

CUBIC THREEFOLDS AND HYPERKÄHLER MANIFOLDS UNIFORMIZED BY THE 10-DIMENSIONAL COMPLEX BALL

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ABSTRACT. We first prove an isomorphism between the moduli space of smooth cubic threefolds and the moduli space of hyperkähler fourfolds of $K3^{[2]}$ -type with a non-symplectic automorphism of order three, whose invariant lattice has rank one and is generated by a class of square 6; both these spaces are uniformized by the same 10-dimensional arithmetic complex ball quotient. We then study the degeneration of the automorphism along the loci of nodal or chordal degenerations of the cubic threefold, showing the birationality of these loci with some moduli spaces of hyperkähler fourfolds of $K3^{[2]}$ -type with non-symplectic automorphism of order three belonging to different families. Finally, we construct a cyclic Pfaffian cubic fourfold to give an explicit construction of a non-natural automorphism of order three on the Hilbert square of a K3 surface.

1. INTRODUCTION

The aim of this paper is to study certain moduli spaces of 4-dimensional irreducible holomorphic symplectic manifolds endowed with a non-symplectic group action of prime order, constructed in full generality by the authors in [15], which are uniformized by the same 10-dimensional arithmetic ball quotient as the smooth cubic threefolds [4], and to explore some deep consequences of this isomorphism.

Arithmetic complex ball quotients have attracted great interest in the last decades since they have been proven to be ubiquitous targets of period maps, allowing to show unexpected isomorphisms between different moduli spaces. For example, an arithmetic quotient of the complement of a hyperplane arrangement Δ in the 4-dimensional complex ball \mathbb{B}^4 arising as period domain of smooth cubic surfaces in [3] has been studied by Dolgachev–van Geemen–Kondō [21], who show that $\mathbb{B}^4 \setminus \Delta$ and degenerations to Δ can also be interpreted as moduli spaces of K3 surfaces endowed with certain non-symplectic automorphisms of order three.

Another famous example, which is the core of the present paper, is given by the quotient of the 10-dimensional complex ball which contains as a Zariski open set the moduli space of smooth cubic threefolds, as shown by Allcock–Carlson–Toledo [4] and by Looijenga–Swierstra [34]. We know nowadays that the hyperplane arrangements corresponding respectively to nodal and chordal degenerations of cubic threefolds are in fact birational to two other arithmetic complex ball quotients (see [18]), first studied by Kondō [30] and appearing as the moduli spaces of K3 surfaces endowed with two different kinds of non-symplectic automorphisms of order three.

The main result of the first part of this paper is to establish an isomorphism between the moduli space $\mathcal{C}_3^{\text{sm}}$ of smooth cubic threefolds and the moduli space $\mathcal{N}_{(6)}^{\rho, \xi}$ of

fourfolds deformation equivalent to the Hilbert square of a K3 surface endowed with a special non-symplectic automorphism of order three, following [15, Section 7.1]. We recall in Section 2 the construction of Fano varieties of cyclic cubic fourfolds and of their automorphism, as done in [14, Example 6.4]; then in Section 3 we prove our first main result:

Theorem 1.1. *The moduli spaces $\mathcal{C}_3^{\text{sm}}$ and $\mathcal{N}_{(6)}^{\rho, \xi}$ are isomorphic.*

This result will be proven by using the period map of irreducible holomorphic symplectic manifolds and Allcock–Carlson–Toledo’s period map for cubic threefolds. One of the consequences of Theorem 1.1 is Corollary 5.2: although *a priori* inside $\mathcal{N}_{(6)}^{\rho, \xi}$ we could have deformations which are non-isomorphic to Fano varieties of cyclic cubic fourfolds, this is not the case and every element in this moduli space is isomorphic to such a Fano variety of lines.

The rest of the paper is devoted to the exploration of more consequences of this isomorphism, in particular we give a geometric description of the degenerations of the non-symplectic automorphism of order three. Indeed, the main part of [4] is devoted to the extension of the period map to singular cubics, either nodal or chordal. In Section 4 we will use this extension to understand geometrically the degenerations of the automorphism along the corresponding hyperplanes in the arithmetic ball quotient. The final outcome of this study are Propositions 4.6 and 4.10, which can be summarized as follows:

Theorem 1.2. *The stable discriminant locus (corresponding to nodal degenerations), respectively the stable chordal locus (corresponding to stable chordal degenerations) are birational to 9-dimensional moduli spaces of fourfolds which are deformations of a Hilbert square of a K3 surface endowed with a non-symplectic automorphism of order three, having respectively invariant lattice isometric to $U(3) \oplus \langle -2 \rangle$ and to $U \oplus \langle -2 \rangle$.*

Finally, in Section 5, looking at Hassett divisors on the moduli space of cubic fourfolds, we produce a Pfaffian cyclic cubic fourfold, thus we show:

Corollary 1.3. *There exists a smooth complex K3 surface whose Hilbert square admits a non-natural non-symplectic automorphism of order three, with fixed locus isomorphic to the Fano surface of a cubic threefold and invariant lattice isometric to $\langle 6 \rangle$.*

We further explicitly describe this automorphism in terms of the Pfaffian geometry of the K3 surface, in analogy with Beauville’s construction of the non-natural non-symplectic involution on the Hilbert scheme of a general quartic surface [10].

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2. FANO VARIETIES OF LINES OF CYCLIC CUBIC FOURFOLDS

We work over the field of complex numbers.

2.1. Cyclic cubic fourfolds. A general element of the linear system $|\mathcal{O}_{\mathbb{P}^4}(3)|$ defines a smooth cubic threefold C in \mathbb{P}^4 . The ramified triple covering $Y \rightarrow \mathbb{P}^4$ branched along C is then a cyclic cubic fourfold in \mathbb{P}^5 . We denote by σ a generator of the fundamental group of the covering, it is a biregular automorphism of order three of Y .

By Hassett [26, Proposition 2.1.2] the lattice $H^4(Y, \mathbb{Z})$ endowed with the Poincaré pairing $\langle \alpha, \beta \rangle_Y := \int_Y \alpha \beta$ is an odd unimodular lattice isometric to

$$\Lambda := \langle 1 \rangle^{\oplus 21} \oplus \langle -1 \rangle^{\oplus 2},$$

where $\langle m \rangle$ denotes the rank one lattice with quadratic form taking the value $m \in \mathbb{Z}$ on the generator. Denote by $h \in H^2(\mathbb{P}^5, \mathbb{Z})$ the class of a hyperplane section. The class $\theta_Y := h^2_Y$ in $H^4(Y, \mathbb{Z})$ has square 3 and the primitive cohomology space $H^4_{\circ}(Y, \mathbb{Z})$ is by definition the orthogonal complement of θ_Y in $H^4(Y, \mathbb{Z})$. Then $H^4_{\circ}(Y, \mathbb{Z})$ is an even lattice of signature $(20, 2)$, isometric to the lattice

$$S := U^{\oplus 2} \oplus E_8^{\oplus 2} \oplus A_2,$$

where U is the hyperbolic plane and E_8, A_2 are the positive definite root lattices (see also [4, proof of Theorem 1.7] or [34, Section 1] for another point of view). We fix for the rest of this paper a polarization class $\kappa \in \Lambda$ of square 3 and a primitive embedding $S \hookrightarrow \Lambda$ such that $(\mathbb{Z}\kappa)^{\perp} = S$.

Denote by $f(x_0, \dots, x_4) = 0$ an equation of C . Then Y has equation

$$F(x_0, \dots, x_5) := f(x_0, \dots, x_4) + x_5^3$$

and the automorphism σ maps (x_0, \dots, x_4, x_5) to $(x_0, \dots, x_4, \xi x_5)$, where ξ is a primitive third root of the unity that we fix for the rest of this paper. By Griffiths residue theorem, the Hodge group $H^{3,1}(Y)$ is one-dimensional, generated by $\text{Res}(\frac{\omega_{\mathbb{P}^5}}{F^2})$, where $\omega_{\mathbb{P}^5} = \sum_{i=0}^5 (-1)^i x_i dx_0 \wedge \dots \wedge \widehat{dx_i} \wedge \dots \wedge dx_5$. Since the residue map $\text{Res}: H^5(\mathbb{P}^5 \setminus Y, \mathbb{C}) \rightarrow H^4(Y, \mathbb{C})$ is σ^* -equivariant, we compute $\sigma^* \text{Res}(\frac{\omega_{\mathbb{P}^5}}{F^2}) = \xi \text{Res}(\frac{\omega_{\mathbb{P}^5}}{F^2})$. Denoting by $H^4_{\circ}(Y, \mathbb{C})_{\xi}$ the eigenspace of σ^* associated to the eigenvalue ξ , the line $H^{3,1}(Y) \subset H^4_{\circ}(Y, \mathbb{C})_{\xi}$ is called, following Allcock–Carlson–Toledo [4], the *period of the cubic threefold* C .

2.2. Fano variety of lines on cyclic cubic fourfolds. Consider the Fano variety of lines $F(Y)$, which parametrizes the projective lines contained in Y . By Beauville–Donagi [12], the variety $F(Y)$ is irreducible holomorphic symplectic, deformation equivalent to the Hilbert square of a K3 surface. As a consequence, the second cohomology group with integer coefficients $H^2(F(Y), \mathbb{Z})$, endowed with the Beauville–Bogomolov–Fujiki bilinear form $\langle -, - \rangle_{F(Y)}$ is isometric to the lattice

$$L := U^{\oplus 3} \oplus E_8(-1)^{\oplus 2} \oplus \langle -2 \rangle.$$

The automorphism σ of Y leaves globally invariant the lines of Y contained in the cubic C , so the fixed locus of the action of σ induced on $F(Y)$ is isomorphic to the Fano variety of lines $F(C)$, which is a surface of general type with Hodge numbers $h^{1,0} = 5$, $h^{2,0} = 10$ and $h^{1,1} = 25$ (see [14, Example 6.4], [19, Formula (0.7)]). It follows from the lattice-theoretical classification of non-symplectic automorphisms in [14, Example 6.4] that the invariant lattice $H^2(F(Y), \mathbb{Z})^{\sigma}$ is isometric to $\langle 6 \rangle$, with orthogonal complement in $H^2(F(Y), \mathbb{Z})$, denoted by $H^2_{\circ}(F(Y), \mathbb{Z})$, isometric to the lattice $S(-1) = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \oplus A_2(-1)$.

The apparent coincidence with the results of the preceding section is due to the Abel–Jacobi map. Denote by $Z \subset F(Y) \times Y$ the universal family and by p , *resp.* q , the projection to $F(Y)$, *resp.* Y . By [12, Proposition 4] the Abel–Jacobi map

$$A := p_*q^* : H^4(Y, \mathbb{Z}) \longrightarrow H^2(F(Y), \mathbb{Z})$$

is an isomorphism of Hodge structures. Denote by $\theta_{F(Y)} \in H^2(F(Y), \mathbb{Z})$ the class of a hyperplane section in the Plücker embedding. It is easy to see geometrically that $A(\theta_Y) = \theta_{F(Y)}$. Since $Y/\langle\sigma\rangle \cong \mathbb{P}^4$, we can directly deduce that the invariant lattice $H^4(Y, \mathbb{Z})^\sigma$ has rank one, hence it is generated by θ_Y . By σ -equivariance of A we observe that the invariant lattice $H^2(F(Y), \mathbb{Z})^\sigma$ is generated by $\theta_{F(Y)}$.

Moreover, by [12, Proposition 6] we have $A(H_\circ^4(Y, \mathbb{Z})) = H_\circ^2(F(Y), \mathbb{Z})$ and

$$\langle A(\alpha), A(\beta) \rangle_{F(Y)} = -\langle \alpha, \beta \rangle_Y \quad \forall \alpha, \beta \in H_\circ^4(Y, \mathbb{Z}).$$

Since $A(H^{3,1}(Y)) = H^{2,0}(F(Y))$, by σ -equivariance of A we deduce from the results of the previous section that the line $H^{2,0}(Y)$, which is the *period of the irreducible holomorphic symplectic manifold* $F(Y)$, lives in the eigenspace $H_\circ^2(F(Y), \mathbb{C})_\xi$ of $H_\circ^2(F(Y), \mathbb{C})$ associated to the eigenvalue ξ . The transition from the lattice S to the opposite lattice $S(-1)$ is explained by the Abel–Jacobi map A being an anti-isometry at the level of the primitive cohomology.

3. OCCULT PERIOD MAPS AND UNEXPECTED ISOMORPHISMS

3.1. Two useful lemmas. We state two classical results of Nikulin [42] on lattice theory which will be used several times below. For any integral lattice S , we denote by S^* the dual lattice and by $D_S := S^*/S$ its discriminant group, endowed with its quadratic form q_S . The *length* $\ell(S)$ of D_S is its minimal number of generators.

