REMARKS ON THE FRACTAL DIMENSION OF BI-SPACES GLOBAL AND EXPONENTIAL ATTRACTORS

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Abstract. Bi-spaces global and exponential attractors for the time continuous dynamical systems are considered and the bounds on their fractal dimension are discussed in the context of the smoothing properties of the system between appropriately chosen function spaces. A unified analytic semigroup approach to abstract parabolic equations is described and applications to the sample problems are given.

Sunto. In questo lavoro sono considerate le nozioni di attrattori globali ed esponenziali "bi-spaces" per sistemi dinamici continui, e discusse limitazioni relative alla loro dimensione frattale in spazi di funzioni opportuni. Inoltre, viene presentato un approccio unificato per lo studio di problemi parabolici astratti, nella teoria dei semigruppi analitici, ed alcuni esempi sono mostrati.

1. Introductory notes

Attractors for the dynamical systems governed by partial differential equations in infinite dimensional Banach spaces have been considered by several authors within past few decades; see [3, 8, 11, 16, 18, 28, 1, 10, 12, 14, 21] and references therein. A general approach leading to the description of their topological dimension have essentially been developed, allowing to obtain much of the relevant information about the asymptotic behavior of the systems corresponding to a number of physical equations in both Hilbert and Banach function space setting.

In this article we reconstruct the results of [1, 3, 11, 12] concerning the bi-spaces global and exponential attractors and provide simultaneously certain bounds on their fractal dimension, which are a straightforward consequence of the smoothing properties of the dynamical system acting between appropriately chosen function spaces. These mentioned properties are known to be typical for problems that fall into a class of abstract parabolic equations in Banach spaces. Since the latter class contains many relevant physical equations, it is reasonable to describe the specific 'dissipative and smoothing mechanism' that leads to the existence of the global and exponential attractors and provides the bounds on their fractal dimension within the unified analytic semigroup approach of [17].

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The article is organized as follows. In Section 2 we obtain some estimate concerning fractal dimension of a precompact invariant set. We reconstruct the existence of a bi-spaces global attractor, derive a bound for its fractal dimension and investigate the existence of a (finite dimensional) exponential attractor. Next, in Section 3, we consider global attractors with bounded fractal dimension for a semigroup $\{S(t)\}$ governed by an abstract semilinear parabolic equation $u_t + Au = F(u)$ in a Banach space X. We consider then some specific applications. These concern a strongly damped wave equation, including the case when the resolvent operators corresponding to a linear operator A are non-compact and nonlinear term satisfies a critical growth condition, reaction diffusion equations with subquadratically growing gradient term, and the higher order parabolic problems involving $2m-$th order elliptic operators in the main part and fast growing nonlinearities. Another specific application involving a (non-parabolic) evolution problem is also discussed in the context of a conserved phase-field system with thermal memory.

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2. Fractal dimension of invariant sets and bi-spaces attractors

In this section we estimate fractal dimension of a precompact invariant set proving a certain generalization of [21, Lemma 1.3]. We discuss next the existence of a suitable notion of attractor and obtain some bounds of its fractal dimension.

Throughout this section V denotes a metric space. Recall that $\emptyset \neq C \subset V$ is precompact in V iff each sequence of elements from C possesses a Cauchy subsequence or, equivalently, for any $\varepsilon > 0$ there exist certain $n \in \mathbb{N}$ and $u_1, \ldots, u_n \in C$ such that $C \subset \bigcup_{i=1}^n B^V(u_i, \varepsilon)$. Recall also that if $C \neq \emptyset$ is precompact in V then its *fractal dimension* is

$$d_f^V(C) = \limsup_{\varepsilon \to 0} \log_{\frac{1}{\varepsilon}} N_\varepsilon^V(C),$$

where $N_\varepsilon^V(C)$ denotes the smallest number of ε–balls in V needed to cover C.

Note that $d_f^V(C) = d_f^V(cl_V C)$ and consider numbers

$$\widetilde{d}_f^V(C) = \limsup_{\varepsilon \to 0} \log_{\frac{1}{\varepsilon}} \widetilde{N}_\varepsilon^V(C) \quad \text{and} \quad d_f^C(C) = \limsup_{\varepsilon \to 0} \log_{\frac{1}{\varepsilon}} N_\varepsilon^C(C),$$

where $\widetilde{N}_\varepsilon^V(C)$ denotes the smallest number of ε–balls in V with centers in C needed to cover C and $N_\varepsilon^C(C)$ is the smallest number of ε–balls in C (with respect to the inherited metric) needed to cover C. We remark that

$$N_\varepsilon^V(C) \leq N_\varepsilon^C(C) = \widetilde{N}_\varepsilon^V(C) \leq N_{\frac{\varepsilon}{2}}^V(C),$$

which implies that for a nonvoid set C precompact in V

$$d_f^V(C) = d_f^C(C) = \widetilde{d}_f^V(C).$$

Furthermore, if a subset $V_0 \subset V$ is considered with the inherited metric and $C \subset V_0 \subset V$, then C is precompact in V_0 and

$$d_f^{V_0}(C) = d_f^V(C).$$

The following result is a generalization of [21, Lemma 1.3].

Lemma 2.1. *Let V, W be normed spaces such that V is compactly embedded in W and let $C \neq \emptyset$ be a precompact subset of W, invariant under a map S, i.e. $S(C)=C$. Assume that for each $u_0 \in C$ the map S has the following decomposition*

$$S = P(u_0) + M(u_0), \text{ where } P(u_0)\colon C \to W,\ M(u_0)\colon C \to V, \tag{2.1}$$

and

$$\exists_{0\le\delta<\frac{1}{2}}\ \exists_{\varepsilon_0>0}\ \forall_{u_0,u_1\in C, \|u_1-u_0\|_W\le\varepsilon_0}\ \|P(u_0)u_1 - P(u_0)u_0\|_W \le \delta \|u_1-u_0\|_W, \tag{2.2}$$

$$\exists_{\kappa>0}\ \forall_{u_0,u_1\in C}\ \|M(u_0)u_1 - M(u_0)u_0\|_V \le \kappa \|u_1-u_0\|_W. \tag{2.3}$$

Under the above assumptions we have for any $\nu \in (0, \frac{1}{2}-\delta)$

$$d_f^W(C) \le \log_{\frac{1}{2(\delta+\nu)}} N^W_{\frac{\nu}{\kappa}}(B^V(0,1)). \tag{2.4}$$

Proof. For $\nu \in (0, \frac{1}{2}-\delta)$ let $N = N^W_{\frac{\nu}{\kappa}}(B^V(0,1))$ and consider $0 < \varepsilon \le \varepsilon_0$. Note that

$$C \subset \bigcup_{1\le i\le \widetilde{N}^W_\varepsilon(C)} B^W(u_i,\varepsilon) \quad \text{with certain } u_1,\dots,u_{\widetilde{N}^W_\varepsilon(C)} \in C.$$

If $u \in C$, then $u \in B^W(u_i,\varepsilon)$ and from (2.3) we obtain

$$\|M(u_i)u - M(u_i)u_i\|_V \le \kappa \|u-u_i\|_W < \kappa\varepsilon,$$

ensuring that

$$\frac{1}{\kappa\varepsilon}(M(u_i)u - M(u_i)u_i) \in B^V(0,1) \subset \bigcup_{1\le j\le N} B^W(w_j, \frac{\nu}{\kappa}).$$

Thus we have $\|M(u_i)u - M(u_i)u_i - \kappa\varepsilon w_j\|_W < \nu\varepsilon$ and we use (2.2) to get

$$\|P(u_i)u - P(u_i)u_i\|_W + \|M(u_i)u - M(u_i)u_i - \kappa\varepsilon w_j\|_W \le (\delta+\nu)\varepsilon.$$

Since (2.1) holds and $S(C) = C$, what was said above ensures that

$$C \subset \bigcup_{1\le i\le \widetilde{N}^W_\varepsilon(C)} \bigcup_{1\le j\le N} B^W(S(u_i) + \kappa\varepsilon w_j, (\delta+\nu)\varepsilon).$$

If we now increase the radius of the balls twice we will obtain

$$C \subset \bigcup_{1\le i\le \widetilde{N}^W_\varepsilon(C)} \bigcup_{1\le j\le N} B^W(\tilde{w}_{ij}, 2(\delta+\nu)\varepsilon) \quad \text{with centers } \tilde{w}_{ij} \in C,$$

which shows that for every $0 < \varepsilon \le \varepsilon_0$ we have

$$\widetilde{N}^W_{2(\delta+\nu)\varepsilon}(C) \le \widetilde{N}^W_\varepsilon(C) \cdot N. \tag{2.5}$$

Induction argument ensures that for any $k \in \mathbb{N}$

$$\widetilde{N}^W_{[2(\delta+\nu)]^k\varepsilon_0}(C) \le \widetilde{N}^W_{\varepsilon_0}(C) \cdot N^k.$$

Since for small $\varepsilon \in (0, \varepsilon_0]$ there is $k \in \mathbb{N}$ such that $[2(\delta+\nu)]^{k+1}\varepsilon_0 < \varepsilon \leq [2(\delta+\nu)]^k\varepsilon_0$ and

$$\log_{\frac{1}{\varepsilon}} \widetilde{N}_\varepsilon^W(C) \leq \log_{\frac{1}{[2(\delta+\nu)]^k\varepsilon_0}} \widetilde{N}_{\varepsilon_0}^W(C) \cdot N^{k+1},$$

the estimate (2.4) now follows easily. □

In what follows $\{S(t)\}$ denotes a family of maps such that

$$S(t)\colon V \to V,\ t \geq 0,\ \ S(0) = Id \ \text{ and } \ S(t)S(s) = S(t+s) \ \text{ for } s, t \geq 0. \tag{2.6}$$

It is reasonable to consider the situation when orbits of bounded sets eventually enter another metric space W and discuss some minimal smoothness conditions on $\{S(t)\}$ that lead to the existence of an attracting compact invariant set.

