

1
2 **ON A CRITERION OF LOCAL INVERTIBILITY AND CONFORMALITY**
3 **FOR SLICE-REGULAR QUATERNIONIC FUNCTIONS**

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ABSTRACT. A new criterion for local invertibility of slice-regular quaternionic functions is obtained. This paper is motivated by the need of finding a geometrical interpretation for analytic conditions on the real Jacobian associated to a slice-regular function f . The criterion involves Spherical and Cullen derivatives of f and gives rise to several geometric implications including an application to related conformality properties.

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5 1. PRELIMINARIES AND INTRODUCTION

6 We denote by \mathbb{H} the algebra of quaternions. Let \mathbb{S} be the sphere of imaginary quaternions,
7 i.e. the set of quaternions I such that $I^2 = -1$. Let $\Omega \subseteq \mathbb{H}$ be a domain.

8 **Definition 1.1.** *We say that Ω is*

- 9
 - an axially symmetric domain if, for all $x + Iy \in \Omega$, with $I \in \mathbb{S}$, the whole sphere
 - 10 $x + \mathbb{S}y$ is contained in Ω ;
 - 11 • a slice domain if $\Omega \cap \mathbb{R}$ is non-empty and if given any $I \in \mathbb{S}$ the complex line
 - 12 $\mathbb{C}_I = \mathbb{R} + \mathbb{R}I$ intersected with Ω is a domain in \mathbb{C}_I .

13 It is possible (see [4]) to introduce a notion of regularity for functions defined in any open
14 ball $B(0, r) = \{q \in \mathbb{H} : |q| < r\}$ (and, more in general, in some axially symmetric slice
15 domains of \mathbb{H}) which extends the one of holomorphicity in the complex case.

16 **Definition 1.2.** *If Ω is an axially symmetric slice domain in \mathbb{H} , a real differentiable function*
17 *$f : \Omega \rightarrow \mathbb{H}$ is said to be slice-regular if, for every $I \in \mathbb{S}$, its restriction f_I to the complex line*
18 *$\mathbb{C}_I = \mathbb{R} + \mathbb{R}I$ passing through the origin and containing 1 and I is holomorphic on $\Omega \cap \mathbb{C}_I$.*

19 We recall that the notion of slice regularity was first introduced in [4]; the theory of slice-
20 regular functions has been significantly developed in the last decade by many authors (a
21 short list of contributions can be found in the references of [2]).

22 **Remark 1.3.** *It can be proved that a function $f : B(0, r) \rightarrow \mathbb{H}$ is slice-regular in $B(0, r) \subset \mathbb{H}$*
23 *if and only if there exists a converging power series $\sum_n q^n a_n$ in $B(0, r)$, with $a_n \in \mathbb{H}$ for any*
24 *$n \in \mathbb{N}$, such that $f(q) = \sum_n q^n a_n$ with $q \in B(0, r)$.*

25 As a direct computation on the real components of a slice-regular function, one immedi-
26 ately obtains (see [4])

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1 **Lemma 1.4.** *If f is a slice-regular function on an axially symmetric slice domain $\Omega \subset \mathbb{H}$,*
2 *then for every $I \in \mathbb{S}$ and for any $J \in \mathbb{S}$, $J \perp I$, there exist two holomorphic functions*
3 *$F_1, F_2 : \Omega \cap \mathbb{C}_I \rightarrow \mathbb{C}_I$ such that $f_I(z) = F_1(z) + F_2(z)J$ with $z = x + Iy$.*

4 For the sequel it will be important to recall a natural notion of product of polynomials
5 (then extended to power series) which turns out to provide a “regular” multiplication of
6 slice-regular functions when represented by converging regular power series.

7 **Definition 1.5.** *Let $f(q) = \sum_{n=0}^{+\infty} q^n a_n$ and $g(q) = \sum_{n=0}^{+\infty} q^n b_n$ be given power series with coeffi-*
8 *cients in \mathbb{H} whose radii of convergence are greater than r . We define the regular product of*
9 *f and g as the series $f * g(q) = \sum_{n=0}^{+\infty} q^n c_n$, where $c_n = \sum_{k=0}^n a_k b_{n-k}$ for all n , which is convergent*
10 *in $B(0, r)$.*

11 It is not difficult to see that $f * g$ is a slice-regular function defined in the open ball $B(0, r)$.
12 Furthermore, the regular product is extended for slice-regular functions defined on a general
13 axially symmetric domain Ω in the following way

$$(1.1) \quad f * g(q) = \begin{cases} 0 & \text{if } f(q) = 0 \\ f(q)g(f(q)^{-1}qf(q)) & \text{otherwise.} \end{cases}$$

