Università degli Studi di Milano
Scuola di Dottorato di Ricerca in Matematica

Université de Picardie Jules Verne
Ecole Doctorale en Sciences, Technologie et Santé (ED 585)

Tesi di Dottorato in Cotutela

DIPARTIMENTO DI MATEMATICA *“FEDERIGO ENRIQUES”*
CORSO DI DOTTORATO IN MATEMATICA, CICLO XXXI

LABORATOIRE AMIÉNOIS DE MATHÉMATIQUE FONDAMENTALE ET APPLIQUÉE (LAMFA)

Minimization Problems Involving Nonlocal Functionals: Nonlocal Minimal Surfaces and a Free Boundary Problem

MAT/05

DOTTORANDO
Luca LOMBARDINI

TUTOR
Enrico VALDINOCI (Milano)

Alberto FARINA (Amiens)

COORDINATORE DEL DOTTORATO (Milano)
Vieri MASTROPIETRO

(ANNO ACCADEMICO 2017-2018)

Minimization Problems Involving Nonlocal Functionals: Nonlocal Minimal Surfaces and a Free Boundary Problem

Luca Lombardini

E-mail address: luca.lombardini@unimi.it

Università degli Studi di Milano, Dipartimento di Matematica, Via Cesare Saldini 50, 20133 Milano, Italy, and Université de Picardie Jules Verne, Faculté des Sciences, 33 Rue Saint Leu, 80039 Amiens CEDEX 1, France

Abstract. This doctoral thesis is devoted to the analysis of some minimization problems that involve nonlocal functionals. We are mainly concerned with the s-fractional perimeter and its minimizers, the s-minimal sets. We investigate the behavior of sets having (locally) finite fractional perimeter and we establish existence and compactness results for (locally) s-minimal sets. We study the s-minimal sets in highly nonlocal regimes, that correspond to small values of the fractional parameter s. We introduce a functional framework for studying those s-minimal sets that can be globally written as subgraphs. In particular, we prove existence and uniqueness results for minimizers of a fractional version of the classical area functional and we show the equivalence between minimizers and various notions of solution of the fractional mean curvature equation. We also prove a flatness result for entire nonlocal minimal graphs having some partial derivatives bounded from either above or below.
Moreover, we consider a free boundary problem, which consists in the minimization of a functional defined as the sum of a nonlocal energy, plus the classical perimeter. Concerning this problem, we prove uniform energy estimates and we study the blow-up sequence of a minimizer—in particular establishing a Weiss-type monotonicity formula.

Contents

Introduction

0.1. Summary

This doctoral thesis is devoted to the analysis of some minimization problems that involve nonlocal functionals. Nonlocal operators have attracted an increasing attention in the latest years, both because of their mathematical interest and for their applications—e.g., in modelling anomalous diffusion processes or long-range phase transitions. We refer the interested reader to [17] for an introduction to nonlocal problems.

In this thesis, we are mainly concerned with the s-fractional perimeter—which can be considered as a fractional and nonlocal version of the classical perimeter introduced by De Giorgi and Caccioppoli—and its minimizers, the s-minimal sets, that were first considered in [21]. The boundaries of these s-minimal sets are usually referred to as nonlocal minimal surfaces. In particular:

- we investigate the behavior of sets having (locally) finite fractional perimeter, proving the density of smooth open sets, an optimal asymptotic result for $s \to 1^-$, and studying the connection existing between the fractional perimeter and sets having fractal boundaries.
- We establish existence and compactness results for minimizers of the fractional perimeter, that extend those proved in [21].
- We study the s-minimal sets in highly nonlocal regimes, that correspond to small values of the fractional parameter s. We show that, in this case, the minimizers exhibit a behavior completely different from that of their local counterparts—the (classical) minimal surfaces.
- We introduce a functional framework for studying those s-minimal sets that can be globally written as subgraphs. In particular, we prove existence and uniqueness results for minimizers of a fractional version of the classical area functional and we prove a rearrangement inequality that implies that the subgraphs of these minimizers are minimizing for the fractional perimeter. We refer to the boundaries of such minimizers as nonlocal minimal graphs. We also show the equivalence between minimizers and various notions of solution—namely, weak solutions, viscosity solutions and smooth pointwise solutions—of the fractional mean curvature equation.
- We prove a flatness result for entire nonlocal minimal graphs having some partial derivatives bounded from either above or below—thus, in particular, extending to the fractional framework classical theorems due to Bernstein and Moser.

We also consider a free boundary problem, which consists in the minimization of a functional defined as the sum of a nonlocal energy, plus the classical perimeter of the interface of separation between the two phases. Concerning this problem:

- we prove the existence of minimizers and we introduce an equivalent minimization problem which has a "local nature"—through the extension technique of [23].

- We prove uniform energy estimates and we study the blow-up sequence of a minimizer. In particular, we establish a monotonicity formula that implies that blow-up limits are homogeneous.
- We investigate the regularity of the free boundary in the case in which the perimeter has a dominant role over the nonlocal energy.

We also mention that the last chapter of the thesis consists in a paper that provides a mathematical model which describes the formation of groups of penguins on the shore at sunset. During the occasion of a research trip at the University of Melbourne, we observed the Phillip Island penguin parade and we were so fascinated by the peculiar behavior of the little penguins that we decided to try and describe it mathematically.

The thesis is divided into seven chapters, each of which is based on one of the following research articles, that I have written—together with collaborators—during my PhD:

(1) *Fractional perimeters from a fractal perspective*, published in Advanced Nonlinear Studies—see [77].
(2) *Approximation of sets of finite fractional perimeter by smooth sets and comparison of local and global s-minimal surfaces*, published in Interfaces and Free Boundaries—see [76].
(3) *Complete stickiness of nonlocal minimal surfaces for small values of the fractional parameter*, joint work with C. Bucur and E. Valdinoci, published in Annales de l'Institut Henri Poincaré Analyse Non Linéaire—see [16].
(4) *On nonlocal minimal graphs*, joint work with M. Cozzi, currently in preparation.
(5) *Bernstein-Moser-type results for nonlocal minimal graphs*, joint work with M. Cozzi and A. Farina, published in Communications in Analysis and Geometry—see [31].
(6) A partial, preliminary, version of the article *A free boundary problem: superposition of nonlocal energy plus classical perimeter*, joint work with S. Dipierro and E. Valdinoci, currently in preparation.
(7) *The Phillip Island penguin parade (a mathematical treatment)*, joint work with S. Dipierro, P. Miraglio and E. Valdinoci, published in ANZIAM Journal—see [41].

The Appendix contains some auxiliary results that have been exploited throughout the thesis.

0.2. A more detailed introduction

We now proceed to give a detailed description of the contents and main results of this thesis. We observe that each topic has its own, more in-depth, presentation, at the beginning of the corresponding chapter. Moreover, each chapter has its own table of contents, to help the reader navigate through the sections.

0.2.1. Sets of (locally) finite fractional perimeter. The s-fractional perimeter and its minimizers, the s-minimal sets, were introduced in [21], in 2010, mainly motivated by applications to phase transition problems in the presence of long-range interactions. In the subsequent years, they have attracted a lot of interest, especially concerning the regularity theory and the qualitative behavior of the boundaries of the s-minimal sets, which are the so-called nonlocal minimal surfaces. We refer the interested reader to [98] and [17, Chapter 6] for an introduction, and to the survey [47] for some recent developments.

In particular, we mention that, even if finding the optimal regularity of nonlocal minimal surfaces is still an engaging open problem, it is known that nonlocal minimal surfaces are $(n-1)$-rectifiable. More precisely, they are smooth, except possibly for a singular set of Hausdorff dimension at most equal to $n-3$ (see [21], [92] and [58]). As a consequence, an s-minimal set has (locally) finite perimeter (in the sense of De Giorgi and Caccioppoli)—and actually some uniform estimates for the (classical) perimeter of s-minimal sets are available (see [28]).

On the other hand, the boundary of a generic set E having finite s-perimeter can be very irregular and indeed it can be "nowhere rectifiable", like in the case of the von Koch snowflake.

Actually, the s-perimeter can be used (following the seminal paper [99]) to define a "fractal dimension" for the measure theoretic boundary

$$\partial^- E := \{x \in \mathbb{R}^n \,|\, 0 < |E \cap B_r(x)| < \omega_n r^n \text{ for every } r > 0\},$$

of a set $E \subseteq \mathbb{R}^n$.

Before going on, we recall the definition of the s-perimeter. Given a fractional parameter $s \in (0,1)$, we define the interaction

$$\mathcal{L}_s(A,B) := \int_A \int_B \frac{1}{|x-y|^{n+s}}\,dx\,dy,$$

for every couple of disjoint sets A, $B \subseteq \mathbb{R}^n$. Then the *s-perimeter* of a set $E \subseteq \mathbb{R}^n$ in an open set $\Omega \subseteq \mathbb{R}^n$ is defined as

$$\operatorname{Per}_s(E,\Omega) := \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\cap\Omega) + \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\setminus\Omega) + \mathcal{L}_s(E\setminus\Omega, \mathcal{C}E\cap\Omega).$$

We simply write $\operatorname{Per}_s(E) := \operatorname{Per}_s(E,\mathbb{R}^n)$.

We say that a set $E \subseteq \mathbb{R}^n$ has *locally finite s-perimeter* in an open set $\Omega \subseteq \mathbb{R}^n$ if

$$\operatorname{Per}_s(E,\Omega') < \infty \qquad \text{for every open set } \Omega' \Subset \Omega.$$

We observe that we can rewrite the s-perimeter as

$$\operatorname{Per}_s(E,\Omega) = \frac{1}{2} \iint_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\,dx\,dy. \tag{0.1}$$

Formula (0.1) shows that the fractional perimeter is, roughly speaking, the Ω-contribution to the $W^{s,1}$-seminorm of the characteristic function χ_E.

This functional is nonlocal, in that we need to know the set E in the whole of $\mathbb{R}^n$ even to compute its s-perimeter in a small bounded domain Ω (contrary to what happens with the classical perimeter or the $\mathcal{H}^{n-1}$ measure, which are local functionals). Moreover, the s-perimeter is "fractional", in the sense that the $W^{s,1}$-seminorm measures a fractional order of regularity.

We also observe that we can split the s-perimeter as

$$\operatorname{Per}_s(E,\Omega) = \operatorname{Per}_s^L(E,\Omega) + \operatorname{Per}_s^{NL}(E,\Omega),$$

where

$$\operatorname{Per}_s^L(E,\Omega) := \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\cap\Omega) = \frac{1}{2}[\chi_E]_{W^{s,1}(\Omega)}$$

can be thought of as the "local part" of the fractional perimeter, and

$$\begin{aligned}\operatorname{Per}_s^{NL}(E,\Omega) &:= \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\setminus\Omega) + \mathcal{L}_s(E\setminus\Omega, \mathcal{C}E\cap\Omega)\\ &= \int_\Omega \int_{\mathcal{C}\Omega} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\,dx\,dy,\end{aligned}$$

can be thought of as the "nonlocal part".

0.2.1.1. ***Fractal boundaries***. In 1991, in the paper [99] the author suggested using the index s of the fractional seminorm $[\chi_E]_{W^{s,1}(\Omega)}$ (and more general continuous families of functionals satisfying appropriate generalized coarea formulas) as a way to measure the codimension of the measure theoretic boundary $\partial^- E$ of the set E in Ω. He proved that the fractal dimension obtained in this way,

$$\mathrm{Dim}_F(\partial^- E, \Omega) := n - \sup\{s \in (0,1) \,|\, [\chi_E]_{W^{s,1}(\Omega)} < \infty\},$$

is less than or equal to the (upper) Minkowski dimension.

The relationship between the Minkowski dimension of the boundary of E and the fractional regularity (in the sense of Besov spaces) of the characteristic function χ_E was investigated also in [94], in 1999. In particular, in [94, Remark 3.10], the author proved that the dimension Dim_F of the von Koch snowflake S coincides with its Minkowski dimension, exploiting the fact that S is a John domain.

The Sobolev regularity of a characteristic function χ_E was further studied in [52], in 2013, where the authors consider the case in which the set E is a quasiball. Since the von Koch snowflake S is a typical example of quasiball, the authors were able to prove that the dimension Dim_F of S coincides with its Minkowski dimension.

In Chapter 1 we compute the dimension Dim_F of the von Koch snowflake S in an elementary way, using only the roto-translation invariance and the scaling property of the s-perimeter and the "self-similarity" of S. More precisely, we show that

$$\mathrm{Per}_s(S) < \infty, \qquad \forall\, s \in \Big(0, 2 - \frac{\log 4}{\log 3}\Big),$$

and

$$\mathrm{Per}_s(S) = \infty, \qquad \forall\, s \in \Big[2 - \frac{\log 4}{\log 3}, 1\Big).$$

The proof can be extended in a natural way to all sets which can be defined in a recursive way similar to that of the von Koch snowflake. As a consequence, we compute the dimension Dim_F of all such sets, without having to require them to be John domains or quasiballs.

Furthermore, we show that we can easily obtain a lot of sets of this kind by appropriately modifying well known self-similar fractals like e.g. the von Koch snowflake, the Sierpinski triangle and the Menger sponge. An example is depicted in Figure 1.

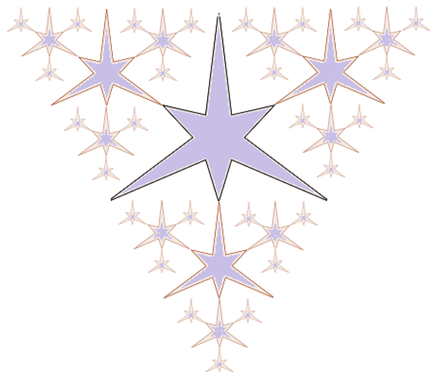


FIGURE 1. *Example of a "fractal" set constructed exploiting the structure of the Sierpinski triangle (seen at the fourth iterative step).*

0.2.1.2. ***Asymptotics*** $s \to 1^-$. The previous discussion shows that the s-perimeter of a set E with an irregular, eventually fractal, boundary can be finite for s below some threshold, $s < \sigma$, and infinite for $s \in (\sigma, 1)$. On the other hand, it is well known that sets with a regular boundary have finite s-perimeter for every s and actually their s-perimeter converges, as s tends to 1, to the classical perimeter, both in the classical sense (see, e.g., [24]) and in the Γ-convergence sense (see, e.g., [5] and also [85] for related results).

In Chapter 1 we exploit [35, Theorem 1] to prove an optimal version of this asymptotic property for a set E having finite classical perimeter in a bounded open set with Lipschitz boundary. More precisely, we prove that if E has finite classical perimeter in a neighborhood of Ω, then

$$\lim_{s\to 1}(1-s)\operatorname{Per}_s(E,\Omega) = \omega_{n-1}\operatorname{Per}(E,\overline{\Omega}).$$

We observe that we lower the regularity requested in [24], where the authors required the boundary ∂E to be $C^{1,\alpha}$, to the optimal regularity (asking E to have only finite perimeter). Moreover, we do not have to ask E to intersect $\partial\Omega$ "transversally", i.e. we do not require

$$\mathcal{H}^{n-1}(\partial^* E \cap \partial\Omega) = 0,$$

with $\partial^* E$ denoting the reduced boundary of E.

Indeed, we prove that the nonlocal part of the s-perimeter converges to the perimeter on the boundary of Ω, i.e. we prove that

$$\lim_{s\to 1}(1-s)\operatorname{Per}_s^{NL}(E,\Omega) = \omega_{n-1}\mathcal{H}^{n-1}(\partial^* E \cap \partial\Omega),$$

which is, to the best of the author's knowledge, a new result.

0.2.1.3. ***Approximation by smooth open sets***. As we have observed in Section 0.2.1.1, sets having finite fractional perimeter can have a very rough boundary, which may indeed be a nowhere rectifiable fractal (like the von Koch snowflake).

This represents a dramatic difference between the fractional and the classical perimeter, since Caccioppoli sets have a "big" portion of the boundary, the so-called reduced boundary, which is $(n-1)$-rectifiable (by De Giorgi's structure Theorem).

Still, we prove in the first part of Chapter 2 that a set has (locally) finite fractional perimeter if and only if it can be approximated (in an appropriate way) by smooth open sets. More precisely, we prove the following:

THEOREM 0.2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. A set $E \subseteq \mathbb{R}^n$ has locally finite s-perimeter in Ω if and only if there exists a sequence $E_h \subseteq \mathbb{R}^n$ of open sets with smooth boundary and $\varepsilon_h \to 0^+$ such that*

$$(i)\quad E_h \xrightarrow{loc} E, \qquad \sup_{h\in\mathbb{N}} \operatorname{Per}_s(E_h,\Omega') < \infty \quad \textit{for every } \Omega' \Subset \Omega,$$
$$(ii)\quad \lim_{h\to\infty}\operatorname{Per}_s(E_h,\Omega') = \operatorname{Per}_s(E,\Omega') \quad \textit{for every } \Omega' \Subset \Omega,$$
$$(iii)\quad \partial E_h \subseteq N_{\varepsilon_h}(\partial E).$$

Moreover, if $\Omega = \mathbb{R}^n$ and the set E is such that $|E| < \infty$ and $\operatorname{Per}_s(E) < \infty$, then

$$|E_h \Delta E| \to 0, \qquad \lim_{h\to\infty}\operatorname{Per}_s(E_h) = \operatorname{Per}_s(E),$$

and we can require each set E_h to be bounded (instead of asking (iii)).

Here above, $N_\delta(\partial E)$ denotes the tubular δ-neighborhood of ∂E.

Such a result is well known for Caccioppoli sets (see, e.g., [79]) and indeed this density property can be used to define the (classical) perimeter functional as the relaxation—with respect to L^1_{loc} convergence—of the $\mathcal{H}^{n-1}$ measure of boundaries of smooth open sets, that is

$$\begin{aligned}\operatorname{Per}(E,\Omega)=\inf\Big\{\liminf_{k\to\infty}\mathcal{H}^{n-1}(\partial E_h\cap\Omega)\,\Big|\,E_h\subseteq\mathbb{R}^n\text{ open with smooth}\\ \text{boundary, s.t. } E_h\xrightarrow{loc}E\Big\}.\end{aligned}\tag{0.2}$$

It is interesting to observe that in [47] the authors have proved, by exploiting the divergence Theorem, that if $E\subseteq\mathbb{R}^n$ is a bounded open set with smooth boundary, then

$$\operatorname{Per}_s(E)=c_{n,s}\int_{\partial E}\int_{\partial E}\frac{2-|\nu_E(x)-\nu_E(y)|^2}{|x-y|^{n+s-2}}d\mathcal{H}^{n-1}_x d\mathcal{H}^{n-1}_y,\tag{0.3}$$

where ν_E denotes the external normal of E and

$$c_{n,s}:=\frac{1}{2s(n+s-2)}.$$

By exploiting equality (0.3), the lower semicontinuity of the s-perimeter and Theorem 0.2.1, we find that, if $E\subseteq\mathbb{R}^n$ is such that $|E|<\infty$, then

$$\begin{aligned}\operatorname{Per}_s(E)=\inf\Big\{\liminf_{h\to\infty}c_{n,s}\int_{\partial E_h}\int_{\partial E_h}\frac{2-|\nu_{E_h}(x)-\nu_{E_h}(y)|^2}{|x-y|^{n+s-2}}d\mathcal{H}^{n-1}_x d\mathcal{H}^{n-1}_y\,\Big|\\ E_h\subseteq\mathbb{R}^n\text{ bounded open set with smooth boundary, s.t. } E_h\xrightarrow{loc}E\Big\}.\end{aligned}$$

This can be thought of as an analogue of (0.2) in the fractional setting.

We also mention that in Section 4.7 we will prove that a subgraph having locally finite s-perimeter in a cylinder $\Omega\times\mathbb{R}$ can be approximated by the subgraphs of smooth functions—and not just by arbitrary smooth open sets.

0.2.2. Nonlocal minimal surfaces. The second part of Chapter 2 is concerned with sets minimizing the fractional perimeter. The boundaries of these minimizers are often referred to as nonlocal minimal surfaces and naturally arise as limit interfaces of long-range interaction phase transition models. In particular, in regimes where the long-range interaction is dominant, the nonlocal Allen-Cahn energy functional Γ-converges to the fractional perimeter (see, e.g., [91]) and the minimal interfaces of the corresponding Allen-Cahn equation approach locally uniformly the nonlocal minimal surfaces (see, e.g., [93]).

We now recall the definition of minimizing sets introduced in [21].

DEFINITION 0.2.2. *Let $\Omega\subseteq\mathbb{R}^n$ be an open set and let $s\in(0,1)$. We say that a set $E\subseteq\mathbb{R}^n$ is* s-minimal *in Ω if $\operatorname{Per}_s(E,\Omega)<\infty$ and*

$$\operatorname{Per}_s(E,\Omega)\le\operatorname{Per}_s(F,\Omega)\quad\textit{for every } F\subseteq\mathbb{R}^n \textit{ s.t. } F\setminus\Omega=E\setminus\Omega.$$

Among the many results, in [21] the authors have proved that, if $\Omega\subseteq\mathbb{R}^n$ is a bounded open set with Lipschitz boundary, then for every fixed set $E_0\subseteq\mathcal{C}\Omega$ there exists a set $E\subseteq\mathbb{R}^n$ which is s-minimal in Ω and such that $E\setminus\Omega=E_0$. The set E_0 is sometimes referred to as *exterior data* and the set E is said to be s-minimal in Ω with respect to the exterior data E_0.

We extend the aforementioned existence result, by proving that, in a generic open set Ω, there exists an s-minimal set with respect to some fixed exterior data $E_0\subseteq\mathcal{C}\Omega$ if and only if there exists a competitor having finite s-perimeter in Ω. More precisely:

THEOREM 0.2.3. *Let $s \in (0,1)$, let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $E_0 \subseteq \mathcal{C}\Omega$. Then, there exists a set $E \subseteq \mathbb{R}^n$ which is s-minimal in Ω, with $E \setminus \Omega = E_0$, if and only if there exists a set $F \subseteq \mathbb{R}^n$ such that $F \setminus \Omega = E_0$ and $\mathrm{Per}_s(F,\Omega) < \infty$.*

As a consequence, we observe that if $\mathrm{Per}_s(\Omega) < \infty$, then there always exists an s-minimal set with respect to the exterior data E_0, for every set $E_0 \subseteq \mathcal{C}\Omega$.

Let us now turn the attention to the case in which the domain of minimization is not bounded. In this situation, it is convenient to introduce the notion of local minimizer.

DEFINITION 0.2.4. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $s \in (0,1)$. We say that a set $E \subseteq \mathbb{R}^n$ is* locally s-minimal *in Ω if E is s-minimal in every open set $\Omega' \Subset \Omega$.*

Notice in particular that we are only requiring E to be of locally finite s-perimeter in Ω and not to have finite s-perimeter in the whole domain. Indeed, the main reason for the introduction of locally s-minimal sets is given by the fact that, in general, the s-perimeter of a set is not finite in unbounded domains.

We have seen in Theorem 0.2.3 that the only obstacle to the existence of an s-minimal set, with respect to some fixed exterior data $E_0 \subseteq \mathcal{C}\Omega$, is the existence of a competitor having finite s-perimeter. On the other hand, we prove that a locally s-minimal set always exists, no matter what the domain Ω and the exterior data are.

THEOREM 0.2.5. *Let $s \in (0,1)$, let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $E_0 \subseteq \mathcal{C}\Omega$. Then, there exists a set $E \subseteq \mathbb{R}^n$ which is locally s-minimal in Ω, with $E \setminus \Omega = E_0$.*

When Ω is a bounded open set with Lipschitz boundary, we show that the two notions of minimizer coincide. That is, if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with Lipschitz boundary and $E \subseteq \mathbb{R}^n$, then

$$E \text{ is } s\text{-minimal in } \Omega \quad \Longleftrightarrow \quad E \text{ is locally } s\text{-minimal in } \Omega.$$

However, we observe that this is not true in an arbitrary open set Ω, since an s-minimal set—in the sense of Definition 0.2.2—may not exist.

As an example, we consider the situation in which the domain of minimization is the cylinder

$$\Omega^\infty := \Omega \times \mathbb{R} \subseteq \mathbb{R}^{n+1},$$

with $\Omega \subseteq \mathbb{R}^n$ a bounded open set with regular boundary. We are interested in exterior data given by the subgraph of some measurable function $\varphi : \mathbb{R}^n \to \mathbb{R}$. That is, we consider the subgraph

$$\mathcal{S}g(\varphi) := \left\{ (x, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x_{n+1} < \varphi(x) \right\},$$

and we want to find a set $E \subseteq \mathbb{R}^{n+1}$ that minimizes—in some sense—the s-perimeter in the cylinder Ω^∞, with respect to the exterior data $E \setminus \Omega^\infty = \mathcal{S}g(\varphi) \setminus \Omega^\infty$.

A motivation for considering such a minimization problem is given by the recent article [43], where the authors have proved that if such a minimizing set E exists—and if φ is a continuous function—then E is actually a global subgraph. More precisely, there exists a function $u : \mathbb{R}^n \to \mathbb{R}$, with $u = \varphi$ in $\mathbb{R}^n \setminus \overline{\Omega}$ and $u \in C(\overline{\Omega})$ such that

$$E = \mathcal{S}g(u).$$

It is readily seen that if a function $u : \mathbb{R}^n \to \mathbb{R}$ is well behaved in Ω, e.g., if $u \in BV(\Omega) \cap L^\infty(\Omega)$, then the local part of the s-perimeter of the subgraph of u is finite,

$$\mathrm{Per}_s^L(\mathcal{S}g(u), \Omega^\infty) < \infty.$$

On the other hand, the nonlocal part of the s-perimeter, in general, is infinite, even for very regular functions u. Indeed, we prove that if $u \in L^\infty(\mathbb{R}^n)$, then

$$\operatorname{Per}_s^{NL}(\mathcal{S}g(u), \Omega^\infty) = \infty.$$

A first consequence of this observation—and of the apriori bound on the "vertical variation" of a minimizing set provided by [43, Lemma 3.3]—is the fact that, if $\varphi \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, then there can not exist a set E which is s-minimal in Ω^∞—in the sense of Definition 0.2.2—with respect to the exterior data $\mathcal{S}g(\varphi) \setminus \Omega^\infty$.

Nevertheless, Theorem 0.2.5 guarantees the existence of a set $E \subseteq \mathbb{R}^{n+1}$ that is locally s-minimal in Ω^∞ and such that $E \setminus \Omega^\infty = \mathcal{S}g(\varphi) \setminus \Omega^\infty$. Therefore, Theorem 0.2.5 and [43, Theorem 1.1] together imply the existence of subgraphs (locally) minimizing the s-perimeter, that is, namely, nonparametric nonlocal minimal surfaces.

A second consequence consists in the fact that we can not define a naive fractional version of the classical area functional as

$$\mathscr{A}_s(u, \Omega) := \operatorname{Per}_s(\mathcal{S}g(u), \Omega^\infty),$$

since this would be infinite even for a function $u \in C_c^\infty(\mathbb{R}^n)$. In Chapter 4 we will get around this issue by introducing an appropriate functional setting for working with subgraphs.

0.2.3. Stickiness effects for small values of s. Chapter 3 is devoted to the study of s-minimal sets in highly nonlocal regimes, i.e. in the case in which the fractional parameter $s \in (0,1)$ is very small. We prove that in this situation the behavior of s-minimal sets, in some sense, degenerates.

Let us first recall some known results concerning the asymptotics as $s \to 1^-$. We have already observed in Section 0.2.1.2 that the s-perimeter converges to the classical perimeter as $s \to 1^-$. Moreover, as $s \to 1^-$, s-minimal sets converge to minimizers of the classical perimeter, both in a "uniform sense" (see [24, 25]) and in the Γ-convergence sense (see [5]). As a consequence, one is able to prove (see [25]) that for s sufficiently close to 1, nonlocal minimal surfaces have the same regularity of classical minimal surfaces. See also [47] for a recent and quite comprehensive survey of the properties of s-minimal sets when s is close to 1.

Furthermore, we observe that also the fractional mean curvature converges, as $s \to 1^-$, to its classical counterpart. To be more precise, let us first recall that the s-fractional mean curvature of a set E at a point $q \in \partial E$ is defined as the principal value integral

$$H_s[E](q) := \text{P.V.} \int_{\mathbb{R}^n} \frac{\chi_{\mathcal{C}E}(y) - \chi_E(y)}{|y-q|^{n+s}}\, dy,$$

that is

$$H_s[E](q) := \lim_{\varrho \to 0^+} H_s^\varrho[E](q), \qquad \text{where} \qquad H_s^\varrho[E](q) := \int_{\mathcal{C}B_\varrho(q)} \frac{\chi_{\mathcal{C}E}(y) - \chi_E(y)}{|y-q|^{n+s}}\, dy.$$

Let us remark that it is indeed necessary to interpret the above integral in the principal value sense, since the integrand is singular and not integrable in a neighborhood of q. On the other hand, if there is enough cancellation between E and $\mathcal{C}E$ in a neighborhood of q—e.g., if ∂E is of class C^2 around q—then the integral is well defined in the principal value sense.

