal and disjoint cones: criteria for marked bases

In the usual reduction procedure w.r.t. a set of marked polynomials, one admits to rewrite any multiple of x^α with the marked polynomial f_α whose head is x^α. In our language, every term in $\mathcal{T}$ is considered as multiplicative for each $x^\alpha \in M$: these are the structures we call *with maximal cones*.

If such an RS $\mathcal{J} = (M, \lambda, \tau)$ is Noetherian we already remarked that it must be necessarily coherent with a term ordering by Theorem 5.10. Then the marked bases over $\mathcal{J}$ are Groebner bases. Moreover, for a set $\mathcal{F}$ marked over $\mathcal{J}$ the fact of being a marked basis and the confluency of $\to^+_{\mathcal{F}\mathcal{J}}$ are equivalent, since $(\mathcal{F})$ and $\langle \tau\mathcal{F} \rangle$ coincide by construction.

It is a well known fact that in the Groebner case, in order to check whether a marked set is a marked basis (*id est* a Groebner basis) it is sufficient to perform a finite number of controls which can be deduced by the given data, namely Buchberger test/completion result states that a basis (in our language: a marked set) $\mathcal{F}$ is Groebner (in our language: a marked basis) if and only if each element in the set of all S-polynomials

$$\left\{ S(f_{\alpha'}, f_\alpha) := \frac{\mathrm{lcm}(x^\alpha, x^{\alpha'})}{x^\alpha} f_\alpha - \frac{\mathrm{lcm}(x^\alpha, x^{\alpha'})}{x^{\alpha'}} f_{\alpha'} : x^\alpha, x^{\alpha'} \in M \right\}$$

between two elements of $\mathcal{F}$, reduces to 0.

Thus we do not need to check any of their multiples.

Being a well known theory, we do not treat it in the usual way, but we change our point of view.

As underlined in Remark 4.6, the concept of marked basis depends only on $\mathcal{F}$ and it does not depend on the RS over which we consider it as a marked set. In order to characterize the marked bases over $\mathcal{J}$, a substructure $\mathcal{J}'$ of $\mathcal{J}$ having disjoint cones (if it exists) could be useful; when, as in Groebner theory, $\mathcal{J}$ has maximal cones, such a substructure exists. We propose here one of the possible ways to construct it.

Lemma 8.1. *If $\mathcal{J} = (M, \lambda, \tau)$ is an RS with maximal cones, then there is a substructure $\mathcal{J}' = (M, \lambda, \tau')$ with disjoint cones.*

Proof. Consider the set $M = \{x^{\alpha_1}, \dots, x^{\alpha_s}\}$ and suppose that its terms are ordered in such a way that none of the x^{α_i} is multiple of a term with an index $< i$.

First of all, set $\tau'_{\alpha_1} := \mathcal{T}$ then $\tau'_{\alpha_2} x^{\alpha_2} := x^{\alpha_2}\mathcal{T} \setminus x^{\alpha_1}\mathcal{T}$. Notice that τ'_{α_2} is an order ideal (in particular $1 \in \tau'_{\alpha_2}$) since $x^{\alpha_1} \nmid x^{\alpha_2}$. By induction, after determining the multiplicative sets of the first r terms of M, set $x^{\alpha_{r+1}}\tau'_{\alpha_{r+1}} := x^{\alpha_{r+1}}\mathcal{T} \setminus \bigcup_{i=1}^{r} x^{\alpha_i}\tau'_{\alpha_i}$. ☐

In the Groebner case, $\mathcal{J}$ has maximal cones and is Noetherian, i.e. coherent with a term ordering $\prec$ (Theorem 5.10), then every substructure $\mathcal{J}'$ of $\mathcal{J}$ with disjoint cones is Noetherian, coherent with $\prec$, and confluent.

We prove now that the well known criteria to check if a marked set is a basis that appear in the Groebner theory are sufficient also in a more general setting that only assume a proper subset of the above conditions.

In the last part of this section, we will study the properties of Noetherian RSs with disjoint cones, for which the following condition on the well founded order < holds:

$$\forall x^{\delta}, x^{\delta'}, x^{\varepsilon} \in \mathcal{T}: \ x^{\delta} > x^{\delta'} \Rightarrow x^{\delta+\varepsilon} > x^{\delta'+\varepsilon} \geq x^{\varepsilon}. \tag{1}$$

This condition clearly holds if $\mathcal{J}'$ is coherent with a term ordering $\prec$ and < is exactly this term ordering.

Proposition 8.2. *Let*

$$\mathcal{J}' = (M, \lambda, \tau')$$

be a Noetherian RS with disjoint cones and suppose also that (1) holds.

If $\mathcal{F}$ is a marked set over $\mathcal{J}'$ and x^{β} is a term, the following are equivalent:

i) for each $x^{\alpha} \in M$, $x^{\eta} \notin \tau'_{\alpha}$ s.t. $x^{\eta+\alpha} < x^{\beta}$, it holds $x^{\eta} f_{\alpha} \in \langle \tau' \mathcal{F} \rangle$

ii) for each $x^{\alpha} \in M$, $x^{\eta} \notin \tau'_{\alpha}$ s.t. $x^{\eta+\alpha} < x^{\beta}$, $x^{\eta} f_{\alpha}$ has a $x^{\beta}-$SLRep.

iii) for each S-polynomial $S(f_{\alpha'}, f_{\alpha})$ s.t.

$$lcm(x^{\alpha}, x^{\alpha'}) \in cone(x^{\alpha'}) \setminus cone(x^{\alpha}) \text{ and } lcm(x^{\alpha}, x^{\alpha'}) < x^{\beta},$$

$S(f_{\alpha'}, f_{\alpha})$ has a $x^{\beta}-$SLRep.

iv) in the same hypotheses as iii) it holds $S(f_{\alpha'}, f_{\alpha}) \rightarrow^{+}_{\mathcal{F}\mathcal{J}} 0 \downarrow$.

Proof. First of all we observe that in our hypotheses if $x^{\eta+\alpha} < x^{\beta}$ then also $x^{\delta} < x^{\beta}$ for every term $x^{\delta} \in \mathrm{Supp}(x^{\eta} f_{\alpha})$.

i) ⇔ *ii)* follows by Corollary 6.3.

ii) ⇒ *iii)* consider an element $x^{\eta} f_{\alpha}$ satisfying the conditions of *ii)*. Since *i)* holds, it has a $x^{\beta}-$SLRep; summing $-x^{\eta'} f_{\alpha'}$ we get a $x^{\beta}-$SLRep of $x^{\eta} f_{\alpha} - x^{\eta'} f_{\alpha'}$.

iii) ⇔ *iv)* comes trivially from Corollary 6.3.

iii) + *iv)* ⇒ *ii)* suppose by contradiction that the assertion is false and that x^{β} is a term with x^{β} minimal among the ones not satisfying the condition. Consider $x^{\eta} f_{\alpha}$ with $x^{\eta+\alpha} < x^{\beta}$. Notice that, by hypothesis, the assertion is true in particular for $x^{\beta'} := x^{\eta+\alpha}$.

The assertion in *ii)* would immediately follow by *iii)* if for the only $x^{\eta'} f_{\alpha'}$ s.t. $x^{\eta+\alpha} = x^{\eta'+\alpha'} \in \mathrm{cone}(x^{\alpha'})$ one has $\mathrm{lcm}(x^{\alpha}, x^{\alpha'}) = x^{\eta+\alpha}$. So we must have $\mathrm{lcm}(x^{\alpha}, x^{\alpha'}) = x^{\varepsilon+\alpha} = x^{\varepsilon'+\alpha'}$ with x^{ε} proper divisor of x^{η}. We can then apply *iv)* getting $x^{\varepsilon} f_{\alpha} - x^{\varepsilon'} f_{\alpha'} =$

$S(f_{\alpha'}, f_\alpha) \to^+_{\mathcal{F}\mathcal{J}} 0 \downarrow$. Notice that by (1) for each term x^γ in the support of $x^\varepsilon f_\alpha$ and of $x^{\varepsilon'} f_{\alpha'}$ it holds $x^\gamma < x^{\varepsilon+\alpha}$. By Corollary 6.3, we have then $x^\varepsilon f_\alpha - x^{\varepsilon'} f_{\alpha'}$ has a $x^{\varepsilon+\alpha}-$SLRep, i.e. $x^\varepsilon f_\alpha - x^{\varepsilon'} f_{\alpha'} = \sum c_i x^{\gamma_i} f_{\alpha_i}$ with $x^{\gamma_i+\alpha_i} < x^{\varepsilon+\alpha}$. Multiply this representation by $x^{\eta-\varepsilon}$. For each summand $x^{\eta-\varepsilon+\gamma_i} f_{\alpha_i}$ it holds $x^{\eta-\varepsilon+\gamma_i+\alpha_i} < x^{\eta-\varepsilon+\varepsilon+\alpha} = x^{\eta+\alpha}$. By the assumption on the truth of our assertion with $x^{\beta'} = x^{\eta+\alpha}$, each polynomial $x^{\eta-\varepsilon+\gamma_i} f_{\alpha_i}$ has a $x^{\eta+\alpha}$ − SLRep. We then get a $x^{\eta+\alpha}$ − SLRep of $x^\eta f_\alpha - x^{\eta'} f_{\alpha'}$ so, summing to the two members $x^{\eta'} f_{\alpha'}$ we get a $x^{\eta+\alpha}$ − LRep of $x^\eta f_\alpha$ since $x^{\eta'+\alpha'} \in \tau'_{\alpha'}$. We conclude noticing that by hypothesis $x^{\eta+\alpha} < x^\beta$. □

By the previous results and by Theorems 7.7 and 7.6 follows

Corollary 8.3. *Let $\mathcal{J}'$ be a Noetherian RS with disjoint cones and order $<$. Suppose that* (1) *holds. Then for a marked set $\mathcal{F}$ over $\mathcal{J}'$ the following are equivalent:*

i) $\mathcal{F}$ is a marked basis
ii) $\forall x^\alpha, x^{\alpha'} \in M$ s.t. lcm$(x^\alpha, x^{\alpha'}) \in$ cone$(x^{\alpha'})$ it holds $S(f_{\alpha'}, f_\alpha) \to^+_{\mathcal{F}\mathcal{J}'} 0 \downarrow$
iii) $\forall x^\alpha, x^{\alpha'} \in M$ s.t. $x^{\gamma+\alpha} =$ lcm$(x^\alpha, x^{\alpha'}) \in$ cone$(x^{\alpha'})$ it holds $x^\gamma f_\alpha \to^+_{\mathcal{F}\mathcal{J}'} 0 \downarrow$.