14 In the spirit of Gateaux, a notion of derivative is well-defined for slice-regular functions,
15 namely (see [4])

Definition 1.6. *Let Ω be an axially symmetric slice domain in \mathbb{H} and let $f : \Omega \rightarrow \mathbb{H}$ be a*
slice-regular function. For any $I \in \mathbb{S}$ and any point $q = x + yI$ in Ω (with $x = \Re q$ and
 $y = \Im q$) we define the Cullen derivative of f at q as

$$\partial_C f(x + yI) = f'(x + yI) := \frac{1}{2} \left(\frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) f_I(x + yI)$$

16 Since in \mathbb{H} one can choose different imaginary units, it is also worth considering the
17 following

Definition 1.7. *Let Ω be an axially symmetric slice domain in \mathbb{H} and let $f : \Omega \rightarrow \mathbb{H}$ be a*
slice-regular function. We define the spherical derivative of f at q as

$$\partial_S f(q) := (q - \bar{q})^{-1} [f(q) - f(\bar{q})].$$

18 It is well known that the possibility of locally inverting a holomorphic function heavily
19 depends on the non vanishing of the derivative; it is also clear that a holomorphic function
20 which is locally invertible turns out to be conformal. The aim of the present paper is to
21 investigate a generalization of these facts for quaternionic slice-regular functions.

22 2. A LOCAL INVERTIBILITY CRITERION

23 Let Ω be an axially symmetric slice domain in \mathbb{H} and $f : \Omega \rightarrow \mathbb{H}$ be a slice-regular
24 function. If $q_0 \in \Omega$ and $q_0 \notin \mathbb{R}$, take $I \in \mathbb{S}$ so that $q_0 \in \mathbb{C}_I$ and let $J \in \mathbb{S}$ such that $I \perp J$
25 as vectors in \mathbb{R}^3 . According to this choice of local coordinates, consider the corresponding
26 splittings

$$f_I = F_1 + F_2 J \quad \text{and} \quad R_{q_0} f = R_1 + R_2 J,$$

27 where $R_{q_0} f$ is defined by

$$f(q) - f(q_0) = (q - q_0) * R_{q_0} f(q).$$

1 We also recall here that

$$R_{q_0} f(q_0) = \partial_C f(q_0) \quad \text{and} \quad R_{q_0} f(\overline{q_0}) = \partial_S f(q_0).$$

2 Furthermore, from Theorem 8.16 in [2] and using the local coordinates as above, the
3 (complex) Jacobian of f at q_0 can be written as

$$df_{q_0} = \begin{pmatrix} R_1(q_0) & -\overline{R_2(\overline{q_0})} \\ R_2(q_0) & \overline{R_1(\overline{q_0})} \end{pmatrix}$$

4 We observe first that if f is a slice-preserving function (i.e. if f maps $\mathbb{C}_I \cap \Omega$ into \mathbb{C}_I) then,
5 in local coordinates, $f = F_1$ and $R_f = R_1$, and hence

$$df_{q_0} = \begin{pmatrix} R_1(q_0) & 0 \\ 0 & R_1(q_0) \end{pmatrix}$$

6 which means that the complex Jacobian is invertible if and only if $R_1(q_0) \neq 0$ or, equivalently,
7 if and only if $\partial_C f(q_0) \neq 0$. In general, for a 2×2 quaternionic matrix, its invertibility depends
8 on the non vanishing of its Dieudonné determinant $\det_{\mathbb{H}}$ which is defined as follows

$$\det_{\mathbb{H}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} -bc & \text{if } a = 0 \\ ad - aba^{-1}c & \text{if } a \neq 0 \end{cases}$$

9 In the case of the Jacobian of f at q_0 , we observe that $R_1, R_2, \overline{R_1}$ and $\overline{R_2}$ are all self-maps
10 of $\Omega \cap \mathbb{C}_I$ and hence their products commute; in other words

$$\det_{\mathbb{H}} \begin{pmatrix} R_1(q_0) & -\overline{R_2(\overline{q_0})} \\ R_2(q_0) & \overline{R_1(\overline{q_0})} \end{pmatrix} = R_1(q_0)\overline{R_1(\overline{q_0})} + R_2(q_0)\overline{R_2(\overline{q_0})}.$$