We denote by ρ_W (resp. d_W) the metric (resp. the Hausdorff semidistance) in W. If there is B_0 bounded in V which *absorbs* bounded subsets of V, that is

$$S(t)B \subset B_0 \ \text{ for each } B \text{ bounded in } V \text{ and all } t \geq t_B > 0,$$

then following [1] we will consider the two *asymptotic smoothness conditions*:

$$\begin{array}{c}\text{any sequence } \{S(t_n)v_n\} \text{ with } t_n \to \infty \text{ and } \{v_n\} \subset B_0 \\ \text{has a subsequence convergent in the metric of } W,\end{array} \tag{2.7}$$

$$\begin{array}{l}\text{for each sequence } \{S(t_n)v_n\} \text{ with } t_n \to \infty, \{v_n\} \subset B_0 \text{ there} \\ \text{is a subsequence } \{S(t_{n_k})v_{n_k}\} \text{ and a certain point } w \in V \cap W \\ \text{such that } \lim\limits_{k\to\infty} \rho_W(S(t)S(t_{n_k})v_{n_k}, S(t)w) = 0 \ \text{ for all } t \geq 0.\end{array} \tag{2.8}$$

Proposition 2.2. *(*[1]*) Suppose B_0 is bounded in V, absorbs bounded subsets of V and W is a metric space containing B_0.*

(i) *If* (2.7) *holds, then there is a nonvoid set $\mathbf{A} \subset cl_W B_0$, compact in W and attracting sets bounded in V with respect to the Hausdorff semidistance d_W.*
(ii) *If* (2.8) *holds, then $\mathbf{A}$ is additionally a subset of V invariant under $\{S(t)\}$. Moreover, $\mathbf{A}$ is closed in V provided that*

$$\{w_m\} \subset V \cap W,\ \rho_V(w_m, w) \to 0 \ \text{ and } \ \rho_W(w_m, \tilde{w}) \to 0 \ \text{ imply } \ w = \tilde{w}.$$

Proof. Following [1], the result is a consequence of the definition of

$$\mathbf{A} = \{w \in W \colon S(t_n)v_n \to w \text{ in } W \text{ for some } \{v_n\} \subset B_0 \text{ and } t_n \to \infty\}. \tag{2.9}$$ □

Coming back to [3] (see also [1] and references therein) we recall that a nonvoid set $\mathbf{A} \subset V \cap W$, which is invariant under $\{S(t)\}$, closed in V, compact in W and attracts bounded subsets of V with respect to the Hausdorff semidistance d_W is called a *global* $(V-W)$ *attractor for* $\{S(t)\}$. Proposition 2.2 then implies

Corollary 2.3. *Let W be a metric space such that $V \subset W$ and $\rho_V(w_m, w) \to 0$ implies $\rho_W(w_m, w) \to 0$. Suppose that B_0 is bounded in V, B_0 absorbs bounded subsets of V and* (2.8) *holds.*

Under these assumptions there exists a global $(V-W)$ attractor $\mathbf{A}$ for $\{S(t)\}$ and if, in addition, V is a reflexive Banach space and W is a normed space, then $\mathbf{A}$ is bounded in V.

Proof. By Proposition 2.2 the set $\mathbf{A}$ in (2.9) is a $(V-W)$ global attractor and $\mathbf{A}$ is bounded in V as a consequence of weak convergence properties. □

If $V = W$, then a global $(V-V)$ attractor is a notion of a *global attractor* in [16] and of a *compact invariant B-attractor* in [18]. In this case Proposition 2.2 implies

Corollary 2.4. *Suppose B_0 is bounded in V and absorbs bounded subsets of V.*

(i) *If* (2.8) *holds with $W = V$, then there exists a global attractor (a compact invariant B-attractor) for $\{S(t)\}$.*

(ii) *If* (2.7) *holds with $W = V$ and the map $[0,\infty)\times V \ni (t,v) \to S(t)v \in V$ is continuous, then there is a global attractor (a compact invariant B-attractor) $\mathbf{A}$ for $\{S(t)\}$ and $\mathbf{A}$ is a stable set (cf.* [8, Observation 1.1.1]*).*

With further assumptions on $\{S(t)\}$ fractal dimension of $\mathbf{A}$ can also be estimated and boundedness of $\mathbf{A}$ in V will follow even if V is not reflexive. In the light of Lemma 2.1 and Corollary 2.3 the following results hold.

Theorem 2.5. *If V, W are normed spaces and the assumptions of Corollary 2.3 hold, then there exists a global $(V-W)$ attractor $\mathbf{A}$.*

If, in addition, V is compactly embedded in W and there exists $t_0 > 0$ such that

$$S(t_0) = P(t_0)(u_0) + M(t_0)(u_0),\ u_0 \in \mathbf{A},$$

where $P(t_0)(u_0)\colon \mathbf{A} \to W$, $M(t_0)(u_0)\colon \mathbf{A}\to V$ satisfy

$$\forall_{u_0,u_1\in\mathbf{A},\|u_1-u_0\|_W\le\varepsilon_0}\ \|P(t_0)(u_0)u_1 - P(t_0)(u_0)u_0\|_W \le \delta\,\|u_1-u_0\|_W\,, \tag{2.10}$$

$$\forall_{u_0,u_1\in\mathbf{A}}\ \|M(t_0)(u_0)u_1 - M(t_0)(u_0)u_0\|_V \le \kappa\,\|u_1-u_0\|_W\,, \tag{2.11}$$

with some $0\le\delta<\frac12$, $\varepsilon_0>0$ and $\kappa>0$, then for every $\nu\in(0,\frac12-\delta)$ we have

$$d_f^W(\mathbf{A}) \le \log_{\frac{1}{2(\delta+\nu)}} N^W_{\frac{\nu}{\kappa}}(B^V(0,1)). \tag{2.12}$$

Corollary 2.6. *If V, W are normed spaces and the assumptions of Corollary 2.3 hold, then there exists a global $(V-W)$ attractor $\mathbf{A}$.*

If, in addition, V is compactly embedded in W and

$$\exists_{t_0>0}\ \exists_{\kappa>0}\ \forall_{u_1,u_2\in\mathbf{A}}\ \|S(t_0)u_1 - S(t_0)u_2\|_V \le \kappa\,\|u_1-u_2\|_W\,, \tag{2.13}$$

then $\mathbf{A}$ is compact in V and satisfies

$$d_f^V(\mathbf{A}) \le d_f^W(\mathbf{A}) \le \log_{\frac{1}{2\nu}} N^W_{\frac{\nu}{\kappa}}(B^V(0,1)),\ \nu\in(0,\frac12). \tag{2.14}$$

In the similar vein, following the ideas of [9, 12, 24], we prove

Proposition 2.7. *Let V be a Banach space compactly embedded in a Banach space W. Assume that V_0 is a subset of V and let B_0 be a bounded set of V_0 absorbing bounded subsets of V_0. Suppose that there exists $t_0\ge t_{B_0}$ such that*

$$S(t_0) = P(t_0) + M(t_0), \tag{2.15}$$

where $P(t_0)\colon B_0\to W$, $M(t_0)\colon B_0\to V$ satisfy with some $0\le\delta<\frac12$, $\kappa>0$, $0<\theta<1$ and $\mu>0$ the following estimates

$$\forall_{u_1,u_2\in B_0}\ \|P(t_0)u_1 - P(t_0)u_2\|_W \le \delta\,\|u_1-u_2\|_W\,, \tag{2.16}$$

$$\forall_{u_1,u_2\in B_0}\ \|M(t_0)u_1 - M(t_0)u_2\|_V \le \kappa \|u_1 - u_2\|_W, \tag{2.17}$$

$$\forall_{t_1,t_2\in[t_0,2t_0]}\ \forall_{u_1,u_2\in B_0}\ \|S(t_1)u_1 - S(t_2)u_2\|_W \le \mu(|t_1 - t_2|^\theta + \|u_1 - u_2\|_W). \tag{2.18}$$

Then for any $\nu \in \left(0, \frac{1}{2} - \delta\right)$ there exists a nonvoid set $\widehat{\mathcal{M}} = \widehat{\mathcal{M}}_\nu \subset B_0$, positively invariant under $\{S(t)\}$, precompact in W and satisfying conditions

(i) $\exists_{\omega>0}\forall_{B \text{ bounded in } V_0}\ \lim_{t\to\infty} e^{\omega t} d_W(S(t)B, \widehat{\mathcal{M}}) = 0$,
(ii) $d_f^W(\widehat{\mathcal{M}}) \le \frac{1}{\theta}\left(1 + \log_{\frac{1}{2(\delta+\nu)}} N^W_{\frac{\nu}{\kappa}}(B^V(0,1))\right)$.

Proof. Fix $\nu \in \left(0, \frac{1}{2} - \delta\right)$ and note that $S(t)B_0 \subset B_0$ for $t \ge t_0$. Since B_0 is bounded in W, from [12, Proposition 1] we deduce that there is a nonvoid set $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_\nu \subset B_0$, precompact in W satisfying $S(t_0)\widetilde{\mathcal{M}} \subset \widetilde{\mathcal{M}}$, $d_f^W(\widetilde{\mathcal{M}}) \le \log_{\frac{1}{2(\delta+\nu)}} N^W_{\frac{\nu}{\kappa}}(B^V(0,1))$ and such that, for certain $C, \xi > 0$, we have $d_W(S(nt_0)B_0, \widetilde{\mathcal{M}}) \le Ce^{-\xi n}$ for each $n \in \mathbb{N}$. Setting $\widehat{\mathcal{M}} = \widehat{\mathcal{M}}_\nu := \bigcup_{s\in[t_0,2t_0]} S(s)\widetilde{\mathcal{M}} \subset B_0$ we then have that $S(t)\widehat{\mathcal{M}} \subset \widehat{\mathcal{M}}$ for $t \ge 0$. As a result of (2.18) the set $\widehat{\mathcal{M}}$ is also precompact in W and

$$d_f^W(\widehat{\mathcal{M}}) \le \frac{1}{\theta} d_f^{\mathbb{R}\times W}([t_0, 2t_0] \times \widetilde{\mathcal{M}}) \le \frac{1}{\theta}(1 + d_f^W(\widetilde{\mathcal{M}})).$$

If B is a bounded subset of V_0, then $S(t_B)B \subset B_0$. If furthermore $t \ge t_B + 3t_0$, then $t - t_B = n_t t_0 + 2t_0 + r_t$ with certain $r_t \in [0, t_0]$, $n_t \in \mathbb{N}$, and via (2.18) we get

$$\begin{aligned} d_W(S(t)B, \widehat{\mathcal{M}}) &= d_W(S(t-t_B)S(t_B)B, \widehat{\mathcal{M}}) \le d_W(S(t-t_B)B_0, \widehat{\mathcal{M}}) \\ &\le d_W(S(t_0)S(n_t t_0)B_0, S(t_0)\widetilde{\mathcal{M}}) \le \mu d_W(S(n_t t_0)B_0, \widetilde{\mathcal{M}}) \le C_B e^{-\frac{\xi}{t_0}t}, \end{aligned}$$

which completes the proof. □

Generalizing the notion of an exponential attractor in [11] we will say that a nonvoid set $\mathcal{M} \subset V \cap W$ is an *exponential* $(V-W)$ *attractor* for $\{S(t)\}$ if $\mathcal{M}$ is positively invariant under $\{S(t)\}$, closed in V, compact in W, $d_f^W(\mathcal{M}) < \infty$ and

$$\exists_{\omega>0}\forall_{B \text{ bounded in } V}\ \lim_{t\to\infty} e^{\omega t} d_W(S(t)B, \mathcal{M}) = 0. \tag{2.19}$$