For such RSs we can improve the characterization of marked bases given in Corollary 8.3 similarly to what done for Groebner bases. We can verify that also in this context some of the known simplifications hold.
The "strategy" presented here exploits a substructure of $\mathcal{J}$ with disjoint cones. Such a structure is inspired by (and generalizes) Gebauer-Moeller's *Staggered linear bases* [49].

9. Criteria

Throughout this section, for notation simplicity, we will assume that the finite set M is enumerated as $\{x^{\alpha_1}, \dots, x^{\alpha_s}\}$ and we will relabel each element f_{α_i} in the related marked set

$$\mathcal{F} = \{f_\alpha\}_{x^\alpha \in M} = \{f_{\alpha_i}, 1 \le i \le s\}$$

as $f_i := f_{\alpha_i}, 1 \le i \le s$.

We will further assume to have performed the construction outlines in Lemma 8.1; in particular we have

$$\tau'_{\alpha_1} = \mathcal{T} \text{ and } x^{\alpha_{r+1}}\tau'_{\alpha_{r+1}} := x^{\alpha_{r+1}}\mathcal{T} \setminus \bigcup_{i=1}^{r} x^{\alpha_i}\tau'_{\alpha_i} \text{ for all } i;$$

Further we will assume that the elements of M are ordered so that

$$x^{\alpha_i} \mid x^{\alpha_j} \implies i < j. \tag{2}$$

We moreover denote

- for each $i, 1 \le i \le s, \mathbf{T}(i) := x^{\alpha_i}$,
- for each $i, j, 1 \le i \le s$,
$$\mathbf{T}(i,j) := \text{lcm}(\mathbf{T}(i), \mathbf{T}(j)) = \text{lcm}(x^{\alpha_i}, x^{\alpha_j})$$
and
- $S(i,j) := S(f_i, f_j) = \frac{\mathbf{T}(i,j)}{\mathbf{T}(j)} f_j - \frac{\mathbf{T}(i,j)}{\mathbf{T}(i)} f_i$;
- for each $i, j, k : 1 \le i, j, k \le s$,
$$\mathbf{T}(i,j,k) := \text{lcm}(\mathbf{T}(i), \mathbf{T}(j), \mathbf{T}(k)) = \text{lcm}(x^{\alpha_i}, x^{\alpha_j}, x^{\alpha_k}).$$

Lemma 9.1 (Möller). [84] *For each $i, j, k : 1 \leq i, j, k \leq s$ it holds*

$$\frac{\mathbf{T}(i,j,k)}{\mathbf{T}(i,k)}S(i,k) - \frac{\mathbf{T}(i,j,k)}{\mathbf{T}(i,j)}S(i,j) + \frac{\mathbf{T}(i,j,k)}{\mathbf{T}(k,j)}S(k,j) = 0.$$

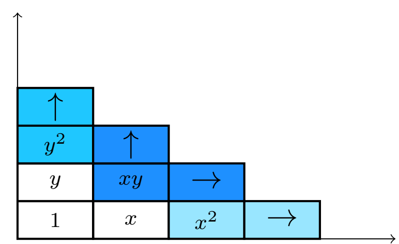
Buchberger test/completion result states that a basis (in our language: a marked set) $\mathcal{F}$ is Groebner (in our language: a marked basis) if and only if each S-polynomial $S(i,j), i, j, 1 \leq i \leq s$, between two elements of $\mathcal{F}$, reduces to 0 and gave two criteria [25] to detect S-pairs which are "useless" in the sense that theoretical results prove that they reduce to 0, thus making useless the normal form computation. The First Criterion (Propostion 9.3) is based on a direct reformulation of trivial syzygies, the Second is a direct application Lemma 9.1.

We remark that the test/completion result given by Proposition 8.2.iv) allows to remove many useless S-pairs.

In fact, an S-polynomial $S(i,j)$ is not to be tested, and thus considered "useless", if $\mathbf{T}(i,j) \notin \mathrm{cone}(\mathbf{T}(i)) \cup \mathrm{cone}(\mathbf{T}(j))$.

Example 9.2. Let us consider $M := \{x^{\alpha_i} : 1 \leq i \leq 3\}$ with

- $x^{\alpha_1} = \mathbf{T}(1) = xy, \tau_{\alpha_1} = \mathcal{T}$,
- $x^{\alpha_2} = \mathbf{T}(2) = y^2, \tau_{\alpha_2} = \{y^i : i \in \mathbb{N}\}$,
- $x^{\alpha_3} = \mathbf{T}(3) = x^2, \tau_{\alpha_3} = \{x^i : i \in \mathbb{N}\}$



and remark that

$$S(2,3) = yS(1,3) - xS(1,2).$$

Note that

$$\mathbf{T}(2,3) = x^2y^2 \notin \mathrm{cone}(\mathbf{T}(2)) \cup \mathrm{cone}(\mathbf{T}(3)) = \{y^{i+2} : i \in \mathbb{N}\} \cup \{x^{i+2} : i \in \mathbb{N}\}$$

while

$$\frac{\mathbf{T}(1,2)}{\mathbf{T}(1)} = y \in \tau_{\alpha_1} = \mathcal{T} \ni y\frac{\mathbf{T}(1,3)}{\mathbf{T}(1)}$$

so we detect the "useless" pair $S(2,3)$.

Naturally, we can prove in our setting Buchberger Second Criterion; we also can prove Buchberger First Criterion

Proposition 9.3. [25] *(Buchberger First Criterion) Under the hypotheses of Corollary 8.3 for $\mathcal{F}$ being a marked basis it is not necessary to check whether the S-polynomials $S(f_{\alpha'}, f_\alpha)$ s.t. lcm$(x^\alpha, x^{\alpha'}) = x^{\alpha+\alpha'}$ reduce to 0.*

Proof. Suppose $\mathrm{lcm}(x^\alpha, x^{\alpha'}) = x^{\alpha+\alpha'}$. Apply Proposition 8.2 choosing $x^\beta = x^{\alpha+\alpha'}$. If some of the requested controls is negative, $\mathcal{F}$ is not a marked basis and we can conclude it without using $S(f_{\alpha'}, f_\alpha)$. Otherwise, all the polynomials $x^\epsilon f_{\alpha''}$ with $x^{\epsilon+\alpha''} < x^{\alpha'+\alpha}$ belong to $\langle \tau\mathcal{F} \rangle$.

Denoted $f_\alpha = x^\alpha - g_\alpha$ and $f_{\alpha'} = x^{\alpha'} - g_{\alpha'}$, it holds $x^{\alpha'}f_\alpha - x^\alpha f_{\alpha'} = g_{\alpha'}f_\alpha - g_\alpha f_{\alpha'}$. By definition of ordered RS, all the terms x^δ in the support of g_α are s.t. $x^\delta < x^\alpha$, so by (1) we have $x^{\delta+\alpha'} < x^{\alpha+\alpha'}$. Then $g_\alpha f_{\alpha'} \in \langle \tau\mathcal{F} \rangle$. Similarly we get $g_{\alpha'}f_\alpha \in \langle \tau\mathcal{F} \rangle$ and we conclude that their difference $S(f_{\alpha'}, f_\alpha)$ is in $\langle \tau\mathcal{F} \rangle$. □

Differently from Groebner bases, it is not always true that the S-polynomial of two polynomials with coprime heads reduces to 0.

Example 9.4. Consider the RS with $M = \{x, y, xz\} \subset \mathcal{P} = A[x, y, z]$, $\tau_x = \mathcal{T}[x, y]$, $\tau_y = \tau_{xz} = \mathcal{T}$. Take $\mathcal{F} = \{f_x = x, f_y = y - z, f_{xz} = xz - z^2\}$. We will have then $yf_x, xf_y \in \tau\mathcal{F}$, but the only reduction of the S-polynomial $S(f_y, f_x) = yf_x - xf_y = xz \to_{\mathcal{F}\mathcal{J}} z^2 \downarrow$ (by means of f_{xz}) does not produce 0. The point, of course, is that (1) is not satisfied

Proposition 9.5. [25] *(Buchberger Second Criterion) Under the hypotheses of Corollary 8.3, for $\mathcal{F}$ being a marked basis it is not necessary to control that $S(f_{\alpha'}, f_{\alpha''})$ reduces to 0 if we already checked $S(f_{\alpha'}, f_\alpha)$ and $S(f_{\alpha''}, f_\alpha)$, and $x^\alpha \mid lcm(x^{\alpha'}, x^{\alpha''})$.*

Proof. By hypothesis and Lemma 9.1 $S(f_{\alpha'}, f_{\alpha''}) = x^{\varepsilon'} S(f_{\alpha'}, f_\alpha) - x^{\varepsilon''} S(f_{\alpha''}, f_\alpha)$ for some $x^{\varepsilon'}, x^{\varepsilon''} \in \mathcal{T}$. Apply Proposition 8.2 choosing $x^\beta = \mathrm{lcm}(x^{\alpha'}, x^{\alpha''})$. If some of the requested controls is negative, $\mathcal{F}$ is not a marked basis and we can conclude it without using $S(f_{\alpha'}, f_{\alpha''})$. Otherwise, we know that all the polynomials $x^\epsilon f_\gamma$ with $x^{\epsilon+\gamma} < x^{\alpha'+\alpha}$ are in $\langle \tau\mathcal{F} \rangle$. By hypothesis we also know that $S(f_{\alpha'}, f_\alpha) \in \langle \tau\mathcal{F} \rangle$; so we can write it by a $\mathrm{lcm}(x^{\alpha+\alpha'})$-SLRep since for each term x^δ in the support of $S(f_{\alpha'}, f_\alpha)$ one has $x^\delta < \mathrm{lcm}(x^\alpha, x^{\alpha'})$. Then, multiplying the summands $x^{\eta_i} f_{\alpha_i}$ of this representation by $x^{\varepsilon'}$, we get polynomials $x^{\varepsilon'+\eta_i} f_{\alpha_i}$ belonging to $\langle \tau'\mathcal{F} \rangle$ since $x^{\varepsilon'+\eta_i+\alpha_i} < x^{\varepsilon'}\mathrm{lcm}(x^\alpha, x^{\alpha'}) = \mathrm{lcm}(x^{\alpha'}, x^{\alpha''})$.