11 Therefore, according to the previous positions, one can write

$$\partial_C f(q_0) = \begin{pmatrix} R_1(q_0) \\ R_2(q_0) \end{pmatrix} \quad \partial_S f(q_0) = \begin{pmatrix} R_1(\overline{q_0}) \\ R_2(\overline{q_0}) \end{pmatrix}$$

12 and hence, using the (standard) Hermitian product $\langle \cdot | \cdot \rangle$ in \mathbb{C}^2 , one obtains that

$$\det_{\mathbb{H}} \begin{pmatrix} R_1(q_0) & -\overline{R_2(\overline{q_0})} \\ R_2(q_0) & \overline{R_1(\overline{q_0})} \end{pmatrix} = \langle \partial_C f(q_0) | \partial_S f(q_0) \rangle.$$

13 **Remark 2.1.** *The usual quaternionic Hermitian product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ is defined as*

$$\langle q, w \rangle_{\mathbb{H}} = q \cdot \overline{w}.$$

14 *If one considers $q = a + bJ$ and $w = c + dJ$, with $a, b, c, d \in \mathbb{C}_I$ and $J \perp I$, then an
15 easy and direct computation shows that $\langle q, w \rangle_{\mathbb{H}}$ splits as the sum of a component along \mathbb{C}_I
16 (namely $a\overline{c} + b\overline{d}$) and another component in \mathbb{C}_I^\perp . The component along \mathbb{C}_I coincides with the
17 Hermitian product $\langle \cdot | \cdot \rangle$ defined above.*

18 **Remark 2.2.** *We recall that in [5] the same Hermitian product of the Cullen and Spherical
19 derivatives of a slice-regular function f appears in conditions which guarantee starlikeness
20 for the function f .*

21 We summarize our considerations by stating the following criterion of local invertibility

1 **Proposition 2.3.** *With the above–given notation,*

$$df_{q_0} \text{ is locally invertible } \iff \langle \partial_C f(q_0) | \partial_S f(q_0) \rangle \neq 0.$$

2 **Remark 2.4.** *The previous proposition can be interpreted in terms of Remark 2.1. Thus if*
 3 $q_0 \in \mathbb{C}_I$, *with the above notation, df_{q_0} is not invertible if and only if the $\langle \partial_C f(q_0) | \partial_S f(q_0) \rangle_{\mathbb{H}}$*
 4 *belongs to \mathbb{C}_I^\perp in accordance with the results in [1] which generalize the ones in [3].*

5 An immediate consequence of the previous proposition is the obvious result that f is
 6 not locally invertible if $\partial_C f$ or $\partial_S f$ vanish. The fact that $\partial_C f(q_0) = 0$ implies non-local
 7 invertibility for f is well-known and clear as in the holomorphic case.

8 On the other hand, if $q_0 = x_0 + Iy_0$ and given any $q = x_0 + Jy_0$ with $J \in \mathbb{S}$ (see [2]), it
 9 turns out that

$$f(q) = c + Jb$$

10 with the same b, c for any $q \in \mathbb{S}_{q_0} := \{x_0 + Jy_0 : J \in \mathbb{S}\}$ and $b = \partial_S f(q_0)$; then it
 11 clearly follows that if $\partial_S f(q_0) = 0$, the function f is constant on the sphere \mathbb{S}_{q_0} , and so f
 12 is not invertible. In order to provide an example of a slice–regular function whose Cullen
 13 and spherical derivatives don't vanish at q_0 but their Hermitian product does, we recall the
 14 following fact (see [2]): the Jacobian of f is not invertible at $q_0 = x_0 + Iy_0$ if and only if
 15 there exist $\tilde{q}_0 = x_0 + I_1y_0$ and a slice–regular function g such that

$$f(q) - f(q_0) = (q - q_0) * (q - \tilde{q}_0) * g(q).$$

16 The previous formula equivalently says that the Jacobian of f is not invertible at q_0 if and
 17 only if

$$R_{q_0} f(q) = (q - \tilde{q}_0) * g(q).$$

18 Assume now that (with the usual frame associated to the choice of $J \perp I$) we choose
 19 the restriction of the slice–regular function g along the slice \mathbb{C}_I to be $g_I(q) = q + q^2J$ and
 20 take $\tilde{q}_0 = x_0 + Jy_0$; thus, in this case, the restriction of the slice–regular function $R_{q_0} f(q)$
 21 along the slice \mathbb{C}_I is $(q - \tilde{q}_0) * (q + q^2J)$ and, in particular, $R_1(q) = q^2 - qx_0 + q^2y_0$ and
 22 $R_2(q) = q^3 - qy_0 - q^2x_0$. Hence

$$R_1(q_0) = q_0^2 - q_0x_0 + q_0^2y_0 = -y_0^2 - y_0^3 + x_0^2y_0 + (x_0y_0 + 2x_0y_0^2)I$$

