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Superspace computations in SUSY and SUGRA theories

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Abstract

In this thesis, we present two current topics in theoretical high energy physics: We construct the Lagrangian of a deformed supergravity theory on a manifold with a non-trivial spacetime boundary by using the geometric (or rheonomic) approach and we discuss and analyze the supersymmetry invariance of the theory. Separately, we compute some scattering amplitudes in a supersymmetric conformal field theory with the Superspace formalism and Feynman superdiagrams. These two different topics conceptually achieve a contact point through the so-called AdS/CFT duality, which is actually one of the most flourishing fields in theoretical physics today. In the supergravity limit of string theory, this duality outlines a one-to-one correspondence between operators in the CFT on the boundary and the fields of the supergravity theory in the bulk. It could constitute a possible theoretical path towards the building of a theory of quantum gravity, which is the last piece needed to complete the puzzle of unified fundamental interactions. Precisely, we first give an overview of some aspects about supersymmetry, supergravity and AdS/CFT duality, in order to introduce the main two parts of the thesis: On one hand, the study of a particular supergravity theory, which will be referred to as $D = 4$ generalized AdS-Lorentz deformed supergravity theory, in the presence of a non-trivial boundary (that is when the boundary of spacetime is not thought as set at infinity); on the other hand, the computation of a 1-loop MHV reduced amplitude in $\mathcal{N} = 2$ SCQCD (in $D = 4$). In the first topic discussed in this thesis, supersymmetry is understood as a local symmetry; indeed, we are dealing with a supergravity theory. On the converse, in the second part of this dissertation, supersymmetry is a global symmetry.

Contents

Chapter 1

A small vision of Nature

"*Nature is a book written in mathematical characters*": This is one of the most famous aphorisms by Galileo Galilei (1564 – 1642). From ancient times to the present day, the study of Nature has seen several evolutions and some critical revolutions in the main concepts and in the researching methods applied to understand the laws underlying physical observable phenomena.

Most of the phenomena involving extended objects could be explained with a sufficient accuracy through classical physical theories such as newtonian mechanics and electromagnetism; when an object belongs to a length scale equal or smaller than $10^{-10}\, m$, a quantum description of its dynamics is necessary. Today, we can contemplate a large amount of scientific achievements (both sperimental and theoretical) in the field of physics of matter and fundamental interactions. From a microscopic point of view, quantum mechanics (and so its generalization quantum field theory) describes matter as made of different elementary particles, called fermions, which are organized in gradually bigger and more complicated structures (i.e. hadrons, atoms, molecules,...). These aggregations of particles are allowed by the presence of what we call fundamental interactions; the latter can be seen, in a microscopic description, as a different set of particles, called bosons. In particular, we can identify some bosons as "the messengers" of the known four forces in Nature we now list: They are the electromagnetic, the strong nuclear, the weak nuclear and the gravitational forces. Everything we can observe in Nature is the macroscopic result of the microscopic interactions between bosons and fermions, bosons and bosons, fermions and fermions. These two different types of particles could be distinguished from their spin: If we take $n \in \mathbb{N}$, a boson carries a spin value $n\hbar$, while on the other side the spin value of a fermion is $\frac{2n+1}{2}\hbar$, being $\hbar = \frac{h}{2\pi}$ where h is the Planck constant. Now we set $\hbar = c = 1$ (c is the light speed in vacuum) for convenience, in order to work in natural units.

Nowadays, this corpuscular vision of Nature finds an effective description in what is commonly called "Standard Model". This model takes its theoretical structure from a quantum gauge theory of fields in $D = 4$ spacetime dimensions with an internal symmetry group $SU(3) \times SU(2) \times U(1)$, where $SU(3)$ is the color symmetry of the strong nuclear force, $SU(2)$ is the isospin symmetry of the weak nuclear force and $U(1)$ is the charge symmetry of the electromagnetic force. Each particle is associated with a corresponding massless field; through the so-called "Higgs mechanism", in which the $SU(3) \times SU(2) \times U(1)$ symmetry breaks into $SU(3) \times U(1)$, each field interacting with the Higgs boson acquires a proper mass value. The particle content of the Standard Model is briefly summarized in the table below.

Name of particles	**Number of particles**	**Spin**
Gluon	1 (8 color states)	1
W bosons	2	1
Z boson	1	1
Photon	1	1
Higgs boson	1	0
Leptons and antileptons	$6 + \bar{6}$	$\frac{1}{2}$
Quarks and antiquarks	$6 + \bar{6}$	$\frac{1}{2}$

Most of the main predictions of the Standard Model have been confirmed through several experiments of particle collisions in different research centers such as CERN, Fermilab or many others. From a theoretical point of view, the study of particle physics gave rise to the Feynman formalism in quantum field theory that is already used to compute scattering amplitudes of particles. It is possible to derive Feynman rules from the action of the theory and the computation of scatternig amplitude becomes the perturbative computation of all the Feynman diagrams involved in the process. Loop corrections suffer from divergences and this fact undermines the predictivity of the theory. Fortunately, theories such as $SU(3) \times SU(2) \times U(1)$ can be renormalized by adding some counterterms which cancel the divergences and made the results finite. Some references about Feynman diagrams, renormalization and quantum field theory in general are [1, 2, 3, 4]. We will deepen more some topics in the following chapters.

The harmonic union of the $SU(3) \times SU(2) \times U(1)$ gauge theory and its phenomenology makes the Standard Model the theory which better describes the observable physics of all the particles found so far. Actually, this model requires 26 phenomenological parameters to be fixed in order to be predictive; furthermore, some important phenomena such as the confinement of quarks and their asymptotic freedom do not find any precise interpretation. From a theoretical point of view, it presents different formal problems like naturalness (discussed in the next section), the hierarchy problem, the meaning of the Yukawa coupling and other phenomenological aspects like the quantization of charge and so on (see [5]). As a consequence, the Standard Model is not sufficient to describe the fundamental interactions of Nature; another prove of this statement is the complete absence of gravity in the treatment.

The Einstein's General Relativity is a mathematical theory which describes the gravitational interaction of extended bodies (objects, planets, stars, galaxies,...). It includes the newtonian theory of gravity, which offers a good approximation of gravitational phenomena characterized by small velocities (i.e. much smaller than the light speed) and long distances (i.e. much bigger than $10^{-10}\ m$); in addiction to that, if we follow the full formalism, we can compute relativistic corrections (i.e. when the velocities are next to the light speed) to the newtonian results with high accuracy. General Relativity presents an elegant and complete geometric theory of spacetime, where its curvature and eventually its torsion are strictly related to gravity. The metric tensor is in a way identified with the gravitational field; given that the metric tensor has two spacetime indices, the boson associated with the gravitational force in quantum regime has spin 2 and it is called graviton. Although this theory successfully describes the dynamics of celestial bodies and consequently it represents a valuable starting point for the study of cosmology, its extension of a quantum version of gravitation drastically fails. In fact, if we try to build a quantum field theory of gravity based on General Relativity, we find out that the theory is not renormalizable: In other words, quantum corrections to scattering amplitudes of gravitons diverge quadratically and there is no way to add proper counterterms to save the predictivity of the theory. In this sense, we can look at the Einstein's theory as an effective theory of gravity, belonging to a more generic theory.

To sum up, on one hand we looked at Standard Model as an effective theory which succeeds in describing physical phenomena involving only electromagnetism and nuclear forces, but it fails in giving a complete view over quantum characteristics of Nature because it excludes gravity and the formal problems stated before undermine its theoretical solidity; on the other hand, we saw that General Relativity provides a classical description of gravity which can not be extended to a quantistic one and consequently there is no way to include it in the Standard Model.

As we said before, what we can see in Nature are the macroscopic effects of microscopic phenomena: In this sense, there is an undefined point of continuity between quantum physics and classical physics. Furthermore, the corpuscular essence of Nature suggests that there must be a unified description of physics; in other words, there must be a theory which unifies all the fundamental interactions and which offers a complete vision of Nature. The unification of fundamental interactions is one of the most interesting and challenging problems in theoretical physics; lots of different attempts have been proposed from the middle of the XX century to the present day. We consider briefly one of the most promising theories that combines both gravity and gauge theories: String Theory. Unexpectly born in 1968 from one of the attemps [6] by Veneziano to explain the behavior of hadrons, from 1974 onwards this theory was understood as a unified theory of quantum physics; we now move into a brief qualitative description of it.

String theory makes the assumption that the fundamental elements of Nature are vibrating strings; a particle like an electron is nothing but a vibration mode of a string and the same is for each particle we mentioned before. A string is a 1-dimensional object that can be closed if the two extremes coincide, or open if the two extremes are disjointed. The tension of a string only depends on the string parameter α' (otherwise Regge slope), with length dimension $[\alpha'] = L^2$. We can identify open string solutions as the solutions coming from gauge theories, while on the other hand closed string solutions describe some possible scenarios of quantum gravity. The first version of this theory is what we know as the bosonic string theory: A string is represented with a bosonic field that lives in a $(1+1)$-dimensional worldsheet and describes the dynamics of a D-dimensional spacetime (in this case $D = 26$). The bosonic string theory is not appropriate to be a "theory of everything" because it does not include fermionic states; moreover, the bosonic case is unstable because it allows tachion solutions (i.e. states with imaginary mass). It is possible to include fermions in the theory by introducing a fermionic field into the worldsheet and so implementing what we call supersymmetry: The final result is the superstring theory with $D = 10$. We will give a detailed introduction of supersymmetry in Chapter 2.

At a first sight, superstring theory seems not so interesting in the unified study of fundamental interactions: In fact, the world we experience has three spatial dimensions and one time dimension while superstring theory introduces so much extra-dimensions. Furthermore, it is improbable to get an experimental confirmation of the computations in superstring theory because we need to go next to the Planck scale $M_P = 10^{19}\ GeV$ and the highest energy reached till now is about 10^4 GeV. The first problem is solved through the concept of compactified dimensions: In other words, 10-dimensional spacetime could be considered as a spacetime with 4 extended dimensions and 6 extra-dimensions compactified on a sphere or in a torus or in any other compact manifold. Interesting relations between gauge theories and gravity can be found from the elegant mathematics of strings and branes (extended objects in which open strings are connected). One example is the set of KLT (Kawai, Lewellen, Tye) equations which derive from the mathematics of the theory and relate an amplitude of n closed strings with the square of a corresponding amplitude of n open strings (with $n \in \mathbb{N}$); broadly speaking, at each perturbative level gravity is the square of a gauge theory (some references could be found in [7, 8, 9]). We do not deal with this topic, but it represents a possible way to build a theory of quantum gravity by using the known properties of quantum field theory. Another possible road to quantum gravity is currently the most beated research direction: The Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence.

Now we can move to a more specific introduction of the main topics mentioned till now.

Chapter 2

Supersymmetry and Superspace

2.1 A possible solution of the problem of Naturalness

In the previous chapter, we gave a qualitative overview of the Standard Model, mainly focusing on the particle content and the elegant internal gauge symmetries of the corresponding field theory. We finally stated that the Standard Model is not sufficient in order to find a complete quantum vision of Nature and this fact is due to many reasons, both experimental and theoretical. Now we consider a particular problem of the Standard Model: Naturalness. The main concepts and examples presented here to introduce the problem of Naturalness are taken from [5]. We consider a $D = 4$ flat spacetime and we assume a mostly-minus $(+, -, -, -)$ Minkowskian metric tensor only in this section.

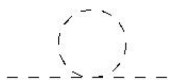
The central role of this discussion is played by a fundamental scalar field; dealing with the Standard Model, our complex scalar field is the Higgs doublet. For our discussion, it is sufficient to consider a toy model of a generic complex scalar ϕ, whose equations of motion derive from the following Lagrangian

$$\mathcal{L}_\phi = \partial^\mu \phi^\dagger \partial_\mu \phi + m^2 \phi^\dagger \phi - \lambda \left(\phi^\dagger \phi\right)^2 , \tag{2.1}$$

with m^2 and λ real parameters. It is possible to derive the Feynman rules directly from $\mathcal{L}_\phi$: In this case, we have the propagator of the field ϕ and the unique vertex of interaction is a 4-legs vertex. In order to get a predictive theory, we can renormalize the Lagrangian in eq.(2.1); in particular, we assume to know the value of the mass of ϕ from the experiments, so the parameters of $\mathcal{L}_\phi$ must be reorganized to reproduce the correct physics. We rename $m = m_0$ and we refer to it as the bare mass, while we call m_R the renormalized mass and it takes the experimental value. The relation between m_R and m_0 is the following one

$$m_R^2 = m_0^2 + \delta m^2. \tag{2.2}$$

In the case we are analyzing, δm^2 is computed with the following 1-loop diagram (called 1-loop self-energy correction)



$$\delta m^2 = \lambda \int_0^\Lambda \frac{d^4k}{(2\pi)^4} \frac{1}{k^2} = \frac{\lambda \Lambda^2}{16\pi^2},$$

where k^μ is the loop momentum ($\mu = 0, \ldots, 3$) and Λ is a cutoff parameter. Then eq.(2.2) becomes

$$m_R^2 = m_0^2 + \frac{\lambda \Lambda^2}{16\pi^2}. \tag{2.3}$$

We can rewrite eq.(2.3) as

$$\frac{m_0^2}{\Lambda^2} = \frac{m_R^2}{\Lambda^2} - \frac{\lambda}{16\pi^2} \tag{2.4}$$

and, if we accept that the Standard Model describes physics at energies smaller than the Planck scale, we can set $m_R \sim 100\ GeV$ and $\Lambda \sim M_P$ and we realize that $\frac{m_0^2}{\Lambda^2}$ must be adjusted to more than 30 orders of magnitude; this fine tuning can not be considered natural. We define Naturalness as the property that the dimensionless ratios between free parameters or physical constants appearing in a physical theory should take values of order 1 and that free parameters are not fine tuned. If we consider the case of the Standard Model, we have to consider the complete set of Feynman rules for the Higgs doublet and the computation of the 1-loop self-energy correction leads to a sum of 1-loop diagrams like the previous one, where we find scalar internal legs for one of them and fermionic internal legs for the remaining ones; actually, at the end of the computation, we find the same situation of fine tuning present before.

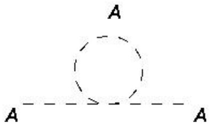
One of the possible paths to follow in order to solve the problem of Naturalness is to find a way to cancel the quadratic divergences that we find in eq.(2.3) if we set $\Lambda \to \infty$; for this purpose, we now consider another toy model, usually called the Wess-Zumino model. We have a complex scalar field ϕ and a fermion field described by a Majorana spinor ψ ($\psi^C = C\bar{\psi}^T = \psi$, see Appendix A.2 for details). We can rewrite $\phi = \frac{1}{\sqrt{2}}(A + iB)$, with A and B real scalar fields and we define $\psi_L = P_L\psi$ and $\psi_R = P_R\psi$, with $P_L = \frac{1-\gamma_5}{2}$ and $P_R = \frac{1+\gamma_5}{2}$. The Lagrangian of this model is

$$\mathcal{L}_{WZ} = \partial^\mu\phi^*\partial_\mu\phi + \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi - \left|\frac{dW}{d\phi}\right|^2 - \frac{1}{2}\left(\frac{d^2W}{d\phi^2}\bar{\psi}_R\psi_L + \frac{d^2W^*}{d\phi^{*2}}\bar{\psi}_L\psi_R\right), \quad (2.5)$$

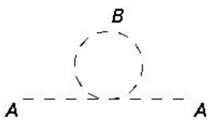
where $\partial\!\!\!/ = \gamma^\mu\partial_\mu$ and $W(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{3}\lambda\phi^3$, with m and λ real parameters. More explicitly, we have

$$\begin{aligned}\mathcal{L}_{WZ} =& \frac{1}{2}\partial^\mu A\partial_\mu A + \frac{1}{2}\partial^\mu B\partial_\mu B - \frac{1}{2}m^2\left(A^2 + B^2\right) + \frac{i}{2}\bar{\psi}\partial\!\!\!/\psi - \frac{1}{2}m\bar{\psi}\psi \\ & - \frac{m\lambda}{\sqrt{2}}A\left(A^2+B^2\right) - \frac{\lambda^2}{4}\left(A^2+B^2\right)^2 - \frac{\lambda}{\sqrt{2}}\bar{\psi}\left(A - iB\gamma_5\right)\psi.\end{aligned} \quad (2.6)$$

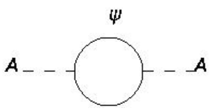
If we want to know the 1-loop self-energy correction for the real field A, we have to compute the following diagrams



$$(a) = -i\frac{\lambda^2}{4}4\cdot 3\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2},$$



$$(b) = -i\frac{\lambda^2}{2}2\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2},$$



$$(c) = -\left(-\frac{i\lambda}{\sqrt{2}}\right)^2 2\int\frac{d^4k}{(2\pi)^4}\,\mathrm{tr}\left(\frac{i}{k\!\!\!/ - m}\frac{i}{k\!\!\!/ - p\!\!\!/ - m}\right),$$

where, in order to avoid confusion, we wrote the corresponding fields next to their propagators in the diagrams. These three contributes can be rewritten as

$$\begin{aligned}(a) &= 3\lambda^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}, \\ (b) &= \lambda^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2}, \\ (c) &= -\lambda^2\int\frac{d^4k}{(2\pi)^4}\frac{\mathrm{tr}\left((k\!\!\!/ + m)(k\!\!\!/ - p\!\!\!/ + m)\right)}{(k^2-m^2)((k-p)^2-m^2)},\end{aligned} \quad (2.7)$$

where k^μ is the loop momentum and p^μ is the momentum of the external leg. With the identities $\mathrm{tr}\left((\not{k}+m)(\not{k}-\not{p}+m)\right) = 4\left(k\cdot(k-p)+m^2\right)$ and the fact that $4\left(k\cdot(k-p)+m^2\right) = 2\left((k^2-m^2)+(k-p)^2-m^2-p^2+4m^2\right)$, the 1-loop self-energy correction for the field A takes the form

$$
\begin{aligned}
(a)+(b)+(c) = 2\lambda^2\Bigg(&\int \frac{d^4k}{(2\pi)^4}\frac{1}{k^2-m^2} - \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k-p)^2-m^2} \\
&+ \int \frac{d^4k}{(2\pi)^4}\frac{p^2-4m^2}{(k^2-m^2)((k-p)^2-m^2)}\Bigg),
\end{aligned}
\tag{2.8}
$$

so, the quadratic divergences still present in the first two terms cancel. However, this fact does not imply that the model considered is finite: The cancellation proved before involves only the quadratic divergences, whereas logarithmic divergences in the scale of higher energy physics (referred to the UV cutoff Λ) are still present. In this case, the problem of logarithmic divergences could be solved through the standard process of renormalization in Quantum Field Theory ([2] and references therein).

In this toy model, we see the presence of a scalar field and a fermion field, as we can find also in the Standard Model; actually, in the Standard Model there is a spinor field for each fermion, but at this level of discussion we can conceptually simplify by thinking of a single spinor field with a flavor index running on all the fermions of the theory. In this point of view, the two models are similar, but only one of them suffers from the problem of Naturalness: In the Wess-Zumino model, the cancellation of the quadratic divergences in eq.(2.8) is due to the fact that in the Lagrangian of eq.(2.6) the field A (or in general ϕ) and the field ψ are associated with the same mass m. Since the two fields have the same mass, it is allowed to think about them as two component fields of a larger structure called chiral supermultiplet; the reason of this name is the invariance under supersymmetric transformations, which mix bosonic and fermionic degrees of freedom. As we can see from a supersymmetric theory such as the Wess-Zumino model, supersymmetry is an elegant solution to the problem of Naturalness; in fact, because of the presence of more Feynman diagrams with fermionic internal legs carrying different signs, it allows the cancellation of all the quadratic divergences. It is possible to prove this cancellation also in more complicated supersymmetric theories.

2.2 Supersymmetry

We choose the convention of a mostly-plus $(-,+,+,+)$ Minkowskian metric tensor in a $D=4$ flat spacetime. We can give a more precise introduction of supersymmetry. First of all, supersymmetry is a spacetime symmetry between bosonic and fermionic fields: Given Q a generator of supersymmetry, a generic physical state $|m,s\rangle$ of mass m and spin s is transformed by Q in the following way

$$Q\,|m,s\rangle = |m,|s\pm 1/2|\rangle\,. \tag{2.9}$$

A boson is transformed in a fermion and a fermion is transformed in a boson; as a consequence, Q must be a fermionic generator and it must carry a spinorial index: This is the reason why supersymmetry is a spacetime symmetry. A generic supersymmetric quantum field theory is associated with a number $\mathcal{N}\in\mathbb{N}$ which specifies the number of generators of supersymmetry; the Wess-Zumino model is an example of a $\mathcal{N}=1$ supersymmetric theory. Obviously, the case $\mathcal{N}=0$ is the non-supersymmetric one and it is not considered.

We define Q_α and $\bar{Q}_{\dot\alpha}$ (with $\alpha=+,-$ and $\dot\alpha=\dot{+},\dot{-}$) as Weyl spinors in order that the structure $\left(\begin{smallmatrix} Q_\alpha \\ \bar{Q}_{\dot\alpha}\end{smallmatrix}\right)$ is a Majorana spinor; Q_α and $\bar{Q}_{\dot\alpha}$ are commonly called supercharges. In order to avoid confusions, in a theory with $\mathcal{N}=1$ there is one generator of supersymmetry and it is represented by a Majorana spinor of four supercharges, which are usually arranged into two Weyl spinors. An example of supersymmetric algebra is the $\mathcal{N}=1$ super-Poincaré algebra in $D=4$

$$\begin{aligned}
&[J_{\mu\nu},J_{\rho\lambda}] = \eta_{\mu\lambda}J_{\nu\rho}+\eta_{\nu\rho}J_{\mu\lambda}-\eta_{\mu\rho}J_{\nu\lambda}-\eta_{\nu\lambda}J_{\mu\rho},\\
&[J_{\mu\nu},P_\rho] = \eta_{\nu\rho}P_\mu-\eta_{\mu\rho}P_\nu,\\
&[P_\mu,P_\nu]=0,\\
&[P_\mu,Q_\alpha]=0, \qquad \left[P_\mu,\bar{Q}_{\dot\alpha}\right]=0,\\
&[J_{\mu\nu},Q_\alpha]=\frac{1}{2}(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta, \qquad \left[J_{\mu\nu},\bar{Q}_{\dot\alpha}\right]=\frac{1}{2}(\bar\sigma_{\mu\nu})_{\dot\alpha}{}^{\dot\beta}\bar{Q}_{\dot\beta},\\
&\{Q_\alpha,\bar{Q}_{\dot\alpha}\}=(\sigma^\mu)_{\alpha\dot\alpha}P_\mu,\\
&\{Q_\alpha,Q_\beta\}=0, \qquad \{\bar{Q}_{\dot\alpha},\bar{Q}_{\dot\beta}\}=0,
\end{aligned} \tag{2.10}$$

where we have $\{J_{\mu\nu},P_\mu\}$ the generators of the Poincaré group and the supercharges Q_α and $\bar{Q}_{\dot\alpha}$ ($\sigma_{\mu\nu}=\frac{1}{2}\left[\sigma_\mu,\sigma_\nu\right]$ and $\bar\sigma_{\mu\nu}$ has the same definition, with only a change of sign for the Pauli matrices). For a generic $\mathcal{N}\geq 1$ supersymmetry in $D=4$, we have the supercharges $Q^i{}_\alpha$ and $\bar{Q}^i{}_{\dot\alpha}$ with $i=1,\ldots,\mathcal{N}$ and the previous relations are modified only with some Kronecker deltas and with the introduction of central charges in some relations.

In the super-Poincaré algebra of eq.(2.10), we read the first three commuting relations which describe the Poincaré algebra; the remaining commuting relations and the anticommuting relations define the enlargement of the symmetry of spacetime. The anticommuting relations between supercharges are fundamental to build a supersymmetry because they allow to deceive the Coleman-Mandula theorem (also called "no-go theorem"). This theorem states that the only conserved charges that transform as tensors under the Lorentz group are the generators of translations and the generators of the Lorentz transformations. As a consequence, any other conserved charge is necessarily a Lorentz scalar and its commuting relations with the generators of Lorentz and of translations are trivial: This implies that it is impossible to enlarge the symmetry of spacetime. Actually, supersymmetry succeeds in doing that because of the previous anticommuting relations.

With the introduction of supercharges in the algebra, another symmetry comes out; it is easy to verify that the algebra of eq.(2.10) is invariant under the following transformation

$$\begin{cases} Q_\alpha \to e^{i\theta} Q_\alpha \\ \bar{Q}_{\dot{\alpha}} \to e^{-i\theta} \bar{Q}_{\dot{\alpha}} \end{cases}$$

with $\theta \in \mathbb{R}$ a constant arbitrary parameter. As a consequence, we can include an extra generator R so that

$$[R, Q_\alpha] = Q_\alpha, \qquad \left[R, \bar{Q}_{\dot{\alpha}}\right] = -\bar{Q}_{\dot{\alpha}}, \qquad [R, J_{\mu\nu}] = 0, \qquad [R, P_\mu] = 0, \tag{2.11}$$

and these commuting relations introduce the R-symmetry; in that case, the R-symmetry of the $\mathcal{N} = 1$ supersymmetry is $U(1)$. In general, R-symmetry is a symmetry of supercharges; we can see it as the group of transformations which rotate the supercharges into each other.

We enlarged the symmetry of a $D = 4$ flat spacetime with the introduction of supersymmetry; now we have to organize particles into representations of the supersymmetric algebra, also called supermultiplets. We use the formalism of bispinorial indices [10] that is summarized in Appendix A (see in particular from eq.(A.20) to eq.(A.30)). For simplicity, we consider a toy model of a scalar ϕ and a spinor ψ^α; their physics is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial^{\alpha\dot{\alpha}}\bar{\phi}\,\partial_{\alpha\dot{\alpha}}\phi - \frac{i}{2}\bar{\psi}^{\dot{\alpha}}\partial^{\alpha}{}_{\dot{\alpha}}\psi_\alpha. \tag{2.12}$$

This Lagrangian is invariant under the $\mathcal{N}=1$ supersymmetric transformation

$$\begin{cases} \delta\phi = \frac{1}{\sqrt{2}}\,\epsilon^{\alpha}\psi_{\alpha} \\ \delta\bar{\phi} = \frac{1}{\sqrt{2}}\,\bar{\epsilon}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}} \\ \delta\psi_{\alpha} = \sqrt{2}\,i\partial_{\alpha}{}^{\dot{\alpha}}\phi\,\bar{\epsilon}_{\dot{\alpha}} \\ \delta\bar{\psi}_{\dot{\alpha}} = \sqrt{2}\,i\partial^{\alpha}{}_{\dot{\alpha}}\bar{\phi}\,\epsilon_{\alpha} \end{cases}, \tag{2.13}$$

(i.e. $\delta\mathcal{L}=0$), with ϵ^{α} generic spinorial parameter (and $\bar{\epsilon}^{\dot{\alpha}}$ its conjugate).
We want to know whether the set (ϕ,ψ^{α}) realizes a representation of the supersymmetric algebra. For that purpose, we consider two different supersymmetric transformations like eq.(2.13) with corresponding spinorial parameters ${\epsilon_1}^{\alpha}$ and ${\epsilon_2}^{\alpha}$ and we compute the commutator of these two transformations applied to ϕ and to ψ^{α}; we expect to find a translation of the two fields. Actually, we find that

$$\begin{aligned} [\delta_1,\delta_2]\,\phi &= \left({\epsilon_1}^{\alpha}{\bar{\epsilon}_2}^{\dot{\alpha}} - {\epsilon_2}^{\alpha}{\bar{\epsilon}_1}^{\dot{\alpha}}\right) i\partial_{\alpha\dot{\alpha}}\phi, \\ [\delta_1,\delta_2]\,\psi_{\alpha} &= \left({\epsilon_1}^{\beta}{\bar{\epsilon}_2}^{\dot{\alpha}} - {\epsilon_2}^{\beta}{\bar{\epsilon}_1}^{\dot{\alpha}}\right) i\left(\partial_{\beta\dot{\alpha}}\psi_{\alpha} + \mathcal{C}_{\alpha\beta}\partial^{\gamma}{}_{\dot{\alpha}}\psi_{\gamma}\right), \end{aligned} \tag{2.14}$$

where we use the identity $\partial_{\alpha\dot{\alpha}}\psi_{\beta} = \partial_{\beta\dot{\alpha}}\psi_{\alpha} + \mathcal{C}_{\alpha\beta}\partial^{\gamma}{}_{\dot{\alpha}}\psi_{\gamma}$ and $\mathcal{C}_{\alpha\beta}$ is defined by eq.(A.21). If we rewrite $i\partial_{\alpha\dot{\alpha}} = p_{\alpha\dot{\alpha}}$, we find what we expect

$$\begin{aligned} [\delta_1,\delta_2]\,\phi &= \left({\epsilon_1}^{\alpha}{\bar{\epsilon}_2}^{\dot{\alpha}} - {\epsilon_2}^{\alpha}{\bar{\epsilon}_1}^{\dot{\alpha}}\right) p_{\alpha\dot{\alpha}}\phi, \\ [\delta_1,\delta_2]\,\psi_{\alpha} &= \left({\epsilon_1}^{\beta}{\bar{\epsilon}_2}^{\dot{\alpha}} - {\epsilon_2}^{\beta}{\bar{\epsilon}_1}^{\dot{\alpha}}\right) p_{\beta\dot{\alpha}}\psi_{\alpha}, \end{aligned} \tag{2.15}$$

only if we impose $i\partial^{\gamma}{}_{\dot{\alpha}}\psi_{\gamma}=0$, which are the equations of motion for ψ^{α}. As a consequence, in our toy model described by eq.(2.12), the supersymmetric algebra closes on (ϕ,ψ^{α}) only on-shell (i.e. with the constraints of the equations of motion). If we include an extra scalar field F into the Lagrangian that becomes

$$\mathcal{L} = \frac{1}{2}\partial^{\alpha\dot{\alpha}}\bar{\phi}\,\partial_{\alpha\dot{\alpha}}\phi - \frac{i}{2}\bar{\psi}^{\dot{\alpha}}\partial^{\alpha}{}_{\dot{\alpha}}\psi_{\alpha} - \frac{1}{4}\bar{F}F, \tag{2.16}$$

which is invariant under the following supersymmetric transformation

$$\begin{cases} \delta\phi = \frac{1}{\sqrt{2}}\,\epsilon^{\alpha}\psi_{\alpha} \\ \delta\bar{\phi} = \frac{1}{\sqrt{2}}\,\bar{\epsilon}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}} \\ \delta\psi_{\alpha} = \sqrt{2}\,i\partial_{\alpha}{}^{\dot{\alpha}}\phi\,\bar{\epsilon}_{\dot{\alpha}} + \frac{1}{\sqrt{2}}\,\epsilon_{\alpha}F \\ \delta\bar{\psi}_{\dot{\alpha}} = \sqrt{2}\,i\partial^{\alpha}{}_{\dot{\alpha}}\bar{\phi}\,\epsilon_{\alpha} + \frac{1}{\sqrt{2}}\,\bar{\epsilon}_{\dot{\alpha}}\bar{F} \\ \delta F = -\sqrt{2}\,i\partial^{\alpha\dot{\alpha}}\psi_{\alpha}\,\bar{\epsilon}_{\dot{\alpha}} \\ \delta\bar{F} = -\sqrt{2}\,\epsilon^{\alpha}\,i\partial_{\alpha}{}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}} \end{cases}, \tag{2.17}$$

in place of eq.(2.15) we find

$$
\begin{aligned}
[\delta_1, \delta_2]\, \phi &= \left(\epsilon_1{}^{\alpha} \bar{\epsilon}_2^{\dot{\alpha}} - \epsilon_2{}^{\alpha} \bar{\epsilon}_1^{\dot{\alpha}}\right) p_{\alpha\dot{\alpha}} \phi, \\
[\delta_1, \delta_2]\, \psi_\alpha &= \left(\epsilon_1{}^{\beta} \bar{\epsilon}_2^{\dot{\alpha}} - \epsilon_2{}^{\beta} \bar{\epsilon}_1^{\dot{\alpha}}\right) p_{\beta\dot{\alpha}} \psi_\alpha, \\
[\delta_1, \delta_2]\, F &= \left(\epsilon_1{}^{\alpha} \bar{\epsilon}_2^{\dot{\alpha}} - \epsilon_2{}^{\alpha} \bar{\epsilon}_1^{\dot{\alpha}}\right) p_{\alpha\dot{\alpha}} F,
\end{aligned}
\tag{2.18}
$$

where it is not necessary to impose any constraint. The field F satisfies the equation of motion $F = 0$, so it is not a physical field; it is called auxiliary field. In conclusion, in our toy model, with the introduction of an auxiliary field, we find (ϕ, ψ^α, F) as an off-shell representation of the supersymmetric algebra. Actually, for a generic $\mathcal{N} \geq 1$ supersymmetric theory, the introduction of auxiliary fields brings the advantage of having an off-shell supersymmetric theory and, as we will see soon, it allows to perform calculations with the help of Superspace techniques.

It is important to specify which values of $\mathcal{N}$ are allowed for a generic supersymmetric theory. According to [11], the largest number of supercharges for a free field theory without gravity is 16: With more supercharges, the free supermultiplet includes fields whose spin is larger than 1 and there is no consistent theory without gravity. As we will specify later, when supersymmetry is local we move to supergravity and the maximum spin is 2: For a consistent supergravity theory, the largest number of real supercharges is 32. Talking about supercharges, we are dealing with the components of the generators of supersymmetry, which are Majorana spinors or, in other words, irreducible representations of the Lorentz group. Given a generic D-dimensional spacetime, if we call $d_L(D)$ the dimension of a spinor (i.e. the number of real components of a spinor) in D dimensions, the maximum numbers of generators of supersymmetry allowed for a theory without gravity (subscript "SUSY") and for a theory with gravity (subscript "SUGRA") are respectively

$$
\mathcal{N}^{max}_{SUSY} = \frac{16}{d_L(D)}, \qquad \mathcal{N}^{max}_{SUGRA} = \frac{32}{d_L(D)}. \tag{2.19}
$$

A quantum field theory with $\mathcal{N} = \mathcal{N}^{max}$ is maximally supersymmetric; some known examples are summarized in the following table.

	$D=3$	$D=4$	$D=10$
$\mathcal{N}^{max}_{SUSY}$	8	4	1
$\mathcal{N}^{max}_{SUGRA}$	16	8	2

Now we move to a generic construction of supersymmetric representations of $\mathcal{N} \geq 1$. Because of the structure of the anticommuting relations between supercharges, $\forall i = 1, \ldots, \mathcal{N}$ it is easy to prove that $Q_i^3 = \bar{Q}_i^3 = 0$: So, the supermultiplet we build up by applying supercharges to a starting state is a finite tower of states with the same mass (as we can read from eq.(2.9)). A generic particle can be massive or massless, so we consider separately these two different cases.

Massive representation of supersymmetry. In the massive representation of $\mathcal{N} \geq 1$ supersymmetry in $D = 4$, the momentum of a particle of mass m is $p_\mu = (m, 0, 0, 0)$ (with $\mu = 0, \ldots, 3$) in order to follow the mass-shell relation $p^2 = -m^2$; as a consequence, the nonvanishing anticommuting relation is

$$\{Q^i_\alpha, \bar{Q}_{j\dot{\alpha}}\} = m\, \delta^i{}_j \delta_{\alpha\dot{\alpha}}. \tag{2.20}$$

We define $|s\rangle$ a vacuum state of spin s by the condition

$$Q^i_\alpha |s\rangle = 0 \quad \forall i = 1, \ldots, \mathcal{N}, \tag{2.21}$$

and we construct the tower of states by repeatedly applying $\bar{Q}_{i\dot{\alpha}}$ to $|s\rangle$; we can choose the convention that $\bar{Q}_{i\dot{+}} |s\rangle \to |s + 1/2\rangle_i$ and $\bar{Q}_{i\dot{-}} |s\rangle \to |s - 1/2\rangle_i$ (the contrary choice leads to the same physics). The massive supermultiplet is represented in the following table.

Massive states	**Number of states**
$\vert s\rangle$	1
$\bar{Q}_{i\dot{\alpha}} \vert s\rangle$	$2\mathcal{N}$
$\bar{Q}_{i\dot{\alpha}}\bar{Q}_{j\dot{\beta}} \vert s\rangle$	$\binom{2\mathcal{N}}{2}$
$\vdots$	$\vdots$
$\left(\prod_{k=1}^{\mathcal{N}} \bar{Q}_{k\dot{\alpha}_k}\right) \vert s\rangle$	$\binom{2\mathcal{N}}{2\mathcal{N}}$

The total number of states in this supermultiplet is $\sum_{k=0}^{2\mathcal{N}} \binom{2\mathcal{N}}{k} = 2^{2\mathcal{N}}$. If we apply a supercharge Q^i_α to the last state of the tower, we find a combination of the previous states: The supermultiplet in the table is a complete set of indipendent states.

Massless representation of supersymmetry. In the massless representation of $\mathcal{N} \geq 1$ supersymmetry in $D = 4$, the momentum of a particle of energy E is $p_\mu = (E, 0, 0, E)$ (with $\mu = 0, \ldots, 3$) in order to follow the mass-shell relation $p^2 = 0$; as a consequence, the nonvanishing anticommuting relation is

$$\{Q^i_\alpha, \bar{Q}_{j\dot{\alpha}}\} = E\ \delta^i{}_j \left(\mathbb{I} + \sigma_3\right)_{\alpha\dot{\alpha}} \quad \to \quad \{Q^i_+, \bar{Q}_{j\dot{+}}\} = 2E\ \delta^i{}_j. \tag{2.22}$$

Massless particles are classified in terms of helicity, which is the projection of the spin onto the direction of momentum. We define $|\lambda\rangle$ a vacuum state of helicity λ by the conditions

$$\begin{aligned} Q^i_- \left|\lambda\right\rangle = 0 \quad \forall i = 1, \ldots, \mathcal{N}, \\ \bar{Q}_{i\dot{-}} \left|\lambda\right\rangle = 0 \quad \forall i = 1, \ldots, \mathcal{N}, \\ Q^i_+ \left|\lambda\right\rangle = 0 \quad \forall i = 1, \ldots, \mathcal{N}, \end{aligned} \tag{2.23}$$

and we construct the tower of states by repeatedly applying $\bar{Q}_{i\dot{+}}$ to $|\lambda\rangle$. The massless supermultiplet is represented in the following table.