Then $x^{\varepsilon'} S(f_{\alpha'}, f_\alpha)$ is in $\langle \tau'\mathcal{F} \rangle$. Similarly we can obtain that $x^{\varepsilon''} S(f_{\alpha''}, f_\alpha)$ is in $\langle \tau'\mathcal{F} \rangle$ and we conclude. □

Let us now enumerate the set of all S-pairs by a well founded order $\prec$ which preserves divisibility (see (1)):

$$\mathbf{T}(i_1, j_1) \mid \mathbf{T}(i_2, j_2) \neq \mathbf{T}(i_1, j_1) \implies (i_1, j_1) \prec (i_2, j_2) \tag{3}$$

Corollary 9.6 (Buchberger)**.** [25][88, II.Lemma 25.1.3] *Let*

$$\mathfrak{B} \subset \{\{i, j\}, 1 \leq i < j \leq s\}$$

be such that for each $\{i, j\}, 1 \leq i < j \leq s$, either

- $\mathbf{T}(i, j) = \mathbf{T}(i)\mathbf{T}(j)$ *or*
- *there is $k, 1 \leq k \leq s$ such that*
  - $\mathbf{T}(k) \mid \mathbf{T}(i, j)$ *and*
  - $\{i, k\} \prec \{i, j\}$
  - $\{k, j\} \prec \{i, j\}$.

Then under the hypotheses of Corollary 8.3 for $\mathcal{F}$ being a marked basis it is sufficient to check whether the S-polynomials belonging to $\{\{i, j\}, 1 \leq i < j \leq s\} \setminus \mathfrak{B}$ for $\mathcal{F}$ reduce to 0.

Proof. The proof is performed by induction according $\prec$: for each $i, j, 1 \leq i < j \leq s$, either

- $\{i, j\} \notin \mathfrak{B}$, and $S(i, j)$ reduces to 0 by assumption, or
- $\mathbf{T}(i)\mathbf{T}(j) = \mathbf{T}(i, j)$ and $S(i, j)$ reduces to 0 by Buchberger's First Criterion, or
- $S(i, j)$ reduces to 0 by Buchberger's Second Criterion, since by inductive assumption both $S(i, k)$ and $S(k, j)$ reduce to 0.

□

The following example shows that Corollary 8.3 can effectively apply the power granted by Möller's Lemma 9.1 and Buchberger's Corollary 9.6 only if the construction outlined in Lemma 8.1 is performed on the elements of M after having preliminarily ordered them so that (2) holds.

Example 9.7. Let $\mathcal{J} = (M = \{xy, xz, yz^2\}, \lambda, \tau)$ be the RS in $\mathcal{T} = \mathcal{T}[x, y, z]$ with disjoint cones given by $\tau_{xy} = \mathcal{T}[x, y]$, $\tau_{xz} = \mathcal{T}[x, z] \cup \mathcal{T}[x, y]$, $\tau_{yz^2} = \mathcal{T}[x, y, z]$ (and tails defined in any way such that $\mathcal{J}$ be Noetherian). In order to decide whether a marked set $\mathcal{F} = \{f_{xy}, f_{xz}, f_{yz^2}\}$ on $\mathcal{J}$ is a basis according with Corollary 8.3 we should check the reductions of the three S-polynomials $S(f_{xz}, f_{xy}) = zf_{xy} - yf_{xz}$, $S(f_{yz^2}, f_{xy}) = z^2 f_{xy} - xf_{yz^2}$, $S(f_{yz^2}, f_{xz}) = yzf_{xz} - xf_{yz^2}$. However, by Proposition 9.5 it is sufficient to check the first and either the second or the third pair, as both xy and xz divide $\mathrm{lcm}(xy, yz^2) = \mathrm{lcm}(xz, yz^2) = xyz^2$.

Note that we have

$$\begin{aligned} S(f_{yz^2}, f_{xy}) - zS(f_{xz}, f_{xy}) + S(f_{yz^2}, f_{xy}) &= \\ \left(z^2 f_{xy} - \mathbf{x f_{yz^2}}\right) - z\left(zf_{xy} - \mathbf{y f_{xz}}\right) + \left(yzf_{xz} - xf_{yz^2}\right) &= 0 \end{aligned}$$

where $xf_{yz^2}, yzf_{xz} \notin \langle \tau' \mathcal{F} \rangle$ while $xf_{yz^2}, yf_{xz} \in \langle \tau' \mathcal{F} \rangle$; as a consequence we have

$$g := yf_{xz} \in \langle \tau' \mathcal{F} \rangle \not\Longrightarrow zg = yzf_{xz} \in \langle \tau' \mathcal{F} \rangle$$

We further remark that the ordering of the elements of M which follows the construction proposed by Janet [64] has the negative aspect that the first element yz^2 to which, according the Staggered Basis construction outlined in Lemma 8.1, we associate $\tau_{yz^2} = \mathcal{T}$ is of higher degree than the other two elements.

This is the reason why we fail here to obtain the full effect of Möller's Lemma 9.1.

It is well-known that the need of storing and ordering all pairs $\{i, j\}, 1 \leq i < j \leq s$, in order to extract $\mathfrak{B}$ produces a bottleneck and is the weakness of Buchberger's Corollary 9.6; all efficient implementation of Buchberger Criteria have the ability of storing only "useful" pairs; our approach based on Corollary 8.3 shares the same property.

10. Applications to Hilbert schemes

As we already remarked in the introduction, Groebner deformations are flat families, but in general they do not give suitable flat families for studying Hilbert schemes; in fact in general, given a point in the Hilbert scheme, defined by an ideal J, Groebner deformations do not allow to build open neighbourhoods of the deformed ideal J in the Hilbert scheme. Similar difficulties can occur even in simpler cases.

Example 10.1. Consider the Hilbert scheme, $\mathrm{Hilb}^2_{3,A}$ parametrizing zero-dimensional subschemes of degree 3 in the projective plane over the field A. Its closed points are defined by homogeneous ideals in $A[x_0, x_1, x_2]$ such that their quotients have Hilbert polynomial equal to 3. Let us consider the following three ideals: $J_a = (x_2, x_1^3)$, $J_b = (x_1, x_2^3)$, $J_c = (x_0, x_1^3)$. From a geometrical point of view, they are perfectly *equivalent*, since they describe points obtained, one from the other, via a change of coordinates; however their Groebner deformations with respect to the DegLex ordering induced by $x_0 > x_1 > x_2$ form families $\mathcal{G}_i$ which are substantially different.

J_a has no non-trivial deformations since $\mathcal{G}_a$ consists of the single root of J_a itself, thus having dimension 0.

As regard the deformations of J_b, they are all and only those of the form (x_1+Tx_2, x_2^3), for every $T \in A$; thus $\mathcal{G}_b$ is isomorphic to $\mathbb{A}^1_A$.

The deformations of J_c are all and only those of the form $(x_0 + T_1x_1 + T_2x_2, x_1^3 + S_1x_1^2x_2 + S_2x_1x_2^2 + S_3x_2^3)$; thus $\mathcal{G}_c$ is isomorphic to $\mathbb{A}^5_A$.

However no $\mathcal{G}_i$, $i = a, b, c$, is an open neighbourhood of J_i in Hilb^2_3, since Hilb^2_3 has dimension 6 and is irreducible.

To obtain an open neighbourhood of the point corresponding to J_c in Hilb^2_3, we can consider the ideal $J_{c,3}$ generated by the monomials of degree 3 in J_c: even if J_c and $J_{c,3}$ have the same saturation and thus define the same point of Hilb^2_3, the family of Groebner deformations of $J_{c,3}$ has dimension 6 thus being an open of the Hilbert scheme.

As regards the previous example we can observe that J_c is the generic ideal of both J_a and J_b and gives a larger family of Groebner deformations. This is not just by chance and actually the connection is evident: if the monomial ideal J is in "generic position", its Groebner deformations are marked bases in a reduction structure which is coherent with the chosen term ordering and where M is the monomial basis of J, the tail sets have more elements than the analogous geometric case, which algebraically speaking are not in generic position (i.e. they are not stable with respect the performance of a generic change of coordinates and the computation of the related initial ideal). For a complete treatment of this problem for particular reduction structures, we point to [35, 72].

We can also remark that passing from J_c to $J_{c,3}$ is not made by chance. Actually, it is a general fact that a saturated monomial ideal J is always obtained considering J_r, i.e. cutting J at degree r, the regularity of J. For a complete discussion of this remark see [73], and in particular section 5.

Anyway, more complex examples show that in general the two cautions above are not sufficient to grant that the deformation family of J gives an open neighbourhood of J in the Hilbert scheme. In order to obtain neighbourhoods of a point defined by a monomial ideal J in the Hilbert scheme, Groebner deformations are not sufficient even if the two conditions above are satisfied.