23 and

$$R_2(q_0) = q_0^3 - q_0y_0 - q_0^2x_0 = -x_0y_0 - 2x_0y_0^2 + (-y_0^2 - y_0^3 + x_0^2y_0)I$$

24 or

$$R_2(q_0) = -IR_1(q_0).$$

25 Furthermore, as easily seen from direct computations,

$$R_1(\overline{q_0}) = -y_0^2 - y_0^3 + x_0^2y_0 - (x_0y_0 + 2x_0y_0^2)I = \overline{R_1(q_0)}$$

26 and

$$R_2(\overline{q_0}) = -x_0y_0 - 2x_0y_0^2 + (y_0^2 + y_0^3 - x_0^2y_0)I = \overline{R_2(q_0)}$$

27 so that

$$\det_{\mathbb{H}} \begin{pmatrix} R_1(q_0) & -\overline{R_2(\overline{q_0})} \\ R_2(q_0) & \overline{R_1(\overline{q_0})} \end{pmatrix} = R_1(q_0)\overline{R_1(\overline{q_0})} + R_2(q_0)\overline{R_2(\overline{q_0})} =$$

$$= \langle \partial_C f(q_0) | \partial_S f(q_0) \rangle = R_1(q_0)^2 + R_2(q_0)^2 = R_1(q_0)^2 - R_1(q_0)^2 = 0$$

28 even though neither $\partial_C f(q_0)$ nor $\partial_S f(q_0)$ vanish if q_0 is not real or an imaginary unit.

$$\det_{\mathbb{H}} \begin{pmatrix} R_1(\overline{q_0}) & R_1(q_0) \\ R_2(\overline{q_0}) & R_2(q_0) \end{pmatrix} = \mathcal{B}.$$

1 It then turns out that $\mathcal{B} = 0$ if and only if $\partial_C f(q_0)$ and $\partial_S f(q_0)$ are linearly dependent. If
 2 one assumes that $\partial_C f(q_0) \neq 0$ and $\partial_S f(q_0) \neq 0$ and $\langle \partial_C f(q_0) | \partial_S f(q_0) \rangle = 0$, then $\partial_C f(q_0)$
 3 and $\partial_S f(q_0)$ are linearly independent, so that $\mathcal{B} \neq 0$. Furthermore, we state the following

4 **Proposition 3.1.** *Let f be a slice-regular function on an axially symmetric slice domain Ω
 5 and let q_0 be in Ω , If $\partial_C f(q_0) \neq 0$ and $\partial_S f(q_0) \neq 0$, then (with the notations introduced so
 6 far) the following conditions are equivalent*

- 7 • f is not locally invertible in (a neighborhood of) q_0 ;
- 8 • the matrix associated to df_{q_0} is not invertible or singular;
- 9 • $\langle \partial_C f(q_0) | \partial_S f(q_0) \rangle = 0$
- 10 • the Hermitian matrix

$$\begin{pmatrix} |\partial_C f(q_0)|^2 & \overline{\mathcal{B}} \\ \mathcal{B} & |\partial_S f(q_0)|^2 \end{pmatrix}$$

11 *is singular or the associated Hermitian product is degenerate.*

12 *Proof.* The first two conditions are clearly equivalent and they both are equivalent to the
 13 third condition, thanks to Proposition 2.3. Thus, under the assumptions $\partial_C f(q_0) \neq 0$ and
 14 $\partial_S f(q_0) \neq 0$, the condition $0 = \langle \partial_C f(q_0) | \partial_S f(q_0) \rangle = R_1(q_0)\overline{R_1(\overline{q_0})} + R_2(q_0)\overline{R_2(\overline{q_0})}$ implies
 15 that at least one of the following identities holds

$$R_1(q_0) = \frac{-R_2(q_0)\overline{R_2(\overline{q_0})}}{\overline{R_1(\overline{q_0})}} \quad R_2(q_0) = \frac{-R_1(q_0)\overline{R_1(\overline{q_0})}}{\overline{R_2(\overline{q_0})}}.$$