The fractional mean curvature was introduced in [21], where the authors proved that it is the Euler-Lagrange operator appearing in the minimization of the s-perimeter. Indeed, if $E \subseteq \mathbb{R}^n$ is s-minimal in an open set Ω, then

$$H_s[E] = 0 \quad \text{on } \partial E,$$

in an appropriate viscosity sense—for more details see, e.g., Appendix C.2.

It is known (see, e.g., [2, Theorem 12] and [25]) that if $E \subseteq \mathbb{R}^n$ is a set with C^2 boundary, and $n \geq 2$, then for any $x \in \partial E$ one has that

$$\lim_{s \to 1}(1-s)H_s[E](x) = \varpi_{n-1} H[E](x).$$

Here above H denotes the classical mean curvature of E at the point x—with the convention that we take H such that the curvature of the ball is a positive quantity—and

$$\varpi_k := \mathcal{H}^{k-1}(\{x \in \mathbb{R}^k \,|\, |x| = 1\}),$$

for every $k \geq 1$. Let us also define $\varpi_0 := 0$. We observe that for $n = 1$, we have that

$$\lim_{s \to 1}(1-s)H_s[E](x) = 0,$$

which is consistent with the notation $\varpi_0 = 0$—see also Remark 3.5.6.

As $s \to 0^+$, the asymptotics are more involved and present some surprising behavior. This is due to the fact that as s gets smaller, the nonlocal contribution to the s-perimeter becomes more and more important, while the local contribution loses influence. Some precise results in this sense were achieved in [40]. There, in order to encode the behavior at infinity of a set, the authors have introduced the quantity

$$\alpha(E) = \lim_{s \to 0^+} s \int_{\mathcal{C}B_1} \frac{\chi_E(y)}{|y|^{n+s}}\, dy,$$

which appears naturally when looking at the asymptotics as $s \to 0^+$ of the fractional perimeter. Indeed, in [40] the authors proved that, if Ω is a bounded open set with $C^{1,\gamma}$ boundary, for some $\gamma \in (0,1]$, $E \subseteq \mathbb{R}^n$ has finite s_0-perimeter in Ω, for some $s_0 \in (0,1)$, and $\alpha(E)$ exists, then

$$\lim_{s \to 0^+} s \operatorname{Per}_s(E, \Omega) = \alpha(\mathcal{C}E)|E \cap \Omega| + \alpha(E)|\mathcal{C}E \cap \Omega|.$$

On the other hand, the asymptotic behavior for $s \to 0^+$ of the fractional mean curvature is studied in Chapter 3 (see also [47] for the particular case in which the set E is bounded). First of all, since the quantity $\alpha(E)$ may not exist—see [40, Example 2.8 and 2.9]—we define

$$\overline{\alpha}(E) := \limsup_{s \to 0^+} s \int_{\mathcal{C}B_1} \frac{\chi_E(y)}{|y|^{n+s}}\, dy \quad \text{and} \quad \underline{\alpha}(E) := \liminf_{s \to 0^+} s \int_{\mathcal{C}B_1} \frac{\chi_E(y)}{|y|^{n+s}}\, dy.$$

We prove that, when $s \to 0^+$, the s-fractional mean curvature becomes completely indifferent to the local geometry of the boundary ∂E, and indeed the limit value only depends on the behavior at infinity of the set E. More precisely, if $E \subseteq \mathbb{R}^n$ and $p \in \partial E$ is such that ∂E is $C^{1,\gamma}$ near p, for some $\gamma \in (0,1]$, then

$$\liminf_{s \to 0^+} s\, H_s[E](p) = \varpi_n - 2\overline{\alpha}(E), \tag{0.4}$$

and

$$\limsup_{s \to 0^+} s\, H_s[E](p) = \varpi_n - 2\underline{\alpha}(E).$$

We remark in particular that if E is bounded, then $\alpha(E)$ exists and $\alpha(E) = 0$. Hence, if $E \subseteq \mathbb{R}^n$ is a bounded open set with $C^{1,\gamma}$ boundary, the asymptotics is simply

$$\lim_{s \to 0^+} s\, H_s[E](p) = \varpi_n,$$

for every $p \in \partial E$—see also [47, Appendix B].

In Section 3.4 we compute the contribution from infinity $\alpha(E)$ of some sets. To have a few examples in mind, we mention here the following cases:

- let $S\subseteq\mathbb{S}^{n-1}$ and consider the cone
$$C:=\{t\sigma\in\mathbb{R}^n\,|\,t\geq 0,\,\sigma\in S\}.$$
Then, $\alpha(C)=\mathcal{H}^{n-1}(S)$.
- If $u\in L^\infty(\mathbb{R}^n)$, then $\alpha(\mathcal{S}g(u))=\varpi_{n+1}/2$. More in general, if $u:\mathbb{R}^n\to\mathbb{R}$ is such that
$$\lim_{|x|\to\infty}\frac{|u(x)|}{|x|}=0,$$
then $\alpha(\mathcal{S}g(u))=\varpi_{n+1}/2$.
- Let $u:\mathbb{R}^n\to\mathbb{R}$ be such that $u(x)\leq-|x|^2$, for every $x\in\mathbb{R}^n\setminus B_R$, for some $R>0$. Then $\alpha(\mathcal{S}g(u))=0$.

Roughly speaking, from the above examples we see that $\alpha(E)$ does not depend on the local geometry or regularity of E, but only on its behavior at infinity.

Now we observe that, as $s\to 0^+$, s-minimal sets exhibit a rather unexpected behavior.

For instance, in [45, Theorem 1.3] it is proved that if we fix the first quadrant of the plane as exterior data, then, quite surprisingly, when s is small enough the s-minimal set in $B_1\subseteq\mathbb{R}^2$ is empty in B_1. The main results of Chapter 3 take their inspiration from this result.

Heuristically, in order to generalize [45, Theorem 1.3] we want to prove that, if $\Omega\subseteq\mathbb{R}^n$ is a bounded and connected open set with smooth boundary and if we fix as exterior data a set $E_0\subseteq\mathcal{C}\Omega$ such that $\overline{\alpha}(E_0)<\varpi_n/2$, then there is a contradiction between the Euler-Lagrange equation of an s-minimal set and the asymptotics of the s-fractional mean curvature as $s\to 0^+$.

To motivate why we expect such a contradiction, we observe that the asymptotics (0.4) seems to suggest that, if s is small enough, then an s-minimal set E having exterior data E_0 and such that $\partial E\cap\Omega\neq\varnothing$ should have some point $p\in\partial E\cap\Omega$ such that $H_s[E](p)>0$—which would contradict the Euler-Lagrange equation. To avoid such a contradiction, we would then conclude that $\partial E=\varnothing$ in Ω, meaning that either $E\cap\Omega=\Omega$ or $E\cap\Omega=\varnothing$.

In order to turn this idea into a rigorous argument, we first prove that we can estimate the fractional mean curvature from below uniformly with respect to the radius of an exterior tangent ball to E. More precisely:

THEOREM 0.2.6. *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set. Let $E_0\subseteq\mathcal{C}\Omega$ be such that*
$$\overline{\alpha}(E_0)<\frac{\varpi_n}{2},$$
and let
$$\beta=\beta(E_0):=\frac{\varpi_n-2\overline{\alpha}(E_0)}{4}.$$
We define
$$\delta_s=\delta_s(E_0):=e^{-\frac{1}{s}\log\frac{\varpi_n+2\beta}{\varpi_n+\beta}},$$
for every $s\in(0,1)$. Then, there exists $s_0=s_0(E_0,\Omega)\in(0,\frac{1}{2}]$ such that, if $E\subseteq\mathbb{R}^n$ is such that $E\setminus\Omega=E_0$ and E has an exterior tangent ball of radius (at least) δ_σ, for some $\sigma\in(0,s_0)$, at some point $q\in\partial E\cap\overline{\Omega}$, then
$$\liminf_{\varrho\to 0^+}H_s^\varrho[E](q)\geq\frac{\beta}{s}>0,\qquad\forall\,s\in(0,\sigma].$$

Let us now introduce the following definition.

DEFINITION 0.2.7. *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set. We say that a set E is δ-*dense *in Ω, for some fixed $\delta>0$, if $|B_\delta(x)\cap E|>0$ for any $x\in\Omega$ for which $B_\delta(x)\Subset\Omega$.*

By exploiting a careful geometric argument and Theorem 0.2.6, we can then pursue the heuristic idea outlined above and prove the following classification result:

THEOREM 0.2.8. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and connected open set with C^2 boundary. Let $E_0 \subseteq \mathcal{C}\Omega$ such that*

$$\overline{\alpha}(E_0) < \frac{\varpi_n}{2}.$$

Then, the following two results hold true.
A) Let s_0 and δ_s be as in Theorem 0.2.6. There exists $s_1 = s_1(E_0, \Omega) \in (0, s_0]$ such that if $s < s_1$ and E is an s-minimal set in Ω with exterior data E_0, then either

$$(A.1)\ E \cap \Omega = \varnothing \quad \text{or} \quad (A.2)\ E \text{ is } \delta_s - \text{dense in } \Omega.$$

B) Either
(B.1) there exists $\tilde{s} = \tilde{s}(E_0, \Omega) \in (0, 1)$ such that if E is an s-minimal set in Ω with exterior data E_0 and $s \in (0, \tilde{s})$, then

$$E \cap \Omega = \varnothing,$$

or
(B.2) there exist $\delta_k \searrow 0$, $s_k \searrow 0$ and a sequence of sets E_k such that each E_k is s_k-minimal in Ω with exterior data E_0 and for every k

$$\partial E_k \cap B_{\delta_k}(x) \neq \varnothing \quad \text{for every } B_{\delta_k}(x) \Subset \Omega.$$

Roughly speaking, either the s-minimal sets are empty in Ω when s is small enough, or we can find a sequence E_k of s_k-minimal sets, with $s_k \searrow 0$, whose boundaries tend to (topologically) fill the domain Ω in the limit $k \to \infty$.

We point out that the typical behavior consists in being empty. Indeed, if the exterior data $E_0 \subseteq \mathcal{C}\Omega$ does not completely surround the domain Ω, we have the following result:

THEOREM 0.2.9. *Let Ω be a bounded and connected open set with C^2 boundary. Let $E_0 \subseteq \mathcal{C}\Omega$ such that*

$$\overline{\alpha}(E_0) < \frac{\varpi_n}{2},$$

and let s_1 be as in Theorem 0.2.8. Suppose that there exists $R > 0$ and $x_0 \in \partial\Omega$ such that

$$B_R(x_0) \setminus \Omega \subseteq \mathcal{C}E_0.$$

Then, there exists $s_3 = s_3(E_0, \Omega) \in (0, s_1]$ such that if $s < s_3$ and E is an s-minimal set in Ω with exterior data E_0, then

$$E \cap \Omega = \varnothing.$$

We observe that the condition $\overline{\alpha}(E_0) < \varpi_n/2$ is somehow optimal. Indeed, when $\alpha(E_0)$ exists and

$$\alpha(E_0) = \frac{\varpi_n}{2},$$

several configurations may occur, depending on the position of Ω with respect to the exterior data $E_0 \setminus \Omega$—we provide various examples in Chapter 3.

Moreover, notice that when E is s-minimal in Ω with respect to E_0, then $\mathcal{C}E$ is s-minimal in Ω with respect to $\mathcal{C}E_0$. Also,

$$\underline{\alpha}(E_0) > \frac{\varpi_n}{2} \qquad \Longrightarrow \qquad \overline{\alpha}(\mathcal{C}E_0) < \frac{\varpi_n}{2}.$$

Thus, in this case we can apply Theorems 0.2.6, 0.2.8 and 0.2.9 to $\mathcal{C}E$ with respect to the exterior data $\mathcal{C}E_0$. For instance, if E is s-minimal in Ω with exterior data E_0 with

$$\underline{\alpha}(E_0) > \frac{\varpi_n}{2},$$

and $s < s_1(\mathcal{C}E_0, \Omega)$, then either

$$E \cap \Omega = \Omega \qquad \text{or} \qquad \mathcal{C}E \;\text{ is }\; \delta_s(\mathcal{C}E_0) - \text{dense}.$$

The analogues of the just mentioned Theorems can be obtained similarly.

Therefore, from our main results and the above observations, we have a complete classification of nonlocal minimal surfaces when s is small, whenever

$$\alpha(E_0) \neq \frac{\varpi_n}{2}.$$

We point out that the stickiness phenomena described in [45] and in Chapter 3 are specific for nonlocal minimal surfaces—since classical minimal surfaces cross transversally the boundary of a convex domain.

Interestingly, these stickiness phenomena are not present in the case of the fractional Laplacian, where the boundary datum of the Dirichlet problem is attained continuously under rather general assumptions, see [89], though solutions of s-Laplace equations are in general no better than C^s at the boundary, hence the uniform continuity degenerates as $s \to 0^+$.

On the other hand, in case of fractional harmonic functions, a partial counterpart of the stickiness phenomenon is, in a sense, given by the boundary explosive solutions constructed in [1,57] (namely, in this case, the boundary of the subgraph of the fractional harmonic function contains vertical walls).

We also mention that stickiness phenomena for nonlocal minimal graphs—eventually in the presence of obstacles—will be studied in the forthcoming article [15].

In the final part of Chapter 3 we prove that the fractional mean curvature is continuous with respect to all variables.

To simplify a little the situation, suppose that E_k, $E \subseteq \mathbb{R}^n$ are sets with $C^{1,\gamma}$ boundaries, for some $\gamma \in (0,1]$, such that the boundaries ∂E_k locally converge in the $C^{1,\gamma}$ sense to the boundary of E, as $k \to \infty$. Then we prove that, if we have a sequence of points $x_k \in \partial E_k$ such that $x_k \to x \in \partial E$ and a sequence of indexes s_k, $s \in (0,\gamma)$ such that $s_k \to s$, it holds

$$\lim_{k\to\infty} H_{s_k}[E_k](x_k) = H_s[E](x).$$

Furthermore, we appropriately extend this convergence result in order to cover also the cases in which $s_k \to 1$ or $s_k \to 0$.

In particular, let us consider a set $E \subseteq \mathbb{R}^n$ such that $\alpha(E)$ exists and ∂E is of class C^2. Then, if we define

$$\tilde{H}_s[E](x) := \begin{cases} s(1-s)H_s[E](x), & \text{for } s \in (0,1) \\ \varpi_{n-1} H[E](x), & \text{for } s = 1 \\ \varpi_n - 2\alpha(E), & \text{for } s = 0, \end{cases}$$

the function

$$\tilde{H}_{(\,\cdot\,)}[E](\,\cdot\,) : [0,1] \times \partial E \longrightarrow \mathbb{R}, \qquad (s,x) \longmapsto \tilde{H}_s[E](x),$$

is continuous. It is interesting to observe that the fractional mean curvature at a fixed point $q \in \partial E$ may change sign as s varies from 0 to 1. Also—as a consequence of the continuity in the fractional parameter s—in such a case there exists an index $\sigma \in (0,1)$ such that $H_\sigma[E](q) = 0$.

0.2.4. Nonparametric setting. In Chapter 4 we introduce a functional framework to study minimizers of the fractional perimeter which can be globally written as the subgraph

$$\mathcal{S}g(u) = \left\{(x, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x_{n+1} < u(x)\right\},$$

of some measurable function $u : \mathbb{R}^n \to \mathbb{R}$. We refer to the boundaries of such minimizers as *nonlocal minimal graphs.*

We define a fractional version of the classical area functional and we study its functional and geometric properties. Then we focus on minimizers and we prove existence and uniqueness results with respect to a large class of exterior data, which includes locally bounded functions.

Furthermore, one of the main contributions of Chapter 4 consists in proving the equivalence of:

- minimizers of the fractional area functional,
- minimizers of the fractional perimeter,
- weak solutions of the fractional mean curvature equation,
- viscosity solutions of the fractional mean curvature equation,
- smooth functions solving pointwise the fractional mean curvature equation.

Before giving a detailed overview of the main results, let us recall the definition of the classical area functional. Given a bounded open set $\Omega \subseteq \mathbb{R}^n$ with Lipschitz boundary, the area functional is defined as

$$\mathscr{A}(u, \Omega) := \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx = \mathcal{H}^n\left(\left\{(x, u(x)) \in \mathbb{R}^{n+1} \,|\, x \in \Omega\right\}\right),$$

for every Lipschitz function $u : \overline{\Omega} \to \mathbb{R}$. One then extends this functional, by defining the relaxed area functional of a function $u \in L^1(\Omega)$ as

$$\mathscr{A}(u, \Omega) := \inf\left\{\liminf_{k\to\infty} \mathscr{A}(u_k, \Omega) \,|\, u_k \in C^1(\overline{\Omega}),\ \|u - u_k\|_{L^1(\Omega)} \to 0\right\}.$$

It is readily seen that, if $u \in L^1(\Omega)$, then

$$\mathscr{A}(u, \Omega) < \infty \quad \Longleftrightarrow \quad u \in BV(\Omega), \tag{0.5}$$

in which case

$$\mathscr{A}(u, \Omega) = \mathrm{Per}\left(\mathcal{S}g(u), \Omega \times \mathbb{R}\right). \tag{0.6}$$

Roughly speaking, the functions of bounded variation are precisely those integrable functions whose subgraphs have finite perimeter—for the details see, e.g., [65, 68].

We could thus be tempted to try and define a fractional version of the area functional, by considering the s-perimeter in place of the classical perimeter, setting, for a measurable function $u : \mathbb{R}^n \to \mathbb{R}$,

$$\mathscr{A}_s(u, \Omega) := \mathrm{Per}_s(\mathcal{S}g(u), \Omega \times \mathbb{R}).$$

However, as we observed in the end of Section 0.2.2, such a definition can not work, because

$$\mathrm{Per}_s^{NL}(\mathcal{S}g(u), \Omega \times \mathbb{R}) = \infty,$$

even if $u \in C_c^\infty(\mathbb{R}^n)$.

Before going on, a couple of observations are in order. Even if the nonlocal part of the fractional perimeter in the cylinder $\Omega^\infty := \Omega \times \mathbb{R}$ is infinite, we recall that we know—see the end of Section 0.2.2—that the local part is finite, provided the function u is regular enough in Ω.

If the function u is bounded in Ω, then we can consider the fractional perimeter in the "truncated cylinder" $\Omega^M := \Omega \times (-M, M)$, with $M \geq \|u\|_{L^\infty(\Omega)}$, instead of in the whole cylinder Ω^∞. As we will see below, by pursuing this idea we obtain a family of fractional area functionals $\mathcal{F}_s^M(\,\cdot\,,\Omega)$.

On the other hand, there is another possibility to come up with a definition of a fractional area functional. In [25], the authors have observed that when $E \subseteq \mathbb{R}^{n+1}$ is the subgraph of a function u, its fractional mean curvature can be written as an integrodifferential operator acting on u. More precisely, letting $u : \mathbb{R}^n \to \mathbb{R}$ be a function of, say, class $C^{1,1}$ in a neighborhood of a point $x \in \mathbb{R}^n$, we have that

$$H_s[\mathcal{S}g(u)](x, u(x)) = \mathscr{H}_s u(x),$$

with

$$\mathscr{H}_s u(x) := 2\,\mathrm{P.V.} \int_{\mathbb{R}^n} G_s\left(\frac{u(x) - u(y)}{|x - y|}\right) \frac{dy}{|x - y|^{n+s}},$$

and

$$G_s(t) := \int_0^t g_s(\tau)\, d\tau, \qquad g_s(t) := \frac{1}{(1 + t^2)^{\frac{n+1+s}{2}}} \quad \text{for } t \in \mathbb{R}.$$

We now show that $\mathscr{H}_s$ is the Euler-Lagrange operator associated to a (convex) functional $\mathcal{F}_s(\,\cdot\,,\Omega)$, which we will then consider as the s-fractional area functional.

Let us begin by remarking that, when u is not regular enough around x, the quantity $\mathscr{H}_s u(x)$ is in general not well-defined, due to the lack of cancellation required for the principal value to converge. Nevertheless, we can understand the operator $\mathscr{H}_s$ as defined in the following weak (distributional) sense. Given a measurable function $u : \mathbb{R}^n \to \mathbb{R}$, we set

$$\langle \mathscr{H}_s u, v\rangle := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G_s\left(\frac{u(x) - u(y)}{|x - y|}\right) \big(v(x) - v(y)\big) \frac{dx\, dy}{|x - y|^{n+s}}$$

for every $v \in C_c^\infty(\mathbb{R}^n)$. More generally, it is immediate to see— by taking advantage of the fact that G_s is bounded—that this definition is well-posed for every $v \in W^{s,1}(\mathbb{R}^n)$. Indeed, one has that

$$|\langle \mathscr{H}_s u, v\rangle| \le \frac{\Lambda_{n,s}}{2}\, [v]_{W^{s,1}(\mathbb{R}^n)},$$

where

$$\Lambda_{n,s} := \int_{\mathbb{R}} g_s(t)\, dt < \infty.$$

Hence, $\mathscr{H}_s u$ can be interpreted as a continuous linear functional $\langle \mathscr{H}_s u, \cdot\,\rangle \in (W^{s,1}(\mathbb{R}^n))^*$. Remarkably, this holds for every measurable function $u : \mathbb{R}^n \to \mathbb{R}$, regardless of its regularity.

We now set

$$\mathcal{G}_s(t) := \int_0^t G_s(\tau)\, d\tau \quad \text{for } t \in \mathbb{R},$$

and, given a measurable function $u : \mathbb{R}^n \to \mathbb{R}$ and an open set $\Omega \subseteq \mathbb{R}^n$, we define the *$s$-fractional area functional*

$$\mathcal{F}_s(u, \Omega) := \iint_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \mathcal{G}_s\left(\frac{u(x) - u(y)}{|x - y|}\right) \frac{dx\, dy}{|x - y|^{n-1+s}}.$$

Then, at least formally, we have

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{F}_s(u + \varepsilon v, \Omega) = \langle \mathscr{H}_s u, v\rangle \quad \text{for every } v \in C_c^\infty(\Omega).$$

We remark that in Chapter 4 we will actually consider more general functionals of fractional area-type—by taking in the above definitions a continuous and even function $g : \mathbb{R} \to (0, 1]$ satisfying an appropriate integrability condition, and the corresponding functions G and $\mathcal{G}$, in place of g_s, G_s and $\mathcal{G}_s$, respectively. However, for simplicity in this introduction we stick to the "geometric case" corresponding to the choice $g = g_s$.

Let us now get to the functional properties of $\mathcal{F}_s(\,\cdot\,,\Omega)$ and to its relationship with the fractional perimeter.

From now on, we fix $n \geq 1$, $s \in (0,1)$ and a bounded open set $\Omega \subseteq \mathbb{R}^n$ with Lipschitz boundary.

It is convenient to split the fractional area functional as the sum of its local and nonlocal parts, that is

$$\mathcal{F}_s(u,\Omega) = \mathcal{A}_s(u,\Omega) + \mathcal{N}_s(u,\Omega),$$

with

$$\mathcal{A}_s(u,\Omega) := \int_\Omega \int_\Omega \mathcal{G}_s\left(\frac{u(x)-u(y)}{|x-y|}\right) \frac{dx\,dy}{|x-y|^{n-1+s}}$$

and

$$\mathcal{N}_s(u,\Omega) := 2\int_\Omega \int_{\mathcal{C}\Omega} \mathcal{G}_s\left(\frac{u(x)-u(y)}{|x-y|}\right) \frac{dx\,dy}{|x-y|^{n-1+s}}.$$

Let us first mention the following interesting observation—see, e.g., Lemma D.1.2. If $u : \Omega \to \mathbb{R}$ is a measurable function, then

$$[u]_{W^{s,1}(\Omega)} < \infty \quad \Longrightarrow \quad \|u\|_{L^1(\Omega)} < \infty.$$

Concerning the local part of the fractional area functional, we prove that, if $u : \Omega \to \mathbb{R}$ is a measurable function, then

$$\begin{aligned} \mathcal{A}_s(u,\Omega) < \infty \quad &\Longleftrightarrow \quad u \in W^{s,1}(\Omega) \\ &\Longleftrightarrow \quad \mathrm{Per}_s^L(\mathcal{S}g(u), \Omega \times \mathbb{R}) < \infty. \end{aligned}$$

Moreover, if $u \in W^{s,1}(\Omega)$, then

$$\mathrm{Per}_s^L(\mathcal{S}g(u), \Omega \times \mathbb{R}) = \mathcal{A}_s(u,\Omega) + c,$$

for some constant $c = c(n,s,\Omega) \geq 0$. These results can be thought of as the fractional counterparts of (0.5) and (0.6).

On the other hand, in order for the nonlocal part to be finite, we have to impose some integrability condition on u at infinity, namely

$$\int_\Omega \left(\int_{\mathcal{C}\Omega} \frac{|u(y)|}{|x-y|^{n+s}}\,dy \right) dx < \infty. \tag{0.7}$$

Such a condition is satisfied, e.g., if u is globally bounded in $\mathbb{R}^n$ and, in general, it implies that the function u must grow strictly sublinearly at infinity. It is thus a very restrictive condition.

Indeed, we remark that the operator $\mathscr{H}_s u$ is well-defined at a point x—provided u is regular enough in a neighborhood of x—without having to impose any condition on u at infinity. Moreover, as we have observed in Section 0.2.2, by Theorem 0.2.5 and [**43**, Theorem 1.1] we know that, fixed any continuous function $\varphi : \mathbb{R}^n \to \mathbb{R}$, there exists a function $u : \mathbb{R}^n \to \mathbb{R}$ such that $u = \varphi$ in $\mathbb{R}^n \setminus \overline{\Omega}$, $u \in C(\overline{\Omega})$ and $\mathcal{S}g(u)$ is locally s-minimal in Ω^∞. Let us stress that no condition on φ at infinity is required.

For these reasons, condition (0.7) seems to be unnaturally restrictive in our framework—even if at first glance it looks necessary, since it is needed to guarantee that $\mathcal{F}_s$ is well-defined.

In order to avoid imposing condition (0.7), we define—see (4.23)—for every $M \geq 0$, the "truncated" nonlocal part $\mathcal{N}_s^M(u,\Omega)$ and the truncated area functional

$$\mathcal{F}_s^M(u,\Omega) := \mathcal{A}_s(u,\Omega) + \mathcal{N}_s^M(u,\Omega).$$

Roughly speaking, the idea consists in adding, inside the double integral defining the nonlocal part, a term which balances the contribution coming from outside Ω. For example, in the simplest case $M = 0$, we have

$$\mathcal{N}_s^0(u,\Omega) = 2\int_\Omega \left\{ \int_{\mathcal{C}\Omega} \left[\mathcal{G}_s\left(\frac{u(x)-u(y)}{|x-y|}\right) - \mathcal{G}_s\left(\frac{u(y)}{|x-y|}\right)\right] \frac{dy}{|x-y|^{n-1+s}} \right\} dx.$$

Remarkably, given a measurable function $u : \mathbb{R}^n \to \mathbb{R}$, we have

$$|\mathcal{N}_s^M(u,\Omega)| < \infty \quad \text{if } u|_\Omega \in W^{s,1}(\Omega),$$

regardless of the behavior of u in $\mathcal{C}\Omega$. However, we remark that, in general, the truncated nonlocal part can be negative, unless we require u to be bounded in Ω and we take $M \geq \|u\|_{L^\infty(\Omega)}$. From a geometric point of view, the truncated area functionals correspond to considering the fractional perimeter in the truncated cylinder Ω^M.

Indeed, if $u : \mathbb{R}^n \to \mathbb{R}$ is a measurable function such that $u|_\Omega \in W^{s,1}(\Omega) \cap L^\infty(\Omega)$, and $M \geq \|u\|_{L^\infty(\Omega)}$, we have

$$\mathcal{F}_s^M(u,\Omega) = \mathrm{Per}_s\big(\mathcal{S}g(u), \Omega \times (-M,M)\big) + c_M,$$

for some constant $c_M = c_M(n,s,\Omega) \geq 0$.

We now proceed to study the minimizers of the fractional area functional.