Massless states	**Number of states**
$\lvert\lambda\rangle$	1
$\bar{Q}_{i\dot{+}} \lvert\lambda\rangle$	$\mathcal{N}$
$\bar{Q}_{i\dot{+}}\bar{Q}_{j\dot{+}} \lvert\lambda\rangle$	$\binom{\mathcal{N}}{2}$
$\vdots$	$\vdots$
$\left(\prod_{k=1}^{\mathcal{N}} \bar{Q}_{k\dot{+}}\right) \lvert\lambda\rangle$	$\binom{\mathcal{N}}{\mathcal{N}}$

The total number of states in this supermultiplet is $\sum_{k=0}^{\mathcal{N}} \binom{\mathcal{N}}{k} = 2^{\mathcal{N}}$. Also in this case, if we apply a supercharge Q^i_+ to the last state of the tower, we find a combination of the previous states: The supermultiplet in the table is a complete set of indipendent states.

2.3 $\mathcal{N} = 1$ Superspace

In the previous section, we showed the basis of supersymmetry and its representations for $\mathcal{N} = 1, \ldots, \mathcal{N}^{max}$.

In this section, we consider a $D = 4$ flat spacetime with $\mathcal{N} = 1$ supersymmetry and we construct what is commonly called "Superspace"; it completes the introduction of supersymmetry and it provides the technical tools we need to perform the computations in supersymmetric gauge theories. Superspace is also fundamental for our study of a supergravity theory, but in that case further explanations about the formalism of k-forms are needed and are discussed in the next chapter. We will follow [10] for a review of Superspace techniques.

For simplicity, we denote $\mathcal{G}_{SP}$ the $\mathcal{N} = 1$ super-Poincaré group and $\mathcal{G}_L$ the Lorentz group. The coset $\mathcal{G}_{SP}/\mathcal{G}_L$ is a set of equivalence classes of super-Poincaré elements with equivalence rule defined as

$$\forall g, g' \in \mathcal{G}_{SP} \qquad g' \simeq g \iff \exists h \in \mathcal{G}_L : g' = g \cdot h. \tag{2.24}$$

The group $\mathcal{G}_{SP}$ is generated by the set $\{P_\mu, J_{\mu\nu}, Q_\alpha, \bar{Q}_{\dot{\alpha}}\}$ and the group $\mathcal{G}_L$ is generated by the set $\{J_{\mu\nu}\}$; as a consequence, $\{P_\mu, Q_\alpha, \bar{Q}_{\dot{\alpha}}\}$ are the generators of the group $\mathcal{G}_{SP}/\mathcal{G}_L$. We assume again the bispinorial indices convention in Appendix A and we choose a coset representative

$$L\left(x, \theta, \bar{\theta}\right) = e^{i\left(x^{\alpha\dot{\alpha}} P_{\alpha\dot{\alpha}} + \theta^\beta Q_\beta + \bar{\theta}^{\dot{\beta}} \bar{Q}_{\dot{\beta}}\right)}, \tag{2.25}$$

where θ^α and $\bar{\theta}^{\dot{\alpha}}$ are anticommuting constant spinorial parameters and $x^{\alpha\dot{\alpha}}$ are spacetime coordinates. With the help of the anticommuting relation of supercharges, we can fix the energy dimensions of these quantities:

$$[P] = E^1, \quad [Q] = \left[\bar{Q}\right] = E^{\frac{1}{2}}, \quad [x] = E^{-1}, \quad [\theta] = \left[\bar{\theta}\right] = E^{-\frac{1}{2}}. \tag{2.26}$$

The set $\left(x^{\alpha\dot{\alpha}}, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}\right)$ is a set of coordinates of the $\mathcal{N} = 1$ Superspace, defined as the coset of super-Poincaré and Lorentz. We can compute the multiplication of two representatives like eq.(2.25) by using the Baker-Campbell-Hausdorff formula

$$\forall A, B \qquad e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}; \tag{2.27}$$

in our case, given that $[A,[A,B]] = [B,[A,B]] = 0$, it reduces into $e^{A+B+\frac{1}{2}[A,B]}$. It is easy to verify that

$$L\left(x, \theta, \bar{\theta}\right) \cdot L\left(\xi, \epsilon, \bar{\epsilon}\right) = L\left(x + \xi - \frac{i}{2}\left(\epsilon\bar{\theta} + \bar{\epsilon}\theta\right), \theta + \epsilon, \bar{\theta} + \bar{\epsilon}\right), \tag{2.28}$$

which is the explicit definition of a super-translation in $\mathcal{N}=1$ Superspace. A generic $\Phi\left(x,\theta,\bar{\theta}\right)$, which is a smooth function of the Superspace coordinates, is called "superfield". Given a generic superfield Φ, if we make a super-translation with parameters $\left(\xi^{\alpha\dot{\alpha}},\epsilon^{\alpha},\bar{\epsilon}^{\dot{\alpha}}\right)$, from $\delta\Phi=\Phi\left(x+\xi-\frac{i}{2}\left(\epsilon\bar{\theta}+\bar{\epsilon}\theta\right),\theta+\epsilon,\bar{\theta}+\bar{\epsilon}\right)-\Phi\left(x,\theta,\bar{\theta}\right)$ we find the operatorial definition of the supercharges as

$$Q_{\alpha}=i\partial_{\alpha}+\frac{1}{2}\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}},\qquad \bar{Q}_{\dot{\alpha}}=i\bar{\partial}_{\dot{\alpha}}+\frac{1}{2}\theta^{\alpha}\partial_{\alpha\dot{\alpha}},\tag{2.29}$$

where we introduce spinorial derivatives $\partial_{\alpha}=\frac{\partial}{\partial\theta^{\alpha}}$ and $\bar{\partial}_{\dot{\alpha}}=\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}}$, which follow the relations $\partial^{\alpha}\theta_{\beta}=\delta^{\alpha}{}_{\beta}$ and $\bar{\partial}^{\dot{\alpha}}\bar{\theta}_{\dot{\beta}}=\delta^{\dot{\alpha}}{}_{\dot{\beta}}$. The operatorial definitions of eq.(2.29) are useful to build covariant derivatives which must have null anticommuting relations with Q_{α} and $\bar{Q}_{\dot{\alpha}}$ in order to be invariant under supersymmetric transformations. These covariant derivatives are

$$D_{\alpha}=\partial_{\alpha}+\frac{i}{2}\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}},\qquad \bar{D}_{\dot{\alpha}}=\bar{\partial}_{\dot{\alpha}}+\frac{i}{2}\theta^{\alpha}\partial_{\alpha\dot{\alpha}},\tag{2.30}$$

and they follow the relations collected from eq.(A.31) to eq.(A.34).

Superfields are the fundamental ingredients for all the computations in this thesis: The advantages coming from the use of superfields is the fact that a superfield generally contains all the fields of a given supermultiplet in the form of a series in the spinorial coordinates, and this leads to a more compact formulation of the theory and a consequent simplification of computations. For example, if we look at the toy model of the previous section, described by the Lagrangian of eq.(2.16), we can define the following superfields

$$\begin{aligned}\Phi(y)&=\phi(y)+\theta^{\alpha}\psi_{\alpha}(y)-\theta^{2}F(y),\\ \bar{\Phi}(y)&=\bar{\phi}(y)+\bar{\theta}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}}(y)-\bar{\theta}^{2}\bar{F}(y),\end{aligned}\tag{2.31}$$

(with a shifted variable $y^{\alpha\dot{\alpha}}=x^{\alpha\dot{\alpha}}+\frac{i}{2}\theta^{\alpha}\bar{\theta}^{\dot{\alpha}}$) containing all the fields of the theory. The action of this model

$$\mathcal{S}=\int d^{4}x\,\left(\frac{1}{2}\partial^{\alpha\dot{\alpha}}\bar{\phi}\,\partial_{\alpha\dot{\alpha}}\phi-\frac{i}{2}\bar{\psi}^{\dot{\alpha}}\partial^{\alpha}{}_{\dot{\alpha}}\psi_{\alpha}-\frac{1}{4}\bar{F}F\right),\tag{2.32}$$

can be easily written as

$$\mathcal{S}=\frac{1}{4}\int d^{4}x\;d^{4}\theta\;\bar{\Phi}\Phi,\tag{2.33}$$

where we introduce the main relations of Berezin integration in Appendix A.2 (see in particular eq.(A.35) and eq.(A.36)).

In eq.(2.31) we find two examples of constrained superfields: Φ is a chiral superfield and its peculiarity is to be solution of the equation $\bar{D}_{\dot{\alpha}}\Phi = 0$, which makes it independent of $\bar{\theta}^{\dot{\alpha}}$; on the other hand, $\bar{\Phi}$ is an antichiral superfield (solution of $D_\alpha\bar{\Phi} = 0$) and it is independent of θ^α. As we have seen from this toy model, a superfield can be written as a θ-expansion of component fields only depending of the spacetime coordinates. A real scalar superfield $V\left(x,\theta,\bar{\theta}\right)$ has the expansion

$$\begin{aligned} V\left(x,\theta,\bar{\theta}\right) = {} & C(x) + \theta^\alpha\chi_\alpha(x) + \bar{\theta}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}}(x) - \theta^2 M(x) - \bar{\theta}^2\bar{M}(x) \\ & + \theta^\alpha\bar{\theta}^{\dot{\alpha}}A_{\alpha\dot{\alpha}}(x) - \bar{\theta}^2\theta^\alpha\lambda_\alpha(x) - \theta^2\bar{\theta}^{\dot{\alpha}}\bar{\lambda}_{\dot{\alpha}}(x) + \theta^2\bar{\theta}^2\mathrm{D}'(x), \end{aligned} \tag{2.34}$$

where the component fields are defined with the following projections

$$\begin{aligned} C(x) &= V\left(x,\theta,\bar{\theta}\right)|_{\theta=\bar{\theta}=0}, \\ \chi_\alpha(x) &= iD_\alpha V\left(x,\theta,\bar{\theta}\right)|_{\theta=\bar{\theta}=0}, \\ \bar{\chi}_{\dot{\alpha}}(x) &= -i\bar{D}_{\dot{\alpha}}V\left(x,\theta,\bar{\theta}\right)|_{\theta=\bar{\theta}=0}, \\ M(x) &= D^2V\left(x,\theta,\bar{\theta}\right)|_{\theta=\bar{\theta}=0}, \\ \bar{M}(x) &= \bar{D}^2V\left(x,\theta,\bar{\theta}\right)|_{\theta=\bar{\theta}=0}, \\ A_{\alpha\dot{\alpha}}(x) &= \frac{1}{2}\left[\bar{D}_{\dot{\alpha}}, D_\alpha\right]V\left(x,\theta,\bar{\theta}\right)|_{\theta=\bar{\theta}=0}, \\ \lambda_\alpha(x) &= i\bar{D}^2D_\alpha V\left(x,\theta,\bar{\theta}\right)|_{\theta=\bar{\theta}=0}, \\ \bar{\lambda}_{\dot{\alpha}}(x) &= -iD^2\bar{D}_{\dot{\alpha}}V\left(x,\theta,\bar{\theta}\right)|_{\theta=\bar{\theta}=0}, \\ \mathrm{D}'(x) &= \frac{1}{2}D^\alpha\bar{D}^2D_\alpha V\left(x,\theta,\bar{\theta}\right)|_{\theta=\bar{\theta}=0}. \end{aligned} \tag{2.35}$$

For the rest of the thesis, we usually omit the explicit dependence on coordinates only to use a smart notation.

$\mathcal{N} = 1$ Superspace is one of the most efficient tools used to study supersymmetric gauge theories with $\mathcal{N} \geq 1$. If we try to construct a $\mathcal{N} = 2$ Superspace, we find that it is impossible to close off-shell the supersymmetry; we glean an example from [12] to illustrate this fact. We consider the Fayet-Sohnius matter hypermultiplet, whose on-shell degrees of freedom are organized into a $SU(2)$ doublet ϕ_i of four scalar fields and into two isosinglet spinor fields ψ^α and $\bar{\chi}^{\dot{\alpha}}$. They are incorporated as component fields of an isodoublet superfield Φ_i of $\mathcal{N} = 2$ Superspace. The superfield Φ_i contains a lot of redundant component fields in addition to the physical ones listed above because of the large number of spinorial variables; it is possible to eliminate the extra fields through the constraints

$$D_\alpha^{(i}\Phi^{j)} = 0, \qquad \bar{D}_{\dot{\alpha}}^{(i}\Phi^{j)} = 0, \tag{2.36}$$

where (ij) stands for symmetrization and the covariant derivatives follow the relation $\{D_\alpha^j, \bar{D}_{\dot{\alpha}}^l\} = i\partial_{\alpha\dot{\alpha}}\delta^{jl}$. After having applied these constraints,

$$\Phi_i = \phi_i + \theta_i^\alpha \psi_\alpha + \bar{\theta}_i^{\dot{\alpha}} \bar{\chi}_{\dot{\alpha}} + \textit{derivatives terms}, \tag{2.37}$$

so only the physical fields and their derivatives in the higher terms of the θ-expansion remain in Φ_i. At the same time, the physical fields are on-shell

$$\Box\phi_i = 0, \qquad \partial^{\alpha\dot{\alpha}}\psi_\alpha = 0, \qquad \partial^{\alpha\dot{\alpha}}\bar{\chi}_{\dot{\alpha}} = 0, \tag{2.38}$$

and this is due to the fact that the constraints of eq.(2.36) are not integrable off-shell: The covariant derivatives do not anticommute. If we want to extend this theory off-shell and to introduce an interaction, it is not possible for us to relax one of the constraints of eq.(2.36) in $\mathcal{N} = 2$ and to use a finite set of auxiliary fields because of the "no-go" theorem. The only way to have an off-shell theory is to look for other Superspaces; in [12], the authors build what is called "Harmonic Superspace", a more complicated structure than the one introduced in this section. $\mathcal{N} = 1$ Superspace allows to formulate off-shell supersymmetric theories not only in the $\mathcal{N} = 1$ case: For example, in $D = 4$, it is possible to formulate a $\mathcal{N} = 2$ or a $\mathcal{N} = 4$ theory by using $\mathcal{N} = 1$ superfields. We will consider these two cases respectively in $\mathcal{N} = 2$ SCQCD (super conformal QCD) and in $\mathcal{N} = 4$ SYM (super Yang-Mills).

We move back to $\mathcal{N} = 1$ Superspace and, for simplicity, we rewrite the action of our toy model as

$$\mathcal{S} = \int d^4x \; d^4\theta \; \bar{\Phi}\Phi, \tag{2.39}$$

with Φ and $\bar{\Phi}$ respectively chiral and antichiral superfields defined in eq.(2.31). Given a generic constant parameter $\lambda \in \mathbb{R}$, the term $\bar{\Phi}\Phi$ is invariant under the transformation

$$\begin{cases} \Phi \to e^{i\lambda}\Phi \\ \bar{\Phi} \to \bar{\Phi}e^{-i\lambda} \end{cases}, \tag{2.40}$$

so the action in eq.(2.39) has a $U(1)$ global symmetry. The equations of motion for the superfields are found by differentiating the Lagrangian with respect to the superfields and by taking the results equal to zero; we find

$$D^2\Phi = 0, \qquad \bar{D}^2\bar{\Phi} = 0. \tag{2.41}$$

These equations describe the motion of free massless fields; this case is not interesting, so we choose to add a potential term into the action in order to see interactions between the component fields of Φ. The way to have also interactions with bosons of spin 1 is to construct a theory with a gauge group, which introduces a gauge field. For that purpose, the first step is to see eq.(2.40) as a local transformation and rewrite it in the following way

$$\begin{cases} \Phi \to e^{i\Lambda}\Phi \\ \bar{\Phi} \to \bar{\Phi}e^{-i\bar{\Lambda}} \end{cases}, \tag{2.42}$$

with $\Lambda(x,\theta)$ and $\bar{\Lambda}\left(x,\bar{\theta}\right)$ chiral and antichiral superfields respectively. This time

$$\bar{\Phi}\Phi \to \bar{\Phi}e^{i\left(\Lambda-\bar{\Lambda}\right)}\Phi,$$

so $U(1)$ is not a local symmetry of the action of eq.(2.39). The second step consists in the introduction of a new superfield which can make the action invariant under local transformations of $U(1)$. This superfield required must be a representation of $U(1)$; for that reason, we insert e^V into the action, where V is a scalar superfield whose local transformation under $U(1)$ is

$$V \to V + i\left(\bar{\Lambda} - \Lambda\right), \tag{2.43}$$

so that $\bar{\Phi}e^V\Phi$ is locally invariant under $U(1)$. The scalar superfield V is generally defined by eq.(2.34), while we can explicit

$$\begin{aligned} \Lambda &= \Lambda_1 + \theta^\alpha\Lambda_\alpha - \theta^2\Lambda_2, \\ \bar{\Lambda} &= \bar{\Lambda}_1 + \bar{\theta}^{\dot{\alpha}}\bar{\Lambda}_{\dot{\alpha}} - \bar{\theta}^2\bar{\Lambda}_2, \end{aligned} \tag{2.44}$$

and see the gauge transformation $\delta V = i\left(\bar{\Lambda} - \Lambda\right)$ for each component field defined in eq.(2.35); the result is

$$\begin{cases} \delta C = i\left(\bar{\Lambda}_1 - \Lambda_1\right) = -2i\ \mathtt{Im}(\Lambda_1) \\ \delta\chi_\alpha = \Lambda_\alpha \\ \delta\bar{\chi}_{\dot{\alpha}} = \bar{\Lambda}_{\dot{\alpha}} \\ \delta M = -i\Lambda_2 \\ \delta\bar{M} = i\bar{\Lambda}_2 \\ \delta A_{\alpha\dot{\alpha}} = \frac{1}{2}\partial_{\alpha\dot{\alpha}}\left(\Lambda_1 + \bar{\Lambda}_1\right) = \partial_{\alpha\dot{\alpha}}\mathtt{Re}(\Lambda_1) \\ \delta\lambda_\alpha = 0 \\ \delta\bar{\lambda}_{\dot{\alpha}} = 0 \\ \delta\mathrm{D}' = 0 \end{cases}. \tag{2.45}$$

By adequately setting the parameters of Λ and $\bar{\Lambda}$, it is possible to choose a particular gauge in which $C = \chi_\alpha = \bar{\chi}_{\dot{\alpha}} = M = \bar{M} = 0$ and consequently

$$V = \theta^\alpha \bar{\theta}^{\dot{\alpha}} A_{\alpha\dot{\alpha}} - \bar{\theta}^2 \theta^\alpha \lambda_\alpha - \theta^2 \bar{\theta}^{\dot{\alpha}} \bar{\lambda}_{\dot{\alpha}} + \theta^2 \bar{\theta}^2 \mathrm{D}'; \tag{2.46}$$

this is commonly called the Wess-Zumino gauge. In this gauge, the superfield V contains only the gauge field $A_{\alpha\dot{\alpha}}$, the gaugino field λ_α (and its conjugate) and an auxiliary field D'. In order to correctly add the gauge superfield V into the action of our toy model, it is convenient to introduce a coupling constant g which is related to the intensity of the interaction between the gauge fields and the matter fields by replacing $V \to gV$; moreover, a kinetic term for V is required since $A_{\alpha\dot{\alpha}}$ and λ_α propagate. For that purpose, we write down the super field-strength of V as

$$W_\alpha = g \bar{D}^2 D_\alpha V, \tag{2.47}$$

whose square is the kinetic term for V and our toy model with gauge group $U(1)$ is described by the action

$$\mathcal{S} = \int d^4x \, d^4\theta \, \bar{\Phi} e^{gV} \Phi + \frac{1}{g^2} \int d^4x \, d^2\theta \, W^\alpha W_\alpha. \tag{2.48}$$

We choose to put V into the adjoint representation of the gauge group $U(1)$ and Φ into the fundamental representation. It is possible to prove that $\bar{W}^{\dot{\alpha}} \bar{W}_{\dot{\alpha}}$ leads to the same terms as $W^\alpha W_\alpha$, so it is sufficient to include only one of them.

We considered an abelian gauge group to introduce the main concepts, but in most cases the gauge group describing a fundamental interaction is non-abelian. Given a gauge group $SU(N_c)$ (N_c is the number of "color" values), generated by T_a, the path to follow is the same as the previous one with some differences. The first one is the inclusion of T_a which can be easily performed with the short notation $V = T^a V_a$ (and the same for each component field in V); this must be applied for each superfield in the adjoint representation of $SU(N_c)$. Another difference is the super field-strength

$$W_\alpha = \bar{D}^2 \left(e^{-gV} D_\alpha e^{gV} \right), \tag{2.49}$$

and the kinetic term is written as $\mathrm{tr}\,(W^\alpha W_\alpha)$, where the trace is on color indices.

To sum up, we introduced supersymmetry as a solution to the problem of Naturalness in a quantum field theory and showed that it leads to the cancellation of quadratic divergences in the self-energy corrections of scalars; we pointed out the main features of $\mathcal{N} \geq 1$ supersymmetry and its massive and massless representations. The off-shell formulation of $\mathcal{N} = 1$ supersymmetry allowed us to construct the $\mathcal{N} = 1$ Superspace as the coset of super-Poincaré and Lorentz; this Superspace is suitable to study supersymmetric gauge theories and supergravity theories, as we will see in the next chapter.

Chapter 3

Supergravity

3.1 Local supersymmetry is supergravity

In the previous chapter, there is a detailed introduction of global supersymmetry and gauge theories defined in $\mathcal{N} = 1$ Superspace; this topic and its formalism in particular will be useful for computations in Chapter 5. Now, we are going to introduce briefly the main concepts and tools for the study of a particular theory of supergravity that will be performed in Chapter 4.

First of all, a theory of supergravity is a theory in which supersymmetric invariance is local; in other words, the parameters of a generic supersymmetric transformation depend on the coordinates. It is common to use the verb "to gauge" in order to indicate the passage from a global supersymmetry to a local one. In this section, we show that when we gauge the supersymmetry of a theory, this passage requires the introduction of two new fields: One field is associated with the graviton (the quantum messenger of gravity, with spin 2) and the second one is associated with the gravitino (the superpartner of the graviton, with spin $\frac{3}{2}$).

In order to avoid complicated computations, we consider a theory in $D = 1$, where there is only a time coordinate t; we first analyze the non-supersymmetric case and after that we extend it to the $\mathcal{N} = 1$ case.

Non-supersymmetric case. We consider a model of a free real massless scalar field $\phi(t)$; the action of this theory is

$$\mathcal{S}_0 = \int dt\, \mathcal{L}_0 \qquad with \quad \mathcal{L}_0 = \frac{1}{2}\dot{\phi}^2, \tag{3.1}$$

where $\dot{\phi} = \frac{d\phi}{dt}$. Given a constant parameter ξ, this theory is invariant under the global time translation $t \to t + \xi$, which transforms ϕ into $\phi + \delta\phi$, with

$$\delta\phi = -\xi\dot{\phi}. \tag{3.2}$$

If we look for the Noether current associated with the local time translation with parameter $\xi(t)$, we find that on-shell

$$\forall\xi(t) \quad \delta\mathcal{S}_0 = 0 \iff \frac{dH}{dt} = 0 \qquad with \quad H = \frac{1}{2}\dot{\phi}^2, \tag{3.3}$$

where it is important to remember that, generally when a theory is defined without any boundary, a field is thought as a function which asymptotically vanishes and, for this reason, all the total derivatives in the integrand integrate to zero. In conclusion, the Noether current associated with time translation is the Hamiltonian H; now we want to include gravity to this model. In General Relativity, a theory is invariant under diffeomorphisms: As a consequence, if we want to couple the theory described by $\mathcal{S}_0$ of eq.(3.1) with gravity, we have to make it invariant under diffeomorphisms. We use the Noether method, which consists in adding to the Lagrangian a term that is the Noether current coupled with a field associated with the local transformation; in our case, given a field $A(t)$ associated with local time translations, the action of the theory becomes

$$\mathcal{S} = \int dt\, \left(\mathcal{L}_0 + AH\right) = \int dt\, \left(1 + A\right)\mathcal{L}_0. \tag{3.4}$$

For simplicity, we write $h = 1 + A$ and, by taking $\delta\mathcal{S} = 0$ valid $\forall\xi(t)$, we find the following tensorial transformation rule for h

$$\delta h = h\dot{\xi} - \dot{h}\xi. \tag{3.5}$$

We end up with the Lagrangian

$$\mathcal{L} = \frac{1}{2}h\dot{\phi}^2, \tag{3.6}$$

where h is the gravitational field coupling with the stress-energy tensor (in our case, it has a single component, the Hamiltonian).

$\mathcal{N} = 1$ **supersymmetric case.** In addition to the real massless scalar field $\phi(t)$, we introduce a real massless anticommuting field $\lambda(t)$; the action of this model is

$$\mathcal{S}_0 = \int dt\, \mathcal{L}_0 \qquad with \quad \mathcal{L}_0 = \frac{1}{2}\dot{\phi}^2 + \frac{i}{2}\lambda\dot{\lambda}, \tag{3.7}$$

and it is invariant under the global supersymmetric transformation

$$\begin{cases} \delta\phi = \frac{1}{\sqrt{2}}\epsilon\lambda \\ \delta\lambda = \frac{i}{\sqrt{2}}\dot{\phi}\epsilon \end{cases}, \tag{3.8}$$

with a generic constant spinorial parameter ϵ. It is easy to verify that, if we take two generic global supersymmetric transformations with respective parameters ϵ_1 and ϵ_2 and we compute the commutator of them applied separately to ϕ and to λ, we find the relations

$$\begin{aligned} [\delta_1, \delta_2]\, \phi &= i\epsilon_2\epsilon_1\dot{\phi}, \\ [\delta_1, \delta_2]\, \lambda &= i\epsilon_2\epsilon_1\dot{\lambda}, \end{aligned} \tag{3.9}$$

which mean that two global supersymmetric transformations imply a time translation. We can extend this concept for $D > 1$ and say that global supersymmetry is "the square root" of translations; so, we expect to find that local supersymmetry is the square root of General Relativity. As we did before in the non-supersymmetric case, we find that on-shell

$$\forall\epsilon(t) \quad \delta\mathcal{S}_0 = 0 \iff \frac{dJ}{dt} = 0 \qquad with \quad J = -\frac{1}{\sqrt{2}}\dot{\phi}\lambda, \tag{3.10}$$

so J is the Noether current associated with the local supersymmetry; we note that J is fermionic, so it anticommutes with ϵ and with λ. After some simple steps, the variation of J under a local supersymmetric transformation like eq.(3.8) is

$$\delta J = -i\epsilon\mathcal{L}_0; \tag{3.11}$$

this result is useful for the next computation. In order to make the model invariant under local supersymmetric transformations, we use the Noether method as before in the non-supersymmetric case: we introduce a field ψ associated with local supersymmetric transformations and we couple it to the Noether current J. Since the Lagrangian is a scalar object, the field ψ coupled with a fermionic current must be fermionic too; another way to justify that is the fact that supersymmetry is generated by fermionic generators, so ψ must be itself a fermion because it is related to supersymmetry in the same way as a generator. Now, if we consider $\mathcal{L}_0 + \psi J$ as the new Lagrangian and we compute the variation of it under an infinitesimal local supersymmetric transformation, we find that if $\delta\psi \sim \dot{\epsilon} + \dots$ it is possible to cancel some terms; however, this is not sufficient to restore the supersymmetric invariance locally. We have to introduce another field to the Lagrangian; we denote it with h and we couple it to $\mathcal{L}_0$, writing the action

$$\mathcal{S} = \int dt\, \mathcal{L} \qquad \textit{with} \quad \mathcal{L} = h\left(\frac{1}{2}\dot{\phi}^2 + \frac{i}{2}\lambda\dot{\lambda}\right) + \psi J. \tag{3.12}$$

This is the only way we can introduce h in that model for different reasons. The first reason is the continuity of the supersymmetric extension of this model with its non-supersymmetric version: If we set $\lambda = 0$, that is to say that we remove supersymmetry, we must achieve the Lagrangian of eq.(3.6) and this choice is suitable for that purpose. Another reason is the fact that after having introduced a fermionic field ψ, it is spontaneous to include a bosonic field in order to restore $\mathcal{N} = 1$ supersymmetry. Given the action of eq.(3.12), if we require $\delta\mathcal{S} = 0$, we find the following variations of ψ and h

$$\begin{cases} \delta h = i\psi\epsilon \\ \delta\psi = h\dot{\epsilon} - \frac{1}{2}\dot{h}\epsilon \end{cases}. \tag{3.13}$$

In conclusion, we proved that local supersymmetry requires gravity: h is the gravitational field and ψ is the field associated with the gravitino, which is seen as the gauge field of supersymmetry.

It could be possible to follow the same path in order to prove that local supersymmetry is supergravity for all the possible cases, expecially in some theories of physical interest; however, it is sufficient to have verified this important statement in one simple case. In the next section, we move to an introduction of the formalism we will use in the study of supergravity.

3.2 Gravity in the dual space

General Relativity is an effective theory of gravity; a brilliant mathematical theory based on the theory of Special Relativity and the principle of equivalence, also called the principle of general covariance. Given a metric tensor $g_{\mu\nu}$ with $\mu, \nu = 0, \ldots, D-1$ and mostly-plus signature $(D-1, 1)$, we can collect the definitions of the main mathematical objects of General Relativity

$$\begin{aligned}
&\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\lambda} \left(\partial_\mu g_{\nu\lambda} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu} \right), \\
&R^{\lambda}{}_{\mu\nu\rho} = \partial_\rho \Gamma^{\lambda}_{\mu\nu} - \partial_\nu \Gamma^{\lambda}_{\mu\rho} + \Gamma^{\eta}_{\mu\nu} \Gamma^{\lambda}_{\rho\eta} - \Gamma^{\eta}_{\mu\rho} \Gamma^{\lambda}_{\nu\eta}, \\
&R_{\mu\nu} = g^{\lambda\rho} R_{\lambda\mu\rho\nu}, \\
&R = g^{\mu\nu} R_{\mu\nu},
\end{aligned} \tag{3.14}$$

which are respectively the affine connection, the Riemann-Christoffel curvature tensor, the Ricci tensor and the scalar curvature. It is possible to give a general definition of a connection like $\tilde{\Gamma}^{\rho}_{\mu\nu} = \Gamma^{\rho}_{\mu\nu} + N^{\rho}_{\mu\nu}$, where $\Gamma^{\rho}_{\mu\nu}$ is defined in eq.(3.14) and $N^{\rho}_{\mu\nu}$ is the distortion term ([13] and references therein). Then, another mathematical object we can define is the torsion $T^{\rho}_{\mu\nu} = \tilde{\Gamma}^{\rho}_{\mu\nu} - \tilde{\Gamma}^{\rho}_{\nu\mu}$; we consider a theory where the torsion is set to zero. A generic free theory of gravity in the vacuum is described by the Einstein-Hilbert action

$$\mathcal{S} = \frac{1}{16\pi G_D} \int d^D x \, \sqrt{-det(g)} \left(R - 2\Lambda \right), \tag{3.15}$$

with G_D Newton constant in D dimensions ($[G_D] = L^{D-2}$) and we introduce Λ cosmological constant with dimension $[\Lambda] = E^2$. The Einstein equations of motion in the vacuum for $g_{\mu\nu}$ are

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0, \tag{3.16}$$

and they are the result of the extremization of eq.(3.15) with respect to $g^{\mu\nu}$; further details about General Relativity could be found in [14].

What we used in order to present the main features of General Relativity from the beginning of the section so far is the tensor formalism; another valid way to describe the same physics is through the formalism of k-forms living in the cotangent space, also called "the geometric approach". As we will see in Chapter 4, in the geometric approach, we can easily build a theory by starting from its algebra, through the Maurer-Cartan equations, and with a geometric construction of the Lagrangian. An accurate introduction of the geometric formalism can be found in [15]; moreover, in Appendix A.1, we summarise the main technical details of the formalism. Now, we give a general introduction of the main concepts of the geometric approach to gravity and supergravity, which are essential for the computations in Chapter 4.

Given a Riemannian D-dimensional manifold M, at each point $P \in M$ we define an orthonormal local moving frame $\{u_a\}$ (with $a = 0, \dots, D-1$) spanning a base of the tangent space $T_P(M)$ with Minkowskian metric η_{ab} given by

$$\eta_{ab} = u_a \cdot u_b. \tag{3.17}$$

The relation between the moving frame $\{u_a\}$ and the natural frame $\{\frac{\partial}{\partial x^\mu}\}$ (with $\mu = 0, \dots, D-1$) is

$$\begin{aligned} u_a &= \mathcal{V}^\mu_a \, \frac{\partial}{\partial x^\mu}, \\ \frac{\partial}{\partial x^\mu} &= \mathcal{V}^a_\mu \, u_a, \end{aligned} \tag{3.18}$$

with $\mathcal{V}^\mu_a, \mathcal{V}^a_\mu \in GL(D, \mathbb{R})$ satisfying $\mathcal{V}^a_\mu \mathcal{V}^\mu_b = \delta^a_{\ b}$ and $\mathcal{V}^\mu_a \mathcal{V}^a_\nu = \delta^\mu_{\ \nu}$. On the other hand, in the cotangent space $T^*_P(M)$, the moving frame $\{V^a\}$ is related with the natural frame $\{dx^\mu\}$ by

$$\begin{aligned} V^a &= \mathcal{V}^a_\mu \, dx^\mu, \\ dx^\mu &= \mathcal{V}^\mu_a \, V^a. \end{aligned} \tag{3.19}$$

The 1-form V^a, which is called "vielbein", is dual to the vector u_a; we can express an infinitesimal dispacement δP of a point $P \in M$ as

$$\delta P = V^a u_a. \tag{3.20}$$

In this case, we use δ instead of d because eq.(3.20) in general is not an exact differential since P is not a function of the coordinates. If we consider an infinitesimal translation $P \to P + \delta P$, the infinitesimal change of the moving frame in $T_P(M)$ is

$$\delta u_a = u_b \, \omega^b_{\ a}, \tag{3.21}$$

and, since $\delta(u_a \cdot u_b) = \delta\eta_{ab} = 0$, we find that $\omega_{ab} = -\omega_{ba}$; the 1-form ω_{ab} is called "spin connection". If we take the exterior derivative of both sides of eq.(3.20) and of eq.(3.21) and we replace them where necessary, we get

$$\begin{aligned} d(\delta P) &= \left(dV^a + \omega^a_{\ b} \wedge V^b\right) u_a, \\ d(\delta u_a) &= \left(d\omega^b_{\ a} + \omega^b_{\ c} \wedge \omega^c_{\ a}\right) u_b, \end{aligned} \tag{3.22}$$

and this result leads to two structure equations

$$\begin{aligned} \mathcal{R}^{ab} &= d\omega^{ab} + \omega^{ac} \wedge \omega_c{}^b, \\ R^a &= dV^a + \omega^a{}_c \wedge V^c, \end{aligned} \tag{3.23}$$

which are respectively the curvature 2-form $\mathcal{R}^{ab}$ and the torsion 2-form R^a. When we deal with gravity, it is useful to define a covariant derivative (that is a derivative which transforms like a tensor). Similarly, we define a covariant exterior derivative

$$D_\omega = d + \omega, \tag{3.24}$$

that is understood as $D_\omega A^{(k)} = dA^{(k)} + \omega \wedge A^{(k)}$ for a generic k-form $A^{(k)}$; consequently, the two definitions in eq.(3.23) can be read as $\mathcal{R}^{ab} = D_\omega \omega^{ab}$ and $R^a = D_\omega V^a$. We are talking about a D-dimensional manifold M by studying it in the cotangent space, where we defined the 1-form fields ω^{ab} and V^a and their respective covariant exterior derivatives $\mathcal{R}^{ab}$ and R^a. It is clear that computing the components of $\mathcal{R}^{ab}$ and R^a is easier than computing the components of $R_{\mu\nu\rho\lambda}$ and $T^\rho_{\mu\nu}$; actually, the dual formalism is chosen not for this reason, but because it facilitates us the study of a theory through the properties coming from its symmetries and its algebraic structure. In order to understand the meaning of this argumentation, we have to take a step back towards group theory.

Our manifold M considered can be seen as a Lie group $\mathcal{G}$: A known result of group theory is that left-invariant vector fields on $\mathcal{G}$ form the Lie algebra $\mathfrak{g}$ of the group $\mathcal{G}$. Given $E \in \mathcal{G}$ the identity element, we can think about $\mathfrak{g}$ as $T_E(\mathcal{G})$ because any left-invariant vector is determined by its value at E; obviously, we have $T_E(M) = T_E(\mathcal{G})$ since, in a Lie group, the manifold nature is strictly connected with the group structure. We define a set $\{T_a\}$ of generators on $T_E(\mathcal{G})$ which close the relations

$$[T_i, T_j] = C^k_{ij}\ T_k, \tag{3.25}$$

where C^k_{ij} are the structure constants. T_a also fulfill the Jacobi identities

$$[T_i, [T_j, T_k]] + [T_j, [T_k, T_i]] + [T_k, [T_i, T_j]] = 0, \tag{3.26}$$

also written in a more compact way $C^k_{i[j}\ C^i_{lm]} = 0$. We can rewrite all these informations in the cotangent space $T^*_E(\mathcal{G})$, where, instead of left-invariant vectors, there are left-invariant 1-forms. Given a basis $\{\sigma^a\}$ of 1-form generators, which are related with the previous ones through

$$\sigma^a(T_b) = \delta^a{}_b, \tag{3.27}$$

it is possible to show (see [15]) that eq.(3.25) becomes

$$d\sigma^i + \frac{1}{2} C^i_{jk}\, \sigma^j \wedge \sigma^k = 0, \tag{3.28}$$

commonly called “Maurer-Cartan equations”. These equations uniquely determine the algebraic structure of $\mathcal{G}$; as a consequence, given a theory with symmetry group $\mathcal{G}$, we can read its Lie algebra in the cotangent space through the Maurer-Cartan equations eq.(3.28). The relation $d^2 = 0$ for the exterior derivative is the dual version of the Jacobi identities in the cotangent space. In the table below are summarized the main parallelisms between tangent space and cotangent space.