16 Let us assume that the first one holds, so that, after substitution,

$$\mathcal{B} = \frac{R_2(q_0)}{\overline{R_1(\overline{q_0})}} |\partial_S f(q_0)|^2$$

17 and hence

$$|df_{q_0}(u + vJ)|^2 = 0 \iff u \frac{R_2(q_0)}{\overline{R_1(\overline{q_0})}} \overline{v} + v \frac{\overline{R_2(q_0)}}{\overline{R_1(\overline{q_0})}} \overline{u} + |u|^2 \frac{|\partial_C f(q_0)|^2}{|\partial_S f(q_0)|^2} + |v|^2 = 0.$$

18 This equation can be regarded as the equation which describes $\ker df_{q_0}$. Another way to
 19 equivalently write this equation for $\ker df_{q_0}$ is to consider

$$(3.1) \quad (u, v) \begin{pmatrix} |\partial_C f(q_0)|^2 & \overline{\mathcal{B}} \\ \mathcal{B} & |\partial_S f(q_0)|^2 \end{pmatrix} \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix} = 0$$

20 We observe that the matrix in (3.1) is Hermitian, so that it defines a Hermitian product.
 21 Therefore, there exists a pair $(u, v) \neq (0, 0)$ such that $df_{q_0}(u + vJ) = 0$ (or, equivalently,
 22 df_{q_0} is not invertible) if and only if the Hermitian product introduced in (3.1) is degenerate.
 23 Indeed, this is equivalent to saying that

$$\det_{\mathbb{H}} \begin{pmatrix} |\partial_C f(q_0)|^2 & \overline{\mathcal{B}} \\ \mathcal{B} & |\partial_S f(q_0)|^2 \end{pmatrix} = |\partial_C f(q_0)|^2 |\partial_S f(q_0)|^2 - |\mathcal{B}|^2 = 0.$$

1 Now, in general, one has

$$\begin{aligned} |\mathcal{B}|^2 &= [R_2(q_0)R_1(\bar{q}_o) - R_1(q_0)R_2(\bar{q}_o)][\overline{R_1(\bar{q}_o) R_2(q_0)} - \overline{R_2(\bar{q}_o) R_1(q_0)}] = \\ &= |R_2(q_0)|^2|R_1(\bar{q}_o)|^2 + |R_1(q_0)|^2|R_2(\bar{q}_o)|^2 - R_2(q_0)R_1(\bar{q}_o)\overline{R_2(\bar{q}_o) R_1(q_0)} + \\ &- R_1(q_0)R_2(\bar{q}_o)\overline{R_1(\bar{q}_o) R_1(q_0)} \end{aligned}$$

2 If $0 = \langle \partial_C f(q_0) | \partial_S f(q_0) \rangle = R_1(q_0)\overline{R_1(\bar{q}_o)} + R_2(q_0)\overline{R_2(\bar{q}_o)}$ then $R_1(q_0)\overline{R_1(\bar{q}_o)} = -R_2(q_0)\overline{R_2(\bar{q}_o)}$
 3 and since R_1, \bar{R}_1, R_2 and \bar{R}_2 commute, one can equivalently write

$$|\mathcal{B}|^2 = |R_2(q_0)|^2|R_1(\bar{q}_o)|^2 + |R_1(q_0)|^2|R_2(\bar{q}_o)|^2 + |R_1(q_0)|^2|R_1(\bar{q}_o)|^2 + |R_2(q_0)|^2|R_2(\bar{q}_o)|^2$$

4 so that

$$|\mathcal{B}|^2 = |\partial_C f(q_0)|^2 |\partial_S f(q_0)|^2$$

5 which implies that

$$|df_{q_0}(u + vJ)|^2 = 0$$

6 has a solution $(u, v) \neq (0, 0)$ or df_{q_0} is singular, as desired. \square

7 We conclude this paper by providing an explicit description of $\ker df_{q_0}$ under the assump-
 8 tions $\partial_C f(q_0) \neq 0$, $\partial_S f(q_0) \neq 0$ and $\langle \partial_C f(q_0) | \partial_S f(q_0) \rangle = 0$. It is known (see [2]) that in
 9 general the rank of df_{q_0} (regarded as a 4×4 real matrix) can be 0, 2 or 4. We'll show in
 10 detail that under the assumptions $\partial_C f(q_0) \neq 0$, $\partial_S f(q_0) \neq 0$ and $\langle \partial_C f(q_0) | \partial_S f(q_0) \rangle = 0$, the
 11 rank of df_{q_0} is precisely 2.