The entire section 6 of [73], is completely devoted to the detailed discussion of an example related with this third remark.

In a series of papers, these weaknesses of Groebner deformation families are overcome by the introduction of marked families which are not coherent with a term ordering. We recall in chronological order, [36, 20, 17, 16, 32].

In all these papers the ideals corresponding to deformations of a given monomial ideal J are always built fixing a set of generators G of J and assigning a set of polynomials marked by G, where the coefficient of each monomial is a parameter or parametrized function. Some of these parameters or parametrized functions can be considered free while other ones are required to satisfy constraints given by some polynomial equations.

This makes necessary to build J-marked ideals (and thus reduction structures) with coefficients in a ring instead of just in a field; we call *marked family* a marked set whose coefficients belong to an A-algebra (for instance the A-algebra obtained extending A with the involved parameters).

Specializing the parameters in a marked family, given using parameters or parametric functions, plugging in constant values (chosen among the elements of A in such a way that the imposed constraints are satisfied), one obtains marked sets defining ideals deforming J and related to points in a locally closed subscheme of the Hilbert scheme.

Such construction is however delicate and, if improperly applied, can produced significant mistakes, as we can easily see reconsidering the previous example.

Example 10.2. Consider the marked family $(T_0x_1 + T_1x_2, x_2^3)$ over J_c with coefficients in the ring $A[T_0, T_1]$ obtained extending A with two indeterminates. All possible specializations of T_0, T_1 in A produce an ideal defining a point in Hilb_3^2, but setting $T_0 = 0$ e $T_1 = t \in A$ the ideal has a different Hilbert polynomial. Clearly the problem occurs when a marked monomial vanishes through the specialization.

A less trivial and more surprising example is discussed in [16]; example 6.15 gives a particular set of marked polynomials over a monomial ideal J whose coefficients depend on a parameter T; all the ideals obtained specializing T to any value t in A define points of the same Hilbert scheme. However, such points do not describe, as one should expect, a curve of deformations of J within the Hilbert scheme parametrized by $\mathbb{A}_A^1$, but a locus with two irreducible components, one consisting of an isolated point P_0, corresponding to the value $T = 0$ of the parameter, and the other parametrized by $\mathbb{A}_A^1 - \{P_0\}$.

What is missing in the quoted examples is the *flatness of the marked family*, the crucial property in the whole theory of Hilbert schemes. Flatness of a family depending by parameters grants that Hilbert polynomial is stable in each specialization of the parameters, but, as shown by the example above, it is more than that. In order for the family to be flat, it is necessary (and sufficient) that for each possible specialization of the parameters (not only in A, but also in each A-algebra) what we obtain is still a family marked over J; so no specializations of the parameters should make the leading coefficients of the marked monomials vanish. To avoid that it is necessary that all leading coefficients are invertible: in this case there is no restriction assuming them equal to 1_A.

The whole theory discussed in this paper is perfectly suitable for satisfying the requirements related to flatness described above. We stress in fact that, though for simplicity we have assumed the coefficients to belong to a field A, if we simply substitute the assumption that A is a field with the requirements that

(1) A is a commutative ring with identity and
(2) each marked polynomial is monic

none of the presented results fail nor any proof is not complete.

Under such more general assumptions, any marked basis with coefficients in an A-algebra B produces a *family* with the meaning that such term assumes in algebraic geometric, id est is a flat family corresponding to a map from $\mathrm{Spec}(B)$ to the Hilbert scheme. For a complete proof of flatness of the marked bases defined in this paper we point to the appendix.

The good property of some particular term ordering free marked families have been initially studied in a series of papers quoted at the beginning of this section and the obtained results have being applied for solving problems related to Hilbert scheme which are open by years and which are object of conjectures. We remember [43] related to the components of the Hilbert scheme whose points correspond to Arithmetically Cohen-Macaulay curves in $\mathbb{P}^3$; [18], which introduced efficient computational methods for detecting components of the Hilbert scheme; [15, 14] which construct subschemes of the Hilbert scheme parametrizing liftings of a given scheme; [19], which gives an affermative answer to a conjecture related to Gorenstein algebras and finds new elementary components of the Hilbert scheme of points;[22, 9] where marked families give explicit equations of each Hilbert scheme and for the loci of the Hibert scheme of points with upper bounded regularity.

A more general construction of marked sets with respect to that presented in this paper and a deeper study of the conditions granting the use of efficient resolution algorithms, could allow to attempt and, hopefully, solve more problems. There is a current study of a generalization to modules of the present results for ideals with applications to the Quot schemes, which are a generalization of the Hilbert Schemes [1].

11. Stably ordered reduction structures

Another case in which the control proving whether a marked set is a marked basis can be performed via a finite number of predetermined reductions is the case of *stably ordered RSs* that now we introduce.

In the following Section 12, we will examine some significant examples that are included in this case, such as border bases and Pommaret bases; we will see that for each of them we can consider term ordering free versions.

Definition 11.1. Let $\mathcal{J} = (M, \lambda, \tau)$ be an RS. We will say that $\mathcal{J}$ is *stably ordered* by a well founded order < if taken $x^\alpha, x^{\alpha'} \in M$ and $x^\eta, x^{\eta'}, x^\epsilon \in \mathcal{T}$:

StOr1: $x^\eta > 1$ for each term $x^\eta \neq 1$
StOr2: $x^{\eta'} > x^\eta$ iff $x^{\eta'+\epsilon} > x^{\eta+\epsilon}$
StOr3: if $x^{\eta+\alpha} = x^{\eta'+\alpha'} \in \text{cone}(x^{\alpha'})$ and $x^\alpha \neq x^{\alpha'}$, then $x^\eta > x^{\eta'}$
StOr4: if $x^\gamma \in \lambda_\alpha$, $x^\eta \in \tau_\alpha$ and $x^{\eta+\gamma} = x^{\alpha'+\eta'} \in \text{cone}(x^{\alpha'})$ then $x^\eta > x^{\eta'}$.

Lemma 11.2. *A stably ordered RS $\mathcal{J}$ has reduced tails and disjoint cones.*

Proof. Let $\mathcal{J} = (M, \lambda, \tau)$ be stably ordered by the well founded order <. If for some $x^\alpha, x^{\alpha'} \in M$ there is $x^\gamma \in \lambda_\alpha \cap \text{cone}(x^{\alpha'})$, then by StOrd4 (with $x^\eta = 1$) we get $1 > x^{\gamma-\alpha'}$ in contradiction with StOrd1. Hence $\mathcal{J}$ has reduced tails.

If there is a term $x^\delta \in \text{cone}(x^\alpha) \cap \text{cone}(x^{\alpha'})$ with $x^\alpha \neq x^{\alpha'}$, by StOrd3 we would get the contradiction $x^{\delta-\alpha} > x^{\delta-\alpha'}$ and also $x^{\delta-\alpha'} > x^{\delta-\alpha}$. Therefore $\mathcal{J}$ has disjoint cones. □

Due to the previous lemma it makes sense the following definition:

Definition 11.3. Let $\mathcal{J} = (M, \lambda, \tau)$ be an RS stably ordered by < and $\varphi \colon J \to \mathcal{T}$ be the function given by $\varphi(x^\beta) := x^{\beta-\alpha}$ where x^α is the unique term in M such that $x^\beta \in \text{cone}(x^\alpha)$. We will denote by $<_{\boldsymbol{\varphi}}$ the following relation in $\mathcal{T}$

$$x^\beta >_{\boldsymbol{\varphi}} x^\delta \text{ iff either } x^\beta \in J,\ x^\delta \in \mathsf{N}(J) \text{ or } x^\beta, x^\delta \in J \text{ and } \varphi(x^\beta) > \varphi(x^\delta).$$

Proposition 11.4. *If $\mathcal{J}$ is stably ordered by the well founded order <, then it is Noetherian, ordered by $<_{\boldsymbol{\varphi}}$, and confluent.*

Proof. Clearly, $<_\varphi$ is a well founded order in $\mathcal{T}$, since $<$ is.

Moreover, for every $x^\alpha \in M$, $x^\gamma \in \lambda_\alpha$, $x^\eta \in \tau_\alpha$ we have either $x^{\eta+\gamma} \in \mathsf{N}(J)$ or $x^{\eta+\gamma} = x^{\alpha'+\eta'} \in \mathrm{cone}(x^{\alpha'})$; in both cases $x^{\alpha+\gamma} >_\varphi x^{\eta+\gamma}$: by definition of $<_\varphi$ in the first case, by StOrd4 in the second one.

The Noetherianity of $\mathcal{J}$ follows from the fact that it is ordered (Theorem 5.9) and the confluence by the fact that it has disjoint cones (Lemma 11.2 and Remark 7.2). $\square$

Let $\mathcal{J}$ be stably ordered RS and let $x^{\overline{\alpha}} \in M$ be such that $\tau_{\overline{\alpha}} = \mathcal{T}$: such an element always exists (Lemma 3.3), and is unique since the cones are disjoint. We can reformulate the conditions StOrd1-StOrd4 in terms of $<_\varphi$ as follows: for every $x^\alpha \in M$

StOr1′: $x^{\eta+\overline{\alpha}} >_\varphi x^{\overline{\alpha}}$ for each term $x^\eta \neq 1$;
StOr2′: $x^{\eta'+\overline{\alpha}} >_\varphi x^{\eta+\overline{\alpha}}$ iff $x^{\eta'+\epsilon+\overline{\alpha}} >_\varphi x^{\eta+\epsilon+\overline{\alpha}}$ for each term x^ϵ;
StOr3′: if $x^\eta \notin \tau_\alpha$, then $x^{\eta+\overline{\alpha}} >_\varphi x^{\eta+\alpha}$;
StOr4′: if $x^\eta \in \tau_\alpha$ and $x^\gamma \in \lambda_\alpha$, then $x^{\eta+\overline{\alpha}} >_\varphi x^{\eta+\gamma}$ and $x^{\eta+\alpha} >_\varphi x^{\eta+\gamma}$.

