Quasi periodic breathers in Hamiltonian lattices with symmetries

Dario Bambusi*, Davide Vella
Dipartimento di Matematica, Universitá di Milano
via Saldini 50, 20133 Milano (Italy)

21/1/2002

Summary. We prove existence of quasiperiodic breathers in Hamiltonian lattices of weakly coupled oscillators having some integrals of motion independent of the Hamiltonian. The proof is obtained by constructing quasiperiodic breathers in the anticontinuoum limit and using a recent theorem by N.N. Nekhoroshev [8] as extended in [5] to continue them to the coupled case. Applications to several models are given.

1 Introduction.

Existence of breathers (i.e. time periodic space localized solutions) in infinite Hamiltonian lattices of weakly coupled oscillators has been proved by Mac Kay and Aubry [6] (see also [3, 15]) starting from the anticontinuoum limit.

Existence of (time) quasiperiodic breathers has also been investigated, but such objects are expected not to exist in generic models due to the presence of linear combinations of the frequencies which fall in the continuous spectrum. As pointed out in [11, 12] such "resonances" are expected to lead to a decay in time of "quasiperiodic breather like solution".

Nevertheless existence of quasiperiodic breathers has been proved in some special models: in particular (1) quasiperiodic breathers with two frequencies have been constructed in two models which have a non trivial symmetry, namely the discrete nonlinear Schrödinger equation [10] and the adiabatic Holstein model [1]; (2) KAM theory has been used to construct breathers with an arbitrary number of independent frequencies in the case where the oscillators are coupled via a first neighborhood potential which is at least cubic [17].

*E-mail: bambusi@mat.unimi.it

In the present paper we start from the anticontinuoum limit and construct quasi periodic breathers in Hamiltonian systems having some integrals of motion independent of the Hamiltonian. Our approach is based on a theorem by N.N. Nekhoroshev [8] that was recently reformulated in a form suitable for our applications in [5]. This theorem is an extension of the Poincaré–Lyapunov theorem of continuation of periodic orbits to the case where some integrals of motion independent of the Hamiltonian exist. It allows to continue families of invariant tori to perturbations of the original system, and to describe precisely the dynamics on them. Applying this theorem one can reproduce the known results on the DNLS and on the Holstein model, and more generally obtain quasiperiodic breathers in any model with symmetries. Here we will explicitly reconstruct the quasiperiodic breathers of [10] for the DNLS, and we will construct a new type of quasiperiodic breathers in the Holstein model. As an example possessing 3–frequencies quasiperiodic breathers we also study the vector discrete nonlinear Schrödinger equation. We think that similar results could be obtained also by the approach of [7, 16].

We point out that the present approach allows also to give more details on the breather's dynamics. In particular breathers will appear as invariant tori, which form smooth families with as many parameters and as many dimensions as the number of independent integrals of motion of the system. In many cases the frequencies can be used to parametrize the family (in particular this is true for the models studied here), so, in particular the set of the tori on which the motion is dense has full measure.

We close this section by mentioning the very recent works [2, 9] in which the authors prove existence of breathers in FPU type systems by new approaches. We did not investigate the possibility of finding quasiperiodic breathers in systems with symmetry using such approaches.

Acknowledgement. We thank Andrea Carati for suggesting the coordinate transformation 3.15, 3.16.

2 The abstract theorem

2.1 Statement

We state here Nekhoroshev's theorem [8] in the improved form of [5]. For the proof see [5].

Let $(\mathcal{B}, \Omega)$ be a (weakly) symplectic Banach space (with symplectic form Ω). Let $\mathbf{H}^\epsilon := \{H_1^\epsilon, ..., H_s^\epsilon\}$ be s real functions on $\mathcal{B}$, defined for ϵ in a neighbourhood E of zero.

Consider the Hamiltonian vector field X_i^ϵ of H_i^ϵ (defined as the unique vector field such that $dH_i^\epsilon X = \Omega(X_i^\epsilon, X)\ \forall X \in \mathcal{B}$).

We assume that there exists an s dimensional compact submanifold $\Lambda \subset \mathcal{B}$, *invariant under the flows of the fields* $X_i^0 \equiv X_i^\epsilon\big|_{\epsilon=0}$ *for all* $i = 1, ..., s$. We also assume that i the functions H_i^ϵ and the vector fields X_i^ϵ are of class C^∞ in $\mathcal{U} \times E$, where $\mathcal{U}$ is a neighborhood of Λ,

ii the vector fields X_i^ϵ are independent on Λ

iii the functions $\mathbf{H}^\epsilon$ are in involution in $\mathcal{U}$, namely

$$\left\{H_i^\epsilon, H_j^\epsilon\right\} \equiv dH_i^\epsilon X_j^\epsilon = 0 \ , \quad \forall i,j = 1,...,s$$

Remark that by our assumptions the manifold Λ is diffeomorphic to an s dimensional torus.