Lemma 3.1. [42, Theorem 1.14.2] *Assume that S is an even indefinite lattice. If $\text{rk}(S) \geq \ell(S) + 2$, then the natural map $O(S) \rightarrow O(D_S)$ is surjective.*

Lemma 3.2. *Let M be an integral non-degenerate lattice, S a primitive sublattice of M and $T := S^\perp$ its orthogonal complement. Let $\phi_S \in O(S)$ and $\phi_T \in O(T)$. The isometry $\phi := (\phi_S, \phi_T)$ of $S \oplus T$ extends to an isometry of M if and only if its action on $D_S \oplus D_T$ leaves globally invariant the subgroup $\frac{M}{S \oplus T}$ and this extension is unique. In particular, if ϕ_S acts by ϵid on D_S , with $\epsilon \in \{-1, 1\}$, then $(\phi_S, \epsilon \text{id}_T)$ extends uniquely to an isometry of M .*

Proof. The inclusions $S \oplus T \subset M \subset M^* \subset S^* \oplus T^*$ produce a subgroup $\frac{M}{S \oplus T}$ of $D_S \oplus D_T$; ϕ extends to an isometry $\tilde{\phi}$ on $S^* \oplus T^*$ and to an isometry $\bar{\phi}$ of $D_S \oplus D_T$. It is easy to see that $\tilde{\phi}$ maps M to itself if and only if $\bar{\phi}$ maps $\frac{M}{S \oplus T}$ to itself. For any $m \in M$, one has $\lambda m = s + t \in S \oplus T$ for some $\lambda \in \mathbb{Z}$, so necessarily $\tilde{\phi}(m) = \frac{\phi_S(s) + \phi_T(t)}{\lambda} \in M$, so the extension is unique. \square

3.2. Moduli space of smooth cubic threefolds. We denote by \mathcal{C}_n^s the GIT moduli space of $\text{PGL}_{n+2}(\mathbb{C})$ -stable points in the linear system $|\mathcal{O}_{\mathbb{P}^{n+1}}(3)|$. Inside this quasi-projective variety, the locus $\mathcal{C}_n^{\text{sm}}$ whose points parametrize projective equivalence classes of smooth cubic n -folds in \mathbb{P}^{n+1} is the open set determined by the nonvanishing of the discriminant (see [40, Chapter 5]).

By Allock–Carlson–Toledo [4], the moduli space $\mathcal{C}_3^{\text{sm}}$ is isomorphic to the quotient by an arithmetic group of the complementary of a hyperplane arrangement in a 10-dimensional complex ball. We briefly recall the main ingredients for later use.

Given $C \in \mathcal{C}_3^{\text{sm}}$ (we choose a representative of the projective equivalence class) we denote by $Y \rightarrow \mathbb{P}^4$ the ramified cyclic triple covering branched along C and by σ the generator of the covering group which acts by multiplication by ξ on the period $H^{3,1}(Y)$. Since $H_o^4(Y, \mathbb{Z})$ has no fixed points for the action of σ , it inherits a natural structure of free module over the ring of Eisenstein integers $\mathcal{E} := \mathbb{Z}[\xi]$, which does not depend on the cubic C . Hence any *marking* (i.e. isometry of \mathbb{Z} -lattices) $\eta: H_o^4(Y, \mathbb{Z}) \rightarrow S$ induces a representation

$$\mathbb{Z}/3\mathbb{Z} \rightarrow \text{O}(S), \quad 1 \mapsto \eta \circ \sigma^* \circ \eta^{-1} =: \rho_0$$

whose isomorphism class does not depend on C by the local topological triviality of the family of cubic threefolds. We call *framing* a marking compatible with the representation, in the sense that $\eta \circ \sigma^* = \rho_0 \circ \eta$. Said differently, a framing $\eta: H_o^4(Y, \mathbb{Z}) \rightarrow S$ is an isometry of \mathbb{Z} -lattices and an isomorphism of \mathcal{E} -modules.

We denote by $\mathcal{F}_3^{\text{sm}}$ the moduli space of framed smooth cubic threefolds: it parametrizes pairs (C, η) , where $C \in \mathcal{C}_3^{\text{sm}}$ and $\eta: H_o^4(Y, \mathbb{Z}) \rightarrow S$ is a framing considered up to composition on the target with the group $\mu_6 := \{\pm 1, \pm \rho_0, \pm \rho_0^2\}$ of 6th roots of the unity, which is also the group of units of \mathcal{E} . We denote by $\Gamma_S^{\rho_0, \xi}$ the group of isometries γ of S such that $\gamma \circ \rho_0 = \rho_0 \circ \gamma$. The group $\Gamma_S^{\rho_0, \xi}$ is equivalently the group of isometries of the \mathcal{E} -valued Hermitian form on S naturally associated to ρ_0 (see Section 3.4). It acts on $\mathcal{F}_3^{\text{sm}}$ on the left by $(\gamma, (C, \eta)) \mapsto (C, \gamma \circ \eta)$ for all $\gamma \in \Gamma_S^{\rho_0, \xi}$, and the units act trivially so we can consider the action of the projective group $\mathbb{P}\Gamma_S^{\rho_0, \xi} := \Gamma_S^{\rho_0, \xi} / \mu_6$. We thus have a natural isomorphism $\mathcal{C}_3^{\text{sm}} \cong \mathbb{P}\Gamma_S^{\rho_0, \xi} \setminus \mathcal{F}_3^{\text{sm}}$.

Any framing $\eta: H_o^4(Y, \mathbb{Z}) \rightarrow S$ induces an isomorphism $\eta: H_o^4(Y, \mathbb{C})_\xi \rightarrow S_\xi$, where S_ξ is the eigenspace of $S \otimes_{\mathbb{Z}} \mathbb{C}$ for the eigenvalue ξ of the isometry ρ_0 . The Poincaré pairing on $H^4(Y, \mathbb{Z})$ extends to a hermitian form $h_Y(\alpha, \beta) := \langle \alpha, \beta \rangle_Y$ on $H^4(Y, \mathbb{C})$. Using the Hodge–Riemann relations [23, p.123] we get that $H^4(Y, \mathbb{C})_\xi$ has signature $(10, 1)$ and that the line $H^{3,1}(Y)$ is negative-definite. Denoting by $h_S(\alpha, \beta) := \langle \alpha, \beta \rangle_S$ the hermitian form on $S \otimes_{\mathbb{Z}} \mathbb{C}$ induced by the integral bilinear form $\langle -, - \rangle_S$ on S , we see that the line $\eta(H^{3,1}(Y))$ defines a point in the 10-dimensional complex ball

$$\Omega_S^{\rho_0, \xi} := \{\omega \in \mathbb{P}(S_\xi) \mid h_S(\omega, \omega) < 0\} \cong \mathbb{B}^{10}.$$

The period map $\mathcal{F}_3^{\text{sm}} \rightarrow \Omega_S^{\rho_0, \xi}$, $(C, \eta) \mapsto \eta(H^{3,1}(Y))$ commutes with the action of $\mathbb{P}\Gamma_S^{\rho_0, \xi}$ and defines a quotient period map

$$\mathcal{P}_S^{\rho_0, \xi}: \mathcal{C}_3^{\text{sm}} \longrightarrow \frac{\mathbb{B}^{10}}{\mathbb{P}\Gamma_S^{\rho_0, \xi}}.$$

By [4, Theorem 1.9] the period map $\mathcal{P}_S^{\rho_0, \xi}$ is an isomorphism onto its image, which is the complementary of the quotient of a hyperplane arrangement (see Section 4).

3.3. Relation with the period map for cubic fourfolds. As initial data, we choose a cubic threefold $\tilde{C} \in \mathcal{C}_3^{\text{sm}}$, its associated cyclic fourfold $\tilde{Y} \in \mathcal{C}_4^{\text{sm}}$ and a framing $\tilde{\eta}: H_o^4(\tilde{Y}, \mathbb{Z}) \rightarrow S$. The period of \tilde{Y} is the line $H^{3,1}(\tilde{Y})$ inside $H_o^4(\tilde{Y}, \mathbb{C})$. The period domain $\Omega_S := \{\omega \in \mathbb{P}(S_{\mathbb{C}}) \mid \langle \omega, \omega \rangle_S = 0, h_S(\omega, \omega) < 0\}$ is isomorphic to the Grassmannian variety of negative definite 2-planes in S , and has two connected components interchanged by complex conjugation. We denote by Ω_S^c the connected component which contains $\tilde{\eta}(H^{3,1}(\tilde{Y}))$. By deformation of the complex structure we get a holomorphic period map from the moduli space of *marked* cubic fourfolds

$\mathcal{C}_{4,\text{mk}}^{\text{sm}} \rightarrow \Omega_S^\circ$. By results of Ebeling and Beauville (see [11, Theorem 2]), the monodromy group of a cubic fourfold is isometric to the group Γ_S of isometries of S which act trivially on the discriminant group D_S and which respect the orientation of a negative-definite 2-plane in S (see Laza [32, §2.2]). Thus Γ_S acts on Ω_S° and by a theorem of Voisin [43], the period map $\mathcal{P}_S: \mathcal{C}_4^{\text{sm}} \rightarrow \frac{\Omega_S^\circ}{\Gamma_S}$ is an open embedding. Note that $\Gamma_S^{\rho_0, \xi}$ is not a subgroup of Γ_S : since $D_S \cong \mathbb{Z}/3\mathbb{Z}$, the elements of $\Gamma_S^{\rho_0, \xi}$ act by ± 1 on D_S (see Lemma 3.1). Observe that $\rho_0 \in \Gamma_S$: the isometry ρ_0 comes from the covering automorphism of Y so it fixes the polarization θ_Y and it acts trivially on the discriminant group $D_S \cong \mathbb{Z}\theta_Y/3\mathbb{Z}\theta_Y$ since the lattice $H^4(Y, \mathbb{Z})$ is unimodular. Thus the $\mathbb{P}\Gamma_S^{\rho_0, \xi}$ -orbits in Ω_S are contained in the corresponding Γ_S -orbits and we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{C}_3^{\text{sm}} & \xrightarrow{\mathcal{P}_S^{\rho_0, \xi}} & \frac{\Omega_S^{\rho_0, \xi}}{\mathbb{P}\Gamma_S^{\rho_0, \xi}} \\ \text{cov} \downarrow & & \downarrow \\ \mathcal{C}_4^{\text{sm}} & \xrightarrow{\mathcal{P}_S} & \frac{\Omega_S^\circ}{\Gamma_S} \end{array}$$

It is well-known that the *covering morphism* cov mapping C to Y is generically injective (see Beauville [9, proof of Theorem 4.6] and references therein for a similar argument).

3.4. The Hermitian module. The ring of Eisenstein integers $\mathcal{E} := \mathbb{Z}[\xi]$ is euclidean with invertible group $\mathcal{E}^* = \{\pm 1, \pm \xi, \pm \xi^2\}$. Its irreducible elements are the prime numbers $p \in \mathbb{Z}$ such that $p \equiv 2 \pmod{3}$ and the elements $x \in \mathcal{E}$ whose norm $x\bar{x}$ is a prime integer. In particular $\theta := \xi - \xi^{-1}$ is irreducible in \mathcal{E} and $3 = \theta\bar{\theta}$. Recall that \mathcal{E} acts on S by $\xi x := \sigma(x)$ for all $x \in S$. Following [4, Chapter 1], we define a Hermitian form on S by

$$H_S(x, y) := \theta(\langle x, \xi y \rangle_S - \xi \langle x, y \rangle_S).$$

It is an easy exercise to check that H_S is a Hermitian form and that it takes values in the prime ideal $\theta\mathcal{E}$ of \mathcal{E} . One has $\langle \delta, \delta \rangle_S = 2$ if and only if $H_S(\delta, \delta) = 3$: this characterizes the *roots* of S . For such a root δ , the image of the \mathcal{E} -linear morphism $H_S(-, \delta): S \rightarrow \mathcal{E}$ is an ideal generated by an element α_δ such that $3\mathcal{E} \subset \alpha_\delta\mathcal{E} \subset \theta\mathcal{E}$. It follows easily that either $H_S(S, \delta) = \theta\mathcal{E}$ (δ is called *nodal*) or $H_S(S, \delta) = 3\mathcal{E}$ (δ is called *chordal*).

3.5. A moduli space of non-symplectic automorphisms. We start again from a ramified cyclic covering $Y \rightarrow \mathbb{P}^4$ branched along the cubic C , with covering automorphism σ acting by multiplication by ξ . Any marking $\eta: H_\circ^4(Y, \mathbb{Z}) \rightarrow S$ induces through the Abel–Jacobi map A an isometry $\eta \circ A^{-1}: H_\circ^2(F(Y), \mathbb{Z}) \rightarrow S(-1)$, which extends to a marking $H^2(F(Y), \mathbb{Z}) \rightarrow L$ as follows. By [14, Theorem 3.7] the lattice $S(-1)$ admits a unique primitive embedding in L , up to an isometry of L , and its orthogonal complement $S(-1)^\perp$ is isometric to the lattice $\langle 6 \rangle$. We implicitly fix this embedding. This determines, up to an isometry of L , a primitive embedding $j: \langle 6 \rangle \hookrightarrow L$ that we fix for the rest of this paper. We extend η to an isometry $H_\circ^2(F(Y), \mathbb{Z}) \oplus \mathbb{Z}\theta_{F(Y)} \rightarrow S(-1) \oplus j(\langle 6 \rangle)$ which extends using Lemma 3.2 to an isometry $\tilde{\eta}: H^2(F(Y), \mathbb{Z}) \rightarrow L$. The action of the automorphism σ defines as above

a representation

$$\mathbb{Z}/3\mathbb{Z} \longrightarrow \mathrm{O}(L), \quad 1 \mapsto \tilde{\eta} \circ \sigma^* \circ \tilde{\eta}^{-1} =: \rho$$

whose isomorphism class does not depend on the cubic C , and whose invariant subrepresentation is $j(\langle 6 \rangle)$. The restriction of ρ to $S(-1)$ is nothing else than ρ_0 viewed as an isometry of $S(-1)$ instead of S . Following Boissière–Camere–Sarti [15], the 4-tuple $(\mathrm{F}(Y), \tilde{\eta}, \sigma, \iota)$ is the initial data of the construction of a *moduli space of (ρ, j) -polarized irreducible holomorphic symplectic manifolds*. We briefly recall the main ingredients for later use.