	Tangent space $T_E(\mathcal{G})$	**Cotangent space** $T^*_E(\mathcal{G})$
Generators	T_a	σ^a
Lie algebra	$[T_i, T_j] = C^k_{ij}\, T_k$	$d\sigma^i + \frac{1}{2} C^i_{jk}\, \sigma^j \wedge \sigma^k = 0$
Jacobi identity	$C^k_{i[j}\, C^i_{lm]} = 0$	$d^2 = 0$

After these considerations, if we look at the definitions in eq.(3.23), $\mathcal{R}^{ab} = 0$ and $R^a = 0$ are nothing but the Maurer-Cartan equations of the Poincaré algebra

$$\begin{aligned}
&[J_{\mu\nu}, J_{\rho\lambda}] = \eta_{\mu\lambda} J_{\nu\rho} + \eta_{\nu\rho} J_{\mu\lambda} - \eta_{\mu\rho} J_{\nu\lambda} - \eta_{\nu\lambda} J_{\mu\rho}, \\
&[J_{\mu\nu}, P_\rho] = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu, \\
&[P_\mu, P_\nu] = 0,
\end{aligned} \tag{3.29}$$

where ω^{ab} is dual to $J_{\mu\nu}$ and V^a is dual to P_μ.

To sum up, we study the D-dimensional manifold M (also seen as a Lie group) through its cotangent space (its Lie algebra), where there are 1-form generators fulfilling the Maurer-Cartan equations. Dealing with the Poincaré algebra of eq.(3.29), that corresponds to $\mathcal{R}^{ab} = 0$ and $R^a = 0$, we can differentiate both sides of the definitions in eq.(3.23) and use the property $d^2 = 0$.

$$\begin{aligned}
d\mathcal{R}^{ab} &= d\omega^{ac} \wedge \omega_c{}^b - \omega^{ac} \wedge d\omega_c{}^b \\
&= \left(\mathcal{R}^{ac} - \omega^{af} \wedge \omega_f{}^c\right) \wedge \omega_c{}^b - \omega^{ac} \wedge \left(\mathcal{R}_c{}^b - \omega_c{}^f \wedge \omega_f{}^b\right) \\
&= \mathcal{R}^{ac} \wedge \omega_c{}^b - \omega^{ac} \wedge \mathcal{R}_c{}^b, \\
dR^a &= d\omega^{ac} \wedge V_c - \omega^{ac} \wedge dV_c \\
&= \left(\mathcal{R}^{ac} - \omega^{af} \wedge \omega_f{}^c\right) \wedge V_c - \omega^{ac} \wedge \left(R_c - \omega_c{}^f \wedge V_f\right) \\
&= \mathcal{R}^{ac} \wedge V_c - \omega^{ac} \wedge R_c.
\end{aligned} \tag{3.30}$$

After some trivial steps, we find that the curvature $\mathcal{R}^{ab}$ and the torsion R^a obey the integrability conditions

$$
\begin{aligned}
&D_\omega \mathcal{R}^{ab} = 0, \\
&D_\omega R^a - \mathcal{R}^{ac} \wedge V_c = 0,
\end{aligned}
\tag{3.31}
$$

which are referred to as the "Bianchi identities". Another step is to assume that $R^a = 0$ in order to study a Riemannian manifold with a Riemannian connection; this assumption is referred to as the on-shell condition.

For completeness, there are two main formulations of gravity, which are respectively the first-order and the second-order formulation. In the first-order formulation of gravity, the vielbein V^a and the spin connection ω^{ab} are in general independent; in the tensor formalism, it means that the affine connection does not depend on the metric. On the other hand, in the second-order formulation, ω^{ab} depends on V^a. Some references about both the formulations could be found in [15, 16]. We defined the mathematical objects of eq.(3.14) in the second-order formulation in the tensor formalism; then, we moved to the geometric approach and we introduced the main definitions through the first-order formulation. The on-shell condition $R^a = 0$ is a torsion-less condition which determines ω^{ab} in terms of V^a and consequently it moves from the first-order to the second-order formalism.

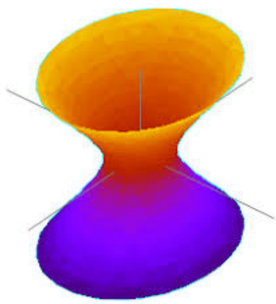
In Section 3.3, we will briefly introduce the AdS spacetime in general and we will use the $D = 4$ case as an example of the passage from the formalism of k-forms to the tensor formalism. In Section 3.4, we will contextualize the general features discussed so far to the $\mathcal{N} = 1$ Superspace in $D = 4$, which will be our working space in Chapter 4.

3.3 Introduction to AdS spacetime

We start from the action of eq.(3.15), which leads to the Einstein equations of eq.(3.16) briefly discussed at the beginning of the previous section. If $\Lambda < 0$, the maximally symmetric solution to eq.(3.16) is the metric tensor describing what is commonly called "the Anti-de Sitter spacetime" in D dimensions (abbreviation AdS_D spacetime). We can visualize the AdS_D spacetime by an isometric embedding in a flat spacetime of one higher dimension that we choose as time-like. So, we consider the flat spacetime $\mathbb{R}_2^{D+1}$ (the subscript indicates that there are two time dimensions) with metric tensor $\eta_{AB} = diag(-1, +1, \ldots, +1, -1)$, described by a set of coordinates ξ^A (with $A = 0, \ldots, D$). We can define the manifold AdS_D as the set of points $\xi^A \in \mathbb{R}_2^{D+1}$ so that

$$\eta_{AB}\, \xi^A \xi^B = -l^2, \tag{3.32}$$

where l is a constant named "AdS radius" ($[l] = L$). Precisely, in $\mathbb{R}_2^{D+1}$ there are two time dimensions whereas in AdS_D there is only one time dimension because eq.(3.32) makes the last time dimension dependent on the other D dimensions. For instance, we show the following image of AdS_2 embedded in $\mathbb{R}_2^3$.



In general, AdS_D is an Einstein D-dimensional spacetime with negative constant curvature, group of isometries $SO(D-1, 2)$ and a negative cosmological constant given by

$$\Lambda = -\frac{(D-1)(D-2)}{4l^2}, \tag{3.33}$$

where we refer to l as the AdS radius. In general, eq.(3.33) is seen with a 2 rather than a 4 in the denominator: The reason why we put 4 is to follow our conventions. Solutions of AdS gravity and black holes in that background are widely studied because of their mathematical properties a their symmetries. In particular, AdS solutions coming from some string theories are strictly connected through their symmetries to respective quantum conformal field theories, outlining what is called AdS/CFT correspondence; we will briefly introduce some of the main concepts about it in Section 3.5.

Now we consider the specific case of $D = 4$ AdS spacetime, whose dynamics in the vacuum is described by the action

$$\mathcal{S} = \frac{1}{16\pi G} \int d^4x \, \sqrt{-det(g)}\, (R - 2\Lambda)\,, \tag{3.34}$$

where G is the Newton constant in 4 dimensions. We define

$$e = \frac{1}{2l}, \tag{3.35}$$

and for the rest of the thesis we will call e the cosmological constant instead of Λ; this choice conforms our notation to most of recent papers dealing with supergravity studied through the geometric (or rheonomic) approach. The group of isometries of AdS_4 is $SO(3,2) \sim Sp(4)$ and its algebra reads

$$\begin{aligned}
&[J_{\mu\nu}, J_{\rho\lambda}] = \eta_{\mu\lambda} J_{\nu\rho} + \eta_{\nu\rho} J_{\mu\lambda} - \eta_{\mu\rho} J_{\nu\lambda} - \eta_{\nu\lambda} J_{\mu\rho}, \\
&[J_{\mu\nu}, P_\rho] = \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu, \\
&[P_\mu, P_\nu] = J_{\mu\nu},
\end{aligned} \tag{3.36}$$

with $J_{\mu\nu}$ generators of Lorentz transformations and P_μ generators of AdS boosts. We move to the geometric formalism introduced in Section 3.2 and we have the spin connection ω^{ab} dual to $J_{\mu\nu}$ and the vielbein V^a dual to P_μ. We fix the length dimensions $[e] = L^{-1}$, $[\omega] = L^0$, $[V] = L^1$ (the d operator is not associated with any length dimension); then the Maurer-Cartan equations corresponding to eq.(3.36) are

$$\begin{aligned}
&\mathcal{R}^{ab} + 4e^2 \, V^a \wedge V^b = D_\omega \omega^{ab} + 4e^2 \, V^a \wedge V^b = 0, \\
&R^a = D_\omega V^a = 0.
\end{aligned} \tag{3.37}$$

The Lagrangian of that theory is defined in the cotangent space as a 4-form, since its integration over the spacetime (that is the action) must be a function (a 0-form). We now demonstrate that this Lagrangian

$$\mathcal{L} = \epsilon_{abcd} \left(\mathcal{R}^{ab} \wedge V^c \wedge V^d + 2e^2 \, V^a \wedge V^b \wedge V^c \wedge V^d \right) \tag{3.38}$$

corresponds to the integrand of eq.(3.34) up to some constant overall factors. The first step consists in expanding the 2-form $\mathcal{R}^{ab}$ in the vielbein basis, so we can write $\mathcal{R}^{ab} = R^{ab}{}_{fg}\, V^f \wedge V^g$ with $R^{ab}{}_{fg}$ parameters; after that, we change the coordinate frame with $V^a = \mathcal{V}^a_\mu\, dx^\mu$.

$$\begin{aligned}
\mathcal{L} &= \epsilon_{abcd}\left(R^{ab}{}_{fg}\, V^f \wedge V^g \wedge V^c \wedge V^d + 2e^2\, V^a \wedge V^b \wedge V^c \wedge V^d\right) \\
&= \epsilon_{abcd}\left(R^{ab}{}_{fg}\, \mathcal{V}^f_\mu \mathcal{V}^g_\nu \mathcal{V}^c_\rho \mathcal{V}^d_\sigma + 2e^2\, \mathcal{V}^a_\mu \mathcal{V}^b_\nu \mathcal{V}^c_\rho \mathcal{V}^d_\sigma\right) dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \\
&= \epsilon_{abcd}\, \epsilon^{\mu\nu\rho\sigma}\, d^4x \left(R^{ab}{}_{fg}\, \mathcal{V}^f_\mu \mathcal{V}^g_\nu \mathcal{V}^c_\rho \mathcal{V}^d_\sigma + 2e^2\, \mathcal{V}^a_\mu \mathcal{V}^b_\nu \mathcal{V}^c_\rho \mathcal{V}^d_\sigma\right) \\
&= \epsilon_{abcd}\, det(\mathcal{V})\, d^4x \left(R^{ab}{}_{fg}\, \epsilon^{fgcd} + 2e^2\, \epsilon^{abcd}\right) \\
&= det(\mathcal{V})\, d^4x \left(4\delta^{fg}_{ab}\, R^{ab}{}_{fg} + 48e^2\right) \\
&= 4\sqrt{-det(g)}\, d^4x \left(R + 12e^2\right) \\
&= 4\sqrt{-det(g)}\, d^4x \left(R - 2\Lambda\right),
\end{aligned} \tag{3.39}$$

where we use the identities $\epsilon_{abcd}\, \epsilon^{fgcd} = 4\delta^{fg}_{ab}$, $\epsilon_{abcd}\, \epsilon^{abcd} = 24$, $\delta^{fg}_{ab}\, R^{ab}{}_{fg} = R$ and $\Lambda = -6e^2$. This equivalence is valid if we choose the first-order formulation and we assume the on-shell condition. This is an explicit example of the way to convert a result got through the geometric formalism into the tensor formalism. It is not necessary to convert into the tensor formalism the other results in that thesis since our purpose is to study the mathematical properties of the theory considered and this goal can be achieved through the geometric approach.

3.4 Geometric approach to supergravity in Superspace

Now, we briefly recall some of the main features of the geometric approach for the description of $\mathcal{N}=1$, $D=4$ pure supergravity (more details can be found in [17, 18, 19]), since this will be useful in the sequel.

In the geometric approach to supergravity [15], the theory is given in terms of 1-form superfields μ^A defined on a Superspace that we will call $\mathcal{M}_{4|4}$ for brevity, where the first number indicates the spacetime dimensions while the second one indicates the spinorial dimensions. In particular, the bosonic 1-form V^a and the fermionic 1-form ψ^α (that is a Majorana spinor) define the supervielbein basis $\{V^a, \psi^\alpha\}$ in $\mathcal{M}_{4|4}$. For a technical overview of the main operations between k-forms, see Appendix A.1.

In this framework, the supersymmetry transformations in spacetime are interpreted as diffeomorphisms in the fermionic directions of Superspace and they are generated by Lie derivatives with fermionic parameter ϵ^α (in Chapter 4 we will remove the fermionic indices for simplicity). Then, the supersymmetry invariance of the theory is fulfilled requiring the Lie derivative of the Lagrangian to vanish for diffeomorphisms in the fermionic directions of Superspace, that is to say:

$$\delta_\epsilon \mathcal{L} = \ell_\epsilon \mathcal{L} = \imath_\epsilon d\mathcal{L} + d(\imath_\epsilon \mathcal{L}) = 0, \tag{3.40}$$

where ϵ is the fermionic parameter along the tangent vector dual to the gravitino.

The contribution $\imath_\epsilon d\mathcal{L}$ in eq.(3.40), which would be identically zero in spacetime, is non-trivial here, in Superspace. On the other hand, the contribution $d(\imath_\epsilon \mathcal{L})$ is a boundary term and does not affect the bulk result. Then, a necessary condition for a supergravity Lagrangian is

$$\imath_\epsilon d\mathcal{L} = 0, \tag{3.41}$$

corresponding to require supersymmetry invariance in the bulk. Under eq.(3.41), the supersymmetry transformation of the action simply reduces to

$$\delta_\epsilon \mathcal{S} = \int_{\mathcal{M}_{4|4}} d(\imath_\epsilon \mathcal{L}) = \int_{\partial \mathcal{M}_{4|4}} \imath_\epsilon \mathcal{L}, \tag{3.42}$$

where $\partial \mathcal{M}_{4|4}$ is the boundary of $\mathcal{M}_{4|4}$. When we consider a Minkowski background (or, generally, a spacetime with boundary thought as set at infinity), the fields asymptotically vanish, so that

$$\imath_\epsilon \mathcal{L}|_{\partial \mathcal{M}_{4|4}} = 0 \tag{3.43}$$

and, consequently,

$$\delta_\epsilon \mathcal{S} = 0. \tag{3.44}$$

Then, we have that, in this case, eq.(3.41) is also a sufficient condition for the supersymmetry invariance of the Lagrangian.

On the other hand, when the background spacetime presents a non-trivial boundary, the condition of eq.(3.43) (modulo an exact differential) becomes non-trivial, and it is necessary to check it explicitly to get supersymmetry invariance of the action, requiring a more subtle treatment.

3.5 AdS/CFT correspondence

In Section 3.3 we gave a brief overview of AdS spacetime and some properties. We mentioned that the great success of the AdS solutions is due to what is called AdS/CFT correspondence (see the first works [20, 21, 22, 23, 24] on this topic and references therein). This is one of the most fascinating discoveries in the modern theoretical physics, which relates gauge theories to gravity theories; we briefly introduce the main concepts about it.

First of all, CFT stands for conformal field theory, which is a D-dimensional field theory invariant under conformal transformations belonging to the group $SO(D, 2)$. In general, conformal theories manifest interesting mathematical properties: For example, it is possible to fix the structure of the 2-point correlation function by only requiring conformal invariance. Another aspect is that the coupling constant does not depend on the energy scale of the theory; as a consequence, the beta function of the theory is zero. A detailed reference dealing with conformal field theory and related topics is [25].

According to the AdS/CFT correspondence, some conformal field theories are related to corresponding superstring theories on curved backgrounds. The AdS/CFT correspondences that are of interest for this thesis are strong/weak dualities. This duality states that, in the parameter range, where one of the two theories is weakly coupled, the other one is strongly coupled and viceversa. On one hand, this peculiarity would allow to investigate the non-perturbative regime of a theory by means of perturbative computations performed on the opposite side of the duality. On the other hand, however, it also makes the correspondence very difficult to prove: In fact, no rigorous proof of the conjecture exists at the moment, even if it has passed several non trivial checks. The strongest version of the AdS/CFT correspondence claims the exact equivalence of the two theories for any values of the parameters. Weaker formulations are more tractable because they concern particular simplified limits. The main example of such weaker versions is represented by the 't Hooft limit, in which $N_c \to \infty$ while the 't Hooft coupling $\lambda = \frac{g^2 N_c}{(4\pi)^2}$ is kept fixed: This is also called "planar limit". In Chapter 5, we will see that, in the planar limit of a field theory, non-planar contributions are suppressed and this fact considerably simplifies perturbative computations by negletting all the Feynman diagrams which can not be drawn in a plane (non-planar diagrams). When the field theory is strongly coupled, the string side can be approximated by a classical theory of supergravity on the bulk. We show a well understood example of such a correspondence with a qualitative discourse: That is the duality between $\mathcal{N} = 4$

SYM in $D = 4$ dimensions and type IIB superstring theory on a $AdS_5 \times S^5$ background [20].

As we mentioned in Section 2.2, $\mathcal{N} = 4$ SYM is a maximally supersymmetric field theory in $D = 4$ dimensions and it is also conformally invariant; one of the several references about it is [26]. The field content of this theory is one vector field, four Weyl spinors and six real scalar fields; all of them are massless. In $\mathcal{N} = 1$ Superspace, these are arranged into a scalar gauge superfield and three chiral superfields. The symmetries of $\mathcal{N} = 4$ SYM are

- superconformal group $SU(2, 2|4)$;
- gauge group $SU(N_c)$;
- conformal group $SO(4, 2)$;
- R-symmetry $SU(4) \sim SO(6)$.

In particular, $SU(2, 2|4)$ includes 16 real supersymmetric generators Q^a_α and 16 real superconformal generators S^α_a. On the other hand, type IIB superstring theory in a $AdS_5 \times S^5$ background is a 10-dimensional theory where five of its dimensions are compactified on a 5-sphere S^5. For a qualitative description of some main ideas about the AdS/CFT correspondence, we consider a set of coincident D-branes in type IIB superstring theory; the theory contains an open string ending on the branes which interacts with closed strings. If we take the low energy limit, where the string lenght goes to zero, the open string does not interact anymore with the closed string and the system is decoupled: As a consequence, we find the 4-dimensional $\mathcal{N} = 4$ SYM theory living on the brane and a free gravity theory outside. It is possible to consider the same system from a different point of view: D-branes are massive charged objects and a set of these massive objects can be thought of as a generalization of a black hole. In the low energy limit, we find again two decoupled pieces, which are free gravity on one side and type IIB supergravity on $AdS_5 \times S^5$ on the other side, which is the low energy limit of type IIB superstring theory. In both points of view, we have found two decoupled theories, and in both case one of them is free gravity: it is so immediate to identify the second system appearing in both description. We are thus led to the conjecture that at all energies $\mathcal{N} = 4$ SYM is dual to type IIB superstring on a $AdS_5 \times S^5$ background. A first obvious check concern symmetries on the two sides: It is easy to see that these two theories share the same symmetry groups. In fact, $SO(4, 2)$ is the conformal group of a $D = 4$ CFT and it is also the isometry group of AdS_5; moreover, global $SO(6)$ is isomorphic to S^5. The 4-dimensional superconformal $\mathcal{N} = 4$ SYM lives on the boundary of AdS_5 space; the matching of the symmetries is an hint of duality. It is also possible to find a matching of the parameters of the theories. The string theory parameters are the radius r of both AdS_5 and S^5 and the string theory coupling g_s, while the CFT ones are the SYM coupling g_{YM} and N_c; we also define the 't Hooft coupling $\lambda = \frac{g_{YM}^2 N_c}{(4\pi)^2}$. These parameters can be matched into each other:

$$\begin{aligned} \frac{r^4}{l_s^4} &= g_{YM}^2 N_c, \\ 4\pi g_s &= \frac{\lambda}{N_c}, \end{aligned} \tag{3.45}$$

where l_s is the string lenght. Note that the perturbation analysis of the field theory can be trusted when the 't Hooft coupling is small while, on the other hand, the classical gravity description becomes reliable when g_s goes to zero and $\frac{r^4}{l_s^4}$ is large. It is clear from eq.(3.45) that these two regime are incompatible: In fact the AdS/CFT is a weak/strong duality. To complete the picture of the correspondence we need a map between the observables in the two theories and a prescription for comparing physical quantities and amplitudes. In AdS/CFT, a field in AdS space is associated with an operator in the CFT with the same quantum numbers and they know about each other via boundary couplings [22].

The original formulation of the AdS/CFT correspondence was later extended to other theories with less symmetries and to theories living in a different number of spacetime dimensions. The most general AdS/CFT correspondence is a duality relating any CFT in D dimensions to a gravity theory on AdS_{D+1}. However, we have to remember that this is only a conjecture: for the moment, the most reasonable way to operate is to separately consider each theory and to study its properties, trying to find some similarities with other models.

In this thesis, we will not study in depth the AdS/CFT correspondence or even look for such dualities: We will focus on a supergravity theory in Chapter 4 and in Chapter 5 we will compute some scattering amplitudes in $\mathcal{N} = 2$ SCQCD. However, the AdS/CFT correspondence could be a very useful tool in order to deeply understand some properties of these theories.

Gravity and supergravity theories in diverse dimensions in the presence of a boundary have been studied in different contexts from the early '70 on [27, 28, 29, 30]. In particular, some works dealing with supergravity studied through the geometric approach mainly focus on the symmetries of the theories and they analyze the way to preserve the supersymmetry invariance when a non-trivial boundary is added. Although the final result shows a supersymmetric theory composed by a bulk and a boundary, so allowing all the possible studies concerning AdS/CFT, this further step is not performed since it is very complicated and also the theories are commonly considered as classical supergravity. However there are some interesting results which can prepare the ground for further studies like these: One example is the so-called “holographic renormalization”, consisting of the inclusion of appropriate counterterms at the boundary of a supergravity theory, with a consequent elimination of the divergences of the bulk metric near the boundary (see for instance [31] and references therein).

Regarding superconformal theories, at strong coupling the dual string description of the theory $\mathcal{N} = 2$ SCQCD seems much more problematic than $\mathcal{N} = 4$ SYM. There are some proposal for the dual string/supergravity background which turn out to be either singular [32] or related to non critical models [33]. Any advancement on the field theory side might help claryfing the correct properties of the gravitational description.

Chapter 4

Generalized $\mathcal{N} = 1$, $D = 4$ AdS-Lorentz deformed supergravity on a manifold with boundary

This chapter shows the most relevant features of the study of a particular supergravity theory in [34], a paper published in the "European Physical Journal Plus". Before going in depth, it is important to briefly contextualize this topic underlining some recent developments.

As we saw in Section 3.5, the study of the relations between the bulk and the boundary of a supergravity theory could be relevant in the context of AdS/CFT. In relevant works such as [35, 36, 37, 38, 39], the inclusion of boundary terms and counterterms to AdS gravity was studied and, on the other hand, many authors [40, 41, 42, 43, 44, 45] considered it in the context of supergravity theories, by adopting different approaches. The results of these works pointed out to the conclusion that, in order to restore all the invariances of a supergravity Lagrangian with cosmological constant on a manifold with a non-trivial boundary (that is when the boundary is not thought as set at infinity), one needs to add topological (i.e. boundary) contributions to the theory, also providing the counterterms necessary for regularizing the action. More recently, in [17] the authors constructed the $\mathcal{N} = 1$ and $\mathcal{N} = 2$, $D = 4$ supergravity theories with negative cosmological constant in the presence a non-trivial boundary in a geometric framework (extending to Superspace the geometric approach of the previous works). In particular, the authors found that the supersymmetry invariance of the full Lagrangian (understood as bulk plus boundary contributions) is recovered with the introduction of a supersymmetric extension of the Gauss-Bonnet term. The final Lagrangian is written down as a sum of quadratic terms in super field-strengths, reproducing the MacDowell-Mansouri action [46]. Lately, in [18] the authors explored the supersymmetry invariance of a particular supergravity theory in the presence of a non-trivial boundary, following the prescription of [17]. Specifically, they presented the explicit construction of a geometric bulk Lagrangian based on an enlarged superalgebra, known as AdS-Lorentz superalgebra, showing that, also in this case, the supersymmetric extension of a Gauss-Bonnet like term is required to restore the supersymmetry invariance of the complete theory. In analogy to the result of [17], they obtained that the full action can be finally written as a MacDowell-Mansouri type action.

Driven by the results of [17, 18, 19], in this chapter we explore the supersymmetry invariance of a supergravity theory we will refer to as $D = 4$ generalized AdS-Lorentz deformed supergravity, in the geometric approach in the presence of a non-trivial boundary. We give a rapid introduction of the main concepts cited before where necessary. Chapter 6 contains our conclusions and possible future developments, while in Appendix A.1 we collect some useful formulas in $D = 4$ spacetime dimensions.

4.1 AdS-Lorentz superalgebras

In this section, we show some features of the AdS-Lorentz superalgebra and of its minimal generalization; the last one will be the physical scenario of our study.

We want to study the supersymmetry invariance of a supergravity theory when a non-trivial boundary is introduced. A non-trivial boundary is a boundary not set to infinity and on it the fields of the theory generally do not vanish. As we saw from Section 3.5, this topic is very interesting for the AdS/CFT, in order to study a supergravity theory and a supersymmetric conformal field theory which are dual to each other. It is reasonable to choose a supergravity theory on an AdS background: In an algebraic point of view, the easiest theory in $D = 4$ is the $\mathcal{N} = 1$ AdS_4, whose Lie superalgebra is called $\mathfrak{osp}(4|1)$. The bosonic subgroup associated with the Lie superalgebra $\mathfrak{osp}(4|1)$ is isomorphic to $Sp(4) \times O(1)$, where $Sp(4) \sim SO(3,2)$ is the isometry group of AdS_4 and $O(1)$ is the R-symmetry. We read from [47] the (anti)commuting relations of $\mathfrak{osp}(4|1)$

$$\begin{aligned}
&[J_{ab}, J_{cd}] = \eta_{ad} J_{bc} + \eta_{bc} J_{ad} - \eta_{ac} J_{bd} - \eta_{bd} J_{ac}, \\
&[J_{ab}, P_c] = \eta_{bc} P_a - \eta_{ac} P_b, \\
&[P_a, P_b] = J_{ab}, \\
&[J_{ab}, Q_\alpha] = -\frac{1}{2} \left(\gamma_{ab} Q\right)_\alpha, \\
&[P_a, Q_\alpha] = -\frac{1}{2} \left(\gamma_a Q\right)_\alpha, \\
&\{Q_\alpha, Q_\beta\} = -\frac{1}{2} \left(\left(\gamma^{ab} C\right)_{\alpha\beta} J_{ab} - 2 \left(\gamma^a C\right)_{\alpha\beta} P_a \right),
\end{aligned} \tag{4.1}$$

where $\{J_{ab}, P_a\}$ are the generators of AdS_4 seen in eq.(3.36) and Q_α is the generator of supersymmetry (for definitions and relations of gamma matrices, see Appendix A.1). In the cotangent space, we have the 1-form fields $\{\omega^{ab}, V^a, \psi^\alpha\}$ which are dual to the previous ones in such a way

$$\omega^{ab}(J_{cd}) = \delta^{ab}_{cd}, \qquad V^a(P_b) = \delta^a_b, \qquad \psi(Q) = \mathbb{1}, \tag{4.2}$$

where we remove the spinor index for convenience. Taken the length dimensions $[\omega^{ab}] = L^0$, $[V^a] = L^1$, $[\psi] = L^{\frac{1}{2}}$ and after introducing the cosmological constant e defined in eq.(3.35), the relations of eq.(4.1) become the following Maurer-

Cartan equations

$$
\begin{aligned}
&D_\omega \omega^{ab} + 4e^2\, V^a \wedge V^b + e\, \bar{\psi} \wedge \gamma^{ab}\psi = 0, \\
&D_\omega V^a - \frac{1}{2}\bar{\psi} \wedge \gamma^a \psi = 0, \\
&D_\omega \psi + e\, V^a \wedge \gamma_a \psi = 0,
\end{aligned}
\tag{4.3}
$$

with the quantities $D_\omega \omega^{ab} = d\omega^{ab} + \omega^{ac} \wedge \omega_c{}^b$, $D_\omega V^a = dV^a + \omega^a{}_c \wedge V^c$ and $D_\omega \psi = d\psi + \frac{1}{4}\omega^{ab} \wedge \gamma_{ab}\psi$. The Maurer-Cartan equations lead to the definitions of super field-strengths

$$
\begin{aligned}
&R^{ab} = \mathcal{R}^{ab} + 4e^2\, V^a \wedge V^b + e\, \bar{\psi} \wedge \gamma^{ab}\psi, \\
&R^a = D_\omega V^a - \frac{1}{2}\bar{\psi} \wedge \gamma^a \psi, \\
&\Psi = \rho + e\, V^a \wedge \gamma_a \psi,
\end{aligned}
\tag{4.4}
$$

where we call $\mathcal{R}^{ab} = D_\omega \omega^{ab}$ and $\rho = D_\omega \psi$. It is possible to construct a Lagrangian of $\mathcal{N} = 1$ AdS_4 by following some geometric rules we will use in Subsection 4.3.1; the bulk Lagrangian of that theory is

$$
\begin{aligned}
\mathcal{L} =& \epsilon_{abcd} \left(\mathcal{R}^{ab} \wedge V^c \wedge V^d + 2e^2\, V^a \wedge V^b \wedge V^c \wedge V^d + 2e\, \bar{\psi} \wedge \gamma^{ab}\psi \wedge V^c \wedge V^d \right) \\
& + 4\bar{\psi} \wedge \gamma_a \gamma_5 \rho \wedge V^a.
\end{aligned}
\tag{4.5}
$$

Specifically, this Lagrangian describes $\mathcal{N} = 1$ AdS_4 theory with Lie superalgebra $\mathfrak{osp}(4|1)$, without a finite boundary: As usual, when a finite boundary is not defined (in other words, the boundary is set to infinity), all the fields of the theory vanish at infinity.

Given $\mathfrak{osp}(4|1)$, we can add a further generator Z_{ab} (with $Z_{ab} = -Z_{ba}$) whose behavior is similar to J_{ab} behavior: The new larger superalgebra obtained is called AdS-Lorentz superalgebra. It is semisimple and historically it was obtained as a deformation of the so-called Maxwell supersymmetries [48, 49]; we do not take care of the Maxwell superalgebra in this thesis. The (anti)commuting relations of the AdS-Lorentz superalgebra read:

$$\begin{aligned}
&[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc},\\
&[J_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc},\\
&[Z_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc},\\
&[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b,\\
&[P_a, P_b] = Z_{ab},\\
&[Z_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b,\\
&[J_{ab}, Q_\alpha] = -\frac{1}{2}\left(\gamma_{ab}Q\right)_\alpha,\\
&[P_a, Q_\alpha] = -\frac{1}{2}\left(\gamma_a Q\right)_\alpha,\\
&[Z_{ab}, Q_\alpha] = -\frac{1}{2}\left(\gamma_{ab}Q\right)_\alpha,\\
&\{Q_\alpha, Q_\beta\} = -\frac{1}{2}\left(\left(\gamma^{ab}C\right)_{\alpha\beta} Z_{ab} - 2\left(\gamma^a C\right)_{\alpha\beta} P_a\right).
\end{aligned} \tag{4.6}$$

We observe that the Lorentz-type algebra generated by $\{J_{ab}, Z_{ab}\}$ is a subalgebra of eq.(4.6). In the cotangent space, we have the 1-forms $\{\omega^{ab}, k^{ab}, V^a, \psi\}$, with k^{ab} dual to Z_{ab} (so that $k^{ab}(Z_{cd}) = \delta^{ab}_{cd}$), with length dimension $[k^{ab}] = L^0$, and the others follow the relations of eq.(4.2). Through the same method used for $\mathfrak{osp}(4|1)$, for the AdS-Lorentz superalgebra we find the super field-strengths

$$\begin{aligned}
&\mathcal{R}^{ab} = D_\omega \omega^{ab},\\
&R^a = D_\omega V^a + k^a{}_c \wedge V^c - \frac{1}{2}\bar{\psi} \wedge \gamma^a \psi,\\
&F^{ab} = D_\omega k^{ab} + k^{ac} \wedge k_c{}^b + 4e^2\, V^a \wedge V^b + e\,\bar{\psi} \wedge \gamma^{ab}\psi,\\
&\Psi = D_\omega \psi + \frac{1}{4}k^{ab} \wedge \gamma_{ab}\psi + e\, V^a \wedge \gamma_a \psi,
\end{aligned} \tag{4.7}$$

with $D_\omega k^{ab} = dk^{ab} + 2\omega^{ac} \wedge k_c{}^b$ and the other exterior covariant derivatives can be read below eq.(4.3).

A formal way to derive the AdS-Lorentz superalgebra is to make a torsion deformation of $\mathfrak{osp}(4|1)$; we show that process in the cotangent space. We start with $\mathfrak{osp}(4|1)$ described by the Maurer-Cartan equations of eq.(4.3). On the same lines of what was done in [50] in the case of $\mathfrak{osp}(32|1)$, we can now exploit the freedom of redefining the Lorentz spin connection in $\mathfrak{osp}(4|1)$ by the addition of a new antisymmetric tensor 1-form B^{ab} (carrying length dimension $[B^{ab}] = L^0$) as follows:

$$\omega^{ab} \to \hat{\omega}^{ab} = \omega^{ab} - B^{ab}. \tag{4.8}$$

We observe that such a redefinition is always possible and also implies a change of the torsion 2-form, that is why we will talk about a "torsion deformation" of $\mathfrak{osp}(4|1)$. After having performed the redefinition eq.(4.8) of the spin connection, if we rename $\hat{\omega}^{ab}$ as ω^{ab}, the Maurer-Cartan equations eq.(4.3) take the following form:

$$
\begin{aligned}
&D_\omega \omega^{ab} + D_\omega B^{ab} + B^{ac} \wedge B_c{}^b + 4e^2\, V^a \wedge V^b + e\, \bar{\psi} \wedge \gamma^{ab}\psi = 0, \\
&D_\omega V^a + B^a{}_c \wedge V^c - \frac{1}{2}\bar{\psi} \wedge \gamma^a \psi = 0, \\
&D_\omega \psi + \frac{1}{4} B^{ab} \wedge \gamma_{ab}\psi + e\, V^a \wedge \gamma_a \psi = 0.
\end{aligned}
\tag{4.9}
$$

Now, if we further require, as an extra condition, the Lorentz spin connection ω^{ab} to satisfy $D_\omega \omega^{ab} = 0$, corresponding to a Minkowski background, then the first equation in eq.(4.9) splits into these two equations

$$
\begin{aligned}
&\mathcal{R}^{ab} = 0, \\
&D_\omega B^{ab} + B^{ac} \wedge B_c{}^b + 4e^2\, V^a \wedge V^b + e\, \bar{\psi} \wedge \gamma^{ab}\psi = 0,
\end{aligned}
\tag{4.10}
$$

where the last one defines the Maurer-Cartan equation for the tensor 1-form field B^{ab}. Observe that the superalgebra obtained from $\mathfrak{osp}(4|1)$ through the procedure written above is not isomorphic to $\mathfrak{osp}(4|1)$ because of the extra constraint $D_\omega \omega^{ab} = 0$, which implies eq.(4.10), imposed on the Maurer-Cartan equations eq.(4.9). On the other hand, renaming B^{ab} as k^{ab}, we can see that the four Maurer-Cartan equations obtained exactly correspond to those of the AdS-Lorentz superalgebra previously introduced, namely the super field-strengths of eq.(4.7) set to zero. We can thus conclude that, at the price of introducing the (torsion) field k^{ab} fulfilling eq.(4.10) renaming B^{ab} as k^{ab}, $\mathfrak{osp}(4|1)$ can be mapped into the AdS-Lorentz superalgebra, where the spin connection ω^{ab} is identified with the Lorentz connection of a 4-dimensional Minkowski spacetime with vanishing Lorentz curvature (albeit with a modification of the supertorsion and of the gravitino super field-strength). Thus, we can say that the AdS-Lorentz superalgebra can also be viewed as a "torsion-deformed" version of $\mathfrak{osp}(4|1)$. This was already observed in [19], but it had not been explicitly derived yet.