Given a measurable function $\varphi : \mathcal{C}\Omega \to \mathbb{R}$, we define the space

$$\mathcal{W}_\varphi^s(\Omega) := \big\{ u : \mathbb{R}^n \to \mathbb{R} \,|\, u|_\Omega \in W^{s,1}(\Omega) \text{ and } u = \varphi \text{ a.e. in } \mathcal{C}\Omega \big\},$$

and we say that $u \in \mathcal{W}_\varphi^s(\Omega)$ is a *minimizer* of $\mathcal{F}_s$ in $\mathcal{W}_\varphi^s(\Omega)$ if

$$\iint_{Q(\Omega)} \left\{ \mathcal{G}_s\left(\frac{u(x)-u(y)}{|x-y|}\right) - \mathcal{G}_s\left(\frac{v(x)-v(y)}{|x-y|}\right)\right\} \frac{dx\,dy}{|x-y|^{n-1+s}} \leq 0$$

for every $v \in \mathcal{W}_\varphi^s(\Omega)$. Here above, we have used the notation $Q(\Omega) := \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2$. Let us stress that such a definition is well-posed without having to impose conditions on the *exterior data* φ, as indeed—thanks to the fractional Hardy-type inequality of Theorem D.1.4—we have

$$\iint_{Q(\Omega)} \left| \mathcal{G}_s\left(\frac{u(x)-u(y)}{|x-y|}\right) - \mathcal{G}_s\left(\frac{v(x)-v(y)}{|x-y|}\right)\right| \frac{dx\,dy}{|x-y|^{n-1+s}} \leq C\,\Lambda_{n,s}\|u-v\|_{W^{s,1}(\Omega)},$$

for every $u,\ v \in \mathcal{W}_\varphi^s(\Omega)$, for some constant $C = C(n,s,\Omega) > 0$.

We prove the existence of minimizers with respect to exterior data satisfying an appropriate integrability condition in a neighborhood of the domain Ω. More precisely, given an open set $\mathcal{O} \subseteq \mathbb{R}^n$ such that $\Omega \Subset \mathcal{O}$, we define the *truncated tail* of $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ at a point $x \in \Omega$ as

$$\mathrm{Tail}_s(\varphi, \mathcal{O}\setminus\Omega; x) := \int_{\mathcal{O}\setminus\Omega} \frac{|\varphi(y)|}{|x-y|^{n+s}}\,dy.$$

We also use the notation

$$\Omega_\varrho := \{x \in \mathbb{R}^n \,|\, d(x,\Omega) < \varrho\},$$

for $\varrho > 0$, to denote the ϱ-neighborhood of Ω. Then, we prove the following:

THEOREM 0.2.10. *There is a constant $\Theta > 1$, depending only on n and s, such that, given any function $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ with $\mathrm{Tail}_s(\varphi, \Omega_{\Theta\,\mathrm{diam}(\Omega)} \setminus \Omega;\,\cdot\,) \in L^1(\Omega)$, there exists a unique minimizer u of $\mathcal{F}_s$ within $\mathcal{W}_\varphi^s(\Omega)$. Moreover, u satisfies*

$$\|u\|_{W^{s,1}(\Omega)} \leq C\left(\big\|\mathrm{Tail}_s(\varphi, \Omega_{\Theta\,\mathrm{diam}(\Omega)}\setminus\Omega;\,\cdot\,)\big\|_{L^1(\Omega)} + 1\right),$$

for some constant $C = C(n,s,\Omega) > 0$.

We observe that the condition on the integrability of the tail is much weaker than (0.7), since we are not requiring anything on the behavior of φ outside $\Omega_{\Theta\,\mathrm{diam}(\Omega)}$.

We also mention that, roughly speaking, the integrability of the tail amounts to the integrability of φ plus some regularity condition near the boundary of $\partial\Omega$. For example, if $\varphi \in L^1(\Omega_{\Theta\,\mathrm{diam}(\Omega)} \setminus \Omega)$ and there exists a $\varrho > 0$ such that, either $\varphi \in W^{s,1}(\Omega_\varrho \setminus \Omega)$ or $\varphi \in L^\infty(\Omega_\varrho \setminus \Omega)$, then $\mathrm{Tail}_s(\varphi, \Omega_{\Theta\,\mathrm{diam}(\Omega)} \setminus \Omega; \cdot\,) \in L^1(\Omega)$.

The uniqueness of the minimizer follows from the strict convexity of $\mathcal{F}_s$. On the other hand, in order to prove the existence, we exploit the (unique) minimizers u_M of the functionals $\mathcal{F}_s^M(\,\cdot\,,\Omega)$—considered within their natural domain. We exploit the hypothesis on the integrability of the tail, to prove a uniform estimate for the $W^{s,1}(\Omega)$ norm of the minimizers u_M, independently on $M \geq 0$. Hence, up to subsequences, u_M converges, as $M \to \infty$, to a limit function u, which is easily proved to minimize $\mathcal{F}_s$.

Moreover, we prove that if u is a minimizer of $\mathcal{F}_s$ within $\mathcal{W}^s_\varphi(\Omega)$, then $u \in L^\infty_{\mathrm{loc}}(\Omega)$. Also, we show that if the exterior data φ is bounded in a big enough neighborhood of Ω, then $u \in L^\infty(\Omega)$, and we establish an apriori bound on the L^∞ norm.

Let us go back to the relationship between the fractional area functional and the fractional perimeter. We show that by appropriately rearranging a set E in the vertical direction we decrease the s-perimeter. More precisely, given a set $E \subseteq \mathbb{R}^{n+1}$, we consider the function $w_E : \mathbb{R}^n \to \mathbb{R}$ defined by

$$w_E(x) := \lim_{R\to+\infty} \left(\int_{-R}^{R} \chi_E(x,t)\,dt - R \right)$$

for every $x \in \mathbb{R}^n$.

Then, we have the following result.

THEOREM 0.2.11. *Let $E \subseteq \mathbb{R}^{n+1}$ be such that $E \setminus \Omega^\infty$ is a subgraph and*

$$\Omega \times (-\infty, -M) \subseteq E \cap \Omega^\infty \subseteq \Omega \times (-\infty, M),$$

for some $M > 0$. Then,

$$\mathrm{Per}_s(\mathcal{S}g(w_E), \Omega^M) \leq \mathrm{Per}_s(E, \Omega^M).$$

The inequality is strict unless $\mathcal{S}g(w_E) = E$.

Exploiting also the local boundedness of a minimizer, we prove that if $u : \mathbb{R}^n \to \mathbb{R}$ is a measurable function such that $u \in W^{s,1}(\Omega)$, then

$$u \text{ minimizes } \mathcal{F}_s \text{ within } \mathcal{W}^s_u(\Omega) \quad \Longrightarrow \quad \mathcal{S}g(u) \text{ is locally } s\text{-minimal in } \Omega^\infty.$$

Theorem 0.2.11 extends to the fractional framework a well known result holding for the classical perimeter—see, e.g., [68, Lemma 14.7]. However, notice that in the fractional framework, due to the nonlocal character of the functionals involved, we have to assume that the set E is already a subgraph outside the cylinder Ω^∞.

We also observe that, since u is locally bounded in Ω and its subgraph is locally s-minimal in the cylinder Ω^∞, by [19, Theorem 1.1] we have that $u \in C^\infty(\Omega)$—that is, minimizers of $\mathcal{F}_s$ are smooth.

Let us now get to the Euler-Lagrange equation satisfied by minimizers. We first introduce the notion of weak solutions.

Let $f \in C(\overline{\Omega})$. We say that a measurable function $u : \mathbb{R}^n \to \mathbb{R}$ is a weak solution of $\mathscr{H}_s u = f$ in Ω if

$$\langle \mathscr{H}_s u, v \rangle = \int_\Omega f v\,dx,$$

for every $v \in C^\infty_c(\Omega)$.

As a consequence of the convexity of $\mathcal{F}_s$, it is easy to prove that, given a measurable function $u : \mathbb{R}^n \to \mathbb{R}$ such that $u \in W^{s,1}(\Omega)$, it holds

$$u \text{ is a minimizer of } \mathcal{F}_s \text{ in } \mathcal{W}^s_u(\Omega) \quad \Longleftrightarrow \quad u \text{ is a weak solution of } \mathscr{H}_s u = 0 \text{ in } \Omega.$$

Another natural notion of solution for the equation $\mathscr{H}_s u = f$ is that of a viscosity solution—we refer to Section 4.3 for the precise definition. One of the main results of Chapter 4 consists in proving that viscosity (sub)solutions are weak (sub)solutions. More precisely:

THEOREM 0.2.12. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let $f \in C(\overline{\Omega})$. Let $u : \mathbb{R}^n \to \mathbb{R}$ be such that u is locally integrable in $\mathbb{R}^n$ and u is locally bounded in Ω. If u is a viscosity subsolution,*

$$\mathscr{H}_s u \leq f \quad \text{in } \Omega,$$

then u is a weak subsolution,

$$\langle \mathscr{H}_s u, v\rangle \leq \int_\Omega f v\, dx, \qquad \forall\, v \in C^\infty_c(\Omega) \text{ s.t. } v \geq 0.$$

Combining the main results of Chapter 4 and exploiting the interior regularity proved in [19], we obtain the following:

THEOREM 0.2.13. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a measurable function such that $u \in W^{s,1}(\Omega)$. Then, the following are equivalent:*

(i) u is a weak solution of $\mathscr{H}_s u = 0$ in Ω,
(ii) u minimizes $\mathcal{F}_s$ in $\mathcal{W}^s_u(\Omega)$,
(iii) $u \in L^\infty_{\mathrm{loc}}(\Omega)$ and $\mathcal{S}g(u)$ is locally s-minimal in $\Omega \times \mathbb{R}$,
(iv) $u \in C^\infty(\Omega)$ and u is a pointwise solution of $\mathscr{H}_s u = 0$ in Ω.

Moreover, if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap W^{s,1}(\Omega)$, then all of the above are equivalent to:

(v) u is a viscosity solution of $\mathscr{H}_s u = 0$ in Ω.

We also point out the following global version of Theorem 0.2.13:

COROLLARY 0.2.14. *Let $u \in W^{s,1}_{\mathrm{loc}}(\mathbb{R}^n)$. Then, the following are equivalent:*

(i) u is a viscosity solution of $\mathscr{H}_s u = 0$ in $\mathbb{R}^n$,
(ii) u is a weak solution of $\mathscr{H}_s u = 0$ in $\mathbb{R}^n$,
(iii) u minimizes $\mathcal{F}_s$ in $\mathcal{W}^s_u(\Omega)$, for every open set $\Omega \Subset \mathbb{R}^n$ with Lipschitz boundary,
(iv) $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathcal{S}g(u)$ is locally s-minimal in $\mathbb{R}^{n+1}$,
(v) $u \in C^\infty(\mathbb{R}^n)$ and u is a pointwise solution of $\mathscr{H}_s u = 0$ in $\mathbb{R}^n$.

Let us also mention that the functional framework introduced above, easily extends to the obstacle problem. Namely, besides imposing the exterior data condition $u = \varphi$ a.e. in $\mathcal{C}\Omega$, we constrain the functions to lie above an obstacle, that is, given an open set $A \subseteq \Omega$ and an obstacle $\psi \in L^\infty(A)$, we restrict ourselves to consider those functions $u \in \mathcal{W}^s_\varphi(\Omega)$ such that $u \geq \psi$ a.e. in A.

In Chapter 4 we briefly cover also this obstacle problem, proving the existence and uniqueness of a minimizer and its relationship with the geometric obstacle problem that involves the fractional perimeter.

Finally, in the last section of Chapter 4, we prove some approximation results for subgraphs having (locally) finite fractional perimeter. In particular, exploiting the surprising density result established in [44], we show that s-minimal subgraphs can be appropriately approximated by subgraphs of σ-harmonic functions, for any fixed $\sigma \in (0, 1)$.

0.2.5. Rigidity results for nonlocal minimal graphs. In Chapter 5 we prove a flatness result for entire nonlocal minimal graphs having some partial derivatives bounded from either above or below. This result generalizes fractional versions of classical theorems due to Bernstein and Moser.

Moreover, we show that entire graphs having constant fractional mean curvature are minimal, thus extending a celebrated result of Chern on classical CMC graphs.

We are interested in subgraphs that locally minimize the s-perimeter in the whole space $\mathbb{R}^{n+1}$. We recall that, as we have seen in Corollary 0.2.14, under very mild assumptions on the function $u : \mathbb{R}^n \to \mathbb{R}$, the subgraph $\mathcal{S}g(u)$ is locally s-minimal in $\mathbb{R}^{n+1}$ if and only if u satisfies the fractional mean curvature equation

$$\mathscr{H}_s u = 0 \quad \text{in } \mathbb{R}^n. \tag{0.8}$$

Moreover, again by Corollary 0.2.14, there are several equivalent notions of solution for the equation (0.8), such as smooth solutions, viscosity solutions, and weak solutions.

In what follows, a solution of (0.8) will always indicate a function $u \in C^\infty(\mathbb{R}^n)$ that satisfies identity (0.8) pointwise. We stress that no growth assumptions at infinity are made on u.

The main contribution of Chapter 5 is the following result.

THEOREM 0.2.15. *Let $n \geq \ell \geq 1$ be integers, $s \in (0,1)$, and suppose that*

$$\textit{there exist no singular } s\textit{-minimal cones in } \mathbb{R}^\ell. \tag{$P_{s,\ell}$}$$

Let u be a solution of $\mathscr{H}_s u = 0$ in $\mathbb{R}^n$, having $n - \ell$ partial derivatives bounded on one side. Then, u is an affine function.

Characterizing the values of s and ℓ for which ($P_{s,\ell}$) is satisfied represents a challenging open problem. Nevertheless, property ($P_{s,\ell}$) is known to hold in the following cases:

- when $\ell = 1$ or $\ell = 2$, for every $s \in (0,1)$;
- when $3 \leq \ell \leq 7$ and $s \in (1 - \varepsilon_0, 1)$ for some small $\varepsilon_0 \in (0,1]$ depending only on ℓ.

Case $\ell = 1$ holds by definition, while $\ell = 2$ is the content of [**92**, Theorem 1]. On the other hand, case $3 \leq \ell \leq 7$ has been established in [**25**, Theorem 2].

As a consequence of Theorem 0.2.15 and the last remarks, we immediately obtain the following result.

COROLLARY 0.2.16. *Let $n \geq \ell \geq 1$ be integers and $s \in (0,1)$. Assume that either*

- *$\ell \in \{1,2\}$, or*
- *$3 \leq \ell \leq 7$ and $s \in (1 - \varepsilon_0, 1)$, with $\varepsilon_0 = \varepsilon_0(\ell) > 0$ as in [**25**, Theorem 2].*

Let u be a solution of $\mathscr{H}_s u = 0$ in $\mathbb{R}^n$, having $n - \ell$ partial derivatives bounded on one side. Then, u is an affine function.

We observe that Theorem 0.2.15 gives a new flatness result for s-minimal graphs, under the assumption that ($P_{s,\ell}$) holds true. It can be seen as a generalization of the fractional De Giorgi-type lemma contained in [**58**, Theorem 1.2], which is recovered here taking $\ell = n$. In this case, we indeed provide an alternative proof of said result.

On the other hand, the choice $\ell = 2$ gives an improvement of [**55**, Theorem 4], when specialized to s-minimal graphs. In light of these observations, Theorem 0.2.15 and Corollary 0.2.16 can be seen as a bridge between Bernstein-type theorems (flatness results in low dimensions) and Moser-type theorems (flatness results under global gradient bounds).

For classical minimal graphs, the counterpart of Corollary 0.2.16 has been recently obtained by A. Farina in [**54**]. In that case, the result is sharp and holds with $\ell =$

$\min\{n, 7\}$. The proof of Theorem 0.2.15 is based on the extension to the fractional framework of a strategy—which relies on a general splitting result for blow-downs of the subgraph $\mathcal{S}g(u)$—devised by A. Farina for classical minimal graphs and previously unpublished. As a result, the ideas contained in Chapter 5 can be used to obtain a different, easier proof of [54, Theorem 1.1]

Let us also mention that, by using the same ideas that lead to Theorem 0.2.15, we can prove the following rigidity result for entire s-minimal graphs that lie above a cone.

THEOREM 0.2.17. *Let $n \geq 1$ be an integer and $s \in (0, 1)$. Let u be a solution of $\mathscr{H}_s u = 0$ in $\mathbb{R}^n$, and assume that there exists a constant $C > 0$ for which*

$$u(x) \geq -C(1 + |x|) \quad \textit{for every } x \in \mathbb{R}^n.$$

Then, u is an affine function.

We remark that in [19] a rigidity result analogous to Theorem 0.2.17 is deduced, under the stronger, two-sided assumption

$$|u(x)| \leq C(1 + |x|) \quad \text{for every } x \in \mathbb{R}^n.$$

Theorem 0.2.17 thus improves [19, Theorem 1.5] directly.

Finally, we prove that if $u : \mathbb{R}^n \to \mathbb{R}$ is such that

$$\langle \mathscr{H}_s u, v \rangle = h \int_{\mathbb{R}^n} v \, dx \quad \text{for every } v \in C_c^\infty(\mathbb{R}^n),$$

for some constant $h \in \mathbb{R}$, then the constant must be $h = 0$.

In particular, recalling Corollary 0.2.14, we see that if $u \in W^{s,1}_{\text{loc}}(\mathbb{R}^n)$ is a weak solution of $\mathscr{H}_s u = h$ in $\mathbb{R}^n$, then the subgraph of u is locally s-minimal in $\mathbb{R}^{n+1}$. This extends to the nonlocal framework a celebrated result of Chern, namely the Corollary of Theorem 1 in [26].

0.2.6. A free boundary problem. In Chapter 6 we study minimizers of the functional

$$\mathcal{N}(u, \Omega) + \operatorname{Per}\big(\{u > 0\}, \Omega\big), \tag{0.9}$$

with $\mathcal{N}(u, \Omega)$ being, roughly speaking, the Ω-contribution to the H^s seminorm of a function $u : \mathbb{R}^n \to \mathbb{R}$, that is

$$\mathcal{N}(u, \Omega) := \iint_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}} \, dx \, dy,$$

for some fixed index $s \in (0, 1)$.

Similar functionals, defined as the superposition of an "elastic energy" plus a "surface tension" term, have already been considered in the following papers:

- Dirichlet energy plus classical perimeter in [6],
- Dirichlet energy plus fractional perimeter in [22],
- the nonlocal energy $\mathcal{N}$ plus the fractional perimeter in [42], and the corresponding one-phase problem in [46].

Studying the functional defined in (0.9) somehow completes this picture.

The main contributions of Chapter 6 consist in establishing a monotonicity formula for the minimizers of the functional (0.9), in exploiting it to investigate the properties of blow-up limits and in proving a dimension reduction result. Moreover, we show that, when $s < 1/2$, the perimeter dominates—in some sense—over the nonlocal energy. As a consequence, we obtain a regularity result for the free boundary $\{u = 0\}$.

As a thechnical note, let us first observe that we can not directly work with the set $\{u > 0\}$. Instead, we consider *admissible pairs* (u, E), with $u : \mathbb{R}^n \to \mathbb{R}$ a measurable function, and $E \subseteq \mathbb{R}^n$ such that

$$u \geq 0 \quad \text{a.e. in } E \quad \text{and} \quad u \leq 0 \quad \text{a.e. in } \mathcal{C}E.$$

The set E is usually referred to as the *positivity set* of u. Then, given an index $s \in (0, 1)$ and a bounded open set with Lipschitz boundary $\Omega \subseteq \mathbb{R}^n$, we define the functional

$$\mathcal{F}_\Omega(u, E) := \mathcal{N}(u, \Omega) + \operatorname{Per}(E, \Omega),$$

for every admissible pair (u, E).

Let us now remark that if $u : \mathbb{R}^n \to \mathbb{R}$ is a measurable function, then

$$\mathcal{N}(u, \Omega) < \infty \quad \Longrightarrow \quad \int_{\mathbb{R}^n} \frac{|u(\xi)|^2}{1 + |\xi|^{n+2s}} \, d\xi < \infty. \tag{0.10}$$

For a proof see, e.g., Lemma D.1.3. As a consequence, we also have that

$$\int_{\mathbb{R}^n} \frac{|u(\xi)|}{1 + |\xi|^{n+2s}} \, d\xi < \infty \quad \text{and} \quad u \in L^2_{\text{loc}}(\mathbb{R}^n).$$

The notion of minimizers that we consider is the following:

DEFINITION 0.2.18. *Given an admissible pair (u, E) such that $\mathcal{F}_\Omega(u, E) < \infty$, we say that a pair (v, F) is an* admissible competitor *if*

$$\begin{aligned} &\mathit{supp}(v - u) \Subset \Omega, \qquad F \Delta E \Subset \Omega, \\ &v - u \in H^s(\mathbb{R}^n) \qquad \mathit{and} \qquad \operatorname{Per}(F, \Omega) < +\infty. \end{aligned} \tag{0.11}$$

We say that the admissible pair (u, E) is minimizing *in Ω if $\mathcal{F}_\Omega(u, E) < \infty$ and*

$$\mathcal{F}_\Omega(u, E) \leq \mathcal{F}_\Omega(v, F),$$

for every admissible competitor (v, F).

Notice that the first line of (0.11) simply says that the pairs (u, E) and (v, F) are equal—in the measure theoretic sense—outside a compact subset of Ω. Then, since $\mathcal{F}_\Omega(u, E) < \infty$, it is readily seen that the second line is equivalent to $\mathcal{F}_\Omega(v, F) < \infty$.

In particular we are interested in the following minimization problem, with respect to fixed "exterior data". Given an admissible pair (u_0, E_0) and a bounded open set $\mathcal{O} \subseteq \mathbb{R}^n$ with Lipschitz boundary, such that

$$\Omega \Subset \mathcal{O}, \qquad \mathcal{N}(u_0, \Omega) < +\infty \quad \text{and} \quad \operatorname{Per}(E_0, \mathcal{O}) < +\infty, \tag{0.12}$$

we want to find an admissible pair (u, E) attaining the following infimum

$$\begin{aligned} \inf \big\{ \mathcal{N}(v, \Omega) + \operatorname{Per}(F, \mathcal{O}) \,|\, (v, F) \text{ admissible pair s.t. } v = u_0 \text{ a.e. in } \mathcal{C}\Omega \\ \text{and } F \setminus \Omega = E_0 \setminus \Omega \big\}. \end{aligned} \tag{0.13}$$

Roughly speaking, as customary when dealing with minimization problems involving the classical perimeter, we are considering a (fixed) neighborhood $\mathcal{O}$ of Ω (as small as we like) in order to "read" the boundary data $\partial E_0 \cap \partial \Omega$.

We prove that, fixed as exterior data any pair (u_0, E_0) satisfying (0.12), there exists a pair (u, E) realizing the infimum in (0.13). Moreover, we show that such a pair (u, E) is also minimizing in the sense of Definition 0.2.18.

A useful result consists in establishing a uniform bound for the energy of minimizing pairs.

THEOREM 0.2.19. *Let (u, E) be a minimizing pair in B_2. Then*

$$\iint_{\mathbb{R}^{2n}\setminus(\mathcal{C}B_1)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \mathrm{Per}(E, B_1) \le C\left(1+\int_{\mathbb{R}^n}\frac{|u(y)|^2}{1+|y|^{n+2s}}\,dy\right),$$

for some $C = C(n, s) > 0$.

In particular, Theorem 0.2.19 is exploited in the proof of the existence of a blow-up limit. For this, we have first to introduce—through the extension technique of [23]—the extended functional associated to the minimization of $\mathcal{F}_\Omega$. We write

$$\mathbb{R}^{n+1}_+ := \{(x, z) \in \mathbb{R}^{n+1} \text{ with } x \in \mathbb{R}^n,\ z > 0\}.$$

Given a function $u : \mathbb{R}^n \to \mathbb{R}$, we consider the function $\overline{u} : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ defined via the convolution with an appropriate Poisson kernel,

$$\overline{u}(\,\cdot\,, z) = u * \mathcal{K}_s(\,\cdot\,, z), \quad \text{where} \quad \mathcal{K}_s(x, z) := c_{n,s}\frac{z^{2s}}{(|x|^2+z^2)^{(n+2s)/2}},$$

and $c_{n,s} > 0$ is an appropriate normalizing constant. Such an extended function $\overline{u}$ is well defined—see, e.g., [75]—provided $u : \mathbb{R}^n \to \mathbb{R}$ is such that

$$\int_{\mathbb{R}^n}\frac{|u(\xi)|}{1+|\xi|^{n+2s}}\,d\xi < \infty.$$

In light of (0.10), we can thus consider the extension function of a minimizer.

We use capital letters, like $X = (x, z)$, to denote points in $\mathbb{R}^{n+1}$. Given a set $\Omega \subseteq \mathbb{R}^{n+1}$, we write

$$\Omega_+ := \Omega \cap \{z > 0\} \qquad \text{and} \qquad \Omega_0 := \Omega \cap \{z = 0\}.$$

Moreover we identify the hyperplane $\{z = 0\} \simeq \mathbb{R}^n$ via the projection function.

Given a bounded open set $\Omega \subseteq \mathbb{R}^{n+1}$ with Lipschitz boundary, such that $\Omega_0 \neq \varnothing$, we define

$$\mathfrak{F}_\Omega(\mathcal{V}, F) := c'_{n,s}\int_{\Omega_+}|\nabla\mathcal{V}|^2 z^{1-2s}\,dX + \mathrm{Per}(F, \Omega_0),$$

for $\mathcal{V} : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ and $F \subseteq \mathbb{R}^n \simeq \{z = 0\}$ the positivity set of the trace of $\mathcal{V}$ on $\{z = 0\}$, that is

$$\mathcal{V}\big|_{\{z=0\}} \ge 0 \quad \text{a.e. in } F \quad \text{and} \quad \mathcal{V}\big|_{\{z=0\}} \le 0 \quad \text{a.e. in } \mathcal{C}F.$$

We call such a pair $(\mathcal{V}, F)$ an *admissible pair* for the extended functional. Then, we introduce the following notion of minimizer for the extended functional.

DEFINITION 0.2.20. *Given an admissible pair $(\mathcal{U}, E)$, such that $\mathfrak{F}_\Omega(\mathcal{U}, E) < \infty$, we say that a pair $(\mathcal{V}, F)$ is an* admissible competitor *if $\mathfrak{F}_\Omega(\mathcal{V}, F) < \infty$ and*

$$\mathit{supp}\,(\mathcal{V}-\mathcal{U}) \Subset \Omega \qquad \mathit{and} \qquad E\Delta F \Subset \Omega_0.$$

We say that an admssible pair $(\mathcal{U}, E)$ is minimal *in Ω if $\mathfrak{F}_\Omega(\mathcal{U}, E) < \infty$ and*

$$\mathfrak{F}_\Omega(\mathcal{U}, E) \le \mathfrak{F}_\Omega(\mathcal{V}, F),$$

for every admissible competitor $(\mathcal{V}, F)$.

An important result consists in showing that an appropriate minimization problem involving the extended functionals is equivalent to the minimization of the original functional $\mathcal{F}_\Omega$. More precisely:

PROPOSITION 0.2.21. *Let (u, E) be an admissible pair for $\mathcal{F}$, s.t. $\mathcal{F}_{B_R}(u.E) < +\infty$. Then, the pair (u, E) is minimizing in B_R if and only if the pair $(\overline{u}, E)$ is minimal for $\mathfrak{F}_\Omega$, in every bounded open set $\Omega \subseteq \mathbb{R}^{n+1}$ with Lipschitz boundary such that $\varnothing \neq \Omega_0 \Subset B_R$.*

One of the main reasons for introducing the extended functional, resides in the fact that it enables us to establish a Weiss-type monotonicity formula for minimizers.

We denote

$$\mathcal{B}_r := \{(x,z) \in \mathbb{R}^{n+1} \,|\, |x|^2 + z^2 < r^2\} \qquad \text{and} \qquad \mathcal{B}_r^+ := \mathcal{B}_r \cap \{z > 0\}.$$

THEOREM 0.2.22 (Weiss-type Monotonicity Formula). *Let (u, E) be a minimizing pair for $\mathcal{F}$ in B_R and define the function $\Phi_u : (0, R) \to \mathbb{R}$ by*

$$\Phi_u(r) := r^{1-n} \left(c'_{n,s} \int_{\mathcal{B}_r^+} |\nabla \overline{u}|^2 z^{1-2s} \, dX + \operatorname{Per}(E, B_r) \right) \\ - c'_{n,s} \Big(s - \frac{1}{2} \Big) r^{-n} \int_{(\partial \mathcal{B}_r)^+} \overline{u}^2 z^{1-2s} \, d\mathcal{H}^n.$$

Then, the function Φ_u is increasing in $(0, R)$. Moreover, Φ_u is constant in $(0, R)$ if and only if the extension $\overline{u}$ is homogeneous of degree $s - \frac{1}{2}$ in $\mathcal{B}_R^+$ and E is a cone in B_R.

Here above, $(\partial \mathcal{B}_r)^+ := \partial \mathcal{B}_r \cap \{z > 0\}$. Let us now introduce the rescaled pairs (u_λ, E_λ). Given $u : \mathbb{R}^n \to \mathbb{R}$ and $E \subseteq \mathbb{R}^n$, we define

$$u_\lambda(x) := \lambda^{\frac{1}{2} - s} u(\lambda x) \qquad \text{and} \qquad E_\lambda := \frac{1}{\lambda} E,$$

for every $\lambda > 0$. We observe that—because of the scaling properties of $\mathcal{F}_\Omega$—a pair (u, E) is minimal in Ω if and only if the rescaled pair (u_λ, E_λ) is minimal in Ω_λ for every $\lambda > 0$.