Denote by \mathcal{M}_L the moduli space of equivalence classes of pairs (X, η) where X is an irreducible holomorphic symplectic manifold deformation equivalent to the Hilbert square of a K3 surface and $\eta: \mathrm{H}^2(X, \mathbb{Z}) \rightarrow L$ is a marking (*i.e.* an isometry of \mathbb{Z} -lattices), where two pairs (X, η) and (X', η') are considered equivalent if there exists a biregular isomorphism $f: X \rightarrow X'$ such that $\eta = \eta' \circ f^*$. Following [35, Section 2] and [15] we fix once and for all a connected component \mathcal{M}_L° of \mathcal{M}_L and we define a (ρ, j) -polarisation of an irreducible holomorphic symplectic manifold X deformation equivalent to the Hilbert square of a K3 surface as the data of:

- a marking η ,
- a primitive embedding $\iota: \langle 6 \rangle \hookrightarrow \mathrm{NS}(X)$ such that $\eta \circ \iota = j$,
- an automorphism $\sigma \in \mathrm{Aut}(X)$ such that η is a framing for $\sigma \in \mathrm{Aut}(X)$ (*i.e.* $\eta \circ \sigma^* = \rho \circ \eta$) and $\sigma|_{\mathrm{H}^{2,0}(X)} = \xi \mathrm{id}$.

If such an automorphism σ exists, then it is uniquely determined by this property (see [38, Lemma 1.2]). We denote by $\mathcal{M}_{\langle 6 \rangle}^{\rho, \xi}$ (the embedding j is implicit in this notation) the set of equivalence classes of such tuples (X, η, σ, ι) , where two tuples (X, η, σ, ι) and $(X', \eta', \sigma', \iota')$ are equivalent if there exists a biregular isomorphism $f: X \rightarrow X'$ such that $\eta' = f^* \circ \eta$, $\sigma' = f \circ \sigma \circ f^{-1}$ and $\iota' = f^* \circ \iota$. We similarly define the space $\mathcal{M}_{\langle 6 \rangle}$ parametrizing j -polarized triples (X, η, ι) up to equivalence. We have then the inclusions $\mathcal{M}_{\langle 6 \rangle}^{\rho, \xi} \subset \mathcal{M}_{\langle 6 \rangle} \subset \mathcal{M}_L^\circ$.

Remark 3.3. The polarisation $\iota(\langle 6 \rangle)$ contains always an ample class: the variety X is projective since it admits a non-symplectic automorphism (see [10, §4]), so if $\ell \in \mathrm{NS}(X)$ is an ample class, the invariant ample class $\ell + \sigma^* \ell + (\sigma^*)^2 \ell$ is necessarily a multiple of the generator of the rank one invariant lattice. Denote $\vartheta \in \iota(\langle 6 \rangle)$ the primitive ample invariant class; since the invariant lattice has rank one, this class is uniquely defined by the automorphism σ . It follows that the map $(X, \sigma, \eta, \iota) \mapsto (X, \eta, \vartheta)$ embeds $\mathcal{M}_{\langle 6 \rangle}^{\rho, \xi}$ in the moduli space of irreducible holomorphic symplectic manifolds polarized by an ample primitive class of square 6 studied by Gritsenko–Hulek–Sankaran [24].

Starting from the initial data $(\mathrm{F}(Y), \tilde{\eta}, \sigma, \iota)$ as above, we consider the monodromy group $\mathrm{Mon}^2(\mathrm{F}(Y)) \subset \mathrm{O}(\mathrm{H}^2(\mathrm{F}(Y), \mathbb{Z}))$ and we define

$$\mathrm{Mon}^2(L) := \tilde{\eta} \circ \mathrm{Mon}^2(\mathrm{F}(Y)) \circ \tilde{\eta}^{-1} \subset \mathrm{O}(L).$$

It is easy to check that this group does not depend on the representative of the equivalence class in $\mathcal{M}_{\langle 6 \rangle}^{\rho, \xi}$ and that it is conjugated to the monodromy group of any variety deformation equivalent to $\mathrm{F}(Y)$. Following [15, Section 6], we define the group $\Gamma_{\langle 6 \rangle}^{\rho, \xi}$ as the image in $\mathrm{O}(S(-1))$ of those elements $\gamma \in \mathrm{Mon}^2(L)$ which act trivially on $j(\langle 6 \rangle)$ and commute with ρ . The group $\Gamma_{\langle 6 \rangle}^{\rho, \xi}$ acts on $\mathcal{M}_{\langle 6 \rangle}^{\rho, \xi}$ by

$(\gamma, (X, \eta, \sigma, \iota)) \mapsto (X, \bar{\gamma} \circ \eta, \sigma, \iota)$, where $\bar{\gamma}$ is the extension of γ to L (see Lemma 3.2). We define $\mathcal{N}_{(6)}^{\rho, \xi} := \Gamma_{(6)}^{\rho, \xi} \setminus \mathcal{M}_{(6)}^{\rho, \xi}$.

Using the properties of the Beauville–Bogomolov–Fujiki quadratic form, we see that the line $\eta(H^{2,0}(X))$ defines a point in the 10-dimensional complex ball

$$\Omega_{(6)}^{\rho, \xi} := \{\omega \in \mathbb{P}(S(-1)_{\xi}) \mid h_{S(-1)}(\omega, \omega) > 0\} \cong \mathbb{B}^{10}.$$

The period map $\mathcal{M}_{(6)}^{\rho, \xi} \rightarrow \Omega_{(6)}^{\rho, \xi}$, $(X, \eta, \sigma, \iota) \mapsto \eta(H^{2,0}(X))$ commutes with the action of $\Gamma_{(6)}^{\rho, \xi}$ and defines a quotient period map

$$\mathcal{P}_{(6)}^{\rho, \xi} : \mathcal{N}_{(6)}^{\rho, \xi} \longrightarrow \frac{\mathbb{B}^{10}}{\Gamma_{(6)}^{\rho, \xi}}$$

By [15, Theorem 4.5, Theorem 5.6, Section 7.1], the moduli space $\mathcal{N}_{(6)}^{\rho, \xi}$ is connected, the period map is an open embedding and its image is the complementary of the quotient of a hyperplane arrangement. Note that there is a tiny difference between this definition of the moduli space and those of [15], since we include σ and ι in the data. Since they are uniquely determined, this is of course equivalent and our period map is, strictly speaking, the composition of a bijective forgetful map and of the period map of [15]. We define the structure of complex manifold of $\mathcal{N}_{(6)}^{\rho, \xi}$ as the one inherited by the one of its period domain.

Theorem 3.4. *The moduli spaces $\mathcal{C}_3^{\text{sm}}$ and $\mathcal{N}_{(6)}^{\rho, \xi}$ are isomorphic.*

Proof. The period domains $\Omega_S^{\rho_0, \xi}$ and $\Omega_{(6)}^{\rho, \xi}$ are clearly identical, we identify them with the complex ball \mathbb{B}^{10} . By [4, Theorem 6.1] the image of the period map \mathcal{P}_C is the complementary of the quotient of the union of the hyperplanes $\delta^{\perp} \cap \mathbb{B}^{10}$ for all roots δ of S , i.e. all elements of norm $\langle \delta, \delta \rangle_S = 2$ (in [4] these have norm 3 for the naturally associated \mathcal{E} -valued Hermitian form on S , see Section 3.4). By [15, Theorem 4.5, Theorem 5.6, Section 7.1], the image of the period map $\mathcal{P}_{(6)}^{\rho, \xi}$ is the complementary of the quotient of the union of the hyperplanes $\delta^{\perp} \cap \mathbb{B}^{10}$ for all $\delta \in S(-1)$ which are *monodromy birationally minimal (MBM)*, see [5]. By [8, 39] those are exactly the classes either of norm -2 or of norm -10 and divisibility 2 (i.e. $\langle S, \delta \rangle_S = 2\mathbb{Z}$). The latter cannot happen since $S(-1)$ has discriminant 3. This shows that both hyperplane arrangements are equal.

By a result of Markman [36, Theorem 1.2], [35, §9] $\text{Mon}^2(\mathbb{F}(Y))$ is the subgroup of $\text{O}(H^2(\mathbb{F}(Y), \mathbb{Z}))$ of isometries whose real extension preserves the orientation of any positive definite 3-dimensional subspace of $H^2(\mathbb{F}(Y), \mathbb{Z})$. Said differently, using the real spinor norm $\text{sn}_L^{\mathbb{R}} : \text{O}(L) \rightarrow \{\pm 1\}$ (see [17, §2.1] and references therein), the elements of $\Gamma_{(6)}^{\rho, \xi}$ are the restrictions to $S(-1)$ of those isometries of L of real spinor norm 1, acting trivially on $j(\langle 6 \rangle)$ and commuting with the representation ρ . Observe that $\Gamma_S^{\rho_0, \xi}$ is contained in the stabilizer of the ball \mathbb{B}^{10} inside the orthogonal group $\text{O}(S)$, hence $\Gamma_S^{\rho_0, \xi}$ preserves the connected component of Ω_S° of Ω_S which contains the ball \mathbb{B}^{10} . This implies that $\text{sn}_S^{\mathbb{R}}(\gamma) = 1$ for every $\gamma \in \Gamma_S^{\rho_0, \xi}$ (see for instance [37, Proposition 11.3]). For any such element, we define $\tilde{\gamma} \in \text{O}(L)$ as the unique extension of $(\epsilon\gamma, \text{id}_{j(\langle 6 \rangle)})$, where $\epsilon \in \{\pm 1\}$ is such that γ acts by ϵid on the discriminant group D_S . Since

$$\text{sn}_L^{\mathbb{R}}(\tilde{\gamma}) = \text{sn}_S^{\mathbb{R}}(\epsilon\gamma) = \text{sn}_S^{\mathbb{R}}(\gamma) = 1,$$

we have a well-defined group homomorphism $\Gamma_S^{\rho_0, \xi} \rightarrow \Gamma_{\langle 6 \rangle}^{\rho, \xi}$ mapping γ to $\tilde{\gamma}|_S$, whose kernel is $\langle \pm \text{id}_S \rangle$. So we have an isomorphism $\Gamma_{\langle 6 \rangle}^{\rho, \xi} \cong \frac{\Gamma_S^{\rho_0, \xi}}{\langle \pm \text{id}_S \rangle}$ and the ball quotients are equal. \square

Concretely, we have a natural map $F: \mathcal{C}_3^{\text{sm}} \rightarrow \mathcal{N}_{\langle 6 \rangle}^{\rho, \xi}$ which maps the equivalence class of a framed cubic (C, η) to the equivalence class of the (ρ, j) -polarized irreducible holomorphic manifold $(F(Y), \tilde{\eta}, \sigma, \iota)$ constructed above, and the theorem says that this map is an isomorphism. We thus have a commutative diagram

$$\begin{array}{ccc} \mathcal{C}_3^{\text{sm}} & \xrightarrow{F} & \mathcal{N}_{\langle 6 \rangle}^{\rho, \xi} \\ & \searrow \sim & \swarrow \sim \\ & \mathcal{P}_S^{\rho_0, \xi} & \mathcal{P}_{\langle 6 \rangle}^{\rho, \xi} \\ & \searrow & \swarrow \\ & \mathbb{B}^{10} \setminus \mathcal{H} & \\ & \underline{\mathbb{P}\Gamma} & \end{array}$$

where $\Gamma := \Gamma_{\langle 6 \rangle}^{\rho, \xi} \cong \frac{\Gamma_S^{\rho_0, \xi}}{\langle \pm \text{id}_S \rangle}$, $\mathbb{P}\Gamma$ is the projective action of Γ and $\mathcal{H} = \mathcal{H}_n \cup \mathcal{H}_c$ is the hyperplane arrangement discussed in the proof of Theorem 3.4, where we denote

$$\begin{aligned} \mathcal{H}_n &:= \bigcup_{\substack{\delta \text{ nodal root} \\ \text{of } S}} \delta^\perp \cap \Omega_S^{\rho_0, \xi}, \\ \mathcal{H}_c &:= \bigcup_{\substack{\delta \text{ chordal root} \\ \text{of } S}} \delta^\perp \cap \Omega_S^{\rho_0, \xi} \end{aligned}$$

the unions of hyperplanes orthogonal respectively to nodal and chordal roots.

Corollary 3.5. *An irreducible holomorphic symplectic manifold X deformation equivalent to the Hilbert square of a K3 surface, polarized by an ample class ϑ of square 6, admits a non-symplectic automorphism of order three with invariant lattice $\mathbb{Z}\vartheta \cong \langle 6 \rangle$ if and only if X is isomorphic to the Fano variety of lines of the ramified cyclic triple covering Y of \mathbb{P}^4 branched along a smooth cubic threefold. In this case, the automorphism σ is induced by the covering automorphism.*

Remark 3.6. Take $(X, \eta, \sigma, \iota) \in \mathcal{M}_{\langle 6 \rangle}^{\rho, \xi}$. A class $d \in \mathbb{H}^2(X, \mathbb{Z})$ is algebraic if and only if it is orthogonal to $\mathbb{H}^{2,0}(X)$, or equivalently if the period $\eta(\mathbb{H}^{2,0}(X))$ is outside the hyperplane orthogonal to d in the period domain $\Omega_{\langle 6 \rangle}^{\rho, \xi}$. Since such hyperplanes form a countable union, their complementary is dense in the period domain, so generically the Néron–Severi group of X is generated by $\iota(\langle 6 \rangle)$. The moduli space $\mathcal{N}_{\langle 6 \rangle}^{\rho, \xi}$ is thus a 10-dimensional moduli space of irreducible holomorphic symplectic manifolds deformation equivalent to the Hilbert square of a K3 surface, admitting a non symplectic automorphism of order three, and whose general Picard number is one. This answers a question asked to the authors by Olivier Debarre [20].

3.6. Relation with the moduli space of genus five principally polarized abelian varieties. The term “occult” in the section title is a reference to a paper of Kudla–Rapoport [31]. As suggested to us by Michael Rapoport and thanks to enlightening discussions with Gregory Sankaran, we can draw direct paths between all the involved period maps, producing in particular a period map from $\mathcal{N}_{\langle 6 \rangle}^{\rho, \xi}$ to a moduli space of principally polarized abelian varieties. The classical period map for cubic threefolds of Clemens–Griffiths [19] maps a cubic $C \in \mathcal{C}_3^{\text{sm}}$ to its intermediate

Jacobian $\text{IJ}(C) := \frac{H^{1,2}(C)}{H^3(C, \mathbb{Z})}$, which gives a point in the 15-dimensional moduli space \mathcal{A}_5 of principally polarized abelian variety of genus 5.