Lemma 11.5. *Let $\mathcal{F}$ be a marked set over a stably ordered RS $\mathcal{J}$ and let $<_\varphi$ and $x^{\overline{\alpha}}$ be as above. Let us consider terms $x^\alpha, x^{\alpha'} \in M$, $x^\eta, x^{\eta'}, x^\delta \in \mathcal{T}$ and polynomials $g, l \in \mathcal{P}$.*

i) If x^δ appears in the support of $x^\eta f_\alpha - x^{\eta+\alpha}$, then $x^{\eta+\overline{\alpha}} >_\varphi x^\delta$.
ii) If $g \to^+_{\mathcal{F}\mathcal{J}} l\downarrow$ and $x^{\eta+\overline{\alpha}} >_\varphi x^\gamma$ for every term x^γ that appears in the support of g, then $g - l$ has a $x^{\eta+\overline{\alpha}}$ − SLRep (w.r.t. $<_\varphi$).
iii) In particular, if $x^\eta \notin \tau_\alpha$, then

$$x^\eta f_\alpha \to^+_{\mathcal{F}\mathcal{J}} 0\downarrow \Longleftrightarrow x^\eta f_\alpha \textit{ has a } x^{\eta+\overline{\alpha}} - \textit{SLRep}.$$

Proof. $i)$ By Proposition 11.4 the marking of $\mathcal{F}$ is coherent with the well founded ordering $>_\varphi$, so that $x^{\eta+\alpha} >_\varphi x^\delta$.

If $x^\eta \in \tau_\alpha$, then $\varphi(x^{\eta+\alpha}) = \varphi(x^{\eta+\overline{\alpha}})$, hence $x^{\eta+\overline{\alpha}} >_\varphi x^\delta$.

If $x^\eta \notin \tau_\alpha$, by StOr3′ we have $x^{\eta+\overline{\alpha}} >_\varphi x^{\eta+\alpha} >_\varphi x^\delta$.

Item $ii)$ follows from Proposition 6.2 iii; item $iii)$ is a consequence of the previous ones and of StOr3′. $\square$

Theorem 11.6. *Let $\mathcal{F}$ be a marked set over a stably ordered RS $\mathcal{J}$. Then, the property for $\mathcal{F}$ of being a marked basis is equivalent to*

(4) $\quad \forall x^\beta \in M,\ \forall x^\varepsilon$ *minimal in $\mathcal{T} \setminus \tau_\beta$ w.r.t. the divisibility, it holds $x^\varepsilon f_\beta \to^+_{\mathcal{F}\mathcal{J}} 0\downarrow$.*

If moreover $\mathcal{J}$ has multiplicative variables, then it is also equivalent to the previous ones:

$$\forall x^\beta \in M,\ \forall x_i \notin \tau_\beta \text{ it holds } x_i f_\beta \to^+_{\mathcal{F}\mathcal{J}} 0\downarrow. \tag{5}$$

Proof. Let $<_\varphi$ and $x^{\overline{\alpha}}$ be as above. Due to Theorem 7.7 it is clear that for a marked basis, (4) and (5) hold. So we only prove the non-obvious implications.

Suppose that (4) holds, but $(\mathcal{F})$ is not contained in $\langle \tau\mathcal{F}\rangle$. Then, the following set is nonempty

$$U := \{x^{\eta+\overline{\alpha}} \mid \exists\, f_\alpha \in \mathcal{F} \text{ s.t. } x^\eta f_\alpha \notin \langle \tau\mathcal{F}\rangle\ \}.$$

Since $<_\varphi$ is a well founded order on $\mathcal{T}$, the set U has at least a minimal element: suppose that such a minimal element is $x^{\gamma+\overline{\alpha}}$ and that $x^\gamma f_\beta \notin \langle \tau\mathcal{F}\rangle$. This is possible only if $x^\gamma \notin \tau_\beta$ and, by the assumption (4), x^γ is not minimal in $\mathcal{T} \setminus \tau_\beta$ w.r.t. the divisbility.

Let x^ε be a divisor of x^γ, minimal in $\mathcal{T} \setminus \tau_\beta$. By hypothesis $x^\varepsilon f_\beta \to^+_{\mathcal{F}\mathcal{J}} 0 \downarrow$, hence it has a $x^{\varepsilon+\overline{\alpha}}$ – SLRep (Lemma 11.5). Multiplying by $x^{\gamma-\varepsilon}$ every polynomial $x^{\eta_i} f_{\alpha_i}$ of this representation, we obtain $x^\gamma f_\beta$ as a sum of polynomials $x^{\gamma-\varepsilon+\eta_i} f_{\alpha_i}$ such that

$$x^{\gamma-\varepsilon+\eta_i+\overline{\alpha}} <_{\boldsymbol{\varphi}} x^{\gamma-\varepsilon} \cdot x^{\varepsilon+\overline{\alpha}} = x^{\gamma+\overline{\alpha}}.$$

By the minimality of $x^{\gamma+\overline{\alpha}}$ in U, we deduce that $x^{\gamma-\varepsilon+\eta_i} f_{\alpha_i} \in \langle \tau\mathcal{F} \rangle$, hence the contradiction $x^\gamma f_\beta \in \langle \tau\mathcal{F} \rangle$.

The second statement directly follows from the first; in fact $\{x_i \notin \tau_\beta\}$ is a minimal basis of $\mathcal{T} \setminus \tau_\beta$. □

Remark 11.7. Consider a polynomial $x^\varepsilon f_\beta$ as stated in Theorem 11.6 and suppose that $x^{\varepsilon+\beta} \in \mathrm{cone}(x^\alpha)$. Then $S(f_\alpha, f_\beta)$ coincides with $x^\varepsilon f_\beta - x^{\varepsilon+\beta-\alpha} f_\alpha$. Indeed by minimality of x^ε in $\mathcal{T} \setminus \tau_\beta$ each proper divisor x^δ of x^ε belongs to τ_β so it cannot also belong to $\mathrm{cone}(x^\alpha)$.

Anyway, the condition concerning the S-polynomials is not sufficient to ensure the minimality of x^γ in $\mathcal{T} \setminus \tau_\beta$. In other words, the conditions required in Theorem 11.6 are weaker than the ones of Corollary 8.3.

12. Specializations

Buchberger reduction, mainly after Reeves-Sturmfels theorem, is associated to the idea of coherence with a term ordering, i.e. the fact that the head terms are bigger than any term in the tails w.r.t. a fixed term ordering.

What, instead, is wrong, is to associate Groebner bases to a Buchberger reduction viewed as, in our language, an RS with maximal cones. In fact all representations (and implementations) of Buchberger reductions assume that the available basis G is given as an ordered set of polynomials and that in each step of reduction the reducible term t is systematically reduced with the first element $g \in G$ whose leading term divides t.

In this paper, the thing we are more interested in, is the reduction procedure $\to^+_{\mathcal{F}\mathcal{J}}$, associated to a marked set $\mathcal{F}$, rather than the marked set (or basis) itself. The reduction depends both on $\mathcal{F}$ and on the RS $\mathcal{J}$, and in particular on the set of multiplicative terms.

Considering a monomial ideal J, a set of generators M and a term ordering $\prec$ we can define the RS $\mathcal{J} = (M, \lambda, \tau)$ setting

- $M = \{x^{\alpha_1}, \dots, x^{\alpha_s}\}$ an ordered set of generators of a monomial ideal J,
- $\lambda_{\alpha_i} = \{x^\gamma \in \mathcal{T} \text{ s.t. } x^\gamma \prec x^{\alpha_i}\}$,
- $\mathrm{cone}(x^{\alpha_i}) := x^{\alpha_i}\mathcal{T} \setminus \bigcup_{j=1}^{i-1} \mathrm{cone}(x^{\alpha_j})$.

and we obtain the RSs coherent with the term ordering $\prec$ and so also Noetherian (with the term ordering $\prec$ as well funded ordering). In this context, Reeves-Sturmfels Theorem (Theorem 5.10) says that an RS is Noetherian iff it is coherent with a term ordering. Thus Groebner bases relative to $\prec$ with initial ideal J are all and only the marked bases over the RS $\mathcal{J}$.

If we alternatively set

- $M = \{x^{\alpha_1}, \dots, x^{\alpha_s}\}$ an ordered set of minimal monomial basis of a monomial ideal J,
- $\lambda_{\alpha_i} = \{x^\gamma \in \mathcal{T} \text{ s.t. } x^\gamma \prec x^{\alpha_i}\} \cap \mathsf{N}(J)$,
- $\mathrm{cone}(x^{\alpha_i}) := x^{\alpha_i}\mathcal{T} \setminus \bigcup_{j=1}^{i-1} \mathrm{cone}(x^{\alpha_j})$

we get all and the only *reduced* Groebner bases.

A marked set $\mathcal{F}$ is a basis iff $\rightarrow^{+}_{\mathcal{F}\mathcal{J}}$ is confluent. Following Buchberger's algorithm, the test can be performed via the reduction of a finte number of S-polynomials among elements of $\mathcal{F}$.
Indeed, the main theorem of Groebner bases Theory [28, 2.2] declares that a generating set $\mathcal{F}$ is a Groebner basis if and only if each S-polynomial between two elements of $\mathcal{F}$ reduces to 0; Gebauer-Möller criteria [48, 84] allow to reduce the number of S-polynomials to be considered.

The importance of Buchberger's Theory as a tool for solving ideal theoretical problems, gave recently interest to alternative tools for producing Groebner bases; of course the milestones of normal forms given as linear combination of elements in the *order ideal* $\mathsf{N}(J)$ and obtained via the (Noetherian) Buchberger reduction are preserved and, after all, were already available to the researchers inspired by Hilbert[5].