We prove the convergence of minimizing pairs under appropriate conditions and we exploit it—together with Theorem 0.2.19—in the particularly important case of the blow-up sequence.

We say that the admissible pair (u, E) is a *minimizing cone* if it is a minimizing pair in B_R, for every $R > 0$, and is such that u is homogeneous of degree $s - \frac{1}{2}$ and E is a cone

THEOREM 0.2.23. *Let $s > 1/2$ and (u, E) be a minimizing pair in B_1, with $0 \in \partial E$. Also assume that $u \in C^{s - \frac{1}{2}}(B_1)$. Then, there exist a minimizing cone (u_0, E_0) and a sequence $r_k \searrow 0$ such that $u_{r_k} \to u_0$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ and $E_{r_k} \xrightarrow{\mathrm{loc}} E_0$.*

The homogeneity properties of the blow-up limit (u_0, E_0) are a consequence of Theorem 0.2.22.

We also point out that we establish appropriate estimates for the tail energies of the functions u_r, that allow us to weaken the assumptions of [42, Theorem 1.3], where the authors ask u to be $C^{s-\frac{1}{2}}$ in the whole of $\mathbb{R}^n$.

We now mention the following dimensional reduction result. Only in the following Theorem, let us redefine

$$\mathcal{F}_\Omega(u, E) := (c'_{n,s})^{-1} \mathcal{N}(u, \Omega) + \operatorname{Per}(E, \Omega).$$

We say that an admissible pair (u, E) is minimizing in $\mathbb{R}^n$ if it minimizes $\mathcal{F}_\Omega$ in any bounded open subset $\Omega \subseteq \mathbb{R}^n$ with Lipschitz boundary.

THEOREM 0.2.24. *Let (u, E) be an admissible pair and define*

$$u^\star(x, x_{n+1}) := u(x) \qquad \textit{and} \qquad E^\star := E \times \mathbb{R}.$$

Then, the pair (u, E) is minimizing in $\mathbb{R}^n$ if and only if the pair $(u^\star, E^\star)$ is minimizing in $\mathbb{R}^{n+1}$.

Finally, we observe that in the case $s < 1/2$ the perimeter is, in some sense, the leading term of the functional $\mathcal{F}_\Omega$. As a consequence, we are able to prove the following regularity result:

THEOREM 0.2.25. *Let $s \in (0, 1/2)$ and let (u, E) be a minimizing pair in Ω. Assume that $u \in L^\infty_{\rm loc}(\Omega)$. Then, E has almost minimal boundary in Ω. More precisely, if $x_0 \in \Omega$ and $d := d(x_0, \Omega)/3$, then, for every $r \in (0, d]$ it holds*

$$\mathrm{Per}(E, B_r(x_0)) \le \mathrm{Per}(F, B_r(x_0)) + C\, r^{n-2s}, \qquad \forall\, F \subseteq \mathbb{R}^n \text{ s.t. } E\Delta F \Subset B_r(x_0),$$

where

$$C = C\left(s, x_0, d, \|u\|_{L^\infty(B_{2d}(x_0))}, \int_{\mathbb{R}^n} \frac{|u(y)|}{1 + |y|^{n+2s}}\, dy\right) > 0.$$

Therefore

(i) $\partial^* E$ is locally $C^{1,\frac{1-2s}{2}}$ in Ω,

(ii) the singular set $\partial E \setminus \partial^* E$ is such that

$$\mathcal{H}^\sigma\big((\partial E \setminus \partial^* E) \cap \Omega\big) = 0, \qquad \text{for every } \sigma > n - 8.$$

We conclude by saying a few words about the one-phase problem, that corresponds to the case in which $u \ge 0$ a.e. in $\mathbb{R}^n$. Even if these results are not included in this thesis, they will be part of the final version of the article on which Chapter 6 is based. Following the arguments of [46], we will prove that if (u, E) is a minimizer of the one-phase problem in B_2, with $s > 1/2$, and if $0 \in \partial E$, then $u \in C^{s-\frac{1}{2}}(B_{1/2})$. Notice in particular that, by Theorem 0.2.23, this ensures the existence of a blow-up limit (u_0, E_0). Moreover, we will establish uniform density estimates for the positivity set E, from both sides.

0.2.7. The Phillip Island penguin parade (a mathematical treatment). The goal of Chapter 7 is to provide a simple, but rigorous, mathematical model which describes the formation of groups of penguins on the shore at sunset.

Penguins are flightless, so they are forced to walk while on land. In particular, they show rather specific behaviours in their homecoming, which are interesting to observe and to describe analytically. We observed that penguins have the tendency to waddle back and forth on the shore to create a sufficiently large group and then walk home compactly together. The mathematical framework that we introduce describes this phenomenon, by taking into account "natural parameters", such as the eye-sight of the penguins and their cruising speed. The model that we propose favours the formation of conglomerates of penguins that gather together, but, on the other hand, it also allows the possibility of isolated and exposed individuals.

The model that we propose is based on a set of ordinary differential equations, with a number of degree of freedom which is variable in time. Due to the discontinuous behaviour of the speed of the penguins, the mathematical treatment (to get existence and uniqueness of the solution) is based on a "stop-and-go" procedure. We use this setting to provide rigorous examples in which at least some penguins manage to safely return home (there are also cases in which some penguins remain isolated). To facilitate the intuition of the model, we also present some simple numerical simulations that can be compared with the actual movement of the penguins parade.

0.3. Résumé

Cette thèse de doctorat est consacrée à l'analyse de quelques problèmes de minimisation impliquant des fonctionnelles non locales. Les opérateurs non locaux ont fait l'objet d'une attention croissante au cours des dernières années, à la fois par leur intérêt mathématique et par leurs applications—par exemple, pour modéliser des processus de diffusion anormaux ou des transitions de phase à longue portée. Pour une introduction aux problèmes non locaux, le lecteur intéressé pourra consulter l'ouvrage [17].

Dans cette thèse, nous nous intéressons principalement au périmètre s-fractionnaire—qui peut être considéré comme une version fractionnaire et non locale du périmètre classique introduit par De Giorgi et Caccioppoli—et ses minimiseurs, les ensembles s-minimaux, qui ont été considérés dans [21] pour la prèmiere fois. Les frontières de ces ensembles s-minimaux sont généralement appelées surfaces minimales non locales. En particulier :

- nous étudions le comportement des ensembles ayant périmètre fractionnaire (localement) fini, en prouvant la densité des ensembles ouverts et lisses, un résultat asymptotique optimal pour $s \to 1^-$, et en étudiant le lien existant entre le périmètre fractionnaire et les ensembles ayant frontières fractales.
- Nous établissons des résultats d'existence et de compacité pour les minimiseurs du périmètre fractionnaire, qui sont une extension de ceux prouvés dans [21].
- Nous étudions les ensembles s-minimaux dans des régimes hautement non locaux, qui correspondent à de petites valeurs du paramètre fractionnaire s. Nous montrons que, dans ce cas, les minimiseurs présentent un comportement complètement différent de celui de leurs homologues locaux—les surfaces minimales (classiques).
- Nous introduisons un cadre fonctionnel pour étudier ces ensembles s-minimaux qui peuvent être écrits globalement en tant que sous-graphes. En particulier, nous prouvons des résultats d'existence et d'unicité pour les minimiseurs d'une version fractionnaire de la fonctionnelle d'aire classique et une inégalité de réarrangement impliquant que les sous-graphes de ces minimiseurs minimisent le périmètre fractionnaire. Nous appelons les frontières de ces minimiseurs des graphes minimaux non locaux. De plus, nous montrons l'équivalence entre les minimiseurs et diverses notions de solution—à savoir, solutions faibles, solutions de viscosité et solutions lisses ponctuelles—de l'équation de courbure moyenne fractionnaire.
- Nous montrons un résultat de platitude pour des graphes minimaux non locaux entiers ayant des dérivés partielles majorées ou minorées—ainsi, en particulier, étendant au cadre fractionnaire des théorèmes classiques dus à Bernstein et Moser.

En outre, nous considérons un problème à frontière libre, qui consiste en la minimisation d'une fonctionnelle définie comme la somme d'une énergie non locale, plus le périmètre classique de l'interface de séparation entre les deux phases. Concernant ce problème :

- nous prouvons l'existence de minimiseurs et introduisons un problème de minimisation équivalent, qui a une "nature locale"—en exploitant la technique d'extension de [23].
- Nous établissons des estimations d'énergie uniformes et étudions la suite de blow-up d'un minimiseur. En particulier, nous prouvons une formule de monotonie qui implique que les limites de blow-up sont homogènes.
- Nous étudions la régularité de la frontière libre dans le cas où le périmètre a un rôle dominant sur l'énergie non locale.

Nous mentionnons que le dernier chapitre de la thèse consiste en un article fournissant un modèle mathématique décrivant la formation de groupes de manchots sur le rivage au coucher du soleil. À l'occasion d'un voyage de recherche à l'Université de Melbourne, nous avons vu le "Phillip Island penguin parade" et nous étions tellement fascinés par le comportement particulier des petits manchots que nous avons décidé de le décrire de manière mathématique.

La thèse est divisée en sept chapitres, chacun reposant sur l'un des articles de recherche suivants, que j'ai écrit—seul ou en collaboration—au cours de mon doctorat :

(1) *Fractional perimeters from a fractal perspective*, publié dans Advanced Nonlinear Studies—voir [77].
(2) *Approximation of sets of finite fractional perimeter by smooth sets and comparison of local and global s-minimal surfaces*, publié dans Interfaces and Free Boundaries—voir [76].
(3) *Complete stickiness of nonlocal minimal surfaces for small values of the fractional parameter*, co-auteur avec C. Bucur et E. Valdinoci, publié dans Annales de l'Institut Henri Poincaré Analyse Non Linéaire—voir [16].
(4) *On nonlocal minimal graphs*, co-auteur avec M. Cozzi, en cours de préparation.
(5) *Bernstein-Moser-type results for nonlocal minimal graphs*, co-auteur avec M. Cozzi et A. Farina, publié dans Communications in Analysis and Geometry—voir [31].
(6) Une version partielle et préliminaire de l'article *A free boundary problem : superposition of nonlocal energy plus classical perimeter*, co-auteur avec S. Dipierro et E. Valdinoci, en cours de préparation.
(7) *The Phillip Island penguin parade (a mathematical treatment)*, co-auteur avec S. Dipierro, P. Miraglio et E. Valdinoci, publié dans ANZIAM Journal—voir [41].

Les annexes contiennent des résultats auxiliaires qui ont été exploités tout au long de la thèse.

0.4. Une présentation plus détaillée

Nous passons maintenant à une description détaillée du contenu et des principaux résultats de cette thèse. Nous observons que chaque sujet a sa propre présentation, plus approfondie, au début du chapitre correspondant. De plus, chaque chapitre a sa propre table des matières, pour aider le lecteur à naviguer entre les sections.

0.4.1. Ensembles de périmètre fractionnaire (localement) fini. Le périmètre s-fractionnaire et ses minimiseurs, les ensembles s-minimaux, ont été introduits dans [21] en 2010, principalement motivés par des applications aux problèmes de transition de phase en prèsence d'interactions à longue portée. Au cours des années suivantes, ils ont suscité un vif intérêt, notamment en ce qui concerne la théorie de la régularité et le comportement qualitatif des frontières des ensembles s-minimaux, qui sont les soi-disant surfaces minimales non locales. Nous invitons le lecteur intéressé à consulter [98] et [17, Chapter 6] pour une introduction, et à l'étude [47] pour quelques développements récents.

En particulier, nous mentionnons que, même si la recherche de la régularité optimale des surfaces minimales non locales reste un problème ouvert et engageant, il est connu que les surfaces minimales non locales sont $(n-1)$-rectifiables. Plus précisément, elles sont lisses, sauf éventuellement pour un ensemble singulier de dimension de Hausdorff au plus égal à $n-3$ (voir [21], [92] et [58]). En conséquence, un ensemble s-minimal a périmètre (au sens de De Giorgi et Caccioppoli) localement fini—et en fait, des estimations uniformes du périmètre (classique) des ensembles s-minimaux sont disponibles (voir [28]).

D'autre part, la frontière d'un ensemble générique E ayant s-périmètre fini peut être très irrégulière et peut même être "nulle part rectifiable", comme dans le cas du flocon de neige de von Koch.

En fait, le s-périmètre peut être utilisé (en suivant l'article fondateur [99]) pour définir une "dimension fractale" pour la frontière, compris au sens de la théorie de la mesure,

$$\partial^- E := \{x \in \mathbb{R}^n \,|\, 0 < |E \cap B_r(x)| < \omega_n r^n \text{ pour chaque } r > 0\},$$

d'un ensemble $E \subseteq \mathbb{R}^n$.

Avant de continuer, nous rappelons la définition du s-périmètre. Étant donné un paramètre fractionnaire $s \in (0,1)$, nous définissons l'interaction

$$\mathcal{L}_s(A,B) := \int_A \int_B \frac{1}{|x-y|^{n+s}}\,dx\,dy,$$

pour chaque couple d'ensembles disjoints A, $B \subseteq \mathbb{R}^n$. Alors, le *s-périmètre* d'un ensemble $E \subseteq \mathbb{R}^n$ dans un ensemble ouvert $\Omega \subseteq \mathbb{R}^n$ est défini comme

$$\mathrm{Per}_s(E,\Omega) := \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\cap\Omega) + \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\setminus\Omega) + \mathcal{L}_s(E\setminus\Omega, \mathcal{C}E\cap\Omega).$$

Nous écrivons simplement $\mathrm{Per}_s(E) := \mathrm{Per}_s(E,\mathbb{R}^n)$.

On dit qu'un ensemble $E \subseteq \mathbb{R}^n$ a *s-périmètre localement fini* dans un ensemble ouvert $\Omega \subseteq \mathbb{R}^n$ si

$$\mathrm{Per}_s(E,\Omega') < \infty \qquad \text{pour chaque ensemble ouvert } \Omega' \Subset \Omega.$$

Nous observons que nous pouvons réécrire le s-périmètre comme

$$\mathrm{Per}_s(E,\Omega) = \frac{1}{2} \iint_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\,dx\,dy. \tag{0.14}$$

La formule (0.14) montre que le périmètre fractionnaire est, approximativement, la Ω-contribution à la seminorme $W^{s,1}$ de la fonction caractéristique χ_E.

Cette fonctionnelle est non locale, au sens qu'il faut connaître l'ensemble E dans tout $\mathbb{R}^n$, même pour calculer son s-périmètre dans un petit domaine borné Ω (contrairement à ce qui se passe avec le périmètre classique ou la mesure $\mathcal{H}^{n-1}$, qui sont des fonctionnelles locales). En plus, le s-périmètre est "fractionnaire", dans le sens où la seminorme $W^{s,1}$ mesure un ordre de régularité fractionnaire.

Nous observons que nous pouvons diviser le s-périmètre comme

$$\mathrm{Per}_s(E,\Omega) = \mathrm{Per}_s^L(E,\Omega) + \mathrm{Per}_s^{NL}(E,\Omega),$$

où

$$\mathrm{Per}_s^L(E,\Omega) := \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\cap\Omega) = \frac{1}{2}[\chi_E]_{W^{s,1}(\Omega)}$$

peut être considéré comme la "partie locale" du périmètre fractionnaire, et

$$\begin{aligned}\mathrm{Per}_s^{NL}(E,\Omega) &:= \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\setminus\Omega) + \mathcal{L}_s(E\setminus\Omega, \mathcal{C}E\cap\Omega)\\ &= \int_\Omega \int_{\mathcal{C}\Omega} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\,dx\,dy,\end{aligned}$$

qui peut être considéré comme la "partie non locale".

0.4.1.1. ***Frontières fractales***. En 1991, dans l'article [99] l'auteur a suggéré d'utiliser le paramètre s de la seminorme fractionnaire $[\chi_E]_{W^{s,1}(\Omega)}$ (et de plus générales familles continues de fonctionnelles satisfaisant des opportunes formules de la co-aire généralisées) comme un moyen de mesurer la codimension de la frontière comprise au sens de la théorie de la mesure, $\partial^- E$, d'un ensemble E dans Ω. Il a prouvé que la dimension fractale obtenue de cette manière,

$$\mathrm{Dim}_F(\partial^- E, \Omega) := n - \sup\{s \in (0,1) \,|\, [\chi_E]_{W^{s,1}(\Omega)} < \infty\},$$

est inférieure ou égale à la dimension (supérieure) de Minkowski.

La relation entre la dimension de Minkowski de la frontière d'un ensemble E et la régularité fractionnaire (dans le sens des espaces de Besov) de la fonction caractéristique χ_E a été étudié aussi dans [94], en 1999. En particulier—voir [94, Remark 3.10]—l'auteur a prouvé que la dimension Dim_F du flocon de neige de von Koch S coïncide avec sa dimension de Minkowski, en exploitant le fait que S est un domaine de John.

La régularité de Sobolev d'une fonction caractéristique χ_E a été approfondie dans [52], en 2013, où les auteurs considèrent le cas dans lequel l'ensemble E est une quasiball. Comme le flocon de neige de von Koch S est un exemple typique de quasiball, les auteurs ont pu prouver que la dimension Dim_F de S coïncide avec sa dimension de Minkowski.

Dans le Chapitre 1, nous calculons la dimension Dim_F du flocon de neige de von Koch S de manière élémentaire, en utilisant uniquement l'invariance par roto-translation et la propriété d'échelle du s-périmètre, et la "auto-similarité" de S. Plus précisément, nous montrons que

$$\mathrm{Per}_s(S) < \infty, \qquad \forall\, s \in \left(0, 2 - \frac{\log 4}{\log 3}\right),$$

et

$$\mathrm{Per}_s(S) = \infty, \qquad \forall\, s \in \left[2 - \frac{\log 4}{\log 3}, 1\right).$$

La démonstration peut être étendue de manière naturelle à tous les ensembles qui peuvent être définis de manière récursive similaire à celle du flocon de von Koch. En conséquence, nous calculons la dimension Dim_F de tous ces ensembles, sans avoir à les obliger à être des domaines de John ou des quasiballs.

De plus, nous montrons que nous pouvons facilement obtenir beaucoup d'ensembles de ce type en modifiant de manière appropriée des fractales auto-similaires bien connues, comme le flocon de neige de von Koch, le triangle de Sierpinski et l'éponge de Menger. Un exemple est illustré dans la Figure 2.

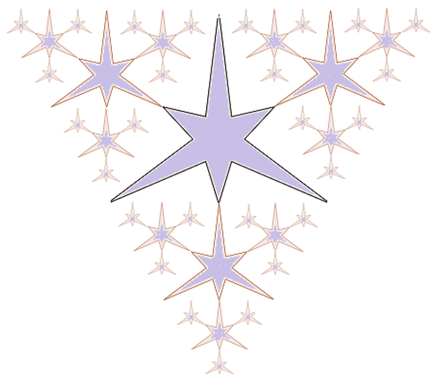


FIGURE 2. *Exemple d'un ensemble "fractal" construit en exploitant la structure du triangle de Sierpinski (visible à la quatrième étape itérative).*

0.4.1.2. ***Asymptotique*** $s \to 1^-$. La discussion précédente montre que le s-périmètre d'un ensemble E ayant frontière irrégulière, éventuellement fractale, peut être fini pour s sous un certain seuil, $s < \sigma$, et infini pour $s \in (\sigma, 1)$. D'autre part, il est bien connu que les ensembles avec une frontière régulière ont s-périmètre fini pour chaque s et leur s-périmètre converge, lorsque s tend vers 1, au périmètre classique, à la fois au sens classique (voir, par exemple, [24]) et au sens de la Γ-convergence (voir, par exemple, [5] et aussi [85] pour des résultats connexes).

Dans le Chapitre 1, nous exploitons [35, Theorem 1] pour prouver une version optimale de cette propriété asymptotique pour un ensemble E ayant périmètre classique fini dans un ensemble ouvert borné avec frontière de classe de Lipschitz. Plus précisément, nous prouvons que, si E a périmètre classique fini dans un voisinage de Ω, alors

$$\lim_{s\to 1}(1-s)\operatorname{Per}_s(E,\Omega) = \omega_{n-1}\operatorname{Per}(E,\overline{\Omega}).$$

Nous observons que nous baissons la régularité demandée dans [24], où les auteurs ont exigé que le frontière ∂E soit $C^{1,\alpha}$, à la régularité optimale (demandent à E seulement d'avoir périmètre fini). En plus, nous n'avons pas à demander à E de croiser $\partial\Omega$ "transversalement", c'est-à-dire que nous n'avons pas besoin que

$$\mathcal{H}^{n-1}(\partial^* E \cap \partial\Omega) = 0,$$

où $\partial^* E$ dénote la frontière réduite de E.

En effet, nous prouvons que la partie non locale du s-périmètre converge au périmètre sur la frontière de Ω, c'est-à-dire que nous prouvons que

$$\lim_{s\to 1}(1-s)\operatorname{Per}_s^{NL}(E,\Omega) = \omega_{n-1}\mathcal{H}^{n-1}(\partial^* E \cap \partial\Omega),$$

qui est, à la connaissance de l'auteur, un nouveau résultat.

0.4.1.3. ***Approximation par ensembles ouverts lisses.*** Comme nous avons observé dans la Section 0.4.1.1, les ensembles ayant périmètre fractionnaire fini peuvent avoir une frontière très rugueuse, qui peut en effet être une fractale nulle part rectifiable (comme le flocon de neige de von Koch).

Cela représente une différence importante entre le périmètre fractionnaire et le périmètre classique, car les ensembles de Caccioppoli ont une partie "grande" de la frontière, dite frontière réduite, qui est $(n-1)$-rectifiable (d'après le Théorème de structure de De Giorgi).

En tout cas, nous prouvons dans la première partie du Chapitre 2 qu'un ensemble a périmètre fractionnaire (localement) fini si et seulement si il peut être approché (de manière appropriée) par des ensembles ouverts lisses. Plus précisément, nous prouvons ce qui suit :

THÉORÈME 0.4.1. *Soit $\Omega \subseteq \mathbb{R}^n$ un ensemble ouvert. Un ensemble $E \subseteq \mathbb{R}^n$ a s-périmètre localement fini dans Ω si et seulement s'il existe une suite $E_h \subseteq \mathbb{R}^n$ de ensembles ouverts ayant frontière lisse et $\varepsilon_h \to 0^+$ tels que*

$$(i)\quad E_h \xrightarrow{loc} E, \qquad \sup_{h\in\mathbb{N}} \operatorname{Per}_s(E_h,\Omega') < \infty \quad \textit{pour chaque } \Omega' \Subset \Omega,$$

$$(ii)\quad \lim_{h\to\infty} \operatorname{Per}_s(E_h,\Omega') = \operatorname{Per}_s(E,\Omega') \quad \textit{pour chaque } \Omega' \Subset \Omega,$$

$$(iii)\quad \partial E_h \subseteq N_{\varepsilon_h}(\partial E).$$

En outre, si $\Omega = \mathbb{R}^n$ et l'ensemble E est tel que $|E| < \infty$ et $\operatorname{Per}_s(E) < \infty$, alors

$$|E_h \Delta E| \to 0, \qquad \lim_{h\to\infty} \operatorname{Per}_s(E_h) = \operatorname{Per}_s(E),$$

et nous pouvons exiger que chaque E_h soit borné (au lieu de demander (iii)).

Ci-dessus, $N_\delta(\partial E)$ dénote le δ-voisinage tubulaire de ∂E.

Un tel résultat est bien connu pour les ensembles de Caccioppoli (voir, par exemple, [79]) et en effet, cette propriété de densité peut être utilisée pour définir la fonctionnelle de périmètre (classique) comme étant la relaxation—par rapport à la convergence L^1_{loc}—de la mesure $\mathcal{H}^{n-1}$ des frontières des ensembles ouverts lisses, c'est-à-dire

$$\mathrm{Per}(E,\Omega) = \inf\Big\{ \liminf_{k\to\infty} \mathcal{H}^{n-1}(\partial E_h \cap \Omega) \,\Big|\, E_h \subseteq \mathbb{R}^n \text{ ouvert ayant frontière lisse, tel que } E_h \xrightarrow{loc} E \Big\}. \tag{0.15}$$

Il est intéressant de noter que, dans [47], les auteurs ont prouvé, en exploitant le théorème de la divergence, que si $E \subsetneq \mathbb{R}^n$ est un ensemble ouvert borné avec frontière lisse, alors

$$\mathrm{Per}_s(E) = c_{n,s} \int_{\partial E}\int_{\partial E} \frac{2 - |\nu_E(x) - \nu_E(y)|^2}{|x-y|^{n+s-2}} d\mathcal{H}^{n-1}_x d\mathcal{H}^{n-1}_y, \tag{0.16}$$

où ν_E dénote la normale externe de E et

$$c_{n,s} := \frac{1}{2s(n+s-2)}.$$

En exploitant la formule (0.16), la semicontinuité inférieure du s-périmètre et le Théorème 0.4.1, nous trouvons que, si $E \subseteq \mathbb{R}^n$ est tel que $|E| < \infty$, alors

$$\mathrm{Per}_s(E) = \inf\Big\{ \liminf_{h\to\infty} c_{n,s} \int_{\partial E_h}\int_{\partial E_h} \frac{2 - |\nu_{E_h}(x) - \nu_{E_h}(y)|^2}{|x-y|^{n+s-2}} d\mathcal{H}^{n-1}_x d\mathcal{H}^{n-1}_y \,\Big|\, E_h \subseteq \mathbb{R}^n \text{ ensemble ouvert borné ayant frontière lisse, tel que } E_h \xrightarrow{loc} E \Big\}.$$

Cela peut être considéré comme un analogue de (0.15) dans le cadre fractionnaire.

Nous mentionnons également que dans la Section 4.7 nous allons prouver qu'un sous-graphe ayant s-périmètre localement fini dans un cylindre $\Omega \times \mathbb{R}$ peut être approché par les sous-graphes de fonctions lisses—et pas seulement par des ensembles ouverts lisses arbitraires.

0.4.2. Surfaces minimales non locales. La deuxième partie du Chapitre 2 concerne les ensembles minimisant le périmètre fractionnaire. Les frontières de ces minimiseurs sont souvent appelés surfaces minimales non locales et apparaissent naturellement comme interfaces limites des modèles de transition de phase à interaction à longue portée. En particulier, dans les régimes où l'interaction à longue portée est dominante, la fonctionnelle de Allen-Cahn non locale Γ-converge au périmètre fractionnaire (voir, par exemple, [91]) et les interfaces minimales de l'équation de Allen-Cahn correspondante approchent localement de manière uniforme les surfaces minimales non locales (voir, par exemple, [93]).

Nous rappelons maintenant la définition des ensembles minimisants introduite dans [21].

Définition 0.4.2. *Soit $\Omega \subseteq \mathbb{R}^n$ un ensemble ouvert et soit $s \in (0,1)$. On dit qu'un ensemble $E \subseteq \mathbb{R}^n$ est s-minimal dans Ω si $\mathrm{Per}_s(E,\Omega) < \infty$ et*

$$\mathrm{Per}_s(E,\Omega) \le \mathrm{Per}_s(F,\Omega) \quad \textit{pour chaque } F \subseteq \mathbb{R}^n \textit{ tel que } F\setminus\Omega = E\setminus\Omega.$$

Parmi les nombreux résultats, dans [21] les auteurs ont prouvé que, si $\Omega \subseteq \mathbb{R}^n$ est un ensemble ouvert borné ayant frontière Lipschitz, alors pour chaque ensemble fixé $E_0 \subseteq \mathcal{C}\Omega$ il existe un ensemble $E \subseteq \mathbb{R}^n$ qui est s-minimal dans Ω et tel que $E\setminus\Omega = E_0$. L'ensemble E_0 est parfois appelé *donné extérieur* et l'ensemble E est dit être s-minimal dans Ω par rapport à la donnée extérieure E_0.

Nous étendons le résultat d'existence susmentionné en prouvant que, dans un ensemble ouvert générique Ω, il existe un ensemble s-minimal par rapport à une certaine donnée extérieure $E_0 \subseteq \mathcal{C}\Omega$ fixée, si et seulement si il existe un concurrent ayant s-périmètre fini dans Ω. Plus précisément :

THÉORÈME 0.4.3. *Soit $s \in (0,1)$, soit $\Omega \subseteq \mathbb{R}^n$ un ensemble ouvert et soit $E_0 \subseteq \mathcal{C}\Omega$. Alors, il existe un ensemble $E \subseteq \mathbb{R}^n$ qui est s-minimal dans Ω et tel que $E \setminus \Omega = E_0$, si et seulement si il existe un ensemble $F \subseteq \mathbb{R}^n$ tel que $F \setminus \Omega = E_0$ et $\mathrm{Per}_s(F,\Omega) < \infty$.*

En conséquence, nous observons que, si $\mathrm{Per}_s(\Omega) < \infty$, alors il existe toujours un ensemble s-minimal par rapport à la donnée extérieure E_0, pour chaque ensemble $E_0 \subseteq \mathcal{C}\Omega$.