Corollary 3.7. *There exists an embedding $\mathcal{N}_{(6)}^{\rho, \xi} \hookrightarrow \mathcal{A}_5$ mapping a (ρ, j) -polarized tuple (X, η, σ, ι) to the Albanese variety $\text{Alb}(X^\sigma)$ of the fixed locus $X^\sigma \subset X$ of the automorphism σ .*

Proof. By [19, (0.8)] the intermediate Jacobian of C is isomorphic to the Albanese variety of the Fano variety of lines on C , i.e. $\text{IJ}(C) \cong \text{Alb}(F(C))$. As observed in Section 2.2, the variety $F(C)$ is the fixed locus in $F(Y)$ of the automorphism σ of the ramified covering map $Y \rightarrow \mathbb{P}^4$ branched along C . Since the morphism $F: \mathcal{C}_3^{\text{sm}} \rightarrow \mathcal{N}_{(6)}^{\rho, \xi}$ mapping C to $(F(V), \sigma)$ (with the appropriate markings and polarizations as above) is an isomorphism, we get the result. \square

4. DEGENERATIONS

Following an idea used by Dolgachev–Kondō [22, §11] to study the degeneration of K3 surfaces with a non-symplectic automorphism of prime order, we study the degenerations of those irreducible holomorphic symplectic manifolds (X, η, σ, ι) parametrized by $\mathcal{M}_{(6)}^{\rho, \xi}$, inside the moduli space $\mathcal{M}_{(6)}$ of j -polarized marked triples. Denote by $\Omega_{(6)}^\circ$ the connected component of $\{\omega \in \mathbb{P}(S_C) \mid h_{S(-1)}(\omega, \omega) > 0\}$ which contains the ball $\Omega_{(6)}^{\rho, \xi}$, it is easy to deduce from the theorem of surjectivity of the period map of Huybrechts [27, Theorem 8.1] that the period map $\mathcal{P}_{(6)}: \mathcal{M}_{(6)} \rightarrow \Omega_{(6)}^\circ$ is surjective. Note that, compared with the notation in Section 3.3, we have $\Omega_{(6)}^\circ = \Omega_S^\circ$ but we use both notations depending on the context. We thus have a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_{(6)}^{\rho, \xi} & \xrightarrow{\mathcal{P}_{(6)}^{\rho, \xi}} & \Omega_{(6)}^{\rho, \xi} \setminus \mathcal{H} \\ \downarrow & & \downarrow \\ \mathcal{M}_{(6)} & \xrightarrow{\mathcal{P}_{(6)}} & \Omega_{(6)}^\circ \end{array}$$

Let $\omega \in \mathcal{H}$ and $(X, \eta, \iota) \in \mathcal{P}_{(6)}^{-1}(\omega)$. By [35, Theorem 1.2], any two points in this fiber of $\mathcal{P}_{(6)}$ correspond to birational manifolds.

We first observe that the isometry $\rho \in O(L)$ is not represented by any automorphism of X . As explained in the proof of Theorem 3.4, this is a special case of [14, Theorem 4.0.8]: if $\omega \in \mathcal{H}$, then ω is orthogonal to some class $\delta \in S$ such that $\eta^{-1}(\delta)$ is a MBM class of X . If ρ was represented by an automorphism of X , this one would be automatically non-symplectic since ω is not in the invariant sublattice of ρ , so as explained in Remark 3.3 the line $\iota(\langle 6 \rangle)$ contains an ample class ϑ . But since $\delta \in S$, the divisor $\eta^{-1}(\delta)$ is orthogonal to ϑ . As explained in [14, Remark 4.0.7 (2)], this is not possible since MBM classes have nonzero intersection with the Kähler cone of X . In particular, in this situation the line $\iota(\langle 6 \rangle)$ contains no ample class any more.

Following [35, Section 1], we denote by $\text{Mon}_{\text{Hdg}}^2(X)$ the subgroup of $\text{Mon}^2(X)$ consisting of those monodromies whose \mathbb{C} -linear extension preserves the Hodge decomposition. For any nonisotropic vector $v \in L$, we denote by $r_v := x \mapsto x - 2 \frac{\langle x, v \rangle}{\langle v, v \rangle} v$ the reflection which leaves v^\perp invariant. We consider the normal subgroup $W_{\text{Exc}}(X)$

of $\text{Mon}_{\text{Hdg}}^2(X)$ generated by the reflections by prime exceptional divisors of X (i.e. reduced and irreducible effective divisors of negative Beauville–Bogolomov–Fujiki degree).

Proposition 4.1. *Let $\omega \in \mathcal{H}$ and $(X, \eta, \iota) \in \mathcal{P}_{(6)}^{-1}(\omega)$. There exists a unique birational map $\beta \in \text{Bir}(X)$ and a unique element $w \in W_{\text{Exc}}(X)$ such that:*

$$\eta^{-1} \circ \rho \circ \eta = w \circ \beta^*.$$

Proof. Since $\rho \in \text{Mon}^2(L)$, the isometry $\psi := \eta^{-1} \circ \rho \circ \eta$ is a monodromy operator of X . Since $\omega \in \mathbb{P}(S(-1)_\xi)$, we have $\psi(\text{H}^{2,0}(X)) = \text{H}^{2,0}(X)$ so $\psi \in \text{Mon}_{\text{Hdg}}^2(X)$. By [35, Theorem 1.6], there exists a unique element $w \in W_{\text{Exc}}(X)$ and a birational map $\beta \in \text{Bir}(X)$ such that $\eta^{-1} \circ \rho \circ \eta = w \circ \beta^*$. The map β is unique in this case since the natural map $\text{Aut}(X) \rightarrow \text{O}(\text{H}^2(X, \mathbb{Z}))$ is injective in this deformation class (see [38, Lemma 1.2]). \square

This results means that the isometry ρ , after composition by a suitable product of reflections, is realized by a birational transformation of X . By a deeper analysis of the degeneracy situations, we will show below that the product of reflections w is not trivial and that ρ itself is in fact realized by another automorphism and not only by a birational map. This means that, when the period point $\omega \in \Omega_{(6)}^{\rho, \xi} \setminus \mathcal{H}$ goes to a point of \mathcal{H} , the non-symplectic automorphism of the family does not degenerate to a birational non-biregular transformation. What happens instead is that the automorphism degenerates by jumping to another one with a bigger invariant sublattice on a different family. We will show this by describing explicitly the new isometry to be considered and by proving that, for a suitable choice of X in the fiber, the birational transformation β extends to an automorphism of X .

This phenomenon is very common in the context of degenerations of automorphisms of K3 surfaces. Take for instance an ample $\langle 2 \rangle$ -polarized K3 surface Σ . It is well-known that Σ carries a non-symplectic involution ι with invariant lattice isometric to $\langle 2 \rangle$ and such that $\Sigma/\iota \cong \mathbb{P}^2$. This exhibits Σ as a ramified double cover of \mathbb{P}^2 branched along a smooth sextic curve. If the sextic acquires ADE singularities then the double cover is singular and its minimal resolution $\tilde{\Sigma}$ is a K3 surface containing several (-2) -curves coming from the desingularisation of the singular points. In this case the K3 surface $\tilde{\Sigma}$ carries again a non-symplectic involution but its invariant lattice has higher rank. A concrete example is when the sextic has a node: generically, the surface $\tilde{\Sigma}$ has a Néron–Severi lattice isometric to $\langle 2 \rangle \oplus \langle -2 \rangle$, which is also the invariant lattice of the non-symplectic involution (see [1]). What happens is that as soon as the sextic admits singular points, the K3 surface $\tilde{\Sigma}$ contains rational curves which do not meet the class of square 2 (the pullback of the class of a line of \mathbb{P}^2 is not an ample polarization any more, it is only big and nef). Then $\tilde{\Sigma}$ admits a non-symplectic involution $\tilde{\iota}$ which is of “different kind” than ι , since their invariant lattices are different, so heuristically the automorphism survives by jumping to a different family.

4.1. Degeneracy lattices. For any $\omega \in \mathcal{H}$ and $(X, \eta, \iota) \in \mathcal{P}_{(6)}^{-1}(\omega)$, the *degeneracy lattice* of (X, η, ι) is the sublattice of $S(-1)$ generated by MBM classes of $S(-1)$ which are orthogonal to ω (see the proof of Theorem 3.4). Here this lattice is simply the lattice generated by the roots δ of $S(-1)$, with $\delta^2 = -2$, such that $\omega \in \delta^\perp$. This lattice is clearly globally ρ -invariant and orthogonal to $j(\langle 6 \rangle)$. Generically,

ω belongs to only one hyperplane $\delta^\perp \cap \mathbb{P}(S(-1)_\xi)$ so the general degeneracy lattice is $R_\delta := \text{Span}(\delta, \rho(\delta))$.

It is easy to compute that R_δ is isometric to the lattice $A_2(-1)$ and that ρ acts trivially on the discriminant group D_{R_δ} . The lattice R_δ is primitive in $S(-1)$. Indeed, using the \mathcal{E} -lattice structure (see Section 3.4) of $S(-1)$ we have an isomorphism of \mathcal{E} -modules $\frac{S(-1)}{R_\delta} \cong \frac{\mathcal{E}^{\oplus 11}}{\mathcal{E}\delta}$. Since δ is a root, with $H_S(\delta) = 3$, it is easy to check that the rank one \mathcal{E} -submodule $\mathcal{E}\delta$ is primitive in $\mathcal{E}^{\oplus 11}$, so R_δ is primitive in $S(-1)$. Using a result of Nikulin [42, Proposition 1.15.1] we find that two degeneration situations appear, depending on the two possible isometry classes of the orthogonal complement of R_δ in $S(-1)$:

$$R_\delta^\perp \cong U^{\oplus 2} \oplus E_8(-1)^{\oplus 2} \text{ or } R_\delta^\perp \cong U \oplus U(3) \oplus E_8(-1)^{\oplus 2}.$$

Recall that following [4] a root δ is called *nodal* if $H_S(S, \delta) = \theta\mathcal{E}$ and *chordal* if $H_S(S, \delta) = 3\mathcal{E}$. The relation with the above dichotomy is explained by the following observation.

Lemma 4.2. *A root δ is chordal if and only if R_δ^\perp is unimodular.*

Proof. It is easy to check that $H_S(R_\delta, \delta) = 3\mathcal{E}$ and $H_S(R_\delta^\perp, \delta) = 0$. If R_δ^\perp is unimodular, then $R_\delta \oplus R_\delta^\perp = S$ so $H_S(S, \delta) = 3\mathcal{E}$ and δ is chordal. Otherwise $\frac{S}{R_\delta \oplus R_\delta^\perp} \cong \frac{\mathbb{Z}}{3\mathbb{Z}}$ is generated by a class $x := \frac{1}{3}(\delta - \rho(\delta) + y)$ with $y \in R_\delta^\perp$ and we get $H_S(x, \delta) = \theta(1 - \xi) \in \theta\mathcal{E}^*$ so $H_S(S, \delta) = \theta\mathcal{E}$ and δ is nodal. \square

4.2. Degeneration of the representation. Let $\delta \in S(-1)$ be a root ($\delta^2 = -2$). Recall that we have chosen primitive embeddings $j: \langle 6 \rangle \hookrightarrow L$ and $S(-1) \subset L$ such that $j(\langle 6 \rangle)^\perp = S(-1)$. This induces an embedding in L of $\langle 6 \rangle \oplus A_2(-1) \cong \langle 6 \rangle \oplus R_\delta$, not necessarily primitive. We thus consider the saturation T_δ of $\langle 6 \rangle \oplus R_\delta$ in L , which is the minimal primitive sublattice of L containing $j(\langle 6 \rangle) \oplus R_\delta$. Observe that the orthogonal complement S_δ of T_δ in L is nothing else than the orthogonal complement R_δ^\perp of R_δ in $S(-1)$ embedded in L .

Lemma 4.3. *Let $\delta \in S(-1)$ be a root.*

- (1) *If δ is nodal, then $\langle 6 \rangle \oplus R_\delta$ is primitive in L and $T_\delta \cong U(3) \oplus \langle -2 \rangle$.*
- (2) *If δ is chordal, then $\langle 6 \rangle \oplus R_\delta$ is not primitive in L and $T_\delta \cong U \oplus \langle -2 \rangle$.*

Proof. Since $\langle 6 \rangle \oplus R_\delta$ has discriminant $3^2 \cdot 2$, its only possible saturation is by adding a 3-divisible class, so either $\text{disc}(T_\delta) = 3^2 \cdot 2$ (primitive case), or $\text{disc}(T_\delta) = 2$ (nonprimitive case). Recall that if M is a nondegenerate lattice and $N \subset M$ a primitive sublattice, denoting by $\phi: M \rightarrow N^*$ the natural map, we have the following formula (see [25, p.47]):

$$\text{disc}(N) \cdot \text{disc}(M) = [N^* : \phi(M)]^2 \cdot \text{disc}(N^\perp).$$

Applying this formula to $M = L$ and either $N = T_\delta$ or $N = S_\delta$, we get that if δ is nodal, then necessarily $\text{disc}(T_\delta) = 3^2 \cdot 2$, and if δ is chordal, then $\text{disc}(T_\delta) = 2$. If δ is nodal, the isomorphism $\langle 6 \rangle \oplus A_2(-1) \cong U(3) \oplus \langle -2 \rangle$ is an easy exercise. If δ is chordal, the lattice T_δ is thus an even lattice of discriminant 2 and signature $(1, 2)$. By Nikulin's classification, such a lattice is unique up to isometry so $T_\delta \cong U \oplus \langle -2 \rangle$. \square

The representation $\mathbb{Z}/3\mathbb{Z} \rightarrow \text{O}(L)$, $1 \mapsto \rho$ has $j(\langle 6 \rangle)$ as fixed sublattice and $S(-1)$ as orthogonal complement. We modify this representation as follows. Recall that r_v denotes the reflection by v^\perp .