The main contribution by Janet is the introduction of the decomposition of the monomial ideal J into cones of multiplicative sets generated by multiplicative variables.

Definition 12.1. (Janet, 1920) [64, pp .75-9] Given a generating set M of a monomial ideal J and given $x^\alpha \in M$, we say that a variable x_j is *multiplicative* for x^α w.r.t. M if in M there is no element x^β s.t. $\deg_i(x^\alpha) = \deg_i(x^\beta)$ for each $i > j$ and $\deg_j(x^\alpha) < \deg_j(x^\beta)$.

The *class of* $x^\alpha \in M$ *is the set* $\{x^\beta x^\gamma, x^\gamma \in \mathcal{T}[\mu_\alpha]\}$ *where* μ_α *is the set of the multiplicative variables for* x^α*.*

Moreover, M *is complete if for each term* $x^\gamma \in M$ *and each non-multiplicative variable* x_j *there is a monomial in* M *whose class contains* $x^\gamma x_j$*.*

Janet bases are the marked bases over RSs of the following form

- M a complete generating set of the monomial ideal J,
- $\lambda_\alpha = \{x^\gamma \in \mathcal{T} \text{ s.t. } x^\gamma \prec x^\alpha\} \cap \mathsf{N}(J)$,
- $\tau_\alpha := \mathcal{T}[\mu_\alpha]$.

These are RSs, which are coherent with a term ordering, have multiplicative variables and disjoint cones. The RSs of the form defined by Janet need to be coherent with a term ordering, in order to satisfy Noetherianity.

The most important difference between Janet's decomposition in cones and our definition is to give a general rule for defining the multiplicative variables for each term in M by considering the inner relation among the elements of M.

This aspect has been inforced in the formulation of Janet's approach proposed by Gerdt and Blinkov [51, 52].

Definition 12.2 (Gerdt—Blinkov)**.** An *involutive division* L or L*-division* on $\mathcal{T}$ is the assignement, for each finite set $U \subset \mathcal{T}$ and each term $u \in U$ of a submonoid $L(u, U) \subset \mathcal{T}$ such that the following holds for each $u, u_1 \in U$ and $t, w \in \mathcal{T}$

(a) $t \in L(u, U), t_1 \mid t \implies t_1 \in L(u, U)$,
(b) if $uL(u, U) \cap u_1 L(u_1, U) \neq \emptyset$ then $u \in u_1 L(u_1, U)$ or $u_1 \in uL(u, U)$;
(c) if $u_1 = uw$ for some $w \in L(u, U)$, then $L(u_1, U) \subseteq L(u, U)$;
(d) if $V \subseteq U$ then $L(u, U) \subseteq L(u, V)$ for each $u \in V$.

[5]Buchberger reduction can be even read already in [56].

Apart from Janet bases, the most important bases defined in terms of involutive divisions considered today are Pommaret bases and Gerdt and Blinkov [53, 54] Janet-like bases. Some adaptations of the theory of Gerdt and Blinkov have been suggested in [5, 31].

13. THE ZERO-DIMENSIONAL CASE AND BORDER BASES

We examine in depth the zero-dimensional case, since it is suitable for many observations.

Let J be a monomial ideal in $A[x_1, \dots, x_n]$ s.t. $\mathsf{N}(J)$ is a finite set. An important concept in many papers on this case is the one of *border*.

Definition 13.1. The border of J (or of $\mathsf{N}(J)$) is the set of terms

$$\mathsf{B}(J) := x_1\mathsf{N}(J) \cup \dots \cup x_n\mathsf{N}(J) \setminus \mathsf{N}(J).$$

Clearly $\mathsf{B}(J)$ contains the monomial basis of J, but in general as a proper subset.

We can characterize the elements of the border as follows:

$$x^\eta \in \mathsf{B}(J) \iff \exists x_j : \; x^\eta / x_j \in \mathsf{N}(J).$$

It follows then that the divisors of an element in the border are all contained in $\mathsf{N}(J) \cup \mathsf{B}(J)$ In many constructions of marked bases over the border, one considers a fixed term ordering and supposes that in each marked polynomial the elements in the tails are smaller than the head w.r.t. such a term ordering; anyway there also exist some bases, marked on $\mathsf{B}(J)$ without this constraint (see [67] and [2]).

Note that the notion of border bases was originally introduced in [82, 83], but in a context with no connection with RSs (actually being a reduction-free approach for computing canonical forms).

Our construction is not compatible with Mourrain improved formulation of border bases in [90] under the notion of *connected to 1*; indeed there it is not required the head terms to be a semigroup ideal nor the escalier to be an order ideal.

We can give a reformulation of these definitions in our language, defining an RS $\mathcal{J}$ as follows. Let $M = \{x^{\alpha_1}, \dots, x^{\alpha_s}\}$ be a list, formed by the elements of $\mathsf{B}(J)$, ordered in an arbitrary way. Then, we associate to each term x^γ in J the last term x^{α_i} of the M dividing x^γ.

TABLE 1. Border bases

M	The border basis $\mathsf{B}(J) = \{x^{\alpha_1}, \dots, x^{\alpha_s}\}$
λ	$\lambda_{\alpha_i} = \mathsf{N}(J)$
τ	$\tau_{\alpha_i} = \{x^\eta \in \mathcal{T} \text{ s.t. } \forall j > i : \; x^{\alpha_j} \nmid x^{\eta+\alpha_i}\}$

Notice that this is actually an RS and that the cones are disjoint, as proved in

Lemma 13.2. *Under the previous hypotheses (and w.r.t. the previous notation)*

i) for each $x^{\alpha_i} \in M$ the set $\tau_{\alpha_i} = \{x^\eta \in \mathcal{T} \text{ s.t. } \forall j > i : \; x^{\alpha_j} \nmid x^{\eta+\alpha_i}\}$ is an order ideal.

ii) setting cone$(x^{\alpha_i}) = x^{\alpha_i}\tau_{\alpha_i}$*, it holds* $\bigcup_{x^{\alpha_i}\in M}$ *cone*$(x^{\alpha_i}) = J$.

Proof. i) Let $x^\eta \in \tau_\alpha$ and $x^\varepsilon | x^\eta$. If some $x^{\alpha'}$ subsequent to x^α in the list divides $x^{\varepsilon+\alpha}$ then it divides also $x^{\eta+\alpha}$ and this contradicts $x^\eta \in \tau_\alpha$.

ii) We prove that for each $x^\beta \in J$ there is a term $x^{\alpha_i} \in \mathsf{B}(J)$ s.t. $x^{\beta-\alpha_i} \in \tau_{\alpha_i}$. This trivially follows by construction and from the fact that $\mathsf{B}(J)$ is a generating set for J. □

In [67] the authors consider a reduction process w.r.t. a marked set $\mathcal{F}$ over $\mathsf{B}(J)$ (called *border pre-basis*) and a procedure of reduction is defined. Roughly speaking, a term x^γ in J is reduced by any element in $\mathcal{F}$ whose head x^α has maximum degree among those in $\mathsf{B}(J)$ dividing x^γ. In order to prove the Noetherianity of this reduction process, a function $ind_{\mathsf{B}(J)} \colon J \cap \mathcal{T} \to \mathbb{N}$ called *index* is defined, associating to each term x^γ in J the degree of $x^{\gamma-\alpha}$.

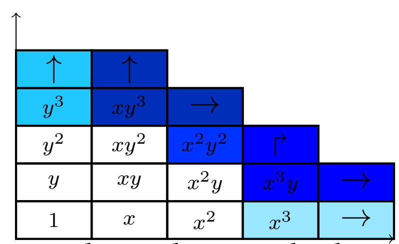
We obtain a special case of this procedure considering $\mathcal{F}$ as a marked set over the RS $\mathcal{J} = (\mathsf{B}(J), \lambda, \tau)$ as in Table 5 where the terms in $\mathsf{B}(J)$ are ordered in increasing order by degree; however the two procedures do not coincide since $\mathcal{J}$ has disjoint cones, while a monomial x^γ may have more than one divisor of maximum degree in $\mathsf{B}(J)$.

Furthermore, if we order the elements of $\mathsf{B}(J)$ in increasing order w.r.t. a term ordering $\prec$ (not necessarily degree compatible), then $\mathcal{J}$ turns out to be stably ordered with well founded order $\prec$: in fact the conditions of Definition 11.1 are obviously consequence of the properties of a term ordering. Therefore, Theorem 11.6 gives us a finite set of reductions to control in order to decide whether $\mathcal{F}$ is a marked basis.

Notice that in [67] there are no characterizations of marked bases using the reduction procedure; the presented one is based, as for Mourrain's work, on the commutativity of multiplication matrices.

We now show in some examples the consequences of modifying the order of the elements of $M = \mathsf{B}(J)$.

Example 13.3. Let $J = (x^3, x^2y^2, y^3) \subset A[x, y]$.



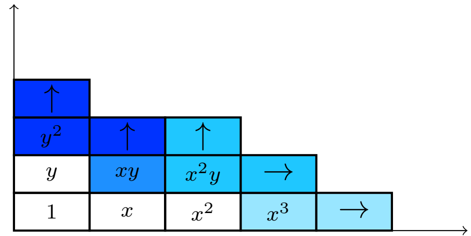
Its border can be written (in increasing order by degree) as $x^3, y^3, x^2y^2, x^3y, xy^3$. The multiplicative sets of the corresponding RS $\mathcal{J}$ are: $\tau_{x^3} = \mathcal{T}[x]$, $\tau_{y^3} = \mathcal{T}[y]$, $\tau_{x^2y^2} = \{1\}$, $\tau_{xy^3} = \mathcal{T}[x, y]$, and $\tau_{x^3y} = \{x^ay^b,\ a \geq 0,\ 0 \leq b \leq 1\}$. Notice that $\mathcal{J}$ is not an RS with multiplicative variables.