Portons maintenant l'attention sur le cas dans lequel le domaine de minimisation n'est pas borné. Dans cette situation, il convient d'introduire la notion de minimiseur local.

DÉFINITION 0.4.4. *Soit $\Omega \subseteq \mathbb{R}^n$ un ensemble ouvert et soit $s \in (0,1)$. On dit qu'un ensemble $E \subseteq \mathbb{R}^n$ est* localement s-minimal *dans Ω si E est s-minimal dans chaque ensemble ouvert $\Omega' \Subset \Omega$.*

Notez en particulier que nous demandons à E seulement d'avoir s-périmètre localement fini dans Ω et pas d'avoir s-périmètre fini dans tout le domaine. En effet, la principale raison de l'introduction des ensembles localement s-minimaux est donnée par le fait qu'en général, le s-périmètre d'un ensemble n'est pas fini dans les domaines non bornés.

Nous avons vu dans le Théorème 0.4.3 que le seul obstacle à l'existence d'un ensemble s-minimal, par rapport à une certaine donnée extérieure $E_0 \subseteq \mathcal{C}\Omega$ fixée, est l'existence d'un concurrent ayant s-périmètre fini. D'autre part, nous prouvons qu'un ensemble localement s-minimal existe toujours, peu importe ce que le domaine Ω et la donnée extérieure sont.

THÉORÈME 0.4.5. *Soit $s \in (0,1)$, soit $\Omega \subseteq \mathbb{R}^n$ un ensemble ouvert et soit $E_0 \subseteq \mathcal{C}\Omega$. Alors, il existe un ensemble $E \subseteq \mathbb{R}^n$ qui est localement s-minimal dans Ω et tel que $E \setminus \Omega = E_0$.*

Quand Ω est un ensemble ouvert borné ayant frontière Lipschitz, nous montrons que les deux notions de minimiseur coïncident. C'est-à-dire, si $\Omega \subseteq \mathbb{R}^n$ est un ensemble ouvert borné ayant frontière Lipschitz et $E \subseteq \mathbb{R}^n$, alors

$$E \text{ est } s\text{-minimal dans } \Omega \quad \Longleftrightarrow \quad E \text{ est localement } s\text{-minimal dans } \Omega.$$

Cependant, nous observons que cela n'est pas vrai dans un ensemble ouvert Ω arbitraire, car un ensemble s-minimal—au sens de la Définition 0.4.2—peut ne pas exister.

A titre d'exemple, nous considérons la situation dans laquelle le domaine de minimisation est le cylindre

$$\Omega^\infty := \Omega \times \mathbb{R} \subseteq \mathbb{R}^{n+1},$$

où $\Omega \subseteq \mathbb{R}^n$ est un ensemble ouvert borné ayant frontière régulière. Nous nous intéressons au cas où la donnée extérieure est le sous-graphe d'une fonction mesurable $\varphi : \mathbb{R}^n \to \mathbb{R}$. C'est-à-dire, nous considérons le sous-graphe

$$\mathcal{S}g(\varphi) := \left\{ (x, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x_{n+1} < \varphi(x) \right\},$$

et nous voulons trouver un ensemble $E \subseteq \mathbb{R}^{n+1}$ qui minimise—dans un certain sens—le s-périmètre dans le cylindre Ω^∞, par rapport à la donnée extérieure $E \setminus \Omega^\infty = \mathcal{S}g(\varphi) \setminus \Omega^\infty$.

Une motivation pour considérer un tel problème de minimisation est donnée par le récent article [**43**], où les auteurs ont prouvé que si un tel ensemble de minimisation E

existe—et si φ est une fonction continue—alors E est en fait un sous-graphe global. Plus précisément, il existe une fonction $u : \mathbb{R}^n \to \mathbb{R}$, telle que $u = \varphi$ dans $\mathbb{R}^n \setminus \overline{\Omega}$ et $u \in C(\overline{\Omega})$, et telle que

$$E = \mathcal{S}g(u).$$

On voit facilement que si une fonction $u : \mathbb{R}^n \to \mathbb{R}$ est assez régulière dans Ω, par exemple, si $u \in BV(\Omega) \cap L^\infty(\Omega)$, alors la partie locale du s-périmètre du sous-graphe de u est finie,

$$\mathrm{Per}_s^L(\mathcal{S}g(u), \Omega^\infty) < \infty.$$

D'autre part, la partie non locale du s-périmètre, en général, est infinie, même pour des fonctions très régulières u. En effet, nous prouvons que si $u \in L^\infty(\mathbb{R}^n)$, alors

$$\mathrm{Per}_s^{NL}(\mathcal{S}g(u), \Omega^\infty) = \infty.$$

Une première conséquence de cette observation—et de l'estimation a priori sur la "variation verticale" d'un ensemble de minimisation fourni par [43, Lemma 3.3]—est le fait que, si $\varphi \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, alors il ne peut pas exister un ensemble E qui est s-minimal dans Ω^∞—au sens de la Définition 0.4.2—par rapport à la donnée extérieure $\mathcal{S}g(\varphi) \setminus \Omega^\infty$.

Toutefois, le Théorème 0.4.5 garantit l'existence d'un ensemble $E \subseteq \mathbb{R}^{n+1}$ qui est localement s-minimal dans Ω^∞ et tel que $E \setminus \Omega^\infty = \mathcal{S}g(\varphi) \setminus \Omega^\infty$. Donc, Théorème 0.4.5 et [43, Theorem 1.1] impliquent ensemble l'existence de sous-graphes minimisant (localement) le s-périmètre, c'est-à-dire, des surfaces minimales non locales non paramétriques.

Une deuxième conséquence consiste dans le fait que nous ne pouvons pas définir une version fractionnaire naïve de la fonctionnelle d'aire classique comme

$$\mathscr{A}_s(u, \Omega) := \mathrm{Per}_s(\mathcal{S}g(u), \Omega^\infty),$$

puisque cela serait infinie même pour une fonction $u \in C_c^\infty(\mathbb{R}^n)$. Au Chapitre 4 nous allons éviter ce problème en introduisant un cadre fonctionnel approprié pour travailler avec des sous-graphes.

0.4.3. Effets de stickiness pour les petits valeurs de s. Le Chapitre 3 est consacré à l'étude des ensembles s-minimaux dans des régimes hautement non locaux, c'est-à-dire dans le cas où le paramètre fractionnaire $s \in (0, 1)$ est très petit. Nous prouvons que, dans cette situation, le comportement des ensembles s-minimaux, d'une certaine manière, dégénère.

Rappelons d'abord quelques résultats connus concernant l'asymptotique $s \to 1^-$.
Nous avons déjà observé dans la Section 0.4.1.2 que le s-périmètre converge vers le périmètre classique lorsque $s \to 1^-$. De plus, quand $s \to 1^-$, les ensembles s-minimaux convergent vers les minimiseurs du périmètre classique, à la fois au "sens uniforme" (voir [24, 25]) et au sens de la Γ-convergence (voir [5]). En conséquence, on peut prouver (voir [25]) que quand s est suffisamment proche de 1, les surfaces minimales non locales ont la même régularité des surfaces minimales classiques. Voir aussi [47] pour une étude récente et assez complète des propriétés des ensembles s-minimaux lorsque s est proche de 1.

De plus, nous observons que la courbure moyenne fractionnaire converge également, comme $s \to 1^-$, vers sa contrepartie classique. Pour être plus précis, rappelons d'abord que la courbure moyenne s-fractionnaire d'un ensemble E en un point $q \in \partial E$ est définie comme l'intégrale au sens de la valeur principale

$$H_s[E](q) := \mathrm{P.V.} \int_{\mathbb{R}^n} \frac{\chi_{\mathcal{C}E}(y) - \chi_E(y)}{|y - q|^{n+s}} \, dy,$$

c'est-à-dire

$$H_s[E](q) := \lim_{\varrho \to 0^+} H_s^\varrho[E](q), \qquad \text{où} \qquad H_s^\varrho[E](q) := \int_{\mathcal{C}B_\varrho(q)} \frac{\chi_{\mathcal{C}E}(y) - \chi_E(y)}{|y-q|^{n+s}}\, dy.$$

Remarquons qu'il est en effet nécessaire d'interpréter l'intégrale ci-dessus au sens de la valeur principale, puisque l'intégrande est singulière et non intégrable dans un voisinage de q. D'autre part, s'il ya suffisamment d'annulation entre E et $\mathcal{C}E$ dans un voisinage de q—par exemple, si ∂E est de classe C^2 autour de q—alors l'intégrale est bien définie au sens de la valeur principale.

La courbure moyenne fractionnaire a été introduite dans [21], où les auteurs ont montré qu'elle est l'opérateur d'Euler-Lagrange apparaissant dans la minimisation du s-périmètre. En effet, si $E \subseteq \mathbb{R}^n$ est s-minimal dans un ensemble ouvert Ω, alors

$$H_s[E] = 0 \quad \text{sur } \partial E,$$

dans un sens de viscosité approprié—pour plus de détails voir, par exemple, l'Annexe C.2.

Il est connu (voir, par exemple, [2, Theorem 12] et [25]) que si $E \subseteq \mathbb{R}^n$ est un ensemble ayant frontière C^2, et $n \geq 2$, alors pour tous $x \in \partial E$ on a que

$$\lim_{s \to 1} (1-s) H_s[E](x) = \varpi_{n-1} H[E](x).$$

Ci-dessus H dénote la courbure moyenne classique de E au point x—selon la convention que nous prenons H tel que la courbure de la boule est une quantité positive—et

$$\varpi_k := \mathcal{H}^{k-1}(\{x \in \mathbb{R}^k \,|\, |x| = 1\}),$$

pour chaque $k \geq 1$. Laissez-nous également définir $\varpi_0 := 0$. Nous observons que pour $n = 1$, nous avons

$$\lim_{s \to 1} (1-s) H_s[E](x) = 0,$$

ce qui est compatible avec la notation $\varpi_0 = 0$—voir aussi Remarque 3.5.6.

Lorsque $s \to 0^+$, les asymptotiques sont plus compliqués et présentent un comportement surprenant. Cela est dû au fait que quand s devient plus petit, la contribution non locale au compteur du s-périmètre devient de plus en plus importante, tandis que la contribution locale perd de son influence. Quelques résultats précis à cet égard ont été obtenus dans [40]. Là, pour encoder le comportement à l'infini d'un ensemble, les auteurs ont introduit la quantité

$$\alpha(E) = \lim_{s \to 0^+} s \int_{\mathcal{C}B_1} \frac{\chi_E(y)}{|y|^{n+s}}\, dy,$$

qui apparaît naturellement quand on regarde l'asymptotique pour $s \to 0^+$ du périmètre fractionnaire. En fait, dans [40] les auteurs ont prouvé que, si Ω est un ensemble ouvert borné ayant frontière $C^{1,\gamma}$, pour quelque $\gamma \in (0,1]$, $E \subseteq \mathbb{R}^n$ a s_0-périmètre fini dans Ω, pour un certain $s_0 \in (0,1)$, et $\alpha(E)$ existe, alors

$$\lim_{s \to 0^+} s \operatorname{Per}_s(E, \Omega) = \alpha(\mathcal{C}E)|E \cap \Omega| + \alpha(E)|\mathcal{C}E \cap \Omega|.$$

D'autre part, le comportement asymptotique lorsque $s \to 0^+$ de la courbure moyenne fractionnaire est étudié au Chapitre 3 (voit aussi [47] pour le cas particulier dans lequel l'ensemble E est borné). Tout d'abord, puisque la quantité $\alpha(E)$ peut ne pas exister—voir [40, Example 2.8 et 2.9]—nous définissons

$$\overline{\alpha}(E) := \limsup_{s \to 0^+} s \int_{\mathcal{C}B_1} \frac{\chi_E(y)}{|y|^{n+s}}\, dy \quad \text{et} \quad \underline{\alpha}(E) := \liminf_{s \to 0^+} s \int_{\mathcal{C}B_1} \frac{\chi_E(y)}{|y|^{n+s}}\, dy.$$

Nous prouvons que, lorsque $s \to 0^+$, la courbure moyenne s-fractionnaire devient complètement indifférente à la géométrie locale de la frontière ∂E, et en effet la valeur limite ne dépend que du comportement à l'infini de l'ensemble E. Plus précisément, si $E \subseteq \mathbb{R}^n$ et $p \in \partial E$ est tel que ∂E est $C^{1,\gamma}$ autour de p, pour un certain $\gamma \in (0,1]$, alors

$$\liminf_{s \to 0^+} s\, H_s[E](p) = \varpi_n - 2\overline{\alpha}(E), \tag{0.17}$$

et

$$\limsup_{s \to 0^+} s\, H_s[E](p) = \varpi_n - 2\underline{\alpha}(E).$$

Nous remarquons en particulier que si E est borné, alors $\alpha(E)$ existe et $\alpha(E) = 0$. Donc, si $E \subseteq \mathbb{R}^n$ est un ensemble ouvert borné ayant frontière $C^{1,\gamma}$, l'asymptotique est simplement

$$\lim_{s \to 0^+} s\, H_s[E](p) = \varpi_n,$$

pour chaque $p \in \partial E$—voir aussi [47, Appendix B].

Dans la Section 3.4 nous calculons la contribution à l'infini $\alpha(E)$ de quelques ensembles. Pour avoir quelques exemples en tête, nous citons ici les cas suivants :

- soit $S \subseteq \mathbb{S}^{n-1}$ et considère le cône
$$C := \{t\sigma \in \mathbb{R}^n \,|\, t \geq 0,\, \sigma \in S\}.$$
Alors, $\alpha(C) = \mathcal{H}^{n-1}(S)$.
- Si $u \in L^\infty(\mathbb{R}^n)$, alors $\alpha(\mathcal{S}g(u)) = \varpi_{n+1}/2$. Plus en général, si $u : \mathbb{R}^n \to \mathbb{R}$ est telle que
$$\lim_{|x| \to \infty} \frac{|u(x)|}{|x|} = 0,$$
alors $\alpha(\mathcal{S}g(u)) = \varpi_{n+1}/2$.
- Soit $u : \mathbb{R}^n \to \mathbb{R}$ telle que $u(x) \leq -|x|^2$, pour chaque $x \in \mathbb{R}^n \setminus B_R$, pour un certain $R > 0$. Alors $\alpha(\mathcal{S}g(u)) = 0$.

Approximativement, à partir des exemples ci-dessus, nous voyons que $\alpha(E)$ ne dépend pas de la géométrie locale ni de la régularité de E, mais seulement de son comportement à l'infini.

Maintenant, nous observons que, lorsque $s \to 0^+$, les ensembles s-minimaux présentent un comportement plutôt inattendu.

Par exemple, en [45, Theorem 1.3] il est prouvé que si nous considérons le premier quadrant du plan comme donnée extérieure, alors, assez étonnamment, si s est assez petit, l'ensemble s-minimal dans $B_1 \subseteq \mathbb{R}^2$ est vide dans B_1. Les principaux résultats du Chapitre 3 s'inspirent de ce résultat.

Heuristiquement, afin de généraliser [45, Theorem 1.3] nous voulons prouver que, si $\Omega \subseteq \mathbb{R}^n$ est un ensemble ouvert borné et connexe ayant frontière lisse et si nous fixons comme donnée extérieure un ensemble $E_0 \subseteq \mathcal{C}\Omega$ tel que $\overline{\alpha}(E_0) < \varpi_n/2$, alors il y a une contradiction entre l'équation d'Euler-Lagrange d'un ensemble s-minimal et l'asymptotique de la courbure moyenne s-fractionnaire pour $s \to 0^+$.

Pour motiver pourquoi nous attendons une telle contradiction, nous observons que l'asymptotique (0.17) semble suggérer que, si s est assez petit, alors un ensemble s-minimal E ayant donnée extérieure E_0 et tel que $\partial E \cap \Omega \neq \varnothing$ devrait avoir un point $p \in \partial E \cap \Omega$ tel que $H_s[E](p) > 0$—qui contredirait l'équation d'Euler-Lagrange. Pour éviter une telle contradiction, nous conclurions alors que $\partial E = \varnothing$ in Ω, c'est-à-dire que soit $E \cap \Omega = \Omega$ ou $E \cap \Omega = \varnothing$.

Afin de transformer cette idée en argument rigoureux, nous montrons d'abord que nous pouvons minorer la courbure moyenne fractionnaire, uniformément par rapport au rayon d'une boule tangente à E extérieurement. Plus précisément :

THÉORÈME 0.4.6. *Soit $\Omega \subseteq \mathbb{R}^n$ un ensemble ouvert borné. Soit $E_0 \subseteq \mathcal{C}\Omega$ tel que*

$$\overline{\alpha}(E_0) < \frac{\varpi_n}{2},$$

et soit

$$\beta = \beta(E_0) := \frac{\varpi_n - 2\overline{\alpha}(E_0)}{4}.$$

Nous définissons

$$\delta_s = \delta_s(E_0) := e^{-\frac{1}{s}\log\frac{\varpi_n+2\beta}{\varpi_n+\beta}},$$

pour chaque $s \in (0,1)$. Alors, il existe $s_0 = s_0(E_0,\Omega) \in (0,\frac{1}{2}]$ tel que, si $E \subseteq \mathbb{R}^n$ est tel que $E \setminus \Omega = E_0$ et E a une boule tangente extérieurement de rayon (au moins) δ_σ, pour un certain $\sigma \in (0,s_0)$, au point $q \in \partial E \cap \overline{\Omega}$, on a

$$\liminf_{\varrho \to 0^+} H_s^\varrho[E](q) \geq \frac{\beta}{s} > 0, \qquad \forall\, s \in (0,\sigma].$$

Introduisons maintenant la définition suivante.

DÉFINITION 0.4.7. *Soit $\Omega \subseteq \mathbb{R}^n$ un ensemble ouvert borné. On dit qu'un ensemble E est δ-*dense *dans Ω, pour un certain $\delta > 0$ fixé, si $|B_\delta(x) \cap E| > 0$ pour chaque $x \in \Omega$ tel que $B_\delta(x) \Subset \Omega$.*

En exploitant un argument géométrique délicat et le Théorème 0.4.6, nous pouvons alors poursuivre l'idée heuristique décrite ci-dessus et prouver le résultat de classification suivant :

THÉORÈME 0.4.8. *Soit $\Omega \subseteq \mathbb{R}^n$ un ensemble ouvert borné et connexe ayant frontière de classe C^2. Soit $E_0 \subseteq \mathcal{C}\Omega$ tel que*

$$\overline{\alpha}(E_0) < \frac{\varpi_n}{2}.$$

Alors, les deux résultats suivants sont vérifiés.
A) Sont s_0 et δ_s comme dans le Théorème 0.4.6. Il existe $s_1 = s_1(E_0,\Omega) \in (0,s_0]$ tel que, si $s < s_1$ et E est un ensemble s-minimal dans Ω ayant donnée extérieure E_0, alors, soit

$$(A.1)\ E \cap \Omega = \varnothing \quad \text{ou} \quad (A.2)\ E \text{ est } \delta_s - \text{dense dans } \Omega.$$

B) Soit
(B.1) il existe $\tilde{s} = \tilde{s}(E_0,\Omega) \in (0,1)$ tel que si E est un ensemble s-minimal dans Ω ayant donnée extérieure E_0 et $s \in (0,\tilde{s})$, alors

$$E \cap \Omega = \varnothing,$$

ou
(B.2) ils existent $\delta_k \searrow 0$, $s_k \searrow 0$ et une suite d'ensembles E_k tels que chaque E_k est s_k-minimal dans Ω par rapport à la donnée extérieure E_0 et pour chaque k

$$\partial E_k \cap B_{\delta_k}(x) \neq \varnothing \quad \text{for every } B_{\delta_k}(x) \Subset \Omega.$$

Approximativement, soit les ensembles s-minimaaux sont vides dans Ω quand s est assez petit, ou nous pouvons trouver une suite E_k d'ensembles s_k-minimaux, pour $s_k \searrow 0$, dont les frontières ont tendance à remplir (topologiquement) le domaine Ω dans la limite $k \to \infty$.

Nous soulignons que le comportement typique consiste à être vide. En fait, si la donnée extérieure $E_0 \subseteq \mathcal{C}\Omega$ n'entoure pas complètement le domaine Ω, nous avons le résultat suivant :

THÉORÈME 0.4.9. *Soit Ω un ensemble ouvert borné et connexe ayant frontière C^2. Soit $E_0 \subseteq \mathcal{C}\Omega$ tel que*

$$\overline{\alpha}(E_0) < \frac{\varpi_n}{2},$$

et soit s_1 comme dans le Théorème 0.4.8. Supposons qu'ils existent $R > 0$ et $x_0 \in \partial\Omega$ tels que

$$B_R(x_0) \setminus \Omega \subseteq \mathcal{C}E_0.$$

Alors, il existe $s_3 = s_3(E_0, \Omega) \in (0, s_1]$ tel que, si $s < s_3$ et E est un ensemble s-minimal dans Ω par rapport à la donnée extérieure E_0, alors

$$E \cap \Omega = \varnothing.$$

Nous observons que la condition $\overline{\alpha}(E_0) < \varpi_n/2$ est en quelque sorte optimale. En effet, lorsque $\alpha(E_0)$ existe et

$$\alpha(E_0) = \frac{\varpi_n}{2},$$

plusieurs configurations peuvent se produire, selon la position de Ω par rapport à la donnée extérieure $E_0 \setminus \Omega$—nous fournissons divers exemples au Chapitre 3.

En outre, notez que lorsque E est s-minimal dans Ω par rapport à E_0, alors $\mathcal{C}E$ est s-minimal dans Ω par rapport à $\mathcal{C}E_0$. En plus,

$$\underline{\alpha}(E_0) > \frac{\varpi_n}{2} \qquad \Longrightarrow \qquad \overline{\alpha}(\mathcal{C}E_0) < \frac{\varpi_n}{2}.$$

Ainsi, dans ce cas, nous pouvons appliquer les Théorèmes 0.4.6, 0.4.8 et 0.4.9 à $\mathcal{C}E$ par rapport à la donnée extérieure $\mathcal{C}E_0$. Par exemple, si E est s-minimal dans Ω par rapport à la donnée extérieure E_0 tel que

$$\underline{\alpha}(E_0) > \frac{\varpi_n}{2},$$

et $s < s_1(\mathcal{C}E_0, \Omega)$, alors, soit

$$E \cap \Omega = \Omega \qquad \text{ou} \qquad \mathcal{C}E \ \text{ est } \ \delta_s(\mathcal{C}E_0) - \text{dense}.$$

Les analogues des Théorèmes mentionnés ci-dessus peuvent être obtenus de la même manière.

Par conséquent, à partir de nos résultats principaux et des observations ci-dessus, nous avons une classification complète des surfaces minimales non locales lorsque s est petit, quand

$$\alpha(E_0) \neq \frac{\varpi_n}{2}.$$

Nous soulignons que les phénomènes de stickiness décrits dans [45] et au Chapitre 3 sont spécifiques aux surfaces minimales non locales, car les surfaces minimales classiques traversent transversalement la frontière d'un domaine convexe.

Fait intéressant, ces phénomènes de stickiness ne sont pas présents dans le cas du Laplacien fractionnaire, où la donnée du problème de Dirichlet est atteint de manière continue sous des hypothèses plutôt générales, voir [89]. Cependant, les solutions des équations de s-Laplace ne sont généralement pas meilleures que C^s à la frontière, donc la continuité uniforme dégénère lorsque $s \to 0^+$.

D'autre part, dans le cas de fonctions harmoniques fractionnaires, une contrepartie partielle du phénomène de stickiness est, en un sens, donnée par les solutions explosives à la frontière construites dans [1, 57] (à savoir, dans ce cas, la frontière du sous-graphe de la fonction harmonique fractionnaire contient des murs verticaux).

Nous mentionnons aussi que des phénomènes de stickiness pour sous-graphes minimaux non locaux—éventuellement en présence d'obstacles—seront étudiés dans le prochain article [15].

Dans la dernière partie du Chapitre 3 nous prouvons que la courbure moyenne fractionnaire est continue pour toutes les variables.

Pour simplifier un peu la situation, supposons que E_k, $E \subseteq \mathbb{R}^n$ sont des ensembles ayant frontières $C^{1,\gamma}$, pour un certain $\gamma \in (0,1]$, tels que les frontières ∂E_k convergent localement au sens $C^{1,\gamma}$ vers la frontière de E, pour $k \to \infty$. Alors, nous prouvons que, si nous avons une séquence de points $x_k \in \partial E_k$ tels que $x_k \to x \in \partial E$ et une suite de paramètres s_k, $s \in (0,\gamma)$ tels que $s_k \to s$, on a

$$\lim_{k\to\infty} H_{s_k}[E_k](x_k) = H_s[E](x).$$

En outre, nous étendons de manière appropriée ce résultat de convergence afin de couvrir également les cas dans lesquels $s_k \to 1$ ou $s_k \to 0$.

En particulier, considérons un ensemble $E \subseteq \mathbb{R}^n$ tel que $\alpha(E)$ existe et ∂E est de classe C^2. Alors, si on définit

$$\tilde{H}_s[E](x) := \begin{cases} s(1-s)H_s[E](x), & \text{pour } s \in (0,1) \\ \varpi_{n-1} H[E](x), & \text{pour } s = 1 \\ \varpi_n - 2\alpha(E), & \text{pour } s = 0, \end{cases}$$

la fonction

$$\tilde{H}_{(\cdot)}[E](\,\cdot\,) : [0,1] \times \partial E \longrightarrow \mathbb{R}, \qquad (s,x) \longmapsto \tilde{H}_s[E](x),$$

est continue. Il est intéressant de noter que la courbure moyenne fractionnaire en un point fixé $q \in \partial E$ peut changer de signe lorsque s varie de 0 à 1. En outre—en conséquence de la continuité dans le paramètre fractionnaire s—dans un tel cas, il existe une valeur $\sigma \in (0,1)$ tel que $H_\sigma[E](q) = 0$.

0.4.4. Cadre non paramétrique. Au Chapitre 4, nous introduisons un cadre fonctionnel pour étudier les minimiseurs du périmètre fractionnaire qui peuvent être écrits globalement en tant que sous-graphes, c'est-à-dire

$$\mathcal{S}g(u) = \left\{(x, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x_{n+1} < u(x)\right\},$$

pour une fonction mesurable $u : \mathbb{R}^n \to \mathbb{R}$. Nous appelons les frontières de ces minimiseurs des *graphes minimaux non locaux.*

Nous définissons une version fractionnaire de la fonctionnelle d'aire classique et nous étudions ses propriétés fonctionnelles et géométriques. Ensuite, nous nous concentrons sur les minimiseurs et nous prouvons des résultats d'existence et d'unicité par rapport à une grande classe de données extérieures, qui inclut les fonctions localement bornées. De plus, l'une des contributions principales du Chapitre 4 consiste à prouver l'équivalence entre :

- minimiseurs de la fonctionnelle d'aire fractionnaire,
- minimiseurs du périmètre fractionnaire,
- solutions faibles de l'équation de courbure moyenne fractionnaire,
- solutions de viscosité de l'équation de courbure moyenne fractionnaire,
- fonctions lisses résolvant ponctuellement l'équation de courbure moyenne fractionnaire.

Avant de donner un aperçu détaillé des principaux résultats, rappelons la définition de la fonctionnelle d'aire classique. Étant donné un ensemble ouvert borné $\Omega \subseteq \mathbb{R}^n$ ayant frontière Lipschitz, la fonctionnelle d'aire est définie comme

$$\mathscr{A}(u,\Omega) := \int_\Omega \sqrt{1+|\nabla u|^2}\,dx = \mathcal{H}^n\left(\left\{(x,u(x)) \in \mathbb{R}^{n+1} \,|\, x \in \Omega\right\}\right),$$

pour chaque fonction Lipschitz $u : \overline{\Omega} \to \mathbb{R}$. On étend alors cette fonctionnelle, en définissant la fonctionnelle d'aire relaxée d'une fonction $u \in L^1(\Omega)$ comme

$$\mathscr{A}(u,\Omega) := \inf\left\{\liminf_{k\to\infty} \mathscr{A}(u_k,\Omega) \,|\, u_k \in C^1(\overline{\Omega}),\ \|u-u_k\|_{L^1(\Omega)} \to 0\right\}.$$

On voit bien que, si $u \in L^1(\Omega)$, alors

$$\mathscr{A}(u,\Omega) < \infty \quad \Longleftrightarrow \quad u \in BV(\Omega), \tag{0.18}$$

dans quel cas

$$\mathscr{A}(u,\Omega) = \mathrm{Per}\left(\mathcal{S}g(u), \Omega\times\mathbb{R}\right). \tag{0.19}$$

Approximativement, les fonctions à variation bornée sont précisément les fonctions intégrables dont les sous-graphes ont périmètre fini—pour les détails, voir, par exemple, [65, 68].