Starting from the AdS-Lorentz superalgebra, we briefly introduce its minimal generalization; it is the object of study in that chapter. The minimal generalization of the AdS-Lorentz superalgebra of eq.(4.6) contains one more spinor charge and also two additional bosonic charges; it can be found in [47]. In the sequel, we will refer to this minimal generalization of the AdS-Lorentz superalgebra as the generalized AdS-Lorentz superalgebra. The generators of the generalized AdS-Lorentz superalgebra are given by the set $\{J_{ab}, P_a, \tilde{Z}_a, \tilde{Z}_{ab}, Z_{ab}, Q_\alpha, \Sigma_\alpha\}$ and they fulfill the following relations:

$$
\begin{aligned}
&[J_{ab}, J_{cd}] = \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac} + \eta_{ad}J_{bc}, && [P_a, P_b] = Z_{ab}, \\
&[Z_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, && \left[\tilde{Z}_a, P_b\right] = \tilde{Z}_{ab}, \\
&[J_{ab}, Z_{cd}] = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, && \left[\tilde{Z}_a, \tilde{Z}_b\right] = Z_{ab}, \\
&\left[J_{ab}, \tilde{Z}_{cd}\right] = \eta_{bc}\tilde{Z}_{ad} - \eta_{ac}\tilde{Z}_{bd} - \eta_{bd}\tilde{Z}_{ac} + \eta_{ad}\tilde{Z}_{bc}, && [J_{ab}, Q_\alpha] = -\frac{1}{2}\left(\gamma_{ab}Q\right)_\alpha, \\
&\left[\tilde{Z}_{ab}, \tilde{Z}_{cd}\right] = \eta_{bc}Z_{ad} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac} + \eta_{ad}Z_{bc}, && [P_a, Q_\alpha] = -\frac{1}{2}\left(\gamma_a\Sigma\right)_\alpha, \\
&\left[\tilde{Z}_{ab}, Z_{cd}\right] = \eta_{bc}\tilde{Z}_{ad} - \eta_{ac}\tilde{Z}_{bd} - \eta_{bd}\tilde{Z}_{ac} + \eta_{ad}\tilde{Z}_{bc}, && \left[\tilde{Z}_{ab}, Q_\alpha\right] = -\frac{1}{2}\left(\gamma_{ab}\Sigma\right)_\alpha, \\
&\{Q_\alpha, Q_\beta\} = -\frac{1}{2}\left(\left(\gamma^{ab}C\right)_{\alpha\beta}\tilde{Z}_{ab} - 2\left(\gamma^a C\right)_{\alpha\beta}P_a\right), && \left[\tilde{Z}_a, Q_\alpha\right] = -\frac{1}{2}\left(\gamma_a Q\right)_\alpha, \\
&\{Q_\alpha, \Sigma_\beta\} = -\frac{1}{2}\left(\left(\gamma^{ab}C\right)_{\alpha\beta}Z_{ab} - 2\left(\gamma^a C\right)_{\alpha\beta}\tilde{Z}_a\right), && [Z_{ab}, Q_\alpha] = -\frac{1}{2}\left(\gamma_{ab}Q\right)_\alpha, \\
&\{\Sigma_\alpha, \Sigma_\beta\} = -\frac{1}{2}\left(\left(\gamma^{ab}C\right)_{\alpha\beta}\tilde{Z}_{ab} - 2\left(\gamma^a C\right)_{\alpha\beta}P_a\right), && [P_a, \Sigma_\alpha] = -\frac{1}{2}\left(\gamma_a Q\right)_\alpha, \\
&[J_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, && [J_{ab}, \Sigma_\alpha] = -\frac{1}{2}\left(\gamma_{ab}\Sigma\right)_\alpha, \\
&[Z_{ab}, P_c] = \eta_{bc}P_a - \eta_{ac}P_b, && \left[\tilde{Z}_a, \Sigma_\alpha\right] = -\frac{1}{2}\left(\gamma_a\Sigma\right)_\alpha, \\
&\left[\tilde{Z}_{ab}, P_c\right] = \eta_{bc}\tilde{Z}_a - \eta_{ac}\tilde{Z}_b, && \left[\tilde{Z}_{ab}, \Sigma_\alpha\right] = -\frac{1}{2}\left(\gamma_{ab}Q\right)_\alpha, \\
&\left[J_{ab}, \tilde{Z}_c\right] = \eta_{bc}\tilde{Z}_a - \eta_{ac}\tilde{Z}_b, && [Z_{ab}, \Sigma_\alpha] = -\frac{1}{2}\left(\gamma_{ab}\Sigma\right)_\alpha. \\
&\left[\tilde{Z}_{ab}, \tilde{Z}_c\right] = \eta_{bc}P_a - \eta_{ac}P_b, && \\
&\left[Z_{ab}, \tilde{Z}_c\right] = \eta_{bc}\tilde{Z}_a - \eta_{ac}\tilde{Z}_b, &&
\end{aligned}
\tag{4.11}
$$

As we can see above, a new Majorana spinor charge appears. The introduction of a second spinorial generator can also be found, for example, in [51, 52, 53, 50, 54] (see also [19]) and [55] in the supergravity and superstrings contexts, respectively. It is possible to show that by setting $\tilde{Z}_a \to 0$ the Jacobi identities of eq.(4.11) are still fulfilled. We also observe, as it was already pointed out in [47], that the generalized AdS-Lorentz algebra $\{J_{ab}, P_a, \tilde{Z}_a, \tilde{Z}_{ab}, Z_{ab}\}$ and the algebra $\{J_{ab}, P_a, Z_{ab}\}$ are bosonic subalgebras of eq.(4.11).

To briefly sum up, we started with $\mathfrak{osp}(4|1)$, which is the first supersymmetric theory in a 4-dimensional AdS background; then we made a torsion deformation by introducing a new field and we consequently got the AdS-Lorentz superalgebra. After that, we added two more bosonic fields and one more fermionic field to the superalgebra and what we got is the generalized AdS-Lorentz superalgebra: Our purpose is to build a bulk supersymmetric Lagrangian of this theory and, after the introduction of a non-trivial boundary, to restore the supersymmetry invariance of the full Lagrangian. The result could be compared with the one of AdS-Lorentz superalgebra.

It is interesting to study a generalization of a theory like AdS-Lorentz and, in particular, to analyze the contributions of the extra fields to the final theory. In other words, when we introduce some extra fields to the theory, they could appear in the boundary terms but also in the bulk Lagrangian. In particular, the presence of the extra fields in the boundary could be useful in the context of the AdS/CFT duality (we can see [56] and references therein).

In the following section, we introduce in detail the generalized AdS-Lorentz superalgebra and its relation with the starting superalgebra $\mathfrak{osp}(4|1)$.

4.2 Relation between the generalized AdS-Lorentz superalgebra and $\mathfrak{osp}(4|1)$

As we have done in the AdS-Lorentz case, we now describe the generalized AdS-Lorentz superalgebra eq.(4.11) in its dual Maurer-Cartan formulation.

We introduce the set of 1-form fields $\{\omega^{ab}, V^a, \tilde{h}^a, \tilde{k}^{ab}, k^{ab}, \psi, \xi\}$ dual to the generators $\{J_{ab}, P_a, \tilde{Z}_a, \tilde{Z}_{ab}, Z_{ab}, Q, \Sigma\}$, that is

$$\begin{aligned} &\omega^{ab}(J_{cd}) = \delta^{ab}_{cd}, \qquad V^a(P_b) = \delta^a_b, \qquad \tilde{h}^a(\tilde{Z}_b) = \delta^a_b, \qquad \tilde{k}^{ab}(\tilde{Z}_{cd}) = \delta^{ab}_{cd}, \\ &k^{ab}(Z_{cd}) = \delta^{ab}_{cd}, \qquad \psi(Q) = \mathbb{1}, \qquad \xi(\Sigma) = \mathbb{1}, \end{aligned} \tag{4.12}$$

where both ψ and ξ are Majorana spinors. These 1-form fields have length dimensions $[\omega^{ab}] = L^0$, $[V^a] = L^1$, $[\tilde{h}^a] = L^1$, $[\tilde{k}^{ab}] = L^0$, $[k^{ab}] = L^0$, $[\psi] = L^{1/2}$, and $[\xi] = L^{1/2}$. The Maurer-Cartan equations describing the generalized AdS-Lorentz superalgebra of eq.(4.11) are:

$$D_\omega \omega^{ab} = 0, \tag{4.13a}$$

$$D_\omega V^a + k^a{}_b \wedge V^b + \tilde{k}^a{}_b \wedge \tilde{h}^b - \frac{1}{2}\bar{\psi} \wedge \gamma^a \psi - \frac{1}{2}\bar{\xi} \wedge \gamma^a \xi = 0, \tag{4.13b}$$

$$D_\omega \tilde{h}^a + \tilde{k}^a{}_b \wedge V^b + k^a{}_b \wedge \tilde{h}^b - \bar{\psi} \wedge \gamma^a \xi = 0, \tag{4.13c}$$

$$D_\omega \tilde{k}^{ab} + 2k^a{}_c \wedge \tilde{k}^{cb} + 8e^2\, V^a \wedge \tilde{h}^b + e\left(\bar{\psi} \wedge \gamma^{ab}\psi + \bar{\xi} \wedge \gamma^{ab}\xi\right) = 0, \tag{4.13d}$$

$$D_\omega k^{ab} + \tilde{k}^a{}_c \wedge \tilde{k}^{cb} + k^a{}_c \wedge k^{cb} + 4e^2\left(V^a \wedge V^b + \tilde{h}^a \wedge \tilde{h}^b\right) + 2e\,\bar{\psi} \wedge \gamma^{ab}\xi = 0, \tag{4.13e}$$

$$D_\omega \psi + \frac{1}{4}k^{ab} \wedge \gamma_{ab}\psi + \frac{1}{4}\tilde{k}^{ab} \wedge \gamma_{ab}\xi + e\left(V^a \wedge \gamma_a \xi + \tilde{h}^a \wedge \gamma_a \psi\right) = 0, \tag{4.13f}$$

$$D_\omega \xi + \frac{1}{4}k^{ab} \wedge \gamma_{ab}\xi + \frac{1}{4}\tilde{k}^{ab} \wedge \gamma_{ab}\psi + e\left(V^a \wedge \gamma_a \psi + \tilde{h}^a \wedge \gamma_a \xi\right) = 0. \tag{4.13g}$$

We can then define the generalized AdS-Lorentz Lie algebra valued 2-form supercurvatures (also called super field-strengths) as follows (see also [47]):

$$\mathcal{R}^{ab} = D_\omega \omega^{ab}, \tag{4.14a}$$

$$R^a = D_\omega V^a + k^a{}_b \wedge V^b + \tilde{k}^a{}_b \wedge \tilde{h}^b - \frac{1}{2}\bar{\psi} \wedge \gamma^a \psi - \frac{1}{2}\bar{\xi} \wedge \gamma^a \xi, \tag{4.14b}$$

$$\tilde{H}^a = D_\omega \tilde{h}^a + \tilde{k}^a{}_b \wedge V^b + k^a{}_b \wedge \tilde{h}^b - \bar{\psi} \wedge \gamma^a \xi, \tag{4.14c}$$

$$\tilde{F}^{ab} = D_\omega \tilde{k}^{ab} + 2k^a{}_c \wedge \tilde{k}^{cb} + 8e^2\, V^a \wedge \tilde{h}^b + e\left(\bar{\psi} \wedge \gamma^{ab}\psi + \bar{\xi} \wedge \gamma^{ab}\xi\right), \tag{4.14d}$$

$$F^{ab} = D_\omega k^{ab} + \tilde{k}^a{}_c \wedge \tilde{k}^{cb} + k^a{}_c \wedge k^{cb} + 4e^2\left(V^a \wedge V^b + \tilde{h}^a \wedge \tilde{h}^b\right) + 2e\, \bar{\psi} \wedge \gamma^{ab}\xi, \tag{4.14e}$$

$$\Psi = D_\omega \psi + \frac{1}{4}k^{ab} \wedge \gamma_{ab}\psi + \frac{1}{4}\tilde{k}^{ab} \wedge \gamma_{ab}\xi + e\left(V^a \wedge \gamma_a \xi + \tilde{h}^a \wedge \gamma_a \psi\right), \tag{4.14f}$$

$$\Xi = D_\omega \xi + \frac{1}{4}k^{ab} \wedge \gamma_{ab}\xi + \frac{1}{4}\tilde{k}^{ab} \wedge \gamma_{ab}\psi + e\left(V^a \wedge \gamma_a \psi + \tilde{h}^a \wedge \gamma_a \xi\right). \tag{4.14g}$$

Now, considering the Maurer-Cartan equations eq.(4.3) of $\mathfrak{osp}(4|1)$, we observe that, by redefining

$$\begin{cases} \omega^{ab} \to \hat{\omega}^{ab} = \omega^{ab} - \tilde{B}^{ab} - B^{ab}, \\ V^a \to \hat{V}^a = V^a - \tilde{B}^a, \\ \psi \to \hat{\psi} = \psi - \eta, \end{cases} \tag{4.15}$$

where both $\tilde{B}^{ab}$ and B^{ab} are antisymmetric tensor 1-forms carrying length dimension zero, $\tilde{B}^a$ is a 1-form carrying length dimension 1, and η is a spinor 1-form carrying length dimension 1/2, if we then rename $\hat{\omega}^{ab} \Rightarrow \omega^{ab}$, $\hat{V}^a \Rightarrow V^a$, and $\hat{\psi} \Rightarrow \psi$, the Maurer-Cartan equations eq.(4.3) take the following form:

$$\begin{aligned} &D_\omega \omega^{ab} + D_\omega \tilde{B}^{ab} + D_\omega B^{ab} + \tilde{B}^a{}_c \wedge \tilde{B}^{cb} + 2B^a{}_c \wedge \tilde{B}^{cb} + B^a{}_c \wedge B^{cb} \\ &+ 4e^2\left(V^a \wedge V^b + 2V^a \wedge \tilde{B}^b + \tilde{B}^a \wedge \tilde{B}^b\right) \\ &+ e\left(\bar{\psi} \wedge \gamma^{ab}\psi + 2\bar{\psi} \wedge \gamma^{ab}\eta + \bar{\eta} \wedge \gamma^{ab}\eta\right) = 0, \end{aligned} \tag{4.16a}$$

$$\begin{aligned} &D_\omega V^a + D_\omega \tilde{B}^a + B^a{}_b \wedge V^b + B^a{}_b \wedge \tilde{B}^b + \tilde{B}^a{}_b \wedge V^b + \tilde{B}^a{}_b \wedge \tilde{B}^b \\ &- \frac{1}{2}\bar{\psi} \wedge \gamma^a \psi - \bar{\psi} \wedge \gamma^a \eta - \frac{1}{2}\bar{\eta} \wedge \gamma^a \eta = 0, \end{aligned} \tag{4.16b}$$

$$\begin{aligned} &D_\omega \psi + D_\omega \eta + \frac{1}{4}B^{ab} \wedge \gamma_{ab}\psi + \frac{1}{4}B^{ab} \wedge \gamma_{ab}\eta + \frac{1}{4}\tilde{B}^{ab} \wedge \gamma_{ab}\psi + \frac{1}{4}\tilde{B}^{ab} \wedge \gamma_{ab}\eta \\ &+ e\left(V^a \wedge \gamma_a \psi + V^a \wedge \gamma_a \eta + \tilde{B}^a \wedge \gamma_a \psi + \tilde{B}^a \wedge \gamma_a \eta\right) = 0. \end{aligned} \tag{4.16c}$$

Then, if we further require the Lorentz spin connection ω^{ab} to satisfy $D_\omega \omega^{ab} = 0$ (corresponding to a Minkowski background), together with the following new extra conditions:

$$D_\omega \tilde{B}^a + \tilde{B}^a{}_b \wedge V^b + B^a{}_b \wedge \tilde{B}^b - \bar{\psi} \wedge \gamma^a \eta = 0, \tag{4.17a}$$

$$D_\omega \tilde{B}^{ab} + 2B^a{}_c \wedge \tilde{B}^{cb} + 8e^2\, V^a \wedge \tilde{B}^b + e\left(\bar{\psi} \wedge \gamma^{ab}\psi + \bar{\eta} \wedge \gamma^{ab}\eta\right) = 0, \tag{4.17b}$$

$$\begin{aligned} &D_\omega B^{ab} + \tilde{B}^a{}_c \wedge \tilde{B}^{cb} + B^a{}_c \wedge B^{cb} \\ &+ 4e^2\left(V^a \wedge V^b + \tilde{B}^a \wedge \tilde{B}^b\right) + 2e\, \bar{\psi} \wedge \gamma^{ab}\eta = 0, \end{aligned} \tag{4.17c}$$

$$D_\omega \eta + \frac{1}{4} B^{ab} \wedge \gamma_{ab}\eta + \frac{1}{4}\tilde{B}^{ab} \wedge \gamma_{ab}\psi + e\left(V^a \wedge \gamma_a \psi + \tilde{B}^a \wedge \gamma_a \eta\right) = 0, \tag{4.17d}$$

which define the Maurer-Cartan equation for the 1-form fields $\tilde{B}^a$, $\tilde{B}^{ab}$, B^{ab}, and η, one can easily prove that, after having redefined $\tilde{B}^a \Rightarrow \tilde{h}^a$, $\tilde{B}^{ab} \Rightarrow \tilde{k}^{ab}$, $B^{ab} \Rightarrow k^{ab}$, and $\eta \Rightarrow \xi$, the superalgebra we end up with is exactly the generalized minimal AdS-Lorentz one, with Maurer-Cartan equations given by eq.(4.13a)-eq.(4.13g). We observe that, again, the superalgebra obtained from $\mathfrak{osp}(4|1)$ through the procedure written above (namely, in this case, the generalized AdS-Lorentz superalgebra) is not isomorphic to $\mathfrak{osp}(4|1)$, because of the extra constraints $D_\omega \omega^{ab} = 0$, eq.(4.17a)-eq.(4.17d) imposed on the Maurer-Cartan equations eq.(4.9). We note that these extra conditions correspond to particular choices performed on eq.(4.16a)-eq.(4.16c). One can then define the AdS-Lorentz super field-strengths as given in eq.(4.14a)-eq.(4.14g). Thus, we can conclude that, at the price of introducing the extra 1-form fields $\tilde{h}^a$, $\tilde{k}^{ab}$, k^{ab}, and ξ (satisfying eq.(4.17a), eq.(4.17b), eq.(4.17c), and eq.(4.17d), respectively), $\mathfrak{osp}(4|1)$ can be mapped into the generalized minimal AdS-Lorentz superalgebra, where the spin connection is identified with the Lorentz connection of a Minkowski spacetime with vanishing Lorentz curvature (furthermore, we also have a modification of the supertorsion and of the gravitino super field-strength). In this sense, the generalized minimal AdS-Lorentz superalgebra can be interpreted as a peculiar torsion deformation of $\mathfrak{osp}(4|1)$.

Some comments are in order. First of all, we observe that the AdS-Lorentz and the generalized minimal AdS-Lorentz superalgebras, which, as we have seen above, correspond to different torsion deformations of $\mathfrak{osp}(4|1)$, can also be both obtained from $\mathfrak{osp}(4|1)$ by performing the so-called S-expansion procedure, as it was done in [47]. It consists in a particular expansion process [57], of the AdS superalgebra of eq.(4.1) [47, 58, 59, 60]. In group theory, the S-expansion method is based on combining the multiplication law of a semigroup S with the structure constants of a Lie algebra $\mathfrak{g}$, in such a way to end up with a new, larger, Lie algebra $\mathfrak{g}_S = S \times \mathfrak{g}$, that is called the S-expanded algebra (see also [61] for an analytic method for performing S-expansion). In particular, the semigroup leading from $\mathfrak{osp}(4|1)$ to the AdS-Lorentz superalgebra eq.(4.6) is the abelian semigroup $S^{(2)}_{\mathcal{M}} = \{\lambda_0, \lambda_1, \lambda_2\}$ (according with the notation of [47]), whose elements obey the multiplication laws

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{if } \alpha+\beta \leq 2, \\ \lambda_{\alpha+\beta-2}, & \text{if } \alpha+\beta > 2. \end{cases} \tag{4.18}$$

Similarly, the semigroup leading from $\mathfrak{osp}(4|1)$ to the generalized minimal AdS-Lorentz superalgebra of eq.(4.11) (again, according with the notation of [47]) is the abelian semigroup $S^{(4)}_{\mathcal{M}} = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4\}$, whose elements obey the following multiplication laws:

$$\lambda_\alpha \lambda_\beta = \begin{cases} \lambda_{\alpha+\beta}, & \text{if } \alpha+\beta \leq 4, \\ \lambda_{\alpha+\beta-4}, & \text{if } \alpha+\beta > 4. \end{cases} \tag{4.19}$$

Then, interestingly enough, we can conclude that semigroups of the type $S^{(2n)}_{\mathcal{M}}$ (with $n \geq 1$) can lead from $\mathfrak{osp}(4|1)$ to different torsion deformations of it. We argue that the same should also occur in higher spacetime dimensions. All the above observations could help to shed some light on the relations occurring among the aforementioned different superalgebras and physical theories based on them.

4.3 Generalized AdS-Lorentz supergravity in the geometric approach

Before analyzing the generalized $D = 4$ AdS-Lorentz deformed supergravity theory in the presence of a non-trivial boundary of spacetime, we study the construction of the bulk Lagrangian and the corresponding supersymmetry transformation laws, on the same lines of [18]. Specifically, we apply the geometric approach to derive the parametrization of the Lorentz-like curvatures involving the extra 1-form fields $\tilde{h}^a$, $\tilde{k}^{ab}$, k^{ab}, and ξ by studying the different sectors of the on-shell Bianchi Identities; this also leads to the supersymmetry transformation laws. Subsequently, we construct a geometric generalized $D = 4$ AdS-Lorentz Lagrangian, showing that it can be written in terms of the aforementioned Lorentz-like supercurvatures. After that, we analyze the supersymmetry invariance of the theory in the presence of a non-trivial spacetime boundary.

Parametrization of the Lorentz-like curvatures. We consider the following Lorentz-type curvatures defined in Superspace:

$$\mathcal{R}^{ab} = D_\omega \omega^{ab}, \tag{4.20a}$$

$$R^a = D_\omega V^a + k^a{}_b \wedge V^b + \tilde{k}^a{}_b \wedge \tilde{h}^b - \frac{1}{2}\bar{\psi} \wedge \gamma^a \psi - \frac{1}{2}\bar{\xi} \wedge \gamma^a \xi, \tag{4.20b}$$

$$\tilde{H}^a = D_\omega \tilde{h}^a + \tilde{k}^a{}_b \wedge V^b + k^a{}_b \wedge \tilde{h}^b - \bar{\psi} \wedge \gamma^a \xi, \tag{4.20c}$$

$$\tilde{\mathcal{F}}^{ab} = D_\omega \tilde{k}^{ab} + 2k^a{}_c \wedge \tilde{k}^{cb}, \tag{4.20d}$$

$$\mathcal{F}^{ab} = D_\omega k^{ab} + \tilde{k}^a{}_c \wedge \tilde{k}^{cb} + k^a{}_c \wedge k^{cb}, \tag{4.20e}$$

$$\rho = D_\omega \psi + \frac{1}{4} k^{ab} \wedge \gamma_{ab} \psi + \frac{1}{4} \tilde{k}^{ab} \wedge \gamma_{ab} \xi, \tag{4.20f}$$

$$\sigma = D_\omega \xi + \frac{1}{4} k^{ab} \wedge \gamma_{ab} \xi + \frac{1}{4} \tilde{k}^{ab} \wedge \gamma_{ab} \psi. \tag{4.20g}$$

Here we use the Greek letters $\tilde{\mathcal{F}}^{ab}$, $\mathcal{F}^{ab}$, ρ, and σ, in order to avoid confusion with the generalized AdS-Lorentz supercurvatures eq.(4.14d)-eq.(4.14g). The supercurvatures eq.(4.20a)-eq.(4.20g) satisfy the Bianchi identities:

$$D_\omega \mathcal{R}^{ab} = 0, \tag{4.21a}$$

$$\begin{aligned} D_\omega R^a &= (\mathcal{R}^a{}_b + \mathcal{F}^a{}_b) \wedge V^b + \tilde{\mathcal{F}}^a{}_b \wedge \tilde{h}^b - k^a{}_b \wedge R^b - \tilde{k}^a{}_b \wedge \tilde{H}^b \\ &\quad + \bar{\psi} \wedge \gamma^a \rho + \bar{\xi} \wedge \gamma^a \sigma, \end{aligned} \tag{4.21b}$$

$$\begin{aligned} D_\omega \tilde{H}^a &= (\mathcal{R}^a{}_b + \mathcal{F}^a{}_b) \wedge \tilde{h}^b + \tilde{\mathcal{F}}^a{}_b \wedge V^b - \tilde{k}^a{}_b \wedge R^b - k^a{}_b \wedge \tilde{H}^b \\ &\quad + \bar{\xi} \wedge \gamma^a \rho + \bar{\psi} \wedge \gamma^a \sigma, \end{aligned} \tag{4.21c}$$

$$D_\omega \tilde{\mathcal{F}}^{ab} = 2\left((\mathcal{R}^a{}_c + \mathcal{F}^a{}_c) \wedge \tilde{k}^{cb} + \tilde{\mathcal{F}}^a{}_c \wedge k^{cb}\right), \tag{4.21d}$$

$$D_\omega \mathcal{F}^{ab} = 2\left((\mathcal{R}^a{}_c + \mathcal{F}^a{}_c) \wedge k^{cb} + \tilde{\mathcal{F}}^a{}_c \wedge \tilde{k}^{cb}\right), \tag{4.21e}$$

$$D_\omega \rho = \frac{1}{4}\left(\left(\mathcal{R}^{ab} + \mathcal{F}^{ab}\right) \wedge \gamma_{ab}\psi + \tilde{\mathcal{F}}^{ab} \wedge \gamma_{ab}\xi - \gamma_{ab}\rho \wedge k^{ab} - \gamma_{ab}\sigma \wedge \tilde{k}^{ab}\right), \tag{4.21f}$$

$$D_\omega \sigma = \frac{1}{4}\left(\left(\mathcal{R}^{ab} + \mathcal{F}^{ab}\right) \wedge \gamma_{ab}\xi + \tilde{\mathcal{F}}^{ab} \wedge \gamma_{ab}\psi - \gamma_{ab}\sigma \wedge k^{ab} - \gamma_{ab}\rho \wedge \tilde{k}^{ab}\right). \tag{4.21g}$$

We write the most general ansatz for the Lorentz-type curvatures in the supervielbein basis $\{V^a, \psi\}$ of Superspace as follows

$$\mathcal{R}^{ab} = \mathcal{R}^{ab}{}_{cd}\, V^c \wedge V^d + \bar{\Theta}^{ab}{}_c\, \psi \wedge V^c + \hat{\alpha}\, \bar{\psi} \wedge \gamma^{ab}\psi, \tag{4.22a}$$

$$R^a = R^a{}_{bc}\, V^b \wedge V^c + \bar{\Theta}^a{}_b\, \psi \wedge V^b + \hat{\beta}\, \bar{\psi} \wedge \gamma^a \psi, \tag{4.22b}$$

$$\tilde{H}^a = \tilde{H}^a{}_{bc}\, V^b \wedge V^c + \bar{\Lambda}^a{}_b\, \psi \wedge V^b + \hat{\gamma}\, \bar{\psi} \wedge \gamma^a \psi, \tag{4.22c}$$

$$\tilde{\mathcal{F}}^{ab} = \tilde{\mathcal{F}}^{ab}{}_{cd}\, V^c \wedge V^d + \bar{\Lambda}^{ab}{}_c\, \psi \wedge V^c + \hat{\delta}\, \bar{\psi} \wedge \gamma^{ab}\psi, \tag{4.22d}$$

$$\mathcal{F}^{ab} = \mathcal{F}^{ab}{}_{cd}\, V^c \wedge V^d + \bar{\Pi}^{ab}{}_c\, \psi \wedge V^c + \hat{\varepsilon}\, \bar{\psi} \wedge \gamma^{ab}\psi, \tag{4.22e}$$

$$\rho = \rho_{ab}\, V^a \wedge V^b + \hat{\lambda}\, \gamma_a \psi \wedge V^a + \Omega_{\alpha\beta}\, \psi^\alpha \wedge \psi^\beta, \tag{4.22f}$$

$$\sigma = \sigma_{ab}\, V^a \wedge V^b + \hat{\mu}\, \gamma_a \psi \wedge V^a + \tilde{\Omega}_{\alpha\beta}\, \psi^\alpha \wedge \psi^\beta. \tag{4.22g}$$

We can expand the curvatures in the directions $V \wedge V$, $V \wedge \psi$, $\psi \wedge \psi$; with this ansatz, from the equations of motion we can find that the outer components (that are the ones in the $V \wedge \psi$ and $\psi \wedge \psi$ directions) are written linear in terms of the inner components. These are the rheonomic on-shell constraints, which eliminate superfluous degrees of freedom. Setting $R^a = 0$, which corresponds to the on-shell condition, we can withdraw some terms appearing in the above ansatz by studying the scaling constraints; since $\tilde{H}^a$ has the same weight of R^a, we also can directly set $\tilde{H}^a = 0$. On the other hand, the coefficients $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, $\hat{\delta}$, $\hat{\varepsilon}$, $\hat{\lambda}$, and $\hat{\mu}$ can be determined from the analysis of the various sectors (which are $\psi \wedge \psi \wedge \psi$, $\psi \wedge \psi \wedge V$, $\psi \wedge V \wedge V$, and $V \wedge V \wedge V$) of the on-shell Bianchi identities in Superspace eq.(4.21a)-eq.(4.21g), with the help of the $D = 4$ Fierz identities for ψ collected in Appendix A.1.

One can then show that the Bianchi identities eq.(4.21a)-eq.(4.21g) are solved by parametrizing on-shell the full set of supercurvatures in the following way

$$\mathcal{R}^{ab} = \mathcal{R}^{ab}{}_{cd}\, V^c \wedge V^d + \bar{\Theta}^{ab}{}_c\, \psi \wedge V^c, \tag{4.23a}$$

$$R^a = 0, \tag{4.23b}$$

$$\tilde{H}^a = 0, \tag{4.23c}$$

$$\tilde{\mathcal{F}}^{ab} = \tilde{\mathcal{F}}^{ab}{}_{cd}\, V^c \wedge V^d + \bar{\Lambda}^{ab}{}_c\, \psi \wedge V^c, \tag{4.23d}$$

$$\mathcal{F}^{ab} = \mathcal{F}^{ab}{}_{cd}\, V^c \wedge V^d + \bar{\Pi}^{ab}{}_c\, \psi \wedge V^c, \tag{4.23e}$$

$$\rho = \rho_{ab}\, V^a \wedge V^b, \tag{4.23f}$$

$$\sigma = \sigma_{ab}\, V^a \wedge V^b, \tag{4.23g}$$

with

$$\begin{aligned} \bar{\Theta}^{ab}{}_c + \bar{\Pi}^{ab}{}_c &= \epsilon^{abde} \left(\bar{\rho}_{cd}\, \gamma_e \gamma_5 + \bar{\rho}_{ec}\, \gamma_d \gamma_5 - \bar{\rho}_{de}\, \gamma_c \gamma_5 \right), \\ \bar{\Lambda}^{ab}{}_c &= \epsilon^{abde} \left(\bar{\sigma}_{cd}\, \gamma_e \gamma_5 + \bar{\sigma}_{ec}\, \gamma_d \gamma_5 - \bar{\sigma}_{de}\, \gamma_c \gamma_5 \right). \end{aligned} \tag{4.24}$$

We have thus found the parametrization of the Lorentz-type curvatures of eq.(4.20a)-eq.(4.20g). This, as we are going to show, also provides us with the supersymmetry transformations laws.

Supersymmetry transformation laws. The parametrizations eq.(4.23a)-eq.(4.23g) we have obtained above allow to derive the supersymmetry transformations in a direct way. Indeed, in the geometric framework we have adopted, the transformations on spacetime are given by

$$\delta \mu^A = (\nabla \epsilon)^A + \imath_\epsilon R^A, \tag{4.25}$$

for all the superfields $\mu^A = \{\omega^{ab}, V^a, \tilde{h}^a, \tilde{k}^{ab}, k^{ab}, \psi^\alpha, \xi^\alpha\}$, where we define the set of parameters $\epsilon^A = (\epsilon^{ab}, \epsilon^a, \tilde{\varepsilon}^a, \tilde{\varepsilon}^{ab}, \varepsilon^{ab}, \epsilon^\alpha, \varepsilon^\alpha)$ ([15, 19, 62] for details). The symbol ∇ in eq.(4.25) denotes the gauge covariant derivative. Then, for a generic

ϵ^α and $\epsilon^{ab} = \epsilon^a = \tilde{\varepsilon}^a = \tilde{\varepsilon}^{ab} = \varepsilon^{ab} = \varepsilon^\alpha = 0$, we have:

$$\imath_\epsilon \mathcal{R}^{ab} = \bar{\Theta}^{ab}{}_c\, \epsilon\, V^c, \tag{4.26a}$$
$$\imath_\epsilon R^a = 0, \tag{4.26b}$$
$$\imath_\epsilon \tilde{H}^a = 0, \tag{4.26c}$$
$$\imath_\epsilon \tilde{\mathcal{F}}^{ab} = \bar{\Lambda}^{ab}{}_c\, \epsilon\, V^c, \tag{4.26d}$$
$$\imath_\epsilon \mathcal{F}^{ab} = \bar{\Pi}^{ab}{}_c\, \epsilon\, V^c, \tag{4.26e}$$
$$\imath_\epsilon \rho = 0, \tag{4.26f}$$
$$\imath_\epsilon \sigma = 0. \tag{4.26g}$$

This provides the following supersymmetry transformation laws for the 1-form fields:

$$\delta_\epsilon \omega^{ab} = \bar{\Theta}^{ab}{}_c\, \epsilon\, V^c, \tag{4.27a}$$
$$\delta_\epsilon V^a = \bar{\epsilon}\gamma^a \psi, \tag{4.27b}$$
$$\delta_\epsilon \tilde{h}^a = \bar{\epsilon}\gamma^a \xi, \tag{4.27c}$$
$$\delta_\epsilon \tilde{k}^{ab} = \bar{\Lambda}^{ab}{}_c\, \epsilon\, V^c, \tag{4.27d}$$
$$\delta_\epsilon k^{ab} = \bar{\Pi}^{ab}{}_c\, \epsilon\, V^c, \tag{4.27e}$$
$$\delta_\epsilon \psi = D_\omega \epsilon + \frac{1}{4}\gamma_{ab}\, \epsilon\, k^{ab}, \tag{4.27f}$$
$$\delta_\epsilon \xi = \frac{1}{4}\gamma_{ab}\, \epsilon\, \tilde{k}^{ab}. \tag{4.27g}$$

We use the on-shell formalism, so we consider supersymmetry transformations like eq.(4.27), which close on-shell on the fields of the multiplet; in other words, we have to impose the equations of motion of the fields to get our theory supersymmetric invariant.

The following step consists in the construction of a geometric bulk Lagrangian.

4.3.1 Geometric construction of the geometric bulk Lagrangian

We now construct a geometric bulk Lagrangian based on the generalized AdS-Lorentz superalgebra. The most general ansatz for the aforementioned Lagrangian can be written as follows:

$$\mathcal{L} = \mu^{(4)} + R^A \wedge \mu^{(2)}_A + R^A \wedge R^B \mu^{(0)}_{AB}, \tag{4.28}$$

where the upper index (p) denotes the degree of the related differential p-forms. Here, the R^A's are the generalized AdS-Lorentz Lie algebra valued supercurvatures defined by eq.(4.14a)-eq.(4.14g), invariant under the rescaling

$$\begin{aligned}
&\omega^{ab} \to \omega^{ab}, \quad V^a \to \omega V^a, \quad \tilde{h}^a \to \omega \tilde{h}^a, \quad \tilde{k}^{ab} \to \tilde{k}^{ab}, \quad k^{ab} \to k^{ab}, \\
&\psi \to \omega^{1/2}\psi, \quad \xi \to \omega^{1/2}\xi.
\end{aligned} \tag{4.29}$$

The Lagrangian must scale with ω^2, being ω^2 the scale-weight of the Einstein-Hilbert term. Thus, due to scaling constraints reasons (see [15]), some of the terms in the ansatz eq.(4.28) disappear. Besides, since we are now constructing the bulk Lagrangian, we can set $R^A \wedge R^B \mu^{(0)}_{AB} = 0$. Nevertheless, these terms will be fundamental for the construction of the boundary contributions needed in order to restore the supersymmetry invariance of the full Lagrangian (understood as bulk plus boundary contributions) in the presence of a non-trivial boundary of spacetime. Then, applying the scaling and the parity conservation law, we are left with the following explicit form for the Lagrangian (written in terms of the generalized AdS-Lorentz 1-form fields and of the super field-strengths eq.(4.14a)-eq.(4.14g)):

$$
\begin{aligned}
\mathcal{L} = \epsilon_{abcd}\Big(& \mathcal{R}^{ab} \wedge V^c \wedge V^d + \alpha_1\, \mathcal{R}^{ab} \wedge V^c \wedge \tilde{h}^d + \alpha_2\, \mathcal{R}^{ab} \wedge \tilde{h}^c \wedge \tilde{h}^d \\
& + \alpha_3\, \tilde{F}^{ab} \wedge V^c \wedge V^d + \alpha_4\, \tilde{F}^{ab} \wedge V^c \wedge \tilde{h}^d + \alpha_5\, \tilde{F}^{ab} \wedge \tilde{h}^c \wedge \tilde{h}^d \\
& + \alpha_6\, F^{ab} \wedge V^c \wedge V^d + \alpha_7\, F^{ab} \wedge V^c \wedge \tilde{h}^d + \alpha_8\, F^{ab} \wedge \tilde{h}^c \wedge \tilde{h}^d\Big) \\
& + \alpha_9\, \bar{\psi} \wedge V^a \gamma_a \gamma_5 \wedge \Psi + \alpha_{10}\, \bar{\psi} \wedge \tilde{h}^a \gamma_a \gamma_5 \wedge \Psi \\
& + \alpha_{11}\, \bar{\psi} \wedge V^a \gamma_a \gamma_5 \wedge \Xi + \alpha_{12}\, \bar{\psi} \wedge \tilde{h}^a \gamma_a \gamma_5 \wedge \Xi \\
& + \alpha_{13}\, \bar{\xi} \wedge V^a \gamma_a \gamma_5 \wedge \Psi + \alpha_{14}\, \bar{\xi} \wedge \tilde{h}^a \gamma_a \gamma_5 \wedge \Psi \\
& + \alpha_{15}\, \bar{\xi} \wedge V^a \gamma_a \gamma_5 \wedge \Xi + \alpha_{16}\, \bar{\xi} \wedge \tilde{h}^a \gamma_a \gamma_5 \wedge \Xi \\
& + e\, \epsilon_{abcd}\Big(\beta_1\, \bar{\psi} \wedge \gamma^{ab}\psi \wedge V^c \wedge V^d + \beta_2\, \bar{\psi} \wedge \gamma^{ab}\psi \wedge V^c \wedge \tilde{h}^d \\
& + \beta_3\, \bar{\psi} \wedge \gamma^{ab}\psi \wedge \tilde{h}^c \wedge \tilde{h}^d + \beta_4\, \bar{\psi} \wedge \gamma^{ab}\xi \wedge V^c \wedge V^d \\
& + \beta_5\, \bar{\psi} \wedge \gamma^{ab}\xi \wedge V^c \wedge \tilde{h}^d + \beta_6\, \bar{\psi} \wedge \gamma^{ab}\xi \wedge \tilde{h}^c \wedge \tilde{h}^d \\
& + \beta_7\, \bar{\xi} \wedge \gamma^{ab}\xi \wedge V^c \wedge V^d + \beta_8\, \bar{\xi} \wedge \gamma^{ab}\xi \wedge V^c \wedge \tilde{h}^d \\
& + \beta_9\, \bar{\xi} \wedge \gamma^{ab}\xi \wedge \tilde{h}^c \wedge \tilde{h}^d\Big) + e^2\, \epsilon_{abcd}\Big(\beta_{10}\, V^a \wedge V^b \wedge V^c \wedge V^d \\
& + \beta_{11}\, V^a \wedge V^b \wedge V^c \wedge \tilde{h}^d + \beta_{12}\, V^a \wedge V^b \wedge \tilde{h}^c \wedge \tilde{h}^d \\
& + \beta_{13}\, V^a \wedge \tilde{h}^b \wedge \tilde{h}^c \wedge \tilde{h}^d + \beta_{14}\, \tilde{h}^a \wedge \tilde{h}^b \wedge \tilde{h}^c \wedge \tilde{h}^d\Big),
\end{aligned}
\tag{4.30}
$$

where, in addition, we have consistently set the coefficient of the first term in eq.(4.30) to 1. The α_i's and the β_j's are constant dimensionless parameters to be determined by studying the field equations.