Corollary 2.8. *If V is reflexive, the assumptions of Proposition 2.7 hold with $V_0 = V$ and $S(t) : W \supset \mathrm{cl}_W B_0 \to W$ is continuous for each $t > 0$, then, for $\nu \in (0, \frac{1}{2} - \delta)$,*

(i) *$\mathcal{M}_\nu := \mathrm{cl}_W \widehat{\mathcal{M}}_\nu$ is an exponential $(V - W)$ attractor bounded in V,*
(ii) *there exists a finite dimensional global $(V - W)$ attractor $\mathbf{A} \subset \mathcal{M}_\nu$.*

Corollary 2.9. *Let V be a Banach space compactly embedded in a Banach space W and let B_0 be a bounded set in V absorbing bounded subsets of V. Suppose that* (2.18) *holds and*

$$\forall_{t>0}\exists_{\kappa(t)>0}\forall_{u_1,u_2\in \mathrm{cl}_V B_0}\ \|S(t)u_1 - S(t)u_2\|_V \le \kappa(t)\|u_1 - u_2\|_W. \tag{2.20}$$

Then for any $\nu \in (0, \frac{1}{2})$

(i) *there exists an exponential $(V - V)$ attractor $\mathcal{M}_\nu \subset \mathrm{cl}_V B_0$ satisfying*

$$d_f^V(\mathcal{M}_\nu) \le \frac{1}{\theta}\left(1 + \log_{\frac{1}{2\nu}} N^W_{\frac{\nu}{\kappa_0}}(B^V(0,1))\right), \tag{2.21}$$

where $\kappa_0 = \kappa(t_0)$,

(ii) *there exists a finite dimensional global* $(V-V)$ *attractor* $\mathbf{A}$ *contained in* $\mathcal{M}_\nu$.

Proof. (i) The argument is the same as in the proof of Proposition 2.7, but we also set $\mathcal{M}_\nu := \operatorname{cl}_V S(1)\widehat{\mathcal{M}}_\nu$ and observe by (2.20) that $\mathcal{M}_\nu \subset \operatorname{cl}_V B_0$ is compact in V and positively invariant under $\{S(t)\}$. Moreover, $d_f^V(\mathcal{M}_\nu) \le d_f^W(\widehat{\mathcal{M}}_\nu)$ and for $t \ge t_B + 3t_0 + 1$ we also have

$$d_V(S(t)B, \mathcal{M}_\nu) = d_V(S(1)S(t-1)B, S(1)\widehat{\mathcal{M}}_\nu) \le \kappa(1) d_W(S(t-1)B, \widehat{\mathcal{M}}_\nu),$$

which ensures that (2.19) holds with $V = W$. This completes the proof of (i).

(ii) Let $t_n \to \infty$, $\{u_n\} \subset B_0$ and consider $\{S(t_n)u_n\}$. Since $\mathcal{M}_\nu$ attracts B_0, there exists a sequence $\{v_n\} \subset \mathcal{M}_\nu$ such that $S(t_n)u_n - v_n \to 0$ in V. By the compactness of $\mathcal{M}_\nu$ there exists $v \in \mathcal{M}_\nu \subset \operatorname{cl}_V B_0$ and a subsequence $\{S(t_{n_k})u_{n_k}\}$ such that it converges to v in V. Using (2.20) we then get $S(t + t_{n_k})u_{n_k} - S(t)v \to 0$ in V for every $t \ge 0$, so that by Corollary 2.4 (i) there is a global $(V - V)$ attractor $\mathbf{A}$. $\square$

3. Applications

In this section we present the results concerning bi-spaces attractors with bounded fractal dimension for the semigroup governed by the Cauchy problem for an abstract parabolic equation. We also consider some specific examples, like a strongly damped wave equation involving a critically growing nonlinearity and a linear main part with non-compact resolvent, reaction-diffusion equations with subquadratically growing gradient term and $2m-$th order parabolic problems with fast growing nonlinearities. We finally discuss a (non-parabolic) evolution problem, which will be a conserved phase-field system with thermal memory.

3.1. **Abstract semilinear parabolic problems.** Throughout this subsection X denotes a Banach space, $A\colon X \supset D(A) \to X$ is a positive sectorial operator in X and X^σ, $\sigma \ge 0$, are the associated fractional power spaces.

It is known that $-A$ generates in $X = X^0$ a C^0 analytic semigroup $\{e^{-At}\}$ and

$$\|e^{-At}\|_{L(X,X^\sigma)} \le c_\sigma \frac{e^{-at}}{t^\sigma}, \qquad \sigma \ge 0,\ t > 0, \tag{3.1}$$

where $a > 0$ is such that $Re\sigma(A) > a$ and c_σ are certain positive constants.

In this subsection we fix $\alpha \in [0,1)$ and assume that $F\colon X^\alpha \to X^0$ satisfies

$$\forall_{B \text{ bounded in } X^\alpha}\ \exists_{L_{\alpha,B}>0}\ \forall_{v,w\in B}\ \|F(v) - F(w)\|_{X^0} \le L_{\alpha,B}\|v - w\|_{X^\alpha}. \tag{3.2}$$

We remark that the inequality in (3.2) is now true with α replaced by any $\beta \in [\alpha, 1)$. We consider the Cauchy problem

$$\begin{cases} u_t + Au = F(u),\ t > 0, \\ u(0) = u_0 \in X^\alpha. \end{cases} \tag{3.3}$$

It is known from [17] that there is a certain $\tau_{u_0} > 0$ such that in $C([0,\tau_{u_0}), X^\alpha)$ there exists a unique mild solution $u(\cdot, u_0)$ of (3.3) and either $\tau_{u_0} = \infty$, that is

$u(\cdot,u_0)$ exists globally in time, or $\tau_{u_0}<\infty$ and $\limsup_{t\to\tau_{u_0}^-}\|u(t,u_0)\|_{X^\alpha}=\infty$. Such $u(\cdot,u_0)$ satisfies variation of constants formula

$$u(t,u_0)=e^{-At}u_0+\int_0^t e^{-A(t-s)}F(u(s,u_0))ds,\ t\in[0,\tau_{u_0}), \tag{3.4}$$

and has additional regularity properties; namely

$$u(\cdot,u_0)\in C([0,\tau_{u_0}),X^\alpha)\cap C^1((0,\tau_{u_0}),X^\alpha)\cap C((0,\tau_{u_0}),X^1).$$

In what follows such $u(\cdot,u_0)$ will be called an X^α *solution of* (3.3).

We will assume throughout this subsection that all X^α solutions of (3.3) exist globally in time, in which case (3.3) defines a C^0 semigroup $\{S(t)\}$ on X^α such that

$$S(t)u_0:=u(t,u_0),\ t\geq 0,\ u_0\in X^\alpha.$$

From (3.1), (3.2) and (3.4) we obtain the following auxiliary relation.

Lemma 3.1. *If $t>0$, $\beta\in[\alpha,1)$, $\mathcal{B}$ is bounded in X^α and $S(s)u_1,S(s)u_2\in\mathcal{B}$ for each $s\in[0,t]$, then*

$$\|S(t)u_1-S(t)u_2\|_{X^\beta}\leq\frac{c_{\beta-\alpha}}{t^{\beta-\alpha}}\|u_1-u_2\|_{X^\alpha}+\int_0^t\frac{c_\beta L_{\beta,\mathcal{B}}}{(t-s)^\beta}\|S(s)u_1-S(s)u_2\|_{X^\beta}\,ds.$$

Theorem 3.2. *Suppose that the resolvent of A is compact, B_0 is bounded in X^α and absorbs bounded subsets of X^α.*

Then, for each $\beta\in(\alpha,1)$,
(i) there exists a global $(X^\alpha-X^\beta)$ attractor $\mathbf{A}$ for $\{S(t)\}$ satisfying

$$d_f^{X^\beta}(\mathbf{A})\leq d_f^{X^\alpha}(\mathbf{A})\leq\log_2 N^{X^\alpha}_{\frac{1}{4\kappa_\beta}}(B^{X^\beta}(0,1)), \tag{3.5}$$

with κ_β specified in (3.7)*,*
(ii) there exists an exponential $(X^\alpha-X^\beta)$ attractor $\mathcal{M}$ containing $\mathbf{A}$.

Proof. Fix $\beta\in(\alpha,1)$ and note that $\{S(t)\}$ is a C^0 semigroup on X^β. By assumption, X^σ is compactly embedded into X^γ for each $\sigma>\gamma$. From [8, Lemma 3.2.1] it follows that $\widetilde{B}_0:=\mathrm{cl}_{X^\beta}S(t_{B_0})B_0$ is compact in X^β and absorbs bounded subsets of X^β.

(i) Corollary 2.4(ii) applies with $V=W=X^\beta$ and there is a global $(X^\beta-X^\beta)$ attractor $\mathbf{A}$, which as a set does not depend on β. We observe that $\mathbf{A}$ is a global $(X^\alpha-X^\beta)$ attractor because $S(t)B\subset\widetilde{B}_0$ for any B bounded in X^α and all t sufficiently large.

Applying Lemma 3.1 with $\mathcal{B}=\mathbf{A}$, $\beta\in(\alpha,1)$, $u_1,u_2\in\mathbf{A}$, and $t=t^*$ satisfying

$$\frac{c_\beta L_{\beta,\mathbf{A}}t^{*1-\beta}}{2^{1-2\beta+\alpha}}\left(\frac{1}{1-\beta+\alpha}+\frac{1}{1-\beta}\right)=\frac{1}{2} \tag{3.6}$$

we obtain via Volterra inequality (see [8, formulas (1.2.21) and (1.2.30)]) that

$$\|S(t^*)u_1-S(t^*)u_2\|_{X^\beta}\leq 2c_{\beta-\alpha}t^{*\alpha-\beta}\|u_1-u_2\|_{X^\alpha}=:\kappa_\beta\|u_1-u_2\|_{X^\alpha}. \tag{3.7}$$

In particular, Corollary 2.6 applies with $V=X^\beta$, $W=X^\alpha$ and with the constant $\kappa=\kappa_\beta=2c_{\beta-\alpha}t^{*\alpha-\beta}$. Part (i) is thus proved.

