# Controllability of quasi-linear Hamiltonian NLS equations

Pietro Baldi, Emanuele Haus, Riccardo Montalto

**Abstract.** We prove internal controllability in arbitrary time, for small data, for quasi-linear Hamiltonian NLS equations on the circle. We use a procedure of reduction to constant coefficients up to order zero and HUM method to prove the controllability of the linearized problem. Then we apply a Nash-Moser-Hörmander implicit function theorem as a black box. *MSC2010:* 35Q55, 35Q93.

*Contents.* 1 Introduction — 2 Reduction of the linearized operator — 3 Observability — 4 Controllability — 5 Proofs — 6 Appendix A. Quadratic Hamiltonians and linear Hamiltonian vector fields — 7 Appendix B. Classical tame estimates — 8 Appendix C. Well-posedness of linear equations — 9 Appendix D. Nash-Moser-Hörmander theorem.

## 1 Introduction

We consider a class of nonlinear Schrödinger equations (NLS) on $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ of the form

$$\partial_t u + \mathrm{i}\partial_{xx} u + \mathcal{N}(x, u, \partial_x u, \partial_{xx} u) = 0\,, \quad x \in \mathbb{T}, \tag{1.1}$$

for the complex-valued unknown $u = u(t,x)$. We assume that $\mathcal{N}$ is a *Hamiltonian, quasi-linear* nonlinearity

$$\mathcal{N}(x, u, u_x, u_{xx}) = -\mathrm{i}\Big(\partial_{\bar z_0} F(x, u, u_x) - \partial_x\{\partial_{\bar z_1} F(x, u, u_x)\}\Big)\,, \tag{1.2}$$

where $u_x, u_{xx}$ denote the partial derivatives $\partial_x u, \partial_{xx} u$, $F : \mathbb{T} \times \mathbb{C}^2 \to \mathbb{R}$ is a real-valued function,

$$F\Big(x, \frac{y_1 + \mathrm{i} y_2}{\sqrt{2}}, \frac{y_3 + \mathrm{i} y_4}{\sqrt{2}}\Big) = G(x, y_1, y_2, y_3, y_4) \quad \text{for some } G \in C^r(\mathbb{T} \times \mathbb{R}^4, \mathbb{R})\,, \tag{1.3}$$

and the differential operators $\partial_{\bar z_0}, \partial_{\bar z_1}$ in (1.2) are defined as

$$\partial_{\bar z_0} = \frac{1}{\sqrt{2}}(\partial_{y_1} + \mathrm{i}\partial_{y_2}), \quad \partial_{\bar z_1} = \frac{1}{\sqrt{2}}(\partial_{y_3} + \mathrm{i}\partial_{y_4}). \tag{1.4}$$

We assume that $G$ satisfies

$$|G(x,y)| \leq C|y|^3 \quad \forall y = (y_1, y_2, y_3, y_4) \in \mathbb{R}^4,\ |y| \leq 1. \tag{1.5}$$

Equation (1.1) is Hamiltonian in the sense that it can be written as

$$\partial_t u = \mathrm{i}\nabla_{\bar u} \mathcal{H}(u)$$

where $\nabla_{\bar u} := \frac{1}{\sqrt{2}}(\nabla_{u_1} + \mathrm{i}\nabla_{u_2})$, $\nabla$ is the $L^2(\mathbb{T})$ gradient, $u = \frac{1}{\sqrt{2}}(u_1 + \mathrm{i} u_2)$, and the real Hamiltonian $\mathcal{H}(u)$ is given by

$$\mathcal{H}(u) = \int_{\mathbb{T}} \big(|u_x|^2 + F(x, u, u_x)\big)\, dx\,. \tag{1.6}$$

We underline that (1.1) is, in fact, the *real* Hamiltonian system

$$\partial_t \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = J \begin{pmatrix} \nabla_{u_1} H(u_1, u_2) \\ \nabla_{u_2} H(u_1, u_2) \end{pmatrix} \tag{1.7}$$

for the real-valued unknowns $u_1, u_2$, where $J := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and

$$H(u_1, u_2) := \mathcal{H}\Big(\frac{u_1 + \mathrm{i}u_2}{\sqrt{2}}\Big) = \frac{1}{2}\int_{\mathbb{T}} \big((\partial_x u_1)^2 + (\partial_x u_2)^2\big)\, dx + \int_{\mathbb{T}} G\big(x, u_1, u_2, \partial_x u_1, \partial_x u_2\big)\, dx. \quad (1.8)$$

As a consequence, the assumption of finite regularity of $G$, i.e. $G \in C^r$ (only finitely many times differentiable) in (1.3) is compatible with the Hamiltonian structure — in particular, no analyticity assumption is needed on the Hamiltonian.

For example, if $G(x, y_1, y_2, y_3, y_4) = \frac{1}{8} a(x)(y_3^2 + y_4^2)^2$, then $\partial_{\bar{z}_1} F(x, u, u_x) = a(x)|u_x|^2 u_x$, and $\mathcal{N}(x, u, u_x, u_{xx}) = \mathrm{i}\partial_x\{a(x)|u_x|^2 u_x\} = \mathrm{i}a_x(x)|u_x|^2 u_x + \mathrm{i}a(x)(u_x^2 \bar{u}_{xx} + 2|u_x|^2 u_{xx})$; if $G = \frac{1}{8}(y_1^2 + y_2^2)^2$, then $\partial_{\bar{z}_0} F(x, u, u_x) = |u|^2 u$, and $\mathcal{N} = -\mathrm{i}|u|^2 u$.

For real $s \geq 0$, let $H^s_x := H^s(\mathbb{T}, \mathbb{C})$ be the usual Sobolev space of complex-valued periodic functions $u(x)$, and let $\|u\|_s := \|u\|_{H^s_x}$ be its norm. The main result of the paper is the following theorem about the exact, internal controllability of equation (1.1).

**Theorem 1.1** (Controllability)**.** *Let $T > 0$, and let $\omega \subset \mathbb{T}$ be a nonempty open set. There exist positive universal constants $r_1, s_1$, with $r_1 > s_1 > 10$, such that, if $G$ in (1.3) is of class $C^{r_1}$ and satisfies (1.5), then there exists a positive constant $\delta_*$ depending on $T, \omega, G$ with the following property.*

*Let $u_{in}, u_{end} \in H^{s_1}(\mathbb{T}, \mathbb{C})$ with*

$$\|u_{in}\|_{s_1} + \|u_{end}\|_{s_1} \leq \delta_*. \quad (1.9)$$

*Then there exists a function $f(t, x)$ satisfying*

$$f(t, x) = 0 \quad \text{for all } x \notin \omega, \text{ for all } t \in [0, T],$$

*belonging to $C([0, T], H^{s_1}_x) \cap C^1([0, T], H^{s_1 - 2}_x) \cap C^2([0, T], H^{s_1 - 4}_x)$ such that the Cauchy problem*

$$\begin{cases} u_t + \mathrm{i}u_{xx} + \mathcal{N}(x, u, u_x, u_{xx}) = f & \forall (t, x) \in [0, T] \times \mathbb{T} \\ u(0, x) = u_{in}(x) \end{cases} \quad (1.10)$$

*has a unique solution $u(t, x)$ belonging to $C([0, T], H^{s_1}_x) \cap C^1([0, T], H^{s_1 - 2}_x) \cap C^2([0, T], H^{s_1 - 4}_x)$, which satisfies*

$$u(T, x) = u_{end}(x), \quad (1.11)$$

*and*

$$\|u, f\|_{C([0,T], H^{s_1}_x)} + \|\partial_t u, \partial_t f\|_{C([0,T], H^{s_1 - 2}_x)} + \|\partial_{tt} u, \partial_{tt} f\|_{C([0,T], H^{s_1 - 4}_x)} \leq C(\|u_{in}\|_{s_1} + \|u_{end}\|_{s_1}) \quad (1.12)$$

*for some $C > 0$ depending on $T, \omega, G$.*

*Moreover the universal constant $\tau_1 := r_1 - s_1 > 0$ has the following property. For all $r \geq r_1$, all $s \in [s_1, r - \tau_1]$, if, in addition to the previous assumptions, $G$ is of class $C^r$ and $u_{in}, u_{end} \in H^s_x$, then $u, f$ belong to $C([0, T], H^s_x) \cap C^1([0, T], H^{s-2}_x) \cap C^2([0, T], H^{s-4}_x)$ and (1.12) holds with another constant $C_s$ instead of $C$, where $C_s > 0$ depends on $s, T, \omega, G$.*

**Remark 1.2.** Theorem 1.1 can be seen as split into two parts: first we fix the "low" regularity thresholds $s_1, r_1$, which are sufficient to prove the existence of a solution to the control problem. Then, in the last paragraph of the theorem, we give a statement about the higher regularity of such a solution.

Note that the smallness assumption (1.9) in Theorem 1.1 is only in the "low" norm: we only assume $\|u_{in}\|_{s_1} + \|u_{end}\|_{s_1} \leq \delta_*$, where the constant $\delta_* > 0$ does not depend on the "high" regularity index $s \in [s_1, r - \tau_1]$. □

Using the same techniques used for proving Theorem 1.1, we also prove the following theorem.

**Theorem 1.3** (Local existence and uniqueness)**.** *There exist positive universal constants $r_0$, $s_0$ with $r_0 > s_0 > 10$, such that, if $G$ in* (1.3) *is of class $C^{r_0}$ and satisfies* (1.5)*, then the following property holds. For all $T > 0$ there exists $\delta_* > 0$ such that for all $u_{in} \in H^{s_0}(\mathbb{T}, \mathbb{C})$ satisfying $\|u_{in}\|_{s_0} \leq \delta_*$, the Cauchy problem*

$$\begin{cases} u_t + \mathrm{i} u_{xx} + \mathcal{N}(x, u, u_x, u_{xx}) = 0, & (t,x) \in [0,T] \times \mathbb{T} \\ u(0,x) = u_{in}(x) \end{cases} \tag{1.13}$$

*has one and only one solution $u \in C([0,T], H^{s_0}_x) \cap C^1([0,T], H^{s_0-2}_x) \cap C^2([0,T], H^{s_0-4}_x)$. Moreover*

$$\|u\|_{C([0,T],H^{s_0}_x)} + \|\partial_t u\|_{C([0,T],H^{s_0-2}_x)} + \|\partial_{tt} u\|_{C([0,T],H^{s_0-4}_x)} \leq C \|u_{in}\|_{s_0} \tag{1.14}$$

*for some $C > 0$ depending on $T, G$.*

*The universal constant $\tau_0 := r_0 - s_0 > 0$ has the following property. For all $r \geq r_0$, all $s \in [s_0, r - \tau_0]$, if, in addition to the previous assumptions, $G$ is of class $C^r$ and $u_{in} \in H^s(\mathbb{T}, \mathbb{C})$, then $u$ belongs to $C([0,T], H^s_x) \cap C^1([0,T], H^{s-2}_x) \cap C^2([0,T], H^{s-4}_x)$ and* (1.14) *holds with another constant $C_s$ instead of $C$, where $C_s > 0$ depends on $s, T, G$.*

## 1.1 Some related literature

There is a vast amount of literature concerning controllability for linear or semilinear Schrödinger equations. Without even trying to be exhaustive, we only cite some relevant contributions to this subject, starting with the early papers by Jaffard [31], Lasiecka and Triggiani [32] and Lebeau [35], which deal with linear Schrödinger equations on bounded domains. Regarding the one-dimensional case, we mention the result of Beauchard and Coron [18] for the controllability of the linear equation by a moving potential well, and the papers by Beauchard, Laurent, Rosier and Zhang [16, 19, 33, 41] about controllability of semilinear Schrödinger equations. For the semilinear case on compact surfaces, we cite the work by Dehman, Gérard and Lebeau [24]. We also mention the recent results by Bourgain, Burq and Zworski [22] and by Anantharaman and Macià [9] concerning linear Schrödinger operators with rough potentials on higher-dimensional tori. More references in control theory for Schrödinger equations can be found in the detailed surveys by Laurent [34] and Zuazua [43].

Concerning controllability theory for quasi-linear PDEs, most known results deal with first order quasi-linear hyperbolic systems of the form $u_t + A(u)u_x = 0$ (see, for example, Coron [23] chapter 6.2 and the many references therein). Recent results for different kinds of quasi-linear PDEs are contained in Alazard, Baldi and Han-Kwan [6] on the internal controllability of gravity-capillary water waves equations, in Alazard [2, 3, 4] on the boundary observability and stabilization of gravity and gravity-capillary water waves, and in Baldi, Floridia and Haus [14, 15] on the internal controllability of quasi-linear perturbations of the Korteweg-de Vries equation.

## 1.2 Strategy of the proof

Because of the presence of two derivatives in the nonlinearity, the controllability of the *quasi-linear* control problem (1.10)-(1.11) cannot be directly deduced by a perturbative argument from the controllability of the corresponding linear problem by applying some fixed point argument or the usual implicit function theorem. A similar difficulty for a quasi-linear control problem was overcome in [6] by using a suitable nonlinear iteration scheme adapted to quasi-linear problems. Such a nonlinear scheme requires solving a linear control problem with variable coefficients at each step of the iteration, with no loss of regularity with respect to the coefficients (i.e., the solution must have the same regularity as the coefficients). In [6] this is achieved by means of paradifferential calculus, together with linear transformations, Ingham-type inequalities and the Hilbert uniqueness method. As an alternative method, in [14] it is used a Nash-Moser approach, which also demands the solving of a linear control problem with variable coefficients, but it requires weaker estimates, allowing some loss of regularity. The proof of such weaker estimates is easier to obtain, and it does not require the use of powerful techniques like paradifferential calculus (for a discussion about pseudo- and paradifferential calculus in connection with the Nash-Moser theorem, see, for example, [29], [8]). The result in [14] is slightly weaker than the one in [6] regarding the regularity of the solution of the nonlinear control problem with respect to the regularity of the data (in [14] for data in $H^s(\mathbb{T})$ both the control and the solution are in $C([0,T],H^{s'}(\mathbb{T}))$ for all $s' < s$, while the result in [6] reaches the corresponding optimal regularity $s' = s$). The version of the Nash-Moser implicit function theorem used in [14] is due to Hörmander [28], and it is the sharpest version in literature regarding the loss of regularity in terms of the coefficients of the linearized problem in several function spaces. As it is observed in [15], the theorem in [28] is the sharpest possible in Hölder class, but it is not optimal in Sobolev spaces (this is the reason for which the optimal regularity $s' = s$ is not obtained in [14]). In [15] the sharpest Hörmander's version of the Nash-Moser theorem has been extended to Sobolev spaces (so that $s' = s$ can be obtained both with the Nash-Moser approach and with the quasi-linear scheme with paradifferential analysis like in [6]). For this reason, in the present paper we use the Nash-Moser theorem in [15].

We mention that Nash-Moser schemes in control problems for PDEs have been used by Beauchard, Coron, Alabau-Boussouira and Olive in [16, 17, 18, 1]. A discussion about Nash-Moser as a method to overcome the problem of the loss of derivatives in the context of controllability for PDEs can be found in [23], Section 4.2.2. Beauchard and Laurent [19] were able to avoid the use of the Nash-Moser theorem in semilinear control problems thanks to a regularizing effect.

We prove Theorem 1.1 by applying the Nash-Moser-Hörmander implicit function theorem of [15] as a black box. To this end, one has to solve the associated linearized control problem (see equation (1.21)), which is a $2\times 2$ real system with variable coefficients at every order, and to prove tame estimates for the solution. Like in [6, 14], we solve the linearized control problem in $L^2(\mathbb{T})$ by applying the Hilbert uniqueness method (HUM), see Lemma (4.1). Then, in Lemma (4.2), we recover the additional regularity of the solution by adapting a method of Dehman-Lebeau [25], also used by Laurent [33] and in [6, 14]. To apply the HUM method, we prove in Section 3 the observability of the linearized operator in (1.29) by a procedure of symmetrization and reduction to constant coefficients up to a bounded remainder (like in [6, 14]) developed in Section 2; then the result follows by applying Ingham inequality (with a further simple argument to deal with double eigenvalues, like in [6]). The procedure of symmetrization and reduction of the linearized operator is an adaptation of the one used by Feola and Procesi [27, 26] in the context of KAM theory for quasi-linear NLS equations. We remark that a similar reduction procedure has been also developed in [30], [10], [11], [12], [13], [5], [6], [20], [38] for water waves, quasi-linear KdV, Benjamin-Ono and Kirchhoff equations.

## 1.3 Functional setting and the linearized problem

Given any open subset $\omega \subset \mathbb{T}$, we introduce a function $\chi_\omega \in C^\infty(\mathbb{T},\mathbb{R})$ whose support is contained in $\omega$, such that $0 \le \chi_\omega(x) \le 1$ for all $x \in \mathbb{T}$, and $\chi_\omega = 1$ on some open interval contained in $\omega$. We write the NLS control problem as a real system, namely, writing $u = \frac{1}{\sqrt{2}}(u_1 + \mathrm{i}u_2)$, $f = \frac{1}{\sqrt{2}}(f_1 + \mathrm{i}f_2)$, with $u_1, u_2, f_1, f_2$ all real-valued functions, the control problem (1.10)-(1.11) becomes the one of finding $(f_1, f_2)$ such that the solution $(u_1, u_2)$ of the Cauchy problem

$$\begin{cases} \partial_t u_1 + \nabla_{u_2} H(u_1,u_2) = \chi_\omega f_1 \\ \partial_t u_2 - \nabla_{u_1} H(u_1,u_2) = \chi_\omega f_2 \\ u_1(0,\cdot) = (u_1)_{in} \\ u_2(0,\cdot) = (u_2)_{in} \end{cases} \quad \text{satisfies} \quad \begin{cases} u_1(T,\cdot) = (u_1)_{end} \\ u_2(T,\cdot) = (u_2)_{end} \end{cases} \tag{1.15}$$

where the real Hamiltonian $H$ is defined in (1.8). We define

$$P(u_1,u_2) := \begin{pmatrix} \partial_t u_1 + \nabla_{u_2} H(u_1,u_2) \\ \partial_t u_2 - \nabla_{u_1} H(u_1,u_2) \end{pmatrix}, \quad \chi_\omega(f_1,f_2) := \begin{pmatrix} \chi_\omega f_1 \\ \chi_\omega f_2 \end{pmatrix}, \tag{1.16}$$

and

$$\Phi(u_1,u_2,f_1,f_2) := \begin{pmatrix} P(u_1,u_2) - \chi_\omega(f_1,f_2) \\ (u_1,u_2)(0,\cdot) \\ (u_1,u_2)(T,\cdot) \end{pmatrix}, \quad z_{data} := \begin{pmatrix} 0 \\ ((u_1)_{in},(u_2)_{in}) \\ ((u_1)_{end},(u_2)_{end}) \end{pmatrix}, \tag{1.17}$$

so that problem (1.15) reads

$$\Phi(u_1, u_2, f_1, f_2) = z_{data}. \tag{1.18}$$

By (1.16) and (1.8), the nonlinear operator $P$ is given by

$$P(u_1, u_2) = \begin{pmatrix} \partial_t u_1 - \partial_{xx} u_2 + (\partial_{y_2} G)(x, u_1, u_2, (u_1)_x, (u_2)_x) - \partial_x \{(\partial_{y_4} G)(x, u_1, u_2, (u_1)_x, (u_2)_x)\} \\ \partial_t u_2 + \partial_{xx} u_1 - (\partial_{y_1} G)(x, u_1, u_2, (u_1)_x, (u_2)_x) + \partial_x \{(\partial_{y_3} G)(x, u_1, u_2, (u_1)_x, (u_2)_x)\} \end{pmatrix}. \tag{1.19}$$

The crucial assumption to verify in order to apply the Nash-Moser theorem is the existence of a right inverse of the linearized operator. The linearized operator $\Phi'(u_1, u_2, f_1, f_2)[h_1, h_2, \varphi_1, \varphi_2]$ at the point $(u_1, u_2, f_1, f_2)$ in the direction $(h_1, h_2, \varphi_1, \varphi_2)$ is given by

$$\Phi'(u_1, u_2, f_1, f_2)[h_1, h_2, \varphi_1, \varphi_2] = \begin{pmatrix} P'(u_1, u_2)[h_1, h_2] - \chi_\omega(\varphi_1, \varphi_2) \\ (h_1, h_2)(0, \cdot) \\ (h_1, h_2)(T, \cdot) \end{pmatrix}. \tag{1.20}$$

Thus we have to prove that, given any $(u_1, u_2, f_1, f_2)$ and any $z = (v_1, v_2, \alpha_1, \alpha_2, \beta_1, \beta_2)$ in a suitable function space, there exists $(h_1, h_2, \varphi_1, \varphi_2)$ such that

$$\Phi'(u_1, u_2, f_1, f_2)[h_1, h_2, \varphi_1, \varphi_2] = z \tag{1.21}$$

(i.e., we have to solve the linearized control problem). The linearized operator $P'(u_1, u_2)[h_1, h_2]$ is

$$\begin{aligned} &P'(u_1, u_2)[h_1, h_2] \qquad (1.22)\\ &= \begin{pmatrix} \partial_t h_1 - \partial_{xx} h_2 + p_2^{(11)} \partial_{xx} h_1 + p_2^{(12)} \partial_{xx} h_2 + p_1^{(11)} \partial_x h_1 + p_1^{(12)} \partial_x h_2 + p_0^{(11)} h_1 + p_0^{(12)} h_2 \\ \partial_t h_2 + \partial_{xx} h_1 + p_2^{(21)} \partial_{xx} h_1 + p_2^{(22)} \partial_{xx} h_2 + p_1^{(21)} \partial_x h_1 + p_1^{(22)} \partial_x h_2 + p_0^{(21)} h_1 + p_0^{(22)} h_2 \end{pmatrix}, \end{aligned}$$

namely

$$\left\{ \partial_t + J\partial_{xx} + \begin{pmatrix} p_2^{(11)} & p_2^{(12)} \\ p_2^{(21)} & p_2^{(22)} \end{pmatrix} \partial_{xx} + \begin{pmatrix} p_1^{(11)} & p_1^{(12)} \\ p_1^{(21)} & p_1^{(22)} \end{pmatrix} \partial_x + \begin{pmatrix} p_0^{(11)} & p_0^{(12)} \\ p_0^{(21)} & p_0^{(22)} \end{pmatrix} \right\} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix} \tag{1.23}$$

where the coefficients of the terms of order 2 are

$$\begin{aligned} p_2^{(11)} &= -(\partial_{y_3 y_4} G), \qquad & p_2^{(12)} &= -(\partial_{y_4 y_4} G), \\ p_2^{(21)} &= (\partial_{y_3 y_3} G), & p_2^{(22)} &= (\partial_{y_3 y_4} G), \end{aligned} \tag{1.24}$$

those of order 1 are

$$\begin{aligned} p_1^{(11)} &= (\partial_{y_2 y_3} G) - (\partial_{y_1 y_4} G) - \partial_x \{(\partial_{y_3 y_4} G)\}, \qquad & p_1^{(12)} &= -\partial_x \{(\partial_{y_4 y_4} G)\}, \\ p_1^{(21)} &= \partial_x \{(\partial_{y_3 y_3} G)\}, & p_1^{(22)} &= -(\partial_{y_1 y_4} G) + (\partial_{y_2 y_3} G) + \partial_x \{(\partial_{y_3 y_4} G)\}, \end{aligned} \tag{1.25}$$

those of order 0 are

$$\begin{aligned} p_0^{(11)} &= (\partial_{y_1 y_2} G) - \partial_x \{(\partial_{y_1 y_4} G)\}, \qquad & p_0^{(12)} &= (\partial_{y_2 y_2} G) - \partial_x \{(\partial_{y_2 y_4} G)\}, \\ p_0^{(21)} &= -(\partial_{y_1 y_1} G) + \partial_x \{(\partial_{y_1 y_3} G)\}, & p_0^{(22)} &= -(\partial_{y_1 y_2} G) + \partial_x \{(\partial_{y_2 y_3} G)\}, \end{aligned} \tag{1.26}$$

and $(\partial_{y_i y_j} G) = (\partial_{y_i y_j} G)(x, u_1, u_2, \partial_x u_1, \partial_x u_2)$ for all $i, j \in \{1, 2, 3, 4\}$.

Consider the transformation

$$\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} = \mathcal{C} \begin{pmatrix} h \\ \bar{h} \end{pmatrix}, \quad \text{where} \quad \mathcal{C} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -i & i \end{pmatrix}, \quad \mathcal{C}^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \tag{1.27}$$

and similarly $(\varphi_1, \varphi_2) = \mathcal{C}(\varphi, \bar{\varphi})$, $(v_1, v_2) = \mathcal{C}(v, \bar{v})$, $(\alpha_1, \alpha_2) = \mathcal{C}(\alpha, \bar{\alpha})$, $(\beta_1, \beta_2) = \mathcal{C}(\beta, \bar{\beta})$. With this "vector complex" notation, the linearized control problem (1.21) becomes

$$\begin{cases} \mathcal{L}[h, \bar{h}] - \chi_\omega(\varphi, \bar{\varphi}) = (v, \bar{v}) \\ (h, \bar{h})(0, \cdot) = (\alpha, \bar{\alpha}) \\ (h, \bar{h})(T, \cdot) = (\beta, \bar{\beta}) \end{cases} \tag{1.28}$$

where $\mathcal{L} := \mathcal{L}(u_1, u_2) := \mathcal{C}^{-1} P'(u_1, u_2)\mathcal{C}$. More explicitly, we calculate

$$\mathcal{L} = \partial_t \mathbb{I}_2 + \mathrm{i}(\Sigma + A_2)\partial_{xx} + \mathrm{i}A_1\partial_x + \mathrm{i}A_0 \,, \tag{1.29}$$

where

$$\mathbb{I}_2 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Sigma := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_k := \begin{pmatrix} a_k & b_k \\ -\bar{b}_k & -\bar{a}_k \end{pmatrix}, \quad k = 0, 1, 2, \tag{1.30}$$

$$a_k := \frac{1}{2}\Big( -\mathrm{i}p_k^{(11)} - p_k^{(12)} + p_k^{(21)} - \mathrm{i}p_k^{(22)} \Big), \quad b_k := \frac{1}{2}\Big( -\mathrm{i}p_k^{(11)} + p_k^{(12)} + p_k^{(21)} + \mathrm{i}p_k^{(22)} \Big), \tag{1.31}$$

and $\bar{a}_k, \bar{b}_k$ are the complex conjugates of the coefficients $a_k, b_k$. By (1.31) and (1.24), (1.25), (1.26), one has

$$a_2 = \bar{a}_2, \quad a_1 = 2\partial_x a_2 - \bar{a}_1, \quad a_0 = \bar{a}_0 + \partial_{xx} a_2 - \partial_x \bar{a}_1, \quad b_1 = \partial_x b_2. \tag{1.32}$$

**Remark 1.4.** The linear system (1.28) is made by three pairs of equations in which the second equation is the complex conjugate of the first one. Hence (1.28) is equivalent to

$$\begin{cases} \mathcal{L}^{(sca)} h - \chi_\omega \varphi = v \\ h(0, \cdot) = \alpha \\ h(T, \cdot) = \beta \end{cases} \tag{1.33}$$

where

$$\mathcal{L}^{(sca)} := \partial_t + \mathrm{i}(1 + a_2 + b_2 \mathfrak{C})\partial_{xx} + \mathrm{i}(a_1 + b_1 \mathfrak{C})\partial_x + \mathrm{i}(a_0 + b_0 \mathfrak{C}), \qquad \mathfrak{C}[h] := \bar{h}. \tag{1.34}$$

The complex conjugate operator $\mathfrak{C} : h \mapsto \bar{h}$ is $\mathbb{R}$-linear, and there is no problem in using it to shorten the notation of the real system (1.21).