We now compute the variation of the Lagrangian with respect to the different fields. Along these calculations, we make use of the formulas given in Appendix A.1. The variation of the Lagrangian with respect to the spin connection ω^{ab}

reads

$$
\begin{aligned}
\delta_\omega \mathcal{L} = 2\, \epsilon_{abcd}\, \delta\omega^{ab} \wedge \Bigg(& D_\omega V^c \wedge V^d + \frac{1}{2}\alpha_1 \left(D_\omega V^c \wedge \tilde{h}^d + D_\omega \tilde{h}^c \wedge V^d \right) \\
& + \alpha_2\, D_\omega \tilde{h}^c \wedge \tilde{h}^d + \alpha_3\, \tilde{k}^c{}_f \wedge V^f \wedge V^d + \alpha_4\, \tilde{k}^c{}_f \wedge V^f \wedge \tilde{h}^d \\
& + \alpha_5\, \tilde{k}^c{}_f \wedge \tilde{h}^f \wedge \tilde{h}^d + \alpha_6\, k^c{}_f \wedge V^f \wedge V^d \\
& + \alpha_7\, k^c{}_f \wedge V^f \wedge \tilde{h}^d + \alpha_8\, k^c{}_f \wedge \tilde{h}^f \wedge \tilde{h}^d \\
& - \frac{1}{8} \Big(\alpha_9\, \bar{\psi} \wedge \gamma^c \psi \wedge V^d + \alpha_{10}\, \bar{\psi} \wedge \gamma^c \psi \wedge \tilde{h}^d \\
& + (\alpha_{11} + \alpha_{13})\, \bar{\psi} \wedge \gamma^c \xi \wedge V^d + (\alpha_{12} + \alpha_{14})\, \bar{\psi} \wedge \gamma^c \xi \wedge \tilde{h}^d \\
& + \alpha_{15}\, \bar{\xi} \wedge \gamma^c \xi \wedge V^d + \alpha_{16}\, \bar{\xi} \wedge \gamma^c \xi \wedge \tilde{h}^d \Big) \Bigg).
\end{aligned}
\tag{4.31}
$$

One can then prove that, if

$$
\begin{aligned}
& \alpha_1 = \alpha_4 = \alpha_7 = 2, \\
& \alpha_2 = \alpha_3 = \alpha_5 = \alpha_6 = \alpha_8 = 1, \\
& \alpha_9 = \alpha_{10} = \alpha_{11} = \alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{15} = \alpha_{16} = 4,
\end{aligned}
\tag{4.32}
$$

$\delta_\omega \mathcal{L} = 0$ yields the following field equation:

$$
\epsilon_{abcd} \left(R^c + \tilde{H}^c \right) \wedge \left(V^d + \tilde{h}^d \right) = 0,
\tag{4.33}
$$

generalizing to $R^c + \tilde{H}^c$ and $V^d + \tilde{h}^d$ the usual equation $\epsilon_{abcd}\, R^c \wedge V^d = 0$ for the supertorsion. The variation of the Lagrangian with respect to $\tilde{k}^{ab}$ and k^{ab} gives the same result, that is it does not imply any additional on-shell constraint. Analogously, one can prove that, by setting

$$
\begin{aligned}
& \beta_1 = \beta_3 = \beta_7 = \beta_9 = -1, \\
& \beta_2 = \beta_4 = \beta_6 = \beta_8 = \beta_{10} = \beta_{14} = -2, \\
& \beta_5 = -4, \\
& \beta_{11} = \beta_{13} = -8, \\
& \beta_{12} = -12,
\end{aligned}
\tag{4.34}
$$

the variation of the Lagrangian with respect to the vielbein V^a can be recast into the following form:

$$\delta_V \mathcal{L} = 4\Bigg(\frac{1}{2}\epsilon_{abcd}\Bigg(\mathcal{R}^{ab} \wedge \left(V^c + \tilde{h}^c\right) + \tilde{F}^{ab} \wedge \left(V^c + \tilde{h}^c\right) + F^{ab} \wedge \left(V^c + \tilde{h}^c\right)\Bigg)$$
$$+ \bar{\psi} \wedge \gamma_d\gamma_5 \Psi + \bar{\psi} \wedge \gamma_d\gamma_5 \Xi + \bar{\xi} \wedge \gamma_d\gamma_5 \Psi + \bar{\xi} \wedge \gamma_d\gamma_5 \Xi\Bigg) \wedge \delta V^d. \tag{4.35}$$

Then, $\delta_V \mathcal{L} = 0$ leads to the generalized equation

$$\epsilon_{abcd}\left(\mathcal{R}^{ab} + \tilde{F}^{ab} + F^{ab}\right) \wedge \left(V^c + \tilde{h}^c\right) + 2\left(\bar{\psi} + \bar{\xi}\,\right) \wedge \gamma_d\gamma_5\left(\Psi + \Xi\right) = 0. \tag{4.36}$$

The variation of the Lagrangian with respect to $\tilde{h}^a$ yields the same result.

Finally, from the variation of the Lagrangian with respect to the gravitino field ψ, we find the generalized field equation

$$2\left(V^a + \tilde{h}^a\right) \wedge \gamma_a\gamma_5\left(\Psi + \Xi\right) + \gamma_a\gamma_5\left(\psi + \xi\,\right) \wedge \left(R^a + \tilde{H}^a\right) = 0. \tag{4.37}$$

The variation with respect to ξ gives the same result.

We have thus completely determined the bulk Lagrangian of the theory, fixing all the coefficients. Interestingly, one can easily prove that the aforementioned geometric bulk Lagrangian can be rewritten in terms of the Lorentz-type curvatures eq.(4.20a)-eq.(4.20g) as follows:

$$
\begin{aligned}
\mathcal{L}_{\text{bulk}} = \epsilon_{abcd}\Big(& \mathcal{R}^{ab} \wedge V^c \wedge V^d + 2\, \mathcal{R}^{ab} \wedge V^c \wedge \tilde{h}^d + \mathcal{R}^{ab} \wedge \tilde{h}^c \wedge \tilde{h}^d \\
& + \tilde{\mathcal{F}}^{ab} \wedge V^c \wedge V^d + 2\, \tilde{\mathcal{F}}^{ab} \wedge V^c \wedge \tilde{h}^d + \tilde{\mathcal{F}}^{ab} \wedge \tilde{h}^c \wedge \tilde{h}^d \\
& + \mathcal{F}^{ab} \wedge V^c \wedge V^d + 2\, \mathcal{F}^{ab} \wedge V^c \wedge \tilde{h}^d + \mathcal{F}^{ab} \wedge \tilde{h}^c \wedge \tilde{h}^d \Big) \\
& + 4\, \bar{\psi} \wedge V^a \gamma_a \gamma_5 \wedge \rho + 4\, \bar{\psi} \wedge \tilde{h}^a \gamma_a \gamma_5 \wedge \rho \\
& + 4\, \bar{\psi} \wedge V^a \gamma_a \gamma_5 \wedge \sigma + 4\, \bar{\psi} \wedge \tilde{h}^a \gamma_a \gamma_5 \wedge \sigma \\
& + 4\, \bar{\xi} \wedge V^a \gamma_a \gamma_5 \wedge \rho + 4\, \bar{\xi} \wedge \tilde{h}^a \gamma_a \gamma_5 \wedge \rho \\
& + 4\, \bar{\xi} \wedge V^a \gamma_a \gamma_5 \wedge \sigma + 4\, \bar{\xi} \wedge \tilde{h}^a \gamma_a \gamma_5 \wedge \sigma \\
& + 2e\, \epsilon_{abcd}\Big(\bar{\psi} \wedge \gamma^{ab} \psi \wedge V^c \wedge V^d + 2\, \bar{\psi} \wedge \gamma^{ab} \psi \wedge V^c \wedge \tilde{h}^d \\
& + \bar{\psi} \wedge \gamma^{ab} \psi \wedge \tilde{h}^c \wedge \tilde{h}^d + 2\, \bar{\psi} \wedge \gamma^{ab} \xi \wedge V^c \wedge V^d \\
& + 4\, \bar{\psi} \wedge \gamma^{ab} \xi \wedge V^c \wedge \tilde{h}^d + 2\, \bar{\psi} \wedge \gamma^{ab} \xi \wedge \tilde{h}^c \wedge \tilde{h}^d \\
& + \bar{\xi} \wedge \gamma^{ab} \xi \wedge V^c \wedge V^d + 2\, \bar{\xi} \wedge \gamma^{ab} \xi \wedge V^c \wedge \tilde{h}^d + \bar{\xi} \wedge \gamma^{ab} \xi \wedge \tilde{h}^c \wedge \tilde{h}^d \Big) \\
& + 2e^2\, \epsilon_{abcd}\Big(V^a \wedge V^b \wedge V^c \wedge V^d + 4\, V^a \wedge V^b \wedge V^c \wedge \tilde{h}^d \\
& + 6\, V^a \wedge V^b \wedge \tilde{h}^c \wedge \tilde{h}^d + 4\, V^a \wedge \tilde{h}^b \wedge \tilde{h}^c \wedge \tilde{h}^d + \tilde{h}^a \wedge \tilde{h}^b \wedge \tilde{h}^c \wedge \tilde{h}^d \Big).
\end{aligned}
\tag{4.38}
$$

The equations of motion of the Lagrangian admit an AdS vacuum solution with cosmological constant (proportional to e^2). We also mention that the Lagrangian in eq.(4.38) has been written as a first-order Lagrangian, and the field equation for the spin connection ω^{ab} implies (up to boundary terms) the vanishing, on-shell, of $R^a + \tilde{H}^a$ (defined in eq.(4.20b) and eq.(4.20c), respectively). This is in agreement with the conditions $R^a = 0$ and $\tilde{H}^a = 0$ we have previously imposed in order to find the on-shell supercurvature parametrizations eq.(4.23a)-eq.(4.23g) by studying the various sectors of the Bianchi identities.

The spacetime Lagrangian eq.(4.38) results to be invariant under the supersymmetry transformations eq.(4.27a)-eq.(4.27g) of the 1-form fields on spacetime, up to boundary terms. As we have already mentioned, if the spacetime background has a non-trivial boundary, we have to check explicitly the condition eq.(3.43).

4.4 Supersymmetry invariance of the theory in the presence of a non-trivial boundary of spacetime

In the following, we analyze the supersymmetry invariance of the Lagrangian in the presence of a non-trivial spacetime boundary and, in particular, we present the explicit boundary terms required to recover the supersymmetry invariance of the full Lagrangian (given by bulk plus boundary contributions), on the same lines of [17, 18] (see also [19]). In the calculations presented in this section, we make extensive use of the formulas given in Appendix A.1.

Thus, we consider the bulk Lagrangian of eq.(4.38). Since we use the on-shell formalism, the bulk Lagrangian eq.(4.38) is invariant under supersymmetry transformations once imposed the torsionless on-shell constraints. Nevertheless, for this theory the boundary invariance of the Lagrangian under supersymmetry is not trivially satisfied, and the condition eq.(3.43) has to be checked in an explicit way in the presence of a non-trivial boundary of spacetime. In fact, we find that, if the fields do not asymptotically vanish at the boundary, we have

$$\imath_\epsilon \mathcal{L}_{\text{bulk}}|_{\partial\mathcal{M}} \neq 0. \tag{4.39}$$

In order to restore the supersymmetry invariance of the theory, it is possible to modify the bulk Lagrangian by adding boundary (i.e. topological) terms, which do not alter the bulk Lagrangian, so that eq.(3.40) is still fulfilled. The only possible boundary contributions (that are topological 4-forms) compatible with parity and Lorentz-like invariance are:

$$\epsilon_{abcd}\, d\left(\tilde{\omega}^{ab} \wedge \mathcal{N}^{cd} + \tilde{\omega}^{a}{}_{f} \wedge \tilde{\omega}^{fb} \wedge \tilde{\omega}^{cd}\right) = \epsilon_{abcd}\, \mathcal{N}^{ab} \wedge \mathcal{N}^{cd}, \tag{4.40a}$$

$$\begin{aligned} d\Big(\left(\bar{\psi} + \bar{\xi}\,\right) \wedge \gamma_5 \left(\rho + \sigma\right)\Big) = {} & \bar{\rho} \wedge \gamma_5 \rho + \bar{\sigma} \wedge \gamma_5 \sigma + 2\, \bar{\rho} \wedge \gamma_5 \sigma \\ & + \frac{1}{8} \epsilon_{abcd} \Big(\mathcal{N}^{ab} \wedge \bar{\psi} \wedge \gamma^{cd} \psi \\ & + 2\, \mathcal{N}^{ab} \wedge \bar{\psi} \wedge \gamma^{cd} \xi \\ & + \mathcal{N}^{ab} \wedge \bar{\xi} \wedge \gamma^{cd} \xi\Big), \end{aligned} \tag{4.40b}$$

where we have defined $\tilde{\omega}^{ab} = \omega^{ab} + \tilde{k}^{ab} + k^{ab}$ and $\mathcal{N}^{ab} = \mathcal{R}^{ab} + \tilde{\mathcal{F}}^{ab} + \mathcal{F}^{ab}$.

Then, the boundary terms in eq.(4.40a) and eq.(4.40b) correspond to the following boundary Lagrangian:

$$
\begin{aligned}
\mathcal{L}_{\text{bdy}} &= d\left(H^{(3)}\right) \\
&= \alpha\, \epsilon_{abcd}\Bigg(\mathcal{R}^{ab} \wedge \mathcal{R}^{cd} + \tilde{\mathcal{F}}^{ab} \wedge \tilde{\mathcal{F}}^{cd} + \mathcal{F}^{ab} \wedge \mathcal{F}^{cd} \\
&\quad + 2\, \mathcal{R}^{ab} \wedge \tilde{\mathcal{F}}^{cd} + 2\, \mathcal{R}^{ab} \wedge \mathcal{F}^{cd} + 2\, \tilde{\mathcal{F}}^{ab} \wedge \mathcal{F}^{cd}\Bigg) \\
&\quad + \beta\Bigg(\bar{\rho} \wedge \gamma_5 \rho + \bar{\sigma} \wedge \gamma_5 \sigma + 2\, \bar{\rho} \wedge \gamma_5 \sigma \\
&\quad + \frac{1}{8}\epsilon_{abcd}\Big(\mathcal{R}^{ab} \wedge \bar{\psi} \wedge \gamma^{cd}\psi + \tilde{\mathcal{F}}^{ab} \wedge \bar{\psi} \wedge \gamma^{cd}\psi + \mathcal{F}^{ab} \wedge \bar{\psi} \wedge \gamma^{cd}\psi \\
&\quad + 2\, \mathcal{R}^{ab} \wedge \bar{\psi} \wedge \gamma^{cd}\xi + 2\, \tilde{\mathcal{F}}^{ab} \wedge \bar{\psi} \wedge \gamma^{cd}\xi + 2\, \mathcal{F}^{ab} \wedge \bar{\psi} \wedge \gamma^{cd}\xi \\
&\quad + \mathcal{R}^{ab} \wedge \bar{\xi} \wedge \gamma^{cd}\xi + \tilde{\mathcal{F}}^{ab} \wedge \bar{\xi} \wedge \gamma^{cd}\xi + \mathcal{F}^{ab} \wedge \bar{\xi} \wedge \gamma^{cd}\xi\Big)\Bigg),
\end{aligned}
\tag{4.41}
$$

where we have defined

$$
\begin{aligned}
H^{(3)} = \alpha\, \epsilon_{abcd}\left(\tilde{\omega}^{ab} \wedge \mathcal{N}^{cd} + \tilde{\omega}^{a}{}_{f} \wedge \tilde{\omega}^{fb} \wedge \tilde{\omega}^{cd}\right) \\
+ \beta\left(\bar{\psi} \wedge \gamma_5 \rho + \bar{\xi} \wedge \gamma_5 \sigma + \bar{\psi} \wedge \gamma_5 \sigma + \bar{\xi} \wedge \gamma_5 \rho\right).
\end{aligned}
\tag{4.42}
$$

Here, α and β are constant parameters. We notice that the structure of a supersymmetric Gauss-Bonnet like term appears in eq.(4.41). Then, we consider the following "full" Lagrangian (thought as bulk plus boundary):

$$
\begin{aligned}
\mathcal{L}_{\text{full}} &= \mathcal{L}_{\text{bulk}} + \mathcal{L}_{\text{bdy}} = \\
&= \epsilon_{abcd}\Big(\mathcal{R}^{ab} \wedge V^c \wedge V^d + 2\,\mathcal{R}^{ab} \wedge V^c \wedge \tilde{h}^d + \mathcal{R}^{ab} \wedge \tilde{h}^c \wedge \tilde{h}^d \\
&\quad + \tilde{\mathcal{F}}^{ab} \wedge V^c \wedge V^d + 2\,\tilde{\mathcal{F}}^{ab} \wedge V^c \wedge \tilde{h}^d + \tilde{\mathcal{F}}^{ab} \wedge \tilde{h}^c \wedge \tilde{h}^d \\
&\quad + \mathcal{F}^{ab} \wedge V^c \wedge V^d + 2\,\mathcal{F}^{ab} \wedge V^c \wedge \tilde{h}^d + \mathcal{F}^{ab} \wedge \tilde{h}^c \wedge \tilde{h}^d\Big) \\
&\quad + 4\,\bar{\psi} \wedge V^a \gamma_a \gamma_5 \wedge \rho + 4\,\bar{\psi} \wedge \tilde{h}^a \gamma_a \gamma_5 \wedge \rho \\
&\quad + 4\,\bar{\psi} \wedge V^a \gamma_a \gamma_5 \wedge \sigma + 4\,\bar{\psi} \wedge \tilde{h}^a \gamma_a \gamma_5 \wedge \sigma \\
&\quad + 4\,\bar{\xi} \wedge V^a \gamma_a \gamma_5 \wedge \rho + 4\,\bar{\xi} \wedge \tilde{h}^a \gamma_a \gamma_5 \wedge \rho \\
&\quad + 4\,\bar{\xi} \wedge V^a \gamma_a \gamma_5 \wedge \sigma + 4\,\bar{\xi} \wedge \tilde{h}^a \gamma_a \gamma_5 \wedge \sigma \\
&\quad + 2e\,\epsilon_{abcd}\Big(\bar{\psi} \wedge \gamma^{ab}\psi \wedge V^c \wedge V^d + 2\,\bar{\psi} \wedge \gamma^{ab}\psi \wedge V^c \wedge \tilde{h}^d \\
&\quad + \bar{\psi} \wedge \gamma^{ab}\psi \wedge \tilde{h}^c \wedge \tilde{h}^d + 2\,\bar{\psi} \wedge \gamma^{ab}\xi \wedge V^c \wedge V^d \\
&\quad + 4\,\bar{\psi} \wedge \gamma^{ab}\xi \wedge V^c \wedge \tilde{h}^d + 2\,\bar{\psi} \wedge \gamma^{ab}\xi \wedge \tilde{h}^c \wedge \tilde{h}^d \\
&\quad + \bar{\xi} \wedge \gamma^{ab}\xi \wedge V^c \wedge V^d + 2\,\bar{\xi} \wedge \gamma^{ab}\xi \wedge V^c \wedge \tilde{h}^d + \bar{\xi} \wedge \gamma^{ab}\xi \wedge \tilde{h}^c \wedge \tilde{h}^d\Big) \\
&\quad + 2e^2\,\epsilon_{abcd}\Big(V^a \wedge V^b \wedge V^c \wedge V^d + 4\,V^a \wedge V^b \wedge V^c \wedge \tilde{h}^d \\
&\quad + 6\,V^a \wedge V^b \wedge \tilde{h}^c \wedge \tilde{h}^d + 4\,V^a \wedge \tilde{h}^b \wedge \tilde{h}^c \wedge \tilde{h}^d + \tilde{h}^a \wedge \tilde{h}^b \wedge \tilde{h}^c \wedge \tilde{h}^d\Big) \\
&\quad + \alpha\,\epsilon_{abcd}\Bigg(\mathcal{R}^{ab} \wedge \mathcal{R}^{cd} + \tilde{\mathcal{F}}^{ab} \wedge \tilde{\mathcal{F}}^{cd} + \mathcal{F}^{ab} \wedge \mathcal{F}^{cd} \\
&\quad + 2\,\mathcal{R}^{ab} \wedge \tilde{\mathcal{F}}^{cd} + 2\,\mathcal{R}^{ab} \wedge \mathcal{F}^{cd} + 2\,\tilde{\mathcal{F}}^{ab} \wedge \mathcal{F}^{cd}\Bigg) \\
&\quad + \beta\Bigg(\bar{\rho} \wedge \gamma_5\rho + \bar{\sigma} \wedge \gamma_5\sigma + 2\,\bar{\rho} \wedge \gamma_5\sigma \\
&\quad + \frac{1}{8}\epsilon_{abcd}\Big(\mathcal{R}^{ab} \wedge \bar{\psi} \wedge \gamma^{cd}\psi + \tilde{\mathcal{F}}^{ab} \wedge \bar{\psi} \wedge \gamma^{cd}\psi + \mathcal{F}^{ab} \wedge \bar{\psi} \wedge \gamma^{cd}\psi \\
&\quad + 2\,\mathcal{R}^{ab} \wedge \bar{\psi} \wedge \gamma^{cd}\xi + 2\,\tilde{\mathcal{F}}^{ab} \wedge \bar{\psi} \wedge \gamma^{cd}\xi + 2\,\mathcal{F}^{ab} \wedge \bar{\psi} \wedge \gamma^{cd}\xi \\
&\quad + \mathcal{R}^{ab} \wedge \bar{\xi} \wedge \gamma^{cd}\xi + \tilde{\mathcal{F}}^{ab} \wedge \bar{\xi} \wedge \gamma^{cd}\xi + \mathcal{F}^{ab} \wedge \bar{\xi} \wedge \gamma^{cd}\xi\Big)\Bigg).
\end{aligned}
\tag{4.43}
$$

We observe that, due to the homogeneous scaling of the Lagrangian, the coefficients α and β must be proportional to e^{-2} and e^{-1} respectively.

Now, the supersymmetry invariance of the full Lagrangian $\mathcal{L}_{\text{full}}$ in eq.(4.43), in the geometric approach, requires

$$\delta_\epsilon \mathcal{L}_{\text{full}} = \ell_\epsilon \mathcal{L}_{\text{full}} = \imath_\epsilon d\mathcal{L}_{\text{full}} + d(\imath_\epsilon \mathcal{L}_{\text{full}}) = 0. \tag{4.44}$$

Since the boundary terms eq.(4.40a) and eq.(4.40b) we have introduced so far are total differentials, the condition for supersymmetry in the bulk, that is $\imath_\epsilon d\mathcal{L}_{\text{full}} = 0$, is trivially satisfied. Then, the supersymmetry invariance of the full Lagrangian $\mathcal{L}_{\text{full}}$ requires just to verify that, for suitable values of α and β, the condition $\imath_\epsilon \mathcal{L}_{\text{full}} = 0$ (modulo an exact differential) holds on the boundary, that is to say $\imath_\epsilon \mathcal{L}_{\text{full}}|_{\partial\mathcal{M}} = 0$. Computing $\imath_\epsilon \mathcal{L}_{\text{full}}$, we get

$$\begin{aligned}
\imath_\epsilon \mathcal{L}_{\text{full}} = {} & \epsilon_{abcd}\, \imath_\epsilon \left(\mathcal{R}^{ab} + \tilde{\mathcal{F}}^{ab} + \mathcal{F}^{ab}\right) \wedge \left(V^c \wedge V^d + 2\, V^c \wedge \tilde{h}^d + \tilde{h}^c \wedge \tilde{h}^d\right) \\
& + 4\, \bar{\epsilon} \left(V^a + \tilde{h}^a\right) \gamma_a \gamma_5 \wedge (\rho + \sigma) \\
& + 4\, \bar{\psi} \wedge V^a \wedge \gamma_a \gamma_5 \imath_\epsilon (\rho) + 4\, \bar{\psi} \wedge V^a \wedge \gamma_a \gamma_5 \imath_\epsilon (\sigma) + 4\, \bar{\psi} \wedge \tilde{h}^a \wedge \gamma_a \gamma_5 \imath_\epsilon (\rho) \\
& + 4\, \bar{\psi} \wedge \tilde{h}^a \wedge \gamma_a \gamma_5 \imath_\epsilon (\sigma) + 4\, \bar{\xi} \wedge V^a \wedge \gamma_a \gamma_5 \imath_\epsilon (\rho) + 4\, \bar{\xi} \wedge V^a \wedge \gamma_a \gamma_5 \imath_\epsilon (\sigma) \\
& + 4\, \bar{\xi} \wedge \tilde{h}^a \wedge \gamma_a \gamma_5 \imath_\epsilon (\rho) + 4\, \bar{\xi} \wedge \tilde{h}^a \wedge \gamma_a \gamma_5 \imath_\epsilon (\sigma) \\
& + 4e\, \epsilon_{abcd}\, \bar{\epsilon} \Big(\gamma^{ab} \psi \wedge V^c \wedge V^d + 2\, \gamma^{ab} \psi \wedge V^c \wedge \tilde{h}^d + \gamma^{ab} \psi \wedge \tilde{h}^c \wedge \tilde{h}^d \\
& + \gamma^{ab} \xi \wedge V^c \wedge V^d + 2\, \gamma^{ab} \xi \wedge V^c \wedge \tilde{h}^d + \gamma^{ab} \xi \wedge \tilde{h}^c \wedge \tilde{h}^d\Big) \\
& + 2\epsilon_{abcd}\, \imath_\epsilon \left(\mathcal{R}^{ab} + \tilde{\mathcal{F}}^{ab} + \mathcal{F}^{ab}\right) \wedge \Bigg(\alpha\, \mathcal{R}^{cd} + \alpha\, \tilde{\mathcal{F}}^{cd} + \alpha\, \mathcal{F}^{cd} \\
& + \frac{\beta}{16} \bar{\psi} \wedge \gamma^{cd} \psi + \frac{\beta}{8} \bar{\psi} \wedge \gamma^{cd} \xi + \frac{\beta}{16} \bar{\xi} \wedge \gamma^{cd} \xi\Bigg) \\
& + \frac{\beta}{4} \epsilon_{abcd} \left(\mathcal{R}^{ab} + \tilde{\mathcal{F}}^{ab} + \mathcal{F}^{ab}\right) \wedge \left(\bar{\epsilon} \gamma^{cd} \psi + \bar{\epsilon} \gamma^{cd} \xi\right) \\
& + 2\beta\, \imath_\epsilon (\bar{\rho}) \wedge \gamma_5 \rho + 2\beta\, \imath_\epsilon (\bar{\sigma}) \wedge \gamma_5 \sigma + 2\beta\, \imath_\epsilon (\bar{\rho}) \wedge \gamma_5 \sigma + 2\beta\, \imath_\epsilon (\bar{\sigma}) \wedge \gamma_5 \rho.
\end{aligned} \tag{4.45}$$

Now, in general, this is not zero, but its projection on the boundary should be. Indeed, in the presence of a non-trivial boundary of spacetime, the field equations in Superspace for the Lagrangian in eq.(4.43) acquire non-trivial boundary contributions, which lead to the following constraints that are valid on the boundary

$$\begin{cases} \left(\mathcal{R}^{ab} + \tilde{\mathcal{F}}^{ab} + \mathcal{F}^{ab}\right)|_{\partial\mathcal{M}} = -\frac{1}{2\alpha}\left(V^a \wedge V^b + 2\, V^a \wedge \tilde{h}^b + \tilde{h}^a \wedge \tilde{h}^b\right) \\ \qquad\qquad -\frac{\beta}{16\alpha}\left(\bar{\psi} \wedge \gamma^{ab}\psi + 2\,\bar{\psi}\wedge\gamma^{ab}\xi + \bar{\xi}\wedge\gamma^{ab}\xi\right), \\ (\rho+\sigma)|_{\partial\mathcal{M}} = -\frac{2}{\beta}\left(V^a\wedge\gamma_a\psi + V^a\wedge\gamma_a\xi + \tilde{h}^a\wedge\gamma_a\psi + \tilde{h}^a\wedge\gamma_a\xi\right). \end{cases} \tag{4.46}$$

We can see that the supercurvatures on the boundary are not dynamical, rather being fixed to constant values. These are values in an enlarged anholonomic basis, meaning that the linear combinations of the supercurvatures on the boundary are fixed in terms of not only the bosonic and fermionic vielbein (V^a and ψ, respectively) but also of the extra bosonic 1-form field $\tilde{h}^a$ and of the extra fermionic one, ξ (that is in terms of 4-dimensional fields). Actually, this should not surprise, since also the Lorentz-like supercurvatures taken as starting point for our geometric construction of the Lagrangian are defined in an enlarged Superspace. Nevertheless, their parametrization results to be well defined in ordinary Superspace. Thus, in our framework the supersymmetry invariance constrains the boundary values of the supercurvatures (Neumann boundary conditions) without fixing the superfields themselves on the boundary.

Then, upon use of eq.(4.46) (and of Fierz identities and gamma matrices formulas reported in Appendix A.1), after some algebraic manipulation, on the boundary we are left with:

$$\begin{aligned} \imath_\epsilon \mathcal{L}_{\text{full}}|_{\partial\mathcal{M}} = \epsilon_{abcd}&\left(4e - \frac{\beta}{8\alpha} - \frac{4}{\beta}\right) \\ &\bar{\epsilon}\left(\gamma^{ab}\psi + \gamma^{ab}\xi\right)\wedge\left(V^c\wedge V^d + 2\,V^c\wedge\tilde{h}^d + \tilde{h}^c\wedge\tilde{h}^d\right). \end{aligned} \tag{4.47}$$

Thus, we find that $\imath_\epsilon \mathcal{L}_{\text{full}}|_{\partial\mathcal{M}} = 0$ if the following relation between α and β holds:

$$\frac{\beta}{4\alpha} + \frac{8}{\beta} = 8e. \tag{4.48}$$

Then, solving eq.(4.48) for β (with $\beta \neq 0$), we obtain

$$\beta = 16e\,\alpha\left(1 \pm \sqrt{1 - \frac{1}{8e^2\,\alpha}}\right). \tag{4.49}$$

Now, we observe that, by setting the square root in eq.(4.49) to zero, which implies

$$\alpha = \frac{1}{8e^2} \quad \Rightarrow \quad \beta = \frac{2}{e}, \tag{4.50}$$

we recover the following 2-form supercurvatures:

$$
\begin{aligned}
N^{ab} = &\mathcal{R}^{ab} + \tilde{\mathcal{F}}^{ab} + \mathcal{F}^{ab} + 8e^2\, V^a \wedge \tilde{h}^b + e\, \bar{\psi} \wedge \gamma^{ab}\psi + e\, \bar{\xi} \wedge \gamma^{ab}\xi \\
&+ 4e^2\, V^a \wedge V^b + 4e^2\, \tilde{h}^a \wedge \tilde{h}^b + 2e\, \bar{\psi} \wedge \gamma^{ab}\xi, \qquad (4.51a)
\end{aligned}
$$

$$
\Omega = \rho + \sigma + e\, V^a \wedge \gamma_a \xi + e\, \tilde{h}^a \wedge \gamma_a \psi + e\, V^a \wedge \gamma_a \psi + e\, \tilde{h}^a \wedge \gamma_a \xi, \qquad (4.51b)
$$

$$
R^a = D_\omega V^a + k^a{}_b \wedge V^b + \tilde{k}^a{}_b \wedge \tilde{h}^b - \frac{1}{2}\bar{\psi} \wedge \gamma^a \psi - \frac{1}{2}\bar{\xi} \wedge \gamma^a \xi, \qquad (4.51c)
$$

$$
\tilde{H}^a = D_\omega \tilde{h}^a + \tilde{k}^a{}_b \wedge V^b + k^a{}_b \wedge \tilde{h}^b - \bar{\psi} \wedge \gamma^a \xi. \qquad (4.51d)
$$

Moreover, eq.(4.51a)-eq.(4.51d) reproduce the generalized AdS-Lorentz supercurvatures, since one can write:

$$
N^{ab} = \mathcal{R}^{ab} + \tilde{F}^{ab} + F^{ab}, \qquad (4.52a)
$$

$$
\Omega = \Psi + \Xi, \qquad (4.52b)
$$

being $\mathcal{R}^{ab}$, $\tilde{F}^{ab}$, F^{ab}, Ψ, and Ξ defined in eq.(4.14a)-eq.(4.14g).

The full Lagrangian of eq.(4.43), written in terms of the 2-form supercurvatures eq.(4.52a) and eq.(4.52b), can be finally recast as a MacDowell-Mansouri like form [46], that is:

$$
\mathcal{L}_{\text{full}} = \frac{1}{8e^2}\epsilon_{abcd}\, N^{ab} \wedge N^{cd} + \frac{2}{e}\bar{\Omega} \wedge \gamma_5 \Omega, \qquad (4.53)
$$

whose boundary term, in particular, corresponds to the following supersymmetric Gauss-Bonnet like term (in the sequel, SUSY GB-like term, that is eq.(4.41) in which we have substituted eq.(4.50)):

$$
\begin{aligned}
\text{SUSY GB-like term} = &\frac{1}{8e^2}\epsilon_{abcd}\, \mathcal{N}^{ab} \wedge \mathcal{N}^{cd} + \frac{2}{e}\Bigg(\bar{\rho} \wedge \gamma_5 \rho + \bar{\sigma} \wedge \gamma_5 \sigma + 2\, \bar{\rho} \wedge \gamma_5 \sigma \\
&+ \frac{1}{8}\epsilon_{abcd}\, \mathcal{N}^{ab} \wedge \left(\bar{\psi} \wedge \gamma^{cd}\psi + 2\, \bar{\psi} \wedge \gamma^{cd}\xi + \bar{\xi} \wedge \gamma^{cd}\xi \right) \Bigg).
\end{aligned} \qquad (4.54)
$$

We observe that considering the square root in eq.(4.49) as different from zero would cause other boundary terms appearing in the MacDowell-Mansouri like Lagrangian. Indeed, defining $f^2 = 1 - \frac{1}{8e^2\,\alpha}$ and considering $f \neq 0$ in eq.(4.49) ($\beta \neq 0 \Rightarrow f \neq -1$), we end up with the following extra contributions:

$$
\begin{aligned}
&-\frac{f^2}{8e^2(f^2-1)} d\left(\tilde{\omega}^{ab}\wedge\mathcal{N}^{cd}+\tilde{\omega}^{a}{}_{g}\wedge\tilde{\omega}^{gb}\wedge\tilde{\omega}^{cd}\right)\epsilon_{abcd} \\
&+16e\;\alpha\;f\;d\left(\bar{\psi}\wedge\gamma_5\rho+\bar{\xi}\wedge\gamma_5\sigma+\bar{\psi}\wedge\gamma_5\sigma+\bar{\xi}\wedge\gamma_5\rho\right). \qquad (4.55)
\end{aligned}
$$

These terms break the off-shell generalized AdS-Lorentz structure of the theory. However, the first term in eq.(4.55) is incompatible with the invariance of the Lagrangian under diffeomorphisms in the bosonic directions of Superspace; on the other hand, considering the second term in eq.(4.55) and using the value of $\rho + \sigma$ at the boundary, given in eq.(4.46), we can easily prove that this term vanishes on-shell. Thus, in view of the fact that the closure of the generalized minimal AdS-Lorentz superalgebra only holds on-shell for a superymmetric theory (in the absence of auxiliary fields), this extra contribution does not play a significant role as far as supersymmetry is concerned.

We have thus shown that the Gauss-Bonnet like term in eq.(4.54) allows to recover the supersymmetry invariance of the on-shell generalized AdS-Lorentz deformed supergravity theory in the presence of a non-trivial boundary. In terms of the newly defined supercurvatures eq.(4.51a) and eq.(4.51b), the boundary conditions on the super field-strengths eq.(4.46) take the following simple form: $N^{ab}|_{\partial\mathcal{M}} = 0$ and $\Omega|_{\partial\mathcal{M}} = 0$. This means, in particular, that the linear combinations $\mathcal{R}^{ab} + \tilde{F}^{ab} + F^{ab}$ and $\Psi + \Xi$ vanish at the boundary.

Chapter 5

Five-point MHV amplitudes in $\mathcal{N} = 2$ SCQCD

In this chapter, we deal with the second main topic of the thesis: We give up supergravity theories studied through the geometric approach and we move to rigid supersymmetric field theories. As we did in Chapter 4, we briefly contextualize this topic underlining some recent developments; we will introduce the most relevant concepts after this short presentation.

The study of scattering amplitudes in supersymmetric field theories has recently unveiled the existence of hidden symmetries and unexpected mathematical properties; one of the richest theories which play a central role in such studies is $\mathcal{N} = 4$ SYM theory in $D = 4$. As we said in Chapter 2, it is a maximally supersymmetric theory and it is also conformally invariant. Its field content belongs to the same $\mathcal{N} = 4$ supermultiplet, so we can see that, in order to renormalize the theory, we find all the renormalization constants equal to 1: $\mathcal{N} = 4$ SYM is a finite theory. Moreover, in the last 15 years unexpected powerful connections between calculations in 4-dimensional $\mathcal{N} = 4$ SYM and integrability techniques typical of condensed matter systems, such as integrable spin chains, has pushed forward the conjecture that $\mathcal{N} = 4$ SYM in $D = 4$ might be a solvable theory (some references of the main topics are in [26, 63, 64] and references therein).

Some recent works mainly focus on mathematical properties evident in scattering amplitudes and in particular in MHV (maximally helicity violation) scattering amplitudes, which are amplitudes that maximally violate the helicity conservation at tree-level; in that perturbative order, they are easily computed through the Parke-Taylor formula ([65] for a rigorous derivation). One of the most surprising novelty is that planar MHV scattering amplitudes of $\mathcal{N} = 4$ SYM theory enjoy an additional dynamical symmetry, which is not present in the Lagrangian formulation and which constrains the form of the amplitudes to be much simpler than a naive analysis might suggest [66, 67]. This hidden symmetry, called dual conformal invariance, can be related to a duality between planar MHV amplitudes and light–like polygonal Wilson loops and was first suggested in the strong coupling string description [68]. The effects of dual conformal symmetry are the fact that the 4-gluons and 5-gluons MHV amplitudes are completely fixed [69, 70] in a form that matches the exponential BDS ansatz [71]. Starting from 6 external particles, dual conformal invariance constrains the amplitudes only up to an undetermined function of the conformal cross ratios which violates the BDS exponentiation [72, 73]; nevertheless the duality with Wilson loops was shown to be preserved.