Proposition 4.4. *The isometry $\text{id}_{T_\delta} \oplus \rho|_{S_\delta} \in \text{O}(T_\delta) \oplus \text{O}(S_\delta)$ extends uniquely to an isometry $\rho_\delta \in \text{O}(L)$ with fixed sublattice T_δ . One has $\rho_\delta = r_{\rho(\delta)} \circ r_\delta \circ \rho$ and $\rho_\delta \in \text{Mon}^2(L)$.*

Proof. We apply Lemma 4.2. If δ is a chordal root, then S_δ is unimodular so the result is clear. If δ is nodal, using [42, Proposition 1.15.1] we know that the discriminant group D_{S_δ} is isometric to the restriction of $(-q_{R_\delta}) \oplus q_{S(-1)}$ to the quotient of a subgroup of $D_{R_\delta} \oplus D_{S(-1)}$. As observed above, ρ acts trivially on D_{R_δ} , so to show that it acts trivially on D_{S_δ} it is enough to show that it acts trivially on $D_{S(-1)}$. By the same argument as above, we can identify $D_{S(-1)}$ with the quotient of a subgroup of $D_{j(\langle 6 \rangle)} \oplus D_L$ by changing a sign on the finite quadratic form. By construction ρ acts trivially on the first factor, and it acts also trivially on $D_L \cong \mathbb{Z}/2\mathbb{Z}$ since ρ has order three. So ρ acts trivially on $D_{S(-1)}$ and the result follows. It is easy to check that $\rho_\delta = r_{\rho(\delta)} \circ r_\delta \circ \rho$. Indeed, both act trivially on R_δ and on $j(\langle 6 \rangle) \subset \delta^\perp$ and act by ρ on $S_\delta = R_\delta^\perp$. By [35, Section 9.1, Theorem 9.1], this shows that $\rho_\delta \in \text{Mon}^2(L)$. \square

4.3. Nodal degenerations. By [4, Theorem 6.1], the period map for cubic threefolds

$$\mathcal{P}_S^{\rho_0, \xi}: \mathcal{C}_3^{\text{sm}} \rightarrow \frac{\mathbb{B}^{10} \setminus (\mathcal{H}_n \cup \mathcal{H}_c)}{\mathbb{P}\Gamma}$$

extends to an isomorphism $\mathcal{P}_S^{\rho_0, \xi}: \mathcal{C}_3^{\text{s}} \rightarrow \frac{\mathbb{B}^{10} \setminus \mathcal{H}_c}{\mathbb{P}\Gamma}$ mapping bijectively the *stable discriminant locus* $\Delta_3^{\text{s}} := \mathcal{C}_3^{\text{s}} \setminus \mathcal{C}_3^{\text{sm}}$ to the $\mathbb{P}\Gamma$ -orbit of the nodal hyperplanes arrangement \mathcal{H}_n . We recall here some aspects of their work needed in the sequel.

A general point in Δ_3^{s} is a cubic threefold C with one nodal (or A_1) singular point. In suitable coordinates, the equation of C writes

$$f(x_0, \dots, x_4) := x_0 q(x_1, \dots, x_4) + g(x_1, \dots, x_4) = 0$$

where q is a nondegenerate quadric surface and g is a homogeneous cubic polynomial. By genericity we may assume that the locus $\{g = 0\}$ is smooth and that it meets the locus $\{q = 0\}$ transversally. The point $p_0 = (1, 0, 0, 0, 0) \in \mathbb{P}^4$ is then the node of C . We put $k(x_1, \dots, x_5) := g(x_1, \dots, x_4) + x_5^3$, so that the cuspidal cyclic cubic fourfold Y associated to C has equation

$$F(x_0, \dots, x_5) := x_0 q(x_1, \dots, x_4) + k(x_1, \dots, x_5)$$

with an A_2 -singularity at $p = (1, 0, 0, 0, 0, 0) \in \mathbb{P}^5$.

We denote by $H \subset \mathbb{P}^5$ the hyperplane of equation $x_0 = 0$ and by Q , *resp.* K , the zero locus of q , *resp.* k , in H . By genericity, the intersection $\Sigma_n := Q \cap K \subset H$ is a smooth K3 surface. By projecting $Y \setminus \{p\}$ to H through the point p , it is easy to check that the subvariety $F(Y, p) \subset F(Y)$ defined by the lines ℓ of Y passing through p maps biholomorphically to Σ_n , sending ℓ to $\ell \cap H$ (see [19, §7]). The Fano variety of lines $F(Y)$ is a singular holomorphic symplectic variety whose singular locus is contained in the surface $F(Y, p)$ (see [33, Proposition 3.5, Theorem 3.6]) and it is birational to $\Sigma_n^{[2]}$. To see it, first observe that the Cartier divisor $\{q = 0\} \cap Y$ in \mathbb{P}^5 is the cone over Σ_n with vertex p . A general line $\ell \subset Y$ intersects the quadric $\{q = 0\}$ in two points which project to two points of Σ_n , yielding a rational map $\mu: F(Y) \rightarrow \Sigma_n^{[2]}$. The birational inverse is given as follows: given two distinct points $x, y \in \Sigma_n$, the plane $\text{Span}(x, y, p)$ cuts Y along the degenerate cubic curve which is

the union of the lines $\text{Span}(p, x)$, $\text{Span}(p, y)$ and a third line $\ell \in F(Y)$ giving the image of $\{x, y\} \in \Sigma_n^{[2]}$.

The covering automorphism σ restricts on H to an order three non-symplectic automorphism τ_n of Σ_n . This family of K3 surfaces has been studied by Artebani–Sarti [6, Proposition 4.7]: the automorphism τ_n fixes a genus 4 curve which, following the discussion above, is isomorphic to the Fano variety $F(C, p_0)$ of lines through p_0 . The invariant lattice in $H^2(\Sigma_n, \mathbb{Z})$ is isometric to $U(3)$ and its orthogonal complement is isometric to $U \oplus U(3) \oplus E_8(-1)^{\oplus 2}$.

Let $\delta \in S(-1)$ be a nodal root (a precise choice is unnecessary since the group Γ acts transitively on the set of nodal roots by [4, Lemma 6.2]). Recall that by Lemma 4.3, the lattice T_δ is isometric to $U(3) \oplus \langle -2 \rangle$. An elementary computation shows that the second summand is generated by the (-2) -class

$$e_n := -\vartheta + 2\delta + 2\rho(\delta),$$

where ϑ is as above the square 6 polarization. Observe that the orthogonal complement of $\mathbb{Z}e_n$ in L is isometric to the K3 lattice $U^{\oplus 3} \oplus E_8(-1)^{\oplus 3}$. We thus obtain an embedding $U(3) \hookrightarrow T_\delta \subset L$ whose image is contained in $(\mathbb{Z}e_n)^\perp$ and whose orthogonal complement in $(\mathbb{Z}e_n)^\perp$ is isometric to $S_\delta = U \oplus U(3) \oplus E_8(-1)^{\oplus 2}$. The isometry $\text{id}_{U(3)} \oplus \rho_\delta|_{S_\delta}$ extends uniquely to an isometry $\tilde{\rho}_\delta$ of $(\mathbb{Z}e_n)^\perp$ which is the restriction of ρ_δ to $(\mathbb{Z}e_n)^\perp \subset L$.

Lemma 4.5. *Let $\delta \in S(-1)$ be a nodal root. The action of τ_n on $H^2(\Sigma_n, \mathbb{Z})$ is conjugated to the isometry $\tilde{\rho}_\delta$ of $(\mathbb{Z}e_n)^\perp$.*

Proof. In a similar manner as in Section 3.5, we introduce the moduli space $\mathcal{K}_{U(3)}^{\tilde{\rho}_\delta, \xi}$ parametrizing K3 surfaces with a non-symplectic automorphism of order three whose action on cohomology is conjugated to $\tilde{\rho}_\delta$, with a period map

$$\mathcal{P}_{U(3)}^{\tilde{\rho}_\delta, \xi} : \mathcal{K}_{U(3)}^{\tilde{\rho}_\delta, \xi} \longrightarrow \Omega_{U(3)}^{\tilde{\rho}_\delta, \xi} := \{\omega \in \mathbb{P}((S_\delta)_\xi) \mid h_{S_\delta}(\omega, \omega) > 0\},$$

where $(S_\delta)_\xi$ is the eigenspace of ρ_δ on $(S_\delta) \otimes_{\mathbb{Z}} \mathbb{C}$ for the eigenvalue ξ and h_{S_δ} is the Hermitian product induced by the bilinear form on S_δ . As observed in the proof of Proposition 4.4, the isometry ρ_δ acts trivially on the discriminant group D_{S_δ} , so $\tilde{\rho}_\delta$ does also, hence following [6, Section 3, Proposition 3.1] there exists a K3 surface with a non-symplectic automorphism of order 3 whose action on the second cohomology space is conjugated to $\tilde{\rho}_\delta$. As explained in *loc.cit.*, the fixed locus of this automorphism, which depends only on lattice theoretical invariants, consists of one genus 4 curve. By considering the linear system associated to this curve, Artebani–Sarti [6, Section 4, Proposition 4.7] prove by geometric arguments that this K3 surface with this automorphism belongs to the same family as the K3 surface Σ_n with automorphism τ_n . This shows that the action of τ_n on $H^2(\Sigma_n, \mathbb{Z})$ is conjugated to $\tilde{\rho}_\delta$. \square

We consider similarly the natural automorphism $\tau_n^{[2]}$ induced on $\Sigma_n^{[2]}$ by τ_n , which satisfies the relation $\mu \circ \sigma = \tau_n^{[2]} \circ \mu$ whenever $\mu: F(Y) \rightarrow \Sigma_n^{[2]}$ is defined. Its invariant lattice is T_δ so it defines a point in the moduli space $\mathcal{N}_{T_\delta}^{\rho_\delta, \xi}$ parametrizing irreducible holomorphic symplectic manifolds deformation equivalent to the Hilbert square of a K3 surface, with an order three automorphism whose action on cohomology is conjugated to ρ_δ . Here we have implicitly fixed the primitive embedding $j_\delta: T_\delta \hookrightarrow L$ discussed above. Since the period map for this moduli space is not

injective, we introduce a smaller moduli space containing the point $(\Sigma_n^{[2]}, \tau_n^{[2]})$. The method follows [15, Section 3], which is based on ideas of Joumaah [29]. Take a marking $\eta: H^2(\Sigma_n^{[2]}, \mathbb{Z}) \rightarrow L$. The image by η of the invariant positive cone $(C_{\Sigma_n^{[2]}})^{\tau_n}$ of $\Sigma_n^{[2]}$ defines a connected component C_{T_δ} of the positive cone of $T_\delta \otimes \mathbb{R}$. The hyperplanes δ^\perp , where $\delta \in T_\delta$ is an MBM class, cut out a chamber decomposition of C_{T_δ} . One of these chambers, denoted $K(T_\delta)$, contains the image by η of the invariant Kähler cone $(K_{\Sigma_n^{[2]}})^{\tau_n}$ of $\Sigma_n^{[2]}$. We denote by $\mathcal{N}_{K(T_\delta)}^{\rho_\delta, \xi}$ the set of points (X, η, σ, ι) in $\mathcal{N}_{T_\delta}^{\rho_\delta, \xi}$ which are $K(T_\delta)$ -general, meaning that $\eta(K_X^\sigma) = K(T_\delta)$. When the surface Σ_n considered in our geometric construction above is a very general point of the moduli space $\mathcal{K}_{U(3)}^{\rho_\delta, \xi}$, we may assume that the Néron–Severi group $\text{NS}(\Sigma_n^{[2]})$ is isometric to T_δ (otherwise stated $\text{NS}(\Sigma_n) \cong U(3)$), so by Lemma [15, Lemma 5.0.13] the point $(\Sigma_n^{[2]}, \tau_n^{[2]})$ is $K(T_\delta)$ -general and lives in $\mathcal{N}_{K(T_\delta)}^{\rho_\delta, \xi}$.

Proposition 4.6. *Let $\delta \in S(-1)$ be a nodal root. The stable discriminant locus Δ_3^s is birational to the moduli space $\mathcal{N}_{K(T_\delta)}^{\rho_\delta, \xi}$.*

Proof. Since the isometry ρ_0 is the restriction of ρ to $S(-1)$ and since ρ_δ is the restriction of ρ to $S_\delta \subset S(-1)$, we have an inclusion of eigenspaces $(S_\delta)_\xi \subset S_\xi$. If $\omega \in \delta^\perp \cap \Omega_S^{\rho_0, \xi}$, it is certainly also orthogonal to $\rho(\delta)$, so the period ω lives in the 9-dimensional complex ball $\Omega_{U(3)}^{\rho_\delta, \xi} = \delta^\perp \cap \Omega_S^{\rho_0, \xi} \subset \mathcal{H}_n$. By [7, Theorem 9.1], the period map considered in the proof of Lemma 4.5 induces a bijection

$$\mathcal{P}_{U(3)}^{\rho_\delta, \xi}: \mathcal{K}_{U(3)}^{\rho_\delta, \xi} \longrightarrow \frac{\Omega_{U(3)}^{\rho_\delta, \xi} \setminus \mathcal{H}_{U(3)}}{\Gamma_{U(3)}^{\rho_\delta, \xi}}$$

where $\mathcal{H}_{U(3)} := \bigcup_{\substack{\xi \in S_\delta \\ \xi^2 = -2}} \varepsilon^\perp$. The group $\Gamma_{U(3)}^{\rho_\delta, \xi}$ is the set of isometries of the K3

lattice $(\mathbb{Z}e_n)^\perp$ which commute with $\tilde{\rho}_\delta$. We thus clearly have an equality of orbit spaces

$$\frac{\Omega_{U(3)}^{\rho_\delta, \xi}}{\Gamma_{U(3)}^{\rho_\delta, \xi}} = \frac{\delta^\perp \cap \Omega_S^{\rho_0, \xi}}{\Gamma_S^{\rho_0, \xi}}.$$

The extension of the period map $\mathcal{P}_S^{\rho_0, \xi}: \mathcal{C}_3^{\text{sm}} \rightarrow \frac{\mathbb{B}^{10} \setminus (\mathcal{H}_n \cup \mathcal{H}_c)}{\mathbb{P}\Gamma}$ to a general nodal cubic C is defined in [4] by using the period of the associated K3 surface Σ_n :

$$\mathcal{P}_S^{\rho_0, \xi}(C) := \mathcal{P}_{U(3)}^{\rho_\delta, \xi}(\Sigma_n) \in \mathcal{H}_n.$$

By studying the limit Hodge structure of the nodal degeneration of a cubic threefold, Allcock–Carlson–Toledo [4, Chapter 6] show that this definition extends holomorphically $\mathcal{P}_S^{\rho_0, \xi}$ to an isomorphism between \mathcal{C}_3^s and $\frac{\mathbb{B}^{10} \setminus \mathcal{H}_c}{\mathbb{P}\Gamma}$ mapping bijectively Δ_3^s to the divisor $\mathcal{H}_n/\mathbb{P}\Gamma$.