(ii) By [8, Corollary 3.3.2] and the properties of $\widetilde{B}_0$, the positive orbit $\gamma^+(\widetilde{B}_0)$ of $\widetilde{B}_0$ is bounded in X^β. Applying Lemma 3.1 with $\mathcal{B} = \gamma^+(\widetilde{B}_0)$, $u_1, u_2 \in \widetilde{B}_0$, $0 < t < T$ and using the Volterra inequality we get

$$\|S(t)u_1 - S(t)u_2\|_{X^\beta} \le t^{\alpha-\beta} const(T, \alpha, \beta, \widetilde{B}_0) \|u_1 - u_2\|_{X^\alpha}, \ 0 < t < T. \tag{3.8}$$

In particular there is a constant $c > 0$ independent of $u_1, u_2 \in \widetilde{B}_0$ and $t \in [\tau_1, \tau_2] \subset (0, \infty)$ such that

$$\|S(t)u_1 - S(t)u_2\|_{X^\alpha} \le c \|u_1 - u_2\|_{X^\alpha}, \ u_1, u_2 \in \widetilde{B}_0, \ t \in [\tau_1, \tau_2]. \tag{3.9}$$

From [8, formula (2.2.3)] it follows for $u \in \widetilde{B}_0$ and $t_1, t_2 \in [\tau_1, \tau_2]$ that

$$\|S(t_1)u - S(t_2)u\|_{X^\alpha} \le \tilde{c} |t_1 - t_2|^\theta, \tag{3.10}$$

where $\theta \in (0, 1)$ and $\tilde{c} > 0$ do not depend on $t_1, t_2 \in [\tau_1, \tau_2]$ and $u \in \widetilde{B}_0$. Therefore, Corollary 2.9 applies with $V = X^\beta$, $W = X^\alpha$ and there exists an exponential $(X^\beta - X^\beta)$ attractor $\mathcal{M}$. Since $S(t)B \subset \widetilde{B}_0$ for any B bounded in X^α and all t sufficiently large, $\mathcal{M}$ is an exponential $(X^\alpha - X^\beta)$ attractor and the proof is complete. □

Repeating the argument leading to (3.7) in the proof of Theorem 3.2 and using Lemma 2.1 with $S = S(t^*)$, $P(u_0) \equiv 0$, $M(u_0) \equiv S(t^*)$, we obtain

Corollary 3.3. *If the resolvent of A is compact and $\mathbf{A}$ is a nonvoid bounded and invariant subset of X^α, then $\mathbf{A}$ is precompact in X^σ, $\sigma \in [\alpha, 1)$, and*

$$d_f^{X^\beta}(\mathbf{A}) \le d_f^{X^\alpha}(\mathbf{A}) \le \log_2 N^{X^\alpha}_{\frac{1}{4\kappa_\beta}}(B^{X^\beta}(0, 1)), \ \beta \in (\alpha, 1), \tag{3.11}$$

with κ_β as in (3.7).

Remark 3.4. We remark that the estimate (3.11) requires the knowledge of the ε-entropy $\log_2 N^{X^\alpha}_\varepsilon(B^{X^\beta}(0, 1))$, for which in the case when abstract fractional power spaces are involved the explicit estimates may not be easily available. However, in applications, fractional power scale X^α, $\alpha \ge 0$, is often characterized with the aid of some known function spaces, for which such estimates appear within the references (see e.g. [29, §4.10.3]).

3.2. **Second order parabolic equations in bounded domains.** Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary and consider the second order problem

$$u_t = \Delta u + f(x, u, \nabla u), \ t > 0, \ x \in \Omega, \quad u(0, x) = u_0(x), \ x \in \Omega, \tag{3.12}$$

with Dirichlet homogeneous boundary condition.

Following [27] (see also [19]) assume for $f \in C(\overline{\Omega} \times \mathbb{R}^{N+1}, \mathbb{R})$ that

$$f \text{ is locally Lipschitz with respect to each variable separately,} \tag{3.13}$$

$$sf(x, s, \mathbf{0}) \le 0 \text{ for all } x \in \overline{\Omega}, \ s \in \mathbb{R}, \ |s| \ge K \text{ with a certain } K > 0, \tag{3.14}$$

and that for some continuous map $h\colon [0, \infty) \to [0, \infty)$ and certain $\gamma \in (0, 2)$

$$|f(x, s, \mathbf{p})| \le h(|s|)(1 + |\mathbf{p}|^\gamma) \ \text{ whenever } \ x \in \overline{\Omega}, \ s \in \mathbb{R}, \ \mathbf{p} \in \mathbb{R}^N. \tag{3.15}$$

We consider in this example $A = -\Delta$ in $X^0 = L^p(\Omega)$, $p > N$, with the domain $D(A) = W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)$ and choose $\alpha \in (\frac{1}{2} + \frac{N}{2p}, 1)$. The resolvent of A is compact and, since $X^\alpha \hookrightarrow C^1(\overline{\Omega})$, the function

$$F\colon X^\alpha \to X^0, \quad F(u)(x) = f(x, u(x), \nabla u(x)) \text{ for } u \in X^\alpha,\ x \in \Omega,$$

fulfils (3.2) with $X^\alpha = [L^p(\Omega), W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)]_\alpha =: W^{2\alpha,p}_0(\Omega)$ and $X^0 = L^p(\Omega)$.

Due to (3.14) and comparison argument we have (see [27] and references therein)

$$\|u(t,u_0)\|_{L^\infty(\Omega)} \leq \max\{\|u_0\|_{L^\infty(\Omega)}, K\}, \quad u_0 \in X^\alpha,\ t \in (0, \tau_{u_0}), \tag{3.16}$$

$$\limsup_{t\to\infty} \sup_{u_0 \in B^{X^\alpha}(0,r)} \|u(t,u_0)\|_{L^\infty(\Omega)} \leq K,\ r > 0. \tag{3.17}$$

Due to (3.15) one obtains as in [2] that, for certain $0 < \eta < 1$ and $\beta > \alpha$,

$$\|F(u(t,u_0))\|_{L^p(\Omega)} \leq c(\|u_0\|_{L^\infty(\Omega)})(1 + \|u(t,u_0)\|^\eta_{X^\beta}),\ t \in (0, \tau_{u_0}). \tag{3.18}$$

From this we infer that all X^α solutions exist globally in time and for the associated C^0 semigroup $\{S(t)\}$ on X^α there is B_0 bounded in X^α and absorbing bounded subsets of X^α (see [8, Corollary 4.1.3]); in particular Theorem 3.2 applies.

Corollary 3.5. *If* (3.13), (3.14), (3.15) *hold and* $\alpha \in (\frac{1}{2} + \frac{N}{2p}, 1)$, $p > N$, *then for each* $\beta \in (\alpha, 1)$ *there is a global* $(W^{2\alpha,p}_0(\Omega) - W^{2\beta,p}_0(\Omega))$ *attractor* $\mathbf{A}$ *satisfying*

$$d_f^{W^{2\beta,p}_0(\Omega)}(\mathbf{A}) \leq d_f^{W^{2\alpha,p}_0(\Omega)}(\mathbf{A}) \leq \log_2 N^{W^{2\alpha,p}_0(\Omega)}_{\frac{1}{4\kappa_\beta}}(B^{W^{2\beta,p}_0(\Omega)}(0,1)), \tag{3.19}$$

which is contained in an exponential $(W^{2\alpha,p}_0(\Omega) - W^{2\beta,p}_0(\Omega))$ *attractor.*

When $f = f(x,u)$ does not depend on the gradient we choose any $p > N$, $\alpha \in (\frac{N}{2p}, \frac{1}{2})$ and note that $\phi(x) \in [-K, K]$ for $\phi \in \mathbf{A}$ and $x \in \Omega$. If L is a (uniform for $x \in \overline{\Omega}$) Lipschitz constant for f with respect to $s \in [-K, K]$ and c_Ω is a constant from the Poincaré inequality, then for $\beta = \frac{1}{2}$ we have $X^\beta = X^{\frac{1}{2}} = W^{1,p}_0(\Omega)$ and

$$\|f(\cdot, u_1(\cdot)) - f(\cdot, u_2(\cdot))\|_{L^p(\Omega)} \leq c_\Omega L \|u_1 - u_2\|_{W^{1,p}_0(\Omega)} =: L_{\frac{1}{2},\mathbf{A}} \|u_1 - u_2\|_{W^{1,p}_0(\Omega)}$$

whenever $u_1, u_2 \in \mathbf{A}$. In this case (3.6)-(3.7) apply with $L_{\beta,\mathbf{A}} = c_\Omega L$ and we get (3.19) with $\kappa_\beta = 2c_{\frac{1}{2}-\alpha}(2^{3-\alpha} c_{\frac{1}{2}} c_\Omega L (1+\alpha)(1+2\alpha)^{-1})^{1-2\alpha}$.

3.3. **Wave equation with damping operator** $(-\Delta_D)^{\frac{1}{2}}$**.** Following [4, 5], we consider[1]

$$\begin{cases} u_{tt} + \eta(-\Delta_D)^{\frac{1}{2}} u_t + (-\Delta_D)u = f(u),\ t > 0,\ x \in \Omega, \\ u(0,x) = u_0(x),\ u_t(0,x) = v_0(x),\ x \in \Omega, \\ u(t,x) = 0,\ t \geq 0,\ x \in \partial\Omega, \end{cases} \tag{3.20}$$

where $\eta > 0$, Ω is a bounded C^2 smooth domain in $\mathbb{R}^N$, $N \geq 3$, and Δ_D is the Dirichlet Laplacian in $L^2(\Omega)$ with the domain $H^2(\Omega) \cap H^1_0(\Omega)$. We assume that

$$|f(s) - f(\bar{s})| \leq c\,|s - \bar{s}|\,(1 + |s|^{\rho-1} + |\bar{s}|^{\rho-1}),\ s, \bar{s} \in \mathbb{R},\ \text{with } \rho \in (1, \tfrac{N+2}{N-2}) \tag{3.21}$$

[1]Instead of $(-\Delta_D)^{\frac{1}{2}}$, chosen here for better clarity of argument, one can consider the damping operator $(-\Delta_D)^\theta$ with $\theta \in [\frac{1}{2}, 1)$, but the analysis would have to be adapted.

and

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} \le 0. \tag{3.22}$$

We can rewrite (3.20) in the form (3.3) in $X^0 = H^1_0(\Omega) \times L^2(\Omega)$ with

$$A\begin{bmatrix}\varphi\\ \psi\end{bmatrix} = \begin{bmatrix}-\psi\\ -\Delta_D\varphi+\eta(-\Delta_D)^{\frac12}\psi)\end{bmatrix} \quad \text{for} \quad \begin{bmatrix}\varphi\\ \psi\end{bmatrix} \in D(A), \qquad F(\begin{bmatrix}\varphi\\ \psi\end{bmatrix}) = \begin{bmatrix}0\\ f(\varphi)\end{bmatrix},$$

where

$$D(A) = X^1 = [H^2(\Omega) \cap H^1_0(\Omega)] \times H^1_0(\Omega).$$

From [7, 4] we know that A is a sectorial positive operator with compact resolvent and bounded imaginary powers. Therefore, we have (see [4, Proposition 3])

$$X^\alpha = [X^0, X^1]_\alpha = D((-\Delta_D)^{\frac{1+\alpha}{2}}) \times D((-\Delta_D)^{\frac{\alpha}{2}}),\ \alpha \in (0, 1).$$

Set $\alpha = \frac{N-2}{N+2}$. By (3.21) it follows from [5, Lemma 2] that F satisfies (3.2). Recall from [5, Theorem 5] that (3.22) ensures the existence of a C^0 semigroup $\{S(t)\}$ of global X^α solutions to (3.20), which possesses a global $(X^\alpha - X^\alpha)$ attractor. Thus Theorem 3.2 applies. Furthermore, it follows from [5] that $\{S(t)\}$ can be considered as a semigroup on X^0 and the attractor attracts bounded subsets of X^0.