12 First of all, we'll write $u = t + sI$, $v = x + yI$ and

$$\mathcal{B} = B_1 + B_2I \quad \bar{\mathcal{B}} = B_1 - B_2I.$$

13 We recall that, under our assumptions,

$$|\mathcal{B}|^2 = B_1^2 + B_2^2 = |\partial_C f(q_0)|^2 |\partial_S f(q_0)|^2.$$

14 Hence the equation of $\ker df_{q_0}$ as in (3.1) becomes

$$(s + tI, x + yI) \begin{pmatrix} |\partial_C f(q_0)|^2 & B_1 - IB_2 \\ B_1 + B_2I & |\partial_S f(q_0)|^2 \end{pmatrix} \begin{pmatrix} s - tI \\ x - yI \end{pmatrix} = 0;$$

15 after some computations, one obtains

$$(t^2 + s^2)|\partial_C f(q_0)|^2 + (x^2 + y^2)|\partial_S f(q_0)|^2 + 2xB_1t - 2yB_2t + 2yB_1s + 2xB_2s = 0$$

16 or

$$(t^2 + s^2) \frac{(B_1^2 + B_2^2)}{|\partial_S f(q_0)|^4} + x^2 + y^2 + 2x \frac{(B_1t + B_2s)}{|\partial_S f(q_0)|^2} + 2y \frac{(B_1s - B_2t)}{|\partial_S f(q_0)|^2} = 0.$$

17 This leads to write

$$\left[x + \frac{(B_1t + B_2s)}{|\partial_S f(q_0)|^2} \right]^2 + \left[y + \frac{(B_1s - B_2t)}{|\partial_S f(q_0)|^2} \right]^2 = 0$$

18 or

$$x = -\frac{(B_1t + B_2s)}{|\partial_S f(q_0)|^2} \quad y = -\frac{(B_1s - B_2t)}{|\partial_S f(q_0)|^2}$$

1 Therefore, the set of pairs (u, v) such that $df_{q_0}(u + vJ) = 0$ is a plane in \mathbb{R}^4 , and so the rank
 2 of the real 4×4 matrix associated to df_{q_0} is 2, as expected.

3 **Remark 3.2.** From the above-given calculations it also follows that a slice-regular quater-
 4 nionic function turns out to be *conformal* at q_0 (in the real sense, as a function from $\Omega \subseteq \mathbb{R}^4$
 5 $\rightarrow \mathbb{R}^4$) if and only if $\mathcal{B} = 0$ and $|\partial_C f(q_0)|^2 = |\partial_S f(q_0)|^2 \neq 0$. This is for instance the case of
 6 a slice-regular function f whose associated (slice-regular) function $R_{q_0}f$ is real analytic (i.e.
 7 $R_{q_0}f(q) = \sum_n q^n a_n$ with $a_n \in \mathbb{R}$ for any $n \in \mathbb{N}$). The real analyticity of $R_{q_0}f$ is clearly a
 8 consequence of the real analyticity of f together with the assumption $q_0 \in \mathbb{R}$ (which implies
 9 that f is a slice-preserving and slice-regular quaternionic function) but one can consider
 10 also other functions such as the following

$$f(q) = J + (q - I) * Exp(q)$$

11 where $q_0 = I \in \mathbb{S}$, $f(I) = J \neq I$, $J \in \mathbb{S}$ and $Exp(q) = \sum_n \frac{q^n}{n!}$. The function f turns out
 12 to be not slice-preserving but conformal at $q_0 = I$. On the other hand, if one drops the
 13 assumptions $q_0 \in \mathbb{R}$ it is not in general true that, for a slice-regular and slice-preserving
 14 function f , the (associated) slice-regular function $R_{q_0}f$ is real analytic, as the function

$$f(q) = q^2 - 2q\Re q_0 + |q_0|^2 = (q - q_0) * (q - \bar{q}_0)$$

15 clearly demonstrates.

16 From the previous remark and considerations, we conclude by stating this interesting
 17 property on the Cullen and Spherical derivatives of a slice-regular function which turns out
 18 to be also conformal.

19 **Corollary 3.3.** *Assume that $f : \Omega \subseteq \mathbb{H} \rightarrow \mathbb{H}$ is a slice-regular function. If f is conformal*
 20 *at $q_0 \in \Omega$ then there exist two unitary quaternions $U, V \in \mathbb{H}$, with $|U| = |V| = 1$, such that*

$$\partial_C f(q_0) = U \partial_S f(q_0) V.$$

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