Turning back to the automorphism $\tau_n^{[2]}$ on $\Sigma_n^{[2]}$, which gives a point in the moduli space $\mathcal{N}_{K(T_\delta)}^{\rho_\delta, \xi}$, the period map constructed in [15, Theorem 5.0.17] produces a bijection

$$\mathcal{P}_{T_\delta}^{\rho_\delta, \xi}: \mathcal{N}_{K(T_\delta)}^{\rho_\delta, \xi} \longrightarrow \frac{\Omega_{T_\delta}^{\rho_\delta, \xi} \setminus (\mathcal{H}_{T_\delta} \cup \mathcal{H}'_{T_\delta})}{\Gamma_{T_\delta}^{\rho_\delta, \xi}}.$$

The notation is similar to those of Section 3.5, it is easy to check that $\Omega_{T_\delta}^{\rho_\delta, \xi} = \widetilde{\Omega}_{U(3)}^{\rho_\delta, \xi}$ and that $\Gamma_{T_\delta}^{\rho_\delta, \xi} = \widetilde{\Gamma}_{U(3)}^{\rho_\delta, \xi}$. The hyperplane arrangements are

$$\mathcal{H}_{T_\delta} = \bigcup_{\substack{\varepsilon \in S_\delta \\ \varepsilon^2 = -2}} \varepsilon^\perp = \mathcal{H}_{U(3)}$$

and \mathcal{H}'_{T_δ} is the union of the hyperplanes ε^\perp which meet $K(T_\delta)$, for those MBM classes $\varepsilon \in L$ which admit a decomposition $\varepsilon = \varepsilon_1 + \varepsilon_2$ with $\varepsilon_1 \in T_\delta \otimes_{\mathbb{Z}} \mathbb{Q}$, $\varepsilon_2 \in S_\delta \otimes_{\mathbb{Z}} \mathbb{Q}$, $\varepsilon_1^2 < 0$, and $\varepsilon_2^2 < 0$.

Since $\mathcal{P}_{U(3)}^{\widetilde{\rho}_\delta, \xi}$ and $\mathcal{P}_{T_\delta}^{\rho_\delta, \xi}$ are bijections and $\frac{\Omega_{U(3)}^{\widetilde{\rho}_\delta, \xi} \setminus \mathcal{H}_{U(3)}}{\Gamma_{U(3)}^{\rho_\delta, \xi}}$ is clearly birational to $\frac{\Omega_{T_\delta}^{\rho_\delta, \xi} \setminus (\mathcal{H}_{T_\delta} \cup \mathcal{H}'_{T_\delta})}{\Gamma_{T_\delta}^{\rho_\delta, \xi}}$, it follows that the moduli spaces Δ_3^s and $\mathcal{K}_{T_\delta}^{\rho_\delta, \xi}$, which are birationally equivalent by [18, Proposition 5.3], are also birational to $\mathcal{N}_{K(T_\delta)}^{\rho_\delta, \xi}$. \square

Remark 4.7. Let us denote by \mathfrak{M}_4 the moduli space of genus 4 curves. From the proof of Proposition 4.6, we get birational isomorphisms

$$\Delta_3^s \sim \mathcal{K}_{U(3)}^{\widetilde{\rho}_\delta, \xi} \sim \mathcal{N}_{K(T_\delta)}^{\rho_\delta, \xi} \sim \mathfrak{M}_4$$

obtained by mapping a general nodal cubic threefold C with node p_0 and associated K3 surface Σ_n respectively to $\Sigma_n, \Sigma_n^{[2]}$ and $F(C, p_0)$.

On a very general period point $\omega \in \mathcal{H}_n$, the birational equivalence between Δ_3^s and $\mathcal{K}_{T_\delta}^{\rho_\delta, \xi}$ can be seen as follows: ω corresponds to a general nodal cubic C whose uniquely associated K3 surface Σ_n is smooth. Conversely, the ramified triple cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along a smooth non-hyperelliptic genus 4 curve of bidegree $(3, 3)$ is a K3 surface living in $\mathcal{K}_{U(3)}^{\widetilde{\rho}_\delta, \xi}$, from which the nodal cubic threefold can be recovered (see Kondō [30, Theorem 1]).

Remark 4.8. To relate with the general statement given in Proposition 4.1, we observe that if we take a very general period point $\omega \in \mathcal{H}_n$, then the variety $\Sigma_n^{[2]}$ lives in the fiber $\mathcal{P}_{(6)}^{-1}(\omega)$ and the birational map $\beta = \tau_n^{[2]}$ is everywhere defined. Observe that $\beta^* = (\tau_n^{[2]})^*$ is conjugated to ρ_δ . Indeed, both have the same action on the K3 lattice $(\mathbb{Z}e_n)^\perp$ and act trivially on the class e_n . Denoting by η a marking, the element $w \in W_{\text{Exc}}(\Sigma_n^{[2]})$ is here $w = r_{\eta^{-1}(\delta)} \circ r_{\eta^{-1}\rho(\delta)}$.

4.4. Chordal degenerations. Consider the Veronese embedding $\nu_4: \mathbb{P}^1 \rightarrow \mathbb{P}^4$, whose image is the standard rational normal quartic curve \mathcal{R} . Its secant variety $T := \text{Sec}(\mathcal{R})$ is a cubic hypersurface in \mathbb{P}^4 , called a *chordal cubic threefold*, which is singular exactly along the curve \mathcal{R} . By [2, Theorem 1.3] the cubic T is a strictly $\text{PGL}_5(\mathbb{C})$ -semistable point of the linear system $|\mathcal{O}_{\mathbb{P}^4}(3)|$. Since the stabilizer of T is $\text{PGL}_2(\mathbb{C})$, its $\text{PGL}_5(\mathbb{C})$ -orbit is 21-dimensional. The closure of this orbit is thus a codimension 13 locus in $|\mathcal{O}_{\mathbb{P}^4}(3)|$. Blowing it up, we get a $\text{PGL}_5(\mathbb{C})$ -equivariant projective morphism $\mathcal{Z} \rightarrow |\mathcal{O}_{\mathbb{P}^4}(3)|$. As explained in [4, Chapter 2], the fiber of this blowup over the chordal cubic T can be identified with an unordered sequence of 12 points in \mathcal{R} as follows. Take a pencil $(C_t)_{t \in \mathbb{B}^1}$ of cubics degenerating to $C_0 = T$, such that the pencil is transverse to the orbit of T . For any $t \neq 0$, the smooth cubic C_t cuts the curve \mathcal{R} in 12 points, whose limit for $t \rightarrow 0$ is an unordered 12-tuple in \mathcal{R} .

We denote by \mathcal{C}_3^s the GIT moduli space of $\mathrm{PGL}_5(\mathbb{C})$ -stable points in \mathcal{Z} (as in [4, p.27] the notation is tricky: \mathcal{C}_3^s is not a blowup of \mathcal{C}_3^s). By [4, Theorem 3.2], the *stable chordal locus* $\Xi_3^s := \mathcal{C}_3^s \setminus \mathcal{C}_3^s$ can be identified with the locus in $\mathcal{R}^{12}/\mathfrak{S}_{12}$ of unordered 12-tuples with no point of multiplicity greater or equal than 6.

A general point (T, μ) in Ξ_3^s is a 12-tuple μ of distinct unordered points in \mathcal{R} . Consider the elliptic K3 surface Σ_c over \mathcal{R} which has a cuspidal fiber over each point of μ . This family of K3 surfaces has been studied by Artebani–Sarti [6, Proposition 4.2]: it admits a non-symplectic automorphism τ_c of order three, preserving the fibration, whose fixed locus consists of the zero section of the fibration and of a genus 5 curve which is the ramified double cover of \mathcal{R} branched along the points μ . The invariant lattice in $H^2(\Sigma_c, \mathbb{Z})$ is isometric to U and its orthogonal complement is isometric to $U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}$.

Let $\delta \in S(-1)$ be a chordal root (as above, an effective choice is unnecessary). Recall that by Lemma 4.3, the saturated lattice T_δ is isometric to $U \oplus \langle -2 \rangle$. An elementary computation shows that the second term is generated by the (-2) -class

$$e_c := \frac{1}{3}(-\vartheta + 4\delta + 2\rho(\delta)).$$

The orthogonal complement of $\mathbb{Z}e_c$ in L is isometric to the K3 lattice. We thus obtain an embedding $U \hookrightarrow T_\delta \subset L$ whose image is contained in $(\mathbb{Z}e_c)^\perp$ and whose orthogonal complement in $(\mathbb{Z}e_c)^\perp$ is isometric to $S_\delta = U^{\oplus 2} \oplus E_8(-1)^{\oplus 2}$. The isometry $\mathrm{id}_U \oplus \rho_{\delta|_{S_\delta}}$ extends uniquely to an isometry $\tilde{\rho}_\delta$ of $(\mathbb{Z}e_c)^\perp$ which is the restriction of ρ_δ to $(\mathbb{Z}e_c)^\perp \subset L$.

Lemma 4.9. *Let $\delta \in S(-1)$ be a chordal root. The action of τ_c on $H^2(\Sigma_c, \mathbb{Z})$ is conjugated to the isometry $\tilde{\rho}_\delta$ of $(\mathbb{Z}e_c)^\perp$.*

Proof. The proof follows the same lines as the proof of Lemma 4.5, we only point out the necessary changes. Consider the moduli space $\mathcal{K}_U^{\tilde{\rho}_\delta, \xi}$ and the period map

$$\mathcal{P}_U^{\tilde{\rho}_\delta, \xi} : \mathcal{K}_U^{\tilde{\rho}_\delta, \xi} \longrightarrow \Omega_U^{\tilde{\rho}_\delta, \xi} := \{\omega \in \mathbb{P}((S_\delta)_\xi) \mid h_{S_\delta}(\omega, \omega) > 0\},$$

and use [6, Theorem 3.3, Proposition 4.2] to relate $\tilde{\rho}_\delta$ to the action of τ_c on the K3 surface Σ_c . \square

We consider similarly the automorphism $\tau_c^{[2]}$ induced on $\Sigma_c^{[2]}$, with invariant lattice T_δ . It defines a point in the moduli space $\mathcal{N}_{T_\delta}^{\rho_\delta, \xi}$. As above we introduce the moduli space $\mathcal{N}_{K(T_\delta)}^{\rho_\delta, \xi}$, which contains the point $(\Sigma_c^{[2]}, \tau_c^{[2]})$.

Proposition 4.10. *Let $\delta \in S(-1)$ be a chordal root. The stable chordal locus Ξ_3^s is birational to the moduli space $\mathcal{N}_{K(T_\delta)}^{\rho_\delta, \xi}$.*

Proof. The proof is similar to that of Proposition 4.6, we only indicate the needed changes. By [7, Theorem 9.1], the period map considered in the proof of Lemma 4.9 induces a bijection

$$\mathcal{P}_U^{\tilde{\rho}_\delta, \xi} : \mathcal{K}_U^{\tilde{\rho}_\delta, \xi} \longrightarrow \frac{\Omega_U^{\tilde{\rho}_\delta, \xi} \setminus \mathcal{H}_U}{\Gamma_U^{\tilde{\rho}_\delta, \xi}}$$

where $\mathcal{H}_U := \bigcup_{\substack{\varepsilon \in S_\delta \\ \varepsilon^2 = -2}} \varepsilon^\perp$. The group $\Gamma_U^{\widetilde{\rho}_\delta, \xi}$ is the set of isometries of the K3 lattice $(\mathbb{Z}e_c)^\perp$ which commute with $\widetilde{\rho}_\delta$. We thus have an equality of orbit spaces

$$\frac{\Omega_U^{\widetilde{\rho}_\delta, \xi}}{\Gamma_U^{\widetilde{\rho}_\delta, \xi}} = \frac{\delta^\perp \cap \Omega_S^{\rho_0, \xi}}{\Gamma_S^{\rho_0, \xi}}.$$

Allcock–Carlson–Toledo [4, Theorem 3.7, Theorem 6.1] extend the period map $\mathcal{P}_S^{\rho_0, \xi}: \mathcal{C}_3^s \rightarrow \frac{\mathbb{B}^{10} \setminus \mathcal{H}_c}{\mathbb{P}\Gamma}$ to an isomorphism $\mathcal{P}_S^{\rho_0, \xi}: \mathcal{C}_3^s \rightarrow \frac{\mathbb{B}^{10}}{\mathbb{P}\Gamma}$ mapping bijectively the stable chordal locus Ξ_3^s to the divisor $\mathcal{H}_c/\mathbb{P}\Gamma$. This extension is unique by Riemann extension theorem, so we may reinterpret generically their construction by using the period of elliptic K3 surfaces: for a general point $(T, \mu) \in \Xi_3^s$ with associated elliptic K3 surface Σ_c , we have:

$$\mathcal{P}_S^{\rho_0, \xi}((T, \mu)) := \mathcal{P}_U^{\widetilde{\rho}_\delta, \xi}(\Sigma_c) \in \mathcal{H}_c.$$

Now consider the isomorphism given by the period map

$$\mathcal{P}_{T_\delta}^{\rho_\delta, \xi}: \mathcal{N}_{K(T_\delta)}^{\rho_\delta, \xi} \longrightarrow \frac{\Omega_{T_\delta}^{\rho_\delta, \xi} \setminus (\mathcal{H}_{T_\delta} \cup \mathcal{H}'_{T_\delta})}{\Gamma_{T_\delta}^{\rho_\delta, \xi}}.$$

As above, here $\Omega_{T_\delta}^{\rho_\delta, \xi} = \Omega_U^{\widetilde{\rho}_\delta, \xi}$ and $\Gamma_{T_\delta}^{\rho_\delta, \xi} = \Gamma_U^{\widetilde{\rho}_\delta, \xi}$. We thus get birational isomorphisms

$$\Xi_3^s \sim \mathcal{K}_U^{\widetilde{\rho}_\delta, \xi} \sim \mathcal{N}_{K(T_\delta)}^{\rho_\delta, \xi} \sim \mathcal{R}^{12}/\mathfrak{S}_{12}.$$

□

Remark 4.11. As in Remark 4.8, if we take a general period point $\omega \in \mathcal{H}_c$ then the variety $\Sigma_c^{[2]}$ lives in the fiber $\mathcal{P}_{(6)}^{-1}(\omega)$ and the birational map $\beta = \tau_c^{[2]}$ is everywhere defined.