Nous pourrions donc être tentés de définir une version fractionnaire de la fonctionnelle d'aire en considérant le s-périmètre à la place du périmètre classique, définissant, pour une fonction mesurable $u : \mathbb{R}^n \to \mathbb{R}$,

$$\mathscr{A}_s(u,\Omega) := \mathrm{Per}_s(\mathcal{S}g(u), \Omega\times\mathbb{R}).$$

Cependant, comme nous l'avons observé à la fin de la Section 0.4.2, une telle définition ne peut pas fonctionner, car

$$\mathrm{Per}_s^{NL}(\mathcal{S}g(u), \Omega\times\mathbb{R}) = \infty,$$

même si $u \in C_c^\infty(\mathbb{R}^n)$.

Avant de poursuivre, quelques observations s'imposent. Même si la partie non locale du périmètre fractionnaire dans le cylindre $\Omega^\infty := \Omega\times\mathbb{R}$ est infinie, nous rappelons que nous savons—voir la fin de la Section 0.4.2—que la partie locale est finie, si la fonction u est assez régulière dans Ω.

Si la fonction u est bornée dans Ω, alors nous pouvons considérer le périmètre fractionnaire dans le "cylindre tronqué" $\Omega^M := \Omega\times(-M,M)$, où $M \geq \|u\|_{L^\infty(\Omega)}$, au lieu du cylindre Ω^∞. Comme nous le verrons plus loin, en poursuivant cette idée, nous obtenons une famille de fonctionnels d'aire fractionnaires $\mathcal{F}_s^M(\,\cdot\,,\Omega)$.

Par ailleurs, il existe une autre possibilité de définir une fonctionnelle d'aire fractionnaire. Dans [25], les auteurs ont observé que lorsque $E \subseteq \mathbb{R}^{n+1}$ est le sous-graphe d'une fonction u, sa courbure moyenne fractionnaire peut être écrite comme un opérateur intégrodifférentiel agissant sur u. Plus précisément, si $u : \mathbb{R}^n \to \mathbb{R}$ est une fonction de classe $C^{1,1}$ dans un voisinage d'un point $x \in \mathbb{R}^n$, nous avons

$$H_s[\mathcal{S}g(u)](x,u(x)) = \mathscr{H}_s u(x),$$

où

$$\mathscr{H}_s u(x) := 2\,\mathrm{P.V.}\int_{\mathbb{R}^n} G_s\left(\frac{u(x)-u(y)}{|x-y|}\right)\frac{dy}{|x-y|^{n+s}},$$

et

$$G_s(t) := \int_0^t g_s(\tau)\,d\tau, \qquad g_s(t) := \frac{1}{(1+t^2)^{\frac{n+1+s}{2}}} \quad \text{pour } t\in\mathbb{R}.$$

Nous montrons maintenant que $\mathscr{H}_s$ est l'opérateur d'Euler-Lagrange associé à une fonctionnelle (convexe) $\mathcal{F}_s(\,\cdot\,,\Omega)$, que nous considérerons alors comme la fonctionnelle d'aire s-fractionnaire.

Commençons par remarquer que, lorsque u n'est pas assez régulier autour de x, la quantité $\mathscr{H}_s u(x)$ n'est généralement pas bien définie, en raison du manque d'annulation requise pour la valeur principale afin de converger. Néanmoins, nous pouvons comprendre l'opérateur $\mathscr{H}_s$ tel que défini dans le sens faible (distributionnel) suivant. Étant donnée une fonction mesurable $u : \mathbb{R}^n \to \mathbb{R}$, nous définissons

$$\langle \mathscr{H}_s u, v \rangle := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_s \left(\frac{u(x) - u(y)}{|x - y|} \right) \big(v(x) - v(y) \big) \frac{dx\, dy}{|x - y|^{n+s}}$$

pour chaque $v \in C_c^\infty(\mathbb{R}^n)$. Plus généralement, il est immédiat de voir—en profitant du fait que G_s est bornée—que cette définition est bien posée pour chaque $v \in W^{s,1}(\mathbb{R}^n)$. En effet, on a

$$|\langle \mathscr{H}_s u, v \rangle| \leq \frac{\Lambda_{n,s}}{2} [v]_{W^{s,1}(\mathbb{R}^n)},$$

où

$$\Lambda_{n,s} := \int_{\mathbb{R}} g_s(t)\, dt < \infty.$$

Partant, $\mathscr{H}_s u$ peut être interprétée comme une forme linéaire et continue $\langle \mathscr{H}_s u, \cdot \rangle \in (W^{s,1}(\mathbb{R}^n))^*$. Remarquablement, cela vaut pour chaque fonction mesurable $u : \mathbb{R}^n \to \mathbb{R}$, quelle que soit sa régularité.

Nous définissons maintenant

$$\mathcal{G}_s(t) := \int_0^t G_s(\tau)\, d\tau \quad \text{pour } t \in \mathbb{R},$$

et, étant donné une fonction mesurable $u : \mathbb{R}^n \to \mathbb{R}$ et un ensemble ouvert $\Omega \subseteq \mathbb{R}^n$, nous définissons la *fonctionnelle d'aire s-fractionnaire*

$$\mathcal{F}_s(u, \Omega) := \iint_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \mathcal{G}_s \left(\frac{u(x) - u(y)}{|x - y|} \right) \frac{dx\, dy}{|x - y|^{n-1+s}}.$$

Ensuite, au moins formellement, nous avons

$$\frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \mathcal{F}_s(u + \varepsilon v, \Omega) = \langle \mathscr{H}_s u, v \rangle \quad \text{pour chaque } v \in C_c^\infty(\Omega).$$

Nous remarquons que dans le Chapitre 4, nous allons en fait considérer des fonctionnelles plus générales du type aire fractionnaire—en prenant dans les définitions ci-dessus une fonction continue et paire $g : \mathbb{R} \to (0, 1]$ satisfaisant une condition d'intégrabilité appropriée, et les fonctions correspondantes G et $\mathcal{G}$, à la place de g_s, G_s et $\mathcal{G}_s$ respectivement. Cependant, pour plus de simplicité dans cette introduction, nous nous en tenons au "cas géométrique" correspondant au choix $g = g_s$.

Voyons maintenant les propriétés fonctionnelles de $\mathcal{F}_s(\,\cdot\,, \Omega)$ et sa relation avec le périmètre fractionnaire.

À partir de maintenant, nous considérons $n \geq 1$, $s \in (0, 1)$ et un ensemble ouvert borné $\Omega \subseteq \mathbb{R}^n$ ayant frontière Lipschitz.

Il est commode de scinder la fonctionnelle d'aire fractionnaire en tant que somme de sa partie locale et de sa partie non locale, c'est-à-dire

$$\mathcal{F}_s(u, \Omega) = \mathcal{A}_s(u, \Omega) + \mathcal{N}_s(u, \Omega),$$

où

$$\mathcal{A}_s(u, \Omega) := \int_\Omega \int_\Omega \mathcal{G}_s \left(\frac{u(x) - u(y)}{|x - y|} \right) \frac{dx\, dy}{|x - y|^{n-1+s}}$$

et

$$\mathcal{N}_s(u, \Omega) := 2 \int_\Omega \int_{\mathcal{C}\Omega} \mathcal{G}_s \left(\frac{u(x) - u(y)}{|x - y|} \right) \frac{dx\, dy}{|x - y|^{n-1+s}}.$$

Mentionnons tout d'abord l'observation intéressante suivante—voir, par exemple, Lemme D.1.2. Si $u : \Omega \to \mathbb{R}$ est une fonction mesurable, alors

$$[u]_{W^{s,1}(\Omega)} < \infty \qquad \Longrightarrow \qquad \|u\|_{L^1(\Omega)} < \infty.$$

En ce qui concerne la partie locale de la fonctionnelle d'aire fractionnaire, nous prouvons que, si $u : \Omega \to \mathbb{R}$ est une fonction mesurable, alors

$$\begin{aligned}\mathcal{A}_s(u,\Omega) < \infty \quad &\Longleftrightarrow \quad u \in W^{s,1}(\Omega)\\ &\Longleftrightarrow \quad \operatorname{Per}_s^L(\mathcal{S}g(u), \Omega\times\mathbb{R}) < \infty.\end{aligned}$$

En outre, si $u \in W^{s,1}(\Omega)$, alors

$$\operatorname{Per}_s^L(\mathcal{S}g(u), \Omega\times\mathbb{R}) = \mathcal{A}_s(u,\Omega) + c,$$

pour une certaine constante $c = c(n, s, \Omega) \geq 0$. Ces résultats peuvent être considérés comme les contreparties fractionnaires de (0.18) et (0.19).

D'autre part, pour que la partie non locale soit finie, nous devons imposer une condition d'intégrabilité sur u à l'infini, à savoir

$$\int_\Omega \left(\int_{\mathcal{C}\Omega} \frac{|u(y)|}{|x-y|^{n+s}}\,dy \right) dx < \infty. \tag{0.20}$$

Une telle condition est remplie, par exemple, si u est globalement bornée dans $\mathbb{R}^n$ et, en général, cela implique que la fonction u doit avoir un comportement sous-linéaire à l'infini. C'est donc une condition très restrictive.

En effet, on remarque que l'opérateur $\mathscr{H}_s u$ est bien défini en un point x—à condition que u soit assez régulier dans un voisinage de x—sans avoir à imposer de conditions à u à l'infini. De plus, comme nous l'avons observé dans la Section 0.4.2, en conséquence du Théorème 0.4.5 et du [**43**, Theorem 1.1] nous savons que, étant donné toute fonction continue $\varphi : \mathbb{R}^n \to \mathbb{R}$, il existe une fonction $u : \mathbb{R}^n \to \mathbb{R}$ telle que $u = \varphi$ dans $\mathbb{R}^n \setminus \overline{\Omega}$, $u \in C(\overline{\Omega})$ et $\mathcal{S}g(u)$ est localement s-minimal dans Ω^∞. Soulignons qu'aucune condition sur φ à l'infini n'est requise.

Pour ces raisons, la condition (0.20) semble être anormalement restrictive dans notre cadre—même si, à première vue, elle semble nécessaire, car elle est nècessaire pour garantir que $\mathcal{F}_s$ soit bien défini.

Afin d'éviter d'imposer la condition (0.20), nous définissons—voir (4.23)—pour chaque $M \geq 0$, la partie non locale "tronquée" $\mathcal{N}_s^M(u,\Omega)$ et la fonctionnelle d'aire fractionnaire tronquée

$$\mathcal{F}_s^M(u,\Omega) := \mathcal{A}_s(u,\Omega) + \mathcal{N}_s^M(u,\Omega).$$

Approximativement, l'idée consiste à ajouter, à l'intérieur de la double intégrale définissant la partie non locale, un terme équilibrant la contribution venant de l'extérieur de Ω. Par exemple, dans le cas le plus simple $M = 0$, on a

$$\mathcal{N}_s^0(u,\Omega) = 2\int_\Omega \left\{ \int_{\mathcal{C}\Omega} \left[\mathcal{G}_s\left(\frac{u(x)-u(y)}{|x-y|}\right) - \mathcal{G}_s\left(\frac{u(y)}{|x-y|}\right)\right] \frac{dy}{|x-y|^{n-1+s}} \right\} dx.$$

Remarquablement, étant donnée une fonction mesurable $u : \mathbb{R}^n \to \mathbb{R}$, on a

$$|\mathcal{N}_s^M(u,\Omega)| < \infty \quad \text{si } u|_\Omega \in W^{s,1}(\Omega),$$

quel que soit le comportement de u dans $\mathcal{C}\Omega$. D'autre part, nous remarquons qu'en général, la partie non locale tronquée peut être négative, sauf si nous exigeons que u soit bornée dans Ω et nous prenons $M \geq \|u\|_{L^\infty(\Omega)}$. D'un point de vue géométrique, les fonctionnelles d'aire fractionnaire tronquées correspondent à la prise en compte du périmètre fractionnaire dans le cylindre tronqué Ω^M.

En fait, si $u : \mathbb{R}^n \to \mathbb{R}$ est une fonction mesurable telle que $u|_\Omega \in W^{s,1}(\Omega) \cap L^\infty(\Omega)$, et $M \geq \|u\|_{L^\infty(\Omega)}$, on a

$$\mathcal{F}_s^M(u,\Omega) = \mathrm{Per}_s\big(\mathcal{S}g(u), \Omega \times (-M,M)\big) + c_M,$$

pour une certaine constante $c_M = c_M(n,s,\Omega) \geq 0$.

Nous passons maintenant à l'étude des minimiseurs de la fonctionnelle d'aire fractionnaire.

Étant donnée une fonction mesurable $\varphi : \mathcal{C}\Omega \to \mathbb{R}$, nous définissons l'espace

$$\mathcal{W}_\varphi^s(\Omega) := \left\{ u : \mathbb{R}^n \to \mathbb{R} \,|\, u|_\Omega \in W^{s,1}(\Omega) \text{ et } u = \varphi \text{ p.p. dans } \mathcal{C}\Omega \right\},$$

et on dit que $u \in \mathcal{W}_\varphi^s(\Omega)$ est un *minimiseur* de $\mathcal{F}_s$ dans $\mathcal{W}_\varphi^s(\Omega)$, si

$$\iint_{Q(\Omega)} \left\{ \mathcal{G}_s\left(\frac{u(x)-u(y)}{|x-y|}\right) - \mathcal{G}_s\left(\frac{v(x)-v(y)}{|x-y|}\right) \right\} \frac{dx\,dy}{|x-y|^{n-1+s}} \leq 0$$

pour chaque $v \in \mathcal{W}_\varphi^s(\Omega)$. Ci-dessus, nous avons utilisé la notation $Q(\Omega) := \mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2$. Soulignons qu'une telle définition est bien posée sans devoir imposer de conditions à la *donnée extérieure* φ, comme en effet—grâce à l'inégalité de type Hardy fractionnaire du Théorème D.1.4—nous avons

$$\iint_{Q(\Omega)} \left| \mathcal{G}_s\left(\frac{u(x)-u(y)}{|x-y|}\right) - \mathcal{G}_s\left(\frac{v(x)-v(y)}{|x-y|}\right) \right| \frac{dx\,dy}{|x-y|^{n-1+s}} \leq C\,\Lambda_{n,s} \|u-v\|_{W^{s,1}(\Omega)},$$

pour chaques u, $v \in \mathcal{W}_\varphi^s(\Omega)$, pour une certaine constante $C = C(n,s,\Omega) > 0$.

Nous prouvons l'existence de minimiseurs par rapport à des données extérieures satisfaisant une condition d'intégrabilité appropriée dans un voisinage du domaine Ω. Plus précisément, étant donné un ensemble ouvert $\mathcal{O} \subseteq \mathbb{R}^n$ tel que $\Omega \Subset \mathcal{O}$, nous définissons la *queue tronquée* de $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ au point $x \in \Omega$ comme

$$\mathrm{Tail}_s(\varphi, \mathcal{O} \setminus \Omega; x) := \int_{\mathcal{O}\setminus\Omega} \frac{|\varphi(y)|}{|x-y|^{n+s}}\,dy.$$

Nous utilisons la notation

$$\Omega_\varrho := \{x \in \mathbb{R}^n \,|\, d(x,\Omega) < \varrho\},$$

pour $\varrho > 0$, pour dénoter le ϱ-voisinage de Ω. Alors, nous prouvons ce qui suit :

Théorème 0.4.10. *Il existe une constante $\Theta > 1$, qui ne dépend que de n et s, telle que, étant donné toute fonction $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ avec $\mathrm{Tail}_s(\varphi, \Omega_{\Theta\,\mathrm{diam}(\Omega)} \setminus \Omega;\,\cdot\,) \in L^1(\Omega)$, il existe un minimiseur unique u de $\mathcal{F}_s$ dans $\mathcal{W}_\varphi^s(\Omega)$. En plus, u satisfait*

$$\|u\|_{W^{s,1}(\Omega)} \leq C\left(\left\| \mathrm{Tail}_s(\varphi, \Omega_{\Theta\,\mathrm{diam}(\Omega)} \setminus \Omega;\,\cdot\,) \right\|_{L^1(\Omega)} + 1 \right),$$

pour une certaine constante $C = C(n,s,\Omega) > 0$.

Nous observons que la condition sur l'intégrabilité de la queue est beaucoup plus faible que (0.20), puisque nous n'exigeons rien du comportement de φ à l'extérieur de $\Omega_{\Theta\,\mathrm{diam}(\Omega)}$.

Nous mentionnons également que, approximativement, l'intégrabilité de la queue équivaut à l'intégrabilité de φ plus certaines conditions de régularité près de la frontière $\partial\Omega$. Par exemple, si $\varphi \in L^1(\Omega_{\Theta\,\mathrm{diam}(\Omega)} \setminus \Omega)$ et il existe $\varrho > 0$ tel que, soit $\varphi \in W^{s,1}(\Omega_\varrho \setminus \Omega)$ ou $\varphi \in L^\infty(\Omega_\varrho \setminus \Omega)$, alors $\mathrm{Tail}_s(\varphi, \Omega_{\Theta\,\mathrm{diam}(\Omega)} \setminus \Omega;\,\cdot\,) \in L^1(\Omega)$.

L'unicité du minimiseur est une conséquence de la stricte convexité de $\mathcal{F}_s$. D'autre part, afin de prouver l'existence, nous exploitons les (uniques) minimiseurs u_M des fonctionnelles $\mathcal{F}_s^M(\,\cdot\,,\Omega)$—considérés dans leur domaine naturel. Nous exploitons l'hypothèse sur l'intégrabilité de la queue pour prouver une estimation uniforme de la norme $W^{s,1}(\Omega)$ des minimiseurs u_M, indépendamment de $M \geq 0$. Donc, quitte à extraire des sous-suites,

u_M converge, lorsque $M \to \infty$, vers une fonction limite u, qui est facilement prouvé être un minimiseur de $\mathcal{F}_s$.

En outre, nous prouvons que, si u est un minimiseur de $\mathcal{F}_s$ dans $\mathcal{W}^s_\varphi(\Omega)$, alors $u \in L^\infty_{\mathrm{loc}}(\Omega)$. De plus, nous montrons que, si la donnée extérieure φ est bornée dans un voisinage assez gros de Ω, alors $u \in L^\infty(\Omega)$, et nous établissons également une estimation a priori pour la norme L^∞.

Revenons à la relation entre la fonctionnelle d'aire fractionnaire et le périmètre fractionnaire. Nous montrons qu'en réarrangeant correctement un ensemble E dans la direction verticale, nous diminuons le s-périmètre. Plus précisément, à partir d'un ensemble $E \subseteq \mathbb{R}^{n+1}$, nous considérons la fonction $w_E : \mathbb{R}^n \to \mathbb{R}$ définie comme

$$w_E(x) := \lim_{R \to +\infty} \left(\int_{-R}^{R} \chi_E(x,t)\, dt - R \right)$$

pour chaque $x \in \mathbb{R}^n$.

Alors, nous avons le résultat suivant :

Théorème 0.4.11. *Soit $E \subseteq \mathbb{R}^{n+1}$ tel que $E \setminus \Omega^\infty$ est un sous-graphe et*

$$\Omega \times (-\infty, -M) \subseteq E \cap \Omega^\infty \subseteq \Omega \times (-\infty, M),$$

pour un certain $M > 0$. Alors,

$$\mathrm{Per}_s(\mathcal{S}g(w_E), \Omega^M) \leq \mathrm{Per}_s(E, \Omega^M).$$

L'ingálité est stricte sauf si $\mathcal{S}g(w_E) = E$.

En exploitant également le fait qu'un minimiseur est localement borné, nous prouvons que, si $u : \mathbb{R}^n \to \mathbb{R}$ est une fonction mesurable telle que $u \in W^{s,1}(\Omega)$, alors

$$u \text{ minimise } \mathcal{F}_s \text{ dans } \mathcal{W}^s_u(\Omega) \quad \Longrightarrow \quad \mathcal{S}g(u) \text{ est localement } s\text{-minimal dans } \Omega^\infty.$$

Le Théorème 0.4.11 étend au cadre fractionnaire un résultat bien connu tenant pour le périmètre classique—voir, par exemple, [68, Lemma 14.7]. Cependant, notez que dans le cadre fractionnaire, en raison du caractère non local des fonctionnelles impliquées, nous devons supposer que l'ensemble E est déjà un sous-graphe à l'extérieur du cylindre Ω^∞.

Nous observons également que, puisque u est localement bornée dans Ω et son sous-graphe est localement s-minimal dans le cylindre Ω^∞, grâce à [19, Theorem 1.1] nous avons $u \in C^\infty(\Omega)$—c'est-à-dire, les minimiseurs de $\mathcal{F}_s$ sont lisses.

Voyons maintenant l'équation d'Euler-Lagrange satisfaite par les minimiseurs. Nous introduisons d'abord la notion de solutions faibles.

Soit $f \in C(\overline{\Omega})$. On dit qu'une fonction mesurable $u : \mathbb{R}^n \to \mathbb{R}$ est une solution faible de $\mathscr{H}_s u = f$ dans Ω, si

$$\langle \mathscr{H}_s u, v \rangle = \int_\Omega f v \, dx,$$

pour chaque $v \in C^\infty_c(\Omega)$.

En conséquence de la convexité de $\mathcal{F}_s$, il est facile de prouver que, étant donnée une fonction mesurable $u : \mathbb{R}^n \to \mathbb{R}$ telle que $u \in W^{s,1}(\Omega)$, on a

$$u \text{ minimise } \mathcal{F}_s \text{ dans } \mathcal{W}^s_u(\Omega) \quad \Longleftrightarrow \quad u \text{ est une solution faible de } \mathscr{H}_s u = 0 \text{ dans } \Omega.$$

Une autre notion naturelle de solution pour l'équation $\mathscr{H}_s u = f$ est celle d'une solution de viscosité—nous nous référons à la Section 4.3 pour la définition précise. Un des principaux résultats du Chapitre 4 consiste à prouver que les (sous-)solutions de viscosité sont des (sous-)solutions faibles. Plus précisément :

THÉORÈME 0.4.12. *Soit $\Omega \subseteq \mathbb{R}^n$ un ensemble ouvert borné et soit $f \in C(\overline{\Omega})$. Soit $u : \mathbb{R}^n \to \mathbb{R}$ localement intégrable et localement borné dans Ω. Si u est une sous-solution de viscosité,*

$$\mathscr{H}_s u \leq f \quad dans\ \Omega,$$

alors u est une sous-solution faible,

$$\langle \mathscr{H}_s u, v \rangle \leq \int_\Omega f v\, dx, \qquad \forall\, v \in C_c^\infty(\Omega)\ telle\ que\ v \geq 0.$$

En combinant les principaux résultats du Chapitre 4 et en exploitant la régularité à la intérieure prouvée dans [19], on obtient ce qui suit :

THÉORÈME 0.4.13. *Soit $u : \mathbb{R}^n \to \mathbb{R}$ une fonction mesurable telle que $u \in W^{s,1}(\Omega)$. Alors, les propositions suivantes sont équivalentes :*
- *(i) u est une solution faible de $\mathscr{H}_s u = 0$ dans Ω,*
- *(ii) u minimise $\mathcal{F}_s$ dans $\mathcal{W}_u^s(\Omega)$,*
- *(iii) $u \in L^\infty_{\mathrm{loc}}(\Omega)$ et $\mathcal{S}g(u)$ est localement s-minimal dans $\Omega \times \mathbb{R}$,*
- *(iv) $u \in C^\infty(\Omega)$ et u est une solution ponctuelle de $\mathscr{H}_s u = 0$ dans Ω.*

En plus, si $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \cap W^{s,1}(\Omega)$, alors les propositions ci-dessus sont équivalentes à :
- *(v) u est une solution de viscosité de $\mathscr{H}_s u = 0$ dans Ω.*

Nous mentionnons également la version globale suivante du Théorème 0.4.13 :

COROLLAIRE 0.4.14. *Soit $u \in W^{s,1}_{\mathrm{loc}}(\mathbb{R}^n)$. Alors, les propositions suivantes sont équivalentes :*
- *(i) u est une solution de viscosité de $\mathscr{H}_s u = 0$ dans $\mathbb{R}^n$,*
- *(ii) u est une solution faible de $\mathscr{H}_s u = 0$ dans $\mathbb{R}^n$,*
- *(iii) u minimise $\mathcal{F}_s$ dans $\mathcal{W}_u^s(\Omega)$, pour chaque ensemble ouvert $\Omega \Subset \mathbb{R}^n$ ayant frontière Lipschitz,*
- *(iv) $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ et $\mathcal{S}g(u)$ est localement s-minimal dans $\mathbb{R}^{n+1}$,*
- *(v) $u \in C^\infty(\mathbb{R}^n)$ et u est une solution ponctuelle de $\mathscr{H}_s u = 0$ dans $\mathbb{R}^n$.*

Nous signalons également que le cadre fonctionnel présenté ci-dessus s'étend facilement au problème avec obstacles. À savoir, en plus d'imposer la condition de la donnée extérieure $u = \varphi$ p.p. dans $\mathcal{C}\Omega$, nous contraignons les fonctions à se trouver au-dessus d'un obstacle, c'est-à-dire, ètant donné un ensemble ouvert $A \subseteq \Omega$ et un obstacle $\psi \in L^\infty(A)$, nous nous bornons à considérer ces fonctions $u \in \mathcal{W}_\varphi^s(\Omega)$ telles que $u \geq \psi$ p.p. dans A.

Au Chapitre 4 nous examinons également brièvement ce problème d'obstacle, prouvant l'existence et l'unicité d'un minimiseur, et sa relation avec le problème d'obstacle géométrique qui concerne le périmètre fractionnaire.

Enfin, dans la dernière Section du Chapitre 4, nous prouvons quelques résultats d'approximation pour les sous-graphes ayant périmètre fractionnaire (localement) fini. En particulier, en exploitant le résultat surprenant de densité établi dans [44], nous montrons que les sous-graphes s-minimaux peuvent être approximés de manière appropriée par des sous-graphes de fonctions σ-harmoniques, pour chaque $\sigma \in (0, 1)$ fixé.

0.4.5. Résultats de rigidité pour les graphes minimaux non locaux. Au Chapitre 5 nous prouvons un résultat de platitude pour des graphes minimaux non locaux entiers ayant des dérivées partielles minorés ou majorés. Ce résultat généralise au cadre fractionnaire des théorèmes classiques dues à Bernstein et Moser.

De plus, nous montrons que les graphes entiers ayant courbure moyenne fractionnaire constante sont minimales, étendant ainsi un résultat célèbre de Chern sur les graphes CMC classiques.

Nous sommes intéressés par les sous-graphes qui minimisent localement le s-périmètre dans tout l'espace $\mathbb{R}^{n+1}$. Nous rappelons que, comme nous l'avons vu dans le Corollaire 0.4.14, sous des hypothèses très faibles sur la fonction $u : \mathbb{R}^n \to \mathbb{R}$, le sous-graphe $\mathcal{S}g(u)$ est localement s-minimal dans $\mathbb{R}^{n+1}$ si et seulement si u satisfait à l'équation de courbure moyenne fractionnaire

$$\mathscr{H}_s u = 0 \quad \text{dans } \mathbb{R}^n. \tag{0.21}$$

En outre, encore une fois grâce au Corollaire 0.4.14, il existe plusieurs notions équivalentes de solution pour l'équation (0.21), telles que solutions lisses, solutions de viscosité et solutions faibles.

Dans ce qui suit, une solution de (0.21) indiquera toujours une fonction $u \in C^\infty(\mathbb{R}^n)$ qui satisfait l'identité (0.21) ponctuellement. Nous soulignons qu'aucune hypothèse de croissance à l'infini n'est faite sur u.

La contribution principale du Chapitre 5 est le résultat suivant :

Théorème 0.4.15. *Soient $n \geq \ell \geq 1$ des entiers, $s \in (0,1)$, et supposons que*

$$(P_{s,\ell}) \qquad \textit{il n'y a pas de cônes singuliers } s\textit{-minimaux dans } \mathbb{R}^\ell.$$

Soit u une solution de $\mathscr{H}_s u = 0$ dans $\mathbb{R}^n$, ayant $n - \ell$ derivées partielles minorés ou majorés. Alors, u est une fonction affine.

La caractérisation des valeurs de s et ℓ pour lesquelles ($P_{s,\ell}$) est satisfaite représente un problème ouvert difficile à résoudre. Néanmoins, il est connu que la propriété ($P_{s,\ell}$) est vraie dans les cas suivants :

- lorsque $\ell = 1$ ou $\ell = 2$, pour chaque $s \in (0,1)$;
- lorsque $3 \leq \ell \leq 7$ et $s \in (1 - \varepsilon_0, 1)$ pour un certain $\varepsilon_0 \in (0,1]$ ne dépendant que de ℓ.