However, instead of the *scalar complex* notation (1.33), in the analysis of the linearized problem we will use the *vector complex* notation (1.28), which is somewhat "more natural" and very common in the literature on the Schrödinger equation. In any case, for linear systems the two notations are, of course, completely equivalent. □

For real $s \geq 0$, we consider the classical Sobolev space

$$H^s(\mathbb{T}) := H^s(\mathbb{T}, \mathbb{C}) := \Big\{ u \in L^2(\mathbb{T}, \mathbb{C}) : \|u\|_s^2 := \sum_{k \in \mathbb{Z}} \langle k \rangle^{2s} |\widehat{u}_k|^2 < \infty \Big\},$$

where $\langle k \rangle := (1 + |k|^2)^{\frac{1}{2}}$ and $u(x) = \sum_{k \in \mathbb{Z}} \widehat{u}_k \, e^{\mathrm{i}kx} \in L^2(\mathbb{T}) := L^2(\mathbb{T}, \mathbb{C})$. We adopt the convention of indicating explicitly $H^s(\mathbb{T}, \mathbb{R})$ the subspace of *real-valued* functions of $H^s(\mathbb{T}, \mathbb{C})$, and to denote, in short, by $H^s(\mathbb{T})$ the whole space $H^s(\mathbb{T}, \mathbb{C})$. The same convention applies to $L^2(\mathbb{T}, \mathbb{R})$ and $L^2(\mathbb{T}) := L^2(\mathbb{T}, \mathbb{C})$. We also consider spaces $H^s(\mathbb{T}, \mathbb{K}^2)$, where $\mathbb{K} = \mathbb{R}, \mathbb{C}$, and for $(u_1, u_2) \in H^s(\mathbb{T}, \mathbb{K}^2)$ we set

$$\|(u_1, u_2)\|_s := \|u_1\|_s + \|u_2\|_s \,.$$

We define the *real subspace* $\mathbf{H}^s(\mathbb{T})$ of $H^s(\mathbb{T}, \mathbb{C}^2)$ as

$$\mathbf{H}^s(\mathbb{T}) := \big\{ \mathbf{u} = (u, \overline{u}) : u \in H^s(\mathbb{T}, \mathbb{C}) \big\} \tag{1.35}$$

where $\bar{u}$ is the complex conjugate of $u$. When there is no ambiguity, we also write, in short, $H^s_x$ to denote $H^s(\mathbb{T}, \mathbb{C})$ or $H^s(\mathbb{T}, \mathbb{R}^2)$, and the same for $L^2_x$, $\mathbf{H}^s_x$ and $\mathbf{L}^2_x$.

We denote by $\langle \cdot, \cdot \rangle_{L^2}$ the standard $L^2$ scalar product in $L^2(\mathbb{T}, \mathbb{C})$, namely

$$\langle u, v \rangle_{L^2} := \int_{\mathbb{T}} u(x)\overline{v}(x) \, dx \quad \forall u, v \in L^2(\mathbb{T}, \mathbb{C}). \tag{1.36}$$

We define the scalar product in $L^2(\mathbb{T}, \mathbb{R}^2)$ as

$$\langle (u_1, u_2), (v_1, v_2) \rangle_{L^2(\mathbb{T}, \mathbb{R}^2)} := \int_{\mathbb{T}} u_1(x) v_1(x) \, dx + \int_{\mathbb{T}} u_2(x) v_2(x) \, dx, \tag{1.37}$$

and the scalar product in $\mathbf{L}^2(\mathbb{T})$ as

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{L}^2} := \int_{\mathbb{T}} u(x)\overline{v}(x)\, dx + \int_{\mathbb{T}} v(x)\overline{u}(x)\, dx\,. \tag{1.38}$$

Note that (1.38) is a real scalar product on $\mathbf{L}^2(\mathbb{T})$, and therefore $(\mathbf{L}^2(\mathbb{T}), \langle \cdot, \cdot \rangle_{\mathbf{L}^2})$ is a real Hilbert subspace of $L^2(\mathbb{T}, \mathbb{C}^2)$.

The transformation $\mathcal{C}$ defined in (1.27) satisfies

$$\langle \mathbf{u}, \mathbf{v} \rangle_{\mathbf{L}^2} = \langle \mathcal{C}\mathbf{u}, \mathcal{C}\mathbf{v} \rangle_{L^2(\mathbb{T},\mathbb{R}^2)} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\mathbb{T}), \tag{1.39}$$

and so $\mathcal{C}$ is a unitary isomorphism between the real Hilbert space $L^2(\mathbb{T}, \mathbb{R}^2)$ equipped with the real scalar product (1.37) and the real Hilbert space $\mathbf{L}^2(\mathbb{T})$ equipped with the scalar product (1.38).

Given a linear operator $R : L^2(\mathbb{T}, \mathbb{C}) \to L^2(\mathbb{T}, \mathbb{C})$, we define the *adjoint* operator $R^*$ as

$$\langle Ru, v \rangle_{L^2} = \langle u, R^* v \rangle_{L^2} \quad \forall u, v \in L^2(\mathbb{T}, \mathbb{C}); \tag{1.40}$$

the *transpose* operator $R^T$ as

$$\int_{\mathbb{T}} (Ru) v\, dx = \int_{\mathbb{T}} u(R^T v)\, dx \quad \forall u, v \in L^2(\mathbb{T}, \mathbb{C}); \tag{1.41}$$

and the *conjugate* operator $\overline{R}$ as

$$\overline{R} u = \overline{(R\overline{u})} \quad \forall u \in L^2(\mathbb{T}, \mathbb{C}). \tag{1.42}$$

For an operator

$$\mathcal{R} := \begin{pmatrix} A & B \\ \overline{B} & \overline{A} \end{pmatrix} : \mathbf{L}^2(\mathbb{T}) \to \mathbf{L}^2(\mathbb{T})\,,$$

we define its adjoint $\mathcal{R}^*$ by

$$\langle \mathcal{R}\mathbf{u}, \mathbf{v} \rangle_{\mathbf{L}^2} = \langle \mathbf{u}, \mathcal{R}^* \mathbf{v} \rangle_{\mathbf{L}^2} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbf{L}^2(\mathbb{T}), \tag{1.43}$$

namely

$$\mathcal{R}^* = \begin{pmatrix} (\overline{A})^T & B^T \\ (\overline{B})^T & A^T \end{pmatrix} = \begin{pmatrix} A^* & B^T \\ B^* & A^T \end{pmatrix} = \begin{pmatrix} A^* & B^T \\ \overline{B^T} & \overline{A^*} \end{pmatrix}. \tag{1.44}$$

For any real $s \geq 0$ and $\mathbf{u} = (u, \overline{u}) \in \mathbf{H}^s(\mathbb{T})$, we set

$$\|\mathbf{u}\|_s := \|u\|_s\,. \tag{1.45}$$

Given a Banach space $(X, \|\cdot\|_X)$, and $T > 0$, we consider the space $C([0,T], X)$ of the continuous functions $u : [0,T] \to X$ equipped with the sup-norm

$$\|u\|_{C([0,T],X)} := \|u\|_{C_T(X)} := \sup_{t \in [0,T]} \|u(t)\|_X\,. \tag{1.46}$$

For $X = H^s(\mathbb{T}, \mathbb{R})$ or $H^s(\mathbb{T}, \mathbb{R}^2)$ or $H^s(\mathbb{T}, \mathbb{C})$ or $H^s(\mathbb{T}, \mathbb{C}^2)$ or $\mathbf{H}^s(\mathbb{T})$, and $u \in C([0,T], X)$, we denote, in short,

$$\|u\|_{T,s} := \sup_{t \in [0,T]} \|u(t)\|_s\,. \tag{1.47}$$

We also define the following notations. Given a Sobolev index $s \geq 0$, we write $A \lesssim_s B$ if there exists a constant $C(s) > 0$ depending on $s$ such that $A \leq C(s)B$. If the constant $C(s)$ is independent of $s$, we simply write $A \lesssim B$.

According to (1.15)-(1.19), Theorem 1.1 follows from the following theorem.

**Theorem 1.5.** *Let $T > 0$, and let $\omega \subset \mathbb{T}$ be a nonempty open set. Let $\chi_\omega$ be a $C^\infty$ function supported in $\omega$, with $0 \le \chi_\omega \le 1$ on $\mathbb{T}$ and $\chi_\omega = 1$ on some open interval contained in $\omega$. There exist positive universal constants $r_1, s_1$ such that, if $G$ in* (1.3) *is of class $C^{r_1}$ and satisfies* (1.5)*, then there exists a positive constant $\delta_*$ depending on $T, \omega, G$ with the following property. Let $(u_1)_{in}, (u_1)_{end}, (u_2)_{in}, (u_2)_{end} \in H^{s_1}(\mathbb{T}, \mathbb{R})$ with*

$$\|(u_i)_{in}\|_{s_1} + \|(u_i)_{end}\|_{s_1} \le \delta_*, \quad i = 1, 2\,.$$

*Then there exist functions*

$$f_1, f_2 \in C([0,T], H^{s_1}(\mathbb{T},\mathbb{R})) \cap C^1([0,T], H^{s_1-2}(\mathbb{T},\mathbb{R})) \cap C^2([0,T], H^{s_1-4}(\mathbb{T},\mathbb{R}))$$

*such that the Cauchy problem*

$$\begin{cases} \partial_t u_1 + \nabla_{u_2} H(u_1,u_2) = \chi_\omega f_1 \\ \partial_t u_2 - \nabla_{u_1} H(u_1,u_2) = \chi_\omega f_2 \\ u_1(0,\cdot) = (u_1)_{in} \\ u_2(0,\cdot) = (u_2)_{in} \end{cases} \tag{1.48}$$

*has a unique solution $(u_1, u_2)$ with*

$$u_1, u_2 \in C([0,T], H^{s_1}(\mathbb{T},\mathbb{R})) \cap C^1([0,T], H^{s_1-2}(\mathbb{T},\mathbb{R})) \cap C^2([0,T], H^{s_1-4}(\mathbb{T},\mathbb{R}))\,,$$

*which satisfies*

$$u_1(T,x) = (u_1)_{end}(x), \qquad u_2(T,x) = (u_2)_{end}(x) \tag{1.49}$$

*and for $i = 1, 2$*

$$\|u_i, f_i\|_{T,s_1} + \|\partial_t u_i, \partial_t f_i\|_{T,s_1-2} + \|\partial_{tt} u_i, \partial_{tt} f_i\|_{T,s_1-4} \le C(\|(u_1)_{in}, (u_2)_{in}\|_{s_1} + \|(u_1)_{end}, (u_2)_{end}\|_{s_1}) \tag{1.50}$$

*for some $C > 0$ depending on $T, \omega, G$.*

*Moreover the universal constant $\tau_1 := r_1 - s_1 > 0$ has the following property. For all $r \ge r_1$, all $s \in [s_1, r - \tau_1]$, if, in addition to the previous assumptions, $G$ is of class $C^r$ and $(u_1)_{in}$, $(u_2)_{in}$, $(u_1)_{end}$, $(u_2)_{end} \in H^s(\mathbb{T},\mathbb{R})$, then $u, f$ belong to $C([0,T], H^s(\mathbb{T},\mathbb{R})) \cap C^1([0,T], H^{s-2}(\mathbb{T},\mathbb{R})) \cap C^2([0,T], H^{s-4}(\mathbb{T},\mathbb{R}))$ and* (1.50) *holds with another constant $C_s$ instead of $C$, where $C_s > 0$ depends on $s, T, \omega, G$.*

Similarly, Theorem 1.3 follows from the following theorem.

**Theorem 1.6.** *Let $T > 0$. There exist positive universal constants $r_0, s_0$ such that, if $G$ in* (1.3) *is of class $C^{r_0}$ in its arguments and satisfies* (1.5)*, then there exists a positive constant $\delta_*$ depending on $T, G$ with the following property. Let $(u_1)_{in}, (u_2)_{in} \in H^{s_0}(\mathbb{T},\mathbb{R})$ with*

$$\|(u_1)_{in}\|_{s_0} + \|(u_2)_{in}\|_{s_0} \le \delta_*\,.$$

*Then the Cauchy problem*

$$\begin{cases} \partial_t u_1 + \nabla_{u_2} H(u_1,u_2) = 0 \\ \partial_t u_2 - \nabla_{u_1} H(u_1,u_2) = 0 \\ u_1(0,\cdot) = (u_1)_{in} \\ u_2(0,\cdot) = (u_2)_{in} \end{cases} \tag{1.51}$$

*has a unique solution $(u_1, u_2)$ with*

$$u_1, u_2 \in C([0,T], H^{s_0}(\mathbb{T},\mathbb{R})) \cap C^1([0,T], H^{s_0-2}(\mathbb{T},\mathbb{R})) \cap C^2([0,T], H^{s_0-4}(\mathbb{T},\mathbb{R}))$$

*and*

$$\|u_i\|_{T,s_0} + \|\partial_t u_i\|_{T,s_0-2} + \|\partial_{tt} u_i\|_{T,s_0-4} \le C(\|(u_1)_{in}\|_{s_0} + \|(u_2)_{in}\|_{s_0})\,, \quad i = 1, 2 \tag{1.52}$$

*for some $C > 0$ depending on $T, G$.*

*Moreover the universal constant $\tau_0 := r_0 - s_0 > 0$ has the following property. For all $r \geq r_0$, all $s \in [s_0, r - \tau_0]$, if, in addition to the previous assumptions, $G$ is of class $C^r$ and $(u_1)_{in}, (u_2)_{in} \in H^s(\mathbb{T}, \mathbb{R})$, then $u$ belongs to $C([0,T], H^s(\mathbb{T}, \mathbb{R})) \cap C^1([0,T], H^{s-2}(\mathbb{T}, \mathbb{R})) \cap C^2([0,T], H^{s-4}(\mathbb{T}, \mathbb{R}))$ and (1.52) holds with another constant $C_s$ instead of $C$, where $C_s > 0$ depends on $s, T, G$.*

## 2 Reduction of the linearized operator

In view of the application of the Nash-Moser scheme, we will consider linear operators of the same form as $\mathcal{L} = \mathcal{L}(u_1, u_2)$ given in (1.29). The aim of this section is to conjugate such operators to constant coefficients up to a bounded remainder, adapting the procedure described in [26, 27]. We first fix some notation.

Let $u_1, u_2 \in C^0\big([0,T], H^{s+4}(\mathbb{T}, \mathbb{R})\big) \cap C^1\big([0,T], H^{s+2}(\mathbb{T}, \mathbb{R})\big) \cap C^2\big([0,T], H^s(\mathbb{T}, \mathbb{R})\big)$. We define

$$M_T(s; u_1, u_2) := \max_{k=1,2} \sup_{t \in [0,T]} \big( \|u_k(t,\cdot)\|_{s+4} + \|\partial_t u_k(t,\cdot)\|_{s+2} + \|\partial_{tt} u_k(t,\cdot)\|_s \big) . \tag{2.1}$$

We recall the notation defined in (1.47): given a function $v \in C\big([0,T], H^s(\mathbb{T}, \mathbb{R})\big)$, we denote $\|v\|_{T,s} := \sup_{t \in [0,T]} \|v(t,\cdot)\|_{H^s_x}$. Also, if $\mathbf{v} = (v, \overline{v}) \in C\big([0,T], \mathbf{H}^s(\mathbb{T})\big)$, we set

$$\|\mathbf{v}\|_{T,s} := \|v\|_{T,s} .$$

In the next Lemma we provide some estimates on the coefficients $a_i$, $b_i$, $i = 0, 1, 2$.

**Lemma 2.1.** *Let $r \geq 6$ be the regularity of $G$ in (1.3). There exists $\delta > 0$, depending on $G$, such that, if $M_T(2; u_1, u_2)$ defined in (2.1) satisfies*

$$M_T(2; u_1, u_2) \leq \delta, \tag{2.2}$$

*then for every $s \in [0, r-6]$ one has*

$$\|a_i\|_{T,s}\,,\ \|\partial_t a_i\|_{T,s}, \|\partial_{tt} a_i\|_{T,s}, \|b_i\|_{T,s}\,,\ \|\partial_t b_i\|_{T,s}, \|\partial_{tt} b_i\|_{T,s} \lesssim_s M_T(s+2; u_1, u_2) .$$

*Proof.* The estimates follow from the explicit expressions given in (1.31), (1.24)-(1.26) and by the composition Lemma 7.2. □

We consider operators of the form

$$\mathcal{L} := \partial_t \mathbb{I}_2 + \mathrm{i}(\Sigma + A_2)\partial_{xx} + \mathrm{i}A_1 \partial_x + \mathrm{i}A_0 , \tag{2.3}$$

where

$$\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \quad A_k := \begin{pmatrix} a_k & b_k \\ -\overline{b}_k & -\overline{a}_k \end{pmatrix} , \quad k = 0, 1, 2 . \tag{2.4}$$

We assume that the time dependent vector field $L(t) := \mathrm{i}A_2 \partial_{xx} + \mathrm{i}A_1 \partial_x + \mathrm{i}A_0$ is Hamiltonian, therefore equations (1.32) hold by Lemma 6.2. We assume that for $S \in \mathbb{N}$ large enough

$$a_2, \partial_t a_2, \partial_{tt} a_2, b_2, \partial_t b_2, a_1, \partial_t a_1, b_1, \partial_t b_1, a_0, b_0 \in C\big([0,T], H^S(\mathbb{T})\big) , \tag{2.5}$$

and, for $s \in [0, S]$, we set

$$\begin{aligned} N_T(s) := &\sup_{t \in [0,T]} \max\{\|a_2\|_{H^s}, \|\partial_t a_2\|_{H^s}, \|\partial_{tt} a_2\|_{H^s}, \|a_1\|_{H^s}, \|\partial_t a_1\|_{H^s}, \|a_0\|_{H^s}\} \\ &+ \sup_{t \in [0,T]} \Big( \|b_2\|_{H^s}, \|\partial_t b_2\|_{H^s}, \|b_1\|_{H^s}, \|\partial_t b_1\|_{H^s}, \|b_0\|_{H^s} \Big) . \end{aligned} \tag{2.6}$$

In Sections 2, 3, we will consider constants $\sigma, S$, with $0 < \sigma < S$, and $\eta \in (0,1)$, and assume that

$$N_T(\sigma) \leq \eta . \tag{2.7}$$

The constant $S$ will have the role of a large and fixed regularity index, $\sigma$ will indicate the "loss of regularity" in terms of the coefficients of the linearized operator, and $\eta$ will be small enough.

## 2.1 Symmetrization of $\mathcal{L}$ up to order zero

In this subsection we remove the off-diagonal terms from the order 2, namely we conjugate the linear operator $\mathcal{L}$ in (2.3) to an operator $\mathcal{L}_0$ (see (2.13)-(2.14)) where the coefficient in front of $\partial_{xx}$ is a *diagonal* $2 \times 2$ matrix. As a consequence of the Hamiltonian structure, the transformation that achieves this cancellation also removes the off-diagonal terms from the order 1 (see equation (2.17)). First we consider the $2 \times 2$ matrix valued function

$$\Sigma + A_2(t,x) = \begin{pmatrix} 1 + a_2(t,x) & b_2 \\ -\bar{b}_2 & -1 - a_2(t,x) \end{pmatrix}$$

(recall that $a_2 = \bar{a}_2$ by Lemma 6.2). The eigenvalues of the above matrix are given by $\pm\lambda(t,x) \in \mathbb{R}$, where

$$\lambda(t,x) := \sqrt{(1+a_2)^2 - |b_2|^2} \, . \tag{2.8}$$

Note that, by Sobolev embedding, (2.7) and because $\sigma \geq 1$, one has

$$\|a_2\|_{L^\infty} + \|b_2\|_{L^\infty} \lesssim \|a_2\|_{T,1} + \|b_2\|_{T,1} \lesssim \eta \, ,$$

so that $(1+a_2)^2 - |b_2|^2$ is close to 1 for $\eta \in (0,1)$ small enough. Then we consider the $2 \times 2$ matrix

$$\mathcal{S} = \mathcal{S}(t,x) := \begin{pmatrix} \dfrac{1+a_2+\lambda}{\sqrt{(1+a_2+\lambda)^2 - |b_2|^2}} & -\dfrac{b_2}{\sqrt{(1+a_2+\lambda)^2 - |b_2|^2}} \\ -\dfrac{\bar{b}_2}{\sqrt{(1+a_2+\lambda)^2 - |b_2|^2}} & \dfrac{1+a_2+\lambda}{\sqrt{(1+a_2+\lambda)^2 - |b_2|^2}} \end{pmatrix} . \tag{2.9}$$

The columns of the matrix $\mathcal{S}$ are the eigenvectors corresponding to the eigenvalues $\pm\lambda$ and $\det(\mathcal{S}(t,x)) = 1$. Then the map

$$\mathcal{S}(t): \quad \mathbf{h}(x) \mapsto \mathcal{S}(t,x)\mathbf{h}(x)$$

is symplectic. The above matrix is invertible and its inverse is given by

$$\mathcal{S}^{-1} = \mathcal{S}^{-1}(t,x) := \begin{pmatrix} \dfrac{1+a_2+\lambda}{\sqrt{(1+a_2+\lambda)^2 - |b_2|^2}} & \dfrac{b_2}{\sqrt{(1+a_2+\lambda)^2 - |b_2|^2}} \\ \dfrac{\bar{b}_2}{\sqrt{(1+a_2+\lambda)^2 - |b_2|^2}} & \dfrac{1+a_2+\lambda}{\sqrt{(1+a_2+\lambda)^2 - |b_2|^2}} \end{pmatrix} \tag{2.10}$$

and a direct calculation shows that

$$\mathcal{S} = \mathcal{S}^* \, . \tag{2.11}$$

We compute the conjugation $\mathcal{S}^{-1}\mathcal{L}\mathcal{S}$. Note that

$$\mathcal{S}^{-1}(\Sigma + A_2)\mathcal{S} = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} = \begin{pmatrix} 1 + a_2^{(0)} & 0 \\ 0 & -1 - a_2^{(0)} \end{pmatrix} , \quad a_2^{(0)} := \lambda - 1 \in \mathbb{R} \tag{2.12}$$

and we get the linear operator

$$\mathcal{L}_0 := \mathcal{S}^{-1}\mathcal{L}\mathcal{S} = \partial_t \mathbb{I}_2 + \mathrm{i}(\Sigma + A_2^{(0)})\partial_{xx} + \mathrm{i}A_1^{(0)}\partial_x + \mathrm{i}A_0^{(0)} \, , \tag{2.13}$$

where

$$A_2^{(0)} := \begin{pmatrix} a_2^{(0)} & 0 \\ 0 & -a_2^{(0)} \end{pmatrix} , \tag{2.14}$$

$$A_1^{(0)} := \begin{pmatrix} a_1^{(0)} & b_1^{(0)} \\ -\bar{b}_1^{(0)} & -\bar{a}_1^{(0)} \end{pmatrix} = 2\mathcal{S}^{-1}(\Sigma + A_2)(\partial_x \mathcal{S}) + \mathcal{S}^{-1}A_1\mathcal{S} \, , \tag{2.15}$$

$$A_0^{(0)} := \begin{pmatrix} a_0^{(0)} & b_0^{(0)} \\ -\bar{b}_0^{(0)} & -\bar{a}_0^{(0)} \end{pmatrix} = \mathcal{S}^{-1}(\Sigma + A_2)(\partial_{xx}\mathcal{S}) + \mathcal{S}^{-1}A_1(\partial_x\mathcal{S}) + \mathcal{S}^{-1}(\partial_t\mathcal{S}) + \mathcal{S}^{-1}A_0\mathcal{S} \, . \tag{2.16}$$

Since the linear transformation $\mathcal{S}(t) : \mathbf{h}(x) \mapsto \mathcal{S}(t,x)\mathbf{h}(x)$ is symplectic, the time dependent linear vector field $L_0(t) := \mathrm{i}(\Sigma + A_2^{(0)})\partial_{xx} + \mathrm{i}A_1^{(0)}\partial_x + \mathrm{i}A_0^{(0)}$ is still Hamiltonian. Then, by Lemma 6.2, one has

$$b_1^{(0)} = \partial_x b_2^{(0)} = 0\,, \tag{2.17}$$

hence

$$A_1^{(0)} = \begin{pmatrix} a_1^{(0)} & 0 \\ 0 & -\overline{a}_1^{(0)} \end{pmatrix} = 2\mathcal{S}^{-1}(\Sigma + A_2)(\partial_x \mathcal{S}) + \mathcal{S}^{-1}A_1\mathcal{S}. \tag{2.18}$$

Note that (2.17) can also be proved by a direct calculation.

**Lemma 2.2.** *There exists $\eta \in (0,1)$ small enough, $\sigma > 0$ such that if $N_T(\sigma) \le \eta$, then for any $0 \le s \le S - \sigma$ (where $S$ is defined in (2.5))*

$$\|\mathcal{S}^{\pm 1} - \mathrm{Id}\|_{T,s} \lesssim_s N_T(s+\sigma)\,. \tag{2.19}$$

*As a consequence*

$$\|\big(\mathcal{S}^{\pm 1} - \mathrm{Id}\big)\mathbf{h}\|_{T,s} \lesssim_s \eta\|\mathbf{h}\|_{T,s} + N_T(s+\sigma)\|\mathbf{h}\|_{T,0}\,. \tag{2.20}$$

*Furthermore,*

$$\|a_2^{(0)}\|_{T,s}, \|\partial_t a_2^{(0)}\|_{T,s}, \|\partial_{tt} a_2^{(0)}\|_{T,s} \lesssim_s N_T(s+\sigma)\,, \tag{2.21}$$

$$\|a_1^{(0)}\|_{T,s}, \|\partial_t a_1^{(0)}\|_{T,s}\,, \|a_0^{(0)}\|_{T,s}, \|b_0^{(0)}\|_{T,s} \lesssim_s N_T(s+\sigma)\,. \tag{2.22}$$

*Proof.* Use definitions (2.9), (2.12), (2.15) and apply Lemmas 7.1, 7.2. □

## 2.2 Change of the space variable

The aim of this subsection is to remove the $x$-dependence from the highest order term of the operator $\mathcal{L}_0$ defined in (2.13) (namely, to conjugate $\mathcal{L}_0$ to an operator where the coefficient of $\partial_{xx}$ does not depend on the space variable $x$). For this purpose, we consider $t$-dependent families of diffeomorphisms of the torus $\mathbb{T}$ of the form

$$x \mapsto x + \alpha(t,x)\,, \quad \alpha : [0,T] \times \mathbb{T} \to \mathbb{R}\,, \qquad |\alpha_x(t,x)| \le 1/2\,.$$

The above diffeomorphism is invertible and its inverse is given by

$$y \mapsto y + \widetilde{\alpha}(t,y)\,.$$

Then we define the linear operator $\mathcal{A}$ as

$$\mathcal{A} := \sqrt{1+\alpha_x}\; A_\alpha\,, \qquad A_\alpha h(t,x) := h(t, x + \alpha(t,x))\,. \tag{2.23}$$

Using the fact that

$$\frac{1}{1 + \alpha_x(t, y + \widetilde{\alpha}(t,y))} = 1 + \widetilde{\alpha}_y(t,y) \tag{2.24}$$

one gets that the inverse of the operator $\mathcal{A}$ has the form

$$\mathcal{A}^{-1} = \mathcal{A}^* = \sqrt{1+\widetilde{\alpha}_y}\; A_{\widetilde{\alpha}}\,, \qquad A_{\widetilde{\alpha}} h(t,y) := A_\alpha^{-1} h(t,y) = h(t, y + \widetilde{\alpha}(t,y))\,. \tag{2.25}$$

A direct calculation shows that $\mathcal{A}\mathbb{I}_2$ is a symplectic map. The conjugation of the differential operators $\partial_t, \partial_x, \partial_{xx}$ and of multiplication operators $a = a(t,x) : h \mapsto ah$ are given by

$$\mathcal{A}^{-1}\partial_t\mathcal{A} = \partial_t + (A_{\widetilde{\alpha}}\alpha_t)\partial_y + \Big(A_{\widetilde{\alpha}}\frac{\alpha_{tx}}{2(1+\alpha_x)}\Big)\,, \qquad \mathcal{A}^{-1}a\mathcal{A} = (A_{\widetilde{\alpha}}a) \tag{2.26}$$

$$\mathcal{A}^{-1}\partial_x\mathcal{A} = [1 + (A_{\widetilde{\alpha}}\alpha_x)]\partial_y + \Big(A_{\widetilde{\alpha}}\frac{\alpha_{xx}}{2(1+\alpha_x)}\Big) \tag{2.27}$$

$$\mathcal{A}^{-1}\partial_{xx}\mathcal{A} = \{A_{\widetilde{\alpha}}[(1+\alpha_x)^2]\}\partial_{yy} + 2(A_{\widetilde{\alpha}}\alpha_{xx})\partial_y + \Big(A_{\widetilde{\alpha}}\frac{2\alpha_{xxx}(1+\alpha_x) - \alpha_{xx}^2}{4(1+\alpha_x)^2}\Big)\,. \tag{2.28}$$

Conjugating the operator $\mathcal{L}_0$ in (2.13) by means of the symplectic map $\mathcal{A}\mathbb{I}_2$ we get the operator

$$\mathcal{L}_1 := \mathcal{A}^{-1}\mathbb{I}_2\mathcal{L}_0\mathcal{A}\mathbb{I}_2 = \partial_t\mathbb{I}_2 + \mathrm{i}A_2^{(1)}\partial_{yy} + \mathrm{i}A_1^{(1)}\partial_y + \mathrm{i}A_0^{(1)}\,, \tag{2.29}$$

where, taking into account (2.17),

$$A_2^{(1)} := \begin{pmatrix} a_2^{(1)} & 0 \\ 0 & -a_2^{(1)} \end{pmatrix}, \qquad A_1^{(1)} := \begin{pmatrix} a_1^{(1)} & 0 \\ 0 & -\overline{a}_1^{(1)} \end{pmatrix}, \quad A_0^{(1)} := \begin{pmatrix} a_0^{(1)} & b_0^{(1)} \\ -\overline{b}_0^{(1)} & -\overline{a}_0^{(1)} \end{pmatrix}$$

and

$$a_2^{(1)} := A_{\widetilde{\alpha}}[(1+a_2^{(0)})(1+\alpha_x)^2]\,, \tag{2.30}$$

$$a_1^{(1)} := A_{\widetilde{\alpha}}[2(1+a_2^{(0)})\alpha_{xx} + a_1^{(0)}(1+\alpha_x) - \mathrm{i}\alpha_t]\,, \tag{2.31}$$

$$a_0^{(1)} := A_{\widetilde{\alpha}}\Big\{\frac{(1+a_2^{(0)})[2\alpha_{xxx}(1+\alpha_x) - \alpha_{xx}^2]}{4(1+\alpha_x)^2} + \frac{a_1^{(0)}\alpha_{xx} - \mathrm{i}\alpha_{tx}}{2(1+\alpha_x)} + a_0^{(0)}\Big\}\,, \tag{2.32}$$

$$b_0^{(1)} := A_{\widetilde{\alpha}}b_0^{(0)}\,. \tag{2.33}$$

Our purpose is to find $\alpha : [0,T]\times\mathbb{T}\to\mathbb{R}$ and a function $m_2 : [0,T]\to\mathbb{R}$ so that

$$a_2^{(1)}(t,y) = m_2(t)\,, \qquad \forall (t,y)\in[0,T]\times\mathbb{T}\,. \tag{2.34}$$

Thus, we have to solve

$$(1+a_2^{(0)})(1+\alpha_x)^2 = m_2\,. \tag{2.35}$$

Since $a_2^{(0)}$ is a real-valued function, the solutions are given by

$$m_2 := \Big(\frac{1}{2\pi}\int_{\mathbb{T}}\frac{dx}{(1+a_2^{(0)})^{\frac12}}\Big)^{-2}, \quad \alpha := \partial_x^{-1}\Big(m_2^{\frac12}(1+a_2^{(0)})^{-\frac12} - 1\Big)\,, \tag{2.36}$$

where $\partial_x^{-1}$ is the Fourier multiplier $\partial_x^{-1}e^{\mathrm{i}jx} = (1/\mathrm{i}j)e^{\mathrm{i}jx}$ for $j\in\mathbb{Z}$, $j\neq 0$, and $\partial_x^{-1}1 = 0$. Note that $m_2 : [0,T]\to\mathbb{R}$ is a real-valued function. The operator $\mathcal{L}_1$ in (2.29) has then the form

$$\mathcal{L}_1 = \partial_t\mathbb{I}_2 + \mathrm{i}m_2\Sigma\partial_{yy} + \mathrm{i}A_1^{(1)}\partial_y + \mathrm{i}A_0^{(1)}\,, \tag{2.37}$$

where $\Sigma$ is defined in (2.4).