Then we have to add the nonresonance assumption which extends that by the Poincaré Lyapunov theorem to the quasiperiodic case. To this end it is useful to restrict to the case where there exists a system of canonical coordinates (J, ψ, p, q) with $\psi \in \mathbf{T}^s$ and (J, p, q) in an open set of a suitable Banach space, in which the manifold Λ has equation $p = q = J = 0$, and the functions $H_i^0 \equiv H_i^\epsilon\big|_{\epsilon=0}$ take the form

$$H_i^0 = \sum_{j=1}^{s} \omega_j^{(i)} \, J_j + \sum_{j \geq 1} \nu_j^{(i)} \frac{p_j^2 + q_j^2}{2} + \tilde{H}_i^0 \tag{2.1}$$

where $\tilde{H}_i^0$ denotes a function with the property that the coefficients of its Taylor expansion in (J, p, q) are at least quadratic in J, p, q if they depend on J, while coefficients independent of J are at least cubic in p, q. As proved by Kuksin [13], such coordinates exist in quite general situations.

Consider the matrix $A \equiv \left\{\omega_j^{(i)}\right\}_{j=1,...,s}^{i=1,...,s}$ constituted by the frequencies of motion in the invariant torus Λ, and the matrix $B \equiv \left\{\nu_j^{(i)}\right\}_{j=1,...,\infty}^{i=1,...,s}$ constituted by the frequencies of small oscillations in the transversal directions to the invariant torus.

We will denote by $A^{(k;j)}$ the matrix obtained from A by substituting its k-th column with the j-th column of B. In the forthcoming theorem we will denote by $|A|$ the determinant of the matrix A.

Theorem 2.2 *In the above situation, assume that there exists* $n \equiv (n_1, n_2, ..., n_s) \in \mathbf{Z}^s$ *and* $\gamma > 0$ *such that*

$$\left| \sum_{k=1}^{s} \ n_k \, |A^{(k;j)}| \ - \ N \ |A| \right| \geq \gamma \ \ \forall N \in \mathbf{Z} \ , \quad \forall j = 1, ..., \infty \ . \tag{2.3}$$

Then there exists $\epsilon_* > 0$*, such that, for all* $\epsilon \in E_0$*,* $E_0 := (-\epsilon_*, \epsilon_*)$*, the following holds true:*
(1) *there exists a family of symplectic submanifolds* N^ϵ*, of dimension* $2s$*, which is the union of* s*–dimensional tori* $\mathbf{T}^\epsilon_\beta$*, with* $\beta \in \Re^s$ *small. For each* $\epsilon \in E_0$*, the tori* $\mathbf{T}^\epsilon_\beta$ *are invariant under the flow of* X_i^ϵ*; the tori* $\mathbf{T}^\epsilon_\beta$ *and the manifold* N^ϵ *depend in a* C^∞ *way on* $\epsilon \in E_0$*; one has* $\Lambda = \mathbf{T}^0_0$.

(2) *There exist symplectic action angle coordinates* $(I_1^\epsilon, ..., I_s^\epsilon; \varphi_1^\epsilon, ..., \varphi_s^\epsilon)$ *in* N^ϵ, *such that the functions* $H_i^\epsilon\big|_{N^\epsilon}$ *depend only on the actions* I_j^ϵ, *namely*

$$H_i^\epsilon\big|_{N^\epsilon} \equiv H_i^\epsilon\big|_{N^\epsilon}(I_1^\epsilon, ..., I_s^\epsilon) \ ,$$

with $i = 1, ..., s$. *The coordinates* $(I^\epsilon; \varphi^\epsilon)$ *depend in a* C^∞ *way on* $\epsilon \in E_0$, *and so do the functions* $H_i^\epsilon\big|_{N^\epsilon}(I_1^\epsilon, ..., I_s^\epsilon)$.

We remark that while statement (1) ensures the existence of the invariant tori and describes their shape, statement (2) describes completely the motion on the invariant tori. Indeed, in the coordinates I, φ (where we drop ϵ) the equations of motion of H_i^ϵ, in N^ϵ, have the form

$$\dot{I}_k = 0 \ , \quad \dot{\varphi}_k = \frac{\partial H_i^\epsilon}{\partial I_k}(I) \ ,$$

which shows that on each of the tori the motion is quasiperiodic. Moreover, the fact that the invariant tori, the coordinates, and the Hamiltonians depend in a smooth way on ϵ ensures that the frequencies of the perturbed system are close to those of the unperturbed one. Moreover, if for example one has (for a fixed i)

$$\left| \frac{\partial^2 H_i^0}{\partial I_l \partial I_k} \right| \neq 0$$

then the same holds for the perturbed system. In particular in this case one has that the action to frequency map is one to one, and therefore the tori can be parametrized by the frequencies.