Summarizing, we obtain

Corollary 3.6. *If $\rho \in (1, \frac{N+2}{N-2})$, $\alpha = \frac{N-2}{N+2}$ and* (3.21),(3.22) *hold, then for each $\beta \in (\alpha, 1)$ there exists a global $(X^0 - X^\beta)$ attractor $\mathbf{A}$ contained in an exponential $(X^0 - X^\beta)$ attractor and satisfying*

$$d_f^{X^\beta}(\mathbf{A}) \le d_f^{X^\alpha}(\mathbf{A}) \le \log_2 N^{X^\alpha}_{\frac{1}{4\kappa_\beta}}(B^{X^\beta}(0, 1)). \tag{3.23}$$

3.4. **Wave equation with damping operator $-\Delta_D$.** Following [5, 24, 25] we will consider in this subsection the problem

$$\begin{cases} u_{tt} - \Delta_D u_t - \Delta_D u = f(u),\ t > 0,\ x \in \Omega, \\ u(0, x) = u_0(x),\ u_t(0, x) = v_0(x),\ x \in \Omega, \\ u(t, x) = 0,\ t \ge 0,\ x \in \partial\Omega, \end{cases} \tag{3.24}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$ and $f \in C^2(\mathbb{R}, \mathbb{R})$ satisfies the (critical) growth condition

$$\exists_{c>0}\ |f''(s)| \le c(1 + |s|^3),\ s \in \mathbb{R}, \tag{3.25}$$

and the dissipativeness condition

$$\limsup_{|s|\to\infty} \frac{f(s)}{s} < \lambda_1, \tag{3.26}$$

with λ_1 being the first eigenvalue of the negative Dirichlet Laplacian $-\Delta_D$ in $L^2(\Omega)$.

As shown in [4] the problem (3.24) can be viewed as the Cauchy problem for an abstract parabolic equation of the form (3.3) with initial data in a product space $X^0 = H^1_0(\Omega) \times L^2(\Omega)$. Here, defining in X^0

$$A\begin{bmatrix}\varphi\\ \psi\end{bmatrix} = \begin{bmatrix}-\psi\\ -\Delta_D(\varphi+\psi)\end{bmatrix} \quad \text{for} \quad \begin{bmatrix}\varphi\\ \psi\end{bmatrix} \in D(A),$$

and

$$D(A) = X^1 = \{ \left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix} \right] \in H^1_0(\Omega) \times H^1_0(\Omega) \colon \varphi + \psi \in H^2(\Omega) \cap H^1_0(\Omega) \},$$

we remark that A with the domain X^1 is a positive sectorial operator in X^0 possessing bounded imaginary powers but the resolvent of A is non-compact.

The approach of [5], which uses a suitable decomposition of f and a nonlinear variation of constants formula (so called Alekseev's formula), ensures that a C^0 semigroup $\{S(t)\}$ is associated to (3.24) in $H^1_0(\Omega) \times L^2(\Omega)$ and $\{S(t)\}$ possesses a global attractor $\mathbf{A}$.

In [25] a significant progress was made, since for f satisfying (3.25) and (3.26) a higher regularity of the attractor was proved. It was merely mentioned in [25, Remark 3.2] that $\mathbf{A}$ has finite fractal dimension, which appeared in connection with the previous work [24], where dimension $d_f^{H^1_0(\Omega)\times L^2(\Omega)}(\mathcal{M})$ of an exponential attractor $\mathcal{M}$ containing $\mathbf{A}$ was estimated from above in a subcritical case by $1 + \log_2 N^{H^1_0(\Omega)\times L^2(\Omega)}_{\frac{1}{8\kappa}}(B^{H^2(\Omega)\cap H^1_0(\Omega)\times H^1_0(\Omega)}(0,1))$. In what follows the estimate of this type will be derived for $d_f^{H^1_0(\Omega)\times L^2(\Omega)}(\mathbf{A})$ in a critical case within the analytic semigroup approach.

We first establish the auxiliary result involving the damped wave operator A.

Lemma 3.7. *$-A$ with the domain $Z^1 := H^2(\Omega) \cap H^1_0(\Omega) \times H^2(\Omega) \cap H^1_0(\Omega)$ generates a C^0 analytic semigroup in $Z^0 := H^2(\Omega) \cap H^1_0(\Omega) \times L^2(\Omega)$, which is exponentially decaying. Furthermore, the associated fractional power spaces Z^σ, $\sigma \geq 0$, satisfy the embedding inequality*

$$\| \left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix} \right] \|_{H^2(\Omega)\cap H^1_0(\Omega)\times H^1_0(\Omega)} \leq b_\sigma \| \left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix} \right] \|_{Z^\sigma}, \left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix} \right] \in Z^\sigma, \text{ whenever } \sigma > \frac{1}{2}.$$

Proof. The first assertion comes from [31, Proposition 2.2]. As for the embedding property we observe that, for each $\left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix} \right] \in Z^1$,

$$\|\psi\|_{H^1_0(\Omega)} \leq c \|\Delta_D \psi\|^{\frac{1}{2}}_{L^2(\Omega)} \|\psi\|^{\frac{1}{2}}_{L^2(\Omega)} \leq \hat{c} \|A \left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix} \right] \|^{\frac{1}{2}}_{Z^0} \| \left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix} \right] \|^{\frac{1}{2}}_{Z^0}$$

and

$$\begin{aligned} \|\Delta_D \varphi\|_{L^2(\Omega)} &= \left(\|\Delta_D \psi\|_{L^2(\Omega)} + \|\Delta_D \varphi\|_{L^2(\Omega)} - \|\Delta_D \psi\|_{L^2(\Omega)} \right)^{\frac{1}{2}} \|\Delta \varphi\|^{\frac{1}{2}}_{L^2(\Omega)} \\ &\leq \left(\|\Delta_D \psi\|_{L^2(\Omega)} + \|\Delta_D(\varphi + \psi)\|_{L^2(\Omega)} \right)^{\frac{1}{2}} \|\Delta_D \varphi\|^{\frac{1}{2}}_{L^2(\Omega)} \leq \tilde{c} \|A \left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix} \right] \|^{\frac{1}{2}}_{Z^0} \| \left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix} \right] \|^{\frac{1}{2}}_{Z^0}. \end{aligned}$$

Hence, using [17, Exercise 1.4.11], we obtain for $\sigma > \frac{1}{2}$

$$\| \begin{bmatrix} -\Delta_D & 0 \\ 0 & (-\Delta_D)^{\frac{1}{2}} \end{bmatrix} \left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix} \right] \|_{L^2(\Omega)\times L^2(\Omega)} \leq C \| \left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix} \right] \|_{Z^\sigma}, \ \left[\begin{smallmatrix} \varphi \\ \psi \end{smallmatrix} \right] \in Z^\sigma, \tag{3.27}$$

which completes the proof. □

Theorem 3.8. *If* (3.25)-(3.26) *hold, then the global* $(H^1_0(\Omega) \times L^2(\Omega) - H^1_0(\Omega) \times L^2(\Omega))$ *attractor* $\mathbf{A}$ *for the semigroup associated to* (3.24) *in* $H^1_0(\Omega) \times L^2(\Omega)$ *is a bounded subset of* $H^2(\Omega) \cap H^1_0(\Omega) \times H^1_0(\Omega)$ *and*

$$d_f^{H^1_0(\Omega)\times L^2(\Omega)}(\mathbf{A}) \leq \log_2 N^{H^1_0(\Omega)\times L^2(\Omega)}_{\frac{1}{8\kappa}}(B^{H^2(\Omega)\cap H^1_0(\Omega)\times H^1_0(\Omega)}(0,1)),$$

where κ *is given in* (3.33).

Proof. Boundedness of the attractor in $H^2(\Omega)\cap H^1_0(\Omega)\times H^1_0(\Omega)$ comes from [25].

If $\{e^{-At}\}$ is an analytic semigroup generated by $-A$ in X^0, then $\{e^{-At}\}$ restricted to Z^0 (which will be denoted the same) coincides with the analytic semigroup generated by $-A$ in Z^0. In both settings $\{e^{-At}\}$ is exponentially decaying, so that

$$\|e^{-At}\|_{L(X^0)}\le c_0e^{-at},\quad \|e^{-At}\|_{L(Z^0)}\le c_0e^{-at},\ t\ge 0,\tag{3.28}$$

for certain $c_0\ge 1$ and $a>0$. In particular, in both settings $Re\sigma(A)>0$.