Le cas $\ell = 1$ est vrai par définition, alors que le cas $\ell = 2$ est le contenu de [**92**, Theorem 1]. D'autre part, le cas $3 \leq \ell \leq 7$ a été établi en [**25**, Theorem 2].

En conséquence du Théorème 0.4.15 et des dernières remarques, nous obtenons immédiatement le résultat suivant :

Corollaire 0.4.16. *Soient $n \geq \ell \geq 1$ des entiers et $s \in (0,1)$. Supposons que*

- *$\ell \in \{1,2\}$, ou*
- *$3 \leq \ell \leq 7$ et $s \in (1 - \varepsilon_0, 1)$, où $\varepsilon_0 = \varepsilon_0(\ell) > 0$ est comme en [**25**, Theorem 2].*

Soit u une solution de $\mathscr{H}_s u = 0$ dans $\mathbb{R}^n$, ayant $n - \ell$ derivées partielles minorés ou majorés. Alors, u est une fonction affine.

Nous observons que le Théorème 0.4.15 est un nouveau résultat de platitude pour les graphes s-minimaux, en supposant que ($P_{s,\ell}$) est vrai. Cela peut être vu comme une généralisation du lemme de type De Giorgi fractionnaire contenu dans [**58**, Theorem 1.2], qui est récupéré ici en prenant $\ell = n$. Dans ce cas, nous fournissons en effet une preuve alternative dudit résultat.

D'autre part, le choix $\ell = 2$ donne une amélioration de [**55**, Theorem 4], quand spécialisé aux graphes s-minimaux. À la lumière de ces observations, le Théorème 0.4.15 et le Corollaraire 0.4.16 peuvent être vus comme un pont entre les théorèmes de type Bernstein (résultats de platitude dans les dimensions basses) et les théorèmes de type Moser (résultats de platitude en conséquence des estimations globales du gradient).

Pour les graphes minimaux classiques, la contrepartie de Corollaire 0.4.16 a récemment été obtenue par A. Farina dans [**54**]. Dans ce cas, le résultat est optimal et tient avec $\ell = \min\{n, 7\}$. La preuve du Théorème 0.4.15 est basée sur l'extension au cadre fractionnaire d'une stratégie—qui repose sur un résultat de splitting général pour les blow-downs du sous-graphe $\mathcal{S}g(u)$—conçu par A. Farina pour les graphes minimaux classiques et inédit. En conséquence, les idées contenues dans le Chapitre 5 peuvent être utilisées pour obtenir une preuve différente, plus simple, de [54, Theorem 1.1]

Signalons également que, en utilisant les mêmes idées que celles qui conduisent au Théorème 0.4.15, nous pouvons prouver le résultat de rigidité suivant pour ces graphes s-mimimaux entiers qui sont situés au-dessus d'un cône.

THÉORÈME 0.4.17. *Sont $n \geq 1$ un entier et $s \in (0,1)$. Soit u une solution de $\mathscr{H}_s u = 0$ dans $\mathbb{R}^n$, et supposons qu'il existe une constante $C > 0$ telle que*

$$u(x) \geq -C(1 + |x|) \quad \textit{pour chaque } x \in \mathbb{R}^n.$$

Alors, u est une fonction affine.

Nous remarquons que dans [19] on en déduit un résultat de rigidité analogue au Théorème 0.4.17, sous l'hypothèse plus forte et bilatérale

$$|u(x)| \leq C(1 + |x|) \quad \text{pour chaque } x \in \mathbb{R}^n.$$

Le Théorème 0.4.17 améliore donc [19, Theorem 1.5] directement.

Enfin, nous prouvons que, si $u : \mathbb{R}^n \to \mathbb{R}$ est telle que

$$\langle \mathscr{H}_s u, v \rangle = h \int_{\mathbb{R}^n} v\, dx \quad \text{pour chaque } v \in C_c^\infty(\mathbb{R}^n),$$

pour une certaine constante $h \in \mathbb{R}$, alors la constante doit être $h = 0$.

En particulier, en rappelant le Corollaire 0.4.14, on voit que, si $u \in W^{s,1}_{\text{loc}}(\mathbb{R}^n)$ est une solution faible de $\mathscr{H}_s u = h$ in $\mathbb{R}^n$, alors le sous-graphe de u est localement s-minimal dans $\mathbb{R}^{n+1}$. Cela étend au cadre non local un résultat célèbre de Chern, à savoir le corollaire du Théorème 1 de [26].

0.4.6. Un problème à frontière libre. Au Chapitre 6 nous étudions les minimiseurs de la fonctionnelle

$$\mathcal{N}(u, \Omega) + \operatorname{Per}\big(\{u > 0\}, \Omega\big), \tag{0.22}$$

où $\mathcal{N}(u, \Omega)$ est, approximativement, la Ω-contribution à la seminorme H^s de la fonction $u : \mathbb{R}^n \to \mathbb{R}$, c'est-à-dire

$$\mathcal{N}(u, \Omega) := \iint_{\mathbb{R}^{2n} \setminus (\mathcal{C}\Omega)^2} \frac{|u(x) - u(y)|^2}{|x - y|^{n+2s}}\, dx\, dy,$$

pour un certain paramètre $s \in (0, 1)$ fixé.

Des fonctionnelles similaires, définies comme la superposition d'un terme "énergie élastique" et d'une "tension de surface", ont déjà été examinées dans les articles suivants :

- energie de Dirichlet plus périmètre dans [6],
- energie de Dirichlet plus périmètre fractionnaire dans [22],
- l'energie non locale $\mathcal{N}$ plus le périmètre dans [42], et le problème à une phase correspondant dans [46].

L'étude de la fonctionnelle définie dans (0.22) complète en quelque sorte cette situation.

Les contributions principales du Chapitre 6 consistent à établir une formule de monotonie pour les minimiseurs de la fonctionnelle (0.22), à l'exploiter pour étudier les propriétés des limites de blow-up et à fournir un résultat de réduction de la dimension. De plus, nous montrons que, lorsque $s < 1/2$, le périmètre domine l'energie non locale. En conséquence, nous obtenons un résultat de régularité pour la frontière libre $\{u = 0\}$.

En guise de note technique, observons d'abord que nous ne pouvons pas travailler directement avec l'ensemble $\{u > 0\}$. Au lieu de cela, nous considérons des *paires admissibles* (u, E), où $u : \mathbb{R}^n \to \mathbb{R}$ est une fonction mesurable, et $E \subseteq \mathbb{R}^n$ est tel que

$$u \geq 0 \quad \text{p.p. dans } E \quad \text{et} \quad u \leq 0 \quad \text{p.p. dans } \mathcal{C}E.$$

L'ensemble E est généralement appelé *ensemble de positivité* de u. Alors, étant donnée une valeur $s \in (0,1)$ et un ensemble ouvert ayant frontière Lipschitz $\Omega \subseteq \mathbb{R}^n$, nous définissons la fonctionnelle

$$\mathcal{F}_\Omega(u, E) := \mathcal{N}(u, \Omega) + \operatorname{Per}(E, \Omega),$$

pour chaque paire admissible (u, E).

Remarquons maintenant que, si $u : \mathbb{R}^n \to \mathbb{R}$ est une fonction mesurable, alors

$$\mathcal{N}(u, \Omega) < \infty \quad \Longrightarrow \quad \int_{\mathbb{R}^n} \frac{|u(\xi)|^2}{1 + |\xi|^{n+2s}} \, d\xi < \infty. \tag{0.23}$$

Pour une preuve, voir par exemple, Lemme D.1.3. En conséquence, nous avons aussi

$$\int_{\mathbb{R}^n} \frac{|u(\xi)|}{1 + |\xi|^{n+2s}} \, d\xi < \infty \quad \text{et} \quad u \in L^2_{\text{loc}}(\mathbb{R}^n).$$

La notion de minimiseurs que nous considérons est la suivante :

Définition 0.4.18. *Étant donnée une paire admissible (u, E) telle que $\mathcal{F}_\Omega(u, E) < \infty$, on dit que une paire (v, F) est un* concurrent admissible *si*

$$\begin{aligned} &\textit{supp}(v - u) \Subset \Omega, && F \Delta E \Subset \Omega, \\ &v - u \in H^s(\mathbb{R}^n) \quad \textit{et} && \operatorname{Per}(F, \Omega) < +\infty. \end{aligned} \tag{0.24}$$

On dit que une paire admissible (u, E) est minimisante *dans Ω si $\mathcal{F}_\Omega(u, E) < \infty$ et*

$$\mathcal{F}_\Omega(u, E) \leq \mathcal{F}_\Omega(v, F),$$

pour chaque concurrent admissible (v, F).

Notez que la première ligne de (0.24) dit simplement que les paires (u, E) et (v, F) sont égales—au sens théorique de la mesure—en dehors d'un sous-ensemble compact de Ω. Donc, puisque $\mathcal{F}_\Omega(u, E) < \infty$, on voit facilement que la deuxième ligne est équivalente à $\mathcal{F}_\Omega(v, F) < \infty$.

En particulier, nous nous intéressons au problème de minimisation suivant, par rapport à la "donnée extérieure" fixée. Étant donnée une paire admissible (u_0, E_0) et un ensemble ouvert borné $\mathcal{O} \subseteq \mathbb{R}^n$ ayant frontière Lipschitz, tels que

$$\Omega \Subset \mathcal{O}, \qquad \mathcal{N}(u_0, \Omega) < +\infty \quad \text{et} \quad \operatorname{Per}(E_0, \mathcal{O}) < +\infty, \tag{0.25}$$

nous voulons trouver une paire admissible (u, E) atteignant l'infimum suivant

$$\begin{aligned} \inf \big\{ \mathcal{N}(v, \Omega) + \operatorname{Per}(F, \mathcal{O}) \,|\, (v, F) &\text{ paire admissible t.q. } v = u_0 \text{ p.p. dans } \mathcal{C}\Omega \\ &\text{et } F \setminus \Omega = E_0 \setminus \Omega \big\}. \end{aligned} \tag{0.26}$$

Approximativement, comme d'habitude lorsqu'il s'agit de problèmes de minimisation impliquant le périmètre classique, nous envisageons un voisinage (fixe) $\mathcal{O}$ de Ω (aussi petit que nous le souhaitons) afin de "lire" la donnée sur la frontière, $\partial E_0 \cap \partial \Omega$.

Nous prouvons que, étant fixée une donnée extérieure (u_0, E_0) satisfaisant (0.25), il existe une paire (u, E) réalisant l'infimum (0.26). De plus, nous montrons qu'une telle paire (u, E) minimise aussi au sens de la Définition 0.4.18.

Un résultat utile consiste à établir une estimation uniforme de l'énergie des paires minimisantes.

THÉORÈME 0.4.19. *Soit* (u, E) *une paire minimisante dans* B_2. *Alors*

$$\iint_{\mathbb{R}^{2n}\setminus(\mathcal{C}B_1)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \mathrm{Per}(E,B_1) \le C\left(1+\int_{\mathbb{R}^n}\frac{|u(y)|^2}{1+|y|^{n+2s}}\,dy\right),$$

pour une certaine $C = C(n,s) > 0$.

En particulier, le Théorème 0.4.19 est exploité dans la preuve de l'existence d'une limite de blow-up. Pour cela, nous devons d'abord introduire—par la technique d'extension de [23]—la fonctionnelle étendue associée à la minimisation de $\mathcal{F}_\Omega$. Nous écrivons

$$\mathbb{R}^{n+1}_+ := \{(x,z)\in\mathbb{R}^{n+1}\,|\,x\in\mathbb{R}^n,\ z>0\}.$$

Étant donnée une fonction $u:\mathbb{R}^n\to\mathbb{R}$, nous considérons la fonction $\overline{u}:\mathbb{R}^{n+1}_+\to\mathbb{R}$ définie via la convolution avec un noyau de Poisson approprié,

$$\overline{u}(\,\cdot\,,z) = u * \mathcal{K}_s(\,\cdot\,,z), \quad \text{où} \quad \mathcal{K}_s(x,z) := c_{n,s}\frac{z^{2s}}{(|x|^2+z^2)^{(n+2s)/2}},$$

et $c_{n,s} > 0$ est une constante de normalisation appropriée. Une telle fonction étendue $\overline{u}$ est bien définie—voir, par exemple, [75]—à condition que $u:\mathbb{R}^n\to\mathbb{R}$ est telle que

$$\int_{\mathbb{R}^n}\frac{|u(\xi)|}{1+|\xi|^{n+2s}}\,d\xi < \infty.$$

À la lumière de (0.23), nous pouvons donc considérer la fonction d'extension d'un minimiseur.

Nous utilisons des lettres majuscules, comme $X = (x,z)$, pour désigner les points dans $\mathbb{R}^{n+1}$. Étant donné un ensemble $\Omega \subseteq \mathbb{R}^{n+1}$, nous écrivons

$$\Omega_+ := \Omega\cap\{z>0\} \qquad \text{et} \qquad \Omega_0 := \Omega\cap\{z=0\}.$$

De plus, nous identifions l'hyperplan $\{z=0\} \simeq \mathbb{R}^n$ via la fonction de projection. Étant donné un ensemble ouvert borné $\Omega \subseteq \mathbb{R}^{n+1}$ ayant frontière Lipschitz, tel que $\Omega_0 \neq \varnothing$, nous définissons

$$\mathfrak{F}_\Omega(\mathcal{V},F) := c'_{n,s}\int_{\Omega_+}|\nabla\mathcal{V}|^2 z^{1-2s}\,dX + \mathrm{Per}(F,\Omega_0),$$

pour $\mathcal{V}:\mathbb{R}^{n+1}_+\to\mathbb{R}$ et $F\subseteq\mathbb{R}^n\simeq\{z=0\}$ l'ensemble de positivité de la trace de $\mathcal{V}$ sur $\{z=0\}$, c'est-à-dire

$$\mathcal{V}\big|_{\{z=0\}} \ge 0 \quad \text{p.p. dans } F \quad \text{et} \quad \mathcal{V}\big|_{\{z=0\}} \le 0 \quad \text{p.p. dans } \mathcal{C}F.$$

Nous appellons une telle paire $(\mathcal{V},F)$ une *paire admissible* pour la fonctionnelle étendue. Alors, nous introduisons la notion suivante de minimiseur pour la fonctionnelle étendue.

DÉFINITION 0.4.20. *Étant donnée une paire admissible* $(\mathcal{U},E)$, *telle que* $\mathfrak{F}_\Omega(\mathcal{U},E) < \infty$, *on dit qu'une paire* $(\mathcal{V},F)$ *est un* concurrent admissible, *si* $\mathfrak{F}_\Omega(\mathcal{V},F)<\infty$ *et*

$$supp\,(\mathcal{V}-\mathcal{U}) \Subset \Omega \qquad et \qquad E\Delta F \Subset \Omega_0.$$

On dit qu'une paire admissible $(\mathcal{U},E)$ *est* minimale *dans* Ω *si* $\mathfrak{F}_\Omega(\mathcal{U},E)<\infty$ *et*

$$\mathfrak{F}_\Omega(\mathcal{U},E) \le \mathfrak{F}_\Omega(\mathcal{V},F),$$

pour chaque concurrent admissible $(\mathcal{V},F)$.

Un résultat important consiste à montrer qu'un problème de minimisation approprié impliquant les fonctionnelles étendues équivaut à la minimisation de la fonctionnelle d'origine $\mathcal{F}_\Omega$. Plus précisément :

Proposition 0.4.21. *Soit (u, E) une paire admissible pour $\mathcal{F}$, telle que $\mathcal{F}_{B_R}(u.E) < +\infty$. Alors, la paire (u, E) est minimisante dans B_R si et seulement si la paire $(\overline{u}, E)$ est minimale pour $\mathfrak{F}_\Omega$, dans chaque ensemble ouvert borné $\Omega \subseteq \mathbb{R}^{n+1}$ ayant frontière Lipschitz tel que $\varnothing \neq \Omega_0 \Subset B_R$.*

L'une des principales raisons d'introduire la fonctionnelle étendue réside dans le fait qu'elle nous permet d'établir une formule de monotonie de type Weiss pour les minimiseurs.

Nous notons

$$\mathcal{B}_r := \{(x, z) \in \mathbb{R}^{n+1} \,|\, |x|^2 + z^2 < r^2\} \qquad \text{et} \qquad \mathcal{B}_r^+ := \mathcal{B}_r \cap \{z > 0\}.$$

Théorème 0.4.22 (Formule de Monotonie de type Weiss). *Soit (u, E) une paire minimisante pour $\mathcal{F}$ dans B_R et définissons la fonction $\Phi_u : (0, R) \to \mathbb{R}$ comme*

$$\Phi_u(r) := r^{1-n} \left(c'_{n,s} \int_{\mathcal{B}_r^+} |\nabla \overline{u}|^2 z^{1-2s} \, dX + \operatorname{Per}(E, B_r) \right) - c'_{n,s}\Big(s - \frac{1}{2}\Big) r^{-n} \int_{(\partial \mathcal{B}_r)^+} \overline{u}^2 z^{1-2s} \, d\mathcal{H}^n.$$

Alors, la fonction Φ_u est croissante dans $(0, R)$. En outre, Φ_u est constante dans $(0, R)$ si et seulement si l'extension $\overline{u}$ est homogène de degré $s - \frac{1}{2}$ dans $\mathcal{B}_R^+$ et E est un cône dans B_R.

Ci-dessus, $(\partial \mathcal{B}_r)^+ := \partial \mathcal{B}_r \cap \{z > 0\}$. Présentons maintenant les paires redimensionnées (u_λ, E_λ). Étant donné $u : \mathbb{R}^n \to \mathbb{R}$ et $E \subseteq \mathbb{R}^n$, nous définissons

$$u_\lambda(x) := \lambda^{\frac{1}{2} - s} u(\lambda x) \qquad \text{et} \qquad E_\lambda := \frac{1}{\lambda} E,$$

pour chaque $\lambda > 0$. Nous observons que—en raison des propriétés d'échelle de $\mathcal{F}_\Omega$—une paire (u, E) est minimale dans Ω si et seulement si la paire redimensionnée (u_λ, E_λ) est minimale dans Ω_λ, pour chaque $\lambda > 0$.

Nous prouvons la convergence des paires minimisantes dans les conditions appropriées et nous l'exploitons—en même temps que le Théorème 0.4.19—dans le cas particulièrement important de la suite de blow-up.

On dit qu'une paire admissible (u, E) est un *cône minimisant* si elle est une paire minimisante dans B_R, pour chaque $R > 0$, et elle est telle que u est homogène de degré $s - \frac{1}{2}$ et E est un cône

Théorème 0.4.23. *Soit $s > 1/2$ et soit (u, E) une paire minimisante dans B_1, avec $0 \in \partial E$. Supposons également que $u \in C^{s - \frac{1}{2}}(B_1)$. Alors, il existe un cône minimisant (u_0, E_0) et une séquence $r_k \searrow 0$ tels que $u_{r_k} \to u_0$ dans $L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ et $E_{r_k} \xrightarrow{\mathrm{loc}} E_0$.*

Les propriétés d'homogénéité de la limite de blow-up (u_0, E_0) sont une conséquence du Théorème 0.4.22.

Nous soulignons également que nous établissons des estimations appropriées pour les énergies de queue des fonctions u_r, ce qui nous permet d'affaiblir les hypothèses de [42, Theorem 1.3], où les auteurs demandent à u d'être $C^{s-\frac{1}{2}}$ dans tout $\mathbb{R}^n$.

Nous mentionnons maintenant le résultat de réduction dimensionnelle suivant. Seulement dans le Théorème suivant, redéfinissons

$$\mathcal{F}_\Omega(u, E) := (c'_{n,s})^{-1} \mathcal{N}(u, \Omega) + \operatorname{Per}(E, \Omega).$$

On dit qu'une paire admissible (u, E) est minimisante dans $\mathbb{R}^n$ si cela minimise $\mathcal{F}_\Omega$ dans chaque ensemble ouvert borné $\Omega \subseteq \mathbb{R}^n$ ayant frontière Lipschitz.

Théorème 0.4.24. *Soit* (u, E) *une paire admissible et définissons*

$$u^\star(x, x_{n+1}) := u(x) \qquad et \qquad E^\star := E \times \mathbb{R}.$$

Alors, la paire (u, E) *est minimisante dans* $\mathbb{R}^n$ *si et seulement si la paire* $(u^\star, E^\star)$ *est minimisante dans* $\mathbb{R}^{n+1}$.

Enfin, nous observons que dans le cas $s < 1/2$, le périmètre est en quelque sorte le terme principal de la fonctionnelle $\mathcal{F}_\Omega$. En conséquence, nous pouvons prouver le résultat de régularité suivant :

Théorème 0.4.25. *Soit* $s \in (0, 1/2)$ *et soit* (u, E) *une paire minimisante dans* Ω. *Suppose que* $u \in L^\infty_{\rm loc}(\Omega)$. *Alors,* E *a frontière presque minimale dans* Ω. *Plus précisément, si* $x_0 \in \Omega$ *et* $d := d(x_0, \Omega)/3$, *alors, pour chaque* $r \in (0, d]$ *on a*

$$\mathrm{Per}(E, B_r(x_0)) \leq \mathrm{Per}(F, B_r(x_0)) + C\, r^{n-2s}, \qquad \forall\, F \subseteq \mathbb{R}^n \ t.q. \ E \Delta F \Subset B_r(x_0),$$

où

$$C = C\left(s, x_0, d, \|u\|_{L^\infty(B_{2d}(x_0))}, \int_{\mathbb{R}^n} \frac{|u(y)|}{1 + |y|^{n+2s}}\, dy\right) > 0.$$

Donc

(i) $\partial^* E$ *est localement* $C^{1,\frac{1-2s}{2}}$ *dans* Ω,

(ii) l'ensemble singulier $\partial E \setminus \partial^* E$ *est tel que*

$$\mathcal{H}^\sigma\big((\partial E \setminus \partial^* E) \cap \Omega\big) = 0, \qquad pour\ chaque\ \sigma > n - 8.$$

Nous concluons en disant quelques mots sur le problème à une phase, qui correspond au cas dans lequel $u \geq 0$ p.p. dans $\mathbb{R}^n$. Même si ces résultats ne sont pas inclus dans cette thèse, ils feront partie de la version finale de l'article sur lequel est basé le Chapitre 6. En suivant les arguments de [46], nous allons prouver que si (u, E) est un minimiseur du problème à une phase dans B_2, pour $s > 1/2$, et si $0 \in \partial E$, alors $u \in C^{s-\frac{1}{2}}(B_{1/2})$. Notez en particulier que, par le Théorème 0.4.23, ceci garantit l'existence d'une limite de blow-up (u_0, E_0). De plus, nous établirons des estimations de densité uniforme pour l'ensemble de positivité E, des deux côtés.

0.4.7. La parade de manchots à Phillip Island (traitement mathématique). Le Chapitre 7 a pour but de fournir un modèle mathématique simple, mais rigoureux, décrivant la formation de groupes de manchots sur le rivage au coucher du soleil.

Les manchots sont incapables de voler, donc ils sont obligés de marcher lorsqu'ils sont à terre. En particulier, ils présentent des comportements assez spécifiques dans leur retour aux tanières, qu'il est intéressant d'observer et de décrire analytiquement. Nous avons observé que les manchots ont tendance à se dandiner sur le rivage pour former un groupe suffisamment grand, puis à marcher de manière compacte chez eux. Le cadre mathématique que nous introduisons décrit ce phénomène en prenant en compte des “paramètres naturels”, tels que la vue des manchots et leur vitesse de croisière. Le modèle que nous proposons favorise la formation de conglomérats de manchots qui se rassemblent, mais permet également des individus isolés et exposés.

Le modèle que nous proposons repose sur un ensemble d'équations différentielles ordinaires, avec un nombre de degrés de liberté variable dans le temps. En raison du comportement discontinu de la vitesse des manchots, le traitement mathématique (pour obtenir l'existence et l'unicité de la solution) est basé sur une procédure “stop-and-go”. Nous utilisons ce cadre pour fournir des exemples rigoureux dans lesquels au moins certains manchots parviennent à rentrer chez eux en toute sécurité (il existe aussi des cas dans lesquels certains manchots restent isolés).

Pour faciliter l'intuition du modèle, nous présentons également quelques simples simulations numériques, qui peuvent être comparées au mouvement réel de la parade des manchots.

Notation and assumptions

For the convenience of the reader, we collect some of the notation and assumptions used throughout the thesis.

- Unless otherwise stated, Ω and Ω' will always denote open sets.
- Given a set $A \subseteq \mathbb{R}^n$, we use the notation $\mathcal{C}A$ to denote the complement of A in $\mathbb{R}^n$, that is $\mathcal{C}A := \mathbb{R}^n \setminus A$.
- We write χ_E to denote the characteristic function of a set $E \subseteq \mathbb{R}^n$.
- We write $A \Subset B$ to mean that the closure of A is compact in $\mathbb{R}^n$ and $\overline{A} \subseteq B$.
- In $\mathbb{R}^n$ we will usually write $|E| = \mathcal{L}^n(E)$ for the n-dimensional Lebesgue measure of a set $E \subseteq \mathbb{R}^n$.
- We write $\mathcal{H}^d$ for the d-dimensional Hausdorff measure, for any $d \geq 0$.
- We define the dimensional constants

$$\omega_d := \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}+1\right)}, \qquad d \geq 0.$$

  In particular, we remark that $\omega_0 = 1$ and, if $k \in \mathbb{N}$, $k \geq 1$, then $\omega_k = \mathcal{L}^k(B_1)$ is the volume of the k-dimensional unit ball $B_1 \subseteq \mathbb{R}^k$ and $k\,\omega_k = \mathcal{H}^{k-1}(\mathbb{S}^{k-1})$ is the surface area of the $(k-1)$-dimensional sphere

$$\mathbb{S}^{k-1} := \partial B_1 = \{x \in \mathbb{R}^k \,|\, |x| = 1\}.$$

  Furthermore, in Chapter 3 we will make use of the notation

$$\varpi_n := \mathcal{H}^{n-1}(\mathbb{S}^{n-1}) = n\,\omega_n \quad \text{ and } \quad \varpi_0 := 0.$$

- By $A_h \xrightarrow{loc} A$ we mean that $\chi_{A_h} \to \chi_A$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, i.e. for every bounded open set $\Omega \subseteq \mathbb{R}^n$ we have $|(A_h \Delta A) \cap \Omega| \to 0$.
- Since

$$|E\Delta F| = 0 \quad \Longrightarrow \quad \mathrm{Per}(E,\Omega) = \mathrm{Per}(F,\Omega) \quad \text{and} \quad \mathrm{Per}_s(E,\Omega) = \mathrm{Per}_s(F,\Omega),$$

  unless otherwise stated, we implicitly identify sets up to sets of negligible Lebesgue measure. Moreover, whenever needed we will implicitly choose a particular representative for the class of χ_E in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, as in Remark MTA.
  We will not make this assumption in Section 1.3, since the Minkowski content can be affected even by changes in sets of measure zero, that is, in general

$$|\Gamma_1 \Delta \Gamma_2| = 0 \quad \not\Rightarrow \quad \overline{\mathcal{M}}^r(\Gamma_1,\Omega) = \overline{\mathcal{M}}^r(\Gamma_2,\Omega)$$

  (see Section 1.3 for a more detailed discussion).
- Given a set $F \subseteq \mathbb{R}^n$, the signed distance function $\bar{d}_F$ from ∂F, negative inside F, is defined as

$$\bar{d}_F(x) := d(x,F) - d(x,\mathcal{C}F) \qquad \text{for every } x \in \mathbb{R}^n,$$

  where

$$d(x,A) = \mathrm{dist}(x,A) := \inf_{y \in A} |x-y|,$$

denotes the usual distance from a set $A \subseteq \mathbb{R}^n$. For every $r \in \mathbb{R}$ we define the set

$$F_r := \left\{ x \in \mathbb{R}^n \,|\, \bar{d}_F(x) < r \right\}.$$

We also consider the open tubular ϱ-neighborhood of ∂F,

$$N_\varrho(\partial F) := \{ x \in \mathbb{R}^n \,|\, d(x, \partial F) < \varrho \} = \left\{ |\bar{d}_F| < \varrho \right\},$$

for every $\varrho > 0$. Given a bounded open set $\Omega \subseteq \mathbb{R}^n$, the constant

$$r_0 = r_0(\Omega) > 0$$

will have two different meanings, depending on the regularity of $\partial\Omega$:

- if Ω has Lipschitz boundary, then r_0 has the same meaning as in Proposition B.1.1. Namely, for every $r \in (-r_0, r_0)$ the bounded open set Ω_r has Lipschitz boundary and the perimeters are uniformly bounded;
- if Ω has C^2 boundary, then r_0 has the same meaning as in Remark B.1.3. Namely, the set Ω satisfies a strict interior and a strict exterior ball condition of radius $2r_0$ at every point of the boundary.