Another mathematical property studied is the trascendentality: For example, in the dimensional regularization scheme of $\mathcal{N} = 4$ SYM, assigning transcendentality -1 to the dimensional regularization parameter, one obtains L-loop corrections with uniform degree of transcendentality $2L$. It is still unclear whether this property has to be ascribed to the special diagrammatics [74] associated to either dual conformal symmetry or supersymmetry, or if it is a unique feature of the model. The investigation on the origin of such properties has led to study theories with less amount of supersymmetry: In $D = 4$, one of these theories is $\mathcal{N} = 2$ SCQCD, our scenario in this chapter. In particular, this theory was studied compared with $\mathcal{N} = 4$ SYM, in order to individuate which features of a maximally supersymmetric theory persist in a non-maximally supersymmetric one. In [75] the authors computed all the 4-point scalar amplitudes at 1 loop and classified them into three sectors; moreover, they explored the behavior of divergent terms up to 2 loops in a specific sector.

Some papers like [33] study the integrable properties of correlation functions in that theory. For example, E. Pomoni with L. Rastelli and others have shown that SCQCD arises as a limit of a $\mathcal{N} = 2$ $SU(N_c) \times SU(N_c)$ elliptic quiver gauge model that interpolates between a $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ SCQCD. Further qualitative explanations about this topic could be found in [76].

The new objects of study are the 5-point scattering amplitudes in $\mathcal{N} = 2$ SCQCD. In particular, we choose to consider 5-point MHV scattering amplitudes: We can easily find them in [77]. We choose to use the $\mathcal{N} = 1$ Superspace formalism of Feynman superdiagrams [10] since we can directly see some intermediate cancellations between ultraviolet divergences. Actually a process of 5 particles is described by a number of Feynman diagrams which is greater than the number of diagrams with 4 external legs; moreover, in general, each 5-point diagram is more complicated than a 4-point one. Given these two reasons, we do not show a final result of the main computation of this chapter; in fact, the project of 5-point scattering amplitudes in $\mathcal{N} = 2$ SCQCD is still open. We focus on some necessary preliminaries of a chosen computation. In Appendix A.2, some useful relations are summarized.

5.1 Some properties of scattering amplitudes

In this section, we introduce two of the interesting features that we expect to find in a 5-point scattering amplitude in $\mathcal{N}=2$ SCQCD: They are the BDS ansatz and the dual conformal symmetry.

Infrared divergences and exponential BDS ansatz. In general, when we deal with a L-loop amplitude, we can find two types of divergences from the integrals, respectively the UV (ultraviolet) and IR (infrared) divergences. The first ones could be reabsorbed by the renormalized parameters of the theory; on the other hand, it is possible to regularize the IR divergences through a chosen regularization scheme; we choose a smart scheme, called “dimensional regularization” ([2] and references therein). Given a parameter ϵ, we set the dimensions of the theory to $D=4-2\epsilon$ and we introduce the IR energy scale μ_{IR}: In that way, IR divergences take the form of poles in ϵ. On-shell loop amplitudes in massless theories always contain IR divergences, due to exchange of soft vectors or virtual collinear splittings. The general structure of IR divergences is well understood, since it was discovered that soft and collinear divergences have a universal form ([78] and references therein are some possible references). A n-point (with $n\geq 3$) 1-loop scattering amplitude $\mathcal{A}_n^{(1)}(\{p_i\})$ $(i=1,\dots,n)$ shows the following structure

$$\mathcal{A}_n^{(1)}(\{p_i\})=\mathcal{I}^{(1)}(\epsilon,\{p_i\})\,\mathcal{A}_n^{(0)}(\{p_i\})+\mathcal{A}_n^{(1fin)}(\{p_i\})\,, \tag{5.1}$$

where $\mathcal{A}_n^{(0)}(\{p_i\})$ is the corresponding tree-level amplitude, $\mathcal{A}_n^{(1fin)}(\{p_i\})$ is a 1-loop contibution finite for $\epsilon\to 0$ and all the divergence is collected in $\mathcal{I}^{(1)}(\epsilon,\{p_i\})$. The expression of the IR divergence is universally determined and its expansion in ϵ poles is

$$\mathcal{I}^{(1)}(\epsilon,\{p_i\})=-\frac{1}{2\epsilon^2}\sum_{i=1}^{n}\eta_i+\mathcal{O}\left(\frac{1}{\epsilon}\right), \tag{5.2}$$

where $\eta_i=\frac{N_c^2-1}{2N_c}$ if the i-th particle is a fermion and $\eta_i=N_c$ if the i-th particle is a vector. We can now consider some results of the well known $\mathcal{N}=4$ SYM theory in the planar limit; for simplicity, we can remove from eq.(5.2) the indication of the dependence on momenta (just preserving the ϵ dependence) and define the L-loop n-point reduced scattering amplitude as

$$\mathcal{M}_n^{(L)}=\frac{\mathcal{A}_n^{(L)}}{\mathcal{A}_n^{(0)}}. \tag{5.3}$$

Since a special property of MHV loop amplitudes is that all their leading singularities are proportional to the MHV tree-level amplitude [79], the interesting quantity is the reduced amplitude $\mathcal{M}_n^{(L)}$. The general structure of the 1-loop n-point reduced amplitude in planar $\mathcal{N} = 4$ SYM is

$$\mathcal{M}_n^{(1)} = \lambda\, \mathcal{I}_n^{(1)}(\epsilon) + \mathcal{M}_n^{(1fin)}, \tag{5.4}$$

where λ is the 't Hooft coupling constant and $\mathcal{M}_n^{(1fin)}$ is finite for $\epsilon \to 0$. Once more again, the IR divergences of $\mathcal{M}_n^{(1)}$ are collected in

$$\mathcal{I}_n^{(1)}(\epsilon) = -\frac{1}{\epsilon^2}\sum_{i=1}^{n}\left(\frac{\mu_{IR}^2}{s_{i,i+1}}\right)^{\epsilon} \tag{5.5}$$

expressed in terms of Mandelstam variables $s_{i,i+1} = (p_i + p_{i+1})^2$, where of course $s_{n,n+1} = s_{n,1}$. It is possible to write the general all-loop structure of IR divergences of $\mathcal{M}_n^{(L)}$ of N = 4 SYM in a very compact way, thanks to the simple structure of amplitudes: It is found that the loop corrections exhibit an iterative structure [79], which can be summarized in the following expression:

$$\mathcal{M}_n|_{IR} = e^{\sum_{l=1}^{\infty}\lambda^l f^{(l)}(\epsilon)\mathcal{I}_n^{(1)}(l\epsilon)} \tag{5.6}$$

with $f^{(l)}(\epsilon) = \Gamma_{cusp}^{(l)} + l\,\epsilon\,\Gamma_{coll}^{(l)}$, where $\Gamma_{coll}^{(l)}$ is the collinear anomalous dimension and $\Gamma_{cusp}^{(l)}$ is the cusp anomalous dimension. This infrared behavior is very interesting because it throws light on a mathematical property of scattering amplitudes, which is not manifest in the action of the theory. In fact, the cusp anomalous dimension comes out as the UV divergence of a Wilson loop with light-like cusps. We will define it in the next paragraph. In [80] and in further works of the authors, it is possible to notice that a MHV reduced amplitude up to 3 loops suggests an exponential structure for the complete amplitude: This consideration leads to what is called the BDS (Bern, Dixon, Smirnov) ansatz. It states that a generic MHV reduced amplitude in planar $\mathcal{N} = 4$ SYM has the form

$$\mathcal{M}^{BDS}(\epsilon) = e^{\sum_{l=1}^{\infty}\left(\lambda^{l-1} f^{(l)}(\epsilon)\,\mathcal{M}^{(1)}(l\epsilon) + \lambda^l C^{(l)} + \mathcal{O}(\epsilon)\right)} \tag{5.7}$$

with the scaling function $f^{(l)}(\epsilon) = f_0^{(l)} + f_1^{(l)}\epsilon + f_2^{(l)}\epsilon^2$ and $C^{(l)}$ is a finite part. We note that the coefficients do not depend on the number of external legs. A fundamental element of the BDS ansatz is the dual conformal symmetry that we will discuss soon. The way one would go about testing the BDS ansatz is by direct calculation of the n-point L-loop amplitudes. It has been shown numerically that $\mathcal{M}_n^{(L)}$ fulfils the BDS ansatz up to $n = 5$ and $L = 2$ (see [81] and references therein). Something new happens when $n \geq 6$: While the BDS ansatz matches the IR divergent structure, it does not produce the correct finite part. In those cases the BDS ansatz determines the finite part of the reduced amplitude only up to a function, which is called "remainder function", of dual conformal cross-ratios of the external momenta: This function is defined as

$$r_n^{(L)} = \mathcal{M}_n^{(L)} - \mathcal{M}_n^{BDS}, \tag{5.8}$$

where $\mathcal{M}_n^{BDS}$ is the $\mathcal{O}(\lambda^L)$ terms of eq.(5.7). Actually, it is known numerically that $r_n^{(L)} = 0$ for $n \leq 5$ and then $r_n^{(L)} \neq 0$ for $n \geq 6$; in these cases, the IR divergence is matched while the finite part is achieved up to $r_n^{(L)}$. It is interesting to prove the BDS ansatz by diagrammatic computations of scattering amplitudes (MHV or not) in a non-maximally supersymmetric theory such as $\mathcal{N} = 2$ SCQCD. A computation via Feynman superdiagrams could clearly show some intemediate cancellations of UV divergent terms and in some cases it could be a more direct way to operate. We choose the 4-dimensional theory $\mathcal{N} = 2$ SCQCD also because we mentioned that a key ingredient of the BDS ansatz is a symmetry called "dual conformal symmetry"; actually, works such as [75] show that 4-point amplitudes in $\mathcal{N} = 2$ SCQCD present such a inner symmetry.

Dual conformal invariance and duality with Wilson loops. We briefly explain the main concepts of an interesting mathematical properties of the scattering amplitudes we are dealing with; some references could be found in [81, 82] and in some of their cited papers. We consider a generic n-point planar tree-level amplitude $\mathcal{A}_n^{(0)}$ in $\mathcal{N} = 4$ SYM, even if the same discussion could be made for $\mathcal{N} = 2$ SCQCD. In our convention, we consider $\mathcal{A}_n^{(0)}$ with n outgoing states. In the momentum space, the invariance under translations corresponds to the momentum conservation. We can define a dual space with coordinates $\hat{x}_i^{\alpha\dot{\alpha}}$ (with $i = 1, \ldots, n$), where the i-th momentum $p_i^{\alpha\dot{\alpha}}$ is translated into $\hat{x}_i^{\alpha\dot{\alpha}} - \hat{x}_{i+1}^{\alpha\dot{\alpha}}$. It is easy to prove that the momentum conservation $\sum_{i=1}^{n} p_i^{\alpha\dot{\alpha}} = 0$ in the momentum space becomes the periodicity condition $\hat{x}_{n+1}^{\alpha\dot{\alpha}} = \hat{x}_1^{\alpha\dot{\alpha}}$ in the dual space; as a consequence, in the dual space, the dynamical parameters of the process form a closed polygonal path C_n. If we further introduce dual fermionic coordinates and we study the action of a generic superconformal transformation, we find that $\mathcal{A}_n^{(0)}$ does not change also in the dual space: This is what is called "dual superconformal invariance" and it is valid for all the perturbative orders and also for the reduced amplitude. Now we come back to the dual space we have defined. Since the theory is conformal invariant, we have no mass scales and then we have massless particles. Given a MHV amplitude $\mathcal{A}_n$, it is possible to define proper dual coordinates for each of the n points and so join them in a closed polygonal path C_n. Each point $\hat{x}_i$ (with $i = 1, \ldots, n$) is connected to the next one $\hat{x}_{i+1}$ by a light-like line because of the fact that this line represents a massless momentum in the momentum space. As a consequence, we have a light-like polygon C_n; a mathematical object that can be naturally associated with C_n in a gauge theory is a Wilson loop. Given a theory with gauge field A_μ and g coupling constant and defined a polygon C_n, a Wilson loop $\mathcal{W}_n$ is a gauge-invariant object which is equal to the trace of a path-ordered exponential of A_μ transported along C_n:

$$\mathcal{W}_n = \mathrm{tr}\left(\mathcal{P}\; e^{ig \oint_{C_n} dx^\mu A_\mu}\right). \tag{5.9}$$

If we compute the expectation value of $\mathcal{W}_n$, we find UV divergences due to the cusps in the points $\hat{x}_i$. These UV divergences have the same structure of the IR divergences of $\mathcal{A}_n$, so there is a relation of duality between MHV amplitudes and light-like polygonal Wilson loops; this duality is valid in each perturbative order for a generic number of points n for $\mathcal{N} = 4$ SYM in $D = 4$. This is another interesting feature that could be studied in a superconformal theory like $\mathcal{N} = 2$ SCQCD.

We now show the preliminary results achieved in order to compute a 5-point scattering amplitude in $\mathcal{N} = 2$ SCQCD.

5.2 $\mathcal{N} = 2$ SCQCD in $\mathcal{N} = 1$ Superspace

In this section, we give a synthetic introduction of $\mathcal{N} = 2$ superconformal quantum chromodynamics (SCQCD) in $D = 4$ spacetime dimensions ([75] and references therein).

We consider a flat 4-dimensional Euclidean spacetime $\mathcal{E}_4$, which is a Wick rotation of a Minkowskian spacetime $\mathcal{M}_4$; as explained in Chapter 2, we build a $\mathcal{N} = 1$ Superspace $\mathcal{E}_{4|4}$ and we define a set of coordinates $\{x^{\alpha\dot{\alpha}}, \theta^{\alpha}, \bar{\theta}^{\dot{\alpha}}\}$ (with $\alpha = +, -$ and $\dot{\alpha} = \dot{+}, \dot{-}$), where $x^{\alpha\dot{\alpha}}$ are spacetime coordinates and θ^{α} are Weyl spinor coordinates (with $\bar{\theta}^{\dot{\alpha}} = (\theta^{\alpha})^{\dagger}$); we use the bispinorial indices formalism and the relations collected in Appendix A.2.

$\mathcal{N} = 2$ SCQCD is a superconformal gauge theory with gauge group $SU(N_c)$ (with N_c color number) and coupling constant g; we can define the 't Hooft coupling $\lambda = \frac{g^2 N_c}{(4\pi)^2}$. Since our computations will be preparatory for further comparisons with $\mathcal{N} = 4$ SYM, it is necessary to consider the most symmetric set for the theory: This could be achieved in the planar perturbative limit, where $g \to 0$, $N_c \to \infty$ and λ is finite and fixed. The field content of $\mathcal{N} = 2$ SCQCD can be conveniently expressed in terms of $\mathcal{N} = 1$ superfields. In particular, we have a scalar superfield $V = T^a V_a$ and a chiral superfield $\Phi = T^a \Phi_a$ (and the antichiral one $\bar{\Phi} = T^a \bar{\Phi}_a$) belonging to the adjoint representation of $SU(N_c)$, with T^a ($a = 1, \ldots, N_c^2 - 1$) generators of $SU(N_c)$ which close the following algebra

$$\left[T^a, T^b\right] = f^{ab}{}_c \, T^c, \tag{5.10}$$

with $f^{ab}{}_c$ structure constants. These superfields have the following θ-expansions

$$\begin{aligned}
\Phi &= \phi + \theta^{\alpha}\psi_{\alpha} - \theta^2 F, \\
\bar{\Phi} &= \bar{\phi} + \bar{\theta}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}} - \bar{\theta}^2 \bar{F}, \\
V &= \theta^{\alpha}\bar{\theta}^{\dot{\alpha}} A_{\alpha\dot{\alpha}} - \bar{\theta}^2\theta^{\alpha}\lambda_{\alpha} - \theta^2\bar{\theta}^{\dot{\alpha}}\bar{\lambda}_{\dot{\alpha}} + \theta^2\bar{\theta}^2 \mathrm{D}',
\end{aligned} \tag{5.11}$$

where ϕ, $\bar{\phi}$, ψ_{α}, $\bar{\psi}_{\dot{\alpha}}$, λ_{α}, $\bar{\lambda}_{\dot{\alpha}}$, and $A_{\alpha\dot{\alpha}}$ are physical massless component fields and where F, $\bar{F}$, and D' are auxiliary fields. These component fields are so defined

$$
\begin{aligned}
&\phi = \Phi|_{\theta=\bar{\theta}=0}, && \bar{\phi} = \bar{\Phi}|_{\theta=\bar{\theta}=0}, \\
&\psi_\alpha = iD_\alpha \Phi|_{\theta=\bar{\theta}=0}, && \bar{\psi}_{\dot{\alpha}} = -i\bar{D}_{\dot{\alpha}}\bar{\Phi}|_{\theta=\bar{\theta}=0}, \\
&F = D^2\Phi|_{\theta=\bar{\theta}=0}, && \bar{F} = \bar{D}^2\bar{\Phi}|_{\theta=\bar{\theta}=0}, \\
&A_{\alpha\dot{\alpha}} = \frac{1}{2}\left[\bar{D}_{\dot{\alpha}}, D_\alpha\right] V|_{\theta=\bar{\theta}=0}, \\
&\lambda_\alpha = i\bar{D}^2 D_\alpha V|_{\theta=\bar{\theta}=0}, \\
&\bar{\lambda}_{\dot{\alpha}} = -iD^2\bar{D}_{\dot{\alpha}} V|_{\theta=\bar{\theta}=0}, \\
&\mathrm{D}' = \frac{1}{2} D^\alpha \bar{D}^2 D_\alpha V|_{\theta=\bar{\theta}=0}.
\end{aligned}
\tag{5.12}
$$

The two $\mathcal{N} = 1$ superfields V and Φ are combined into an $\mathcal{N} = 2$ vector supermultiplet. The theory is coupled to N_f hypermultiplets in the fundamental representation of $SU(N_c)$; since the theory is conformally invariant, its beta function $\beta(g) = \mu\frac{\partial g}{\partial\mu}$ (with μ energy scale) must be equal to zero and this is possible if the number of hypermultiplets is fixed to $N_f = 2N_c$. We denote with Q_I and $\tilde{Q}^I$ (with $I = 1, \ldots, N_f$) the chiral superfields respectively in the fundamental and antifundamental representation of the gauge group; they have the following θ-expansions (omitting the indices for convenience)

$$
\begin{aligned}
Q &= q + \theta^\alpha \xi_\alpha - \theta^2 G, \\
\tilde{Q} &= \tilde{q} + \theta^\alpha \tilde{\xi}_\alpha - \theta^2 \tilde{G}, \\
\bar{Q} &= \bar{q} + \bar{\theta}^{\dot{\alpha}} \bar{\xi}_{\dot{\alpha}} - \bar{\theta}^2 \bar{G}, \\
\bar{\tilde{Q}} &= \bar{\tilde{q}} + \bar{\theta}^{\dot{\alpha}} \bar{\tilde{\xi}}_{\dot{\alpha}} - \bar{\theta}^2 \bar{\tilde{G}},
\end{aligned}
\tag{5.13}
$$

where q, $\tilde{q}$, $\bar{q}$, $\bar{\tilde{q}}$, ξ_α, $\tilde{\xi}_\alpha$, $\bar{\xi}_{\dot{\alpha}}$, and $\bar{\tilde{\xi}}_{\dot{\alpha}}$ are physical massless component fields and where G, $\tilde{G}$, $\bar{G}$, and $\bar{\tilde{G}}$ are auxiliary fields. Together, these superfields in the fundamental representation form an $\mathcal{N} = 2$ hypermultiplet. To avoid confusion, in this chapter it is not useful for our purpose to consider the superalgebra of the theory and the generators of supersymmetry: as a consequence, we can use the character "Q" to label the fundamental superfields instead of the supercharges.

The classical action of $\mathcal{N}=2$ SCQCD is

$$\mathcal{S} = \int d^4x\, d^4\theta \left(\operatorname{tr}\left(e^{-gV}\bar{\Phi}e^{gV}\Phi\right) + \bar{Q}^I e^{gV} Q_I + \tilde{Q}^I e^{-gV}\bar{\tilde{Q}}_I \right) \\ + \frac{1}{g^2}\int d^4x\, d^2\theta\, \operatorname{tr}\left(W^\alpha W_\alpha\right) + ig\int d^4x\, d^2\theta\, \tilde{Q}^I \Phi Q_I - ig\int d^4x\, d^2\bar{\theta}\, \bar{Q}^I\bar{\Phi}\bar{\tilde{Q}}_I, \tag{5.14}$$

where $W_\alpha = i\bar{D}^2\left(e^{-gV}D_\alpha e^{gV}\right)$ and we find a definition of the covariant spinor derivatives D_α and $\bar{D}_{\dot{\alpha}}$ in Appendix A.2. The symmetries of this action are the gauge group $SU(N_c)$, the conformal group $SO(6)$ and a global symmetry group $U(N_f)\times SU(2)\times U(1)$, where $U(N_f)$ is the flavor symmetry and $SU(2)\times U(1)$ is the R-symmetry.

For computations of scattering amplitudes, we first need to quantize the theory; the most common way to do that is the functional quantization. We show a simple example present in [10] of ordinary non-supersymmetric quantum field theory is order to highlight the main definitions. Given a generic theory of a field ϕ described by a classical action $\mathcal{S}(\phi)$, one can write the functional generator

$$\mathcal{Z}(J) = \int \mathfrak{D}\phi\; e^{S(\phi)+\int d^4x\; J\phi}, \tag{5.15}$$

which is a path integral, otherwise an integral over all the possible paths of the field ϕ; J is a source with the same characteristics of ϕ. If ϕ is a gauge field, we have to quantize it by introducing gauge-fixing terms and Faddeev-Popov ghosts [1]. The functional generator could be conceptually seen as a partition function of the theory; $\mathcal{Z}(J)$ is the generator of Green functions of the theory, which are correlation functions between the fields. The following step is to consider only the connected Green functions: They are generated by

$$\mathcal{W}(J) = \ln(\mathcal{Z}(J)). \tag{5.16}$$

We can compute the expectation value of the field ϕ by

$$\langle\phi\rangle(J) = \frac{\delta\mathcal{W}(J)}{\delta J}, \tag{5.17}$$

where we have a functional derivative, and after we can invert the relation in order to find $J(\langle\phi\rangle)$. Finally, with a Legendre transformation, we get the effective action

$$\Gamma(\phi) = \mathcal{W}(J(\phi)) - \int d^4x\; J(\phi)\phi, \tag{5.18}$$

which is the functional generator of one-particle irreducible diagrams, the building blocks for the Feynman diagrams. Through the effective action, we can derive the Feynman rules of the theory: They are the propagators of the fields and the vertices of the interactions. This example could be extended to any supersymmetric field theory.

We now come back to $\mathcal{N} = 2$ SCQCD. After the quantization of the action eq.(5.14), where the superghosts c, c' and their conjugates are introduced, we expand the quantum action as far as we need for our computation; we finally get the Euclidean action

$$\begin{aligned}\mathcal{S} = \int d^4x\, d^4\theta \Bigg(& \mathrm{tr}\Big(\bar{\Phi}\Phi + g\left(\bar{\Phi}V\Phi - \bar{\Phi}\Phi V\right) + \frac{g^2}{2}\left(\bar{\Phi}\Phi VV + \bar{\Phi}VV\Phi - 2\,\bar{\Phi}V\Phi V\right) \\ & + \frac{g^3}{6}\left(\bar{\Phi}VVV\Phi - VVV\bar{\Phi}\Phi + 3\,VV\bar{\Phi}V\Phi - 3\,V\bar{\Phi}VV\Phi\right)\Big) + \bar{Q}^I Q_I + \tilde{Q}^I \bar{\tilde{Q}}_I \\ & + g\left(\bar{Q}^I V Q_I - \tilde{Q}^I V \bar{\tilde{Q}}_I\right) + \frac{g^2}{2}\left(\bar{Q}^I VV Q_I + \tilde{Q}^I VV \bar{\tilde{Q}}_I\right) \\ & + \frac{g^3}{6}\left(\bar{Q}^I VVV Q_I - \tilde{Q}^I VVV \bar{\tilde{Q}}_I\right) + \mathrm{tr}\Big(-\frac{1}{2}V\Box V + \frac{g}{2}V\{D^\alpha V, \bar{D}^2 D_\alpha V\} \\ & + \frac{g^2}{8}\,[V, D^\alpha V]\,\bar{D}^2\,[V, D_\alpha V] + \bar{c}'c + c'\bar{c} + \frac{g}{2}\,(c' + \bar{c}')\,[V, (c + \bar{c})] \\ & + \frac{g^2}{12}\,(c' + \bar{c}')\,[V, [V, (c - \bar{c})]]\Big) + \dots\Bigg) \\ & + ig\int d^4x\, d^2\theta\, \tilde{Q}^I \Phi Q_I - ig\int d^4x\, d^2\bar{\theta}\, \bar{Q}^I \bar{\Phi} \bar{\tilde{Q}}_I .\end{aligned} \tag{5.19}$$

Since this action is Euclidean, there are no $i = \sqrt{-1}$ factors in the functional derivatives used to extract the correlation functions from the action, so the vertices of the theory can be immediately read from it (our convention consists in reading the legs of a vertex counterclockwise); for convenience, we choose to work in the momentum space. The Feynman rules for the propagators of superfields propagating from a point 1 to a point 2 with momentum p are the following ones:

$$
\begin{aligned}
&\langle V^a V^b\rangle = -\frac{1}{p^2}\delta\left(\theta_1-\theta_2\right)\delta^{ab}, \\
&\langle \Phi^a \bar{\Phi}^b\rangle = \frac{1}{p^2}\delta\left(\theta_1-\theta_2\right)\delta^{ab}, \\
&\langle Q_{iI}\bar{Q}^{jJ}\rangle = \langle \bar{\tilde{Q}}_{iI}\tilde{Q}^{jJ}\rangle = \frac{1}{p^2}\delta\left(\theta_1-\theta_2\right)\delta_i{}^j\delta_I{}^J, \\
&\langle \bar{c}'^a c^b\rangle = -\langle c'^a \bar{c}^b\rangle = \frac{1}{p^2}\delta\left(\theta_1-\theta_2\right)\delta^{ab}.
\end{aligned}
\tag{5.20}
$$

In the formalism of Feynman superdiagrams, we assign a wavy line to each propagator $\langle VV\rangle$, a continuous line to each propagator $\langle \Phi\bar{\Phi}\rangle$, a dashed line to each propagator $\langle Q\bar{Q}\rangle$ or $\langle \bar{\tilde{Q}}\tilde{Q}\rangle$, and a dotted line to each superghost propagator.

5.3 How to compute a scattering amplitude in $\mathcal{N} = 2$ SCQCD

In Section 5.2 we outlined the main features of $\mathcal{N} = 2$ SCQCD in $\mathcal{N} = 1$ Superspace and we derived the Feynman rules for vertices and propagators of the theory: These tools are essential to perform a diagrammatic computation of an amplitude. Here we summarize the key steps to follow for that purpose.

1. The first thing to do is to choose the amplitude to compute in terms of the external component fields and the fixed color structure; in our convention, we consider all the legs of the amplitude as external states. Then, we assign to each external leg the respective superfield depending on a proper outgoing momentum.

2. The amplitude must be computed in each L-loop perturbative levels, with $L = 0, \dots, L^{(ch)}$ where $L = 0$ is the tree-level and $L = L^{(ch)}$ is the highest loop level we are interested in. We consider the perturbative level L.

3. Following the Feynman rules, we construct all the possible planar L-loop superdiagrams of the chosen fixed structure of step 1.

4. For each L-loop superdiagram we compute the overall factor given by the Feynman rules and any combinatorial factors.

5. For each L-loop superdiagram, we perform the D-algebra following the rules explained in [10]. This step consists in integration by parts of the D and $\bar{D}$ operators present in the internal lines of the superdiagram, in order to bring them to the external legs, with the appropriate relations and identities summarized in Appendix A.2. At the end of this step, the D-algebra is closed and we translate each final diagrammatic term into a proper algebraic string of terms, with its sign (which is fixed through the rules explained in [10]) multiplied by the respective prefactor found in step 4; each final string of terms is multiplied by a $D = 4 - 2\epsilon$ dimensional regularized integral on internal momenta if there are any in the final diagrams.

6. Given the Superspace result of step 5, we have to make the projection $\int d^4\theta(\dots) = \bar{D}^2 D^2(\dots)|_{\theta=\bar{\theta}=0}$ on it by applying the covariant spinor derivatives with the Leibniz rule and the relations collected in Appendix A.2. In this step, we move from the superdiagram with closed D-algebra to all the possible diagrams with component fields we find through the

projection; at the end, we hold only the diagrams with the external component fields we are interested in.

7. We sum all the contributions found in the previous steps: We end up with a linear combination of standard bosonic integrals with numerators, which can be simplified by completion of squares and using on-shell symmetries.

8. We express the final L-loop result, using the integration by part reduction technique explained in [83], as a linear combination of master integrals. This passage could be automated through the algorithm FIRE running on Wolfram Mathematica [84, 85].

9. The master integrals found in step 8 are expanded in terms of ϵ and the L-loop result is presented as a series in the IR divergences poles.

10. We repeat the procedure from step 2 to step 9 one time for each value of L, from 0 to $L^{(ch)}$.

We can follow the first 6 steps with direct computations by hand and the steps 7, 8, 9 with appropriate algorithms in Wolfram Mathematica: If this stategy is carefully planned, it can help to avoid mistakes due to the considerable length of the preliminary results.

5.4 Computation of 5-point amplitudes

In [75], the authors computed 1-loop and 2-loop 4-point scattering amplitudes in planar $\mathcal{N}=2$ SCQCD and made a complete classification of the amplitudes, which can be divided in three independent sectors according to the color representation of the external particles. In particular, in the adjoint subsector, where all the processes have only superfields belonging to the adjoint representation as external legs, after having fixed a color structure, they considered the process $\mathrm{tr}\left(\Phi\Phi\bar{\Phi}\bar{\Phi}\right)$ and its non-ciclic permutation $\mathrm{tr}\left(\Phi\bar{\Phi}\Phi\bar{\Phi}\right)$ (belonging to the adjoint subsector) and they obtained exactly the same expressions of the corresponding $\mathcal{N}=4$ SYM amplitudes up to 1-loop, demonstrating the presence of dual conformal symmetry and maximal transcendentality.

According to [77], an amplitude with 4 scalar component fields is a MHV amplitude; moreover, the first non-trivial 5-point MHV amplitude constructed from the previous one has 4 scalar component fields and a gauge vector component field with positive helicity as external legs. We thus consider the 5-point process $\mathrm{tr}\left(\Phi\Phi\bar{\Phi}\bar{\Phi}V\right)$ in Superspace and, in particular, we are interested in computing the tree-level and 1-loop amplitude of the projection $\mathrm{tr}\left(\phi\phi\bar{\phi}\bar{\phi}A_{\alpha\dot{\alpha}}\right)$; the final goal is to obtain the 1-loop reduced amplitude of the process. We choose this particular color structure in order to minimize the number of superdiagrams: As it is seen in [75], it is possible to show that the amplitude depends on the order of the superfields, but the reduced amplitude is invariant under such a change of order.

Definition of Mandelstam variables and some useful relations. We are dealing with the process $\mathrm{tr}\left(\phi(p_1)\phi(p_2)\bar{\phi}(p_3)\bar{\phi}(p_4)A_{\alpha\dot{\alpha}}(p_5)\right)$ where we assign a proper momentum p_i to each external leg. Since in our convention all the final legs of the diagrams are understood as outgoing states, the total momentum conservation is written as

$$p_1^{\alpha\dot{\alpha}}+p_2^{\alpha\dot{\alpha}}+p_3^{\alpha\dot{\alpha}}+p_4^{\alpha\dot{\alpha}}+p_5^{\alpha\dot{\alpha}}=0. \tag{5.21}$$

These momenta belong to massless states, so we have the following on-shell conditions

$$p_1^2=0, \qquad p_2^2=0, \qquad p_3^2=0, \qquad p_4^2=0, \qquad p_5^2=0. \tag{5.22}$$

Thinking about the gauge component field $A_{\alpha\dot{\alpha}}$, it is known from ordinary Quantum Field Theory that a gauge field of spin 1 has 4 degrees of freedom, but only 2 of them are physical: In fact, the longitudinal degrees of freedom lead to states with negative norm, which are not physical states. In order to consider only physical states, we have to allow only the 2 transversal degrees of freedom of $A_{\alpha\dot{\alpha}}$; this is possible if $A_{\alpha\dot{\alpha}}$ is transversal to its momentum $p_5^{\alpha\dot{\alpha}}$, so we impose

$$A \cdot p_5 = 0. \tag{5.23}$$

We define the Mandelstam variables in order to describe the amplitude: The generic definition of a Mandelstam variable is $s_{ij} = (p_i + p_j)^2$, but if we inpose the on-shell conditions of eq.(5.22), in our case we can simplify the definition into

$$s_{ij} = 2p_i \cdot p_j \qquad for\ \ i, j = 1, \ldots, 5\ \ and\ \ i \neq j. \tag{5.24}$$

Since $s_{ij} = s_{ji}$, we have 10 non-vanishing Mandelstam variables; however, they are linear dependent because of the total momentum conservation eq.(5.21) and the on-shell conditions eq.(5.22). We can choose 5 linear independent Mandelstam variables and write the remaining 5 as linear combinations of the chosen 5. For computation convenience, we use the Mandelstam variables $\{s_{12}, s_{23}, s_{34}, s_{45}, s_{15}\}$ and we write the remaining ones in the following way:

$$\begin{aligned}
s_{13} &= s_{45} - s_{12} - s_{23}, \\
s_{14} &= s_{23} - s_{15} - s_{45}, \\
s_{24} &= s_{15} - s_{23} - s_{34}, \\
s_{25} &= s_{34} - s_{12} - s_{15}, \\
s_{35} &= s_{12} - s_{34} - s_{45}.
\end{aligned} \tag{5.25}$$

The process we are dealing with leads to a MHV amplitude, as we can see in [77]; however, it is necessary to further constraint the amplitute in order to get the expected Parke-Taylor formula. We can read from [86] that it is sufficient to define $A_{\alpha\dot{\alpha}}$ transversal to its momentum and also transversal to another external momentum: The choice is arbitrary because it reflects gauge invariance. We choose $A_{\alpha\dot{\alpha}}$ to be transversal to $p_4^{\alpha\dot{\alpha}}$, so

$$A \cdot p_4 = 0. \tag{5.26}$$

Some useful relations for the projection. As we mentioned before in Section 5.3, after having closed the D-algebra of a given superdiagram, we have to project it in the component fields: This operation corresponds to move from $\mathcal{N} = 1$ Superspace $\mathcal{E}_{4|4}$ to the Euclidean spacetime $\mathcal{E}_4$ by solving the integration over spinorial degrees of freedom. We know that an integration over spinorial variables corresponds to a derivative over the same variables, so we perform $\int d^4\theta(\dots) = \bar{D}^2 D^2(\dots)|_{\theta=\bar{\theta}=0}$ on the algebraic string of terms of the superdiagram; we are interested only in the contributions of $\mathrm{tr}\left(\phi\phi\bar{\phi}\bar{\phi}A_{\alpha\dot{\alpha}}\right)$. In eq.(5.12) we read that

$$\begin{aligned}
&\phi = \Phi|_{\theta=\bar{\theta}=0},\\
&\bar{\phi} = \bar{\Phi}|_{\theta=\bar{\theta}=0},\\
&A_{\alpha\dot{\alpha}} = \frac{1}{2}\left[\bar{D}_{\dot{\alpha}}, D_\alpha\right] V|_{\theta=\bar{\theta}=0} = \frac{1}{2}\left(2\,\bar{D}_{\dot{\alpha}}D_\alpha - \{\bar{D}_{\dot{\alpha}}, D_\alpha\}\right) V|_{\theta=\bar{\theta}=0} =\\
&\qquad = \left(\bar{D}_{\dot{\alpha}}D_\alpha - \frac{1}{2}p_{\alpha\dot{\alpha}}\right) V|_{\theta=\bar{\theta}=0} = \bar{D}_{\dot{\alpha}}D_\alpha V|_{\theta=\bar{\theta}=0}.
\end{aligned}$$

It is not necessary to write all the possible ways to distribute $\bar{D}^2 D^2$ on a generic superdiagram: It is sufficient to find some universal rules in order to get $\mathrm{tr}\left(\phi\phi\bar{\phi}\bar{\phi}A_{\alpha\dot{\alpha}}\right)$ as a final result. The case of $\mathrm{tr}\left(\Phi\Phi\bar{\Phi}\bar{\Phi}V\right)$ is the first one to be considered. In order to obtain $A_{\alpha\dot{\alpha}}$, only a D operator and a $\bar{D}$ operator must be both applied to V; the remaining ones must be applied both in only one leg, in order to generate a momentum and so keep the chiral legs free from the operators. As a consequence, we have

$$\begin{aligned}
&\bar{D}^2 D^2\Big(\mathrm{tr}\left(\Phi\Phi\bar{\Phi}\bar{\Phi}V\right)\Big)\Big|_{\theta=\bar{\theta}=0} =\\
&= \bar{D}^2\Big(\mathrm{tr}\left(D^\alpha\Phi\Phi\bar{\Phi}\bar{\Phi}D_\alpha V\right) + \mathrm{tr}\left(\Phi D^\alpha\Phi\bar{\Phi}\bar{\Phi}D_\alpha V\right)\Big)\Big|_{\theta=\bar{\theta}=0} =\\
&= -\Big(\mathrm{tr}\left(\bar{D}^{\dot{\alpha}}D^\alpha\Phi\Phi\bar{\Phi}\bar{\Phi}\bar{D}_{\dot{\alpha}}D_\alpha V\right) + \mathrm{tr}\left(\Phi\bar{D}^{\dot{\alpha}}D^\alpha\Phi\bar{\Phi}\bar{\Phi}\bar{D}_{\dot{\alpha}}D_\alpha V\right)\Big)\Big|_{\theta=\bar{\theta}=0} =\\
&= -\left(p_1^{\alpha\dot{\alpha}} + p_2^{\alpha\dot{\alpha}}\right)\mathrm{tr}\left(\phi\phi\bar{\phi}\bar{\phi}A_{\alpha\dot{\alpha}}\right),
\end{aligned}$$

where the sign $-$ appears when a covariant spinor derivative jumps an odd number of covariant spinor derivatives. We can prove that the following characteristics we enumerate do not allow a superdiagram to contribute to $\mathrm{tr}\left(\phi\phi\bar{\phi}\bar{\phi}A_{\alpha\dot{\alpha}}\right)$.