**Lemma 2.3.** *There exists $\eta\in(0,1)$ small enough and $\sigma\in\mathbb{N}$ large enough, such that if $N_T(\sigma)\le\eta$ (see (2.6)), then, for any $0\le s\le S-\sigma$,*

$$\|m_2 - 1\|_{C_T^2} \lesssim \eta\,, \tag{2.38}$$

$$\|\alpha\|_{T,s}, \|\partial_t\alpha\|_{T,s}, \|\partial_{tt}\alpha\|_{T,s}, \|\widetilde{\alpha}\|_{T,s}, \|\partial_t\widetilde{\alpha}\|_{T,s}, \|\partial_{tt}\widetilde{\alpha}\|_{T,s} \lesssim N_T(s+\sigma)\,. \tag{2.39}$$

*The transformations $\mathcal{A}^{\pm1}$ map $C([0,T],H^s(\mathbb{T}))\to C([0,T],H^s(\mathbb{T}))$ and they satisfy the estimate*

$$\|\mathcal{A}^{\pm1}h\|_{T,s} \lesssim_s \|h\|_{T,s} + N_T(s+\sigma)\|h\|_{T,0}\,, \qquad \forall h\in C([0,T],H^s(\mathbb{T}))\,. \tag{2.40}$$

*The functions $a_1^{(1)}, a_0^{(1)}, b_0^{(1)}$ satisfy*

$$\|a_1^{(1)}\|_{T,s}, \|\partial_t a_1^{(1)}\|_{T,s}, \|a_0^{(1)}\|_{T,s}, \|b_0^{(1)}\|_{T,s} \lesssim_s N_T(s+\sigma)\,. \tag{2.41}$$

*Proof.* The Lemma follows by the explicit expressions of the coefficients, applying Lemmas 7.1, 7.5, 7.6. □

## 2.3 Reparametrization of time

In this subsection we remove also the dependence on time from the highest order (namely we conjugate the operator $\mathcal{L}_1$ in (2.37) to an operator where the coefficient of $\partial_{xx}$ is a constant matrix, independent of $(t,x)$, see (2.49)). We consider a diffeomorphism of the time interval $[0,T]$,

$$\beta : [0,T] \to [0,T] \,, \quad \beta(0) = 0 \,, \quad \beta(T) = T \tag{2.42}$$

with inverse $\beta^{-1}$. We define the operators $\mathcal{B}^{\pm 1}$ induced by the diffeomorphisms $\beta^{\pm 1}$ as

$$\mathcal{B}h(t,x) := h(\beta(t),x) \,, \qquad \mathcal{B}^{-1}h(\tau,x) := h(\beta^{-1}(\tau),x) \,. \tag{2.43}$$

The following conjugation rules hold:

$$\mathcal{B}^{-1} a \mathcal{B} = (\mathcal{B}^{-1}a) \,, \qquad \mathcal{B}^{-1}\partial_t \mathcal{B} = (\mathcal{B}^{-1}\beta')\partial_\tau \,, \quad \mathcal{B}^{-1}\partial_x^m \mathcal{B} = \partial_x^m \,, \quad m \in \mathbb{N}. \tag{2.44}$$

Conjugating the operator $\mathcal{L}_1$ in (2.37), we get

$$\mathcal{B}^{-1}\mathbb{I}_2\mathcal{L}_1\mathcal{B}\mathbb{I}_2 = (\mathcal{B}^{-1}\beta')\partial_\tau\mathbb{I}_2 + \mathrm{i}(\mathcal{B}^{-1}m_2)\Sigma\partial_{xx} + \mathrm{i}(\mathcal{B}^{-1}\mathbb{I}_2A_1^{(1)})\partial_x + \mathrm{i}(\mathcal{B}^{-1}\mathbb{I}_2A_0^{(1)}) \,. \tag{2.45}$$

Our aim is to choose $\beta$ so that the coefficients of $\partial_\tau\mathbb{I}_2$ and $\mathrm{i}\Sigma\partial_{xx}$ are proportional, namely we have to look for a diffeomorphism $\beta : [0,T] \to [0,T]$ and a constant $\mu \in \mathbb{R}$ such that

$$\beta'(t) = \frac{1}{\mu} m_2(t) \,, \qquad \forall t \in [0,T]. \tag{2.46}$$

Then, integrating in time from 0 to $T$, by (2.42) we fix the value of $\mu$ and define $\beta(t)$ as

$$\mu := \frac{1}{T}\int_0^T m_2(t)\,dt, \qquad \beta(t) := \frac{1}{\mu}\int_0^t m_2(s)\,ds \,. \tag{2.47}$$

Defining

$$\rho(\tau) := (\mathcal{B}^{-1}\beta')(\tau) = \mu^{-1}(\mathcal{B}^{-1}m_2)(\tau), \qquad \tau \in [0,T] \,, \tag{2.48}$$

we get

$$\mathcal{B}^{-1}\mathbb{I}_2\mathcal{L}_1\mathcal{B}\mathbb{I}_2 = \rho\mathcal{L}_2 \,, \quad \mathcal{L}_2 := \partial_\tau\mathbb{I}_2 + \mathrm{i}\mu\Sigma\partial_{yy} + \mathrm{i}A_1^{(2)}\partial_y + \mathrm{i}A_0^{(2)} \,, \tag{2.49}$$

$$A_1^{(2)} := \begin{pmatrix} a_1^{(2)} & 0 \\ 0 & -\overline{a}_1^{(2)} \end{pmatrix} \,, \quad A_0^{(2)} := \begin{pmatrix} a_0^{(2)} & b_0^{(2)} \\ -\overline{b}_0^{(2)} & -\overline{a}_0^{(2)} \end{pmatrix} \,, \tag{2.50}$$

$$a_1^{(2)} := \rho^{-1}(\mathcal{B}^{-1}a_1^{(1)}) \,, \quad a_0^{(2)} := \rho^{-1}(\mathcal{B}^{-1}a_0^{(1)}) \,, \quad b_0^{(2)} := \rho^{-1}(\mathcal{B}^{-1}b_0^{(1)}) \,. \tag{2.51}$$

Note that the vector field $L_2(t) := \mathrm{i}\mu\Sigma\partial_{yy} + \mathrm{i}A_1^{(2)}\partial_y + \mathrm{i}A_0^{(2)}$ is still Hamiltonian, since reparametrizations of time preserve the Hamiltonian structure. We also remark that, changing the time variable in the integral, one has

$$\int_0^T \langle \mathcal{B}\mathbf{u}(t), \mathbf{v}(t)\rangle_{\mathbf{L}^2}\,dt = \int_0^T \langle \mathbf{u}(\tau), \rho^{-1}(\tau)\mathcal{B}^{-1}\mathbf{v}(\tau)\rangle_{\mathbf{L}^2}\,d\tau \quad \forall \mathbf{u},\mathbf{v} \in \mathbf{L}^2, \tag{2.52}$$

namely the transpose of $\mathcal{B}$ with respect to the *time-space* scalar product $\int_0^T \langle \cdot,,\cdot\rangle_{\mathbf{L}^2}\,dt$ is

$$\mathcal{B}_* = \rho^{-1}\mathcal{B}^{-1}. \tag{2.53}$$

**Lemma 2.4.** *There exists $\eta \in (0,1)$ small enough, $\sigma \in \mathbb{N}$ large enough such that if $N_T(\sigma) \le \eta$, then for any $0 \le s \le S - \sigma$, the following holds:*

$$|\mu - 1| \,, \ \|\beta^{\pm 1} - 1\|_{C_T^3} \lesssim \eta \tag{2.54}$$

$$\|\mathcal{B}^{\pm 1}h\|_{T,s} \lesssim \|h\|_{T,s} \quad \forall h \in C([0,T], H^s(\mathbb{T})) \tag{2.55}$$

$$\|\rho^{\pm 1} - 1\|_{C_T^1} \lesssim \eta \tag{2.56}$$

$$\|a_1^{(2)}\|_{T,s}, \|\partial_t a_1^{(2)}\|_{T,s}, \|a_0^{(2)}\|_{T,s}, \|b_0^{(2)}\|_{T,s} \lesssim N_T(s+\sigma) \,. \tag{2.57}$$

*Proof.* Estimate (2.54) for $\mu$ and $\beta^{\pm 1}$ follows from definitions (2.47) and estimate (2.38) for $m_2$. Estimate (2.55) for $\mathcal{B}^{\pm 1}$ follows directly from definition (2.43), computing the norm $\|\cdot\|_{T,s}$. Estimates (2.56), (2.57) for $\rho^{\pm 1}$ follow by the explicit expressions (2.48), (2.50), applying Lemma 7.1 and estimates (2.38), (2.54), (2.55), (2.41), (2.55), (2.56). $\square$

## 2.4 Translation of the space variable

In this subsection we remove the space average from the order 1 coefficient $a_1^{(2)}$ (namely we conjugate the operator $\mathcal{L}_2$ in (2.49) to an operator where the coefficient in front of $\partial_x$ is a $2\times 2$ diagonal matrix whose entries are functions with zero space average, see (2.66), (2.61)). We consider the change of the space variable $z = y + p(\tau)$, where $p : [0,T] \to \mathbb{R}$, and define the operators

$$\mathcal{T}h(\tau, y) := h(\tau, y + p(\tau)), \qquad \mathcal{T}^{-1}h(\tau, z) = \mathcal{T}^* h(\tau, z) = h(\tau, z - p(\tau)). \tag{2.58}$$

A direct calculation shows that $\mathcal{T}$ is symplectic. Moreover, one has

$$\mathcal{T}^{-1}\partial_\tau \mathcal{T} = \partial_\tau + p'\partial_z, \qquad \mathcal{T}^{-1} a \mathcal{T} = (\mathcal{T}^{-1}a), \qquad \mathcal{T}^{-1}\partial_y^m \mathcal{T} = \partial_z^m, \quad m \in \mathbb{N}. \tag{2.59}$$

Then

$$\mathcal{L}_3 := \mathcal{T}^{-1}\mathbb{I}_2 \mathcal{L}_2 \mathcal{T}\mathbb{I}_2 = \partial_\tau \mathbb{I}_2 + \mathrm{i}\mu\Sigma\partial_{zz} + \mathrm{i}A_1^{(3)}\partial_z + \mathrm{i}A_0^{(3)} \tag{2.60}$$

with

$$A_1^{(3)} := \begin{pmatrix} a_1^{(3)} & 0 \\ 0 & -\overline{a}_1^{(3)} \end{pmatrix}, \qquad A_0^{(3)} := \begin{pmatrix} a_0^{(3)} & b_0^{(3)} \\ -\overline{b}_0^{(3)} & -\overline{a}_0^{(3)} \end{pmatrix}, \tag{2.61}$$

$$a_1^{(3)} := -\mathrm{i}p' + (\mathcal{T}^{-1}a_1^{(2)}), \qquad a_0^{(3)} := (\mathcal{T}^{-1}a_0^{(2)}), \qquad b_0^{(3)} := (\mathcal{T}^{-1}b_0^{(2)}). \tag{2.62}$$

Our aim is to choose the function $p$ so that

$$\int_{\mathbb{T}} a_1^{(3)}(\tau, z)\, dz = 0, \qquad \forall \tau \in [0,T]. \tag{2.63}$$

Performing the change of variable $y = z - p(\tau)$, the above equation becomes (multiplying by i)

$$2\pi p'(\tau) + \mathrm{i}\int_{\mathbb{T}} a_1^{(2)}(\tau, y)\, dy = 0. \tag{2.64}$$

By Lemma 6.2, we have that $a_1^{(2)} = 2(\partial_x \mu) - \overline{a}_1^{(2)} = -\overline{a}_1^{(2)}$ (recall that $\mu$ is a constant), implying that $a_1^{(2)} : [0,T] \times \mathbb{T} \to \mathrm{i}\mathbb{R}$, and then $\mathrm{i}a_1^{(2)} : [0,T]\times\mathbb{T} \to \mathbb{R}$. Hence we can solve equation (2.64) by setting

$$p(\tau) := -\frac{1}{2\pi}\int_0^\tau \int_{\mathbb{T}} \mathrm{i}a_1^{(2)}(\zeta, y)\, dy\, d\zeta, \qquad \tau \in [0,T] \tag{2.65}$$

and we get that $p : [0,T] \to \mathbb{R}$ is a real-valued function. Renaming the variables $\tau = t$, $z = x$ we have

$$\mathcal{L}_3 = \partial_t \mathbb{I}_2 + \mathrm{i}\mu\Sigma\partial_{xx} + \mathrm{i}A_1^{(3)}\partial_x + \mathrm{i}A_0^{(3)}, \qquad \int_{\mathbb{T}} a_1^{(3)}(t,x)\, dx = 0, \qquad \forall t \in [0,T]. \tag{2.66}$$

**Lemma 2.5.** *There exists $\eta \in (0,1)$ small enough and $\sigma \in \mathbb{N}$ large enough such that if $N_T(\sigma) \le \eta$, then for any $0 \le s \le S - \sigma$, the following estimates hold:*

$$\|p\|_{C_T^2} \lesssim \eta. \tag{2.67}$$

$$\tag{2.68}$$

*The transformations $\mathcal{T}^{\pm 1}$ map $C([0,T], H^s(\mathbb{T})) \to C([0,T], H^s(\mathbb{T}))$ and they satisfy*

$$\|\mathcal{T}^{\pm 1}h\|_{T,s} \lesssim \|h\|_{T,s} \quad \forall h \in C([0,T], H^s(\mathbb{T})),\ \forall s \ge 0. \tag{2.69}$$

*Furthermore*

$$\|a_1^{(3)}\|_{T,s}, \|\partial_t a_1^{(3)}\|_{T,s}, \|a_0^{(3)}\|_{T,s}, \|b_0^{(3)}\|_{T,s} \lesssim_s N_T(s+\sigma). \tag{2.70}$$

*Proof.* The lemma follows from definitions (2.58), (2.62), (2.65), applying Lemmas 7.1, 7.5, 7.6 and using estimates (2.57). □

## 2.5 Elimination of order one

In this last subsection, we remove completely the order 1 (namely we conjugate the operator $\mathcal{L}_3$ in (2.66) to an operator where the term $\partial_x$ is not present). We consider the multiplication operator by the matrix valued function

$$\mathcal{M} := \begin{pmatrix} v & 0 \\ 0 & \overline{v} \end{pmatrix}, \qquad v : [0,T] \times \mathbb{T} \to \mathbb{C}\,, \tag{2.71}$$

where $v$ is a function sufficiently close to 1, to be determined. The inverse $\mathcal{M}^{-1}$ and the adjoint $\mathcal{M}^*$ are

$$\mathcal{M}^{-1} = \begin{pmatrix} v^{-1} & 0 \\ 0 & \overline{v}^{-1} \end{pmatrix}, \qquad \mathcal{M}^* = \begin{pmatrix} \overline{v} & 0 \\ 0 & v \end{pmatrix} \tag{2.72}$$

We compute

$$\mathcal{L}_4 := \mathcal{M}^{-1}\mathcal{L}_3\mathcal{M} = \partial_t \mathbb{I}_2 + \mathrm{i}\mu\Sigma\partial_{xx} + \mathrm{i}A_1^{(4)}\partial_x + \mathrm{i}A_0^{(4)} \tag{2.73}$$

with

$$A_1^{(4)} := \begin{pmatrix} a_1^{(4)} & 0 \\ 0 & -\overline{a}_1^{(4)} \end{pmatrix}, \qquad A_0^{(4)} := \begin{pmatrix} a_0^{(4)} & b_0^{(4)} \\ -\overline{b}_0^{(4)} & -\overline{a}_0^{(4)} \end{pmatrix}, \tag{2.74}$$

$$a_1^{(4)} := a_1^{(3)} + 2\mu v^{-1} v_x\,, \qquad a_0^{(4)} := a_0^{(3)} + v^{-1}(\mu v_{xx} + a_1^{(3)} v_x - \mathrm{i} v_t)\,, \qquad b_0^{(4)} := b_0^{(3)}\,. \tag{2.75}$$

To remove the first order term we need to solve the equation

$$a_1^{(3)} + 2\mu v^{-1} v_x = 0\,. \tag{2.76}$$

We look for solutions of the form $v = \exp(q)$ and we get $a_1^{(3)} + 2\mu q_x = 0$, which, recalling (2.63), has the solution $q = -(2\mu)^{-1}\partial_x^{-1} a_1^{(3)}$. Hence we set

$$v := \exp\Big( - \frac{\partial_x^{-1} a_1^{(3)}}{2\mu} \Big)\,, \tag{2.77}$$

which solves (2.76) and gives

$$\mathcal{L}_4 = \partial_t \mathbb{I}_2 + \mathrm{i}\mu\Sigma\partial_{xx} + \mathcal{R}\,, \qquad \mathcal{R} := \mathrm{i}A_0^{(4)}\,. \tag{2.78}$$

We remark that, by the Hamiltonian structure, $a_1^{(3)} = -\overline{a}_1^{(3)}$, therefore

$$\overline{v} = \overline{\exp\Big( - \frac{\partial_x^{-1} a_1^{(3)}}{2\mu} \Big)} = \exp\Big( - \frac{\partial_x^{-1} \overline{a}_1^{(3)}}{2\mu} \Big) = \exp\Big( \frac{\partial_x^{-1} a_1^{(3)}}{2\mu} \Big) = v^{-1}\,.$$

Recalling (2.72) one gets

$$\mathcal{M}^{-1} = \mathcal{M}^*\,. \tag{2.79}$$

**Lemma 2.6.** *There exist $\eta \in (0,1)$ small enough, $\sigma \in \mathbb{N}$ large enough such that, if $N_T(\sigma) \le \eta$, for any $0 \le s \le S - \sigma$, the function $v$ defined in (2.77) satisfies the estimate*

$$\|v^{\pm 1} - 1\|_{T,s}, \|\partial_t v^{\pm 1}\|_{T,s} \lesssim_s N_T(s+\sigma)\,. \tag{2.80}$$

*As a consequence, the transformations $\mathcal{M}^{\pm 1}$ satisfy*

$$\|\mathcal{M}^{\pm 1}\mathbf{h}\|_{T,s} \lesssim_s \Big( \|\mathbf{h}\|_{T,s} + N_T(s+\sigma)\|\mathbf{h}\|_{T,0} \Big)\,, \quad \forall \mathbf{h} = (h, \overline{h}) \in C([0,T], \mathbf{H}_x^s)\,. \tag{2.81}$$

*The multiplication operator*

$$\mathcal{R} = \begin{pmatrix} \mathrm{i}a_0^{(4)} & \mathrm{i}b_0^{(4)} \\ -\mathrm{i}\overline{b}_0^{(4)} & -\mathrm{i}\overline{a}_0^{(4)} \end{pmatrix} := \begin{pmatrix} r_1 & r_2 \\ \overline{r}_1 & \overline{r}_2 \end{pmatrix} \tag{2.82}$$

*satisfies*

$$\|r_1\|_{T,s}, \|r_2\|_{T,s} \lesssim_s N_T(s+\sigma) \,. \tag{2.83}$$

*Proof.* The lemma follows by recalling definitions (2.71), (2.72), (2.77), (2.78), applying Lemma 7.1 and estimates (2.54), (2.70). □

# 3 Observability

In this section we prove the observability for linear operators $\mathcal{L}$ of the form (2.3). The proof is split in several lemmas.

**Lemma 3.1** (Ingham)**.** *Let $T > 0$. Then there exists a constant $C_1(T) > 0$ such that for any $\mu \geq \frac{1}{2}$ and for any $w = (w_j)_{j\in\mathbb{N}} \in \ell^2(\mathbb{N},\mathbb{C})$, one has*

$$\int_0^T \Big| \sum_{j\in\mathbb{N}} w_j e^{\mathrm{i}\mu j^2 t} \Big|^2 \, dt \geq C_1(T) \sum_{j\in\mathbb{N}} |w_j|^2 \,.$$

*Proof.* This result is classical. For a proof, see for instance Theorem 4.3 in Section 4.1 of [37]. To prove that the constant $C_1(T)$ does not depend on $\mu \in [\frac{1}{2}, +\infty)$ it is enough to follow the proof in [37] and use the lower bound $|\mu j^2 - \mu k^2| \geq \frac{1}{2}$ for all pairs of distinct nonnegative integers $j \neq k$. □

**Lemma 3.2** (Observability for $\partial_t + \mathrm{i}\mu\partial_{xx}$)**.** *Let $T > 0$ and $\omega \subset \mathbb{T}$ be a non-empty open set. Then there exists a constant $C_2 := C_2(T,\omega) > 0$ such that for any $\mu \geq \frac{1}{2}$, the following holds: for any $u_T \in L^2(\mathbb{T})$ the solution $u$ of the backward Cauchy problem*

$$\partial_t u + \mathrm{i}\mu\partial_{xx} u = 0 \,, \qquad u(T,\cdot) = u_T(\cdot) \tag{3.1}$$

*satisfies the estimate*

$$\int_0^T \int_\omega |u(t,x)|^2 \, dx \, dt \geq C_2 \|u_T\|_0^2 \,.$$

*Proof.* The proof of this result is standard. For instance, it can be deduced by adapting the proof of Proposition 6.5 in [6] to the present, simpler case. We give here the proof for completeness.

We fix an open interval $\omega_0 = (a,b) \subset \omega$. We choose $b-a$ smaller than a suitable universal constant, so that

$$\left| \frac{\sin(n(b-a))}{n} \right| = (b-a) \left| \frac{\sin(n(b-a))}{n(b-a)} \right| \leq (b-a) \frac{\sin(b-a)}{b-a} = \sin(b-a) \qquad \forall n \geq 1. \tag{3.2}$$

Let $u_T = \sum_{n\in\mathbb{Z}} w_n e^{\mathrm{i}nx}$, so that $\|u_T\|_{L^2_x}^2 = \sum_{n\in\mathbb{Z}} |w_n|^2$. We compute

$$u(t,x) = \sum_{n\in\mathbb{Z}} w_n e^{\mathrm{i}nx} e^{\mathrm{i}\mu n^2 (t-T)} = \sum_{n\in\mathbb{N}} z_n(x) e^{\mathrm{i}\mu n^2 t}$$

where

$$z_n(x) := \begin{cases} e^{-\mathrm{i}\mu n^2 T}(w_n e^{\mathrm{i}nx} + w_{-n} e^{-\mathrm{i}nx}) & \text{for } n \geq 1, \\ w_0 & \text{for } n = 0. \end{cases}$$

By Lemma 3.1 we get

$$\int_0^T \int_\omega |u(t,x)|^2 \, dx \, dt \geq C_1(T) \sum_{n\in\mathbb{N}} \int_{\omega_0} |z_n(x)|^2 \, dx.$$

It remains to prove that

$$\sum_{n\in\mathbb{N}}\int_{\omega_0}|z_n(x)|^2\,dx \ge C(\omega_0)\sum_{n\in\mathbb{Z}}|w_n|^2 \tag{3.3}$$

for some constant $C(\omega_0)$ depending only on $\omega_0$. We have

$$\int_{\omega_0}|z_0(x)|^2\,dx = (b-a)|w_0|^2. \tag{3.4}$$

For $n\ge 1$, we compute

$$\begin{aligned}\int_{\omega_0}|z_n(x)|^2\,dx &= \int_{\omega_0}\big(|w_n|^2+|w_{-n}|^2+w_n\bar{w}_{-n}e^{2\mathrm{i}nx}+\bar{w}_n w_{-n}e^{-2\mathrm{i}nx}\big)dx\\ &\ge (b-a)\{|w_n|^2+|w_{-n}|^2\}-|w_n||w_{-n}|\left(\left|\int_{\omega_0}e^{2\mathrm{i}nx}\,dx\right|+\left|\int_{\omega_0}e^{-2\mathrm{i}nx}\,dx\right|\right)\\ &= (b-a)\{|w_n|^2+|w_{-n}|^2\}-2|w_n||w_{-n}|\left|\frac{\sin(n(b-a))}{n}\right|\\ &\ge \left\{b-a-\left|\frac{\sin(n(b-a))}{n}\right|\right\}\big(|w_n|^2+|w_{-n}|^2\big).\end{aligned}$$

Finally, we use (3.2) and we deduce

$$\int_{\omega_0}|z_n(x)|^2\,dx \ge \big\{b-a-\sin(b-a)\big\}\big(|w_n|^2+|w_{-n}|^2\big). \tag{3.5}$$

Note that $b-a-\sin(b-a)>0$ is a constant depending only on $\omega_0$. Summing (3.5) over $n\in\mathbb{N}$ and adding (3.4), we get (3.3), which concludes the proof. □

**Lemma 3.3** (Observability for $\mathcal{L}_4=\partial_t\mathbb{I}_2+\mathrm{i}\mu\Sigma\partial_{xx}+\mathcal{R}$)**.** *Let $T>0$, $\omega\subset\mathbb{T}$ be a non-empty open set and $\mathcal{L}_4$ the operator defined in (2.73). Then there exist $\eta\in(0,1)$ small enough and $\sigma\in\mathbb{N}$ large enough such that if $N_T(\sigma)\le\eta$ then the following holds: let $\mathbf{u}_T\in\mathbf{L}^2(\mathbb{T})$ and let $\mathbf{u}(t,x)$ be the solution of the backward Cauchy problem*

$$\partial_t\mathbf{u}+\mathrm{i}\mu\Sigma\partial_{xx}\mathbf{u}+\mathcal{R}\mathbf{u}=0\,,\qquad \mathbf{u}(T,\cdot)=\mathbf{u}_T\,. \tag{3.6}$$