According to the criterion presented in Theorem 11.6 in order to know whether a marked set $\mathcal{F} = \{f_{x^3}, f_{y^3}, f_{x^2y^2}, f_{x^3y}, f_{xy^3}\}$ is a marked basis we would control the reduction of the following polynomials yf_{x^3}, xf_{y^3}, $xf_{x^2y^2}$, $yf_{x^2y^2}$, $y^2f_{x^3y}$.

Now we reorder the terms in $\mathsf{B}(J)$ in increasing order w.r.t. DegLex (induced by $x \prec y$) getting $x^3, y^3, x^3y, x^2y^2, xy^3$. In this case the multiplicative sets are $\tau'_{x^3} = \tau'_{x^3y} = \tau'_{x^2y^2} = \mathcal{T}[x]$, $\tau'_{y^3} = \mathcal{T}[y]$ and $\tau'_{xy^3} = \mathcal{T}[x, y]$. We get a stably ordered RS $\mathcal{J}'$ with multiplicative variables. Following Theorem 11.6 to decide whether $\mathcal{F}$ is a marked basis, we only have to check the reduction of yf_{x^3}, xf_{y^3}, yf_{x^3y}, $yf_{x^2y^2}$, all of them of "linear type".

Anyway, we cannot generalize what we observed in the previous example, since reordering the terms of the border w.r.t. DegLex (or a degree compatible term ordering) we do not obtain necessarily an RS with multiplicative variables.

Example 13.4. Ordering the border of $J = (x^3, xy, y^2) \subset A[x, y]$, w.r.t DegLex ($x \prec y$) we obtain xy, y^2, x^3, x^2y with cones $\tau_{xy} = \{1\}$, $\tau_{y^2} = \{x^l y^h,\ l \leq 1,\ h \geq 0\}$, $\tau_{x^3} = \mathcal{T}[x]$, $\tau_{x^2y} = \mathcal{T}[x, y]$, as in the picture. The term x^2 is one of the minimal elements of $\mathcal{T} \setminus \tau_{y^2}$ w.r.t. divisibility. This means that in order to verify that a marked set $\mathcal{F}$ is also a marked basis we have also to reduce $x^2 f_{y^2}$, which is not of "linear type".



The most convenient choice in general is to forget the degree and reorder the terms w.r.t. Lex.

Theorem 13.5. *Let J be a zero-dimensional monomial ideal and let $M = \mathsf{B}(J)$ be its border. Consider M ordered w.r.t. the lexicographic term ordering $\prec_{\text{Lex}}$ and let $\mathcal{J}$ be the associated RS according to Table 1.*

Then $\mathcal{J}$ has multiplicative variables, which for every $x^\alpha \in \mathsf{B}(J)$ coincide with the Janet-multiplicative variables for x^α w.r.t. $\mathsf{B}(J)$, so $\mathsf{B}(J)$ is a Janet complete system.

Proof. Let μ_α the set of Janet-multiplicative variables for $x^\alpha \in \mathsf{B}(J)$. We have to prove that $\tau_\alpha = \mathcal{T}[\mu_\alpha]$.

$\supseteq$ Consider $x^\eta \in \mathcal{T}[\mu_\alpha]$ and verify that $x^\eta \in \tau_\alpha$ i.e. that there is no term $x^\beta \in M$ dividing $x^{\eta+\alpha}$ and s.t. $x^\beta \succ_{Lex} x^\alpha$.
Suppose by contradiction that such a term x^β exists. Let x_j be s.t. $\deg_i(x^\beta) = \deg_i(x^\alpha)$ for each $i > j$ and $\deg_j(x^\beta) > \deg_j(x^\alpha)$. By definition of Janet-multiplicative variable , $x_j \notin \mu_\alpha$. We then get a contradiction, since by $\deg_j(x^{\eta+\alpha}) \geq \deg_j(x^\beta) > \deg_j(x^\alpha)$ follows that $x_j \mid x^\eta$ so, by hypothesis, $x_j \in \mu_\alpha$.

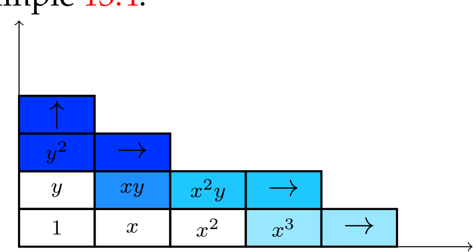
$\subseteq$ It is sufficient to prove that $x_j \notin \mu_\alpha$ implies $x_j \notin \tau_\alpha$.

If $x_j \notin \mu_\alpha$, by the definition of Janet-multiplicative variable there is a term $x^\beta \in \mathsf{B}(J)$ with $\deg_i(x^\beta) = \deg_i(x^\alpha)$ for each $i > j$ and $\deg_j(x^\beta) > \deg_j(x^\alpha)$. We prove then that the border contains also an element $x^{\beta'}$ with $\deg_i(x^{\beta'}) = \deg_i(x^\alpha)$ for each $i > j$ and $\deg_j(x^{\beta'}) = \deg_j(x^\alpha) + 1$, so that $x_j \notin \tau_\alpha$.

Consider the term x^γ obtained by x^α evaluating at 1 the variables x_i, $i < j$. By construction $x_j x^\gamma \mid x^\beta$ which is in the border; thus $x_j x^\gamma \in \mathsf{B}(J) \cup \mathsf{N}(J)$. Moreover $x_j x^\gamma$ also divides $x_j x^\alpha$, which belongs to J. Then, we find the wanted term $x^{\beta'} \in \mathsf{B}(J)$ in the set of the multiples of $x_j x^\gamma$ dividing $x_j x^\alpha$. $\square$

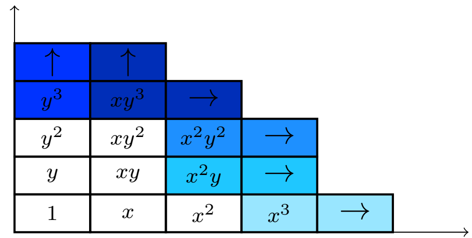
Example 13.6. Consider again the monomial ideal of Example 13.4.

The border of $J = (x^3, xy, y^2)$, ordered w.r.t. Lex is x^3, xy, x^2y, y^2. The multiplicative sets of the corresponding RS $\mathcal{J}''$ are $\tau''_{x^3} = \tau''_{x^2y} = \mathcal{T}[x]$, $\tau''_{xy} = \{1\}$, $\tau''_{y^2} = \mathcal{T}[x, y]$. Thus, $\mathcal{J}''$ is a stably ordered RS with multiplicative variables (coinciding with the Janet-multiplicative ones).



To conclude, we present a monomial ideal J for which the border basis (with terms ordered w.r.t. the Lex term ordering) is not a Pommaret basis, even though J, being quasi stable, has both a Pommaret and a border basis.

Example 13.7. The terms of the border of $J = (x^3, x^2y, y^3) \subset A[x, y]$ ordered w.r.t. Lex are $x^3, x^2y, x^2y^2, y^3, xy^3$. We get: $\tau_{x^3} = \tau_{x^2y} = \tau_{x^2y^2} = \mathcal{T}[x]$, $\tau_{y^3} = \mathcal{T}[y]$, $\tau_{xy^3} = \mathcal{T}[x, y]$.



The set of controls one has to perform in order to decide whether a marked set $\mathcal{F} = \{f_{x^3}, f_{x^2y}, f_{x^2y^2}, f_{y^3}, f_{xy^3}\}$ involves the reduction of $yf_{x^3}, yf_{x^2y}, yf_{x^2y^2}, xf_{y^3}$.

Notice that the sets of multiplicative variables of y^3 and xy^3 do not coincide with the ones w.r.t. Pommaret. Indeed, in the Pommaret basis $\{x^3, x^2y, x^2y^2, y^3\}$ of J there is one term less than in the border basis. At least in this case, in order to determine all the ideals in $A[x, y]$ whose quotient is a free A-vector space with basis $\mathsf{N}(J)$, it would be more convenient to use the Pommaret basis, instead of the border basis. Indeed, the set of controls that are needed involves only three reductions: $yf_{x^3}, yf_{x^2y}, yf_{x^2y^2}$

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APPENDIX: FUNCTORIALITY OF MARKED BASES

In all this paper we consider marked sets and m bases $\mathcal{F}$ over an RS $\mathcal{J}$ as a set of polynomials in the polynomial ring $\mathcal{P}_A$ where A is a field. However, everything holds true if we assume that A is any commutative ring. In fact, the only coefficients in $\mathcal{F}$ that we need to invert performing a reduction procedure are the leading coefficients. It is then natural to ask whether our construction is stable under extension of scalars. In this appendix we give a positive answer to this question.

There are at least two functors from the category of commutative rings to the category of sets that is natural to associate to an RS $\mathcal{J} = (M, \lambda, \tau)$ in $\mathcal{T}$

The functor of marked sets on $\mathcal{J}$

$$\underline{\mathbf{Ms}}_{\mathcal{J}} : \underline{\text{Ring}} \to \underline{\text{Set}}$$

that associates to any ring A the set $\underline{\mathbf{Ms}}_{\mathcal{J}}(A) := \{\mathcal{J} - \text{marked sets } \mathcal{F} \subset \mathcal{P}_A\}$ and to any morphism $\sigma : A \to B$ the map

$$\begin{array}{rccc} \underline{\mathbf{Ms}}_{\mathcal{J}}(\sigma): & \underline{\mathbf{Ms}}_{\mathcal{J}}(A) & \longrightarrow & \underline{\mathbf{Ms}}_{\mathcal{J}}(B) \\ & \mathcal{F} & \longmapsto & \sigma(\mathcal{F}) \end{array}$$

where $\sigma(\mathcal{F})$ is the set of polynomials that we obtain from those in $\mathcal{F}$ replacing each coefficient $a \in A$ with its image $\sigma(a)$. More formally, $\sigma(\mathcal{F})$ is the image of $\mathcal{F}$ under the map $\mathcal{P}_A \to \mathcal{P}_B = \mathcal{P}_A \otimes_\sigma B$

We observe that this functor is well defined since the coefficient of the distinguished term x^α in each marked polynomial $f_\alpha \in \mathcal{F}$ is the unit element and $\sigma(1_A) = 1_B$ for every homomorphism $\sigma\colon A \to B$. Hence $\sigma(\mathcal{F})$ is indeed a $\mathcal{J}$-marked in $\mathcal{P}_B$.