Letting $v=u_t$ note from [4, 5] that the variation of constants formula

$$\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](t,\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right])=e^{-At}\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]+\int_0^t e^{-A(t-s)}\left[\begin{smallmatrix}0\\f(u(s,u_0,v_0))\end{smallmatrix}\right]ds,\ t\ge 0,\tag{3.29}$$

holds in X^0 but also in Z^0 provided that in the latter case $\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\in Z^0$. Therefore, if $\left[\begin{smallmatrix}u_1\\v_1\end{smallmatrix}\right],\left[\begin{smallmatrix}u_2\\v_2\end{smallmatrix}\right]\in\mathbf{A}$ ($\mathbf{A}$ being invariant and bounded in Z^0) and

$$M(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]:=\int_0^t e^{-A(t-s)}\left[\begin{smallmatrix}0\\f(u(s,u_0,v_0))\end{smallmatrix}\right]ds,\ t>0,$$

then

$$\begin{aligned}&\|M(t)\left[\begin{smallmatrix}u_1\\v_1\end{smallmatrix}\right]-M(t)\left[\begin{smallmatrix}u_2\\v_2\end{smallmatrix}\right]\|_{Z^\sigma}\\&\le c_\sigma\int_0^t\frac{1}{(t-s)^\sigma}\left(\|f(u(s,u_1,v_1))-f(u(s,u_2,v_2))\|_{L^2(\Omega)}\right)ds\\&\le c_\sigma L(1-\sigma)^{-1}t^{1-\sigma}\sup_{s\in[0,t]}\left(\|u(s,u_1,v_1)-u(s,u_2,v_2)\|_{L^2(\Omega)}\right)\\&\le c_\sigma L\lambda_1^{-\frac12}(1-\sigma)^{-1}t^{1-\sigma}\sup_{s\in[0,t]}\|\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](s,\left[\begin{smallmatrix}u_1\\v_1\end{smallmatrix}\right])-\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](s,\left[\begin{smallmatrix}u_2\\v_2\end{smallmatrix}\right])\|_{X^0},\ \sigma\in(\frac12,1),\end{aligned}\tag{3.30}$$

where the embedding $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$, local Lipschitz continuity of $f\colon\mathbb{R}\to\mathbb{R}$ and Poincaré inequality $\lambda_1\|\phi\|^2_{L^2(\Omega)}\le\|(-\Delta_D)^{\frac12}\phi\|^2_{L^2(\Omega)}$ were used.[2]

Denoting next

$$P(t)\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]:=e^{-At}\left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right],\ t>0,$$

and using (3.28) we obtain

$$\|P(t)\left[\begin{smallmatrix}u_1\\v_1\end{smallmatrix}\right]-P(t)\left[\begin{smallmatrix}u_2\\v_2\end{smallmatrix}\right]\|_{X^0}\le c_0e^{-at}\|\left[\begin{smallmatrix}u_1\\v_1\end{smallmatrix}\right]-\left[\begin{smallmatrix}u_2\\v_2\end{smallmatrix}\right]\|_{X^0},\ t>0.\tag{3.31}$$

We also estimate $\sup_{s\in[0,t]}\|\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](s,\left[\begin{smallmatrix}u_1\\v_1\end{smallmatrix}\right])-\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](s,\left[\begin{smallmatrix}u_2\\v_2\end{smallmatrix}\right])\|_{X^0}$ using (3.28)-(3.29) to get

$$\begin{aligned}&\|\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](t,\left[\begin{smallmatrix}u_1\\v_1\end{smallmatrix}\right])-\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](t,\left[\begin{smallmatrix}u_2\\v_2\end{smallmatrix}\right])\|_{X^0}\le c_0\|\left[\begin{smallmatrix}u_1\\v_1\end{smallmatrix}\right]-\left[\begin{smallmatrix}u_2\\v_2\end{smallmatrix}\right]\|_{X^0}\\&\qquad+\int_0^t c_0\|f(u(s,u_1,v_1))-f(u(s,u_2,v_2))\|_{L^2(\Omega)}ds\\&\le c_0\|\left[\begin{smallmatrix}u_1\\v_1\end{smallmatrix}\right]-\left[\begin{smallmatrix}u_2\\v_2\end{smallmatrix}\right]\|_{X^0}+\int_0^t c_0L\lambda_1^{-\frac12}\|\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](s,\left[\begin{smallmatrix}u_1\\v_1\end{smallmatrix}\right])-\left[\begin{smallmatrix}u\\v\end{smallmatrix}\right](s,\left[\begin{smallmatrix}u_2\\v_2\end{smallmatrix}\right])\|_{X^0}ds,\ t\ge 0.\end{aligned}\tag{3.32}$$

[2]Due to $H^2(\Omega)\cap H^1_0(\Omega)\times H^1_0(\Omega)$ regularity of $\mathbf{A}$ the set $\{u(s,u_0,v_0)(x),\ x\in\Omega,\ s\ge 0,\ \left[\begin{smallmatrix}u_0\\v_0\end{smallmatrix}\right]\in\mathbf{A}\}$ is contained in a certain interval $[-r,r]$ and $f_{|_{[-r,r]}}\colon[-r,r]\to\mathbb{R}$ is Lipschitz continuous with a Lipschitz constant L.

Since the Gronwall's inequality ensures

$$\sup_{s\in[0,t]} \| \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (s, \left[\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right]) - \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] (s, \left[\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}\right]) \|_{X^0} \le c_0 \| \left[\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}\right] \|_{X^0} e^{c_0 L \lambda_1^{-\frac{1}{2}} t}, \ t > 0,$$

from this and (3.30) we obtain for $\sigma \in (\frac{1}{2}, 1)$

$$\|M(t) \left[\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right] - M(t) \left[\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}\right] \|_{Z^\sigma} \le c_0 c_\sigma L \lambda_1^{-\frac{1}{2}} (1-\sigma)^{-1} t^{1-\sigma} \| \left[\begin{smallmatrix} u_1 \\ v_1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} u_2 \\ v_2 \end{smallmatrix}\right] \|_{X^0} e^{c_0 L \lambda_1^{-\frac{1}{2}} t}, \ t > 0.$$

Hence, Lemma 2.1 applies with $V = H^2(\Omega) \cap H_0^1(\Omega) \times H_0^1(\Omega)$, $W = H_0^1(\Omega) \times L^2(\Omega)$ and $S = S(t^*)$, for which we choose $\delta = \nu = \frac{1}{8}$ and

$$t^* = a^{-1} \ln 8 c_0, \ \kappa = b_\sigma c_0 c_\sigma L \lambda_1^{-\frac{1}{2}} (1-\sigma)^{-1} t^{*1-\sigma} e^{c_0 L \lambda_1^{-\frac{1}{2}} t^*}, \ \sigma \in (\frac{1}{2}, 1). \tag{3.33}$$ □

3.5. **Higher order parabolic problems with fast growing nonlinearities.** Consider the well-known Cahn Hilliard equation

$$u_t = \Delta(-\Delta u + f(u)), \ t > 0, \ x \in \Omega, \tag{3.34}$$

with the initial-boundary conditions

$$\begin{aligned} & u(0,x) = u_0(x), \ x \in \Omega, \\ & \mathcal{B}_0 u = \frac{\partial u}{\partial \mathbf{n}} = 0 \text{ and } \mathcal{B}_1 u = \frac{\partial(\Delta u)}{\partial \mathbf{n}} = 0 \ \text{ on } \ \partial\Omega, \end{aligned} \tag{3.35}$$

in a bounded smooth domain $\Omega \subset \mathbb{R}^N$ where $N \le 3$; see [28] and references therein.

As in [20, 8] we assume that $f \in C^{2+Lip}(\mathbb{R}, \mathbb{R})$ satisfies

$$\int_0^z f(s)ds \ge -M, \quad -f'(z) \le \lambda \quad \text{for } \ z \in \mathbb{R} \tag{3.36}$$

with some positive constants M, λ and

$$\forall_{M' \ge 0} \exists_{\beta > 0} \forall_{|m| \le M'} \forall_{v \in \mathbb{R}} \ v f(v+m) \ge -\beta. \tag{3.37}$$

We remark that one can take a polynomial $f(v) = \sum_{k=1}^{2p-1} a_k v^k$ with a_{2p-1} positive and an arbitrary $p \in \mathbb{N}$.

In this example we consider $A = \Delta^2 + I$ in $X = L^2(\Omega)$ with the domain

$$D(A) = H^4_{2,\{\mathcal{B}_0,\mathcal{B}_1\}}(\Omega) = \{\phi \in H^4_2(\Omega) \colon \mathcal{B}_0 \phi = \mathcal{B}_1 \phi = 0\}.$$

It is known that A is a sectorial positive operator with compact resolvent and

$$X^{\frac{1}{2}} = H^2_{2,\{\mathcal{B}_0\}}(\Omega) \hookrightarrow L^\infty(\Omega) \cap H^1_4(\Omega).$$

In this setting we have that the nonlinear map

$$F(u)(x) = \Delta(f(u(x))) + u(x), \ u \in H^2_{2,\{\mathcal{B}_0\}}(\Omega), \ x \in \Omega,$$

satisfies the Lipschitz condition (3.2) with $\alpha = \frac{1}{2}$, $X^{\frac{1}{2}} = H^2_{2,\{\mathcal{B}_0\}}(\Omega)$ and $X^0 = L^2(\Omega)$.

Recall from [8, §6.4] that to the Cahn-Hilliard problem (3.34)-(3.35) corresponds a C^0 semigroup $\{S(t)\}$ of the global $X^{\frac{1}{2}}$ solutions. Define also a complete metric subspace H_γ of $H^2_{2,\{\mathcal{B}_0\}}(\Omega)$ by

$$H_\gamma = \{\phi \in H^2_{2,\{\mathcal{B}_0\}}(\Omega) \colon \left| \frac{1}{|\Omega|} \int_\Omega \phi(x) dx \right| \le \gamma\}, \ \gamma > 0.$$

As shown in [8, Proposition 6.4.3], for every $\gamma > 0$ there exists a nonvoid invariant set $\mathbf{A}_\gamma$, compact in $H^2_{2,\{\mathcal{B}_0\}}(\Omega)$, and attracting bounded subsets of H_γ with respect to the Hausdorff semidistance in $H^2_{2,\{\mathcal{B}_0\}}(\Omega)$.

Thus Corollary 3.3 applies and we conclude that

Corollary 3.9. *If* (3.36)-(3.37) *hold, then for any* $\beta \in (\frac{1}{2}, 1)$ *the set* $\mathbf{A}_\gamma$ *is compact in* $H^{4\beta}_{2,\{\mathcal{B}_0,\mathcal{B}_1\}}(\Omega)$ *and*

$$d_f^{H^{4\beta}_{2,\{\mathcal{B}_0,\mathcal{B}_1\}}(\Omega)}(\mathbf{A}_\gamma) \le d_f^{H^{2}_{2,\{\mathcal{B}_0\}}(\Omega)}(\mathbf{A}_\gamma) \le \log_2 N^{H^2_{2,\{\mathcal{B}_0\}}(\Omega)}_{\frac{1}{4\kappa_{\beta,\gamma}}}(B^{H^{4\beta}_{2,\{\mathcal{B}_0,\mathcal{B}_1\}}(\Omega)}(0,1)). \tag{3.38}$$

Now let Ω be a bounded domain in $\mathbb{R}^N$, $A = \sum_{|\sigma|\le 2m} a_\sigma D^\sigma$ be a linear $2m-$th order differential operator and let $B_j = \sum_{|\sigma|\le m_j} b^j_\sigma D^\sigma$ $(j = 0, \dots, m-1)$ be boundary operators such that the triple $(A, \{B_j\}, \Omega)$ forms a *regular elliptic boundary value problem* (see e.g. [8, pp. 29-30]). Suppose, in addition, that A in $L^2(\Omega)$ with the domain $H^{2m}_{2,\{B_j\}}(\Omega)$ is selfadjoint and positive definite; e.g. $A = (-\Delta)^m$ and $B_j = \frac{\partial^j}{\partial \nu^j}$ for $j = 0, \dots, m-1$.