2.2 Idea of the proof

First we recall the scheme of the proof of the Poincaré–Lyapunov theorem (corresponding to $s = 1$). In this case the manifold Λ reduces to a periodic orbit. Consider a Poincaré section and the corresponding Poincaré map P^ϵ; then periodic orbits of X_1^ϵ are found as solutions of $P^\epsilon(x) = x$; such solutions are constructed by using the implicit function theorem. The standard condition that the Floquet multiplier 1 have multiplicity 2 ensures that the implicit function theorem applies.

In the case $s \geq 2$ one tries to mimic the above proof. To define the Poincaré map remark that, by the first step of the proof of Arnold's theorem, there exists a function $K^\epsilon(p, q) := \sum_j \alpha_j H_j^\epsilon(p, q)$ with the property that all solution of the Hamilton equations of K^0 with initial data in Λ are periodic with a definite period (actually there exist s different functions with this property). Then define a Poincaré section of one of these periodic orbits and the corresponding Poincaré map P^ϵ. Due to the symmetries of the problem there exists a local foliation which is invariant under the flow of the X_i^ϵ, and moreover it turns out that P^ϵ defines a natural map $\tilde{P}^\epsilon$ from one leaf of the foliation to an other. It turns out that fixed points of $\tilde{P}^\epsilon$ give rise to invariant tori. To find such fixed points one uses again the implicit function theorem: the corresponding invertibility condition is ensured by assuming that 1 is an isolated eigenvalue of multiplicity $2s$ of the Floquet operator. Finally it can be proved that such a condition is equivalent to 2.3.

3 Applications

All the applications will deal with systems of the form

$$H_1^\epsilon := \sum_{k\in\mathbf{Z}} H_{os}(P_k, Q_k) + \epsilon \sum_{k\in\mathbf{Z}} F(P_k - P_{k-1}, Q_k - Q_{k-1}) \ , \qquad (3.4)$$

where $(P_k, Q_k) \in \Re^{2j}$ are canonically conjugated variables, and H_{os} is the Hamiltonian of the on site system that we will assume to have j degrees of freedom. In our cases we will have $j = 1$ or 2. The phase space is defined formally as follows: Fix $\beta > 0$ and define the Banach space ℓ_β of the sequences $P = \{P_k\}$, $P_k \in \Re^j$, such that

$$\|P\|_\beta := \sup_{k\in\mathbf{Z}} \|P_k\| \, e^{\beta|k|} < \infty \qquad (3.5)$$

where $\|P_k\|$ denotes the euclidean norm. The phase space is $(P, Q) \in \ell_\beta \times \ell_\beta$.

The system 3.4 will also have s independent integral of motion in involution given by H_1^ϵ and by $s-1$ more functions $H_2, ..., H_s$, that will turn out to be independent of ϵ. In our cases we will have $s = 2$ or 3.

Then we will proceed by first defining the manifold Λ for the different models and then introducing the coordinates such that the system takes the form 2.1; we will denote by $\omega_1, ..., \omega_s$ the frequencies of motion of the unperturbed breather. We point out that in our cases the unperturbed quasiperiodic breather will be concentrated at the sites $1, ..., s$ of the lattice. Then we will write down explicitly the nonresonance condition 2.3. Actually in all the cases that we will consider it takes a quite simple form very similar to the nonresonance condition of the Poincaré–Lyapunov theorem. To obtain such a simple form we will compute explicitly the determinants involved in condition 2.3 and simplify as much as possible the so obtained expression.

Proposition 3.6 *Assume that the system satisfy a suitable nonresonance condition that depends on the model (see 3.11, 3.14 and 3.18 below); then there exists ϵ_* and a function $\delta_* = \delta_*(\epsilon)$ defined for $|\epsilon| < \epsilon_*$, such that for $|\epsilon| < \epsilon_*$ there exists a $2s$–dimensional manifold N_ϵ invariant under the flows of $X_1^\epsilon, ..., X_s$; moreover one has*

$$N_\epsilon = \bigcup_{|\delta_i|<\delta_*, i=1,...,s} \mathbf{T}^\epsilon_{\delta_1,...,\delta_s}$$

with $\mathbf{T}^\epsilon_{\delta_1,...,\delta_s}$ an s–dimensional torus invariant under flow of X_1^ϵ,..., X_s. On $\mathbf{T}^\epsilon_{\delta_1,...,\delta_s}$ the dynamics of X_1^ϵ (i.e. of the model we are interested in) is quasiperiodic with the frequencies

$$(\omega_1 + \delta_1, ..., \omega_s + \delta_s) \ ,$$

The tori $\mathbf{T}^{\epsilon}_{\delta_1,...,\delta_s}$ depend smoothly on δ_i and on ϵ. In particular there exists a constant C such that, for all points in N_ϵ one has

$$\|P_k\| + \|Q_k\| < C\epsilon e^{-\beta|k|} \ , \quad \forall k \neq 1, ..., s \tag{3.7}$$

We point out that equation 3.7 is a consequence of smooth dependence of the manifold N_ϵ on ϵ in the topology of the phase space. The fact that it is possible to parameterize the tori by the frequencies is a consequence of the fact that in the case $\epsilon = 0$ the application from the actions to the frequencies will turn out to be one to one in all the models we are interested in.