- A superdiagram with $D^2\Phi$ or $\bar{D}^2\bar{\Phi}$ (or both) does not contribute because, in order to remove these operators from the (anti)chiral legs, we generate a p^2 factor which is null on-shell.
- The superdiagram $\mathrm{tr}\left(D^\alpha\Phi D_\alpha\Phi\bar{D}^{\dot{\alpha}}\bar{\Phi}\bar{D}_{\dot{\alpha}}\bar{\Phi}V\right)$ does not contribute because $\bar{D}^2 D^2$ is not sufficient both to remove all the 4 operators and to generate $\bar{D}_{\dot{\alpha}}D_\alpha V$.
- The superdiagrams $\mathrm{tr}\left(\Phi\Phi\bar{\Phi}\bar{\Phi}\bar{D}^2 D^2 V\right)$, $\mathrm{tr}\left(\Phi\Phi\bar{\Phi}\bar{\Phi}D^2\bar{D}^2 V\right)$ (and the superdiagram $\mathrm{tr}\left(\Phi\Phi\bar{\Phi}\bar{\Phi}D^\alpha\bar{D}^2 D_\alpha V\right)$ related to them through eq.(A.32)) do

not contribute because it is impossible to generate $\bar{D}_{\dot{\alpha}} D_\alpha V$ without a $p^2 = 0$ factor.

In general, a superdiagram which does not present anyone of the previous 3 characteristics could contribute to $\mathrm{tr}\left(\phi\phi\bar{\phi}\bar{\phi}A_{\alpha\dot{\alpha}}\right)$. Since it is difficult to project a superdiagram with $\bar{D}^2 D_\alpha V$ or other allowed combinations of operators in the V leg, we can exploit the intrinsic features of chiral and antichiral legs by integrating by parts all the operators present in the V leg; we remind that, besides assigning a $-$ sign for a jumping of a covariant spinor derivative, we have to include one more $-$ sign for the integration by parts. After putting together the integration by parts and the rules enumerated before, we find that each superdiagram which contribute to our process can be written as a linear combination of the following superdiagrams:

- $\mathrm{tr}\left(\Phi\Phi\bar{\Phi}\bar{\Phi}V\right)$,
- $\mathrm{tr}\left(D_\alpha\Phi\Phi\bar{D}_{\dot{\alpha}}\bar{\Phi}\bar{\Phi}V\right)$,
- $\mathrm{tr}\left(D_\alpha\Phi\Phi\bar{\Phi}\bar{D}_{\dot{\alpha}}\bar{\Phi}V\right)$,
- $\mathrm{tr}\left(\Phi D_\alpha\Phi\bar{D}_{\dot{\alpha}}\bar{\Phi}\bar{\Phi}V\right)$,
- $\mathrm{tr}\left(\Phi D_\alpha\Phi\bar{\Phi}\bar{D}_{\dot{\alpha}}\bar{\Phi}V\right)$.

For a smarter notation, in our computation we can define

$$
\begin{aligned}
t &= \mathrm{tr}\left(\Phi\Phi\bar{\Phi}\bar{\Phi}V\right), \\
t^{(13)}_{\alpha\dot{\alpha}} &= \mathrm{tr}\left(D_\alpha\Phi\Phi\bar{D}_{\dot{\alpha}}\bar{\Phi}\bar{\Phi}V\right), \\
t^{(14)}_{\alpha\dot{\alpha}} &= \mathrm{tr}\left(D_\alpha\Phi\Phi\bar{\Phi}\bar{D}_{\dot{\alpha}}\bar{\Phi}V\right), \\
t^{(23)}_{\alpha\dot{\alpha}} &= \mathrm{tr}\left(\Phi D_\alpha\Phi\bar{D}_{\dot{\alpha}}\bar{\Phi}\bar{\Phi}V\right), \\
t^{(24)}_{\alpha\dot{\alpha}} &= \mathrm{tr}\left(\Phi D_\alpha\Phi\bar{\Phi}\bar{D}_{\dot{\alpha}}\bar{\Phi}V\right),
\end{aligned}
\tag{5.27}
$$

and, after some trivial steps, we find these projections

$$
\begin{aligned}
\bar{D}^2 D^2\left(t\right)|_{\theta=\bar{\theta}=0} &= -2\ A\cdot(p_1+p_2), \\
\bar{D}^2 D^2\left(t^{(13)}_{\alpha\dot{\alpha}}\right)|_{\theta=\bar{\theta}=0} &= p_{1\alpha}{}^{\dot{\beta}} p_3{}^{\beta}{}_{\dot{\alpha}} A_{\beta\dot{\beta}}, \\
\bar{D}^2 D^2\left(t^{(14)}_{\alpha\dot{\alpha}}\right)|_{\theta=\bar{\theta}=0} &= p_{1\alpha}{}^{\dot{\beta}} p_4{}^{\beta}{}_{\dot{\alpha}} A_{\beta\dot{\beta}}, \\
\bar{D}^2 D^2\left(t^{(23)}_{\alpha\dot{\alpha}}\right)|_{\theta=\bar{\theta}=0} &= p_{2\alpha}{}^{\dot{\beta}} p_3{}^{\beta}{}_{\dot{\alpha}} A_{\beta\dot{\beta}}, \\
\bar{D}^2 D^2\left(t^{(24)}_{\alpha\dot{\alpha}}\right)|_{\theta=\bar{\theta}=0} &= p_{2\alpha}{}^{\dot{\beta}} p_4{}^{\beta}{}_{\dot{\alpha}} A_{\beta\dot{\beta}},
\end{aligned}
\tag{5.28}
$$

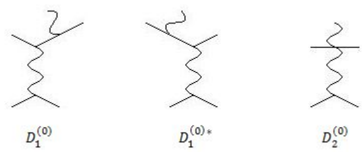
where we define shortly

$$A_{\alpha\dot{\alpha}} = \mathrm{tr}\left(T^a T^b T^c T^d T^e\right) \phi_a \phi_b \bar{\phi}_c \bar{\phi}_d A_{e\alpha\dot{\alpha}}. \tag{5.29}$$

Since the amplitude we are dealing with is a tensorial amplitude, there is a problem in defining the reduced amplitude: In fact, we can not define an object which is a fraction of two tensors. To avoid that formal problem, we consider the scalar amplitude, so we saturate the tensorial (spinor) indices by contracting them with $A_{\alpha\dot{\alpha}}$. Now all is ready for the computation of the MHV amplitude.

5.4.1 $\mathrm{tr}\left(\phi\phi\bar{\phi}\bar{\phi}A_{\alpha\dot{\alpha}}\right)$: tree-level MHV amplitude

Given the process $\mathrm{tr}\left(\phi(p_1)\phi(p_2)\bar{\phi}(p_3)\bar{\phi}(p_4)A_{\alpha\dot{\alpha}}(p_5)\right)$, the tree-level diagrams which contribute to the amplitude of it are collected in the following image.



We remind that, in our convention, we read the external legs of a diagram counterclockwise. We use the sign "$*$" to indicate a superdiagram which is specular (right-left symmetric image) to another superdiagram with the same name and without the sign "$*$". Following the first steps summarized in Section 5.3, we close the D-algebras of these superdiagrams finding

$$\begin{aligned}
D_1^{(0)} &\to \frac{g^3}{s_{23}s_{45}}\Big(-s_{12}\,t + p_1^{\alpha\dot{\alpha}}\,t^{(23)}_{\alpha\dot{\alpha}} + p_2^{\alpha\dot{\alpha}}\,t^{(13)}_{\alpha\dot{\alpha}}\Big), \\
D_1^{(0)*} &\to \frac{g^3}{s_{23}s_{15}}\Big(-s_{34}\,t - p_3^{\alpha\dot{\alpha}}\,t^{(24)}_{\alpha\dot{\alpha}} - p_4^{\alpha\dot{\alpha}}\,t^{(23)}_{\alpha\dot{\alpha}}\Big), \\
D_2^{(0)} &\to \frac{g^3}{s_{23}}t.
\end{aligned} \tag{5.30}$$

We can project these terms with the help of eq.(5.28) and find

$$\begin{aligned}
D_1^{(0)} &= -2g^3\frac{s_{12}}{s_{23}s_{45}}A\cdot p_4, \\
D_1^{(0)*} &= 2g^3\frac{s_{34}}{s_{23}s_{15}}A\cdot p_1, \\
D_2^{(0)} &= -2g^3\frac{1}{s_{23}}A\cdot(p_1+p_2)\,.
\end{aligned} \tag{5.31}$$

If we sum all the terms in eq.(5.31) and impose the MHV condition eq.(5.26), we find the tree-level MHV amplitude

$$\mathcal{A}^{(0)}_{MHV} = \frac{2g^3}{s_{23}}\left(\frac{s_{34}}{s_{15}} A \cdot p_1 + A \cdot p_3\right). \tag{5.32}$$

This result is in agreement with the Parke-Taylor formula

$$\mathcal{A}^{(0)}_{MHV} = Cg^3 \frac{\langle 12\rangle\langle 34\rangle}{\langle 23\rangle\langle 45\rangle\langle 51\rangle}, \tag{5.33}$$

written in the helicity formalism, where C contains the remaining overall factors. In fact, as we can find summarized in [86], if we use the identities

$$\begin{aligned} s_{ij} &= \langle ij\rangle\,[ij]\,, \\ A \cdot p1 &= \frac{1}{\sqrt{2}}\frac{\langle 41\rangle\,[15]}{\langle 45\rangle}, \\ A \cdot p3 &= \frac{1}{\sqrt{2}}\frac{\langle 43\rangle\,[35]}{\langle 45\rangle}, \end{aligned} \tag{5.34}$$

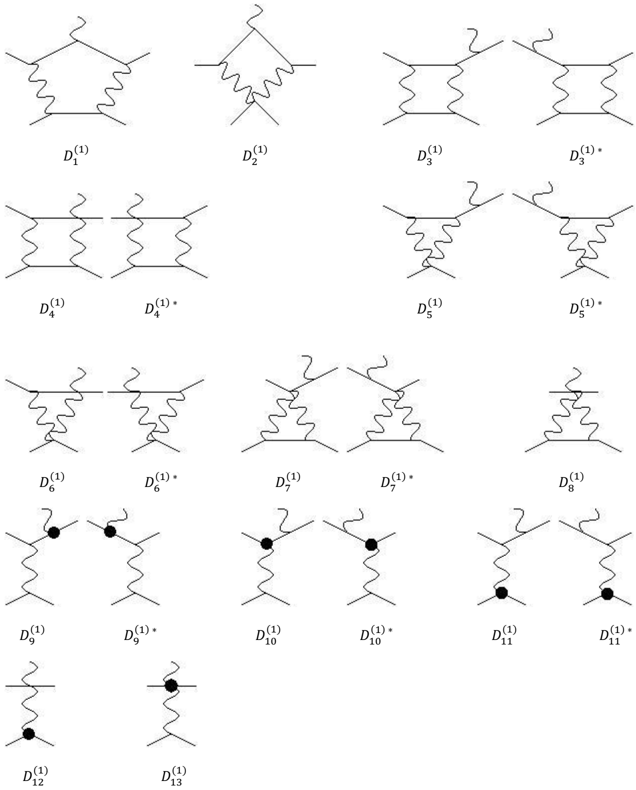
with $\langle ij\rangle = -\langle ji\rangle$ and $[ij] = -\,[ji]$ and the momentum conservation

$$\sum_{i=1}^{5}\langle qi\rangle\,[ik] = 0 \qquad \forall k, q = 1, \ldots, 5, \tag{5.35}$$

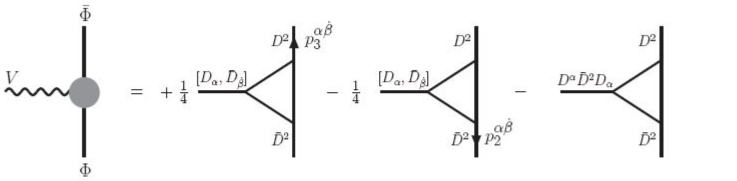
we find that eq.(5.32) is equivalent to eq.(5.33).

5.4.2 tr $\left(\phi\phi\bar{\phi}\bar{\phi}A_{\alpha\dot{\alpha}}\right)$: 1-loop MHV amplitude

We now analyze the 1-loop perturbative order of the process tr $\left(\phi\phi\bar{\phi}\bar{\phi}A_{\alpha\dot{\alpha}}\right)$. Starting from the tree-level superdiagrams, we have to draw all the possible 1-loop internal insertions. It is possible to demonstrate (and this could be found in [75]) that the 1-loop correction of each propagator in $\mathcal{N} = 2$ SCQCD is null. As a consequence, the 1-loop superdiagrams of our process are found from the tree-level ones in two ways: By drawing vector propagators starting from an internal line and ending in another internal line or by inserting 1-loop corrections to the vertices. The 1-loop superdiagrams which contribute to our amplitude are collected in the following image.



The dots in bold stand for 1-loop vertex corrections, which we can compute separately and express as effective 1-loop vertices. One 1-loop off-shell vertex correction of $\mathrm{tr}\left(V(p_1)\Phi(p_2)\bar{\Phi}(p_3)\right)$ (where we make an abuse of notation in labelling momenta with the same previous names) can be expressed in the following smart expression [75]



(5.36)

where an overall factor $g^3 N_c$ is stripped out. The 1-loop off-shell vertex correction of tr $\big(V(p_1)\bar{\Phi}(p_2)\Phi(p_3)\big)$ has the same expression, with the change $D \leftrightarrow \bar{D}$ in the chiral and antichiral legs and with a sign $+$ on the third term. We insert these two effective 1-loop vertices in $\{D_9^{(1)}, D_9^{(1)*}, D_{10}^{(1)}, D_{10}^{(1)*}, D_{11}^{(1)}, D_{11}^{(1)*}, D_{12}^{(1)}\}$ and properly close their D-algebras. More difficulties come out when we approach the superdiagram $D_{13}^{(1)}$: In fact, it presents a 1-loop correction to the 4-leg vertex tr $\big(\Phi V \bar{\Phi} V\big)$, which is not yet computed in works dealing with $\mathcal{N} = 2$ SCQCD.

We start with the computation of the 1-loop amplitude; since the procedure is well explained in Section 5.3, we do not present all the intermediate steps; rather, we show the result of the projection of each superdiagram with its implicit regularized loop integral over the internal momentum $k^{\alpha\dot{\alpha}}$, in $D = 4 - 2\epsilon$, with energy scale μ. For a smarter notation, we define

$$
\begin{aligned}
k_1 &= k, \\
k_2 &= k - p_2, \\
k_3 &= k - p_2 - p_3, \\
k_4 &= k - p_2 - p_3 - p_4, \\
k_5 &= k + p_1.
\end{aligned}
\tag{5.37}
$$

We find the following preliminary results for the first 13 superdiagrams of the previous image:

$$
\begin{aligned}
D_1^{(1)} &= -2g^5 N_c\, \mu^{2\epsilon} s_{12} s_{34} \int \frac{d^D k_1}{(2\pi)^D} \frac{A \cdot k_5}{k_1^2\, k_2^2\, k_3^2\, k_4^2\, k_5^2}, \\
D_2^{(1)} &= \frac{1}{2} g^5 N_c\, \mu^{2\epsilon} \int \frac{d^D k_1}{(2\pi)^D} \frac{(s_{12} + s_{34}) A \cdot k_5 + 2(k_5^2 + k_5 \cdot p_5) A \cdot (p_1 + p_2)}{k_1^2\, k_3^2\, k_4^2\, k_5^2}, \\
D_3^{(1)} &= 2g^5 N_c\, \mu^{2\epsilon} \frac{s_{12}^2}{s_{45}} A \cdot p_4 \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2\, k_2^2\, k_3^2\, k_5^2}, \\
D_3^{(1)*} &= -2g^5 N_c\, \mu^{2\epsilon} \frac{s_{34}^2}{s_{15}} A \cdot p_1 \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2\, k_2^2\, k_3^2\, k_4^2}, \\
D_4^{(1)} &= 2g^5 N_c\, \mu^{2\epsilon} s_{12} A \cdot (p_1 + p_2) \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2\, k_2^2\, k_3^2\, k_5^2}, \\
D_4^{(1)*} &= 2g^5 N_c\, \mu^{2\epsilon} s_{34} A \cdot (p_1 + p_2) \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2\, k_2^2\, k_3^2\, k_4^2}, \\
D_5^{(1)} &= -g^5 N_c\, \mu^{2\epsilon} \frac{s_{12}}{s_{45}} A \cdot p_4 \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2\, k_3^2\, k_5^2}, \\
D_5^{(1)*} &= g^5 N_c\, \mu^{2\epsilon} \frac{s_{34}}{s_{15}} A \cdot p_1 \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2\, k_3^2\, k_4^2}, \\
D_6^{(1)} &= -g^5 N_c\, \mu^{2\epsilon} A \cdot (p_1 + p_2) \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2\, k_3^2\, k_5^2}, \\
D_6^{(1)*} &= -g^5 N_c\, \mu^{2\epsilon} A \cdot (p_1 + p_2) \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2\, k_3^2\, k_4^2}, \\
D_7^{(1)} &= -g^5 N_c\, \mu^{2\epsilon} \frac{s_{12}}{s_{45}} A \cdot p_4 \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2\, k_2^2\, k_3^2}, \\
D_7^{(1)*} &= g^5 N_c\, \mu^{2\epsilon} \frac{s_{34}}{s_{15}} A \cdot p_1 \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2\, k_2^2\, k_3^2}, \\
D_8^{(1)} &= -g^5 N_c\, \mu^{2\epsilon} A \cdot (p_1 + p_2) \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2\, k_2^2\, k_3^2}.
\end{aligned}
\tag{5.38}
$$

We notice that it is possible to define a "mirror duality" as a transformation which assigns a $-$ and exchanges the momenta in the following way:

$$
p_1 \leftrightarrow p_4 \quad p_2 \leftrightarrow p_3 \quad k_1 \leftrightarrow -k_3 \quad k_2 \leftrightarrow -k_2 \quad k_4 \leftrightarrow -k_5 \tag{5.39}
$$

This mirror duality maps

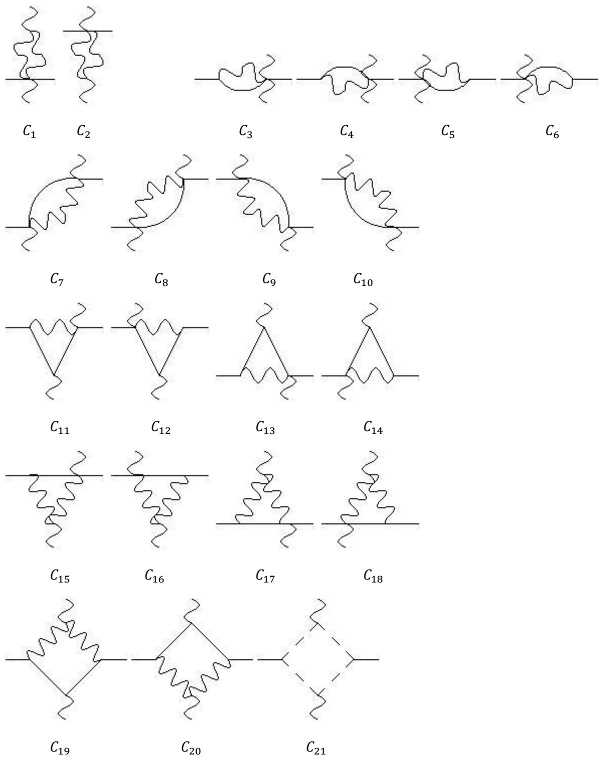
$$
\begin{aligned}
&D_1^{(1)} \to D_1^{(1)}, \quad D_2^{(1)} \to D_2^{(1)}, \quad D_3^{(1)} \to D_3^{(1)*}, \quad D_4^{(1)} \to D_4^{(1)*}, \\
&D_5^{(1)} \to D_5^{(1)*}, \quad D_6^{(1)} \to D_6^{(1)*}, \quad D_7^{(1)} \to D_7^{(1)*}, \quad D_8^{(1)} \to D_8^{(1)}.
\end{aligned} \tag{5.40}
$$

The preliminary results of the superdiagrams with a 1-loop 3-leg vertex correction are collected below:

$$
\begin{aligned}
D_9^{(1)} &= -g^5 N_c\, \mu^{2\epsilon} \frac{s_{12}}{s_{23}} A \cdot p_4 \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_3^2\, k_4^2\, k_5^2}, \\
D_9^{(1)*} &= -g^5 N_c\, \mu^{2\epsilon} \frac{s_{34}}{s_{23}} A \cdot p_1 \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2\, k_4^2\, k_5^2}, \\
D_{10}^{(1)} &= g^5 N_c\, \mu^{2\epsilon} \left(\frac{s_{12}}{s_{23}} + 2\frac{s_{12}}{s_{45}} \right) A \cdot p_4 \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2\, k_3^2\, k_5^2}, \\
D_{10}^{(1)*} &= g^5 N_c\, \mu^{2\epsilon} \left(\frac{s_{34}}{s_{23}} - 2\frac{s_{34}}{s_{15}} \right) A \cdot p_1 \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2\, k_3^2\, k_4^2}, \\
D_{11}^{(1)} &= 2g^5 N_c\, \mu^{2\epsilon} \frac{s_{12}}{s_{45}} A \cdot p_4 \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2\, k_2^2\, k_3^2}, \\
D_{11}^{(1)*} &= -2g^5 N_c\, \mu^{2\epsilon} \frac{s_{34}}{s_{15}} A \cdot p_1 \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2\, k_2^2\, k_3^2}, \\
D_{12}^{(1)} &= 2g^5 N_c\, \mu^{2\epsilon} A \cdot (p_1 + p_2) \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^2\, k_2^2\, k_3^2}.
\end{aligned} \tag{5.41}
$$

It is curious that some of the contributions in eq.(5.41) break partially or totally the mirror duality by showing a different sign. We can think about this fact as there are some diagrams with an intrinsic parity: The fact that in some cases of eq.(5.41) there are not overall signs but internal signs is because these cases are made by different diagrams bringing their respective parity.

Now we focus on the superdiagram $D_{13}^{(1)}$. Its dot in bold hides a lot of non-trivial contributions that we can collect in the following image.



Since a complete derivation of the off-shell 1-loop correction to that 4-leg vertex is not known in the literature, for our purposes it is sufficient to comute separately each insertion of each contribution C_i of the image (with $i = 1, \ldots, 21$);

as a consequence, we have to compute 21 superdiagrams in order to obtain the value of $D_{13}^{(1)}$. We easily find that the insertions of C_1 and C_2 lead to 0. An important check to consider is the absence of UV divergences. An UV divergence rises from an integral like $\int \frac{d^4k}{k^a}$ with $a \leq 4$, so we have to look at what happens to all the loops with only two internal propagators. After closing all the D-algebras, we can conclude that the expression $\sum_{i=3}^{14} C_i + C_{21}$ cancels all the UV divergences if we impose $N_f = 2N_c$, which is the conformal constraint.

After having projected each of the 19 non-vanishing diagrams which form $D_{13}^{(1)}$, we sum them to all the contributions of eq.(5.38) and eq.(5.41), in order to obtain the integrand of the amplitude. In that phase of the computation, it is necessary to develop an algorithmic method in order to avoid algebraic errors: In fact, all the outputs consist in very long strings of mathematical expressions. For convenience, we use Wolfram Mathematica; in particular, we insert the closed D-algebras as input and we perform the projection by replacing the projections of the main structures. We get a very long expression and we further complete the squares of all the numerators in the integrals with the on-shell relations

$$\begin{aligned}
A \cdot k_1 &= \frac{1}{2}\left((A+k_1)^2 - A^2 - k_1^2\right), \\
k_1 \cdot p_1 &= \frac{1}{2}\left(k_5^2 - k_1^2\right), \\
k_1 \cdot p_2 &= \frac{1}{2}\left(k_1^2 - k_2^2\right), \\
k_1 \cdot p_3 &= \frac{1}{2}\left(k_2^2 - k_3^2 + s_{23}\right), \\
k_1 \cdot p_4 &= \frac{1}{2}\left(k_3^2 - k_4^2 + s_{15} - s_{23}\right), \\
k_1 \cdot p_5 &= -k_1 \cdot p_1 - k_1 \cdot p_2 - k_1 \cdot p_3 - k_1 \cdot p_4,
\end{aligned} \tag{5.42}$$

where, for the moment, we can leave the explicit A^2 term. For convenience, we write the partial result with the help of the function

$$F\left(a_1, a_2, a_3, a_4, a_5, a_6\right) = \mu^{2\epsilon} \int \frac{d^D k_1}{(2\pi)^D} \frac{1}{k_1^{2a_1}\, k_2^{2a_2}\, k_3^{2a_3}\, k_4^{2a_4}\, k_5^{2a_5}\, (A+k_1)^{2a_6}}. \tag{5.43}$$

The 1-loop MHV amplitude to compute becomes

$$\begin{aligned}
\mathcal{A}^{(1)}_{MHV} = g^5 N_c \Bigg(& 2\frac{s_{34}}{s_{23}} A\cdot p_1 \; F(1,0,1,1,0,0) - \frac{s_{34}}{s_{15}} A\cdot p_1 \; F(1,0,1,1,0,0) \\
& - 2\frac{s_{34}s_{45}}{s_{23}} A\cdot p_1 \; F(1,0,1,1,1,0) - 2\frac{s_{15}s_{45}}{s_{23}} A\cdot p_3 \; F(1,0,1,1,1,0) \\
& - \frac{s_{34}}{s_{15}} A\cdot p_1 \; F(1,1,1,0,0,0) - 2\frac{s_{34}^2}{s_{15}} A\cdot p_1 \; F(1,1,1,1,0,0) \\
& + s_{12}s_{34} \; F(0,1,1,1,1,0) - A\cdot p_3 \; F(1,1,1,0,0,0) \\
& - 2s_{12} \; A\cdot p_3 \; F(1,1,1,0,1,0) - 2s_{34} \; A\cdot p_3 \; F(1,1,1,1,0,0) \\
& - s_{12}s_{34} \; F(1,1,1,1,1,-1) + s_{12}s_{34} \; A^2 \; F(1,1,1,1,1,0) \\
& - 2s_{12}s_{34} \; A\cdot p_1 \; F(1,1,1,1,1,0) \Bigg).
\end{aligned} \tag{5.44}$$

The following step we have to perform is to compute all the scalar integrals in eq.(5.44). We use an analytic method which reduces our integrals into a linear combination of master integrals whose ϵ-expansion is known [83]. In particular, we automate this method through the "FIRE" package of Wolfram Mathematica [84, 85]. From [87], we know that the 1-loop massless pentagon integral in $D = 4$ could be expanded into a cyclic sum of box integrals with a massive external leg, plus a $\mathcal{O}(\epsilon)$ term that we can neglect. We find the 1-loop MHV amplitude written in terms of the following master integrals:

$$\begin{aligned}
B_1 &= F(0,1,1,1,1,0), \\
B_2 &= F(1,0,1,1,1,0), \\
B_3 &= F(1,1,0,1,1,0), \\
B_4 &= F(1,1,1,0,1,0), \\
B_5 &= F(1,1,1,1,0,0), \\
G_1 &= F(1,0,1,0,0,0), \\
G_2 &= F(1,0,0,1,0,0),
\end{aligned} \tag{5.45}$$

where we can identify them with their ϵ-expansion. The 1-loop MHV amplitude we find is

$$
\begin{aligned}
\mathcal{A}^{(1)}_{MHV} = g^5 N_c \frac{e^{-\gamma\epsilon}}{(4\pi)^{2-\epsilon}} \Bigg(& -\frac{s_{34}s_{45}(s_{34}\ A\cdot p_1 + s_{15}\ A\cdot p_3)}{s_{15}s_{23}} B_1(\epsilon) \\
& - \frac{s_{45}(s_{34}\ A\cdot p_1 + s_{15}\ A\cdot p_3)}{s_{23}} B_2(\epsilon) - \frac{s_{12}(s_{34}\ A\cdot p_1 + s_{15}\ A\cdot p_3)}{s_{23}} B_3(\epsilon) \\
& - \frac{s_{12}(s_{34}\ A\cdot p_1 + s_{15}\ A\cdot p_3)}{s_{15}} B_4(\epsilon) - \frac{s_{34}(s_{34}\ A\cdot p_1 + s_{15}\ A\cdot p_3)}{s_{15}} B_5(\epsilon) \\
& + \frac{(3s_{15} - 2s_{23})s_{34}\ A\cdot p_1 + s_{15}(s_{15} - s_{23}) A\cdot p_3}{s_{15}(s_{15} - s_{23})s_{23}} \frac{2\epsilon - 1}{\epsilon} G_1(\epsilon) \\
& - \frac{(2s_{15} - s_{23})s_{34} A\cdot p_1}{s_{15}(s_{15} - s_{23})s_{23}} \frac{2\epsilon - 1}{\epsilon} G_2(\epsilon) \Bigg),
\end{aligned}
\tag{5.46}
$$

and we can directly compute the reduced amplitude by dividing eq.(5.46) with the thee-level amplitude of eq.(5.32). The final result is

$$
\begin{aligned}
\mathcal{M}^{(1)}_{MHV} = \lambda \frac{e^{-\gamma\epsilon}}{(4\pi)^{-\epsilon}} \Bigg(& -\frac{1}{2} \Big(s_{34}s_{45}B_1(\epsilon) + s_{15}s_{45}B_2(\epsilon) + s_{12}s_{15}B_3(\epsilon) \\
& + s_{12}s_{23}B_4(\epsilon) + s_{23}s_{34}B_5(\epsilon) \Big) \\
& + \frac{(2s_{15} - s_{23})a + 3s_{15}s_{23} - 2s_{23}^2}{(s_{15} - s_{23})s_{23}} \frac{2\epsilon - 1}{2\epsilon} G_1(\epsilon) \\
& - \frac{(2s_{15} - s_{23})(a + s_{23})}{(s_{15} - s_{23})s_{23}} \frac{2\epsilon - 1}{2\epsilon} G_2(\epsilon) \Bigg),
\end{aligned}
\tag{5.47}
$$

where we define $a = -2\frac{A\cdot p_3}{\mathcal{A}^{(0)}_{MHV}}$ for a smarter notation.

In [88] we read that the analogous 5-point process in $\mathcal{N} = 4$ SYM has its reduced amplitude written as a cyclic combination of boxes with a massive leg, our B_i (with $i = 1, \ldots, 5$); since $\mathcal{N} = 2$ SCQCD offers the same results of $\mathcal{N} = 4$ SYM at the perturbative orders $L = 0$ and $L = 1$, we can conclude that the reduced amplitude of eq.(5.47) is not correct. As a consequence, this computation is still an open work; it is easy to think that such a complicated computation could require more suitable techniques and some more time to be finished. However, we can notice that the first terms

$$s_{34}s_{45}B_1(\epsilon) + s_{15}s_{45}B_2(\epsilon) + s_{12}s_{15}B_3(\epsilon) + s_{12}s_{23}B_4(\epsilon) + s_{23}s_{34}B_5(\epsilon)$$

appear in a symmetric cyclic form. As a consequence, we achieve the right IR behavior of our process if we limit the result to the first 5 terms; a more accurate computation of the amplitude could lead to a cancellation of the terms

$$\frac{(2s_{15} - s_{23})a + 3s_{15}s_{23} - 2s_{23}^2}{(s_{15} - s_{23})s_{23}} \frac{2\epsilon - 1}{2\epsilon} G_1(\epsilon) - \frac{(2s_{15} - s_{23})(a + s_{23})}{(s_{15} - s_{23})s_{23}} \frac{2\epsilon - 1}{2\epsilon} G_2(\epsilon),$$

which describe a different behavior which does not follow the BDS ansatz.

Chapter 6

Conclusions

Here we collect some of the main concepts achieved in this thesis and some final comments.

In Chapter 4, driven by the results of [17, 18], we have presented the explicit geometric construction of the $D = 4$ generalized (minimal) AdS-Lorentz deformed supergravity bulk Lagragian (based on the generalized minimal AdS-Lorentz superalgebra of [47]). In particular, we have studied the supersymmetry invariance of the Lagrangian in the presence of a non-trivial boundary of space-time, finding that the supersymmetric extension of a Gauss-Bonnet like term is required in order to restore the supersymmetry invariance of the full Lagrangian. In this way, we have also further investigated on the study performed in [47] in the context of AdS-Lorentz superalgebras and generalized supersymmetric cosmological constant terms in $\mathcal{N} = 1$ supergravity.

The presence of the 1-form fields $\tilde{k}^{ab}$, k^{ab}, and ξ in the boundary could be useful in the context of the AdS/CFT correspondence. In particular, as it was shown in [89], the introduction of a topological boundary in a 4-dimensional bosonic action is equivalent to the holographic renormalization procedure in the AdS/CFT context. Then, we conjecture that the presence of $\tilde{k}^{ab}$, k^{ab}, and ξ in the boundary of our theory, allowing to recover the supersymmetry invariance in the geometric approach, could also allow to regularize the deformed supergravity action in the holographic renormalization context.

In Chapter 4, we have also observed that both the AdS-Lorentz and the generalized minimal AdS-Lorentz superalgebras can be viewed as peculiar torsion deformations of $\mathfrak{osp}(4|1)$. This is intriguing, since, on the other hand, the same superalgebras can be obtained through S-expansion from $\mathfrak{osp}(4|1)$ by using semigroups of the type $S_{\mathcal{M}}^{(2n)}$, with $n \geq 1$ ($S_{\mathcal{M}}^{(2)}$ and $S_{\mathcal{M}}^{(4)}$, respectively, see [47] for details). Then, our results could be useful to shed some light on the properties and physical role of these semigroups, also in higher-dimensional cases. Moreover, the form of the MacDowell-Mansouri like action obtained in [47] by considering the generalized minimal AdS-Lorentz superalgebra coincides with the one in eq.(4.53), obtained by adopting a geometric approach. We argue that all the superalgebras which can be obtained through S-expansion from $\mathfrak{osp}(4|1)$ by using semigroups of the type $S_{\mathcal{M}}^{(2n)}$ ($n \geq 1$) can be viewed as particular torsion deformations of $\mathfrak{osp}(4|1)$, in the sense intended, and that they can consequently lead to MacDowell-Mansouri like actions involving supersymmetric extension of Gauss-Bonnet like terms allowing the supersymmetry invariance of the full Lagrangians in the presence of a non-trivial boundary of spacetime.

Another future analysis could consist in investigating the possible relations among the extra 1-form fields appearing in the generalized minimal AdS-Lorentz superalgebra and the extra 1-forms appearing in the hidden superalgebras underlying supergravity theories in higher dimensions [50, 51, 52, 54].

Finally, one could also carry on a further analysis in order to shed some light on the boundary theory produced in our geometric approach. In this context, we stress that in our framework the supersymmetry invariance constrains the boundary values of the supercurvatures (Neumann boundary conditions), without fixing, however, the superfields themselves on the boundary. The boundary conditions obtained within our approach are still written in terms of 4-dimensional fields and give the values of the curvatures on the 3-dimensional boundary, that is on the contour of the 4-dimensional space-time, while in order to discuss the theory living on the boundary (in the spirit of the the AdS/CFT correspondence, where the supergravity fields act as sources for the CFT operators) one should set the boundary at infinity (that is at $r \to \infty$, being r the radial coordinate) and study the asymptotic limit $r \to \infty$ of the $D = 3$ equations on the boundary. The explicit 3-dimensional description of the equations we have found in $D = 4$ would depend on the general symmetry properties of the theory on the boundary, which can be obtained as an effective theory on an asymptotic boundary placed at $r \to \infty$. One should properly choose the boundary behavior of the $D = 4$ fields which relates them to the $D = 3$ ones and perform the asymptotic limit $r \to \infty$. Since such a study goes beyond the aim of Chapter 4 and would require a lot of work and further calculations, we leave it as a future development.

Dealing with $\mathcal{N} = 2$ SCQCD, we have explored the computation of a 5-point MHV scattering amplitude at 1-loop; the final result is not yet achieved, so it is an open work to be completed. There are other analytic techniques to perform it, but there are no reasons to give up the diagrammatic computation or the reduction of integrals: They are two fundamental tools in order to find the final result. Certainly, the problem is hidden inside the many steps of the computation; in order to solve it, a good idea is to plan a different algorithm to support the computation. Wolfram Mathematica could help even if it is recommended to use the last version in order to avoid to fill in some shortcomings by non-automatic steps. A good algorithm should improve the control over the varius steps; the lack of explicit control has been a serious problem for our computation.

Once found the way to compute exactly, a direct development of this work would consist in comparing the amplitude in the adjoint sector with the corresponding amplitude in the fundamental sector and in the mixed sector, following the order of [75] in the classification.

Appendix A

Notation

In this thesis, each formula is expressed in natural units, with $\hbar = c = 1$. We split the natation in two sections; the first one is dedicated to the conventions adopted in supergravity, while the second one collects all the useful relations when we deal with supersymmetric gauge theories.

A.1 Notation in supergravity

We consider a generic $(3+1)$-dimensional spacetime and we define a spacetime index labeled with Greek letters from the center of the alphabet onwards $(\mu, \nu, \dots)$, with possible values $\mu = 0, \dots, 3$ ($\mu = 0$ corresponds to a time index); we adopt the Einstein convention of sum over contracted indices. In absence of gravity, we have a Minkowskian spacetime with metric tensor

$$\eta_{\mu\nu} = diag(-1, +1, +1, +1). \tag{A.1}$$

with metric signature $(3, 1)$. Spacetime coordinates are denoted as x^μ; spacetime indices are raised and lowered with the respective relations

$$\begin{aligned} x^\mu &= \eta^{\mu\nu} x_\nu \\ x_\mu &= \eta_{\mu\nu} x^\nu. \end{aligned} \tag{A.2}$$

In a curved spacetime, which is seen as a differential manifold M, the metric tensor is $g_{\mu\nu}$. We adopt the dual formulation of gravity, with the formalism of

k-forms in the cotangent space $T^*(M)$. Given a basis of 1-forms $\{dx^\mu\}$, which satisfy the relation

$$dx^\mu \left(\frac{d}{dx^\nu} \right) = \delta^\mu_\nu \tag{A.3}$$

being $\{\frac{d}{dx^\mu}\}$ a vector basis and δ^μ_ν the Kronecker delta, each 1-form is a linear combination of 1-forms belonging to the basis $\{dx^\mu\}$. A generic k-form $\omega^{(k)}$ is completely antisymmetric and it is written as follows

$$\omega^{(k)} = a_{\mu_1 \dots \mu_k} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_k} \tag{A.4}$$

where $\wedge$ is the wedge product and $a_{\mu_1 \dots \mu_k}$ is completely antisymmetric. Sometimes, where it is necessary, we explicitly antisymmetrize indices with square brackets. Given a k-form $\omega^{(k)}$ and a q-form $\omega^{(q)}$, their wedge product follows the relation

$$\omega^{(k)} \wedge \omega^{(q)} = (-1)^{kq}\, \omega^{(q)} \wedge \omega^{(k)}, \tag{A.5}$$

$\forall k, q \in \mathbb{N}$; moreover, the basic properties of the wedge product are associativity and distributivity. We denote $T^{*(k)}(M)$ the subsector of the cotangent space in which are collected all the k-forms.