5. PFAFFIAN CUBICS WITH TRIPLE COVERING

Recall that $\mathcal{P}_S: \mathcal{C}_4^{\text{sm}} \rightarrow \frac{\Omega_S^\circ}{\Gamma_S}$ is the period map for cubic fourfolds (see Section 3.3), $\Lambda = \langle 1 \rangle^{\oplus 21} \oplus \langle -1 \rangle^{\oplus 2}$, $\kappa \in \Lambda$ is a polarization class of square 3 such that $S = (\mathbb{Z}\kappa)^\perp$. For any rank two lattice M primitively embedded in Λ and containing κ , we define a hyperplane

$$\mathcal{H}_M := \{\omega \in \Omega_S \mid \omega \perp M\}.$$

By Hassett [26, Propositions 3.2.2&3.2.4] (see also Laza [32, §2]), for any d the hyperplanes \mathcal{H}_M where $\det(M) = d$ form a single Γ_S -orbit. We denote by $\mathcal{H}_d \subset \frac{\Omega_S^\circ}{\Gamma_S}$ the corresponding irreducible hypersurface, which is nonempty if and only if $d \equiv 0, 2 \pmod{6}$ and $d > 6$ [26, Theorem 1.0.1]. By Hassett [26] and Laza [32, Theorem 1.1], the image of the period map \mathcal{P}_S is precisely $\frac{\Omega_S^\circ}{\Gamma_S} \setminus (\mathcal{H}_2 \cup \mathcal{H}_6)$. Of particular interest in this section are the divisor \mathcal{H}_8 parametrizing cubic fourfolds containing a plane, and the divisor $\mathcal{P}\mathcal{f}$ of Pfaffian cubic fourfolds, whose closure is \mathcal{H}_{14} . Proposition 5.1 below shows that $\mathcal{P}\mathcal{f} \setminus \mathcal{H}_8$ contains a cyclic cubic fourfold.

Let V be a 6-dimensional rational vector space and $(Z \subset W)$ be general subspaces in the flag variety $\text{Flag}(5, 6; \wedge^2 V^*)$. The Pfaffian locus $\mathcal{P}\mathcal{f} \subset \mathbb{P}(\wedge^2 V^*)$ is the hypersurface of degenerate skew-symmetric 2-forms (by convention $\mathbb{P}(-)$ denotes the projective space of lines). To the pair $(Z \subset W)$ we associate the cubic threefold $C := \mathcal{P}\mathcal{f} \cap \mathbb{P}(Z)$ and the cubic fourfold $Y := \mathcal{P}\mathcal{f} \cap \mathbb{P}(W)$. Using the

Plücker embedding we consider the Grassmannian $\text{Grass}(2, V)$ as the subvariety of decomposable tensors in $\mathbb{P}(\wedge^2 V)$. The orthogonal flag $W^\circ \subset Z^\circ$ in $\text{Flag}(9, 10; \wedge^2 V)$ defines a Fano threefold $X := \text{Grass}(2, V) \cap \mathbb{P}(Z^\circ)$ of index one and genus 8 and a K3 surface $\Sigma := \text{Grass}(2, V) \cap \mathbb{P}(W^\circ)$ of genus 14 as anticanonical section of X (see Iliev–Manivel [28]). We call the pair (Σ, X) the *dual* of the pair (Y, C) . By Beauville–Donagi [12, Proposition 5], there exists a birational map between the Fano variety of lines $F(Y)$ and the Hilbert square $\Sigma^{[2]}$, which is an isomorphism if Y contains no plane and if Σ contains no line.

Proposition 5.1. *There exists a smooth Pfaffian cubic fourfold Y , defined over \mathbb{Q} , containing no plane, which is the triple cover of \mathbb{P}^4 along a smooth Pfaffian cubic threefold C , and whose dual K3 surface Σ is smooth and contains no line.*

Proof. Let V be a 6-dimensional rational vector space. We fix an isomorphism $\wedge^6 V^* \cong \mathbb{Q}$. The Pfaffian of a skew-symmetric 2-form $\varphi \in \wedge^2 V^*$ is $\text{Pf}(\varphi) = \frac{1}{6}\varphi^3$. We fix a skew-symmetric 2-form of maximal rank $\varphi_0 \in \wedge^2 V^*$. Without loss of generality, we can assume that $\text{Pf}(\varphi_0) = 1$. We define on $\wedge^2 V^*$ a linear form h and a quadratic form q by the decomposition

$$\text{Pf}(x_0\varphi_0 + \varphi) = x_0^3 + x_0^2 h(\varphi) + x_0 q(\varphi) + \text{Pf}(\varphi) \quad \forall \varphi \in \wedge^2 V^*,$$

where $h(\varphi) = \frac{1}{2}\varphi_0^2 \wedge \varphi$ and $q(\varphi) = \frac{1}{2}\varphi_0 \wedge \varphi^2$. Since $q(\varphi_0) = 3$, the hyperplane $H := \ker(h)$ is regular for q . A computation shows that the rational quadratic form q has Witt index 7 and that its restriction q_H to H has Witt index 6. Let Z be a general 5-dimensional totally isotropic vector subspace of H and $W := \mathbb{C}\varphi_0 \oplus Z$. The cubic fourfold

$$Y_Z := \{[\varphi] \in \mathbb{P}(W) \mid \text{Pf}(\varphi) = 0\}$$

is by construction a ramified cyclic triple covering of $\mathbb{P}(Z)$ branched along the cubic threefold

$$C_Z := \{[\varphi] \in \mathbb{P}(Z) \mid \text{Pf}(\varphi) = 0\}.$$

We show that the isotropic space Z can be chosen in such a way that the general elements Y_Z and C_Z of this family are smooth by exhibiting an example in Appendix A.

Assume that Y_Z contains a plane $P \subset \mathbb{P}(W)$. This plane cannot be contained in $\mathbb{P}(Z)$ otherwise it would be contained in C_Z , which would be singular. Thus P meets $\mathbb{P}(Z)$ along a line $\text{Span}(\varphi_2, \varphi_3)$ with $\varphi_2, \varphi_3 \in Z$. Take $\varphi_1 \in W \setminus Z$ such that $P = \text{Span}(\varphi_1, \varphi_2, \varphi_3)$ and denote by $\varphi'_1 \in \mathbb{P}(Z)$ the projection of φ_1 on $\mathbb{P}(Z)$ from the point φ_0 . We may assume that $\varphi_1 = \varphi_0 + \varphi'_1$. For any $x_1, x_2, x_3 \in \mathbb{C}$ we have

$$\begin{aligned} 0 &= \text{Pf}(x_1\varphi_1 + x_2\varphi_2 + x_3\varphi_3) = \text{Pf}(x_1\varphi_0 + (x_1\varphi'_1 + x_2\varphi_2 + x_3\varphi_3)) \\ &= x_1^3 + \text{Pf}(x_1\varphi'_1 + x_2\varphi_2 + x_3\varphi_3). \end{aligned}$$

In particular $\text{Pf}(\varphi'_1) \neq 0$ and $\text{Pf}(x_2\varphi_2 + x_3\varphi_3) = 0$ for any x_2, x_3 . This shows that $\varphi'_1 \notin C_Z$ and that P meets C_Z along the triple line $\text{Span}(\varphi_2, \varphi_3)$.

To prove that C_Z has no triple line, we make a local analysis of the Fano surface of lines $F(C_Z)$, following [41, §1]. Denote by $p_{i,j}$, $0 \leq i < j \leq 4$ the Plücker coordinates of the Grassmannian of lines $\text{Grass}(1, \mathbb{P}(Z)) \subset \mathbb{P}^9$ and denote by x_0, \dots, x_4 the coordinates on $\mathbb{P}(Z)$. On the affine chart $p_{0,1} = 1$, the variety $\text{Grass}(1, \mathbb{P}(Z))$ is isomorphic to \mathbb{C}^6 and a point $\ell \in \text{Grass}(1, \mathbb{P}(Z))$ in this chart corresponds to a line generated by two vectors $\ell_0 = (1, 0, -p_{1,2}, -p_{1,3}, -p_{1,4})$ and $\ell_1 = (0, 1, p_{0,2}, p_{0,3}, p_{0,4})$.

Any plane $P \supset \ell$ cuts the plane $x_0 = x_1 = 0$ at a unique point $p = (0, 0, \alpha_2, \alpha_3, \alpha_4)$ and $P = \text{Span}(\ell, p)$. The plane cubic $P \cap C_Z$ has equation $F(u\ell_0 + v\ell_1 + tp) = 0$ where $(u : v : t)$ are the projective coordinates of the plane P . Expanding in t we get:

$$F(u\ell_0 + v\ell_1) + t \sum_{i=2}^4 \frac{\partial F}{\partial x_i} (u\ell_0 + v\ell_1) \alpha_i + t^2 \sum_{2 \leq i, j \leq 4} \frac{\partial^2 F}{\partial x_i \partial x_j} (u\ell_0 + v\ell_1) \alpha_i \alpha_j + t^3 F(p) = 0.$$

The line ℓ of equation $t = 0$ in P is contained in C_Z if and only if $F(u\ell_0 + v\ell_1) = 0$ for every u, v . Write $\frac{\partial F}{\partial x_i} = u^2 \phi_i^{2,0} + uv \phi_i^{1,1} + v^2 \phi_i^{0,2}$ where $\phi_i^{j,k}$ are functions of the local Plücker coordinates. The plane P is tangent at C_Z along ℓ (*i.e.* $P \cap C_Z = 2\ell + \ell'$) if and only if $\det \left(\phi_i^{j,k} \right)_{i,j,k} = 0$. As explained in [41, Corollary 1,9], this defines a reducible curve in $F(C_Z)$ with smooth irreducible components (which may intersect with each other). Finally, ℓ is a triple line (*i.e.* $P \cap C_Z = 3\ell$) if and only if the residual plane cubic equation is a multiple of t^3 . The locus of triple lines is thus easy to compute on each local chart of $\text{Grass}(1, \mathbb{P}(Z))$ and with the help of a computer program we find that C_Z contains none.

To conclude the proof, a computer program shows that the dual K3 surface Σ contains no line: to reduce the computation we use the embedding of Σ in $\mathbb{P}(W^\circ)$ and we compute the equations of its Fano variety of lines $F(\Sigma)$ in $\mathbb{P}(\wedge^2(W^\circ))$ to check that it is empty. \square

The proof of the global Torelli theorem of Huybrechts–Markman–Verbitsky has made possible to show the existence of hyperkähler manifolds admitting an automorphism with a given action on cohomology (see for instance [14, Theorem 5.5]), but it is still hard to construct automorphisms on a given hyperkähler manifold without deforming it (see [16] for related results on involutions of Hilbert squares of general K3 surfaces). In particular, it is very challenging to show the existence of a non-natural automorphism on the Hilbert square of a K3 surface (*i.e.* an automorphism which does not come from an automorphism of the underlying K3 surface, see [13]). Until now, the only such geometric realization was the involution found by Beauville on the Hilbert square of a general quartic in \mathbb{P}^3 . Proposition 5.1 gives a new geometric example with an order three automorphism.