3.1 Discrete nonlinear Schrödinger equation

Consider the discrete nonlinear Schrödinger equation

$$i\dot{\psi}_k = \psi_k \left(\frac{|\psi_k|^2}{2}\right)^{n-1} + \epsilon\left[(\psi_{k+1} - \psi_k) + (\psi_{k-1} - \psi_k)\right] \ , \quad k \in \mathbf{Z} \tag{3.8}$$

where $n \geq 2$ is an arbitrary integer. The Hamiltonian is

$$H_1^\epsilon = \sum_{k\in\mathbf{Z}} \frac{1}{n}\left(\frac{p_k^2 + q_k^2}{2}\right)^n + \epsilon \sum_{k\in\mathbf{Z}} \frac{(q_{k+1} - q_k)^2 + (p_{k+1} - p_k)^2}{2} \ , \tag{3.9}$$

with $p_k + iq_k = \psi_k$. The second integral of motion is given by

$$H_2(p,q) := \sum_{k\in\mathbf{Z}} \frac{p_k^2 + q_k^2}{2} \ .$$

In analogy with the standard Schrödinger such a quantity can be called electron probability.

Fix positive ω_1, ω_2, then the manifold Λ is defined by

$$\Lambda := \left\{\frac{p_1^2 + q_1^2}{2} = \omega_1^{1/(n-1)} \ , \quad \frac{p_2^2 + q_2^2}{2} = \omega_2^{1/(n-1)} \ , \quad p_k = q_k = 0 \ , \quad \forall k \neq 1, 2\right\}$$

and introduce action variables at the sites 1,2 by $I_1 = (p_1^2 + q_1^2)/2, I_2 = (p_2^2 + q_2^2)/2$, and the corresponding angles. To introduce the coordinates we need in order to apply theorem 2.2 define $J_1 := I_1 - \omega_1^{1/(n-1)}$, and $J_2 := I_2 - \omega_2^{1/(n-1)}$ so that the Hamiltonians take the form

$$H_1^0 = \omega_1 J_1 + \omega_2 J_2 + \tilde{H}_1 \ , \quad H_2 = J_1 + J_2 + \sum_{k\neq 1,2} \frac{p_k^2 + q_k^2}{2}$$

with $\tilde{H}_1$ having the same meaning as in 2.1.

We remark that here the manifold $N^\epsilon\big|_{\epsilon=0}$ is just the phase space of the first two oscillators, and the restriction of H_1^0 to N^0 is simply given by

$$\frac{I_1^n}{n} + \frac{I_2^n}{n} \ .$$

Lemma 3.10 *The nonresonance condition 2.3 takes here the form*

$$\frac{\omega_1}{\omega_2 - \omega_1} \notin \mathbf{Z} \tag{3.11}$$

Proof. The matrices A and B are given by

$$A := \begin{pmatrix} \omega_1 & \omega_2 \\ 1 & 1 \end{pmatrix} \ , \quad B := \begin{pmatrix} 0 & 0 & 0 & ... \\ 1 & 1 & 1 & ... \end{pmatrix}$$

from which one has

$$A^{(1,j)} = \begin{pmatrix} 0 & \omega_2 \\ 1 & 1 \end{pmatrix} \ , \quad A^{(2,j)} = \begin{pmatrix} \omega_1 & 0 \\ 1 & 1 \end{pmatrix} \ , \quad \forall j \geq 1$$

and thus the nonresonance condition 2.3 takes the form 'there exist $(n_1, n_2) \in \mathbf{Z}^2$ such that

$$-n_1\omega_2 + n_2\omega_1 \neq N(\omega_1 - \omega_2) \ ,$$

for all $N \in \mathbf{Z}$'. This can be rewritten as

$$(-n_1 + n_2)\omega_2 + n_2(\omega_1 - \omega_2) \neq N(\omega_1 - \omega_2) \ ,$$

from which it is evident that the second term at left hand side (l.h.s.) does not affect the condition and can be dropped. So the condition is equivalent to 'there exist $n \in \mathbf{Z}$ such that

$$n\frac{\omega_2}{\omega_1 - \omega_2} \neq N \ , \tag{3.12}$$

for all $N \in \mathbf{Z}$', and in turn this is equivalent to 3.11, since, in case the fraction at l.h.s. of 3.12 is an integer, the l.h.s. of 3.12 is an integer for any choice of n, while in case the fraction is not an integer, just choose $n = 1$. So equation 3.11 is the wanted nonresonance condition under which the theorem 3.6 applies. $\triangle$

So we have that proposition 3.6 holds for DNLS.