*Then there exists a constant $C_3:=C_3(T,\omega)>0$ (independent of $\mathbf{u}_T$) such that*

$$\int_0^T\int_\omega|\mathbf{u}(t,x)|^2\,dx\,dt\ge C_3\|\mathbf{u}_T\|_0^2\,.$$

*Proof.* Let $\mathbf{u}_1$ be the solution of

$$\partial_t\mathbf{u}_1+\mathrm{i}\mu\Sigma\partial_{xx}\mathbf{u}_1=0\,,\qquad \mathbf{u}_1(T,\cdot)=\mathbf{u}_T\,.$$

If $\mathbf{u}_1=(u_1,\overline{u}_1)$ and $\mathbf{u}_T=(u_T,\overline{u}_T)$, then $u_1$ solves (3.1). Therefore

$$\int_0^T\int_\omega|\mathbf{u}_1(t,x)|^2\,dx\,dt\ge C_2\|\mathbf{u}_T\|_0^2\qquad\text{and}\qquad\|\mathbf{u}_1\|_{T,0}=\|\mathbf{u}_T\|_0\,. \tag{3.7}$$

Then the function $\mathbf{u}_2:=\mathbf{u}-\mathbf{u}_1$ solves the Cauchy problem

$$\partial_t\mathbf{u}_2+\mathrm{i}\mu\Sigma\partial_{xx}\mathbf{u}_2+\mathcal{R}\mathbf{u}_2=-\mathcal{R}\mathbf{u}_1\,,\qquad \mathbf{u}_2(T,\cdot)=0\,.$$

By Lemma 8.2, (2.83), (3.7), since $N_T(\sigma)\le\eta$,

$$\|\mathbf{u}_2\|_{T,0}\lesssim\|\mathcal{R}\mathbf{u}_1\|_{T,0}\lesssim N_T(\sigma)\|\mathbf{u}_T\|_0\lesssim\eta\|\mathbf{u}_T\|_0\,. \tag{3.8}$$

Therefore, using the elementary inequality $(a+b)^2 \geq \frac{1}{2}a^2 - b^2$,

$$
\begin{aligned}
\int_0^T \int_\omega |\mathbf{u}(t,x)|^2 \, dx \, dt &\geq \frac{1}{2} \int_0^T \int_\omega |\mathbf{u}_1(t,x)|^2 \, dx \, dt - \int_0^T \int_\omega |\mathbf{u}_2(t,x)|^2 \, dx \, dt \\
&\overset{(3.7)}{\geq} \frac{C_2}{2} \|\mathbf{u}_T\|_0^2 - \int_0^T \int_{\mathbb{T}} |\mathbf{u}_2(t,x)|^2 \, dx \, dt \\
&\geq \frac{C_2}{2} \|\mathbf{u}_T\|_0^2 - T \|\mathbf{u}_2\|_{T,0}^2 \\
&\overset{(3.8)}{\geq} \frac{C_2}{2} \|\mathbf{u}_T\|_0^2 - T\eta^2 \|\mathbf{u}_T\|_0^2 \geq \frac{C_2}{4} \|\mathbf{u}_T\|_0^2
\end{aligned}
$$

by taking $\eta \in (0,1)$ small enough, then the claimed inequality holds by taking $C_3 := C_2/4$. □

**Lemma 3.4** (Observability for $\mathcal{L}_3 = \partial_t \mathbb{I}_2 + \mathrm{i}\mu\Sigma\partial_{xx} + \mathrm{i}A_1^{(3)}\partial_x + \mathrm{i}A_0^{(3)}$)**.** *Let $T > 0$, $\omega \subset \mathbb{T}$ be a non-empty open set and $\mathcal{L}_3$ be the operator defined in (2.66). Then there exist $\eta \in (0,1)$ small enough and $\sigma \in \mathbb{N}$ large enough such that if $N_T(\sigma) \leq \eta$ then the following holds: let $\mathbf{u}_T \in \mathbf{L}^2(\mathbb{T})$ and $\mathbf{u}(t,x)$ be the solution of the backward Cauchy problem*

$$
\partial_t \mathbf{u} + \mathrm{i}\mu\Sigma\partial_{xx}\mathbf{u} + \mathrm{i}A_1^{(3)}(t,x)\partial_x \mathbf{u} + \mathrm{i}A_0^{(3)}(t,x)\mathbf{u} = 0 \,, \qquad \mathbf{u}(T,\cdot) = \mathbf{u}_T \,. \tag{3.9}
$$

*Then there exists a constant $C_4 := C_4(T,\omega) > 0$ (independent of $\mathbf{u}_T$) such that*

$$
\int_0^T \int_\omega |\mathbf{u}(t,x)|^2 \, dx \, dt \geq C_4 \|\mathbf{u}_T\|_0^2 \,.
$$

*Proof.* Lemma 8.3 guarantees that if $\mathbf{u}_T \in \mathbf{L}^2(\mathbb{T})$, then the Cauchy problem (3.9) admits a unique solution $\mathbf{u} \in C([0,T], \mathbf{L}^2(\mathbb{T}))$. In Section 2.5, we have proved that the operator $\mathcal{L}_3$ in (2.66) is conjugated to the operator $\mathcal{L}_4$ in (2.73) by using the operator $\mathcal{M}$ defined in (2.71). Therefore $\mathbf{u}$ solves the Cauchy problem

$$
\mathcal{L}_3 \mathbf{u} = 0 \,, \qquad \mathbf{u}(T,\cdot) = \mathbf{u}_T
$$

if and only if $\widetilde{\mathbf{u}}(t,\cdot) := \mathcal{M}^{-1}(t)\mathbf{u}(t,\cdot)$ solves the Cauchy problem

$$
\mathcal{L}_4 \widetilde{\mathbf{u}} = 0 \,, \qquad \widetilde{\mathbf{u}}(T,\cdot) = \mathcal{M}^{-1}(T)\mathbf{u}_T \,.
$$

By Lemma 3.3 we get the inequality for $\widetilde{\mathbf{u}}$

$$
\int_0^T \int_\omega |\widetilde{\mathbf{u}}(t,x)|^2 \, dx \, dt \geq C_3 \|\widetilde{\mathbf{u}}_T\|_0^2 \,. \tag{3.10}
$$

By estimate (2.80) of Lemma 2.6, using that $C([0,T] \times \mathbb{T})$ is embedded into $C([0,T], H^1(\mathbb{T}))$ one has that, for some $\sigma \in \mathbb{N}$ large enough, the function $v(t,x)$, defined in (2.77) and determining the operator $\mathcal{M}$, satisfies

$$
\|v^{\pm 1} - 1\|_{L_T^\infty L_x^\infty} \lesssim N_T(\sigma) \lesssim \eta \,.
$$

Hence, for any function $\mathbf{h} = (h, \overline{h}) : [0,T] \times \mathbb{T} \to \mathbb{C}^2$, for $\eta$ small enough, we get for any $(t,x) \in [0,T] \times \mathbb{T}$

$$
|\mathcal{M}^{-1}(t)\mathbf{h}(t,x)| \leq (1 + \|v^{-1} - 1\|_{L_T^\infty L_x^\infty})|\mathbf{h}(t,x)| \leq (1 + C\eta)|\mathbf{h}(t,x)| \leq 2|\mathbf{h}(t,x)| \,, \tag{3.11}
$$

$$
|\mathcal{M}^{-1}(t)\mathbf{h}(t,x)| \geq |\mathbf{h}(t,x)| - \|v^{-1} - 1\|_{L_T^\infty L_x^\infty}|\mathbf{h}(t,x)| \geq (1 - C\eta)|\mathbf{h}(t,x)| \geq \frac{1}{2}|\mathbf{h}(t,x)| \,. \tag{3.12}
$$

Using that $\widetilde{\mathbf{u}}(t,x) = \mathcal{M}^{-1}(t)\mathbf{u}(t,x)$, the two inequalities above imply

$$
\int_0^T \int_\omega |\widetilde{\mathbf{u}}(t,x)|^2 \, dx \, dt \leq 4 \int_0^T \int_\omega |\mathbf{u}(t,x)|^2 \, dx \, dt \,, \quad \|\widetilde{\mathbf{u}}_T\|_0^2 \geq \frac{1}{4} \|\mathbf{u}_T\|_0^2 \,,
$$

and then the claimed inequality follows by (3.10) and by setting $C_4 := C_3/16$. □

**Lemma 3.5** (Observability for $\mathcal{L}_2 = \partial_t \mathbb{I}_2 + \mathrm{i}\mu\Sigma\partial_{xx} + \mathrm{i}A_1^{(2)}\partial_x + \mathrm{i}A_0^{(2)}$)**.** *Let $T > 0$, let $\omega \subset \mathbb{T}$ be a non-empty open set and $\mathcal{L}_2$ be the operator defined in* (2.49)*. Then there exist $\eta \in (0,1)$ small enough and $\sigma \in \mathbb{N}$ large enough such that if $N_T(\sigma) \leq \eta$ then the following holds: let $\mathbf{u}_T \in \mathbf{L}^2(\mathbb{T})$ and $\mathbf{u}(t,x)$ be the solution of the backward Cauchy problem*

$$\partial_t \mathbf{u} + \mathrm{i}\mu\Sigma\partial_{xx}\mathbf{u} + \mathrm{i}A_1^{(2)}(t,x)\partial_x\mathbf{u} + \mathrm{i}A_0^{(2)}(t,x)\mathbf{u} = 0\,, \qquad \mathbf{u}(T,\cdot) = \mathbf{u}_T\,. \tag{3.13}$$

*Then there exists a constant $C_5 := C_5(T,\omega) > 0$ (independent of $\mathbf{u}_T$) such that*

$$\int_0^T \int_\omega |\mathbf{u}(t,x)|^2\, dx\, dt \geq C_5 \|\mathbf{u}_T\|_0^2\,.$$

*Proof.* Lemma 8.4 guarantees that if $\mathbf{u}_T \in \mathbf{L}^2(\mathbb{T})$ then there exists a unique solution $\mathbf{u} \in C([0,T], \mathbf{L}^2(\mathbb{T}))$ of the Cauchy problem (3.13). In Section 2.4, we have proved that the transformation $\mathcal{T}$ defined in (2.58) conjugates the operator $\mathcal{P}_4$ defined in (2.49) to the operator $\mathcal{P}_5$ given in (2.66), hence $\mathbf{u}$ solves the Cauchy problem

$$\mathcal{L}_2\mathbf{u} = 0\,, \qquad \mathbf{u}(T,\cdot) = \mathbf{u}_T$$

if and only if $\widetilde{\mathbf{u}}(t,x) := \mathcal{T}^{-1}(t)\mathbf{u}(t,x)$ solves the Cauchy problem

$$\mathcal{L}_3\widetilde{\mathbf{u}} = 0\,, \qquad \widetilde{\mathbf{u}}(T,\cdot) = \mathcal{T}^{-1}(T)\mathbf{u}_T\,.$$

Then by Lemma 3.4, applied to a time interval $\omega_1 := (\alpha_1, \beta_1) \subset \omega$, the function $\widetilde{\mathbf{u}}$ satisfies the property

$$\int_0^T \int_{\omega_1} |\widetilde{\mathbf{u}}(t,x)|^2\, dx\, dt \geq C_4(T,\omega_1)\|\widetilde{\mathbf{u}}_T\|_0^2\,. \tag{3.14}$$

Performing the change of variables $y = x - p(T)$ (where $p(t)$, defined in (2.65), is the function determining the operator $\mathcal{T}$), one has

$$\|\widetilde{\mathbf{u}}_T\|_0^2 = \int_{\mathbb{T}} |\mathbf{u}_T(x - p(T))|^2\, dx = \int_{\mathbb{T}} |\mathbf{u}_T(y)|^2\, dy = \|\mathbf{u}_T\|_0^2\,. \tag{3.15}$$

By the change of variables $y = x - p(t)$,

$$\int_0^T \int_{\omega_1} |\widetilde{\mathbf{u}}(t,x)|^2\, dx\, dt = \int_0^T \int_{\omega_1} |\mathbf{u}(t, x - p(t))|^2\, dx\, dt = \int_0^T \int_{\alpha_1 - p(t)}^{\beta_1 - p(t)} |\mathbf{u}(t,y)|^2\, dy\, dt\,. \tag{3.16}$$

By estimate (2.67), for all $t \in [0,T]$, $[\alpha_1 - p(t), \beta_1 - p(t)] \subseteq [\alpha_1 - C\eta, \beta_1 + C\eta] \subset \omega$ if $\eta$ is small enough. Therefore, by (3.16),

$$\int_0^T \int_{\omega_1} |\widetilde{\mathbf{u}}(t,x)|^2\, dx\, dt \leq \int_0^T \int_\omega |\mathbf{u}(t,x)|^2\, dx\, dt\,. \tag{3.17}$$

The claimed inequality follows by (3.14), (3.15), (3.17), with $C_5 := C_4(T,\omega_1)$. □

**Lemma 3.6** (Observability for $\mathcal{L}_1 = \partial_t \mathbb{I}_2 + \mathrm{i}m_2\Sigma\partial_{yy} + \mathrm{i}A_1^{(1)}\partial_y + \mathrm{i}A_0^{(1)}$)**.** *Let $T > 0$, $\omega \subset \mathbb{T}$ be a non-empty open set and $\mathcal{L}_1$ be the operator defined in* (2.37)*. Then there exist $\eta \in (0,1)$ small enough and $\sigma \in \mathbb{N}$ large enough such that if $N_T(\sigma) \leq \eta$ then the following holds: let $\mathbf{u}_T \in \mathbf{L}^2(\mathbb{T})$ and $\mathbf{u}(t,x)$ be the solution of the backward Cauchy problem*

$$\partial_t \mathbf{u} + \mathrm{i}m_2(t)\Sigma\partial_{xx}\mathbf{u} + \mathrm{i}A_1^{(1)}(t,x)\partial_x\mathbf{u} + \mathrm{i}A_0^{(1)}(t,x)\mathbf{u} = 0\,, \qquad \mathbf{u}(T,\cdot) = \mathbf{u}_T(\cdot)\,. \tag{3.18}$$

*Then there exists a constant $C_6 := C_6(T,\omega) > 0$ (independent of $\mathbf{u}_T$) such that*

$$\int_0^T \int_\omega |\mathbf{u}(t,x)|^2\, dx\, dt \geq C_6 \|\mathbf{u}_T\|_0^2\,.$$

*Proof.* Lemma 8.5 guarantees that if $\mathbf{u}_T \in \mathbf{L}^2(\mathbb{T})$ then there exists a unique solution $\mathbf{u} \in C([0,T], \mathbf{L}^2(\mathbb{T}))$ of the Cauchy problem (3.18). In Section 2.3, we have proved that the transformation $\mathcal{B}$ defined in (2.43) conjugates the operator $\mathcal{L}_1$ defined in (2.37) to the operator $\rho\mathcal{L}_2$ where the function $\rho$ is defined by (2.48) and the operator $\mathcal{L}_2$ is given in (2.49). Hence $\mathbf{u}$ solves the Cauchy problem

$$\mathcal{L}_1\mathbf{u} = 0\,, \qquad \mathbf{u}(T,\cdot) = \mathbf{u}_T$$

if and only if $\widetilde{\mathbf{u}}(t,x) := \mathcal{B}^{-1}\mathbf{u}(t,x)$ solves

$$\mathcal{L}_2\widetilde{\mathbf{u}} = 0\,, \qquad \widetilde{\mathbf{u}}(T,\cdot) = \mathbf{u}_T$$

(we use that $\mathcal{B}^{-1}\mathbf{u}_T = \mathbf{u}_T$ since $\mathcal{B}$ acts only in time). Then, by Lemma 3.5, the function $\widetilde{\mathbf{u}}$ satisfies

$$\int_0^T \int_\omega |\widetilde{\mathbf{u}}(t,x)|^2\, dx\, dt \geq C_5 \|\mathbf{u}_T\|_0^2\,. \tag{3.19}$$

Performing the change of the time variable $\tau = \beta^{-1}(t)$ (recall (2.42)), we get for $\eta$ small enough

$$\begin{aligned}\int_0^T \int_\omega |\widetilde{\mathbf{u}}(t,x)|^2\, dx\, dt &= \int_0^T \int_\omega |\mathbf{u}(\beta^{-1}(t),x)|^2\, dx\, dt = \int_0^T \int_\omega |\mathbf{u}(\tau,x)|^2 \beta'(\tau)\, dx\, d\tau \\ &\overset{(2.54)}{\leq} (1+C\eta)\int_0^T \int_\omega |\mathbf{u}(\tau,x)|^2\, dx\, d\tau \leq 2\int_0^T \int_\omega |\mathbf{u}(\tau,x)|^2\, dx\, d\tau\,. \end{aligned} \tag{3.20}$$

The claimed inequality follows by (3.19), (3.20) and setting $C_6 := C_5/2$. □

**Lemma 3.7** (Observability for $\mathcal{L}_0 = \partial_t \mathbb{I}_2 + \mathrm{i}(\Sigma + A_2^{(0)})\partial_{xx} + \mathrm{i}A_1^{(0)}\partial_x + \mathrm{i}A_0^{(0)}$)**.** *Let $T > 0$, let $\omega \subset \mathbb{T}$ be a non-empty open set and $\mathcal{L}_0$ be the operator defined in* (2.13)*. Then there exist $\eta \in (0,1)$ small enough and $\sigma \in \mathbb{N}$ large enough such that if $N_T(\sigma) \leq \eta$ then the following holds: let $\mathbf{u}_T \in \mathbf{L}^2(\mathbb{T})$ and $\mathbf{u}(t,x)$ be the solution of the backward Cauchy problem*

$$\partial_t \mathbf{u} + \mathrm{i}(\Sigma + A_2^{(0)})\partial_{xx}\mathbf{u} + \mathrm{i}A_1^{(0)}\partial_x \mathbf{u} + \mathrm{i}A_0^{(0)}\mathbf{u} = 0\,, \qquad \mathbf{u}(T,\cdot) = \mathbf{u}_T\,. \tag{3.21}$$

*Then there exists a constant $C_7 := C_7(T,\omega) > 0$ (independent of $\mathbf{u}_T$) such that*

$$\int_0^T \int_\omega |\mathbf{u}(t,x)|^2\, dx\, dt \geq C_7 \|\mathbf{u}_T\|_0^2\,.$$

*Proof.* Lemma 8.6 guarantees that if $\mathbf{u}_T \in \mathbf{L}^2(\mathbb{T})$ then there exists a unique solution $\mathbf{u} \in C([0,T], \mathbf{L}^2(\mathbb{T}))$ of the Cauchy problem (3.21). In Section 2.2, we have proved that the transformation $\mathcal{A}$ defined in (2.23) conjugates the operator $\mathcal{L}_0$ defined in (2.13) to the operator $\mathcal{L}_1$ defined in (2.37). Hence $\mathbf{u}$ solves the Cauchy problem

$$\mathcal{L}_0\mathbf{u} = 0\,, \qquad \mathbf{u}(T,\cdot) = \mathbf{u}_T$$

if and only if $\widetilde{\mathbf{u}}(t,x) := \mathcal{A}^{-1}\mathbf{u}(t,x)$ solves $\mathcal{L}_1\widetilde{\mathbf{u}} = 0$, $\widetilde{\mathbf{u}}(T,\cdot) = \mathcal{A}^{-1}(T)\mathbf{u}_T$. Applying Lemma 3.6 to the time interval $\omega_1 := (\alpha_1, \beta_1) \subset \omega$ one gets

$$\int_0^T \int_{\omega_1} |\widetilde{\mathbf{u}}(t,x)|^2\, dx\, dt \geq C_6(T,\omega_1) \|\widetilde{\mathbf{u}}_T\|_0^2\,. \tag{3.22}$$

Recalling (2.24), (2.25) and performing the change of variable $x = y + \widetilde{\alpha}(T,y)$, one has

$$\begin{aligned}\|\widetilde{\mathbf{u}}_T\|_0^2 &= \int_{\mathbb{T}} (1 + \widetilde{\alpha}_y(T,y)) |\mathbf{u}_T(y + \widetilde{\alpha}(T,y))|^2\, dy \\ &= \int_{\mathbb{T}} \Big(1 + \widetilde{\alpha}_y(T, x + \alpha(T,x))\Big)\Big(1 + \alpha_x(T,x)\Big) |\mathbf{u}_T(x)|^2\, dx = \int_{\mathbb{T}} |\mathbf{u}_T(x)|^2\, dx = \|\mathbf{u}_T\|_0^2\,. \end{aligned} \tag{3.23}$$

By (2.39) (applied with $s_0 \geq 1$), and using the standard Sobolev embedding, we get that for some $\sigma \in \mathbb{N}$ large enough

$$\|\widetilde{\alpha}\|_{L^\infty_T L^\infty_x} \lesssim N_T(\sigma) \lesssim \eta.$$

Hence, for some constant $C > 0$,

$$\big\{(t, y + \tilde{\alpha}(t,y)) : t \in [0,T]\,,\ y \in \omega_1\big\} \subset [0,T] \times [\alpha_1 - C\eta, \beta_1 + C\eta] \subset [0,T] \times \omega$$

for $\eta \in (0,1)$ small enough. Then, using the change of variables $x = y + \widetilde{\alpha}(t,y)$ and (2.24),

$$\begin{aligned}\int_0^T \int_{\omega_1} |\widetilde{\mathbf{u}}(t,y)|^2 \, dy \, dt &= \int_0^T \int_{\omega_1} (1 + \widetilde{\alpha}_y(t,y)) |\mathbf{u}(t, y + \widetilde{\alpha}(t,y))|^2 \, dy \, dt \\ &\leq \int_0^T \int_{\omega} |\mathbf{u}(t,x)|^2 \, dx \, dt\,. \end{aligned} \tag{3.24}$$

The claimed inequality follows by (3.22), (3.23), (3.24) by choosing $C_7 := C_6(T, \omega_1)$. ☐

**Lemma 3.8** (Observability for $\mathcal{L} = \partial_t \mathbb{I}_2 + \mathrm{i}(\Sigma + A_2)\partial_{xx} + \mathrm{i}A_1\partial_x + \mathrm{i}A_0$)**.** *Let $T > 0$, let $\omega \subset \mathbb{T}$ be a non-empty open set and let $\mathcal{L}$ be the operator defined in (2.3). Then there exist $\eta \in (0,1)$ small enough and $\sigma \in \mathbb{N}$ large enough such that if $N_T(\sigma) \leq \eta$ then the following holds: let $\mathbf{u}_T \in \mathbf{L}^2(\mathbb{T})$ and $\mathbf{u}(t,x)$ be the solution of the backward Cauchy problem*

$$\partial_t \mathbf{u} + \mathrm{i}(\Sigma + A_2)\partial_{xx}\mathbf{u} + \mathrm{i}A_1\partial_x \mathbf{u} + \mathrm{i}A_0 \mathbf{u} = 0\,, \qquad \mathbf{u}(T,\cdot) = \mathbf{u}_T(\cdot)\,. \tag{3.25}$$

*Then there exists a constant $C_8 := C_8(T,\omega) > 0$ (independent of $\mathbf{u}_T$) such that*

$$\int_0^T \int_\omega |\mathbf{u}(t,x)|^2 \, dx \, dt \geq C_8 \|\mathbf{u}_T\|_0^2\,.$$

*Proof.* Lemma 8.7 guarantees that if $\mathbf{u}_T \in \mathbf{L}^2(\mathbb{T})$ then there exists a unique solution $\mathbf{u} \in C([0,T], \mathbf{L}^2(\mathbb{T}))$ of the Cauchy problem (3.25). In Section 2.1 we have proved that the transformation $\mathcal{S}$ defined in (2.10) conjugates the operator $\mathcal{L}$ defined in (2.3) to the operator $\mathcal{L}_0$ defined in (2.13). Hence $\mathbf{u}$ solves the Cauchy problem

$$\mathcal{L}\mathbf{u} = 0\,, \qquad \mathbf{u}(T,\cdot) = \mathbf{u}_T$$

if and only if $\widetilde{\mathbf{u}}(t,x) := \mathcal{S}^{-1}(t)\mathbf{u}(t,x)$ solves $\mathcal{L}_0 \widetilde{\mathbf{u}} = 0$, $\widetilde{\mathbf{u}}(T,\cdot) = \mathcal{S}^{-1}(T)\mathbf{u}_T$. By Lemma 3.7,

$$\int_0^T \int_\omega |\widetilde{\mathbf{u}}(t,x)|^2 \, dx \, dt \geq C_7 \|\widetilde{\mathbf{u}}_T\|_0^2\,. \tag{3.26}$$

Applying (2.19) and the ansatz (2.7), together with Sobolev embeddings, there exists $\sigma \in \mathbb{N}$ large enough such that

$$\|\mathcal{S}^{\pm 1} - \mathbb{I}_2\|_{L^\infty} \lesssim N_T(\sigma) \leq C\eta\,, \quad \|\mathcal{S}^{\pm 1}\|_{L^\infty} \leq 2 \tag{3.27}$$

for $\eta \in (0,1)$ small enough. Therefore, recalling (2.25) and performing the change of variable $x = y + \widetilde{\alpha}(T,y)$, provided that $\eta$ is small enough, one has

$$\begin{aligned}\|\widetilde{\mathbf{u}}_T\|_0^2 &= \int_{\mathbb{T}} |\mathcal{S}^{-1}(T,x)\mathbf{u}_T(x)|^2 \, dx \\ &\overset{(3.27)}{\geq} (1 - C^2\eta^2) \int_{\mathbb{T}} |\mathbf{u}_T(x)|^2 \, dx \geq \frac{1}{2}\|\mathbf{u}_T\|_0^2\,. \end{aligned} \tag{3.28}$$

Moreover, using again (3.27),

$$\int_0^T \int_\omega |\widetilde{\mathbf{u}}(t,y)|^2 \, dy \, dt = \int_0^T \int_\omega |\mathcal{S}^{-1}(t,x)\mathbf{u}(t,x)|^2 \, dy \, dt \leq 2 \int_0^T \int_\omega |\mathbf{u}(t,x)|^2 \, dx \, dt\,. \tag{3.29}$$

The claimed inequality follows by (3.26), (3.28), (3.29) and taking $C_8 := C_7/4$. ☐

# 4 Controllability

In this Section we prove the controllability of linear operators $\mathcal{L}$ of the form (2.3), namely

$$\mathcal{L} = \partial_t \mathbb{I}_2 + \mathrm{i}(\Sigma + A_2)\partial_{xx} + \mathrm{i}A_1\partial_x + \mathrm{i}A_0$$

where the vector field $L(t) = -\mathrm{i}\big((\Sigma + A_2)\partial_{xx} + A_1\partial_x + A_0\big)$ is Hamiltonian and $A_2, A_1, A_0$ satisfy hypotheses (2.4)-(2.7). We define the operator $\mathcal{L}^*$ as

$$\mathcal{L}^* := -\partial_t \mathbb{I}_2 - \mathrm{i}(\Sigma + [A_2]^*)\partial_{xx} - \mathrm{i}\widetilde{A}_1\partial_x - \mathrm{i}\widetilde{A}_0\,, \tag{4.1}$$

where

$$\widetilde{A}_1 := 2\partial_x[A_2]^* - [A_1]^*\,, \quad \widetilde{A}_0 := \partial_{xx}[A_2]^* + \partial_x[A_1]^*\,. \tag{4.2}$$

We point out that by Lemma 6.3, the time-dependent vector field $L_2^*(t) := -\mathrm{i}[A_2]^*\partial_{xx} - \mathrm{i}\widetilde{A}_1\partial_x - \mathrm{i}\widetilde{A}_0$ is still a Hamiltonian operator. Note that

$$\max\{\|\widetilde{A}_1\|_{T,s_0-1}, \|\partial_t\widetilde{A}_1\|_{T,s_0-1}, \|\widetilde{A}_0\|_{T,s_0-2}\} \lesssim N_T(s_0)\,,$$

so that the operator $\mathcal{L}^*$ satisfies the same hyphotheses as $\mathcal{L}$ and the reduction procedure of Section 2 can be applied also to $\mathcal{L}^*$.