Corollary 5.2. *There exists a smooth complex K3 surface whose Hilbert square admits a non-natural non-symplectic automorphism of order three, with fixed locus isomorphic to the Fano surface of lines on a cubic threefold, and with invariant lattice isometric to $\langle 6 \rangle$.*

Proof. The cyclic Pfaffian cubic fourfold Y constructed in Proposition 5.1 has a non-symplectic automorphism σ of order three. Since Y contains no plane and since its dual K3 surface Σ contains no line, the isomorphism $F(Y) \cong \Sigma^{[2]}$ produces a non-symplectic automorphism of order three on $\Sigma^{[2]}$ which we still denote by σ . Since the invariant lattice $H^2(S^{[2]}, \mathbb{Z})^\sigma$ is isometric to $\langle 6 \rangle$, the automorphism σ is certainly non-natural (see [14, Section 6.1]). \square

Looking at the geometric description of the isomorphism $F(Y) \cong \Sigma^{[2]}$ in [12, proof of Proposition 5], we get a geometric description of the action of σ on $\Sigma^{[2]}$ as follows. Let $\zeta = \{P, Q\} \in \Sigma^{[2]}$ with $P, Q \in \text{Grass}(2, V) \cap \mathbb{P}(W^\circ)$ that we assume

to be distinct for simplicity. Consider the 4-plane $K = P + Q$, the 5-dimensional linear space $M = \text{Span}(\text{Grass}(2, K)) \subset \mathbb{P}(\wedge^2 V)$ and the line

$$\ell_{P,Q} := \{\varphi \in \mathbb{P}(W) \mid \varphi|_K = 0\} \in \mathbb{F}(Y).$$

This line is a Kronecker pencil of skew-symmetric 2-forms whose unique bilagrangian is K . The points P, Q can be recovered from K as follows. First observe that $\ell_{P,Q} = M^\circ \cap \mathbb{P}(W)$, then by dimension counting $M \cap \mathbb{P}(W^\circ)$ is a line, which cuts the quadric $\text{Grass}(2, K)$ exactly in the points P, Q . If the line $\ell_{P,Q}$ is contained in $\mathbb{P}(Z)$, it is invariant by σ , so ζ is invariant by σ . Otherwise, this line cuts $\mathbb{P}(Z)$ in a point $\varphi_{P,Q}$. The plane $\text{Span}(\varphi_0, \ell_{P,Q})$ cuts Y along a cubic curve with $\varphi_{P,Q}$ as a triple point, it is thus the union of three lines passing through $\varphi_{P,Q}$, which are precisely $\ell_{P,Q}, \sigma(\ell_{P,Q})$ and $\sigma^2(\ell_{P,Q})$. We denote the corresponding bilagrangians by $K, \sigma K$ and $\sigma^2 K$, which are three 4-planes of V meeting along the 2-plane $\ker(\varphi_{P,Q})$. As explained above, the 4-plane σK characterizes a length two subscheme of Σ which is the image of ζ by σ .

APPENDIX A. A SMOOTH PFAFFIAN CUBIC FOURFOLD WITH TRIPLE COVERING

Let M be the skew-symmetric matrix of linear forms on \mathbb{Q}^6 :

$$\begin{pmatrix} 0 & u_1 & u_2 & u_3 - x_5 & u_4 & u_5 \\ -u_1 & 0 & u_6 & u_7 & u_8 - x_5 & u_9 \\ -u_2 & -u_6 & 0 & u_{10} & u_{11} & u_{12} - x_5 \\ -u_3 + x_5 & -u_7 & -u_{10} & 0 & u_{13} & u_{14} \\ -u_4 & -u_8 + x_5 & -u_{11} & -u_{13} & 0 & u_{15} \\ -u_5 & -u_9 & -u_{12} + x_5 & -u_{14} & -u_{15} & 0 \end{pmatrix}$$

with:

$$\begin{aligned} u_1 &= 2x_0 + 3x_1 + 2x_2 + 3x_3 + 2x_4, & u_2 &= x_0 + x_2 + x_3, \\ u_3 &= 2x_0 + 3x_1 + 2x_2 + 2x_3 + x_4, & u_4 &= 3x_0 + x_1 + 2x_2 + x_3 + x_4, \\ u_5 &= 2x_0 + 4x_1 + 3x_2 + 3x_3 + x_4, & u_6 &= 3x_0 + 4x_1 + 4x_2 + 4x_3 + x_4, \\ u_7 &= x_1 + x_2 + x_3, & u_8 &= 2x_0 + 4x_1 + 3x_2 + 3x_3 + x_4, \\ u_9 &= 2x_0 + 2x_1 + x_2 + 2x_3 + 2x_4, & u_{10} &= 2x_1 + x_2 + 2x_3 + x_4, \\ u_{11} &= x_0 - x_1 + x_2 - x_3 - x_4, & u_{12} &= -4x_0 - 7x_1 - 5x_2 - 5x_3 - 2x_4, \\ u_{13} &= -4x_0 - 4x_1 - 6x_2 - 4x_3, & u_{14} &= 6x_0 + 11x_1 + 7x_2 + 8x_3 + 4x_4, \\ u_{15} &= -4x_0 - 8x_1 - 5x_2 - 5x_3 - 2x_4. \end{aligned}$$

The Pfaffian of A defines a cubic fourfold $Y \subset \mathbb{P}^5$ of equation

$$\begin{aligned} &x_5^3 + (-42x_0^3 - 179x_0^2x_1 - 187x_0x_1^2 - 32x_1^3 - 161x_0^2x_2 - 391x_0x_1x_2 \\ &\quad - 155x_1^2x_2 - 193x_0x_2^2 - 196x_1x_2^2 - 74x_2^3 - 138x_0^2x_3 - 280x_0x_1x_3 \\ &\quad - 41x_1^2x_3 - 291x_0x_2x_3 - 205x_1x_2x_3 - 142x_2^2x_3 - 99x_0x_3^2 - x_1x_3^2 \\ &\quad - 62x_2x_3^2 + 9x_3^3 - 46x_0^2x_4 - 104x_0x_1x_4 - 27x_1^2x_4 - 91x_0x_2x_4 \\ &\quad - 71x_1x_2x_4 - 36x_2^2x_4 - 80x_0x_3x_4 - 20x_1x_3x_4 - 40x_2x_3x_4 + 4x_3^2x_4 \\ &\quad - 24x_0x_4^2 - 17x_1x_4^2 - 19x_2x_4^2 - 11x_3x_4^2 - 3x_4^3) = 0 \end{aligned}$$

which is the ramified triple covering of \mathbb{P}^4 branched over a cubic threefold C . It is easy to check that both C and Y are nonsingular. Following the method explained in the proof of Proposition 5.1, we show that the dual K3 surface contains no line.

REFERENCES

- [1] V. Alexeev and V. V. Nikulin. *Del Pezzo and K3 surfaces*, volume 15 of *MSJ Memoirs*. Mathematical Society of Japan, Tokyo, 2006.
- [2] D. Allcock. The moduli space of cubic threefolds. *J. Algebraic Geom.*, 12(2):201–223, 2003.
- [3] D Allcock, J. A. Carlson, and D. Toledo. The complex hyperbolic geometry of the moduli space of cubic surfaces. *J. Algebraic Geom.*, 11(4):659–724, 2002.
- [4] D. Allcock, J. A. Carlson, and D. Toledo. The moduli space of cubic threefolds as a ball quotient. *Mem. Amer. Math. Soc.*, 209(985):xii+70, 2011.
- [5] E. Amerik and M. Verbitsky. Rational Curves on Hyperkähler Manifolds. *International Mathematics Research Notices*, 2015(23):13009–13045, 2015.
- [6] M. Artebani and A. Sarti. Non-symplectic automorphisms of order 3 on K3 surfaces. *Math. Ann.*, 342(4):903–921, 2008.
- [7] M. Artebani, A. Sarti, and S. Taki. K3 surfaces with non-symplectic automorphisms of prime order. *Math. Z.*, 268(1-2):507–533, 2011. With an appendix by Shigeyuki Kondō.
- [8] A. Bayer, B. Hassett, and Y. Tschinkel. Mori cones of holomorphic symplectic varieties of K3 type. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(4):941–950, 2015.
- [9] A. Beauville. Moduli of cubic surfaces and Hodge theory (after Allcock, Carlson, Toledo). In *Géométries à courbure négative ou nulle, groupes discrets et rigidités*, volume 18 of *Sémin. Congr.*, pages 445–466.
- [10] A. Beauville. Some remarks on Kähler manifolds with $c_1 = 0$. In *Classification of algebraic and analytic manifolds (Katata, 1982)*, volume 39 of *Progr. Math.*, pages 1–26.
- [11] A. Beauville. Le groupe de monodromie des familles universelles d’hypersurfaces et d’intersections complètes. In *Complex analysis and algebraic geometry (Göttingen, 1985)*, volume 1194 of *Lecture Notes in Math.*, pages 8–18. Springer, Berlin, 1986.
- [12] A. Beauville and R. Donagi. La variété des droites d’une hypersurface cubique de dimension 4. *C. R. Acad. Sci. Paris Sér. I Math.*, 301(14):703–706, 1985.
- [13] S. Boissière. Automorphismes naturels de l’espace de Douady de points sur une surface. *Canad. J. Math.*, 64(1):3–23, 2012.
- [14] S. Boissière, C. Camere, and A. Sarti. Classification of automorphisms on a deformation family of hyper-Kähler four-folds by p -elementary lattices. *Kyoto J. Math.*, 56(3):465–499, 2016.
- [15] S. Boissière, C. Camere, and A. Sarti. Complex ball quotients from manifolds of $K3^{[n]}$ -type. *J. Pure Appl. Algebra*, 2018.
- [16] S. Boissière, A. Cattaneo, M. Nieper-Wisskirchen, and A. Sarti. The automorphism group of the Hilbert scheme of two points on a generic projective K3 surface. In *K3 surfaces and their moduli*, volume 315 of *Progr. Math.*, pages 1–15. Birkhäuser/Springer, [Cham], 2016.
- [17] C. Camere. Some remarks on moduli spaces of lattice polarized holomorphic symplectic manifolds. *Commun. Contemp. Math.*, 20(4):1750044, 29, 2018.
- [18] S. Casalaina-Martin, D. Jensen, and R. Laza. The geometry of the ball quotient model of the moduli space of genus four curves. In *Compact moduli spaces and vector bundles*, volume 564 of *Contemp. Math.*, pages 107–136. Amer. Math. Soc., Providence, RI, 2012.
- [19] C. H. Clemens and P. A. Griffiths. The intermediate Jacobian of the cubic threefold. *Ann. of Math. (2)*, 95:281–356, 1972.
- [20] O. Debarre and E. Macrì. On the period map for polarized hyperkähler fourfolds. *International Mathematics Research Notices*, 2018.
- [21] I. Dolgachev, B. van Geemen, and S. Kondō. A complex ball uniformization of the moduli space of cubic surfaces via periods of K3 surfaces. *J. Reine Angew. Math.*, 588:99–148, 2005.
- [22] I. V. Dolgachev and S. Kondō. Moduli of K3 surfaces and complex ball quotients. In *Arithmetic and geometry around hypergeometric functions*, volume 260 of *Progr. Math.*, pages 43–100. Birkhäuser, Basel, 2007.
- [23] P. Griffiths and J. Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1994. Reprint of the 1978 original.

- [24] V. Gritsenko, K. Hulek, and G. K. Sankaran. Moduli spaces of irreducible symplectic manifolds. *Compos. Math.*, 146(2):404–434, 2010.
- [25] V. Gritsenko, K. Hulek, and G. K. Sankaran. Moduli of K3 surfaces and irreducible symplectic manifolds. In *Handbook of moduli. Vol. I*, volume 24 of *Adv. Lect. Math. (ALM)*, pages 459–526. Int. Press, Somerville, MA, 2013.
- [26] B. Hassett. Special cubic fourfolds. *Compositio Math.*, 120(1):1–23, 2000.
- [27] D. Huybrechts. Compact hyper-Kähler manifolds: basic results. *Invent. Math.*, 135(1):63–113, 1999.
- [28] A. Iliev and L. Manivel. Pfaffian lines and vector bundles on Fano threefolds of genus 8. *J. Algebraic Geom.*, 16(3):499–530, 2007.
- [29] M. Joumaah. Non-symplectic involutions of irreducible symplectic manifolds of $K3^{[n]}$ -type. *Math. Z.*, 283(3-4):761–790, 2016.
- [30] S. Kondō. The moduli space of curves of genus 4 and Deligne-Mostow’s complex reflection groups. In *Algebraic geometry 2000, Azumino (Hotaka)*, volume 36 of *Adv. Stud. Pure Math.*, pages 383–400. Math. Soc. Japan, Tokyo, 2002.
- [31] S. Kudla and M. Rapoport. On occult period maps. *Pacific J. Math.*, 260(2):565–581, 2012.
- [32] R. Laza. The moduli space of cubic fourfolds via the period map. *Ann. of Math. (2)*, 172(1):673–711, 2010.
- [33] C. Lehn. Twisted cubics on singular cubic fourfolds - on Starr’s fibration. *Mathematische Zeitschrift*, 290(1):379–388, 2018.
- [34] E. Looijenga and R. Swierstra. The period map for cubic threefolds. *Compos. Math.*, 143(4):1037–1049, 2007.
- [35] E. Markman. A survey of Torelli and monodromy results for holomorphic-symplectic varieties. In *Complex and differential geometry*, volume 8 of *Springer Proc. Math.*, pages 257–322.
- [36] E. Markman. Integral constraints on the monodromy group of the hyperKähler resolution of a symmetric product of a K3 surface. *Internat. J. Math.*, 21(2):169–223, 2010.
- [37] R. Miranda and D. R. Morrison. Embeddings of integral quadratic forms, 2009. Preliminary draft.
- [38] G. Mongardi. On natural deformations of symplectic automorphisms of manifolds of $K3^{[n]}$ type. *C. R. Math. Acad. Sci. Paris*, 351(13-14):561–564, 2013.
- [39] G. Mongardi. A note on the Kähler and Mori cones of hyperkähler manifolds. *Asian J. Math.*, 19(4):583–591, 2015.
- [40] S. Mukai. *An introduction to invariants and moduli*, volume 81 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2003. Translated from the 1998 and 2000 Japanese editions by W. M. Oxbury.
- [41] J. P. Murre. Algebraic equivalence modulo rational equivalence on a cubic threefold. *Compositio Math.*, 25:161–206, 1972.
- [42] V. V. Nikulin. Integral symmetric bilinear forms and some of their applications. *Math. USSR Izv.*, 14:103–167, 1980.
- [43] C. Voisin. Théorème de Torelli pour les cubiques de \mathbf{P}^5 . *Invent. Math.*, 86(3):577–601, 1986.

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