In this case the quasiperiodic breather is a solution in which the electron probability is essentially concentrated at two lattice cites.

3.2 Adiabatic Holstein model

The equations of motion of the adiabatic Holstein model [1] are given by

$$-i\dot{\psi}_i = -q_i\psi_i - \epsilon[(\psi_i - \psi_{i-1}) + (\psi_i - \psi_{i+1})]$$

$$\ddot{q}_i = -\omega_0^2 q_i + |\psi_i|^2 \ . \qquad i \in \mathbf{Z}$$

which are Hamiltonian with Hamiltonian function

$$H_1^\epsilon \equiv H := \sum_i H_{os}(p_i, q_i, x_i, y_i) + \frac{1}{2}\epsilon \sum_i \left[(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2\right] \ ,$$

where

$$H_{os}(p_i, q_i, x_i, y_i) := \frac{p_i^2 + \omega_0^2 q_i^2}{2} + q_i \frac{x_i^2 + y_i^2}{2} \ ,$$

(p_k, q_k), (x_i, y_i) are canonically conjugated variables, and one has $\psi_j = x_j + iy_j$. The additional integral of motion is

$$H_2 := \sum_i \frac{x_i^2 + y_i^2}{2} \ .$$

To construct the manifold Λ we proceed as follows: Following [4] we introduce action angle variables for H_{os} at the sites 1,2 by first defining the variables (I_i, φ_i) by

$$x_i = \sqrt{I_i} \cos \varphi_i \ , \quad y_i = \sqrt{I_i} \sin \varphi_i \ ; \quad i = 1, 2$$

and then performing the canonical transformation

$$\xi_i = q_i + \frac{I_i}{\omega_0^2} \ , \quad \eta_i = p_i \ , \quad I_i' = I_i, \quad \varphi_i' = \varphi_i + \frac{\eta_i}{\omega_0^2} \ , \quad i = 1, 2$$

which gives H_{os} the form

$$\frac{\eta_i^2 + \omega_0^2 \xi_i^2}{2} - \frac{1}{2\omega_0^2} I_i^2$$

where we omitted the prime from I.

Fix now two positive frequencies ω_1 and ω_2, and define

$$\Lambda := \left\{ I_1 = \omega_1 \omega_0^2 \ , \quad I_2 = \omega_2 \omega_0^2 \ , \quad \xi_i = \eta_i = p_k = q_k = 0 \ , \quad \forall i = 1, 2, \ k \neq 1, 2 \right\}$$

Defining $J_1 := I_1 - \omega_1 \omega_0^2$ and $J_2 := I_2 - \omega_2 \omega_0^2$ one has that the Hamiltonians of the system take the form we need, namely

$$H_1^0 = -\omega_1 J_1 - \omega_2 J_2 + \sum_{i=1,2} \frac{\eta_i^2 + \omega_0^2 \xi_i^2}{2} + \sum_{i \neq 1,2} \frac{p_i^2 + \omega_0^2 q_i^2}{2} + \tilde{H}_1$$

$$H_2 = J_1 + J_2 + \sum_{i \neq 1,2} \frac{x_i^2 + y_i^2}{2} \ .$$

Lemma 3.13 *The nonresonance condition 2.3 takes here the form*

$$\frac{\omega_1}{\omega_2 - \omega_1} \notin \mathbf{Z} \text{ and } \frac{\omega_0}{\omega_2 - \omega_1} \notin \mathbf{Z} \tag{3.14}$$

Proof. One has

$$A = \begin{pmatrix} -\omega_1 & -\omega_2 \\ 1 & 1 \end{pmatrix} \ , \quad B := \begin{pmatrix} \omega_0 & \omega_0 & \omega_0 & 0 & \omega_0 & 0 & \omega_0 & 0 & \dots \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & \end{pmatrix}$$

where the first two columns of the matrix B refer to the variables ξ, η at the lattice sites 1,2. The $2i-3$ column refers to the variables (p_i, q_i) $(i \geq 3)$, and the $2i-2$ columns refer to the variables (x_i, y_i) $(i \geq 3)$. We thus obtain the matrices $A^{(k,j)}$, and from them the nonresonance conditions which have the form 'there exists $(n_1, n_2) \in \mathbf{Z}^2$ such that

$$m(\omega_2 - \omega_1) \neq n_1\omega_0 - n_2\omega_0 \ , \quad m(\omega_2 - \omega_1) \neq n_1\omega_2 - n_2\omega_1 \ ,$$

for all $m \in \mathbf{Z}$'. In turn this is equivalent to the nonresonance conditions 3.14. $\triangle$

The kind of breathers we constructed here are new and consist of solutions in which the electron probability is essentially concentrated at two lattice sites, and the oscillators are at rest. Notice that the rest position of the oscillators at the sites where the breather is localized are translated with respect to the unperturbed ones.