**Lemma 4.1.** *Let $T > 0$, let $\omega \subset \mathbb{T}$ be an open set. Let $\mathcal{L}^*$ be the operator defined by* (4.1)*. There exists $\eta \in (0,1)$ small enough and $\sigma \in \mathbb{N}$ large enough such that, if $N_T(\sigma) \le \eta$, then for any $\mathbf{h}_{in}, \mathbf{h}_{end} \in \mathbf{L}^2(\mathbb{T})$, $\mathbf{q} \in C([0,T], \mathbf{L}^2(\mathbb{T}))$ there exists a unique function $\mathbf{f} \in C([0,T], \mathbf{L}^2(\mathbb{T}))$ that solves $\mathcal{L}^*\mathbf{f} = 0$ such that the only solution $\mathbf{h} \in C([0,T], \mathbf{L}^2(\mathbb{T}))$ of the Cauchy problem*

$$\begin{cases} \mathcal{L}\mathbf{h} = \chi_\omega \mathbf{f} + \mathbf{q} \\ \mathbf{h}(0,\cdot) = \mathbf{h}_{in} \end{cases} \tag{4.3}$$

*satisfies $\mathbf{h}(T,\cdot) = \mathbf{h}_{end}$. Furthermore*

$$\|\mathbf{f}\|_{T,0} \lesssim \|\mathbf{h}_{in}\|_0 + \|\mathbf{h}_{end}\|_0 + \|\mathbf{q}\|_{T,0}\,.$$

*Proof. (Existence).* For any $\mathbf{f}_1, \mathbf{g}_1 \in \mathbf{L}^2(\mathbb{T})$, applying Lemma 8.7, we consider the unique solutions $\mathbf{f}, \mathbf{g} \in C([0,T], \mathbf{L}^2(\mathbb{T}))$ of the Cauchy problems

$$\begin{cases} \mathcal{L}^*\mathbf{f} = 0 \\ \mathbf{f}(T,\cdot) = \mathbf{f}_1\,, \end{cases} \qquad \begin{cases} \mathcal{L}^*\mathbf{g} = 0 \\ \mathbf{g}(T,\cdot) = \mathbf{g}_1 \end{cases} \tag{4.4}$$

and we define the bilinear form

$$B(\mathbf{f}_1, \mathbf{g}_1) := \int_0^T \langle \chi_\omega \mathbf{f}, \mathbf{g}\rangle_{\mathbf{L}^2} dt$$

and the linear form

$$\Lambda(\mathbf{g}_1) := \langle \mathbf{h}_{end}, \mathbf{g}(T,\cdot)\rangle_{\mathbf{L}^2} - \langle \mathbf{h}_{in}, \mathbf{g}(0,\cdot)\rangle_{\mathbf{L}^2} - \int_0^T \langle \mathbf{q}(t,\cdot), \mathbf{g}(t,\cdot)\rangle_{\mathbf{L}^2}\, dt\,,$$

where the real scalar product $\langle\cdot,\cdot\rangle_{\mathbf{L}^2}$ is defined in (1.38). By (4.4) and Lemma 8.7 we have

$$|B(\mathbf{f}_1, \mathbf{g}_1)| \lesssim \|\mathbf{f}_1\|_0 \|\mathbf{g}_1\|_0\,, \quad |\Lambda(\mathbf{g}_1)| \lesssim (\|\mathbf{h}_{in}\|_0 + \|\mathbf{h}_{end}\|_0 + \|\mathbf{q}\|_{T,0})\|\mathbf{g}_1\|_0\,.$$

By Lemma 3.8, the bilinear form $B$ is coercive and therefore, by Riesz representation theorem (or Lax-Milgram lemma), there exists a unique $\mathbf{f}_1 \in \mathbf{L}^2(\mathbb{T})$ such that

$$B(\mathbf{f}_1, \mathbf{g}_1) = \Lambda(\mathbf{g}_1) \quad \forall \mathbf{g}_1 \in \mathbf{L}^2(\mathbb{T}), \tag{4.5}$$

satisfying $\|\mathbf{f}_1\|_0 \lesssim \|\Lambda\|_{\mathcal{L}(\mathbf{L}^2,\mathbb{C})} \lesssim \|\mathbf{h}_{in}\|_0 + \|\mathbf{h}_{end}\|_0 + \|\mathbf{q}\|_{T,0}$. Now let $\mathbf{f}_1$ be the only solution of (4.5) and let $\mathbf{h}$ be the solution of the Cauchy problem (4.3) (whose existence follows by Lemma 8.7). We have

$$
\begin{aligned}
0 &= B(\mathbf{f}_1, \mathbf{g}_1) - \Lambda(\mathbf{g}_1) \\
&= \int_0^T \langle \chi_\omega \mathbf{f}, \mathbf{g}\rangle_{\mathbf{L}^2} dt - \langle \mathbf{h}_{end}, \mathbf{g}(T,\cdot)\rangle_{\mathbf{L}^2} + \langle \mathbf{h}_{in}, \mathbf{g}(0,\cdot)\rangle_{\mathbf{L}^2} + \int_0^T \langle \mathbf{q}(t,\cdot), \mathbf{g}(t,\cdot)\rangle_{\mathbf{L}^2}\, dt \\
&\overset{(4.3)}{=} \int_0^T \langle \mathcal{L}\mathbf{h}, \mathbf{g}\rangle_{\mathbf{L}^2}\, dt - \langle \mathbf{h}_{end}, \mathbf{g}(T,\cdot)\rangle_{\mathbf{L}^2} + \langle \mathbf{h}_{in}, \mathbf{g}(0,\cdot)\rangle_{\mathbf{L}^2} \\
&= \int_0^T \langle \mathbf{u}, \mathcal{L}^*\mathbf{g}\rangle_{\mathbf{L}^2}\, dt + \langle \mathbf{h}(T,\cdot), \mathbf{g}(T,\cdot)\rangle_{\mathbf{L}^2} - \langle \mathbf{h}(0,\cdot), \mathbf{g}(0,\cdot)\rangle_{\mathbf{L}^2} - \langle \mathbf{h}_{end}, \mathbf{g}(T,\cdot)\rangle_{\mathbf{L}^2} + \langle \mathbf{h}_{in}, \mathbf{g}(0,\cdot)\rangle_{\mathbf{L}^2} \\
&\overset{(4.4)}{=} \langle \mathbf{h}(T,\cdot) - \mathbf{h}_{end}, \mathbf{g}_1\rangle_{\mathbf{L}^2}\,.
\end{aligned}
$$

Then for any $\mathbf{g}_1 \in \mathbf{L}^2(\mathbb{T})$ we have that $\langle \mathbf{h}(T,\cdot) - \mathbf{h}_{end}, \mathbf{g}_1\rangle_{\mathbf{L}^2} = 0$, implying that $\mathbf{h}(T,\cdot) = \mathbf{h}_{end}$ and then the lemma follows.

*(Uniqueness).* Assume that $\widetilde{\mathbf{f}} \in C([0,T], \mathbf{L}^2(\mathbb{T}))$ satisfies $\mathcal{L}^*\widetilde{\mathbf{f}} = 0$, and that the solution $\mathbf{h}$ of the Cauchy problem $\mathcal{L}\mathbf{h} = \chi_\omega \widetilde{\mathbf{f}} + \mathbf{q}$, $\mathbf{h}(0,\cdot) = \mathbf{h}_{in}$ satisfies $\mathbf{h}(T,\cdot) = \mathbf{h}_{end}$. Setting $\widetilde{\mathbf{f}}_1 := \widetilde{\mathbf{f}}(T,\cdot)$ and arguing as above, one sees that $B(\widetilde{\mathbf{f}}_1, \mathbf{g}_1) = \Lambda(\mathbf{g}_1)$ for all $\mathbf{g}_1 \in \mathbf{L}^2(\mathbb{T})$, and then, by uniqueness of the solution $\mathbf{f}_1$ of (4.5), we deduce $\widetilde{\mathbf{f}}_1 = \mathbf{f}_1$. □

**Lemma 4.2** (Higher regularity)**.** *Assume the hypotheses of Lemma 4.1, and $N_T(\sigma+2) \le 1$. Let $s \in [0, S-\sigma-1]$, and assume that $N_T(s+1+\sigma) < \infty$. If $\mathbf{h}_{in}, \mathbf{h}_{end} \in \mathbf{H}^s(\mathbb{T})$, $\mathbf{q} \in C([0,T], \mathbf{H}^s(\mathbb{T}))$, then $\mathbf{h}, \mathbf{f} \in C([0,T], \mathbf{H}^s(\mathbb{T}))$ and*

$$
\|\mathbf{f}\|_{T,s}, \|\mathbf{h}\|_{T,s} \lesssim_s \|\phi\|_{T,s} + N_T(s+\sigma)\|\phi\|_{T,0}\,, \qquad \phi := (\mathbf{q}, \mathbf{h}_{in}, \mathbf{h}_{end})\,.
$$

*Furthermore, if $\mathbf{h}_{in}, \mathbf{h}_{end} \in \mathbf{H}^{s+4}(\mathbb{T})$, $\mathbf{q} \in C\big([0,T], \mathbf{H}^{s+4}(\mathbb{T})\big) \cap C^1([0,T], \mathbf{H}^s(\mathbb{T}))$, then*

$$
\mathbf{h}, \mathbf{f} \in C([0,T], \mathbf{H}^{s+4}(\mathbb{T})) \cap C^1([0,T], \mathbf{H}^{s+2}(\mathbb{T})) \cap C^2([0,T], \mathbf{H}^s(\mathbb{T}))\,,
$$

*and*

$$
\|\mathbf{h}, \mathbf{f}\|_{T,s+4}, \|\partial_t \mathbf{h}, \partial_t \mathbf{f}\|_{T,s+2}, \|\partial_{tt}\mathbf{h}, \partial_{tt}\mathbf{f}\|_{T,s} \lesssim_s \|\phi\|_{T,s+4} + \|\partial_t \mathbf{q}\|_{T,s} + N_T(s+\sigma)\|\phi\|_{T,4}\,. \tag{4.6}
$$

*Proof.* Assume that $\mathbf{h}, \mathbf{f} \in C([0,T], \mathbf{L}^2(\mathbb{T}))$ are the solutions of

$$
\begin{cases}
\mathcal{L}\mathbf{h} = \chi_\omega \mathbf{f} + \mathbf{q} \\
\mathbf{h}(0,\cdot) = \mathbf{h}_{in} \\
\mathbf{h}(T,\cdot) = \mathbf{h}_{end}\,,
\end{cases}
\qquad \mathcal{L}^*\mathbf{f} = 0\,. \tag{4.7}
$$

By the results of Section 2, one has that

$$
\mathcal{L} = \Phi \mathcal{L}_4 \Psi, \quad \Phi := \mathcal{S}(\mathcal{A}\mathbb{I}_2)(\mathcal{B}\mathbb{I}_2)\rho(\mathcal{T}\mathbb{I}_2)\mathcal{M}, \quad \Psi := \mathcal{M}^{-1}(\mathcal{T}^{-1}\mathbb{I}_2)(\mathcal{B}^{-1}\mathbb{I}_2)(\mathcal{A}^{-1}\mathbb{I}_2)\mathcal{S}^{-1}, \tag{4.8}
$$

with $\mathcal{L}_4 = \partial_t \mathbb{I}_2 + \mathrm{i}\mu\Sigma\partial_{xx} + \mathcal{R}$, and $\mathcal{R} \in C([0,T], \mathbf{H}^s(\mathbb{T}))$ is the multilplication operator given by (2.82). We define the *adjoint operator*

$$
\mathcal{L}_4^* := -\partial_t - \mathrm{i}\mu\Sigma\partial_{xx} + \mathcal{R}^*\,,
$$

where $\mathcal{R}^*$ is the adjoint of the multiplication operator $\mathcal{R}$ with respect to the scalar product $\langle\cdot,\cdot\rangle_{\mathbf{L}^2}$, namely, recalling (2.82),

$$
\mathcal{R}^* = \begin{pmatrix} \overline{r}_1 & r_2 \\ \overline{r}_2 & r_1 \end{pmatrix}. \tag{4.9}
$$

Now we define

$$\widetilde{\mathbf{h}} := \Psi h \qquad \widetilde{\mathbf{h}}_{in} := \Psi|_{t=0}\,\mathbf{h}_{in} \qquad \widetilde{\mathbf{h}}_{end} := \Psi|_{t=T}\,\mathbf{h}_{end} \tag{4.10}$$

$$\widetilde{\mathbf{q}} := \Phi^{-1}\mathbf{q} \qquad \widetilde{\mathbf{f}} := \Phi_*\mathbf{f} \qquad K := \Phi^{-1}\chi_\omega(\Phi_*)^{-1}, \tag{4.11}$$

where $\Phi_*$ is the adjoint of $\Phi$ with respect to the *time-space* scalar product $\langle\cdot,\cdot\rangle_{(t,x)} := \int_0^T \langle\cdot,\cdot\rangle_{\mathbf{L}^2}\,dt$. We call "time-space adjoint" the adjoint of an operator with respect to $\langle\cdot,\cdot\rangle_{(t,x)}$. By (2.11), (2.25), (2.58), (2.79), the adjoint operators (with respect to the $\mathbf{L}^2$ scalar product) of $\mathcal{S},\mathcal{A},\mathcal{T},\mathcal{M}$ are

$$\mathcal{S}^* = \mathcal{S}, \quad \mathcal{A}^* = \mathcal{A}^{-1}, \quad \mathcal{T}^* = \mathcal{T}^{-1}, \quad \mathcal{M}^* = \mathcal{M}^{-1} \tag{4.12}$$

at each fixed $t \in [0,T]$, and therefore, integrating over $[0,T]$, the equalities in (4.12) also hold for the time-space adjoint operators $\mathcal{S}_*,\mathcal{A}_*,\mathcal{T}_*,\mathcal{M}_*$. The time-space adjoint of $\mathcal{B}$ satisfies $\mathcal{B}_* = \rho^{-1}\mathcal{B}^{-1}$ (see (2.53)), and therefore, from the definitions of $\Phi,\Psi$ in (4.8), we calculate $\Phi_* = \mathcal{M}^{-1}(\mathcal{T}^{-1}\mathbb{I}_2)(\mathcal{B}^{-1}\mathbb{I}_2)(\mathcal{A}^{-1}\mathbb{I}_2)\mathcal{S}$. We also calculate

$$K = \mathcal{M}^{-1}(\mathcal{T}^{-1}\mathbb{I}_2)\rho^{-1}(\mathcal{B}^{-1}\mathbb{I}_2)(\mathcal{A}^{-1}\mathbb{I}_2)\mathcal{S}^{-1}\chi_\omega\mathcal{S}(\mathcal{A}\mathbb{I}_2)(\mathcal{B}\mathbb{I}_2)(\mathcal{T}\mathbb{I}_2)\mathcal{M}\,.$$

Since $[\mathcal{S},\chi_\omega\mathbb{I}_2] = 0$ and $[\mathcal{M},k\mathbb{I}_2] = 0$ for all real-valued functions $k(t,x)$, using the conjugation rules (2.26), (2.44), (2.59), and recalling also (2.23)-(2.25), one can easily see that $K$ is the multiplication operator

$$K = k(t,x)\mathbb{I}_2, \quad k(t,x) := (\mathcal{T}^{-1}\rho^{-1})(t)\,(\mathcal{T}^{-1}\mathcal{B}^{-1}A_\alpha^{-1}\chi_\omega)(t,x). \tag{4.13}$$

By the estimates of Section 2, we get

$$\|K\mathbf{h}\|_{T,s} \lesssim_s \|\mathbf{h}\|_{T,s} + N_T(s+\sigma)\|\mathbf{h}\|_{T,0} \quad \forall \mathbf{h} \in C([0,T],\mathbf{H}^s(\mathbb{T}))\,. \tag{4.14}$$

Note that, by the estimates of Section 2, one has that if $\mathbf{h}_{in},\mathbf{h}_{end} \in \mathbf{H}^s(\mathbb{T})$, $\mathbf{q} \in C([0,T],\mathbf{H}^s(\mathbb{T}))$, then $\widetilde{\mathbf{h}}_{in},\widetilde{\mathbf{h}}_{end} \in \mathbf{H}^s(\mathbb{T})$, $\widetilde{\mathbf{q}} \in C([0,T],\mathbf{H}^s(\mathbb{T}))$. Moreover using that $\mathbf{h},\mathbf{f} \in C([0,T],\mathbf{L}^2(\mathbb{T}))$, one has that also $\widetilde{\mathbf{h}},\widetilde{\mathbf{f}},K\widetilde{\mathbf{f}} \in C([0,T],\mathbf{L}^2(\mathbb{T}))$. By construction, $\widetilde{\mathbf{h}},\widetilde{\mathbf{f}}$ satisfy

$$\begin{cases} \mathcal{L}_4\widetilde{\mathbf{h}} = K\widetilde{\mathbf{f}} + \widetilde{\mathbf{q}} \\ \widetilde{\mathbf{h}}(0,\cdot) = \widetilde{\mathbf{h}}_{in} \\ \widetilde{\mathbf{h}}(T,\cdot) = \widetilde{\mathbf{h}}_{end}\,, \end{cases} \qquad \mathcal{L}_4^*\widetilde{\mathbf{f}} = 0\,. \tag{4.15}$$

To prove that $\mathcal{L}_4^*\widetilde{\mathbf{f}} = 0$ it is enough to write it in its weak form, namely

$$\langle\widetilde{\mathbf{f}}(T,\cdot),\mathbf{v}(T,\cdot)\rangle_{\mathbf{L}^2} - \langle\widetilde{\mathbf{f}}(0,\cdot),\mathbf{v}(0,\cdot)\rangle_{\mathbf{L}^2} = \int_0^T \langle\widetilde{\mathbf{f}},\mathcal{L}_4\mathbf{v}\rangle_{\mathbf{L}^2}\,dt \quad \forall \mathbf{v} \in C^\infty([0,T]\times\mathbb{T})$$

and to apply the changes of coordinates in the integrals.

Now we show that $\widetilde{\mathbf{h}},\widetilde{\mathbf{f}} \in C([0,T],\mathbf{H}^s(\mathbb{T}))$. We adapt an argument used by Dehman-Lebeau [25], also used in [33], [6], [14]. We split the proof into two parts.

Proof in the case $\mathbf{h}_{end} = 0$, $\mathbf{q} = 0$. Define the map

$$S : \mathbf{L}^2(\mathbb{T}) \to \mathbf{L}^2(\mathbb{T})\,, \qquad S\widetilde{\mathbf{f}}_1 := \widetilde{\mathbf{h}}(0,\cdot)\,, \tag{4.16}$$

where $\widetilde{\mathbf{f}}$ and $\widetilde{\mathbf{h}}$ are the solutions of the Cauchy problems

$$\begin{cases} \mathcal{L}_4^*\widetilde{\mathbf{f}} = 0 \\ \widetilde{\mathbf{f}}(T,\cdot) = \widetilde{\mathbf{f}}_1\,, \end{cases} \qquad \begin{cases} \mathcal{L}_4\widetilde{\mathbf{h}} = K\widetilde{\mathbf{f}} \\ \widetilde{\mathbf{h}}(T,\cdot) = 0\,. \end{cases} \tag{4.17}$$

By existence and uniqueness in Lemma 4.1, it follows that $S$ is a linear isomorphism. Then for every initial datum $\widetilde{\mathbf{h}}_{in} \in \mathbf{L}^2(\mathbb{T})$ there exists a unique $\widetilde{\mathbf{f}}_1 \in \mathbf{L}^2(\mathbb{T})$ such that $S\widetilde{\mathbf{f}}_1 = \widetilde{\mathbf{h}}_{in}$. Note that $\|\Lambda^s\widetilde{\mathbf{f}}_1\|_{L^2_x} \lesssim \|S\Lambda^s\widetilde{\mathbf{f}}_1\|_{L^2_x}$, since $S : \mathbf{L}^2(\mathbb{T}) \to \mathbf{L}^2(\mathbb{T})$ is an isomorphism, where $\Lambda := \mathrm{Op}\big((1+\xi^2)^{\frac{1}{2}}\big)$.

To study the commutator $[\Lambda^s, S]$, we have to compare $(\Lambda^s\widetilde{\mathbf{u}}, \Lambda^s\widetilde{\mathbf{f}})$ with $(\underline{\mathbf{h}}, \underline{\mathbf{f}})$ solving the Cauchy problems

$$\begin{cases} \mathcal{L}_4^*\underline{\mathbf{f}} = 0 \\ \underline{\mathbf{f}}(T,\cdot) = \Lambda^s\widetilde{\mathbf{f}}_1\,, \end{cases} \qquad \begin{cases} \mathcal{L}_4\underline{\mathbf{h}} = K\underline{\mathbf{f}} \\ \underline{\mathbf{h}}(T,\cdot) = 0\,. \end{cases} \tag{4.18}$$

Since $[\mathcal{L}_4^*, \Lambda^s] = [\mathcal{R}^*, \Lambda^s]$, the difference $\Lambda^s\widetilde{\mathbf{f}} - \underline{\mathbf{f}}$ satisfies

$$\begin{cases} \mathcal{L}_4^*\big(\Lambda^s\widetilde{\mathbf{f}} - \underline{\mathbf{f}}\big) = [\mathcal{R}^*, \Lambda^s]\widetilde{\mathbf{f}} \\ \big(\Lambda^s\widetilde{\mathbf{f}} - \underline{\mathbf{f}}\big)(T,\cdot) = 0\,. \end{cases}$$

By Lemma 8.2, and then using Lemma 7.3, (4.9), (2.83), one gets the estimate

$$\|\Lambda^s\widetilde{\mathbf{f}} - \underline{\mathbf{f}}\|_{T,0} \lesssim \|\,[\mathcal{R}^*, \Lambda^s]\widetilde{\mathbf{f}}\,\|_{T,0} \lesssim_s N_T(s+\sigma)\|\widetilde{\mathbf{f}}\|_{T,0} + \|\widetilde{\mathbf{f}}\|_{T,s-1}\,, \tag{4.19}$$

for some constant $\sigma > 0$, where we have used that $N_T(\sigma) \lesssim 1$. The difference $\Lambda^s\widetilde{\mathbf{h}} - \underline{\mathbf{h}}$ satisfies the Cauchy problem

$$\begin{cases} \mathcal{L}_4\big(\Lambda^s\widetilde{\mathbf{h}} - \underline{\mathbf{h}}\big) = K\big(\Lambda^s\widetilde{\mathbf{f}} - \underline{\mathbf{f}}\big) + [\mathcal{R}, \Lambda^s]\widetilde{\mathbf{h}} + [\Lambda^s, K]\widetilde{\mathbf{f}} \\ (\Lambda^s\widetilde{\mathbf{h}} - \underline{\mathbf{h}})(T,\cdot) = 0\,. \end{cases}$$

Arguing as in (4.19) one gets

$$\|[\mathcal{R}, \Lambda^s]\widetilde{\mathbf{h}}\|_{T,0} \lesssim_s N_T(s+\sigma)\|\widetilde{\mathbf{h}}\|_{T,0} + \|\widetilde{\mathbf{h}}\|_{T,s-1}\,.$$

Since $K$ is a multiplication operator (see (4.13)), the commutator $[\Lambda^s, K]$ is of order $s-1$. By (4.14), using again Lemma 7.3, we deduce that

$$\|K(\Lambda^s\widetilde{\mathbf{f}} - \underline{\mathbf{f}})\|_{T,0} \lesssim \|\Lambda^s\widetilde{\mathbf{f}} - \underline{\mathbf{f}}\|_{T,0}\,, \qquad \|[\Lambda^s, K]\widetilde{\mathbf{f}}\|_{T,0} \lesssim \|\widetilde{\mathbf{f}}\|_{T,s-1} + N_T(s+\sigma)\|\widetilde{\mathbf{f}}\|_{T,0}\,.$$

Therefore, by Lemma 8.2,

$$\begin{aligned} \|\Lambda^s\widetilde{\mathbf{h}} - \underline{\mathbf{h}}\|_{T,0} &\lesssim \|[\mathcal{R}, \Lambda^s]\widetilde{\mathbf{h}}\|_{T,0} + \|K\big(\Lambda^s\widetilde{\mathbf{f}} - \underline{\mathbf{f}}\big)\|_{T,0} + \|[\Lambda^s, K]\widetilde{\mathbf{f}}\|_{T,0} \\ &\lesssim N_T(s+\sigma)\|\widetilde{\mathbf{h}}\|_{T,0} + \|\widetilde{\mathbf{h}}\|_{T,s-1} + \|\Lambda^s\widetilde{\mathbf{f}} - \underline{\mathbf{f}}\|_{T,0} + \|\widetilde{\mathbf{f}}\|_{T,s-1} + N_T(s+\sigma)\|\widetilde{\mathbf{f}}\|_{T,0} \\ &\overset{(4.19)}{\lesssim} \|\widetilde{\mathbf{h}}\|_{T,s-1} + \|\widetilde{\mathbf{f}}\|_{T,s-1} + N_T(s+\sigma)\big(\|\widetilde{\mathbf{h}}\|_{T,0} + \|\widetilde{\mathbf{f}}\|_{T,0}\big)\,. \end{aligned} \tag{4.20}$$

Applying Lemma 8.2 to the Cauchy problems (4.17), and using also (4.14), we have

$$\begin{aligned} &\|\widetilde{\mathbf{f}}\|_{T,s} \lesssim \|\widetilde{\mathbf{f}}_1\|_s + N_T(s+\sigma)\|\widetilde{\mathbf{f}}_1\|_0\,, \\ &\|\widetilde{\mathbf{h}}\|_{T,s} \lesssim \|K\widetilde{\mathbf{f}}\|_{T,s} + N_T(s+\sigma)\|K\widetilde{\mathbf{f}}\|_{T,0} \lesssim \|\widetilde{\mathbf{f}}_1\|_s + N_T(s+\sigma)\|\widetilde{\mathbf{f}}_1\|_0\,. \end{aligned} \tag{4.21}$$

Hence estimates (4.19), (4.20) become

$$\|\Lambda^s\widetilde{\mathbf{f}} - \underline{\mathbf{f}}\|_{T,0}, \|\Lambda^s\widetilde{\mathbf{h}} - \underline{\mathbf{h}}\|_{T,0} \lesssim_s \|\widetilde{\mathbf{f}}_1\|_{s-1} + N_T(s+\sigma)\|\widetilde{\mathbf{f}}_1\|_0\,. \tag{4.22}$$

By the definition of the map $S$ in (4.16), one has $\underline{\mathbf{h}}(0,\cdot) = S\Lambda^s\widetilde{\mathbf{f}}_1$. Also recall that we have fixed $S\widetilde{\mathbf{f}}_1 = \widetilde{\mathbf{h}}_{in} = \widetilde{\mathbf{h}}(0,\cdot)$. Using (4.22) and triangular inequality,

$$\begin{aligned} \|S\Lambda^s\widetilde{\mathbf{f}}_1\|_0 &\lesssim \|\Lambda^s\widetilde{\mathbf{h}}(0,\cdot)\|_0 + \|\Lambda^s\widetilde{\mathbf{h}}(0,\cdot) - \underline{\mathbf{h}}(0,\cdot)\|_0 \\ &\lesssim \|\widetilde{\mathbf{h}}_{in}\|_s + \|\Lambda^s\widetilde{\mathbf{h}} - \underline{\mathbf{h}}\|_{T,0} \lesssim \|\widetilde{\mathbf{h}}_{in}\|_s + \|\widetilde{\mathbf{f}}_1\|_{s-1} + N_T(s+\sigma)\|\widetilde{\mathbf{f}}_1\|_0\,. \end{aligned} \tag{4.23}$$

Since $S : \mathbf{L}^2(\mathbb{T}) \to \mathbf{L}^2(\mathbb{T})$ is a linear isomorphism, we have $\|\widetilde{\mathbf{f}}_1\|_s \simeq \|\Lambda^s\widetilde{\mathbf{f}}_1\|_0 \lesssim \|S\Lambda^s\widetilde{\mathbf{f}}_1\|_0$ and therefore, by (4.23),

$$\|\widetilde{\mathbf{f}}_1\|_s \lesssim \|\widetilde{\mathbf{h}}_{in}\|_s + \|\widetilde{\mathbf{f}}_1\|_{s-1} + N_T(s+\sigma)\|\widetilde{\mathbf{f}}_1\|_0\,.$$

Using again that $S : \mathbf{L}^2(\mathbb{T}) \to \mathbf{L}^2(\mathbb{T})$ is an isomorphism, we have $\|\widetilde{\mathbf{f}}_1\|_0 \lesssim \|\widetilde{\mathbf{h}}_{in}\|_0$, and the above inequality becomes

$$\|\widetilde{\mathbf{f}}_1\|_s \lesssim \|\widetilde{\mathbf{h}}_{in}\|_s + N_T(s+\sigma)\|\widetilde{\mathbf{h}}_{in}\|_0 + \|\widetilde{\mathbf{f}}_1\|_{s-1} \,. \tag{4.24}$$

If $0 < s \le 1$, then $\|\widetilde{\mathbf{f}}_1\|_{s-1} \le \|\widetilde{\mathbf{f}}_1\|_0$, and, as already observed, $\|\widetilde{\mathbf{f}}_1\|_0 \lesssim \|\widetilde{\mathbf{h}}_{in}\|_0$, whence

$$\|\widetilde{\mathbf{f}}_1\|_s \lesssim \|\widetilde{\mathbf{h}}_{in}\|_s + N_T(s+\sigma)\|\widetilde{\mathbf{h}}_{in}\|_0 \,. \tag{4.25}$$