Following [6, 30], consider an initial-boundary value problem of the form

$$\begin{cases} u_t + Au = f(u),\ t > 0,\ x \in \Omega \subset \mathbb{R}^N,\ N > 2m > 2, \\ B_0 u = \ldots = B_{m-1} u = 0,\ t > 0,\ x \in \partial\Omega, \\ u(0,x) = u_0 \in H^m_{2,\{B_j\}}(\Omega), \end{cases} \tag{3.39}$$

where the nonlinear term satisfies (3.22) and

$$f \in C^1(\mathbb{R},\mathbb{R}), \quad \lim_{|s|\to\infty} \frac{|f'(s)|}{|s|^{\frac{4m}{N-2m}}} = 0. \tag{3.40}$$

It was shown in [6, Theorem 4.1] that under the assumptions (3.22) and (3.40) the problem (3.39) is globally well-posed in $H^m_{2,\{B_j\}}(\Omega)$ and the associated semigroup $\{S(t)\}$ has a global $(H^m_{2,\{B_j\}}(\Omega) - H^m_{2,\{B_j\}}(\Omega))$ attractor $\mathbf{A} \subset H^{2m}_{2,\{B_j\}}(\Omega)$.

We now refer to the specific condition in [6, Theorem 2.1 (ii)], which implies that there are certain $\tau_0, C', \varsigma > 0$ such that for each $u_1, u_2 \in \mathbf{A}$

$$\|S(\tau_0)u_1 - S(\tau_0)u_2\|_{H^{m+2m\varsigma}_{2,\{B_j\}}(\Omega)} \le C'\tau_0^{-\varsigma}\|u_1 - u_2\|_{H^m_{2,\{B_j\}}(\Omega)}, \tag{3.41}$$

where $\varsigma > 0$ can be chosen less but arbitrarily close to $\frac{1}{2}$ accordingly to the results of [6, Lemma 4.1 and Theorem 4.1]. The following conclusion is now straightforward.

Corollary 3.10. *If* (3.22) *and* (3.40) *hold, then there is a global* $(H^m_{2,\{B_j\}}(\Omega) - H^m_{2,\{B_j\}}(\Omega))$ *attractor for* (3.39) *satisfying the estimate*

$$d_f^{H^m_{2,\{B_j\}}(\Omega)}(\mathbf{A}) \le \log_2 N^{H^m_{2,\{B_j\}}(\Omega)}_{\frac{\tau_0^\varsigma}{4C'}}\big(B^{H^{m+2m\varsigma}_{2,\{B_j\}}(\Omega)}(0,1)\big).$$

3.6. **A problem with memory.** The conserved phase-field system with thermal memory for the *temperature variation field* ϑ and for the *order parameter* χ reads as follows

$$(\vartheta+\chi)_t = \int_0^\infty k(s)\Delta\vartheta(t-s)ds,\ t>0,\ x\in\Omega, \tag{3.42}$$

$$\chi_t = \Delta(-\Delta\chi+\alpha\chi_t+f(\chi)-\vartheta),\ t>0,\ x\in\Omega, \tag{3.43}$$

with the initial-boundary conditions

$$\begin{aligned}&\vartheta(0,x)=\vartheta_0(x),\ \chi(0,x)=\chi_0(x) \text{ and } \vartheta(-s,x)=\vartheta_1(s,x),\ s>0,\ x\in\Omega,\\ &\mathcal{B}_0\vartheta=\mathcal{B}_0\chi=0 \text{ and } \mathcal{B}_0(-\Delta\chi+\alpha\chi_t+f(\chi)-\vartheta)=0 \text{ on } \partial\Omega,\end{aligned} \tag{3.44}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $N\le 3$, and $\mathcal{B}_0$ is the operator introduced in (3.35). Here $\alpha>0$ is the *viscosity parameter* and k is the smooth, nonnegative and summable *memory kernel*, which accounts for the thermal memory. The presence of the memory requires the introduction of the given past history $\vartheta_1\colon\Omega\times(0,\infty)\to\mathbb{R}$. Throughout this section we assume that

$$f\in C^2(\mathbb{R},\mathbb{R}),\quad rf(r)\ge c_0r^4-c_1 \text{ and } |f''(r)|\le c_2(1+|r|),\ r\in\mathbb{R}, \tag{3.45}$$

for some $c_0>0$ and $c_1,c_2\ge 0$. We recall that in this case any double well potential derivative, i.e. $f(r)=r^3-cr$, $c>0$, fulfils the above assumptions. In order to prove that our problem generates a dynamical system, following the approach in [22], we need to introduce an additional variable η, usually called the *summed past history*, and defined as

$$\eta(t)(s) = -\int_0^s \Delta\left(e(t-\tau)-\chi(t-\tau)\right)d\tau,\ (t,s)\in[0,\infty)\times[0,\infty),\ x\in\Omega,$$

where $e=\vartheta+\chi$ is the *enthalpy density.* It is immediate to check that η formally satisfies the first order hyperbolic equation

$$\eta_t(t)(s) = -\eta_s(t)(s)-\Delta\left(e(t)-\chi(t)\right),\ (t,s)\in(0,\infty)\times(0,\infty),\ x\in\Omega.$$

Concerning the boundary and initial conditions to associate with the equation above, on account of (3.44), we deduce

$$\begin{aligned}&\eta(0)(s) = -\int_0^s \Delta\vartheta_1(\tau,x)d\tau,\ s\ge 0,\ x\in\Omega,\\ &\mathcal{B}_0\eta(t)(s)=0 \text{ on } \partial\Omega,\ (t,s)\in(0,\infty)\times(0,\infty).\end{aligned} \tag{3.46}$$

Moreover, we observe that, under reasonable assumptions on the past history and the memory kernel, a formal integration by parts yields

$$-\int_0^\infty k(s)\Delta\left(e(t-s)-\chi(t-s)\right)ds = \int_0^\infty \mu(s)\eta(t)(s)ds,\ t>0,\ x\in\Omega,$$

where we have set

$$\mu(s)=-k'(s),\ s>0.$$

The assumptions on $\mu\in C^1((0,\infty),\mathbb{R})\cap L^1((0,\infty),\mathbb{R})$ are the following

$$\mu(s)\ge 0,\quad k_0=\int_0^\infty \mu(s)ds>0,\quad \mu'(s)\le 0,\ s>0, \tag{3.47}$$

$$\exists_{\sigma>0}\forall_{s>0}\ \mu'(s)+\sigma\mu(s)\le 0. \tag{3.48}$$

Notice in particular that (3.48) entails the exponential decay of μ as $s\to\infty$. In order to deepen the dissipative properties of our system, following [9] (see also [23]), it is necessary to introduce the *rescaled kernel*

$$\tilde{\mu}(s)=\frac{1}{\varepsilon_0^2}\mu\left(\frac{s}{\varepsilon_0}\right),\ s>0,$$

where $\varepsilon_0>0$ is a sufficiently small number called a *relaxation time* of the system. It is important to point out that for the function $\tilde{\mu}$ assumptions (3.47)-(3.48) still hold, by replacing k_0 and σ by $\frac{k_0}{\varepsilon_0}$ and $\frac{\sigma}{\varepsilon_0}$, respectively.

We next consider the operator $B=-\Delta$ in $H=\{u\in L^2(\Omega)\colon \int_\Omega u(x)dx=0\}$ with the domain $D(B)=\{u\in H^2_{2,\{\mathcal{B}_0\}}(\Omega)\colon \int_\Omega u(x)dx=0\}$. It is known that B is a strictly positive operator, so that we can define $V=D(B^{1/2})$.

We introduce the memory space defined as the weighted Lebesgue space

$$\mathcal{M}=L^2_{\tilde{\mu}}((0,\infty),V'),$$

where V' is the adjoint space of V. Moreover, we let T be the linear operator on $\mathcal{M}$ with domain

$$D(T)=\{\eta\in\mathcal{M}\colon \eta_s\in\mathcal{M},\ \eta(0)=0\},$$

defined by

$$T\eta=-\eta_s, \tag{3.49}$$

where η_s is the distributional derivative of η.

Then we can reformulate the original initial-boundary value problem as the following integro-partial differential system in the variables $(e,\chi,\eta)=(e(t),\chi(t),\eta(t))$

$$e_t+\int_0^\infty\tilde{\mu}(s)\eta(s)ds=0,\ t>0,\ x\in\Omega, \tag{3.50}$$

$$\chi_t=\Delta\left(-\Delta\chi+\alpha\chi_t+f(\chi)-e+\chi\right),\ t>0,\ x\in\Omega, \tag{3.51}$$

$$\eta_t=T\eta-\Delta\left(e-\chi\right),\ t>0,\ x\in\Omega, \tag{3.52}$$

subjected to the boundary and initial conditions (3.44) and (3.46). The proper phase space for our problem will be then

$$\mathcal{H}=L^2(\Omega)\times H^1_2(\Omega)\times\mathcal{M}.$$

As showed in [23, Theorem 3.4], phase-field problem (3.50)-(3.52) generates a C^0 semigroup $\{S(t)\}$ of global $\mathcal{H}$ solutions.

In order to prove dissipativeness, we need to restrict the phase-space. Taking into account the mass conservation for e and χ, we introduce the complete metric space

$$\mathcal{H}_{\beta,\gamma}=\left\{(e,\chi,\eta)\in\mathcal{H}\colon \frac{1}{|\Omega|}\left|\int_\Omega e(x)dx\right|\le\beta \text{ and } \frac{1}{|\Omega|}\left|\int_\Omega \chi(x)dx\right|\le\gamma\right\},\ \beta,\gamma\ge 0.$$

Moreover, it is essential to construct a compactly embedded subspace of $\mathcal{H}_{\beta,\gamma}$. To this purpose we also define for any $\eta\in\mathcal{M}$ the *tail* of η in $\mathcal{M}$, that is the function

$$\mathcal{T}_\eta\colon [1,\infty)\ \to\ [0,\infty)$$

given by

$$\mathcal{T}_\eta(\tau) = \varepsilon_0 \int_{(0,\frac{1}{\tau})\cup(\tau,\infty)} \tilde{\mu}(s)\|\eta(s)\|^2_{V'} ds,\ \tau \geq 1.$$

Then we introduce the vector space

$$\mathcal{L} = \left\{ \eta \in L^2_{\tilde{\mu}}((0,\infty), H) \colon \eta \in D(T),\ \sup_{\tau\geq 1} \tau \mathcal{T}_\eta(\tau) < \infty \right\}.$$

As proved in [15], $\mathcal{L}$ is a Banach space endowed with the norm

$$\|\eta\|_{\mathcal{L}} = \left(\|\eta\|^2_{L^2_{\tilde{\mu}}((0,\infty),H)} + \varepsilon_0 \|T\eta\|^2_{\mathcal{M}} + \sup_{\tau\geq 1} \tau \mathcal{T}_\eta(\tau) \right)^{\frac{1}{2}},\ \eta \in \mathcal{L}.$$

Thus, on account of an immediate generalization of [26, Lemma 5.5], we have

$$\mathcal{Z} = H^1_2(\Omega) \times H^2_{2,\{\mathcal{B}_0\}}(\Omega) \times \mathcal{L} \hookrightarrow \mathcal{H} \text{ and } \mathcal{Z}_{\beta,\gamma} = \mathcal{Z} \cap \mathcal{H}_{\beta,\gamma} \hookrightarrow \mathcal{H}_{\beta,\gamma}$$

with compact embeddings. Moreover, $\mathcal{Z}_{\beta,\gamma}$ is positively invariant under $\{S(t)\}$. On account of the above results, we are now in a position to proceed in the asymptotic analysis of the problem. As showed in [23, Corollary 5.2], since ε_0 is sufficiently small, we have

Lemma 3.11. *There exists a ball $B_1 = B_1(\beta,\gamma)$ in $\mathcal{Z}_{\beta,\gamma}$, which absorbs bounded sets in $\mathcal{Z}_{\beta,\gamma}$ under $\{S(t)\}$.*

Consequently (see [23, Theorem 9.2]), we can establish the next lemma, which provides conditions of Proposition 2.7.