3.3 Vector discrete nonlinear Schrödinger equation

Consider the vector DNLS equation whose equations of motion are

$$i\dot{u}_k = u_k \left(\frac{|u_k|^2 + |v_k|^2}{2}\right)^{n-1} + \epsilon\left(u_{k-1} + u_{k+1} - 2u_k\right)$$

$$i\dot{v}_k = v_k \left(\frac{|u_k|^2 + |v_k|^2}{2}\right)^{n-1} + \epsilon\left(v_{k-1} + v_{k+1} - 2v_k\right)$$

with $k \in \mathbf{Z}$. Introducing the real and imaginary parts of u_k, v_k as new variables, namely

$$u_k = x_{k,1} + iy_{k,1} \ , \quad v_k = x_{k,2} + iy_{k,2}$$

one sees that the system is Hamiltonian with Hamiltonian function

$$H_1^\epsilon = \sum_{k\in\mathbf{Z}} \frac{1}{n}\left(\frac{x_{k,1}^2 + y_{k,1}^2}{2} + \frac{x_{k,2}^2 + y_{k,2}^2}{2}\right)^n$$

$$+\epsilon\frac{1}{2}\sum_{k\in\mathbf{Z}}\left[(x_{k,1} - x_{k-1,1})^2 + (y_{k,1} - y_{k-1,1})^2 + (x_{k,2} - x_{k-1,2})^2 + (y_{k,2} - y_{k-1,2})^2\right] \ .$$

The additional integrals of motion are due to the phase shift symmetries, and to the rotation invariance in the plane of u and v. They are given by

$$F_2 = \sum_{k\in\mathbf{Z}}\left(\frac{x_{k,1}^2 + y_{k,1}^2}{2}\right) \ , \quad F_3 := \sum_{k\in\mathbf{Z}}\left(\frac{x_{k,2}^2 + y_{k,2}^2}{2}\right) \ ,$$

and by the angular momentum

$$F_4 := \sum_{k\in\mathbf{Z}}\left(x_{k,1}y_{k,2} - x_{k,2}y_{k,1}\right) \ ,$$

which are independent but not in involution. As functions independent and in involution on which we will base our construction we choose

$$H_2 := F_2 + F_3 \ , \quad H_3 := F_4 \ .$$

Obviously other choices are possible, and they would lead to different kinds of breathers. For example one could choose

$$H_2 := F_2 \ , \quad H_3 := F_3 \ ,$$

and obtain a kind of breathers that exist also in the anisotropic vector DNLS. Our choice is motivated by the fact that a construction very similar to ours can be performed also in other interesting models, like an infinite lattice of three dimensional oscillators interacting via a spherically symmetric potential.

In order to continue the analysis it is useful to perform the following change of variables

$$p_{1,k} := \frac{1}{\sqrt{2}}\left(x_{k,1} + y_{k,2}\right) \ , \quad q_{k,1} = \frac{1}{\sqrt{2}}\left(x_{k,2} - y_{k,1}\right) \tag{3.15}$$

$$p_{2,k} := \frac{1}{\sqrt{2}}\left(x_{k,2} + y_{k,1}\right) \ , \quad q_{k,2} = \frac{1}{\sqrt{2}}\left(x_{k,1} - y_{k,2}\right) \tag{3.16}$$

the functions H_i^{ϵ}, for $\epsilon = 0$ take the form

$$H_1^0 = \sum_{k\in\mathbf{Z}} \frac{1}{n}\left(\frac{p_{k,1}^2 + q_{k,1}^2}{2} + \frac{p_{k,2}^2 + q_{k,2}^2}{2}\right)^n$$

$$H_2 = \sum_{k\in\mathbf{Z}} \frac{p_{k,1}^2 + q_{k,1}^2}{2} + \frac{p_{k,2}^2 + q_{k,2}^2}{2}$$

$$H_3 = \sum_{k\in\mathbf{Z}} \frac{p_{k,1}^2 + q_{k,1}^2}{2} - \frac{p_{k,2}^2 + q_{k,2}^2}{2}$$