If $s > 1$, bound (4.25) is proved by induction on $s$, applying (4.24) repeatedly. Hence, by (4.21),

$$\|\widetilde{\mathbf{h}}\|_{T,s}, \|\widetilde{\mathbf{f}}\|_{T,s} \lesssim_s \|\widetilde{\mathbf{h}}_{in}\|_s + N_T(s+\sigma)\|\widetilde{\mathbf{h}}_{in}\|_0 \,. \tag{4.26}$$

Finally, recalling (4.8), (4.10)-(4.11), (4.12) and the estimates (2.19), (2.40), (2.55), (2.56), (2.69), (2.81) of Section 2, we obtain the claimed estimate for $\mathbf{h}$ and $\mathbf{f}$, namely

$$\|\mathbf{h}\|_{T,s}, \|\mathbf{f}\|_{T,s} \lesssim_s \|\mathbf{h}_{in}\|_s + N_T(s+\sigma)\|\mathbf{h}_{in}\|_0 \,. \tag{4.27}$$

Proof of the general case. Now we remove the hypothesis that $\mathbf{h}_{end}$ and $\mathbf{q}$ are zero. Assume that $\mathbf{h}, \mathbf{f}$ solve (4.7) and let $\mathbf{w}$ be the solution of the backward Cauchy problem

$$\mathcal{L}\mathbf{w} = \mathbf{q}, \quad \mathbf{w}(T,\cdot) = \mathbf{h}_{end} \,. \tag{4.28}$$

Since $\mathbf{h}_{end} \in \mathbf{H}^s(\mathbb{T})$ and $\mathbf{q} \in C([0,T], \mathbf{H}^s(\mathbb{T}))$, by Lemma 8.7 one has $\mathbf{w} \in C([0,T], \mathbf{H}^s(\mathbb{T}))$ with

$$\|\mathbf{w}\|_{T,s} \lesssim_s \|\mathbf{q}\|_{T,s} + \|\mathbf{h}_{end}\|_s + N_T(s+\sigma)\|\mathbf{h}_{end}\|_0 \,. \tag{4.29}$$

Let $\mathbf{v} := \mathbf{h} - \mathbf{w}$. Hence

$$\mathcal{L}\mathbf{v} = \chi_\omega \mathbf{f}, \quad \mathbf{v}(0,\cdot) = \mathbf{h}_{in} - \mathbf{w}(0,\cdot), \quad \mathbf{v}(T,\cdot) = 0 \tag{4.30}$$

and therefore $\mathbf{v}, \mathbf{f}$ solve (4.7) where $(\mathbf{h}_{in}, \mathbf{h}_{end}, \mathbf{q})$ are replaced by $(0, \mathbf{h}_{in} - \mathbf{w}(0,\cdot), 0)$. Hence we can apply to $\mathbf{v}, \mathbf{f}$ the estimate (4.27) proved in the previous step, obtaining that

$$\begin{aligned}\|\mathbf{v}\|_{T,s}\,,\ \|\mathbf{f}\|_{T,s} &\lesssim_s \|\mathbf{h}_{in} - \mathbf{w}(0,\cdot)\|_s + N_T(s+\sigma)\|\mathbf{h}_{in} - \mathbf{w}(0,\cdot)\|_0 \\ &\lesssim_s \|\mathbf{h}_{in}\|_s + \|\mathbf{w}(0,\cdot)\|_s + N_T(s+\sigma)\big(\|\mathbf{h}_{in}\|_0 + \|\mathbf{w}(0,\cdot)\|_0\big) \\ &\lesssim_s \|\mathbf{h}_{in}\|_s + \|\mathbf{w}\|_{T,s} + N_T(s+\sigma)\big(\|\mathbf{h}_{in}\|_0 + \|\mathbf{w}\|_{T,0}\big) \,. \end{aligned} \tag{4.31}$$

Therefore (4.29), (4.31) imply that

$$\|\mathbf{v}\|_{T,s}\,,\ \|\mathbf{f}\|_{T,s} \lesssim_s \|\mathbf{h}_{in}\|_s + \|\mathbf{h}_{end}\|_s + \|\mathbf{q}\|_{T,s} + N_T(s+\sigma)\big(\|\mathbf{h}_{in}\|_0 + \|\mathbf{h}_{end}\|_0 + \|\mathbf{q}\|_{T,0}\big) \,. \tag{4.32}$$

The estimate for $\mathbf{h} = \mathbf{v} + \mathbf{w}$ follows by triangular inequality and by (4.29) and (4.32). Estimate (4.6) is deduced from the fact that $\mathbf{h}, \mathbf{f}$ solve the equations $\mathcal{L}\mathbf{h} = \chi_\omega \mathbf{f} + \mathbf{q}$ and $\mathcal{L}^*\mathbf{f} = 0$. □

For any $s \in \mathbb{R}$, we consider the space

$$C([0,T], H^s(\mathbb{T},\mathbb{R}^2)) = C([0,T], H^s(\mathbb{T},\mathbb{R})) \times C([0,T], H^s(\mathbb{T},\mathbb{R}))$$

and for $u = (u_1, u_2) \in C([0,T], H^s(\mathbb{T},\mathbb{R}^2))$ we set

$$\|u\|_{T,s} := \|u_1\|_{T,s} + \|u_2\|_{T,s} \,.$$

We define

$$E_s := X_s \times X_s, \tag{4.33}$$

$$X_s := C([0,T], H^{s+4}(\mathbb{T},\mathbb{R}^2)) \cap C^1([0,T], H^{s+2}(\mathbb{T},\mathbb{R}^2)) \cap C^2([0,T], H^s(\mathbb{T},\mathbb{R}^2)) \,, \tag{4.34}$$

and (recall notations in (1.20)-(1.21)),

$$\begin{aligned} F_s := \Big\{ z := (v,\alpha,\beta) = (v_1, v_2, \alpha_1, \alpha_2, \beta_1, \beta_2) : \\ v \in C([0,T], H^{s+4}(\mathbb{T},\mathbb{R}^2)) \cap C^1([0,T], H^s(\mathbb{T},\mathbb{R}^2)),\ \alpha, \beta \in H^{s+4}(\mathbb{T},\mathbb{R}^2) \Big\} \end{aligned} \tag{4.35}$$

equipped with the norms

$$\|(u,f)\|_{E_s} := \|u\|_{X_s} + \|f\|_{X_s}, \quad \|u\|_{X_s} := \|u\|_{T,s+4} + \|\partial_t u\|_{T,s+2} + \|\partial_{tt} u\|_{T,s}, \tag{4.36}$$

and

$$\|z\|_{F_s} := \|v\|_{T,s+4} + \|\partial_t v\|_{T,s} + \|\alpha\|_{s+4} + \|\beta\|_{s+4}. \tag{4.37}$$

With this notation, we have proved the following linear inversion result.

**Theorem 4.3** (Right inverse of the linearized operator)**.** *Let $T > 0$, and let $\omega \subset \mathbb{T}$ be an open set. There exist constants $\tau \geq 6$, $\sigma \geq 3$ (independent of $T, \omega$) and $\delta_* > 0$ (depending on $T, \omega$) with the following property.*

*Let $s \in [0, r - \tau]$, where $r$ is the regularity of the nonlinearity in* (1.3)*. Let $z = (v, \alpha, \beta) \in F_s$. If $(u, f) \in E_{s+\sigma}$, with $\|u\|_{X_\sigma} \leq \delta_*$, then there exists $(h, \varphi) := \Psi(u, f)[z] \in E_s$, such that*

$$P'(u)[h] - \chi_\omega \varphi = v, \quad h(0, \cdot) = \alpha, \quad h(T, \cdot) = \beta, \tag{4.38}$$

*and*

$$\|h, \varphi\|_{E_s} \leq C(s)\big(\|z\|_{F_s} + \|u\|_{X_{s+\sigma}} \|z\|_{F_0}\big) \tag{4.39}$$

*where the constant $C(s) > 0$ depends on $s, T, \omega$.*

*Proof.* Using the transformation $\mathcal{C}$ defined in (1.27), the linear control problem (4.38) for the operator $P'(u_1, u_2)$ is transformed into the linear control problem (1.28) for the operator $\mathcal{L}(u_1, u_2) = \mathcal{C}^{-1} P'(u_1, u_2) \mathcal{C}$, where the operator $\mathcal{L} = \mathcal{L}(u_1, u_2)$ is given in (1.29). We apply Lemma 4.2 to the control problem (1.28), since by definition (2.6) and Lemma 2.1 the smallness condition $\|\mathbf{u}\|_{X_\sigma} \leq \delta_*$ implies that $N_T(\sigma') \lesssim \delta_*$, for some $\sigma' < \sigma$. Then the lemma follows by noticing that the map $\mathcal{C} : \mathbf{H}^s(\mathbb{T}) \to H^s(\mathbb{T}, \mathbb{R}^2)$ is a unitary isomorphism. □

# 5 Proofs

In this section we prove Theorems 1.1, 1.5 and 1.3, 1.6. As explained in Section 1.3, Theorems 1.1 and 1.3 follow by Theorems 1.5, 1.6.

## 5.1 Proof of Theorems 1.1, 1.5

We check that all the assumptions of Theorem 9.1 are verified. The spaces $E_s, F_s$ defined in (4.33)-(4.37), with $s \geq 0$, form scales of Banach spaces. We define the smoothing operators $S_j$, $j = 0, 1, 2, \ldots$ as

$$S_j u(x) := \sum_{|k| \leq 2^j} \widehat{u}_k \, e^{\mathrm{i}kx} \qquad \text{where} \quad u(x) = \sum_{k \in \mathbb{Z}} \widehat{u}_k \, e^{\mathrm{i}kx} \in L^2(\mathbb{T}).$$

The definition of $S_j$ extends in the obvious way to functions $u(t, x) = \sum_{k \in \mathbb{Z}} \widehat{u}_k(t) \, e^{\mathrm{i}kx}$ depending on time. Since $S_j$ and $\partial_t$ commute, the smoothing operators $S_j$ are defined on the spaces $E_s$, $F_s$ defined in (4.33)-(4.35) by setting $S_j(u, f) := (S_j u, S_j f)$ and similarly on $z = (v, \alpha, \beta)$. One easily verifies that $S_j$ satisfies (9.1)-(9.5) and (9.8) on $E_s$ and $F_s$.

By (1.17), observe that $\Phi(u, f) := (P(u) - \chi_\omega f, \, u(0), \, u(T))$ belongs to $F_s$ when $(u, f) \in E_{s+2}$, $s \in [0, r - 4]$, with $\|u\|_{T,3} \leq 1$. Its second derivative in the directions $(h, \varphi) = (h_1, h_2, \varphi_1, \varphi_2)$ and $(w, \psi) = (w_1, w_2, \psi_1, \psi_2)$ is

$$\Phi''(u, f)[(h, \varphi), (w, \psi)] = \begin{pmatrix} P''(u)[h, w] \\ 0 \\ 0 \end{pmatrix}.$$

For $u$ in a fixed ball $\|u\|_{X_1} \leq \delta_0$, with $\delta_0$ small enough, one has

$$\|P''(u)[h, w]\|_{F_s} \lesssim_s \big(\|h\|_{X_1} \|w\|_{X_{s+2}} + \|h\|_{X_{s+2}} \|w\|_{X_1} + \|u\|_{X_{s+2}} \|h\|_{X_1} \|w\|_{X_1}\big) \tag{5.1}$$

for all $s \in [0, r-4]$. We fix $V = \{(u,f) \in E_2 : \|(u,f)\|_{E_2} \leq \delta_0\}$, $\delta_1 = \delta_*$,

$$a_0 = 1, \quad \mu = 2, \quad a_1 = \sigma, \quad \alpha = \beta > 2\sigma, \quad a_2 > 2\alpha - a_1, \tag{5.2}$$

where $\delta_*, \sigma, \tau$ are given by Theorem 4.3, and $r \geq r_1 := a_2 + \tau$ is the regularity of $G$ in Theorem 1.5. The right inverse $\Psi$ in Theorem 4.3 satisfies the assumptions of Theorem 9.1. Let $u_{in}, u_{end} \in H^{\beta+4}(\mathbb{T}, \mathbb{R}^2)$, with $\|u_{in}, u_{end}\|_{H_x^{\beta+4}}$ small enough. Let $g := (0, u_{in}, u_{end})$, so that $g \in F_\beta$ and $\|g\|_{F_\beta} \leq \delta$. Since $g$ does not depend on time, it satisfies (9.12).

Thus by Theorem 9.1 there exists a solution $(u,f) \in E_\alpha$ of the equation $\Phi(u,f) = g$, with $\|u, f\|_{E_\alpha} \leq C\|g\|_{F_\beta}$ (and recall that $\beta = \alpha$). We fix $s_1 := \alpha + 4$, and (1.50) is proved.

We have found a solution $(u,f)$ of the control problem (1.48)-(1.49). Now we prove that $u$ is the unique solution of the Cauchy problem (1.48), with that given $f$. Let $u, v$ be two solutions of (1.10) in $E_{s_1-4}$. We calculate

$$P(u) - P(v) = \int_0^1 P'(v + \lambda(u-v))\, d\lambda\, [u-v].$$

Conjugating the operator $P'(v + \lambda(u-v))$ by means of the unitary isomorphism $\mathcal{C} : \mathbf{H}^s(\mathbb{T}) \to H^s(\mathbb{T}, \mathbb{R}^2)$ defined in (1.27), one gets

$$\mathcal{C}^{-1} P'(v + \lambda(u-v))\, \mathcal{C} = \mathcal{L}(v + \lambda(u-v)),$$

where $\mathcal{L}$ has the form (1.29). Hence

$$\mathcal{C}^{-1} \int_0^1 P'(v + \lambda(u-v))\, d\lambda\, \mathcal{C} = \widetilde{\mathcal{L}},$$

where

$$\widetilde{\mathcal{L}} := \partial_t + \mathrm{i}(\Sigma + \tilde{A}_2(t,x))\partial_{xx} + \mathrm{i}\tilde{A}_1(t,x)\partial_x + \mathrm{i}\tilde{A}_0(t,x),$$
$$\tilde{A}_i(t,x) := \int_0^1 A_i(v + \lambda(u-v))(t,x)\, d\lambda, \quad i = 0,1,2,$$

and $A_i(u)$ is defined in (1.30)-(1.31). Setting $\mathbf{u} := \mathcal{C}^{-1}u$, $\mathbf{v} := \mathcal{C}^{-1}v$ one has that the difference $\mathbf{u} - \mathbf{v}$ satisfies $\widetilde{\mathcal{L}}(\mathbf{u} - \mathbf{v}) = 0$, $(\mathbf{u} - \mathbf{v})(0) = 0$. We apply Lemma 8.7 to the operator $\widetilde{\mathcal{L}}$, and we obtain $\mathbf{u} - \mathbf{v} = 0$. Then $u - v = 0$. This completes the proof of Theorem 1.5, and therefore of Theorem 1.1. □

## 5.2 Proof of Theorems 1.3, 1.6

We define

$$E_s := C([0,T], H^{s+4}(\mathbb{T}, \mathbb{R}^2)) \cap C^1([0,T], H^{s+2}(\mathbb{T}, \mathbb{R}^2)) \cap C^2([0,T], H^s(\mathbb{T}, \mathbb{R}^2)), \tag{5.3}$$
$$F_s := \{(v, \alpha) : v \in C([0,T], H^{s+4}(\mathbb{T}, \mathbb{R}^2)) \cap C^1([0,T], H^s(\mathbb{T}, \mathbb{R}^2)), \alpha \in H^{s+4}(\mathbb{T}, \mathbb{R}^2)\} \tag{5.4}$$

equipped with norms

$$\|u\|_{E_s} := \|u\|_{T,s+4} + \|\partial_t u\|_{T,s+2} + \|\partial_{tt} u\|_{T,s} \tag{5.5}$$
$$\|(v,\alpha)\|_{F_s} := \|v\|_{T,s+4} + \|\partial_t v\|_{T,s} + \|\alpha\|_{s+4}, \tag{5.6}$$

and $\Phi(u) := (P(u), u(0))$, where $P$ is defined in (1.16). Given $g := (0, u_{in}) \in F_{s_0}$, the Cauchy problem (1.51) writes $\Phi(u) = g$. We fix $V := \{u \in E_2 : \|u\|_{E_2} \leq \delta_0\}$, where $\delta_0$ is the same as in subsection 5.1; we fix $a_0, \mu, a_1, \alpha, \beta, a_2$ like in (5.2), where the constants $\sigma, \tau$ are now given in Lemma 8.7, $r \geq r_0 := a_2 + \tau$ is the regularity of $G$ in Theorem 1.6, and $\delta_1$ is small enough to satisfy both assumption (2.2) in Lemma 2.1 and $N_T(\sigma) \leq \eta$ in Lemma 8.7.

Assumption (9.11) about the right inverse of the linearized operator is satisfied by Lemmas 8.7 and 2.1. We fix $s_0 := \alpha + 4$. Then Theorem 9.1 applies, giving the existence part of Theorem 1.6. The uniqueness of the solution is proved exactly as in Subsection 5.1. This completes the proof of Theorem 1.6, and therefore of Theorem 1.3. □

# 6 Appendix A. Quadratic Hamiltonians and linear Hamiltonian vector fields

Dealing with linear Hamiltonian equations, we develop Hamiltonian formalism only for quadratic Hamiltonians. We consider real quadratic Hamiltonians $\mathcal{H} : \mathbf{H}^s(\mathbb{T}) \to \mathbb{R}$ of the form

$$\mathcal{H}(u, \overline{u}) = \int_{\mathbb{T}} R_1[u]\overline{u}\, dx + \frac{1}{2}\int_{\mathbb{T}} R_2[u]u\, dx + \frac{1}{2}\int_{\mathbb{T}} \overline{R_2}[\overline{u}]\, \overline{u}\, dx\,, \tag{6.1}$$

where $R_1, R_2 : H^s(\mathbb{T}) \to H^{s-2}(\mathbb{T})$ and

$$R_1 = R_1^*\,, \qquad R_2 = R_2^T\,. \tag{6.2}$$

the Hamiltonian equation associated to $\mathcal{H}$ is given by

$$\partial_t \mathbf{u} = \mathrm{i}J\nabla_{\mathbf{u}}\mathcal{H}(\mathbf{u})\,, \qquad \mathbf{u} = (u, \overline{u}) \in \mathbf{H}^s(\mathbb{T})$$

where

$$\nabla_{\mathbf{u}}\mathcal{H} := (\nabla_u\mathcal{H}, \nabla_{\overline{u}}\mathcal{H})\,, \qquad J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Note that the Hamiltonian vector field associated to the Hamiltonian $\mathcal{H}$ has the form

$$\mathcal{R} = \mathrm{i}J\nabla_{\mathbf{u}}\mathcal{H} = \mathrm{i}\begin{pmatrix} R_1 & R_2 \\ -\overline{R}_2 & -\overline{R}_1 \end{pmatrix}, \qquad R_1 = R_1^*\,, \quad R_2 = R_2^T\,. \tag{6.3}$$

The symplectic form on the phase space $\mathbf{L}^2(\mathbb{T})$ is defined as

$$\mathcal{W}[\mathbf{u}_1, \mathbf{u}_2] = \mathrm{i}\int_{\mathbb{T}} (u_1\overline{u}_2 - \overline{u}_1 u_2)\, dx\,, \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in \mathbf{L}^2(\mathbb{T})\,. \tag{6.4}$$

**Definition 6.1.** *Let $\Phi_i = \Phi_i : H^s(\mathbb{T}) \to H^s(\mathbb{T})$, $i = 1, 2$. We say that the map*

$$\Phi = \begin{pmatrix} \Phi_1 & \Phi_2 \\ \overline{\Phi_2} & \overline{\Phi_1} \end{pmatrix},$$

*is symplectic if*

$$\mathcal{W}[\Phi[\mathbf{u}_1], \Phi[\mathbf{u}_2]] = \mathcal{W}[\mathbf{u}_1, \mathbf{u}_2]\,, \quad \forall \mathbf{u}_1, \mathbf{u}_2 \in \mathbf{L}^2(\mathbb{T})\,,$$

*or equivalently $\Phi^T J \Phi = J$.*

It is well known that if $\mathcal{R}$ is an operator of the form (6.3), then the operators $\exp(\pm\mathcal{R})$ are symplectic maps. In the next lemma we state some properties of some particular Hamiltonian vector fields.

**Lemma 6.2.** *Let $a_i, b_i \in H^s(\mathbb{T})$, $i = 0, 1, 2$ and*

$$A_i := \begin{pmatrix} a_i & b_i \\ -\overline{b}_i & -\overline{a}_i \end{pmatrix}, \qquad i = 0, 1, 2\,.$$

*If the vector field $\mathcal{R} := \mathrm{i}\big(A_2\partial_{xx} + A_1\partial_x + A_0\big) : \mathbf{H}^s(\mathbb{T}) \to \mathbf{H}^{s-2}(\mathbb{T})$ is Hamiltonian then the following holds:*

$$a_2 = \overline{a}_2\,, \quad a_1 = 2(\partial_x a_2) - \overline{a}_1\,, \quad a_0 = \overline{a}_0 + (\partial_{xx}a_2) - (\partial_x\overline{a}_1)\,, \quad b_1 = (\partial_x b_2)$$

**Lemma 6.3.** *Assume that $\mathcal{R}$ is a Hamiltonian operator of the form (6.3). Then its adjoint $\mathcal{R}^*$ with respect to the complex scalar product $\langle\cdot,\cdot\rangle_{\mathbf{L}^2}$ is still a Hamiltonian operator.*

*Proof.* Let $\mathcal{R}$ be a Hamiltonian operator

$$\mathcal{R} = \mathrm{i}\begin{pmatrix} R_1 & R_2 \\ -\overline{R}_2 & -\overline{R}_1 \end{pmatrix}, \qquad R_1 = R_1^*, \quad R_2 = R_2^T.$$

A direct calculation shows that the adjoint $\mathcal{R}^*$ with respect to the complex scalar product $\langle \cdot, \cdot \rangle_{\mathbf{L}^2}$ is given by

$$\mathcal{R}^* = \mathrm{i}\begin{pmatrix} Q_1 & Q_2 \\ -\overline{Q}_2 & -\overline{Q}_1 \end{pmatrix}, \qquad Q_1 := -\overline{R}_1^T, \qquad Q_2 := R_2^T.$$

using that $R_1$ is selfadjoint and $\overline{R}_1^T = R_1^*$, we get that $Q_1 = -R_1$ and therefore $Q_1 = Q_1^*$. Moreover since $R_2 = R_2^T$, we get that $Q_2 = R_2$ and therefore $Q_2 = Q_2^T$. This implies that

$$\mathcal{R}^* = \mathrm{i}\begin{pmatrix} -R_1 & R_2 \\ -\overline{R}_2 & \overline{R}_1 \end{pmatrix}$$

is still Hamiltonian. □

# 7 Appendix B. Classical tame estimates

In this appendix we recall some classical interpolation estimates used in this paper. We introduce the following notation: given $k \in \mathbb{R}$, we denote

$$\mathbb{Z}_{\geq k} := \{n \in \mathbb{Z} : n \geq k\}, \quad \mathbb{R}_{\geq k} := \{s \in \mathbb{R} : s \geq k\}, \quad \mathbb{R}_{> k} := \{s \in \mathbb{R} : s > k\}.$$

**Lemma 7.1.** (*i*) (Embedding). *For any $s \in \mathbb{Z}_{\geq 0}$, the space $H^{s+1}(\mathbb{T})$ is compactly embedded in $C^s(\mathbb{T})$ and*

$$\|u\|_{C^s} \lesssim_s \|u\|_{s+1} \quad \forall u \in H^{s+1}(\mathbb{T}). \tag{7.1}$$

(*ii*) (Tame product). *Let $s \in \mathbb{R}_{\geq 1}$ and $u_1, u_2 \in H^s(\mathbb{T})$. Then*

$$\|u_1 u_2\|_s \lesssim_s \|u_1\|_1 \|u_2\|_s + \|u_1\|_s \|u_2\|_1. \tag{7.2}$$

*In particular*

$$\|u_1 u_2\|_s \lesssim_s \|u_1\|_s \|u_2\|_s. \tag{7.3}$$

(*iii*) (Interpolation). *Let $a_0, b_0, p, q \in \mathbb{R}_{\geq 0}$. Then*

$$\|u_1\|_{a_0+p} \|u_2\|_{b_0+q} \leq \|u_1\|_{a_0+p+q} \|u_2\|_{b_0} + \|u_1\|_{a_0} \|u_2\|_{b_0+p+q}. \tag{7.4}$$

**Lemma 7.2** (Composition)**.** *Let $s \in \mathbb{R}_{\geq 0}$, $m \in \mathbb{N}$, with $m > s+1$. Let $F : \mathbb{C}^n \to \mathbb{R}$ be a function of $C^m$ class in the real sense. Let $u \in H^s(\mathbb{T}, \mathbb{C}^n) \cap H^1(\mathbb{T}, \mathbb{C}^n)$, with $\|u\|_1 \leq 1$. Then*

$$\|F(u)\|_s \lesssim_s 1 + \|u\|_s. \tag{7.5}$$

*Moreover, if $F(0) = 0$, then*

$$\|F(u)\|_s \lesssim_s \|u\|_s. \tag{7.6}$$

*Proof.* For $s \in \mathbb{N}$ see [39, p. 272–275] and [40, Lemma 7, p. 202–203]. For the more general case of real $s$ see [36, Theorem 5.2.6], [8, Proposition 2.2, p. 87], and [7, Proposition 7.3 *iii*]. The result in [7] is stated in the *uniformly local* Sobolev spaces $H^s_{ul}(\mathbb{R}^d)$, which contain the periodic Sobolev spaces $H^s(\mathbb{T}^d)$. The result in [36] is stated for $F \in C^\infty$, but, in fact, the proof in [36] only uses the assumption that $F$ has derivatives up to order $m > s+1$ that are bounded on compact sets. The proof in [36] is on $\mathbb{R}^d$, but it also holds on the torus $\mathbb{T}$ and, more generally, $\mathbb{T}^d$. The only nontrivial point when adapting that proof to $\mathbb{T}^d$ is equation (5.2.10) of [36], which is also "Bernstein inequality" (4.1.8), which follows from Lemma 4.1.6 of [36].

We explain how to adapt Lemma 4.1.6 of [36] to $\mathbb{T}^d$. Let $\chi \in C^\infty(\mathbb{R}^d, \mathbb{R})$, with $0 \leq \chi \leq 1$, supported on $\{|\xi| \leq 2\}$ and such that $\chi = 1$ on $|\xi| \leq 1$. Let $\mathrm{Op}(\chi_\lambda)$ be the Fourier multiplier of symbol $\chi_\lambda(\xi) := \chi(\xi/\lambda)$, $\lambda \geq 1$. Let $\varphi_\lambda := \mathcal{F}^{-1}_{\mathbb{R}^d}\chi_\lambda$, where $\mathcal{F}^{-1}_{\mathbb{R}^d}$ denotes the inverse Fourier transform on $\mathbb{R}^d$, so that the Fourier transform of $\varphi_\lambda$ is $\widehat{\varphi_\lambda} = \chi_\lambda$. Thus for functions $u \in L^2(\mathbb{R}^d)$ we have

$$\mathrm{Op}(\chi_\lambda)u(x) = \int_{\mathbb{R}^d} \hat{u}(\xi)\chi_\lambda(\xi)e^{\mathrm{i}\xi\cdot x}\,d\xi = \int_{\mathbb{R}^d} u(x-y)\varphi_\lambda(y)\,dy = (u *_{\mathbb{R}^d} \varphi_\lambda)(x),$$

where $\hat{u}$ if the Fourier transform of $u$ and $*_{\mathbb{R}^d}$ denotes the convolution on $\mathbb{R}^d$. Similarly, for periodic functions $u \in L^2(\mathbb{T}^d)$ one has

$$\mathrm{Op}(\chi_\lambda)u(x) = \sum_{k\in\mathbb{Z}^d} \hat{u}_k\chi_\lambda(k)e^{\mathrm{i}k\cdot x} = \int_{\mathbb{T}^d} u(x-y)\psi_\lambda(y)\,dy = (u *_{\mathbb{T}^d} \psi_\lambda)(x),$$

where $\hat{u}_k$ are the Fourier coefficients of $u$, $*_{\mathbb{T}^d}$ denotes the convolution on $\mathbb{T}^d$, and $\psi_\lambda(x) := \sum_{k\in\mathbb{Z}^d}\chi_\lambda(k)e^{\mathrm{i}k\cdot x}$. With elementary calculations (imitating Section 13.4 of [5]), one proves that $\psi_\lambda$ is the periodization of $\varphi_\lambda$, namely

$$\psi_\lambda(x) = \sum_{m\in\mathbb{Z}^d}\varphi_\lambda(x+2\pi m), \quad \text{and} \quad \widehat{(\psi_\lambda)}_k = \widehat{\varphi_\lambda}(k) \quad \forall k\in\mathbb{Z}^d,$$

where $\widehat{(\psi_\lambda)}_k$ are Fourier coefficients, and $\widehat{\varphi_\lambda}(k)$ is the Fourier transform. As a consequence, one proves that, for $u \in L^\infty(\mathbb{T}^d)$, $u *_{\mathbb{R}^d} \varphi_\lambda = u *_{\mathbb{T}^d} \psi_\lambda$ (see equation (13.19) of [5]). We deduce that

$$\|\mathrm{Op}(\chi_\lambda)u\|_{L^\infty(\mathbb{T}^d)} = \|u *_{\mathbb{R}^d} \varphi_\lambda\|_{L^\infty(\mathbb{T}^d)} \leq \|u\|_{L^\infty(\mathbb{T}^d)}\|\varphi_\lambda\|_{L^1(\mathbb{R}^d)}$$

and the bounds for $\varphi_\lambda$ over $\mathbb{R}^d$ proved in [36] can still be used. The periodization trick makes it possible to safely bypass a change of the variable $\xi$ which does not seem to be applicable when $\xi \in \mathbb{Z}^d$. □

We recall also the standard commutator estimate between a multiplication operator and a Fourier multiplier.