We study supergravity in $\mathcal{N} = 1$ $D = 4$ Superspace through the geometric approach; in Superspace, spacetime indices are denoted with latin letters. We choose a new set of 1-forms $\{V^a\}$ (with $a = 0, \dots, 3$) called vielbeins, which are related to the previous set $\{dx^\mu\}$ through

$$V^a = \mathcal{V}^a_\mu\, dx^\mu \tag{A.6}$$

where $\mathcal{V}^a_\mu \in GL(4, \mathbb{R})$ are real invertible matrices satisfying $\mathcal{V}^a_\mu \mathcal{V}^\mu_b = \delta^a_b$ and $\mathcal{V}^\mu_a \mathcal{V}^a_\nu = \delta^\mu_\nu$. In the vielbein basis, the metric tensor is written as

$$g_{\mu\nu} = \mathcal{V}^a_\mu\, \mathcal{V}^b_\nu\, \eta_{ab}. \tag{A.7}$$

$\mathcal{N} = 1$ Superspace coordinates are a set of 1-forms $\{V^a, \psi^\alpha\}$, where V^a is a vielbein and ψ^α is a Majorana spinorial 1-form (with spinorial index $\alpha = 1, \dots, 4$ that will be omitted for simplicity).

The gamma matrices are defined through

$$\begin{aligned} &\{\gamma_a, \gamma_b\} = -2\eta_{ab}, \\ &[\gamma_a, \gamma_b] = 2\gamma_{ab}, \\ &\gamma_5 = -\gamma_0\gamma_1\gamma_2\gamma_3, \end{aligned} \tag{A.8}$$

and they satisfy the algebraic relations

$$
\begin{aligned}
&\gamma_5^2 = -1, \\
&\{\gamma_5, \gamma_a\} = [\gamma_5, \gamma_{ab}] = 0, \\
&\gamma_{ab}\gamma_5 = -\frac{1}{2}\epsilon_{abcd}\gamma^{cd}, \\
&\gamma_a\gamma_b = \gamma_{ab} - \eta_{ab}, \\
&\gamma^{ab}\gamma_{cd} = \epsilon^{ab}{}_{cd}\gamma_5 - 4\delta^{[a}_{[c}\gamma^{b]}{}_{d]} - 2\delta^{ab}_{cd}, \\
&\gamma^{ab}\gamma^c = 2\gamma^{[a}\delta^{b]}_c - \epsilon^{abcd}\gamma_5\gamma_d, \\
&\gamma^c\gamma^{ab} = -2\gamma^{[a}\delta^{b]}_c - \epsilon^{abcd}\gamma_5\gamma_d, \\
&\gamma_m\gamma^{ab}\gamma^m = 0, \\
&\gamma_{ab}\gamma_m\gamma^{ab} = 0, \\
&\gamma_{ab}\gamma_{cd}\gamma^{ab} = 4\gamma_{cd}, \\
&\gamma_m\gamma^a\gamma^m = -2\gamma^a,
\end{aligned}
\tag{A.9}
$$

where ϵ^{abcd} is the Levi-Civita tensor and $\delta^{ab}_{cd} = \frac{1}{2}\left(\delta^a{}_c\delta^b{}_d - \delta^a{}_d\delta^b{}_c\right)$. Furthermore, we have

$$
\begin{aligned}
&(C\gamma_a)^T = C\gamma_a, \quad (C\gamma_{ab})^T = C\gamma_{ab}, \\
&(C\gamma_5)^T = -C\gamma_5, \quad (C\gamma_5\gamma_a)^T = -C\gamma_5\gamma_a,
\end{aligned}
\tag{A.10}
$$

where C is the charge conjugation matrix ($C^T = -C$). We are dealing with Majorana spinors, fulfilling $\bar{\psi} = \psi^T C$. The following identities hold

$$
\begin{aligned}
&\bar{\psi}^{(p)} \wedge \xi^{(q)} = (-1)^{pq}\bar{\xi}^{(q)} \wedge \psi^{(p)}, \\
&\bar{\psi}^{(p)} \wedge S\xi^{(q)} = -(-1)^{pq}\bar{\xi}^{(q)} \wedge S\psi^{(p)}, \\
&\bar{\psi}^{(p)} \wedge A\xi^{(q)} = (-1)^{pq}\bar{\xi}^{(q)} \wedge A\psi^{(p)}
\end{aligned}
\tag{A.11}
$$

for the p-form $\psi^{(p)}$ and q-form $\xi^{(q)}$, being S and A symmetric and antisymmetric matrices, respectively. Finally, we can write the Fierz identities in $D = 4$ for the 1-form spinor ψ:

$$
\begin{aligned}
&\psi \wedge \bar{\psi} = \frac{1}{2}\gamma_a\bar{\psi} \wedge \gamma^a\psi - \frac{1}{8}\gamma_{ab}\bar{\psi} \wedge \gamma^{ab}\psi, \\
&\gamma_a\psi \wedge \bar{\psi} \wedge \gamma^a\psi = 0, \\
&\gamma_{ab}\psi \wedge \bar{\psi} \wedge \gamma^{ab}\psi = 0, \\
&\gamma_{ab}\psi \wedge \bar{\psi} \wedge \gamma^a\psi = \psi \wedge \bar{\psi} \wedge \gamma_b\psi.
\end{aligned}
\tag{A.12}
$$

Now, we collect the definitions of some operators we use in the work: the exterior derivative, the contraction and the Lie derivative. The exterior derivative d is a unique function

$$d : T^{*(k)}(M) \to T^{*(k+1)}(M) \tag{A.13}$$

$\forall k \in \mathbb{N}$ such that for any k-form we hawe

$$\begin{aligned}
&d(\omega^{(k)} + \omega^{(q)}) = d\omega^{(k)} + d\omega^{(q)}, \\
&d(\omega^{(k)} \wedge \omega^{(q)}) = d\omega^{(k)} \wedge \omega^{(q)} + (-1)^k \omega^{(k)} \wedge d\omega^{(q)}, \\
&d^2 = 0.
\end{aligned} \tag{A.14}$$

With eq.(A.4), we can define the exterior derivative of a k-form such that

$$d\omega^{(k)} = \frac{\partial a_{\mu_1 \dots \mu_k}}{\partial x^\rho}\, dx^\rho \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_k}. \tag{A.15}$$

Another useful tool is the operator of contraction $\imath_v$ with parameter v, defined as a function

$$\imath_v : T^{*(k)}(M) \to T^{*(k-1)}(M), \tag{A.16}$$

which fulfills the following relations

$$\begin{aligned}
&\imath_v \imath_u = -\imath_u \imath_v, \\
&\imath_{\alpha u + \beta v} = \alpha \imath_u + \beta \imath_v \qquad \forall \alpha, \beta \in \mathbb{C}, \\
&\imath_v(\omega^{(k)} \wedge \omega^{(q)}) = (\imath_v \omega^{(k)}) \wedge \omega^{(q)} + (-1)^k \omega^{(k)} \wedge (\imath_v \omega^{(q)}),
\end{aligned} \tag{A.17}$$

for all k-forms $\omega^{(k)}$. In this thesis, we often compute the contraction of a k-form in the fermionic direction of Superspace: operatively, $\imath_\epsilon \omega^{(k)}$ stands for the derivation of $\omega^{(k)}$ with respect to the spinorial 1-form ψ and the replacement of one ψ with the parameter ϵ. In particular, we have $\imath_\epsilon(\psi) = \epsilon$ and $\imath_\epsilon(V^a) = 0$. Another example is $\imath_\epsilon(\bar{\psi} \wedge \gamma^a \psi) = 2\bar{\epsilon} \wedge \gamma^a \psi = -2\bar{\psi} \wedge \gamma^a \epsilon$. The last mathematical tool is the Lie derivative ℓ_ϵ. In general, given a function, its Lie derivative is its infinitesimal variation under diffeomorphism; for our specific case, we use the Lie derivative when we compute the variation under supersymmetric transformation, that coincides with the Lie derivative in the fermionic direction. Operatively, the Lie derivative of a k-form $\omega^{(k)}$ with parameter ϵ is defined as

$$\ell_\epsilon \omega^{(k)} = \imath_\epsilon(d\omega^{(k)}) + d(\imath_\epsilon \omega^{(k)}), \tag{A.18}$$

and, as a consequence, it fulfills the following relations

$$\begin{aligned}&\ell_\epsilon d = d\ell_\epsilon,\\&\ell_\epsilon(\omega^{(k)}\wedge\omega^{(q)}) = (\ell_\epsilon\omega^{(k)})\wedge\omega^{(q)} + \omega^{(k)}\wedge(\ell_\epsilon\omega^{(q)}),\end{aligned}\tag{A.19}$$

for all k-forms $\omega^{(k)}$.

A.2 Notation in supersymmetric gauge theories

We consider a flat $(3+1)$-dimensional spacetime with metric tensor of eq.(A.1) and we apply a Wick rotation: The result is a flat 4-dimensional Euclidean spacetime having the 4×4 identity matrix as metric tensor.

For computations in a rigid supersymmetric field theory like $\mathcal{N}=2$ SCQCD, we give up the dual formalism of k-forms and adopt the vector formalism.

We work in a 4-dimensional Euclidean $\mathcal{N}=1$ Superspace described by real commuting spacetime coordinates x^μ (with $\mu = 0,1,2,3$), by complex anti-commuting Weyl spinor coordinates θ^α (with $\alpha = +,-$) and by their complex conjugates $\bar\theta^{\dot\alpha} = (\theta^\alpha)^\dagger$ (with $\dot\alpha = \dot+,\dot-$). To avoid confusion, we indicate $\mu,\nu = 0,1,2,3$ as vector indices (corresponding to the vector representation of the Lorentz group $SO(4)$), $\alpha,\beta = +,-$ and $\dot\alpha,\dot\beta = \dot+,\dot-$ as spinor indices (corresponding to the irreducible representations $(0,\frac{1}{2})$ and $(\frac{1}{2},0)$ of $SU(2)\times SU(2)$, which is isomorphic to the covering group of the Lorentz group).

Given a generic Weyl spinor ψ^α and its complex conjugate $\bar\psi^{\dot\alpha}$, spinor indices are raised and lowered through the following relations

$$\begin{aligned}&\psi^\alpha = \mathcal{C}^{\alpha\beta}\psi_\beta, \quad \psi_\alpha = \psi^\beta\mathcal{C}_{\beta\alpha},\\&\bar\psi^{\dot\alpha} = \mathcal{C}^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}, \quad \bar\psi_{\dot\alpha} = \bar\psi^{\dot\beta}\mathcal{C}_{\dot\beta\dot\alpha},\end{aligned}\tag{A.20}$$

where we define

$$\mathcal{C}^{\alpha\beta} = \mathcal{C}^{\dot\alpha\dot\beta} = -\mathcal{C}_{\alpha\beta} = -\mathcal{C}_{\dot\alpha\dot\beta} = \begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}\tag{A.21}$$

which verify the identities

$$\begin{aligned}&\mathcal{C}^{\alpha\beta}\mathcal{C}_{\gamma\epsilon} = \delta^\alpha{}_\gamma\delta^\beta{}_\epsilon - \delta^\alpha{}_\epsilon\delta^\beta{}_\gamma,\\&\mathcal{C}^{\dot\alpha\dot\beta}\mathcal{C}_{\dot\gamma\dot\epsilon} = \delta^{\dot\alpha}{}_{\dot\gamma}\delta^{\dot\beta}{}_{\dot\epsilon} - \delta^{\dot\alpha}{}_{\dot\epsilon}\delta^{\dot\beta}{}_{\dot\gamma}.\end{aligned}\tag{A.22}$$

Spinors are contracted according to

$$\begin{aligned}
&\psi\xi = \psi^\alpha \xi_\alpha = \xi^\alpha \psi_\alpha = \xi\psi, &&\bar{\psi}\bar{\xi} = \bar{\psi}^{\dot{\alpha}} \bar{\xi}_{\dot{\alpha}} = \bar{\xi}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}} = \bar{\xi}\bar{\psi},\\
&\psi^2 = \frac{1}{2}\psi^\alpha \psi_\alpha, &&\bar{\psi}^2 = \frac{1}{2}\bar{\psi}^{\dot{\alpha}} \bar{\psi}_{\dot{\alpha}},
\end{aligned} \tag{A.23}$$

where we write the raised spinor index before the lowered one.

We define

$$(\sigma_\mu)^{\alpha\dot{\alpha}} = (i\mathbb{I}, \sigma_1, \sigma_2, \sigma_3), \tag{A.24}$$

with $\mathbb{I}$ a 2×2 identity matrix and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

are Pauli matrices.

It is possible to verify that

$$(\sigma_\mu)^{\alpha\dot{\alpha}} (\sigma^\nu)_{\alpha\dot{\alpha}} = 2\delta_\mu{}^\nu, \quad (\sigma_\mu)^{\alpha\dot{\alpha}} (\sigma^\mu)_{\beta\dot{\beta}} = 2\delta^\alpha{}_\beta \delta^{\dot{\alpha}}{}_{\dot{\beta}}, \tag{A.25}$$

and starting from eq.(A.25), we obtain all the following trace identities

$$\mathrm{tr}(\sigma^\mu \sigma^\nu) = -2\delta^{\mu\nu}, \tag{A.26}$$

$$\begin{aligned}
\mathrm{tr}(\sigma^\mu \sigma^\nu \sigma^\rho \sigma^\tau) = \frac{1}{2}\Big(&\mathrm{tr}(\sigma^\mu \sigma^\nu)\,\mathrm{tr}(\sigma^\rho \sigma^\tau) \\
&- \mathrm{tr}(\sigma^\mu \sigma^\rho)\,\mathrm{tr}(\sigma^\nu \sigma^\tau) \\
&+ \mathrm{tr}(\sigma^\mu \sigma^\tau)\,\mathrm{tr}(\sigma^\nu \sigma^\rho)\Big),
\end{aligned} \tag{A.27}$$

$$\begin{aligned}
\mathrm{tr}(\sigma^\mu \sigma^\nu \sigma^\rho \sigma^\tau \sigma^\epsilon \sigma^\omega) = \frac{1}{2}\Big(&\mathrm{tr}(\sigma^\mu \sigma^\nu)\,\mathrm{tr}(\sigma^\rho \sigma^\tau \sigma^\epsilon \sigma^\omega) \\
&- \mathrm{tr}(\sigma^\mu \sigma^\rho)\,\mathrm{tr}(\sigma^\nu \sigma^\tau \sigma^\epsilon \sigma^\omega) \\
&+ \mathrm{tr}(\sigma^\mu \sigma^\tau)\,\mathrm{tr}(\sigma^\nu \sigma^\rho \sigma^\epsilon \sigma^\omega) \\
&- \mathrm{tr}(\sigma^\mu \sigma^\epsilon)\,\mathrm{tr}(\sigma^\nu \sigma^\rho \sigma^\tau \sigma^\omega) \\
&+ \mathrm{tr}(\sigma^\mu \sigma^\omega)\,\mathrm{tr}(\sigma^\nu \sigma^\rho \sigma^\tau \sigma^\epsilon)\Big),
\end{aligned} \tag{A.28}$$

while the trace of a product of an odd number of Pauli matrices vanishes.

Vector and bispinorial indices are exchanged as follows:

$$\begin{aligned}
&\text{coordinates} \qquad && x^\mu = (\sigma^\mu)_{\alpha\dot\alpha} x^{\alpha\dot\alpha}, && x^{\alpha\dot\alpha} = \frac{1}{2}(\sigma_\mu)^{\alpha\dot\alpha} x^\mu, \\
&\text{derivatives} && \partial_\mu = \frac{1}{2}(\sigma_\mu)^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}, && \partial_{\alpha\dot\alpha} = (\sigma^\mu)_{\alpha\dot\alpha}\partial_\mu, \\
&\text{fields} && V_\mu = \frac{1}{\sqrt{2}}(\sigma_\mu)^{\alpha\dot\alpha} V_{\alpha\dot\alpha}, && V_{\alpha\dot\alpha} = \frac{1}{\sqrt{2}}(\sigma^\mu)_{\alpha\dot\alpha} V_\mu,
\end{aligned} \tag{A.29}$$

where we use the short notation

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (-i\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}).$$

The scalar product of two generic vectors p^μ and q^μ is rewritten as

$$p \cdot q = \frac{1}{2}\, p^{\alpha\dot\alpha} q_{\alpha\dot\alpha}. \tag{A.30}$$

In a $\mathcal{N} = 1$ Superspace, covariant spinor derivatives are defined as

$$\begin{aligned}
D_\alpha &= \partial_\alpha + \frac{i}{2}\bar\theta^{\dot\alpha}\partial_{\alpha\dot\alpha}, \qquad & \bar D_{\dot\alpha} &= \bar\partial_{\dot\alpha} + \frac{i}{2}\theta^\alpha\partial_{\alpha\dot\alpha}, \\
D^2 &= \frac{1}{2}D^\alpha D_\alpha, & \bar D^2 &= \frac{1}{2}\bar D^{\dot\alpha}\bar D_{\dot\alpha},
\end{aligned} \tag{A.31}$$

(with ∂_α, $\bar\partial_{\dot\alpha}$ spinor derivatives that obey to $\partial_\alpha\theta^\beta = \delta_\alpha{}^\beta$ and $\bar\partial_{\dot\alpha}\bar\theta^{\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta}$).

Covariant spinor derivatives vanish at the third power: $(D)^3 = (\bar D)^3 = 0$. They satisfy the following anticommutators

$$\begin{aligned}
&\{D_\alpha, \bar D_{\dot\alpha}\} = i\partial_{\alpha\dot\alpha}, && \{D^\alpha, \bar D^{\dot\alpha}\} = i\partial^{\alpha\dot\alpha}, \\
&\{D_\alpha, D_\beta\} = 0, && \{\bar D_{\dot\alpha}, \bar D_{\dot\beta}\} = 0, \\
&\{\bar D_{\dot\alpha}, D^2\} = -D^\alpha \bar D_{\dot\alpha} D_\alpha, && \{D_\alpha, \bar D^2\} = -\bar D^{\dot\alpha} D_\alpha \bar D_{\dot\alpha}, \\
&\{D^2, \bar D^2\} = \Box + D^\alpha \bar D^2 D_\alpha = \Box + \bar D^{\dot\alpha} D^2 \bar D_{\dot\alpha}, &&
\end{aligned} \tag{A.32}$$

(with $\Box = \frac{1}{2}\partial^{\alpha\dot\alpha}\partial_{\alpha\dot\alpha}$) and the following commutators

$$\begin{aligned}
&[D_\alpha, \bar D^2] = -i\partial_{\alpha\dot\alpha}\bar D^{\dot\alpha}, \qquad && [D^\alpha, \bar D^2] = i\partial^{\alpha\dot\alpha}\bar D_{\dot\alpha}, \\
&[\bar D_{\dot\alpha}, D^2] = -i\partial_{\alpha\dot\alpha} D^\alpha, && [\bar D^{\dot\alpha}, D^2] = i\partial^{\alpha\dot\alpha} D_\alpha, \\
&[\Box, D^2] = 0, && [\Box, \bar D^2] = 0.
\end{aligned} \tag{A.33}$$

The following couple of identities

$$D^2\bar{D}^2D^2 = \Box D^2, \qquad \bar{D}^2D^2\bar{D}^2 = \Box\bar{D}^2, \tag{A.34}$$

is useful to simplify the D-algebra of a Feynman superdiagram.

Integration over spinor coordinates follows Berezin rules (see [10]) which define the integration over anticommuting parameters (sometimes called Grassmann variables). In short, integrations over θ and $\bar{\theta}$ are defined as

$$\begin{aligned} \int d^2\theta &= \frac{1}{2}\partial^\alpha\partial_\alpha, \\ \int d^2\bar{\theta} &= \frac{1}{2}\bar{\partial}^{\dot{\alpha}}\bar{\partial}_{\dot{\alpha}}, \\ \int d^4\theta &= \int d^2\theta\, d^2\bar{\theta}. \end{aligned} \tag{A.35}$$

A more practical way to express the definitions of eq.(A.35) consists in the following projections

$$\begin{aligned} \int d^4x\, d^2\theta &= \int d^4x\, D^2|_{\theta=\bar{\theta}=0}, \\ \int d^4x\, d^2\bar{\theta} &= \int d^4x\, \bar{D}^2|_{\theta=\bar{\theta}=0}, \\ \int d^4x\, d^4\theta &= \int d^4x\, \bar{D}^2D^2|_{\theta=\bar{\theta}=0}. \end{aligned} \tag{A.36}$$

Since the space of momenta is well suited for a computation of a scattering amplitude, we replace

$$i\partial_{\alpha\dot{\alpha}} \to p_{\alpha\dot{\alpha}}, \qquad \Box \to -p^2, \tag{A.37}$$

where $p_\mu = (iE, p_1, p_2, p_3)$ is the momentum of a particle with energy E and $p_{\alpha\dot{\alpha}} = (\sigma^\mu)_{\alpha\dot{\alpha}}p_\mu$.

The mass-shell relation for a generic particle with mass m and momentum p is

$$p^2 = -m^2. \tag{A.38}$$

All the particles considered are massless: so, we can fix $m = 0$. We arbitrarily choose to represent the process of n particles as a $0 \to n$ process, with only outgoing particles; consequently, the conservation of total momentum is

$$\sum_{j=1}^{n} p_j^\mu = 0. \tag{A.39}$$

Physical processes with more than 3 final states are described using Mandelstam invariants: We denote a Mandelstam variable as $s_{ij} = (p_i + p_j)^2$ and for massless particles it is simply $s_{ij} = 2p_i \cdot p_j$.

Bibliography

[1] C. Itzykson and J. Zuber, *Quantum Field Theory*. Dover Books on Physics. Dover Publications, 2012.

[2] G. Sterman, *An Introduction to Quantum Field Theory*. Cambridge University Press, 1993.

[3] M. Peskin, *An Introduction To Quantum Field Theory, Student Economy Edition*. CRC Press, 2018.

[4] M. Veltman, C. U. Press, P. Goddard and J. Yeomans, *Diagrammatica: The Path to Feynman Diagrams*. Cambridge Lecture Notes in Physics. Cambridge University Press, 1994.

[5] P. Binetruy, *Supersymmetry*. Oxford Graduate Texts. Oxford University Press, 2012.

[6] G. Veneziano, *Construction of a crossing - symmetric, Regge behaved amplitude for linearly rising trajectories*, *Nuovo Cim.* **A57** (1968) 190–197.

[7] K. Becker, M. Becker, I. V. Melnikov, D. Robbins and A. B. Royston, *Some tree-level string amplitudes in the NSR formalism*, *JHEP* **12** (2015) 010 [1507.02172].

[8] Z. Bern, *Perturbative quantum gravity and its relation to gauge theory*, *Living Rev. Rel.* **5** (2002) 5 [gr-qc/0206071].

[9] S. Stieberger, *Open & Closed vs. Pure Open String Disk Amplitudes*, 0907.2211.

[10] S. J. Gates, M. T. Grisaru, M. Rocek and W. Siegel, *Superspace Or One Thousand and One Lessons in Supersymmetry*, *Front. Phys.* **58** (1983) 1–548 [hep-th/0108200].

[11] N. Seiberg, *Notes on theories with 16 supercharges*, *Nucl. Phys. Proc. Suppl.* **67** (1998) 158–171 [hep-th/9705117]. [,158(1997)].

[12] A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky and E. S. Sokatchev, *Harmonic Superspace.* Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2001.

[13] D. Klemm and L. Ravera, *Einstein manifolds with torsion and nonmetricity*, 1811.11458.

[14] S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity.* New York: Wiley, 1972.

[15] L. Castellani, R. D'Auria and P. Fre, *Supergravity and superstrings: A Geometric perspective. Vol. 2: Supergravity.* 1991.

[16] L. Castellani, R. D'Auria and P. Fre, *Supergravity and superstrings: A Geometric perspective. Vol. 1: Mathematical foundations.* 1991.

[17] L. Andrianopoli and R. D'Auria, *N=1 and N=2 pure supergravities on a manifold with boundary*, *JHEP* **08** (2014) 012 [1405.2010].

[18] M. C. Ipinza, P. K. Concha, L. Ravera and E. K. Rodríguez, *On the Supersymmetric Extension of Gauss-Bonnet like Gravity*, *JHEP* **09** (2016) 007 [1607.00373].

[19] L. Ravera, *Group Theoretical Hidden Structure of Supergravity Theories in Higher Dimensions.* PhD thesis, INFN, Turin, 2018. 1802.06602.

[20] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, *Int. J. Theor. Phys.* **38** (1999) 1113–1133 [hep-th/9711200]. [Adv. Theor. Math. Phys.2,231(1998)].

[21] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, *Large N field theories, string theory and gravity*, *Phys. Rept.* **323** (2000) 183–386 [hep-th/9905111].

[22] E. Witten, *Anti-de Sitter space and holography*, *Adv. Theor. Math. Phys.* **2** (1998) 253–291 [hep-th/9802150].

[23] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, *Gauge theory correlators from noncritical string theory*, *Phys. Lett.* **B428** (1998) 105–114 [hep-th/9802109].

[24] E. D'Hoker and D. Z. Freedman, *Supersymmetric gauge theories and the AdS / CFT correspondence*, in *Strings, Branes and Extra Dimensions: TASI 2001: Proceedings*, pp. 3–158, 2002. hep-th/0201253.

[25] P. H. Ginsparg, *APPLIED CONFORMAL FIELD THEORY*, in *Les Houches Summer School in Theoretical Physics: Fields, Strings, Critical Phenomena Les Houches, France, June 28-August 5, 1988*, pp. 1–168, 1988. hep-th/9108028.

[26] J. A. Minahan, *Review of AdS/CFT Integrability, Chapter I.1: Spin Chains in N=4 Super Yang-Mills*, *Lett. Math. Phys.* **99** (2012) 33–58 [1012.3983].

[27] J. W. York, Jr., *Role of conformal three geometry in the dynamics of gravitation*, *Phys. Rev. Lett.* **28** (1972) 1082–1085.

[28] G. W. Gibbons and S. W. Hawking, *Action integrals and partition functions in quantum gravity*, *Phys. Rev. D* **15** (May, 1977) 2752–2756.

[29] J. D. Brown and J. W. York, Jr., *Quasilocal energy and conserved charges derived from the gravitational action*, *Phys. Rev.* **D47** (1993) 1407–1419 [gr-qc/9209012].

[30] P. Horava and E. Witten, *Eleven-dimensional supergravity on a manifold with boundary*, *Nucl. Phys.* **B475** (1996) 94–114 [hep-th/9603142].

[31] K. Skenderis, *Lecture notes on holographic renormalization*, *Class. Quant. Grav.* **19** (2002) 5849–5876 [hep-th/0209067].

[32] E. O Colgain and B. Stefanski, Jr., *A search for AdS5 X S2 IIB supergravity solutions dual to N = 2 SCFTs*, *JHEP* **10** (2011) 061 [1107.5763].

[33] A. Gadde, E. Pomoni and L. Rastelli, *The Veneziano Limit of N = 2 Superconformal QCD: Towards the String Dual of N = 2 SU(N(c)) SYM with N(f) = 2 N(c)*, 0912.4918.

[34] A. Banaudi and L. Ravera, *Generalized ads-lorentz deformed supergravity on a manifold with boundary*, *The European Physical Journal Plus* **133** (Dec, 2018) 514.

[35] R. Aros, M. Contreras, R. Olea, R. Troncoso and J. Zanelli, *Conserved charges for gravity with locally AdS asymptotics*, *Phys. Rev. Lett.* **84** (2000) 1647–1650 [gr-qc/9909015].

[36] R. Aros, M. Contreras, R. Olea, R. Troncoso and J. Zanelli, *Conserved charges for even dimensional asymptotically AdS gravity theories*, *Phys. Rev.* **D62** (2000) 044002 [hep-th/9912045].

[37] P. Mora, R. Olea, R. Troncoso and J. Zanelli, *Finite action principle for Chern-Simons AdS gravity*, *JHEP* **06** (2004) 036 [hep-th/0405267].

[38] R. Olea, *Mass, angular momentum and thermodynamics in four-dimensional Kerr-AdS black holes*, *JHEP* **06** (2005) 023 [hep-th/0504233].

[39] D. P. Jatkar, G. Kofinas, O. Miskovic and R. Olea, *Conformal Mass in AdS gravity*, *Phys. Rev.* **D89** (2014), no. 12 124010 [1404.1411].

[40] P. van Nieuwenhuizen and D. V. Vassilevich, *Consistent boundary conditions for supergravity*, *Class. Quant. Grav.* **22** (2005) 5029–5051 [hep-th/0507172].

[41] D. V. Belyaev, *Boundary conditions in supergravity on a manifold with boundary*, *JHEP* **01** (2006) 047 [hep-th/0509172].

[42] D. V. Belyaev and P. van Nieuwenhuizen, *Tensor calculus for supergravity on a manifold with boundary*, *JHEP* **02** (2008) 047 [0711.2272].

[43] D. V. Belyaev and P. van Nieuwenhuizen, *Simple d=4 supergravity with a boundary*, *JHEP* **09** (2008) 069 [0806.4723].

[44] D. Grumiller and P. van Nieuwenhuizen, *Holographic counterterms from local supersymmetry without boundary conditions*, *Phys. Lett.* **B682** (2010) 462–465 [0908.3486].

[45] D. V. Belyaev and T. G. Pugh, *The Supermultiplet of boundary conditions in supergravity*, *JHEP* **10** (2010) 031 [1008.1574].

[46] S. W. MacDowell and F. Mansouri, *Unified Geometric Theory of Gravity and Supergravity*, *Phys. Rev. Lett.* **38** (1977) 739. [Erratum: Phys. Rev. Lett.38,1376(1977)].

[47] P. K. Concha, E. K. Rodríguez and P. Salgado, *Generalized supersymmetric cosmological term in $N = 1$ Supergravity*, *JHEP* **08** (2015) 009 [1504.01898].

[48] D. V. Soroka and V. A. Soroka, *Semi-simple extension of the (super)Poincare algebra*, *Adv. High Energy Phys.* **2009** (2009) 234147 [hep-th/0605251].

[49] R. Durka, J. Kowalski-Glikman and M. Szczachor, *Gauged AdS-Maxwell algebra and gravity*, *Mod. Phys. Lett.* **A26** (2011) 2689–2696 [1107.4728].

[50] L. Andrianopoli, R. D'Auria and L. Ravera, *More on the Hidden Symmetries of 11D Supergravity*, *Phys. Lett.* **B772** (2017) 578–585 [1705.06251].

[51] R. D'Auria and P. Fre, *Geometric Supergravity in $d = 11$ and Its Hidden Supergroup*, *Nucl. Phys.* **B201** (1982) 101–140. [Erratum: Nucl. Phys.B206,496(1982)].

[52] L. Andrianopoli, R. D'Auria and L. Ravera, *Hidden Gauge Structure of Supersymmetric Free Differential Algebras*, *JHEP* **08** (2016) 095 [1606.07328].

[53] D. M. Peñafiel and L. Ravera, *On the Hidden Maxwell Superalgebra underlying D=4 Supergravity*, *Fortsch. Phys.* **65** (2017), no. 9 1700005 [1701.04234].

[54] L. Ravera, *Hidden role of Maxwell superalgebras in the free differential algebras of $D = 4$ and $D = 11$ supergravity*, *Eur. Phys. J.* **C78** (2018), no. 3 211 [1801.08860].

[55] M. B. Green, *Supertranslations, Superstrings and Chern-Simons Forms*, *Phys. Lett.* **B223** (1989) 157–164.

[56] D. M. Peñafiel and L. Ravera, *Generalized cosmological term in $D = 4$ supergravity from a new AdS–Lorentz superalgebra*, *Eur. Phys. J.* **C78** (2018), no. 11 945 [1807.07673].

[57] F. Izaurieta, E. Rodriguez and P. Salgado, *Expanding Lie (super)algebras through Abelian semigroups*, *J. Math. Phys.* **47** (2006) 123512 [hep-th/0606215].

[58] J. Diaz, O. Fierro, F. Izaurieta, N. Merino, E. Rodriguez, P. Salgado and O. Valdivia, *A generalized action for (2 + 1)-dimensional Chern-Simons gravity*, *J. Phys.* **A45** (2012) 255207 [1311.2215].

[59] O. Fierro, F. Izaurieta, P. Salgado and O. Valdivia, *(2+1)-dimensional supergravity invariant under the AdS-Lorentz superalgebra*, 1401.3697.

[60] P. K. Concha, R. Durka, N. Merino and E. K. Rodríguez, *New family of Maxwell like algebras*, *Phys. Lett.* **B759** (2016) 507–512 [1601.06443].

[61] M. C. Ipinza, F. Lingua, D. M. Peñafiel and L. Ravera, *An Analytic Method for S-Expansion involving Resonance and Reduction*, *Fortsch. Phys.* **64** (2016), no. 11-12 854–880 [1609.05042].

[62] L. Castellani, *Supergravity in the group-geometric framework: a primer*, 1802.03407.

[63] C. Sieg, *Review of AdS/CFT Integrability, Chapter I.2: The spectrum from perturbative gauge theory*, *Lett. Math. Phys.* **99** (2012) 59–84 [1012.3984].

[64] M. Staudacher, *Review of AdS/CFT Integrability, Chapter III.1: Bethe Ansätze and the R-Matrix Formalism*, *Lett. Math. Phys.* **99** (2012) 191–208 [1012.3990].

[65] F. A. Berends and W. T. Giele, *Recursive Calculations for Processes with n Gluons*, *Nucl. Phys.* **B306** (1988) 759–808.

[66] J. M. Drummond, G. P. Korchemsky and E. Sokatchev, *Conformal properties of four-gluon planar amplitudes and Wilson loops*, *Nucl. Phys.* **B795** (2008) 385–408 [0707.0243].

[67] A. Brandhuber, P. Heslop and G. Travaglini, *MHV amplitudes in N=4 super Yang-Mills and Wilson loops*, *Nucl. Phys.* **B794** (2008) 231–243 [0707.1153].

[68] L. F. Alday and J. M. Maldacena, *Gluon scattering amplitudes at strong coupling*, *JHEP* **06** (2007) 064 [0705.0303].

[69] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, *On planar gluon amplitudes/Wilson loops duality*, *Nucl. Phys.* **B795** (2008) 52–68 [0709.2368].

[70] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, *Conformal Ward identities for Wilson loops and a test of the duality with gluon amplitudes*, *Nucl. Phys.* **B826** (2010) 337–364 [0712.1223].

[71] Z. Bern, L. J. Dixon and V. A. Smirnov, *Iteration of planar amplitudes in maximally supersymmetric Yang-Mills theory at three loops and beyond*, *Phys. Rev.* **D72** (2005) 085001 [hep-th/0505205].

[72] L. F. Alday and J. Maldacena, *Comments on gluon scattering amplitudes via AdS/CFT*, *JHEP* **11** (2007) 068 [0710.1060].

[73] J. M. Drummond, J. Henn, G. P. Korchemsky and E. Sokatchev, *The hexagon Wilson loop and the BDS ansatz for the six-gluon amplitude*, *Phys. Lett.* **B662** (2008) 456–460 [0712.4138].

[74] A. V. Kotikov, *The property of maximal transcendentality: calculation of master integrals*, *Theor. Math. Phys.* **176** (2013) 913–921 [1212.3732].

[75] M. Leoni, A. Mauri and A. Santambrogio, *Four-point amplitudes in $\mathcal{N} = 2$ SCQCD*, *JHEP* **09** (2014) 017 [1406.7283]. [Erratum: JHEP02,022(2015)].

[76] M. Leoni, *Scattering Amplitudes in Superconformal Gauge Theories*. PhD thesis, Milan U., 2015-01-28.

[77] G. Georgiou, E. W. N. Glover and V. V. Khoze, *Non-MHV tree amplitudes in gauge theory*, *JHEP* **07** (2004) 048 [hep-th/0407027].

[78] S. Catani, *The Singular behavior of QCD amplitudes at two loop order*, *Phys. Lett.* **B427** (1998) 161–171 [hep-ph/9802439].

[79] E. W. N. Glover, V. V. Khoze and C. Williams, *Component MHV amplitudes in N=2 SQCD and in N=4 SYM at one loop*, *JHEP* **08** (2008) 033 [0805.4190].

[80] C. Anastasiou, Z. Bern, L. J. Dixon and D. A. Kosower, *Planar amplitudes in maximally supersymmetric Yang-Mills theory*, *Phys. Rev. Lett.* **91** (2003) 251602 [hep-th/0309040].

[81] Z. Bern, L. J. Dixon, D. A. Kosower, R. Roiban, M. Spradlin, C. Vergu and A. Volovich, *The Two-Loop Six-Gluon MHV Amplitude in Maximally Supersymmetric Yang-Mills Theory*, *Phys. Rev.* **D78** (2008) 045007 [0803.1465].

[82] M. Leoni, A. Mauri and A. Santambrogio, *On the amplitude/Wilson loop duality in N=2 SCQCD*, *Phys. Lett.* **B747** (2015) 325–330 [1502.07614].

[83] V. A. Smirnov, *Feynman integral calculus*. 2006.

[84] A. V. Smirnov, *Algorithm FIRE – Feynman Integral REduction*, *JHEP* **10** (2008) 107 [0807.3243].

[85] A. V. Smirnov and V. A. Smirnov, *FIRE4, LiteRed and accompanying tools to solve integration by parts relations*, *Comput. Phys. Commun.* **184** (2013) 2820–2827 [1302.5885].

[86] H. Elvang and Y.-t. Huang, *Scattering Amplitudes*, 1308.1697.

[87] Z. Bern, L. J. Dixon and D. A. Kosower, *Dimensionally regulated pentagon integrals*, *Nucl. Phys.* **B412** (1994) 751–816 [hep-ph/9306240].

[88] Z. Bern, M. Czakon, D. A. Kosower, R. Roiban and V. A. Smirnov, *Two-loop iteration of five-point N=4 super-Yang-Mills amplitudes*, *Phys. Rev. Lett.* **97** (2006) 181601 [hep-th/0604074].

[89] O. Miskovic and R. Olea, *Topological regularization and self-duality in four-dimensional anti-de Sitter gravity*, *Phys. Rev.* **D79** (2009) 124020 [0902.2082].