Lemma 3.12. *For any $\delta \in (0, \frac{1}{2})$ there exist $t_0 \geq t_{B_1}$, $\kappa > 0$ and a decomposition*

$$S(t_0) = P(t_0) + M(t_0), \tag{3.53}$$

such that

$$\|P(t_0)u_1 - P(t_0)u_2\|_{\mathcal{H}} \leq \delta \|u_1 - u_2\|_{\mathcal{H}},\ u_1, u_2 \in B_1, \tag{3.54}$$

$$\|M(t_0)u_1 - M(t_0)u_2\|_{\mathcal{Z}} \leq \kappa \|u_1 - u_2\|_{\mathcal{H}},\ u_1, u_2 \in B_1, \tag{3.55}$$

and there exist $0 < \theta \leq 1$ and $\lambda > 0$ such that

$$\|S(t_1)u_1 - S(t_2)u_2\|_{\mathcal{H}} \leq \lambda(|t_1 - t_2|^\theta + \|u_1 - u_2\|_{\mathcal{H}}) \tag{3.56}$$

for all $t_1, t_2 \in [t_0, 2t_0]$ and $u_1, u_2 \in B_1$.

Thus, Proposition 2.7 applies and we can state the following result.

Proposition 3.13. *For any $\beta, \gamma \geq 0$, $\delta \in (0, \frac{1}{2})$ and $\nu \in \left(0, \frac{1}{2} - \delta\right)$ there exist $\omega > 0$ and a nonvoid set $\widehat{\mathcal{M}} \subset B_1$ such that $\widehat{\mathcal{M}}$ is positively invariant under $\{S(t)\}$, precompact in $\mathcal{H}_{\beta,\gamma}$ with*

$$d_f^{\mathcal{H}_{\beta,\gamma}}(\widehat{\mathcal{M}}) \leq \frac{1}{\theta}\Big(1 + \log_{\frac{1}{2(\delta+\nu)}} N^{\mathcal{H}}_{\frac{\nu}{\kappa}}(B^{\mathcal{Z}}(0,1))\Big),$$

and for any bounded subset B of $\mathcal{Z}_{\beta,\gamma}$

$$\lim_{t\to\infty} e^{\omega t} d_{\mathcal{H}_{\beta,\gamma}}(S(t)B, \widehat{\mathcal{M}}) = 0.$$

Note that by [22, Theorem 4.1] there exists a bounded and closed set $B_0 \subset \mathcal{H}_{\beta,\gamma}$ absorbing bounded subsets of $\mathcal{H}_{\beta,\gamma}$ such that $B_1 \subset B_0$ and with some constants $C > 0$ and $\xi > 0$ we have

$$\|S(t)u_1 - S(t)u_2\|_{\mathcal{H}} \leq Ce^{\xi t} \|u_1 - u_2\|_{\mathcal{H}}, \ u_1, u_2 \in B_0, \ t > 0.$$

Furthermore, it follows from [23, Theorem 6.1] that there exists $\omega_1 > 0$ such that for every bounded subset B of $\mathcal{H}_{\beta,\gamma}$ we have

$$\lim_{t\to\infty} e^{\omega_1 t} d_{\mathcal{H}_{\beta,\gamma}}(S(t)B, B_1) = 0. \tag{3.57}$$

Using (3.57) and the compactness of the embedding $\mathcal{Z}_{\beta,\gamma}$ into $\mathcal{H}_{\beta,\gamma}$, we obtain the following consequence of Corollary 2.4 (i).

Corollary 3.14. *For any $\beta, \gamma \geq 0$, $\{S(t)\}$ has a global $(\mathcal{H}_{\beta,\gamma} - \mathcal{H}_{\beta,\gamma})$ attractor $\mathcal{A}$.*

By (3.57) and the transitivity of exponential attraction (see [13, Theorem 3.1]) we also get

Corollary 3.15. *If $\beta, \gamma \geq 0$, $\delta \in (0, \frac{1}{2})$ and $\nu \in \left(0, \frac{1}{2} - \delta\right)$, then $\mathcal{M} = \mathrm{cl}_{\mathcal{H}} \widehat{\mathcal{M}}$ is an exponential $(\mathcal{H}_{\beta,\gamma} - \mathcal{H}_{\beta,\gamma})$ attractor for $\{S(t)\}$ and contains the finite dimensional global $(\mathcal{H}_{\beta,\gamma} - \mathcal{H}_{\beta,\gamma})$ attractor $\mathcal{A}$.*

References

[1] J. M. Arrieta, J. W. Cholewa, T. Dlotko, A. Rodríguez-Bernal, Dissipative parabolic equations in locally uniform spaces, *Math. Nachr.*, to appear.

[2] H. Amann, Global existence for semilinear parabolic systems, *J. Reine Angew. Math.* **360** (1985), 47–83.

[3] A. V. Babin, M. I. Vishik, *Attractors of Evolution Equations*, North-Holland, Amsterdam, 1992.

[4] A. N. Carvalho, J. W. Cholewa, Local well posedness for strongly damped wave equations with critical nonlinearities, *Bulletin of the Australian Mathematical Society* **66** (2002), 443–463.

[5] A. N. Carvalho, J. W. Cholewa, Attractors for strongly damped wave equations with critical nonlinearities, *Pacific Journal of Mathematics* **207** (2002), 287–310.

[6] A. N. Carvalho, J. W. Cholewa, Continuation and asymptotics of solutions to semilinear parabolic equations with critical nonlinearities, *J. Math. Anal. Appl.* **310** (2005), 557–578.

[7] S. Chen, R. Triggiani, Proof of two conjectures on structural damping for elastic systems: The case $\alpha = \frac{1}{2}$, Lecture Notes in Mathematics **1354**, Springer, 1988, 234–256.

[8] J. W. Cholewa, T. Dlotko, *Global Attractors in Abstract Parabolic Problems*, Cambridge University Press, Cambridge, 2000.

[9] M. Conti, V. Pata, M. Squassina, Singular limit of differential systems with memory, *Indiana Univ. Math. J.* **55** (2006), 169–215.

[10] L. Dung, B. Nicolaenko, Exponential attractors in Banach spaces, *J. Dynam. Differential Equations* **13** (2001), 791–806.

[11] A. Eden, C. Foias, B. Nicolaenko, R. Temam, *Exponential Attractors for Dissipative Evolution Equations*, John Wiley & Sons, Ltd., Chichester, 1994.

[12] M. Efendiev, A. Miranville, S. Zelik, Exponential attractors for a nonlinear reaction-diffusion system in $\mathbb{R}^3$, *C. R. Acad. Sci. Paris Sr. I Math.* **330** (2000), 713–718.

[13] P. Fabrie, C. Galusinski, A. Miranville, S. Zelik, Uniform exponential attractors for a singularly perturbed damped wave equation, *Discrete Cont. Dyn. Systems* **10** (2004), 211–238.

[14] S. Gatti, M. Grasselli, A. Miranville, V. Pata, A construction of a robust family of exponential attractors, *Proc. Amer. Math. Soc.* **134** (2006), 117–127.

[15] S. Gatti, M. Grasselli, V. Pata, Exponential attractors for a phase-field model with memory and quadratic nonlinearities, *Indiana Univ. Math. J.* **53** (2004), 719–754.
[16] J. K. Hale, *Asymptotic Behavior of Dissipative Systems*, AMS, Providence, R.I., 1988.
[17] D. Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, Berlin, 1981.
[18] O. A. Ladyženskaya, *Attractors for Semigroups and Evolution Equations*, Cambridge University Press, Cambridge, 1991.
[19] O. A. Ladyženskaya, V. A. Solonnikov, N. N. Ural'ceva, *Linear and Quasilinear Equations of Parabolic Type*, Translations of Mathematical Monographs, AMS, Providence, R.I., 1967.
[20] De-Sheng Li, Chen-Kui Zhong, Global attractor for the Cahn-Hilliard system with fast growing nonlinearity, *J. Diff. Equations* **149** (1998), 191–210.
[21] J. Málek, D. Pražák, Large time behavior via the method of ℓ-trajectories, *J. Diff. Equations* **181** (2002), 243–279.
[22] G. Mola, Global attractors for a three-dimensional conserved phase-field system with memory, *Commun. Pure Appl. Anal.*, to appear.
[23] G. Mola, Stability of global and exponential attractors for a three-dimensional conserved phase-field system with memory, submitted.
[24] V. Pata, M. Squassina, On the strongly damped wave equation, *Comm. Math. Phys.* **253** (2005), 511–533.
[25] V. Pata, S. Zelik, Smooth attractors for strongly damped wave equations, *Nonlinearity* **19** (2006), 1495–1506.
[26] V. Pata, A. Zucchi, Attractors for a damped hyperbolic equation with linear memory, *Adv. Math. Sci. Appl.* **11** (2001), 505–529.
[27] P. Poláčik, Parabolic equations: asymptotic behavior and dynamics on invariant manifolds, in: Handbook of Dynamical Systems Vol. 2, North-Holland, Amsterdam, 2002, 835–883.
[28] R. Temam, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics,* Springer, New York, 1988.
[29] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, Veb Deutscher, Berlin, 1978.
[30] W. von Wahl, Global solutions to evolution equations of parabolic type, in: *Differential Equations in Banach Spaces, Proceedings, 1985*, Springer-Verlag, Berlin, 1986, 254–266.
[31] G. F. Webb, Existence and asymptotic behavior for a strongly damped nonlinear wave equation, *Can. J. Math.* **32** (1980), 631–643.

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