**Lemma 7.3.** *Let $s \in \mathbb{R}_{>0}$. Let $\varphi_s(D)$ be a Fourier multiplier of order $s$ and $a \in H^{s+1}(\mathbb{T}) \cap H^2(\mathbb{T})$. Then*

$$\|[a, \varphi_s(D)]u\|_0 \lesssim_s \|a\|_{s+1}\|u\|_0 + \|a\|_2\|u\|_{s-1} \quad \forall u \in H^{s-1}(\mathbb{T}) \cap L^2(\mathbb{T}).$$

We now state a lemma on changes of variables induced by diffeomorphisms of the torus.

**Lemma 7.4** (Change of variables)**.** *(i) Let $s \in \mathbb{Z}_{\geq 1}$ and $\alpha \in C^s(\mathbb{T})$, with $\|\alpha\|_{C^1} \leq 1/2$. Then the operator $\mathcal{A}u(x) := u(x+\alpha(x))$ satisfies the estimate*

$$\|\mathcal{A}u\|_0 \lesssim \|u\|_0 \quad \forall u \in L^2(\mathbb{T}), \tag{7.7}$$

$$\|\mathcal{A}u\|_s \lesssim_s \|u\|_s + \|\alpha\|_{C^s}\|u\|_1 \quad \forall u \in H^s(\mathbb{T}), \quad s \in \mathbb{Z}_{\geq 1}\,. \tag{7.8}$$

*Moreover, for any $s \in \mathbb{R}_{\geq 0}$, if $\alpha \in H^{s+2}(\mathbb{T})$, with $\|\alpha\|_2 \leq 1$, then*

$$\|\mathcal{A}u\|_s \lesssim_s \|u\|_s + \|\alpha\|_{s+2}\|u\|_0 \quad \forall u \in H^s(\mathbb{T}), \quad s \in \mathbb{R}_{\geq 0}\,. \tag{7.9}$$

*(ii) Let $s \in \mathbb{Z}_{\geq 1}$ and $\alpha \in C^s(\mathbb{T})$, with $\|\alpha\|_{C^1} \leq 1/2$. The map $\mathbb{T} \to \mathbb{T}$, $x \mapsto x + \alpha(x)$ is invertible and the inverse diffeomorphism $\mathbb{T} \to \mathbb{T}$, $y \mapsto y + \tilde{\alpha}(y)$ satisfies*

$$\|\tilde{\alpha}\|_{C^s} \lesssim_s \|\alpha\|_{C^s}\,, \quad s \in \mathbb{Z}_{\geq 1}\,. \tag{7.10}$$

*(iii) The inverse operator $\mathcal{A}^{-1}$ defined as $\mathcal{A}^{-1}u(y) := u(y + \tilde{\alpha}(y))$ satisfies the same estimates (7.7)-(7.8) as $\mathcal{A}$ in (i). Moreover there exists $\delta \in (0,1)$ such that, for any $s \in \mathbb{R}_{\geq 0}$, if $\alpha \in H^{s+4}(\mathbb{T})$ with $\|\alpha\|_4 \leq \delta$, then*

$$\|\mathcal{A}^{-1}u\|_s \lesssim_s \|u\|_s + \|\alpha\|_{s+4}\|u\|_0 \quad \forall u \in H^s(\mathbb{T}), \quad s \in \mathbb{R}_{\geq 0}\,. \tag{7.11}$$

*Proof.* Proof of $(i)$. Estimates (7.7)-(7.8) are classical; they are proved, e.g., in [10], Lemma B.4. Let us prove (7.9). Applying (7.8) for $s=1$ and recalling (7.7) one has

$$\|\mathcal{A}u\|_0 \lesssim \|u\|_0 \,, \quad \|\mathcal{A}u\|_1 \lesssim \|u\|_1 \,. \tag{7.12}$$

Now let $u \in H^2(\mathbb{T})$ and assume that $\alpha \in H^2(\mathbb{T})$, with $\|\alpha\|_2 \le 1$. Then, using (7.12), (7.3) and the bound $\|\alpha\|_2 \le 1$,

$$\begin{aligned} \|\mathcal{A}u\|_2 \simeq \|\mathcal{A}u\|_0 + \|\partial_x(\mathcal{A}u)\|_1 &\overset{(7.12)}{\lesssim} \|u\|_0 + \|(1+\alpha_x)\mathcal{A}(u_x)\|_1 \\ &\overset{(7.3)}{\lesssim} \|u\|_0 + \|\mathcal{A}(u_x)\|_1(1+\|\alpha\|_2) \overset{(7.12)}{\lesssim} \|u\|_2 \,. \end{aligned} \tag{7.13}$$

By (7.7) and (7.13), using a classical interpolation result, one has

$$\|\mathcal{A}u\|_s \lesssim \|u\|_s \quad \forall u \in H^s(\mathbb{T}), \quad s \in [0,2]. \tag{7.14}$$

Now we argue by induction on $s$. Assume that the claimed estimate holds for $s \in \mathbb{R}_{\ge 1}$ and let us prove it for $s+1$. Using the bound $\|\alpha\|_2 \le 1$, we have

$$\begin{aligned} \|\mathcal{A}u\|_{s+1} \simeq \|\mathcal{A}u\|_0 + \|\partial_x(\mathcal{A}u)\|_s &\overset{(7.7)}{\lesssim} \|u\|_0 + \|(1+\alpha_x)\mathcal{A}(\partial_x u)\|_s \\ &\overset{(7.2)}{\lesssim_s} \|u\|_0 + \|\mathcal{A}(u_x)\|_s + \|\alpha_x\|_s \|\mathcal{A}(u_x)\|_1 \,. \end{aligned}$$

By the inductive hyphothesis, we deduce that

$$\|\mathcal{A}u\|_{s+1} \lesssim_s \|u\|_{s+1} + \|\alpha\|_{s+2}\|u\|_1 + \|\alpha\|_{s+1}\|u\|_2 \,. \tag{7.15}$$

By (7.4), applied with $u_1 = \alpha, u_2 = u$, $a_0 = 2, b_0 = 0, p = s, q = 1$, one gets

$$\|\alpha\|_{s+2}\|u\|_1 \le \|\alpha\|_{s+3}\|u\|_0 + \|\alpha\|_2\|u\|_{s+1} \,. \tag{7.16}$$

Using again (7.4), applied with $u_1 = \alpha, u_2 = u$, $a_0 = 2, b_0 = 0, p = s-1, q = 2$, one gets

$$\|\alpha\|_{s+1}\|u\|_2 \le \|\alpha\|_{s+3}\|u\|_0 + \|\alpha\|_2\|u\|_{s+1} \,. \tag{7.17}$$

Then (7.15)-(7.17), using that $\|\alpha\|_2 \le 1$, imply that

$$\|\mathcal{A}u\|_{s+1} \lesssim_s \|u\|_{s+1} + \|\alpha\|_{s+3}\|u\|_0 \,,$$

which is estimate (7.9) at the Sobolev index $s+1$.

Proof of $(ii)$. It is proved in [10], Lemma B.4.

Proof of $(iii)$. The fact that $\mathcal{A}^{-1}$ satisfies the estimate (7.7)-(7.8) is proved in [10], Lemma B.4. Let us prove (7.11). For any real $s \ge 0$, we denote by $[s]$ the integer part of $s$. One has

$$\|\widetilde{\alpha}\|_{s+2} \le \|\widetilde{\alpha}\|_{[s]+3} \lesssim \|\widetilde{\alpha}\|_{C^{[s]+3}} \overset{(ii)}{\lesssim_s} \|\alpha\|_{C^{[s]+3}} \overset{(7.1)}{\lesssim_s} \|\alpha\|_{[s]+4} \lesssim_s \|\alpha\|_{s+4} \,. \tag{7.18}$$

Hence, for $s=0$, one has $\|\widetilde{\alpha}\|_2 \le C\|\alpha\|_4 \le 1$ by taking $\|\alpha\|_4$ small enough. Therefore we can apply (7.9) to $\mathcal{A}^{-1}$ and the claimed estimate follows by (7.18). □

We also study the action of the operators induced by diffeomorphisms of the torus on the spaces $C([0,T], H^s(\mathbb{T}))$. For any function $\alpha : [0,T]\times\mathbb{T} \to \mathbb{R}$ and any $h : \mathbb{T} \to \mathbb{C}$, we define the $t$-dependent family $\mathcal{A}(t)h(x) := h(x+\alpha(t,x))$. Then, given $h : [0,T]\times\mathbb{T} \to \mathbb{R}$, we define

$$\mathcal{A}h(t,x) := \mathcal{A}(t)h(t,x) = h(t, x+\alpha(t,x)) \,. \tag{7.19}$$

**Lemma 7.5.** *Let $s \in \mathbb{Z}_{\geq 1}$, $\alpha \in C([0,T], C^s(\mathbb{T}))$ with $\|\alpha_x\|_{L^\infty} \leq 1/2$. Let $y \mapsto y + \widetilde{\alpha}(t,y)$ be the inverse diffeomorphism of $x \mapsto x + \alpha(t,x)$. Then $\widetilde{\alpha} \in C([0,T], C^s(\mathbb{T}))$ and*

$$\|\widetilde{\alpha}\|_{C([0,T],C^s(\mathbb{T}))} \lesssim_s \|\alpha\|_{C([0,T],C^s(\mathbb{T}))}, \quad s \in \mathbb{Z}_{\geq 1}. \tag{7.20}$$

*Moreover, for any $s \in \mathbb{R}_{\geq 0}$, if $\alpha \in C([0,T], H^{s+2}(\mathbb{T}))$, then $\widetilde{\alpha} \in C([0,T], H^s(\mathbb{T}))$, with*

$$\|\widetilde{\alpha}\|_{T,s} \lesssim_s \|\alpha\|_{T,s+2}, \quad s \in \mathbb{R}_{\geq 0}. \tag{7.21}$$

*Proof.* Proof of (7.20). Let $y \mapsto y + \widetilde{\alpha}(t,y)$ be the inverse diffeomorphism of $x \mapsto x + \alpha(t,x)$. Since

$$\widetilde{\alpha}(t,y) + \alpha(t, y + \widetilde{\alpha}(t,y)) = 0,$$

one can directly check that if $\alpha \in C([0,T], C^1(\mathbb{T}))$ then also $\widetilde{\alpha} \in C([0,T], C^1(\mathbb{T}))$ and

$$\widetilde{\alpha}_y(t,y) = -\frac{\alpha_x(t, y + \widetilde{\alpha}(t,y))}{1 + \widetilde{\alpha}_y(t,y)}.$$

Using the above formula and a bootstrap argument, one can show that for any integer $s \geq 1$, if $\alpha \in C([0,T], C^s(\mathbb{T}))$, then $\widetilde{\alpha} \in C([0,T], C^s(\mathbb{T}))$. By (7.10), one has $\|\widetilde{\alpha}(t,\cdot)\|_{C^s} \lesssim_s \|\alpha(t,\cdot)\|_{C^s}$. Then (7.20) follows by taking the sup over $t \in [0,T]$.

Proof of (7.21). Let $\alpha \in C([0,T], H^{s+2}(\mathbb{T}))$. Since $[s] \leq s$, one has $C([0,T], H^{s+2}(\mathbb{T})) \subseteq C([0,T], H^{[s]+2}(\mathbb{T}))$. Using (7.1), $C([0,T], H^{[s]+2}(\mathbb{T})) \subseteq C([0,T], C^{[s]+1}(\mathbb{T}))$. As a consequence, $\alpha \in C([0,T], C^{[s]+1}(\mathbb{T}))$, with

$$\|\alpha\|_{C([0,T],C^{[s]+1}(\mathbb{T}))} \lesssim_s \|\alpha\|_{T,s+2}. \tag{7.22}$$

By (7.20), $\widetilde{\alpha} \in C([0,T], C^{[s]+1}(\mathbb{T}))$ and using that $C([0,T], C^{[s]+1}(\mathbb{T})) \subseteq C([0,T], H^s(\mathbb{T}))$, we get that $\widetilde{\alpha} \in C([0,T], H^s(\mathbb{T}))$, with $\|\widetilde{\alpha}\|_{T,s} \lesssim_s \|\widetilde{\alpha}\|_{C([0,T],C^{[s]+1}(\mathbb{T}))}$. The claimed inequality (7.21) follows by recalling (7.22). □

**Lemma 7.6** (Change of variables)**.** *There exists $\delta \in (0,1)$ with the following properties.*

*(i) Let $s \in \mathbb{R}_{\geq 0}$ and $\alpha \in C([0,T], H^{s+2}(\mathbb{T}))$, with $\|\alpha\|_{T,2} \leq \delta$. Then the operator $\mathcal{A}u(t,x) := u(t, x + \alpha(t,x))$ is a linear and continuous operator $C([0,T], H^s(\mathbb{T})) \to C([0,T], H^s(\mathbb{T}))$, with*

$$\|\mathcal{A}u\|_{T,s} \lesssim_s \|u\|_{T,s} + \|\alpha\|_{T,s+2}\|u\|_{T,0} \quad \forall u \in C([0,T], H^s(\mathbb{T})). \tag{7.23}$$

*(ii) Let $s \in \mathbb{R}_{\geq 0}$ and $\alpha \in C([0,T], H^{s+4}(\mathbb{T}))$, with $\|\alpha\|_{T,4} \leq \delta$. Then the inverse operator $\mathcal{A}^{-1}$, defined by $\mathcal{A}^{-1}u(t,y) := u(t, y + \widetilde{\alpha}(t,y))$, maps $C([0,T], H^s(\mathbb{T}))$ into itself, with*

$$\|\mathcal{A}^{-1}u\|_{T,s} \lesssim_s \|u\|_{T,s} + \|\alpha\|_{T,s+4}\|u\|_{T,0} \quad \forall u \in C([0,T], H^s(\mathbb{T})).$$

*Proof.* First, we prove $(i)$. Let $s \in \mathbb{R}_{\geq 0}$ and $u \in C([0,T], H^s(\mathbb{T}))$. We have to prove that $\mathcal{A}u \in C([0,T], H^s(\mathbb{T}))$, namely, for any $t_0 \in [0,T]$, we have to prove that $\|(\mathcal{A}u)(t) - (\mathcal{A}u)(t_0)\|_s \to 0$ as $t \to t_0$. By triangular inequality,

$$\|(\mathcal{A}u)(t) - (\mathcal{A}u)(t_0)\|_s \leq \|\mathcal{A}(t)[u(t) - u(t_0)]\|_s + \|(\mathcal{A}(t) - \mathcal{A}(t_0))[u(t_0)]\|_s \tag{7.24}$$

(where, in short, $u(t)$ means $u(t,\cdot)$). The first term is estimated using (7.9), which gives

$$\|\mathcal{A}(t)[u(t) - u(t_0)]\|_s \lesssim_s \|u(t) - u(t_0)\|_s + \|\alpha\|_{T,s+2}\|u(t) - u(t_0)\|_0 \to 0 \quad (t \to t_0).$$

To prove that the last term in (7.24) also vanishes as $t \to t_0$ is equivalent to prove that, for every $h \in H^s(\mathbb{T})$, the map $[0,T] \to H^s(\mathbb{T})$, $t \mapsto \mathcal{A}(t)h$ is continuous. Let $h \in H^s(\mathbb{T})$, and let $\widehat{h}(k)$ be its Fourier coefficients. Let

$$\Pi_n h(x) := \sum_{|k| \leq n} \widehat{h}(k) e^{\mathrm{i}kx}, \quad \Pi_n^\perp h(x) := (I - \Pi_n)h(x) = \sum_{|k| > n} \widehat{h}(k) e^{\mathrm{i}kx},$$

and

$$f_n(t) := \mathcal{A}(t)\Pi_n h, \quad f(t) := \mathcal{A}(t)h.$$

The sequence $(f_n)$ converges to $f$ uniformly in $t \in [0,T]$ in the space $H^s(\mathbb{T})$, because, using (7.9) and the assumption $h \in H^s(\mathbb{T})$,

$$\sup_{t\in[0,T]} \|f_n(t) - f(t)\|_s = \|f_n - f\|_{T,s} = \|\mathcal{A}\Pi_n^\perp h\|_{T,s} \lesssim_s \|\Pi_n^\perp h\|_s + \|\alpha\|_{T,s+2}\|\Pi_n^\perp h\|_0 \to 0 \quad (n \to \infty).$$

Since continuity is preserved by uniform limits, we have to prove that all $f_n$ are continuous. For any $n$, the function $f_n$ is

$$f_n(t,x) = \mathcal{A}(t)\Pi_n h(x) = \sum_{|k|\le n} \widehat{h}(k)\, \psi_k(t,x), \quad \psi_k(t,x) := e^{\mathrm{i}k(x+\alpha(t,x))} = \mathcal{A}(t)[e^{\mathrm{i}kx}].$$

Hence $f_n$ is a finite linear combination of functions $\psi_k$. It remains to prove that, for all $k \in \mathbb{Z}$, the function $\psi_k$ belongs to $C([0,T], H^s(\mathbb{T}))$. Fix $k \in \mathbb{Z}$, and consider the functions $G(u) := e^{\mathrm{i}ku}$ and $F(u) := e^{\mathrm{i}ku} - 1$. Split

$$\psi_k(t) - \psi_k(t_0) = e^{\mathrm{i}kx} e^{\mathrm{i}k\alpha(t_0,x)} \{e^{\mathrm{i}k[\alpha(t,x)-\alpha(t_0,x)]} - 1\},$$

and estimate each factor. First, $\|e^{\mathrm{i}kx}\|_s = \langle k\rangle^s$. Second, using (7.5) and the assumption $\|\alpha\|_{T,1} \le 1$,

$$\|e^{\mathrm{i}k\alpha(t_0,x)}\|_s = \|G(\alpha(t_0,\cdot))\|_s \le C_{k,s}(1 + \|\alpha(t_0,\cdot)\|_s) \le C_{k,s}(1 + \|\alpha\|_{T,s}).$$

Third, by (7.6),

$$\|e^{\mathrm{i}k[\alpha(t,\cdot)-\alpha(t_0,\cdot)]} - 1\|_s = \|F(\alpha(t,\cdot) - \alpha(t_0,\cdot))\|_s \le C_{k,s}\|\alpha(t,\cdot) - \alpha(t_0,\cdot)\|_s\,.$$

Hence

$$\|\psi_k(t,\cdot) - \psi_k(t_0,\cdot)\|_s \le C_{k,s}(1 + \|\alpha\|_{T,s})\|\alpha(t,\cdot) - \alpha(t_0,\cdot)\|_s \to 0 \quad (t \to t_0)$$

because $\alpha \in C([0,T], H^s(\mathbb{T}))$. Hence, we have proved that $\mathcal{A} : C([0,T], H^s(\mathbb{T})) \to C([0,T], H^s(\mathbb{T}))$. Estimate (7.23) then follows by applying (7.9) at any fixed $t \in [0,T]$ and taking the supremum.

Finally, $(ii)$ follows by $(i)$ and (7.21). □

# 8 Appendix C. Well-posedness of linear equations

**Lemma 8.1.** *Let $T > 0$, $t_0 \in [0,T]$, $\mu \in \mathbb{R}$. Let $S \ge 1$, $\mathbf{h}_{in} \in \mathbf{H}^S(\mathbb{T})$, $\mathbf{g} \in C([0,T], \mathbf{H}^S(\mathbb{T}))$ and let $\mathcal{R}$ be the multiplication operator*

$$\mathcal{R} := \begin{pmatrix} r_1 & r_2 \\ \overline{r}_2 & \overline{r}_1 \end{pmatrix}, \qquad r_1, r_2 \in C([0,T], H^{S+1}(\mathbb{T}))\,. \tag{8.1}$$

*There exists $\eta > 0$ small enough depending on $T$ such that if*

$$\|\mathcal{R}\|_{T,1} = \max\{\|r_1\|_{T,1}, \|r_2\|_{T,1}\} \le \eta\,, \tag{8.2}$$

*then there exists a unique solution $\mathbf{h} \in C([0,T], \mathbf{H}^S(\mathbb{T}))$ of the Cauchy problem*

$$\begin{cases} \partial_t \mathbf{h} + \mathrm{i}\mu\Sigma\partial_{xx}\mathbf{h} + \mathcal{R}\mathbf{h} = \mathbf{g} \\ \mathbf{h}(t_0,\cdot) = \mathbf{h}_{in} \end{cases} \tag{8.3}$$

*satisfying for any $0 \le s \le S$, the estimate*

$$\|\mathbf{h}\|_{T,s} \lesssim_s \|\mathbf{h}_{in}\|_s + \|\mathbf{g}\|_{T,s} + \|\mathcal{R}\|_{T,s+1}\|\mathbf{h}_{in}\|_0\,.$$

*Proof.* Since $\mathbf{h}_0 = (h_0, \overline{h}_0)$, $\mathbf{g} = (g, \overline{g})$, $\mathbf{h} = (h, \overline{h})$ and $\mathcal{R}$ has the form (8.1), it is enough to study the Cauchy problem

$$\begin{cases} \partial_t h + \mathrm{i}\mu \partial_{xx} h + \mathcal{Q}(h) = g \\ h(t_0, \cdot) = h_0 \,, \end{cases} \qquad \mathcal{Q}(h) := r_1 h + r_2 \overline{h} \,. \tag{8.4}$$

Note that for any $0 \le s \le S$, by Lemma 7.1-$(ii)$, applying (7.4), with $v = (r_1, r_2)$, $u = h$, $a_0 = 1$, $b_0 = 0$, $p = s - 1$, $q = 1$ and using the smallness condition (8.2), one gets that

$$\|\mathcal{Q}h\|_{T,s} \lesssim_s \eta \|h\|_{T,s} + \|\mathcal{R}\|_{T,s+1} \|h\|_{T,0} \,, \quad \forall h \in C([0,T], H^s(\mathbb{T})) \,. \tag{8.5}$$

We split in (8.4), $h = v + \varphi$, where

$$\begin{cases} \partial_t v + \mathrm{i}\mu \partial_{xx} v = g \\ v(t_0, \cdot) = h_{in} \,, \end{cases} \qquad \begin{cases} \partial_t \varphi + \mathrm{i}\mu \partial_{xx} \varphi + \mathcal{Q}(\varphi) + \mathcal{Q}(v) = 0 \\ \varphi(t_0, \cdot) = 0 \,. \end{cases} \tag{8.6}$$

The first Cauchy problem in (8.5) can be solved explicitly and since $h_{in} \in H^S(\mathbb{T})$, $g \in C([0,T], H^S(\mathbb{T}))$ there exists a unique solution $v \in C([0,T], H^S(\mathbb{T}))$ satisfying

$$\|v\|_{T,s} \le \|h_{in}\|_{T,s} + T\|g\|_{T,s} \,, \qquad \forall 0 \le s \le S \,. \tag{8.7}$$

Then, we construct iteratively the solution of the second Cauchy problem in (8.6), by setting

$$\varphi_0 := 0 \,, \qquad \varphi_{n+1} := \Phi(\varphi_n) \,, \qquad n \ge 0 \,,$$

where

$$\Phi(\varphi) := -\int_{t_0}^{t} e^{-\mathrm{i}\mu\partial_{xx}(t-\tau)}[\mathcal{Q}(v)(\tau)]\, d\tau - \int_{t_0}^{t} e^{-\mathrm{i}\mu\partial_{xx}(t-\tau)}[\mathcal{Q}(\varphi)(\tau)]\, d\tau \,. \tag{8.8}$$

We prove the following claim: for any $0 \le s \le S$ there exists a constant $K_T(s) > 0$ (depending on $T$ and $s$) such that for any $n \ge 0$, $\varphi_n \in C([0,T], H^s(\mathbb{T}))$ and

$$\|\varphi_n\|_{T,s} \le R(s) \,, \qquad R(s) := K_T(s)\Big(\eta\|v\|_{T,s} + \|\mathcal{R}\|_{T,s+1}\|v\|_{T,0}\Big) \,. \tag{8.9}$$

We argue by induction on $n$. For $n = 0$ the statement is trivial. Then assume that the claim holds for some $n \ge 0$ and let us prove it for $n+1$. By the definition of the map $\Phi$ in (8.8), using the inductive hyphothesis, one has immediately that $\varphi_{n+1} = \Phi(\varphi_n) \in C([0,T], H^s(\mathbb{T}))$, for any $0 \le s \le S$. Moreover, using that for any $t, \tau \in [0,T]$, $\|e^{-\mathrm{i}\mu\partial_{xx}(t-\tau)}\|_{\mathcal{L}(H^s(\mathbb{T}))} \le 1$ and by estimate (8.5), one gets

$$\begin{aligned}
\|\varphi_{n+1}\|_{T,s} &\le C(s)T\big(\eta\|v\|_{T,s} + \|\mathcal{R}\|_{T,s+1}\|v\|_{T,0}\big) + C(s)T\big(\eta\|\varphi_n\|_{T,s} + \|\mathcal{R}\|_{T,s+1}\|\varphi_n\|_{T,0}\big) \\
&\overset{(8.9)}{\le} C(s)T\big(\eta\|v\|_{T,s} + \|\mathcal{R}\|_{T,s+1}\|v\|_{T,0}\big) + C(s)TK_T(s)\eta\Big(\eta\|v\|_{T,s} + \|\mathcal{R}\|_{T,s+1}\|v\|_{T,0}\Big) \\
&\quad + C(s)T\|\mathcal{R}\|_{T,s+1}K_T(0)\big(\eta\|v\|_{T,0} + \|\mathcal{R}\|_{T,1}\|v\|_{T,0}\big) \\
&\overset{(8.2)}{\le} \Big(C(s)T\eta + C(s)K_T(s)T\eta^2\Big)\|v\|_{T,s} \\
&\quad + \Big(C(s)T + C(s)K_T(s)T\eta + 2TC(s)K_T(0)\eta\Big)\|\mathcal{R}\|_{T,s+1}\|v\|_{T,0} \\
&\le K_T(s)\Big(\eta\|v\|_{T,s} + \|\mathcal{R}\|_{T,s+1}\|v\|_{T,0}\Big) \,,
\end{aligned} \tag{8.10}$$

provided that

$$C(s)T + C(s)K_T(s)T\eta \le K_T(s) \,, \qquad C(s)T + C(s)K_T(s)T\eta + 2TC(s)K_T(0)\eta \le K_T(s) \,.$$

The above conditions are fulfilled by taking $K_T(s) > 0$ large enough and $\eta \in (0,1)$ small enough, therefore (8.9) has been proved at the step $n+1$.