To define Λ we fix arbitrary positive quantities $\omega_1, \omega_2, \omega_3$ and put

$$\Lambda := \left\{ \frac{p_{1,1}^2 + q_{1,1}^2}{2} = \omega_1^{1/(n-1)} \ , \quad \frac{p_{2,1}^2 + q_{2,1}^2}{2} = \omega_2^{1/(n-1)} \ , \quad \frac{p_{3,2}^2 + q_{3,2}^2}{2} = \omega_3^{1/(n-1)} \ , \right.$$

$$\left. p_{k,j} = q_{k,j} = 0 \text{ otherwise } \right\}$$

Finally we introduce action angle variables by putting

$$I_1 := \frac{p_{1,1}^2 + q_{1,1}^2}{2} \ , \quad I_2 := \frac{p_{2,1}^2 + q_{2,1}^2}{2} \ , \quad I_3 := \frac{p_{3,2}^2 + q_{3,2}^2}{2} \ ,$$

and define

$$J_i := I_i - \omega_i^{1/(n-1)} \ , \quad i = 1, 2, 3 \ ,$$

so that the three independent integrals of motion take the form

$$H_1^0 = \omega_1 J_1 + \omega_2 J_2 + \omega_3 J_3 + \omega_1 \frac{p_{1,2}^2 + q_{1,2}^2}{2} + \omega_2 \frac{p_{2,2}^2 + q_{2,2}^2}{2} + \omega_3 \frac{p_{3,1}^2 + q_{3,1}^2}{2} + \tilde{H}_1$$

$$H_2 = J_1 + J_2 + J_3 + \frac{p_{1,2}^2 + q_{1,2}^2}{2} + \frac{p_{2,2}^2 + q_{2,2}^2}{2} + \frac{p_{3,1}^2 + q_{3,1}^2}{2} + \sum_{k \neq 1,2,3} \frac{p_{k,1}^2 + q_{k,1}^2}{2} + \frac{p_{k,2}^2 + q_{k,2}^2}{2}$$

$$H_3 = J_1 + J_2 - J_3 - \frac{p_{1,2}^2 + q_{1,2}^2}{2} - \frac{p_{2,2}^2 + q_{2,2}^2}{2} + \frac{p_{3,1}^2 + q_{3,1}^2}{2} + \sum_{k \neq 1,2,3} \frac{p_{k,1}^2 + q_{k,1}^2}{2} - \frac{p_{k,2}^2 + q_{k,2}^2}{2}$$

Then one can apply our theory and obtain a family of three dimensional invariant tori continuing to the coupled case the manifold Λ. In particular one has

Lemma 3.17 *The nonresonance condition 2.3 takes here the form*

$$\frac{\omega_1 - \omega_3}{\omega_1 - \omega_2} \notin \mathbf{Z} \ , \quad \frac{\omega_1}{\omega_1 - \omega_2} \notin \mathbf{Z} \ , \quad \frac{\omega_3}{\omega_1 - \omega_2} \notin \mathbf{Z} \ . \tag{3.18}$$

Proof. A long but straightforward computation (just write down the determinants and compute them) shows that the nonresonance condition has here the form "there exist $(n_1, n_2, n_3) \in \mathbf{Z}^3$ such that

$$n_1(\omega_3 - \omega_1) + n_2(\omega_1 - \omega_3) + n_3(\omega_2 - \omega_1) \neq N(\omega_2 - \omega_1)$$

$$n_1(\omega_3 - \omega_2) + n_2(\omega_2 - \omega_3) + n_3(\omega_2 - \omega_1) \neq N(\omega_2 - \omega_1)$$

$$n_1(\omega_2 - \omega_3) + n_2(\omega_3 - \omega_1) + \neq N(\omega_2 - \omega_1)$$

$$n_1\omega_3 - n_2\omega_1 \neq N(\omega_2 - \omega_1)$$

$$n_1\omega_3 - n_2\omega_3 \neq N(\omega_2 - \omega_1)$$

for all $N \in \mathbf{Z}$'. Introducing the variable $\mu_1 := \omega_2 - \omega_1$ in order to eliminate ω_2, remarking that the last terms at l.h.s. of the first two equations are inessential and denoting $m := n_1 - n_2$ one sees that the above equations are equivalent to

$$m(\omega_3 - \omega_1) \neq N\mu_1$$

$$m(\omega_3 - \omega_1 - \mu_1) \neq N\mu_1$$

$$-(n_2 + m)(\omega_3 - \omega_1 - \mu_1) + n_2(\omega_3 - \omega_1) \neq N\mu_1$$

$$(n_2 + m)(\omega_1 + \mu_1) + n_2\omega_1 \neq N\mu_1$$

$$m\omega_3 \neq N\mu_1$$

then it is clear that the terms containing μ_1 at l.h.s. do not affect the nonresonance conditions, and therefore this is equivalent to the stated conditions.△

4 Discussion

We first discuss briefly the relation with the papers [11, 12, 17].