We will denote by $C_{\mathcal{J}}$ a set of $N := \sum_{x^\alpha \in M} |\lambda_\alpha|$ distinct variables $C_{\alpha,\beta}$ where $x^\alpha \in M$ and $x^\beta \in \lambda_\alpha$. Moreover, $\mathfrak{F}$ will denote the marked set in $\underline{\mathbf{Ms}}_{\mathcal{J}}(\mathbb{Z}[C_{\mathcal{J}}])$ formed by the polynomials $\mathfrak{f}_\alpha := x^\alpha + \sum_{x^\beta \in \lambda_\alpha} C_{\alpha,\beta} x^\beta$.

Lemma .8. *$\underline{\mathbf{Ms}}_{\mathcal{J}}$ is the functor of points of the ring $\mathbb{Z}[C_{\mathcal{J}}]$.*

Proof. For every ring A there is a 1–1 correspondence between $\underline{\mathbf{Ms}}_{\mathcal{J}}(A)$ and $Hom(\mathbb{Z}[C_{\mathcal{J}}], A)$.

In fact we can associate to every homomorphism $\pi : \mathbb{Z}[C_{\mathcal{J}}] \to A$ the marked set $\pi(\mathfrak{F}) \in \underline{\mathbf{Ms}}_{\mathcal{J}}(A)$ and, on the other hand, every marked set $\mathcal{F} = \{f_\alpha := x^\alpha + \sum_{x^\beta \in \lambda_\alpha} c_{\alpha,\beta} x^\beta\} \in \underline{\mathbf{Ms}}_{\mathcal{J}}(A)$ can be obtained in this way considering the homomorphism $\pi_{\mathcal{F}} : \mathbb{Z}[C_{\mathcal{J}}] \to A$ given by $\pi_{\mathcal{F}}(C_{\alpha,\beta}) = c_{\alpha,\beta}$.

Obviously, this 1–1 correspondence commutes with the extension of scalars, since for every homomorphism $\sigma \colon A \to B$ we have: $\sigma(\mathcal{F}) = \{\sigma(f_\alpha) = x^\alpha + \sum_{x^\beta \in \lambda_\alpha} \sigma(c_{\alpha,\beta}) x^\beta\}$, and so $\pi_{\sigma(\mathcal{F})} = \sigma \circ \pi_{\mathcal{F}}$. □

As well know, the category of affine schemes is equivalent to the the category of rings. Therefore, we can also define $\underline{\mathbf{Ms}}_{\mathcal{J}}$ as a contravariant functor $\underline{\text{AfScheme}} \to \underline{\text{Set}}$ and say that it is representable by the scheme $\mathbb{A}^N_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[\mathrm{C}_{\mathcal{J}}])$.

Focusing on the marked bases, we get an even more interesting functor, as a subfunctor of $\underline{\mathbf{Ms}}_{\mathcal{J}}$:

$$\underline{\mathbf{Mf}}_{\mathcal{J}}(A) := \{\mathcal{J}\text{-marked bases } \mathcal{F} \subset \mathcal{P}_A\}.$$

We now prove that this is in fact a functor.

Lemma .9. *Let $\mathcal{F} \in \underline{\mathbf{Mf}}_{\mathcal{J}}(A)$ and let us consider any morphism $\sigma : A \to B$.*

Then $\sigma(\mathcal{F})$ is a marked basis in $\mathcal{P}_B$.

Proof. By definition (Definition 4.3) a $\mathcal{J}$-marked set $\mathcal{G} \in \mathcal{P}_R$ is a basis if and only if $(\mathcal{G})_R \oplus \langle \mathsf{N}(J) \rangle_R = \mathcal{P}_R$.

Therefore, by hypothesis we know that $(\mathcal{F})_A \oplus \langle \mathsf{N}(J) \rangle_A = \mathcal{P}_A$, and applying $- \otimes_\sigma B$ we get $(\sigma(\mathcal{F}))_B \oplus \langle \mathsf{N}(J) \rangle_B = \mathcal{P}_B$. □

Under the additional assumption that $\mathcal{J}$ is weakly Noetherian, also this subfunctor turns out to be representable by a quotient of $\mathbb{Z}[C_{\mathcal{J}}]$, or, equivalently, by an affine subscheme of $\mathbb{A}^N_{\mathbb{Z}}$. Similarly to what has been done in [73], we now show how this subscheme can be obtained.

Let us consider the marked set $\mathfrak{F}$ in $\mathcal{P}_{\mathbb{Z}[C_{\mathcal{J}}]}$ and compute all the complete reductions $x^\eta \mathbf{f}_\alpha \to^{+}_{\mathbf{F}\mathcal{J}} \mathbf{g} \downarrow$ for every $x^\alpha \in M$ and $x^\eta \in \mathcal{T}$ and collect in a set $\mathcal{R} \subset \mathbb{Z}[C_{\mathcal{J}}]$ the coefficients of the monomials $x^\eta \in \mathsf{N}((M))$ of all the reduced polynomials g. By Proposition 7.6 and Theorem 7.7 the marked set $\pi(\mathbf{F})$, where $\pi \colon \mathbb{Z}[C_{\mathcal{J}}] \to \mathbb{Z}[C_{\mathcal{J}}]/(\mathcal{R})$, is in fact a marked basis.

The functor $\underline{\mathbf{Mf}}_{\mathcal{J}}$ is represented by the scheme $\mathbf{Mf}_{\mathcal{J}} := \mathrm{Spec}(\mathbb{Z}[C_{\mathcal{J}}]/(\mathcal{R}))$. For a detailed proof see [73]: the arguments presented there also apply in our, more general, framework.

There are many possible applications of the functorial approach to RSs, first of all to the study of Hilbert schemes since the marked schemes $\mathbf{Mf}_{\mathcal{J}}$ are flat families. In [15] a subfunctor of $\underline{\mathbf{Mf}}_{\mathcal{J}}$ for a suitable RS $\mathcal{J}$ is used to investigate the set of x_n-liftings of a given homogeneous ideal. We conclude with an aplication to the theory of marked bases: for every RS $\mathcal{J}$ we can check whether the $\mathcal{J}$-marked sets are bases performing a finite set of reductions.

Corollary .10. *Let $\mathcal{J} = (M, \lambda, \tau)$ be a weakly Noetherian RS.*

Then, there exists a finite subset $G \subset \mathcal{T} \times M$ such that for every marked set $\mathcal{F}$ on $\mathcal{J}$ TFAE:

1) $\mathcal{F}$ is a marked basis
2) for all $(x^\eta, x^\alpha) \in G$ and for all reduction $x^\eta f_\alpha \to^+_{\mathcal{F}\mathcal{J}} l \downarrow$ it holds $l = 0$.
3) for all $(x^\eta, x^\alpha) \in G$ there is a reduction $x^\eta f_\alpha \to^+_{\mathcal{F}\mathcal{J}} 0 \downarrow$

Proof. By the Noetherianity of the ring $\mathbb{Z}[C]$ there exists a finite set $\mathcal{R}' \subset \mathcal{R}$ that generates the ideal $(\mathcal{R})$. For every element $r \in \mathcal{R}'$ let us choose $x^\eta \in \mathcal{T}$ and $\mathbf{f}_\alpha \in \mathbf{F}$ and a reduction $x^\eta \mathbf{f}_\alpha \to^+_{\mathcal{F}\mathcal{J}} \mathbf{l} \downarrow$ s.t. r is a coefficient in $\mathbf{l}$; then let us collect in G the pairs (x^η, x^α).

The thesis is a direct consequence of the fact that $\mathcal{F} := \{f_\alpha + \sum_{x^\gamma \in \lambda_\alpha} c_{\alpha\gamma} x^\gamma,\ x^\alpha \in M\} \subset \mathcal{P}_A$ is a marked basis on $\mathcal{J}$ if and only if the morphism $\sigma \colon \mathbb{Z}[C] \to A$ given by $\sigma(C_{\alpha\gamma}) = c_{\alpha\gamma}$ factorizes through $\mathbb{Z}[C]/(\mathcal{R}) = \mathbb{Z}[C]/(\mathcal{R}')$. $\square$

In the case of homogeneous structures, due to Gotzmann Persistence [58] and Macaulay Estimate of Growth [59, Theorem 3.3], the controls one has to perform can be limited to the polynomials whose degree is bounded from above by $1+r$, where r is the maximum between the maximal degree of terms in M and the Castelnuovo-Mumford Regularity of the monomial ideal $J = (M)$.

A similar upper bound on the degree of polynomials involved in a sufficient set of controls appears also in the affine case in [16, Theorems 5.1 and 5.4]; indeed, those affine marked sets are marked bases if the following refinement of the condition *ii)* of Theorem 7.7 holds: $(\mathcal{F})_{\leq t} = \langle \tau\mathcal{F} \rangle_{\leq t}$ for some integers $t \leq r+1$.

Finally the recent result proved by [111] gives a further bound:

$$(\mathcal{F})_{\leq t} = \langle \tau\mathcal{F} \rangle_{\leq t} \text{ for all } t \geq 2\left(\frac{d^2}{2} + d\right)^{2^{n-1}} + \sum_{j=0}^{n-1} (ud)^{2^j}$$

where $d = \max \deg(f : f \in \mathcal{F})$ and $u = \#\mathcal{F}$.

In the above quoted cases we should perform a finite (but in general not small) number of controls.

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