Convergence of $\varphi_n$. We prove that for any $0 \le s \le S$, there exists a constant $J_T(s) > 0$ such that for any $n \ge 0$

$$\|\varphi_{n+1} - \varphi_n\|_{T,s} \le J_T(s)\Big(\frac{1}{2^{n+1}}\|v\|_{T,s} + \frac{1}{2^n}\|\mathcal{R}\|_{T,s+1}\|v\|_{T,0}\Big) . \tag{8.11}$$

We argue by induction on $n$. For $n = 0$, since $\varphi_0 = 0$, the estimate follows by (8.9) applied for $n = 1$ and by taking $J_T(s) \ge K_T(s)$ and $\eta \le 1/2$. Now let us assume that (8.11) holds for some $n \ge 0$ and let us prove it for $n+1$. Recalling (8.8) and the definition of $\mathcal{Q}$ in (8.4), one has

$$\varphi_{n+2} - \varphi_{n+1} = \Phi(\varphi_{n+1}) - \Phi(\varphi_n) = -\int_{t_0}^{t} e^{-\mathrm{i}\mu\partial_{xx}(t-\tau)}[\mathcal{Q}(\varphi_{n+1} - \varphi_n)(\tau)]\, d\tau .$$

Using estimates (8.5), (8.2), (8.11), one gets

$$\begin{aligned}
\|\varphi_{n+2} - \varphi_{n+1}\|_{T,s} &\le C(s)T\Big(\eta\|\varphi_{n+1} - \varphi_n\|_{T,s} + \|\mathcal{R}\|_{T,s+1}\|\varphi_{n+1} - \varphi_n\|_{T,0}\Big) \\
&\le C(s)T\eta J_T(s)\Big(\frac{1}{2^{n+1}}\|v\|_{T,s} + \frac{1}{2^n}\|\mathcal{R}\|_{T,s+1}\|v\|_{T,0}\Big) \\
&\quad + C(s)T\|\mathcal{R}\|_{T,s+1}J_T(0)\Big(\frac{1}{2^{n+1}} + \eta\frac{1}{2^n}\Big)\|v\|_{T,0} \\
&\le J_T(s)\Big(\frac{1}{2^{n+2}}\|v\|_{T,s} + \frac{1}{2^{n+1}}\|\mathcal{R}\|_{T,s+1}\|v\|_{T,0}\Big)
\end{aligned}$$

by taking $J_T(s) > 0$ large enough and $\eta \in (0,1)$ small enough. Thus (8.11) at the step $n+1$ has been proved. Using a telescoping argument one has that there exists $\varphi \in C([0,T], H^S(\mathbb{T}))$ such that

$$\varphi_n \to \varphi , \qquad \text{in} \qquad C([0,T], H^s(\mathbb{T})) , \quad \forall 0 \le s \le S .$$

Moreover, $\Phi(\varphi_n) \to \Phi(\varphi)$ in $C([0,T], H^s(\mathbb{T}))$, for any $0 \le s \le S$, implying that $\Phi(\varphi) = \varphi$. Since $\|\varphi\|_{T,s} = \lim_{n\to+\infty}\|\varphi_n\|_{T,s}$, by (8.9) one deduces that $\varphi$ satisfies

$$\|\varphi\|_{T,s} \lesssim_s \eta\|v\|_{T,s} + \|\mathcal{R}\|_{T,s+1}\|v\|_{T,0} . \tag{8.12}$$

Recalling that $h = \varphi + v$ and using estimates (8.7), (8.12), one gets

$$\|h\|_{T,s} \lesssim_s \|h_{in}\|_s + \|g\|_{T,s} + \|\mathcal{R}\|_{T,s+1}\|h_{in}\|_0 ,$$

and the lemma is proved. □

**Lemma 8.2** (Well posedness of the operator $\mathcal{L}_4$ in (2.73))**.** *Let $T > 0$, $t_0 \in [0,T]$ and let $\mathcal{L}_4 = \partial_t\mathbb{I}_2 + \mathrm{i}\mu\partial_{xx}\Sigma + \mathcal{R}$ be the operator defined in* (2.73)*. There exists $\eta \in (0,1)$ small enough and universal constants $\sigma, \tau > 0$ large enough such that if $N_T(\sigma) \le \eta$ (see the definition* (2.6)*), then for any $s \in [0, r-\tau]$, $\mathbf{h}_{in} \in \mathbf{H}^s(\mathbb{T})$, $\mathbf{g} \in C([0,T], \mathbf{H}^s(\mathbb{T}))$, there exists a unique solution $\mathbf{h} \in C([0,T], \mathbf{H}^s(\mathbb{T}))$ such that*

$$\begin{cases} \mathcal{L}_4\mathbf{h} = \mathbf{g} \\ \mathbf{h}(t_0, \cdot) = \mathbf{h}_{in} \end{cases}$$

*satisfying the estimate*

$$\|\mathbf{h}\|_{T,s} \lesssim_s \|\mathbf{h}_{in}\|_{H^s_x} + \|\mathbf{g}\|_{T,s} + N_T(s+\sigma)\|\mathbf{h}_{in}\|_{L^2_x} . \tag{8.13}$$

*Proof.* The lemma follows by applying Lemmas 2.1, 2.6 and 8.1. Indeed, by (2.82)-(2.83), using that $N_T(\sigma) \le \eta$ for some $\eta \in (0,1)$ small enough and $\sigma \in \mathbb{N}$ large enough, the smallness condition (8.2) is fulfilled. □

**Lemma 8.3** (Well posedness of the operator $\mathcal{L}_3$ in (2.66))**.** *Let $T > 0$, $t_0 \in [0,T]$ and let $\mathcal{L}_3 = \partial_t \mathbb{I}_2 + \mathrm{i}\mu\Sigma\partial_{xx} + \mathrm{i}A_1^{(3)}\partial_x + \mathrm{i}A_0^{(3)}$ be the operator defined in* (2.66)*. There exists $\eta \in (0,1)$ small enough and universal constants $\sigma, \tau > 0$ large enough such that if $N_T(\sigma) \le \eta$ (see the definition* (2.6)*), then for any $s \in [0, r-\tau]$, $\mathbf{h}_{in} \in \mathbf{H}^s(\mathbb{T})$, $\mathbf{g} \in C([0,T], \mathbf{H}^s(\mathbb{T}))$, there exists a unique solution $\mathbf{h} \in C([0,T], \mathbf{H}^s(\mathbb{T}))$ such that*

$$\begin{cases} \mathcal{L}_3 \mathbf{h} = \mathbf{g} \\ \mathbf{h}(t_0, \cdot) = \mathbf{h}_{in} \end{cases} \tag{8.14}$$

*satisfying the estimate*

$$\|\mathbf{h}\|_{T,s} \lesssim_s \|\mathbf{h}_{in}\|_{H^s_x} + \|\mathbf{g}\|_{T,s} + N_T(s+\sigma)\|\mathbf{h}_{in}\|_{L^2_x} .$$

*Proof.* Let $\mathcal{M}$ be the transformation defined in (2.71). By (2.73), defining $\widetilde{\mathbf{h}}(t,\cdot) := \mathcal{M}^{-1}(t)\mathbf{h}(t,\cdot)$, $\widetilde{\mathbf{g}} := \mathcal{M}^{-1}(t)\mathbf{g}(t,\cdot)$, the Cauchy problem (8.14) transforms into the Cauchy problem

$$\begin{cases} \mathcal{L}_4 \widetilde{\mathbf{h}} = \widetilde{\mathbf{g}} \\ \widetilde{\mathbf{h}}(t_0, \cdot) = \widetilde{\mathbf{h}}_{in} \end{cases} .$$

Then the statement follows by Lemma 8.2 and by estimate (2.81) on the transformation $\mathcal{M}$. □

**Lemma 8.4** (Well posedness of the operator $\mathcal{L}_2$ in (2.49))**.** *Let $T > 0$, $t_0 \in [0,T]$ and let $\mathcal{L}_2 = \partial_t \mathbb{I}_2 + \mathrm{i}\mu\Sigma\partial_{xx} + \mathrm{i}A_1^{(2)}\partial_x + \mathrm{i}A_0^{(2)}$ be the operator defined in* (2.49)*. There exists $\eta \in (0,1)$ small enough and universal constants $\sigma, \tau > 0$ large enough such that if $N_T(\sigma) \le \eta$ (see the definition* (2.6)*), then for any $s \in [0, r-\tau]$, $\mathbf{h}_{in} \in \mathbf{H}^s(\mathbb{T})$, $\mathbf{g} \in C([0,T], \mathbf{H}^s(\mathbb{T}))$, there exists a unique solution $\mathbf{h} \in C([0,T], \mathbf{H}^s(\mathbb{T}))$ such that*

$$\begin{cases} \mathcal{L}_2 \mathbf{h} = \mathbf{g} \\ \mathbf{h}(t_0, \cdot) = \mathbf{h}_{in} \end{cases} \tag{8.15}$$

*satisfying the estimate*

$$\|\mathbf{h}\|_{T,s} \lesssim_s \|\mathbf{h}_{in}\|_s + \|\mathbf{g}\|_{T,s} + N_T(s+\sigma)\|\mathbf{h}_{in}\|_0 .$$

*Proof.* Let $\mathcal{T}$ be the transformation defined in (2.58). By (2.60), defining $\widetilde{\mathbf{h}}(t,\cdot) := \mathcal{T}^{-1}(t)\mathbf{h}(t,\cdot)$, $\widetilde{\mathbf{g}} := \mathcal{T}^{-1}(t)\mathbf{g}(t,\cdot)$, the Cauchy problem (8.15) transforms into the Cauchy problem

$$\begin{cases} \mathcal{L}_3 \widetilde{\mathbf{h}} = \widetilde{\mathbf{g}} \\ \widetilde{\mathbf{h}}(t_0, \cdot) = \widetilde{\mathbf{h}}_{in} \end{cases} .$$

Then the statement follows by Lemma 8.3 and by estimate (2.69) on the transformation $\mathcal{T}$. □

**Lemma 8.5** (Well posedness of the operator $\mathcal{L}_1$ in (2.37))**.** *Let $T > 0$, $t_0 \in [0,T]$ and let $\mathcal{L}_1 = \partial_t \mathbb{I}_2 + \mathrm{i}m_2\Sigma\partial_{yy} + \mathrm{i}A_1^{(1)}\partial_y + \mathrm{i}A_0^{(1)}$ be the operator defined in* (2.37)*. There exists $\eta \in (0,1)$ small enough and universal constants $\sigma, \tau > 0$ large enough such that if $N_T(\sigma) \le \eta$ (see the definition* (2.6)*), then for any $s \in [0, r-\tau]$, $\mathbf{h}_{in} \in \mathbf{H}^s(\mathbb{T})$, $\mathbf{g} \in C([0,T], \mathbf{H}^s(\mathbb{T}))$, there exists a unique solution $\mathbf{h} \in C([0,T], \mathbf{H}^s(\mathbb{T}))$ such that*

$$\begin{cases} \mathcal{L}_1 \mathbf{h} = \mathbf{g} \\ \mathbf{h}(t_0, \cdot) = \mathbf{h}_{in} \end{cases} \tag{8.16}$$

*satisfying the estimate*

$$\|\mathbf{h}\|_{T,s} \lesssim_s \|\mathbf{h}_{in}\|_s + \|\mathbf{g}\|_{T,s} + N_T(s+\sigma)\|\mathbf{h}_{in}\|_0 .$$

*Proof.* Let $\mathcal{B}$ be the transformation defined in (2.43). By (2.49), defining $\widetilde{\mathbf{h}}(t,\cdot) := \mathcal{B}^{-1}(t)\mathbf{h}(t,\cdot)$, $\widetilde{\mathbf{g}} := \rho^{-1}\mathcal{B}^{-1}(t)\mathbf{g}(t,\cdot)$, the Cauchy problem (8.16) transforms into the Cauchy problem

$$\begin{cases} \mathcal{L}_2 \widetilde{\mathbf{h}} = \widetilde{\mathbf{g}} \\ \widetilde{\mathbf{h}}(t_0, \cdot) = \mathbf{h}_{in} \end{cases}$$

(note that for a function $h(x)$ depending only on the variable $x$, $\mathcal{B}^{-1}h = h$). Then the statement follows by Lemma 8.4 and by estimate (2.55) on the transformation $\mathcal{B}$. $\square$

**Lemma 8.6** (Well posedness of the operator $\mathcal{L}_0$ in (2.13))**.** *Let $T > 0$, $t_0 \in [0,T]$ and let $\mathcal{L}_0 = \partial_t \mathbb{I}_2 + \mathrm{i}(\Sigma + A_2^{(0)})\partial_{xx} + \mathrm{i}A_1^{(0)}\partial_x + \mathrm{i}A_0^{(0)}$ be the operator defined in (2.13). There exists $\eta \in (0,1)$ small enough and universal constants $\sigma, \tau > 0$ large enough such that if $N_T(\sigma) \le \eta$ (see the definition (2.6)), then for any $s \in [0, r-\tau]$, $\mathbf{h}_{in} \in \mathbf{H}^s(\mathbb{T})$, $\mathbf{g} \in C([0,T], \mathbf{H}^s(\mathbb{T}))$, there exists a unique solution $\mathbf{h} \in C([0,T], \mathbf{H}^s(\mathbb{T}))$ such that*

$$\begin{cases} \mathcal{L}_0 \mathbf{h} = \mathbf{g} \\ \mathbf{h}(t_0, \cdot) = \mathbf{h}_{in} \end{cases} \tag{8.17}$$

*satisfying the estimate*

$$\|\mathbf{h}\|_{T,s} \lesssim_s \|\mathbf{h}_{in}\|_s + \|\mathbf{g}\|_{T,s} + N_T(s+\sigma)\|\mathbf{h}_{in}\|_0 .$$

*Proof.* Let $\mathcal{A}$ be the transformation defined in (2.23). By (2.29), defining $\widetilde{\mathbf{h}}(t,\cdot) := \mathcal{A}^{-1}(t)\mathbf{h}(t,\cdot)$, $\widetilde{\mathbf{g}} := \mathcal{A}^{-1}(t)\mathbf{g}(t,\cdot)$, the Cauchy problem (8.17) transforms into the Cauchy problem

$$\begin{cases} \mathcal{L}_1 \widetilde{\mathbf{h}} = \widetilde{\mathbf{g}} \\ \widetilde{\mathbf{h}}(t_0, \cdot) = \widetilde{\mathbf{h}}_{in} . \end{cases}$$

Then the statement follows by Lemma 8.5 and by estimate (2.40) on the transformation $\mathcal{A}$. $\square$

**Lemma 8.7** (Well posedness of the operator $\mathcal{L}$ in (2.3))**.** *Let $T > 0$, $t_0 \in [0,T]$ and let $\mathcal{L} = \partial_t \mathbb{I}_2 + \mathrm{i}(\Sigma + A_2)\partial_{xx} + \mathrm{i}A_1\partial_x + \mathrm{i}A_0$ be the operator defined in (2.3). There exists $\eta \in (0,1)$ small enough and universal constants $\sigma, \tau > 0$ large enough such that if $N_T(\sigma) \le \eta$ (see the definition (2.6)), then for any $s \in [0, r-\tau]$, $\mathbf{h}_{in} \in \mathbf{H}^s(\mathbb{T})$, $\mathbf{g} \in C([0,T], \mathbf{H}^s(\mathbb{T}))$, there exists a unique solution $\mathbf{h} \in C([0,T], \mathbf{H}^s(\mathbb{T}))$ such that*

$$\begin{cases} \mathcal{L} \mathbf{h} = \mathbf{g} \\ \mathbf{h}(t_0, \cdot) = \mathbf{h}_{in} \end{cases} \tag{8.18}$$

*satisfying the estimate*

$$\|\mathbf{h}\|_{T,s} \lesssim_s \|\mathbf{h}_{in}\|_s + \|\mathbf{g}\|_{T,s} + N_T(s+\sigma)\|\mathbf{h}_{in}\|_0 .$$

*Proof.* Let $\mathcal{S}$ be the transformation defined in (2.10). By (2.13), defining $\widetilde{\mathbf{h}}(t,\cdot) := \mathcal{S}^{-1}(t)\mathbf{h}(t,\cdot)$, $\widetilde{\mathbf{g}} := \mathcal{S}^{-1}(t)\mathbf{g}(t,\cdot)$, the Cauchy problem (8.18) transforms into the Cauchy problem

$$\begin{cases} \mathcal{L}_0 \widetilde{\mathbf{h}} = \widetilde{\mathbf{g}} \\ \widetilde{\mathbf{h}}(t_0, \cdot) = \widetilde{\mathbf{h}}_{in} . \end{cases}$$

Then the statement follows by Lemma 8.6 and by estimate (2.20) on the transformation $\mathcal{S}$. $\square$

# 9 Appendix D. Nash-Moser-Hörmander theorem

We state here the Nash-Moser-Hörmander theorem, proved in [15], which we use in Section 5 to prove Theorems 1.1 and 1.3.

Let $(E_a)_{a \ge 0}$ be a decreasing family of Banach spaces with continuous injections $E_b \hookrightarrow E_a$,

$$\|u\|_a \le \|u\|_b \quad \text{for } a \le b. \tag{9.1}$$

Set $E_\infty = \cap_{a \ge 0} E_a$ with the weakest topology making the injections $E_\infty \hookrightarrow E_a$ continuous. Assume that $S_j : E_0 \to E_\infty$ for $j = 0, 1, \dots$ are linear operators such that, with constants $C$ bounded when

$a$ and $b$ are bounded, and independent of $j$,

$$\|S_j u\|_a \le C\|u\|_a \qquad \text{for all } a; \tag{9.2}$$
$$\|S_j u\|_b \le C2^{j(b-a)}\|S_j u\|_a \qquad \text{if } a < b; \tag{9.3}$$
$$\|u - S_j u\|_b \le C2^{-j(a-b)}\|u - S_j u\|_a \qquad \text{if } a > b; \tag{9.4}$$
$$\|(S_{j+1} - S_j)u\|_b \le C2^{j(b-a)}\|(S_{j+1} - S_j)u\|_a \quad \text{for all } a, b. \tag{9.5}$$

Set

$$R_0 u := S_1 u, \qquad R_j u := (S_{j+1} - S_j)u, \quad j \ge 1. \tag{9.6}$$

Thus

$$\|R_j u\|_b \le C2^{j(b-a)}\|R_j u\|_a \quad \text{for all } a, b. \tag{9.7}$$

Bound (9.7) for $j \ge 1$ is (9.5), while, for $j = 0$, it follows from (9.1) and (9.3).

We also assume that

$$\|u\|_a^2 \le C\sum_{j=0}^{\infty}\|R_j u\|_a^2 \quad \forall a \ge 0, \tag{9.8}$$

with $C$ bounded for $a$ bounded (a sort of "orthogonality property" of the smoothing operators).

Now let us suppose that we have another family $F_a$ of decreasing Banach spaces with smoothing operators having the same properties as above. We use the same notation also for the smoothing operators.

**Theorem 9.1.** *Let $a_1, a_2, \alpha, \beta, a_0, \mu$ be real numbers with*

$$0 \le a_0 \le \mu \le a_1, \qquad a_1 + \frac{\beta}{2} < \alpha < a_1 + \beta, \qquad 2\alpha < a_1 + a_2. \tag{9.9}$$

*Let $V$ be a convex neighborhood of $0$ in $E_\mu$. Let $\Phi$ be a map from $V$ to $F_0$ such that $\Phi : V \cap E_{a+\mu} \to F_a$ is of class $C^2$ for all $a \in [0, a_2 - \mu]$, with*

$$\|\Phi''(u)[v, w]\|_a \le C\big(\|v\|_{a+\mu}\|w\|_{a_0} + \|v\|_{a_0}\|w\|_{a+\mu} + \|u\|_{a+\mu}\|v\|_{a_0}\|w\|_{a_0}\big) \tag{9.10}$$

*for all $u \in V \cap E_{a+\mu}$, $v, w \in E_{a+\mu}$. Also assume that $\Phi'(v)$, for $v \in E_\infty \cap V$ belonging to some ball $\|v\|_{a_1} \le \delta_1$, has a right inverse $\Psi(v)$ mapping $F_\infty$ to $E_{a_2}$, and that*

$$\|\Psi(v)g\|_a \le C(\|g\|_{a+\beta-\alpha} + \|g\|_0\|v\|_{a+\beta}) \quad \forall a \in [a_1, a_2]. \tag{9.11}$$

*For all $A > 0$ there exist $\delta, C_1 > 0$ such that, for every $g \in F_\beta$ satisfying*

$$\sum_{j=0}^{\infty}\|R_j g\|_\beta^2 \le A\|g\|_\beta^2, \quad \|g\|_\beta \le \delta, \tag{9.12}$$

*there exists $u \in E_\alpha$, with $\|u\|_\alpha \le C_1\|g\|_\beta$, solving $\Phi(u) = \Phi(0) + g$.*

*Moreover, let $c > 0$ and assume that (9.10) holds for all $a \in [0, a_2 + c - \mu]$, $\Psi(v)$ maps $F_\infty$ to $E_{a_2+c}$, and (9.11) holds for all $a \in [a_1, a_2 + c]$. If $g$ satisfies (9.12) and, in addition, $g \in F_{\beta+c}$ with*

$$\sum_{j=0}^{\infty}\|R_j g\|_{\beta+c}^2 \le A_c\|g\|_{\beta+c}^2 \tag{9.13}$$

*for some $A_c$, then the solution $u$ belongs to $E_{\alpha+c}$, with $\|u\|_{\alpha+c} \le C_{1,c}\|g\|_{\beta+c}$.*

**Acknowledgements**. Baldi and Haus were supported by the European Research Council under FP7 (ERC Project 306414), by PRIN 2012 "Variational and perturbative aspects of nonlinear differential problems", and partially by Programme STAR (UniNA and Compagnia di San Paolo). Montalto was partially supported by the Swiss National Science Foundation.

# References


[1] F. Alabau-Boussouira, J.-M. Coron, G. Olive, *Internal controllability of first order quasilinear hyperbolic systems with a reduced number of controls.* (2015) https://hal.archives-ouvertes.fr/hal-01139980.

[2] T. Alazard, *Boundary observability of gravity water waves.* Preprint 2015 (arXiv:1506.08520).

[3] T. Alazard, *Stabilization of gravity water waves.* Preprint 2016 (arXiv:1603.08541).

[4] T. Alazard, *Stabilization of the water-wave equations with surface tension.* Preprint 2016 (arXiv:1610.07917).

[5] T. Alazard, P. Baldi, *Gravity capillary standing water waves.* Arch. Ration. Mech. Anal. 217 (2015), 741-830.

[6] T. Alazard, P. Baldi, D. Han-Kwan, *Control of water waves.* J. Eur. Math. Soc. (JEMS), in print.

[7] T. Alazard, N. Burq, C. Zuily, *Cauchy theory for the gravity water waves system with non-localized initial data.* Ann. Inst. H. Poincaré (C) Anal. Non Linéaire 33 (2016), 337-395.

[8] S. Alinhac, P. Gérard, *Pseudo-differential operators and the Nash-Moser theorem.* Grad. Studies in Math. 82. Amer. Math. Soc., Providence, RI, 2007.

[9] N. Anantharaman, F. Macià, *Semiclassical measures for the Schrödinger equation on the torus.* J. Eur. Math. Soc. (JEMS) 16 (2014), no. 6, 1253-1288.

[10] P. Baldi, *Periodic solutions of fully nonlinear autonomous equations of Benjamin-Ono type.* Ann. Inst. H. Poincaré (C) Anal. Non Linéaire, 30, no. 1, 33-77, 2013.

[11] P. Baldi, M. Berti, R. Montalto, *KAM for quasi-linear and fully nonlinear forced perturbations of Airy equation.* Math. Annalen 359 (2014), 471-536.

[12] P. Baldi, M. Berti, R. Montalto, *KAM for autonomous quasi-linear perturbations of KdV.* Ann. I. H. Poincaré (C) Anal. Non Linéaire 33 (2016), 1589-1638.

[13] P. Baldi, M. Berti, R. Montalto, *KAM for autonomous quasi-linear perturbations of mKdV.* Bollettino Unione Matematica Italiana, 9, 143-188, 2016.

[14] P. Baldi, G. Floridia, E. Haus, *Exact controllability for quasi-linear perturbations of KdV.* Anal. PDE 10 (2017), 281-322.

[15] P. Baldi, E. Haus, *A Nash-Moser-Hörmander implicit function theorem with applications to control and Cauchy problems for PDEs.* Preprint 2016 (arXiv:1609.00213).

[16] K. Beauchard, *Local controllability of a 1-D Schrödinger equation.* J. Math. Pures Appl. (9) 84 (2005), no. 7, 851-956.

[17] K. Beauchard, *Local controllability of a one-dimensional beam equation.* SIAM J. Control Optim. 47 (2008), no. 3, 1219-1273.

[18] K. Beauchard, J.-M. Coron, *Controllability of a quantum particle in a moving potential well.* J. Funct. Anal. 232 (2006), no. 2, 328-389.

[19] K. Beauchard, C. Laurent, *Local controllability of 1D linear and nonlinear Schrödinger equations with bilinear control.* J. Math. Pures Appl. (9) 94 (2010), no. 5, 520-554.

[20] M. Berti, R. Montalto, *Quasi-periodic standing wave solutions for gravity-capillary water waves.* Preprint 2016 (arXiv:1602.02411v1).

[21] J.-M. Bony, *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires.* Ann. Sci. École Norm. Sup. (4), 14(2):209–246, 1981.

[22] J. Bourgain, N. Burq, M. Zworski, *Control for Schrödinger operators on 2-tori: rough potentials.* J. Eur. Math. Soc. (JEMS) 15 (2013), no. 5, 1597-1628.

[23] J.-M. Coron, *Control and Nonlinearity.* Math. Surveys Monogr. 136, AMS, Providence, RI, 2007.

[24] B. Dehman, P. Gérard, G. Lebeau, *Stabilization and control for the nonlinear Schrödinger equation on a compact surface.* Math. Z. 254 (2006), no. 4, 729-749.

[25] B. Dehman, G. Lebeau, *Analysis of the HUM control operator and exact controllability for semilinear waves in uniform time.* SIAM J. Control Optim. 48 (2009), no. 2, 521-550.

[26] R. Feola, *KAM for quasi-linear forced hamiltonian NLS.* Preprint 2016 (arXiv:1602.01341).

[27] R. Feola, M. Procesi, *Quasi-periodic solutions for fully nonlinear forced reversible Schrödinger equations.* J. Differential Equations 259 (2015), no. 7, 3389-3447.

[28] L. Hörmander, *The boundary problems of physical geodesy.* Arch. Rational Mech. Anal. 62 (1976), no. 1, 1-52.

[29] L. Hörmander, *The Nash-Moser theorem and paradifferential operators.* Analysis, et cetera, 429-449, Academic Press, Boston, MA, 1990.

[30] G. Iooss, P.I. Plotnikov, J. Toland, *Standing waves on an infinitely deep perfect fluid under gravity.* Arch. Ration. Mech. Anal. 177 (2005), no. 3, 367-478.

[31] S. Jaffard, *Contrôle interne exact des vibrations d'une plaque rectangulaire.* Portugal. Math. 47 (1990), no. 4, 423-429.

[32] I. Lasiecka, R. Triggiani, *Optimal regularity, exact controllability and uniform stabilization of Schrödinger equations with Dirichlet control.* Differential Integral Equations 5 (1992), no. 3, 521-535.

[33] C. Laurent, *Global controllability and stabilization for the nonlinear Schrödinger equation on an interval.* ESAIM-COCV 16 (2010), 356-379.

[34] C. Laurent, *Internal control of the Schrödinger equation.* Math. Control Relat. Fields 4 (2014), no. 2, 161-186.

[35] G. Lebeau, *Contrôle de l'équation de Schrödinger.* J. Math. Pures Appl. (9) 71 (1992), no. 3, 267-291.

[36] G. Métivier, *Para-differential calculus and applications to the Cauchy problem for nonlinear systems.* Volume 5 of Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series. Edizioni della Normale, Pisa, 2008.

[37] S. Micu, E. Zuazua, *An introduction to the controllability of partial differential equations.* Quelques questions de théorie du contrôle. In: Sari T. (ed.) Collection Travaux en Cours (2004), 69-157.

[38] R. Montalto, *Quasi-periodic solutions of forced Kirchhoff equation*, Preprint 2016 (arXiv:1602.05093v1).

[39] J. Moser, *A rapidly convergent iteration method and non-linear differential equations. I.* Ann. Scuola Norm. Sup. Pisa (3) 20 (1966), 265-315.

[40] P. Rabinowitz, *Periodic solutions of nonlinear hyperbolic partial differential equations. I.* Comm. Pure Appl. Math. 20 (1967), 145-205.

[41] L. Rosier, B.-Y. Zhang, *Local exact controllability and stabilizability of the nonlinear Schrödinger equation on a bounded interval.* SIAM J. Control Optim. 48 (2009), no. 2, 972-992.

[42] M.E. Taylor, Pseudodifferential operators and nonlinear PDE. Volume 100 of Progress in Mathematics, Birkhäuser Boston Inc., Boston, MA, 1991.

[43] E. Zuazua, *Remarks on the controllability of the Schrödinger equation.* Quantum control: mathematical and numerical challenges, 193-211, CRM Proc. Lecture Notes, 33, Amer. Math. Soc., Providence, RI, 2003.



Pietro Baldi
Dipartimento di Matematica e Applicazioni "R. Caccioppoli"
Università di Napoli Federico II
Via Cintia, 80126 Napoli, Italy
pietro.baldi@unina.it

Emanuele Haus
Dipartimento di Matematica e Applicazioni "R. Caccioppoli"
Università di Napoli Federico II
Via Cintia, 80126 Napoli, Italy
emanuele.haus@unina.it

Riccardo Montalto
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
CH-8057 Zürich
riccardo.montalto@math.uzh.ch