To fix ideas we consider a quasiperiodic breather of the DNLS, and consider the motion of one observable, for example q_1, then one can consider the Fourier expansion of $q_1(t)$, namely write

$$q_1(t) = \sum_{(n_1,n_2)\in\mathbf{Z}^2} c_{n_1 n_2} e^{i(n_1\omega_1+n_2\omega_2)t} \ :$$

generically all coefficients $c_{n_1 n_2}$ are different from zero, i.e. the motion contains all the frequencies $n_1\omega_1 + n_2\omega_2$. In particular some of these frequencies will fall in the continuous spectrum, and as shown by [11, 12] in general this phenomenon creates a coupling between the quasiperiodic motion and the continuous spectrum, and as a consequence the breather begins to radiate energy and therefore to decay.

So at first sight it is quite surprising that quasiperiodic motions exist in the considered models. However, as it is clear from the perturbative construction of quasiperiodic solutions (see e.g. [14]), in order to destroy a quasiperiodic motion two ingredients are needed: the first one is a resonance, and the second one is a coupling term in the nonlinearity. *Generically* all possible coupling terms are present in the nonlinearity, but *a system with symmetry is not generic!* The symmetry actually prevents the existence of such coupling terms. For this reason the mechanism of [11, 12] is not active in this case.

Concerning the relation with the work by Yuan [17] we point out that his situation is completely different from ours, indeed, while in our case radiation is not possible due to the non genericiticy of the nonlinearity, in his case radiation is not possible due to the nongenericiticy of the linear part of the system, indeed in his case there is no continuous spectrum.

In conclusion we have shown that quasiperiodic breathers exist in Hamiltonian lattices having some integrals of motion independent of the Hamiltonian, and that Nekhoroshev's theorem is a powerful tool in order to actually construct such breathers in concrete cases.

We also emphasize that systems with symmetry are exceptional, but at the same time they are quite common and interesting: we think that the same is true for quasiperiodic breathers.

References

[1] S. Aubry, "Breathers in nonlinear lattices: existence, linear stability and quantization". *Phys. D* **103** (1997), 201–250.

[2] S. Aubry, G. Kopidakis, V. Kadelburg "Variational proof for hard discrete breathers in some classes of Hamiltonian dynamical systems". *DCDS-B* **1** (2001), 271-298.

[3] D. Bambusi "Exponential stability of breathers in Hamiltonian networks of weakly coupled oscillators". *Nonlinearity* **9** (1996), 433–457.

[4] D.Bambusi, "On the dynamics of the Holstein model from the anticontinuous limit". *J. Math. Phys.* **40** (1999), 3710–3717.

[5] D. Bambusi, G. Gaeta: "On persistence of invariant tori and a theorem by Nekhoroshev". Preprint 2001, available at http://mpej.unige.ch/mp_arc/mp_arc-home.html, Preprint 01-426.

[6] R.S.MacKay, S.Aubry, "Proof of existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators". *Nonlinearity* **7** (1994), 1623–1643.

[7] J.Montaldi, "Persistance d'orbites périodiques relatives dans les systèmes hamiltoniens symétriques", *C. R. Acad. Sci. Paris Sér. I Math.* **324** (1997), 553–558.

[8] N.N. Nekhoroshev, "The Poincaré-Lyapounov-Liouville-Arnol'd theorem", *Funct. Anal. Appl.* **28** (1994), 128-129

[9] G. James, "Existence of breathers on FPU lattices", *C. R. Acad. Sci. Paris Sér.* I Math. **332** (2001), 581–586.

[10] M. Johansson, S. Aubry, "Existence and stability of quasiperiodic breathers in the discrete nonlinear Schrödinger equation." *Nonlinearity* **10** (1997), 1151–1178.

[11] P.G. Kevrekidis, M.I. Weinstein, "Dynamics of lattice kinks'. *Phys. D* **142** (2000), 113–152.

[12] P.G. Kevrekidis, M.I. Weinstein, "Breathers on a background: Periodic and quasiperiodic Solutions of extended discrete nonlinear wave systems" Preprint 2001.

[13] S.B. Kuksin "An infinitesimal Liouville–Arnold theorem as a criterion of reducibility for variational Hamiltonian equations" *Chaos, Solitons & Fractals* **2** (1992) 259–269.

[14] S.B. Kuksin: Nearly Integrable Infinite-Dimensional Hamiltonian Systems, Lect. Notes Math. 1556 (Springer 1994)

[15] J.A.Sepulchre, R.S.MacKay "Localized oscillations in conservative or dissipative networks of weakly coupled autonomous oscillators." *Nonlinearity* **10** (1997), 679–713.

[16] C.Wulff, M. Roberts, "Hamiltonian systems near relative periodic orbits", *SIAM J. Dyn. Syst.*, to appear.

[17] X. Yuan, "Construction of quasiperiodic breathers via KAM thechnique" *Commun. Math. Phys.*, to appear.