For a more detailed discussion, see Appendix B.1

Remark MTA (Measure theoretic assumption). Let $E \subseteq \mathbb{R}^n$ be a measurable set. Up to modifications in sets of Lebesgue measure zero, we can assume (see Appendix A for a detailed discussion) that E contains its measure theoretic interior, it does not intersect its measure theoretic exterior and is such that the topological boundary coincides with the measure theoretic boundary. More precisely, we define

$$E_{int} := \{ x \in \mathbb{R}^n \,|\, \exists\, r > 0 \text{ s.t. } |E \cap B_r(x)| = \omega_n r^n \},$$
$$E_{ext} := \{ x \in \mathbb{R}^n \,|\, \exists\, r > 0 \text{ s.t. } |E \cap B_r(x)| = 0 \},$$

and the measure theoretic boundary

$$\begin{aligned} \partial^- E &:= \mathbb{R}^n \setminus \left(E_{int} \cup E_{ext} \right) \\ &= \{ x \in \mathbb{R}^n \,|\, 0 < |E \cap B_r(x)| < \omega_n r^n \text{ for every } r > 0 \}. \end{aligned}$$

Then we assume that

$$E_{int} \subseteq E, \qquad E \cap E_{ext} = \varnothing \qquad \text{and} \qquad \partial E = \partial^- E.$$

As detailed in Appendix A, one way to do this consists in identifying the set E with the set $E^{(1)}$ of points of density one.

CHAPTER 1

Fractional perimeters from a fractal perspective

1.1. Introduction and main results

The purpose of this chapter consists in better understanding the fractional nature of the nonlocal perimeters introduced in [21]. Following [99], we exploit these fractional perimeters to introduce a definition of fractal dimension for the measure theoretic boundary of a set.

We calculate the fractal dimension of sets which can be defined in a recursive way and we give some examples of this kind of sets, explaining how to construct them starting from well known self-similar fractals. In particular, we show that in the case of the von Koch snowflake $S \subseteq \mathbb{R}^2$ this fractal dimension coincides with the Minkowski dimension.

We also obtain an optimal result for the asymptotics as $s \to 1^-$ of the fractional perimeter of a set having locally finite (classical) perimeter.

Now we give precise statements of the results obtained, starting with the fractional analysis of fractal dimensions.

1.1.1. Fractal boundaries.

We recall that we implicitly assume that all the sets we consider contain their measure theoretic interior, do not intersect their measure theoretic exterior, and are such that their topological boundary coincides with their measure theoretic boundary—see Remark MTA and Appendix A for the details. We will not make this assumption in Section 1.3, since the Minkowski content can be affected even by changes in sets of measure zero.

We recall that we split the fractional perimeter as the sum

$$\mathrm{Per}_s(E,\Omega) = \mathrm{Per}_s^L(E,\Omega) + \mathrm{Per}_s^{NL}(E,\Omega),$$

where

$$\begin{aligned}\mathrm{Per}_s^L(E,\Omega)&:=\mathcal{L}_s(E\cap\Omega,\mathcal{C}E\cap\Omega)=\frac{1}{2}[\chi_E]_{W^{s,1}(\Omega)},\\ \mathrm{Per}_s^{NL}(E,\Omega)&:=\mathcal{L}_s(E\cap\Omega,\mathcal{C}E\setminus\Omega)+\mathcal{L}_s(E\setminus\Omega,\mathcal{C}E\cap\Omega).\end{aligned}$$

We can think of $\mathrm{Per}_s^L(E,\Omega)$ as the local part of the fractional perimeter, in the sense that if $|(E\Delta F)\cap\Omega|=0$, then $\mathrm{Per}_s^L(F,\Omega)=\mathrm{Per}_s^L(E,\Omega)$.

We usually refer to $\mathrm{Per}_s^{NL}(E,\Omega)$ as the nonlocal part of the s-perimeter.

We say that a set E has locally finite s-perimeter if it has finite s-perimeter in every bounded open set $\Omega\subseteq\mathbb{R}^n$.

When $\Omega=\mathbb{R}^n$, we simply write

$$\mathrm{Per}_s(E):=\mathrm{Per}_s(E,\mathbb{R}^n)=\frac{1}{2}[\chi_E]_{W^{s,1}(\mathbb{R}^n)}.$$

First of all, we prove in Section 1.3.1 that in some sense the measure theoretic boundary ∂^-E is the "right definition" of boundary for working with the s-perimeter.

To be more precise, we show that

$$\partial^-E=\{x\in\mathbb{R}^n\,|\,\mathrm{Per}_s^L(E,B_r(x))>0,\,\forall\,r>0\},$$

and that if Ω is a connected open set, then

$$\mathrm{Per}_s^L(E,\Omega)>0\quad\Longleftrightarrow\quad\partial^-E\cap\Omega\neq\varnothing.$$

This can be thought of as an analogue in the fractional framework of the fact that for a Caccioppoli set E we have $\partial^-E=\mathrm{supp}\,|D\chi_E|$.

Now the idea of the definition of the fractal dimension consists in using the index s of $\mathrm{Per}_s^L(E,\Omega)$ to measure the codimension of $\partial^-E\cap\Omega$,

$$\mathrm{Dim}_F(\partial^-E,\Omega):=n-\sup\{s\in(0,1)\,|\,\mathrm{Per}_s^L(E,\Omega)<\infty\}.$$

As shown in [99] (Proposition 11 and Proposition 13), the fractal dimension Dim_F defined in this way is related to the (upper) Minkowski dimension (whose precise definition we recall in Definition 1.3.4) by

$$\mathrm{Dim}_F(\partial^-E,\Omega)\leq\overline{\mathrm{Dim}}_{\mathcal{M}}(\partial^-E,\Omega).\tag{1.1}$$

For the convenience of the reader we provide a proof of inequality (1.1) in Proposition 1.3.6.

If Ω is a bounded open set with Lipschitz boundary, (1.1) means that

$$\mathrm{Per}_s(E,\Omega)<\infty\qquad\text{for every } s\in\big(0,n-\overline{\mathrm{Dim}}_{\mathcal{M}}(\partial^-E,\Omega)\big),$$

since the nonlocal part of the s-perimeter of any set $E\subseteq\mathbb{R}^n$ is

$$\mathrm{Per}_s^{NL}(E,\Omega)\leq 2\,\mathrm{Per}_s(\Omega)<\infty,\qquad\text{for every } s\in(0,1).$$

We show that for the von Koch snowflake (1.1) is actually an equality.

Namely, we prove the following:

THEOREM 1.1.1 (Fractal dimension of the von Koch snowflake). *Let $S\subseteq\mathbb{R}^2$ be the von Koch snowflake. Then*

$$\mathrm{Per}_s(S)<\infty,\qquad\forall\,s\in\Big(0,2-\frac{\log 4}{\log 3}\Big),\tag{1.2}$$

and

$$\mathrm{Per}_s(S)=\infty,\qquad\forall\,s\in\Big[2-\frac{\log 4}{\log 3},1\Big).\tag{1.3}$$

Therefore

$$\mathrm{Dim}_F(\partial S) = \mathrm{Dim}_{\mathcal{M}}(\partial S) = \frac{\log 4}{\log 3}.$$

Actually, exploiting the self-similarity of the von Koch curve, we have

$$\mathrm{Dim}_F(\partial S, \Omega) = \frac{\log 4}{\log 3},$$

for every Ω such that $\partial S \cap \Omega \neq \varnothing$. In particular, this is true for every $\Omega = B_r(p)$ with $p \in \partial S$ and $r > 0$ as small as we want.

We remark that this represents a deep difference between the classical and the fractional perimeter.
Indeed, if a set E has (locally) finite perimeter, then by De Giorgi's structure Theorem we know that its reduced boundary $\partial^* E$ is locally $(n-1)$-rectifiable. Moreover $\overline{\partial^* E} = \partial^- E$, so the reduced boundary is, in some sense, a "big" portion of the measure theoretic boundary.

On the other hand, we have seen that there are (open) sets, like the von Koch snowflake, which have a "nowhere rectifiable" boundary (meaning that $\partial^- E \cap B_r(p)$ is not $(n-1)$-rectifiable for every $p \in \partial^- E$ and $r > 0$) and still have finite s-perimeter for every $s \in (0, \sigma_0)$.

1.1.1.1. *Self-similar fractal boundaries.* Our argument for the von Koch snowflake is quite general and can be adapted to compute the dimension Dim_F of all sets which can be constructed in a similar recursive way.

To be more precise, we start with a bounded open set $T_0 \subseteq \mathbb{R}^n$ with finite perimeter $\mathrm{Per}(T_0) < \infty$, which is, roughly speaking, our basic "building block".

Then we go on inductively by adding roto-translations of a scaling of the building block T_0, i.e. sets of the form

$$T_k^i = F_k^i(T_0) := \mathcal{R}_k^i\big(\lambda^{-k} T_0\big) + x_k^i,$$

where $\lambda > 1$, $k \in \mathbb{N}$, $1 \leq i \leq ab^{k-1}$, with $a, b \in \mathbb{N}$, $\mathcal{R}_k^i \in SO(n)$ and $x_k^i \in \mathbb{R}^n$. We ask that these sets do not overlap, i.e.

$$|T_k^i \cap T_h^j| = 0, \qquad \text{whenever } i \neq j \text{ or } k \neq h.$$

Then we define

$$T_k := \bigcup_{i=1}^{ab^{k-1}} T_k^i \qquad \text{and} \qquad T := \bigcup_{k=1}^{\infty} T_k. \tag{1.4}$$

The final set E is either

$$E := T_0 \cup \bigcup_{k \geq 1} \bigcup_{i=1}^{ab^{k-1}} T_k^i, \quad \text{or} \quad E := T_0 \setminus \Big(\bigcup_{k \geq 1} \bigcup_{i=1}^{ab^{k-1}} T_k^i \Big).$$

For example, the von Koch snowflake is obtained by adding pieces.
Examples obtained by removing the T_k^i's are the middle Cantor set $E \subseteq \mathbb{R}$, the Sierpinski triangle $E \subseteq \mathbb{R}^2$ and the Menger sponge $E \subseteq \mathbb{R}^3$.

We will consider just the set T and exploit the same argument used for the von Koch snowflake to compute the fractal dimension related to the s-perimeter.
However, we observe that the Cantor set, the Sierpinski triangle and the Menger sponge are such that $|E| = 0$, i.e. $|T_0 \Delta T| = 0$.

Therefore neither the perimeter nor the s-perimeter can detect the fractal nature of the (topological) boundary of T and indeed, since

$$\mathrm{Per}(T)=\mathrm{Per}(T_0)<\infty,$$

we have $\mathrm{Per}_s(T)<\infty$ for every $s\in(0,1)$.

For example, in the case of the Sierpinski triangle, T_0 is an equilateral triangle and $\partial^- T=\partial T_0$, even if ∂T is a self-similar fractal.

The reason of this situation is that the fractal object is the topological boundary of T, while the s-perimeter "measures" the measure theoretic boundary, which is regular. Roughly speaking, the problem is that in these cases there is not room enough to find a small ball $B_k^i=F_k^i(B)\subseteq\mathcal{C}T$ near each piece T_k^i.

Therefore, we will make the additional assumption that

$$\exists\, S_0\subseteq\mathcal{C}T\quad\text{s.t. }|S_0|>0\quad\text{and } S_k^i:=F_k^i(S_0)\subseteq\mathcal{C}T\quad\forall\, k,\,i. \tag{1.5}$$

We remark that it is not necessary to ask that these sets do not overlap.

THEOREM 1.1.2. *Let $T\subseteq\mathbb{R}^n$ be a set which can be written as in (1.4). If $\frac{\log b}{\log\lambda}\in(n-1,n)$ and (1.5) holds true, then*

$$\mathrm{Per}_s(T)<\infty,\qquad\forall\, s\in\Big(0,n-\frac{\log b}{\log\lambda}\Big)$$

and

$$\mathrm{Per}_s(T)=\infty,\qquad\forall\, s\in\Big[n-\frac{\log b}{\log\lambda},1\Big).$$

Thus

$$\mathrm{Dim}_F(\partial^- T)=\frac{\log b}{\log\lambda}.$$

Furthermore, we show how to modify self-similar sets like the Sierpinski triangle, without altering their "structure", to obtain new sets which satisfy the hypothesis of Theorem 1.1.2 (see Remark 1.3.10 and the final part of Section 1.3.4). An example is given in Figure 1 above.

However, we also remark that the measure theoretic boundary of such a new set will look quite different from the original fractal (topological) boundary and in general it will be a mix of smooth parts and unrectifiable parts.

The most interesting examples of this kind of sets are probably represented by bounded sets, because in this case the measure theoretic boundary does indeed have, in some sense, a "fractal nature" (see Remark 1.3.11).
Indeed, if T is bounded, then its boundary $\partial^- T$ is compact. Nevertheless, it has infinite (classical) perimeter and actually $\partial^- T$ has Minkowski dimension strictly greater than $n-1$, thanks to (1.1).

However, even unbounded sets can have an interesting behavior. Indeed we obtain the following

PROPOSITION 1.1.3. *Let $n\geq 2$. For every $\sigma\in(0,1)$ there exists a Caccioppoli set $E\subseteq\mathbb{R}^n$ such that*

$$\mathrm{Per}_s(E)<\infty\qquad\forall\, s\in(0,\sigma)\quad\textit{and}\quad\mathrm{Per}_s(E)=\infty\qquad\forall\, s\in[\sigma,1).$$

Roughly speaking, the interesting thing about this Proposition is the following. Since E has locally finite perimeter, $\chi_E\in BV_{\mathrm{loc}}(\mathbb{R}^n)$, it also has locally finite s-perimeter for every $s\in(0,1)$, but the global perimeter $\mathrm{Per}_s(E)$ is finite if and only if $s<\sigma<1$.

1.1.2. Asymptotics as $s \to 1^-$. In Section 1.1.1 we have shown that sets with an irregular, eventually fractal, boundary can have finite s-perimeter.

On the other hand, if the set E is "regular", then it has finite s-perimeter for every $s \in (0,1)$. Indeed, if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with Lipschitz boundary (or $\Omega = \mathbb{R}^n$), then $BV(\Omega) \hookrightarrow W^{s,1}(\Omega)$. As a consequence of this embedding, we find that

$$\mathrm{Per}(E,\Omega) < \infty \qquad \Longrightarrow \qquad \mathrm{Per}_s(E,\Omega) < \infty \quad \text{for every } s \in (0,1).$$

Actually we can be more precise and obtain a sort of converse, using only the local part of the s-perimeter and adding the condition

$$\liminf_{s \to 1^-} (1-s)\,\mathrm{Per}_s^L(E,\Omega) < \infty.$$

Indeed one has the following result, which is a combination of [14, Theorem 3'] and [35, Theorem 1], restricted to characteristic functions:

THEOREM 1.1.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then $E \subseteq \mathbb{R}^n$ has finite perimeter in Ω if and only if $\mathrm{Per}_s^L(E,\Omega) < \infty$ for every $s \in (0,1)$, and*

$$\liminf_{s \to 1} (1-s)\,\mathrm{Per}_s^L(E,\Omega) < \infty. \tag{1.6}$$

In this case we have

$$\lim_{s \to 1} (1-s)\,\mathrm{Per}_s^L(E,\Omega) = \frac{n\omega_n}{2} K_{1,n}\,\mathrm{Per}(E,\Omega). \tag{1.7}$$

We briefly show how to get this result (and in particular why the constant looks like that) from the two Theorems cited above. Then we compute the constant $K_{1,n}$ in an elementary way, proving that

$$\frac{n\omega_n}{2} K_{1,n} = \omega_{n-1}.$$

Moreover we show the following:

REMARK 1.1.5. Condition (1.6) is necessary. Indeed, there exist bounded sets (see Example 1.1.1) having finite s-perimeter for every $s \in (0,1)$ which do not have finite perimeter. This also shows that in general the inclusion

$$BV(\Omega) \subseteq \bigcap_{s \in (0,1)} W^{s,1}(\Omega)$$

is strict.

EXAMPLE 1.1.1. Let $0 < a < 1$ and consider the open intervals $I_k := (a^{k+1}, a^k)$ for every $k \in \mathbb{N}$. Define $E := \bigcup_{k \in \mathbb{N}} I_{2k}$, which is a bounded (open) set. Due to the infinite number of jumps $\chi_E \notin BV(\mathbb{R})$. However it can be proved that E has finite s-perimeter for every $s \in (0,1)$. We postpone the proof to Section 1.4.

REMARK 1.1.6. For completeness, we also mention a related result contained in [40], where the authors provide an example (Example 2.10) of a bounded set $E \subseteq \mathbb{R}$ which does not have finite s-perimeter for any $s \in (0,1)$. In particular, this example proves that in general the inclusion

$$\bigcup_{s \in (0,1)} W^{s,1}(\Omega) \subseteq L^1(\Omega)$$

is strict.

The main result of Section 1.2 is the following Theorem, which extends the asymptotic convergence of (1.7) to the whole s-perimeter.

THEOREM 1.1.7 (Asymptotics). *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $E \subseteq \mathbb{R}^n$. Then, E has locally finite perimeter in Ω if and only if E has locally finite s-perimeter in Ω for every $s \in (0,1)$ and*

$$\liminf_{s\to 1}(1-s)\operatorname{Per}_s^L(E,\Omega') < \infty, \qquad \forall\,\Omega' \Subset \Omega.$$

If E has locally finite perimeter in Ω, then

$$\lim_{s\to 1}(1-s)\operatorname{Per}_s(E,\mathcal{O}) = \omega_{n-1}\operatorname{Per}(E,\overline{\mathcal{O}}),$$

for every open set $\mathcal{O} \Subset \Omega$ with Lipschitz boundary. More precisely,

$$\lim_{s\to 1}(1-s)\operatorname{Per}_s^L(E,\mathcal{O}) = \omega_{n-1}\operatorname{Per}(E,\mathcal{O})$$

and

$$\lim_{s\to 1}(1-s)\operatorname{Per}_s^{NL}(E,\mathcal{O}) = \omega_{n-1}\operatorname{Per}(E,\partial\mathcal{O}) = \omega_{n-1}\mathcal{H}^{n-1}(\partial^* E\cap\partial\mathcal{O}). \tag{1.8}$$

The proof of Theorem 1.1.7 relies only on [14, Theorem 3'], [35, Theorem 1] and on an appropriate estimate of what happens in a neighborhood of $\partial\mathcal{O}$. The main improvement of the known asymptotics results is the convergence (1.8).

1.2. Asymptotics as $s \to 1^-$

We say that an open set $\Omega \subseteq \mathbb{R}^n$ is an extension domain if there exists a constant $C = C(n,s,\Omega) > 0$ such that for every $u \in W^{s,1}(\Omega)$ there exists $\tilde{u} \in W^{s,1}(\mathbb{R}^n)$ with $\tilde{u}_{|\Omega} = u$ and

$$\|\tilde{u}\|_{W^{s,1}(\mathbb{R}^n)} \le C\|u\|_{W^{s,1}(\Omega)}.$$

Every open set with bounded Lipschitz boundary is an extension domain (see [38] for a proof). By definition we consider $\mathbb{R}^n$ itself as an extension domain.

We begin with the following embedding.

PROPOSITION 1.2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an extension domain. Then there exists a constant $C = C(n,s,\Omega) \ge 1$ such that for every $u : \Omega \longrightarrow \mathbb{R}$*

$$\|u\|_{W^{s,1}(\Omega)} \le C\|u\|_{BV(\Omega)}. \tag{1.9}$$

In particular we have the continuous embedding

$$BV(\Omega) \hookrightarrow W^{s,1}(\Omega).$$

PROOF. The claim is trivially satisfied if the right hand side of (1.9) is infinite, so let $u \in BV(\Omega)$. Let $\{u_k\} \subseteq C^\infty(\Omega)\cap BV(\Omega)$ be an approximating sequence as in [68, Theorem 1.17], that is

$$\|u-u_k\|_{L^1(\Omega)} \longrightarrow 0 \qquad \text{and} \qquad \lim_{k\to\infty}\int_\Omega |\nabla u_k|\,dx = |Du|(\Omega).$$

We only need to check that the $W^{s,1}$-seminorm of u is bounded by its BV-norm. Since Ω is an extension domain, we know (see [38, Proposition 2.2]) that $\exists\, C(n,s) \ge 1$ such that

$$\|v\|_{W^{s,1}(\Omega)} \le C\|v\|_{W^{1,1}(\Omega)}.$$

Then

$$[u_k]_{W^{s,1}(\Omega)} \le \|u_k\|_{W^{s,1}(\Omega)} \le C\|u_k\|_{W^{1,1}(\Omega)} = C\|u_k\|_{BV(\Omega)},$$

and hence, using Fatou's Lemma,

$$\begin{aligned}[u]_{W^{s,1}(\Omega)} &\le \liminf_{k\to\infty}[u_k]_{W^{s,1}(\Omega)} \le C\liminf_{k\to\infty}\|u_k\|_{BV(\Omega)} = C\lim_{k\to\infty}\|u_k\|_{BV(\Omega)}\\ &= C\|u\|_{BV(\Omega)},\end{aligned}$$

proving (1.9). □

Given a set $E \subseteq \mathbb{R}^n$ and $r \in \mathbb{R}$, we denote

$$E_r := \{x \in \mathbb{R}^n \,|\, \bar{d}_E(x) < r\},$$

where $\bar{d}_E$ is the signed distance function from E (see Appendix B.1).

COROLLARY 1.2.2. *(i) If $E \subseteq \mathbb{R}^n$ has finite perimeter, i.e. $\chi_E \in BV(\mathbb{R}^n)$, then E has also finite s-perimeter for every $s \in (0,1)$.*

(ii) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then there exists $r_0 > 0$ such that

$$\sup_{|r|<r_0} \operatorname{Per}_s(\Omega_r) < \infty. \tag{1.10}$$

(iii) If $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with Lipschitz boundary, then

$$\operatorname{Per}_s^{NL}(E, \Omega) \leq 2 \operatorname{Per}_s(\Omega) < \infty$$

for every $E \subseteq \mathbb{R}^n$.

(iv) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then

$$\operatorname{Per}(E, \Omega) < \infty \qquad \Longrightarrow \qquad \operatorname{Per}_s(E, \Omega) < \infty \quad \text{for every } s \in (0,1).$$

PROOF. Claim (i) follows from

$$\operatorname{Per}_s(E) = \frac{1}{2}[\chi_E]_{W^{s,1}(\mathbb{R}^n)}$$

and Proposition 1.2.1 with $\Omega = \mathbb{R}^n$.

(ii) Let r_0 be as in Proposition B.1.1 and notice that

$$\operatorname{Per}(\Omega_r) = \mathcal{H}^{n-1}\big(\{\bar{d}_\Omega = r\}\big),$$

so that

$$\|\chi_{\Omega_r}\|_{BV(\mathbb{R}^n)} = |\Omega_r| + \mathcal{H}^{n-1}\big(\{\bar{d}_\Omega = r\}\big).$$

Thus

$$\sup_{|r|<r_0} \operatorname{Per}_s(\Omega_r) \leq C\Big(|\Omega_{r_0}| + \sup_{|r|<r_0} \mathcal{H}^{n-1}\big(\{\bar{d}_\Omega = r\}\big)\Big) < \infty.$$

(iii) Notice that

$$\mathcal{L}_s(E \cap \Omega, \mathcal{C}E \setminus \Omega) \leq \mathcal{L}_s(\Omega, \mathcal{C}\Omega) = \operatorname{Per}_s(\Omega),$$
$$\mathcal{L}_s(\mathcal{C}E \cap \Omega, E \setminus \Omega) \leq \mathcal{L}_s(\Omega, \mathcal{C}\Omega) = \operatorname{Per}_s(\Omega),$$

and use (1.10) (with $\Omega_0 = \Omega$).

(iv) The nonlocal part of the s-perimeter is finite thanks to (iii). As for the local part, recall that

$$\operatorname{Per}(E, \Omega) = |D\chi_E|(\Omega) \qquad \text{and} \qquad \operatorname{Per}_s^L(E, \Omega) = \frac{1}{2}[\chi_E]_{W^{s,1}(\Omega)},$$

then use Proposition 1.2.1. □

1.2.1. Asymptotics of the local part of the s-perimeter.

We recall the results of [14] and [35], which straightforwardly give Theorem 1.1.4.

THEOREM 1.2.3 (Theorem 3' of [14]). *Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded domain. Let $u \in L^1(\Omega)$. Then $u \in BV(\Omega)$ if and only if*

$$\liminf_{n\to\infty} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x-y|} \varrho_n(x-y)\, dx dy < \infty,$$

and then

$$C_1|Du|(\Omega) \le \liminf_{n\to\infty} \int_\Omega\int_\Omega \frac{|u(x)-u(y)|}{|x-y|}\varrho_n(x-y)\,dxdy$$
$$\le \limsup_{n\to\infty} \int_\Omega\int_\Omega \frac{|u(x)-u(y)|}{|x-y|}\varrho_n(x-y)\,dxdy \le C_2|Du|(\Omega),$$

for some constants C_1, C_2 depending only on Ω.

This result was refined by Dávila:

THEOREM 1.2.4 (Theorem 1 of [35]). *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $u \in BV(\Omega)$. Then*

$$\lim_{k\to\infty} \int_\Omega\int_\Omega \frac{|u(x)-u(y)|}{|x-y|}\varrho_k(x-y)\,dxdy = K_{1,n}|Du|(\Omega),$$

where

$$K_{1,n} = \frac{1}{n\omega_n}\int_{\mathbb{S}^{n-1}} |v\cdot e|\,d\sigma(v),$$

with $e \in \mathbb{R}^n$ any unit vector.

In the above Theorems ϱ_k is any sequence of radial mollifiers i.e. of functions satisfying

$$\varrho_k(x) \ge 0, \quad \varrho_k(x) = \varrho_k(|x|), \quad \int_{\mathbb{R}^n} \varrho_k(x)\,dx = 1 \tag{1.11}$$

and

$$\lim_{k\to\infty} \int_\delta^\infty \varrho_k(r)r^{n-1}dr = 0 \quad \text{for all } \delta > 0. \tag{1.12}$$

In particular, for R big enough, $R > \mathrm{diam}(\Omega)$, we can consider

$$\varrho(x) := \chi_{[0,R]}(|x|)\frac{1}{|x|^{n-1}}$$

and define for any sequence $\{s_k\} \subseteq (0,1)$, $s_k \nearrow 1$,

$$\varrho_k(x) := (1-s_k)\varrho(x)c_{s_k}\frac{1}{|x|^{s_k}},$$

where the c_{s_k} are normalizing constants. Then

$$\int_{\mathbb{R}^n} \varrho_k(x)\,dx = (1-s_k)c_{s_k}n\omega_n\int_0^R \frac{1}{r^{n-1+s_k}}r^{n-1}\,dr$$
$$= (1-s_k)c_{s_k}n\omega_n\int_0^R \frac{1}{r^{s_k}}\,dr = c_{s_k}n\omega_n R^{1-s_k},$$

and hence taking $c_{s_k} := \frac{1}{n\omega_n}R^{s_k-1}$ gives (1.11); notice that $c_{s_k} \to \frac{1}{n\omega_n}$.
Also

$$\lim_{k\to\infty}\int_\delta^\infty \varrho_k(r)r^{n-1}\,dr = \lim_{k\to\infty}(1-s_k)c_{s_k}\int_\delta^R \frac{1}{r^{s_k}}\,dr$$
$$= \lim_{k\to\infty} c_{s_k}(R^{1-s_k} - \delta^{1-s_k}) = 0,$$

giving (1.12). With this choice we obtain

$$\int_\Omega\int_\Omega \frac{|u(x)-u(y)|}{|x-y|}\varrho_k(x-y)\,dxdy = c_{s_k}(1-s_k)[u]_{W^{s_k,1}(\Omega)}.$$

Then, if $u \in BV(\Omega)$, Dávila's Theorem gives

$$\begin{aligned} \lim_{s\to 1}(1-s)[u]_{W^{s,1}(\Omega)} &= \lim_{s\to 1}\frac{1}{c_s}(c_s(1-s)[u]_{W^{s,1}(\Omega)}) \\ &= n\omega_n K_{1,n}|Du|(\Omega). \end{aligned} \tag{1.13}$$

1.2.2. Proof of Theorem 1.1.7. We split the proof of Theorem 1.1.7 into several steps, which we believe are interesting on their own.

1.2.2.1. *The constant* ω_{n-1}. We need to compute the constant $K_{1,n}$. Notice that we can choose e in such a way that $v \cdot e = v_n$.
Then using spheric coordinates for $\mathbb{S}^{n-1}$ we obtain $|v\cdot e| = |\cos\theta_{n-1}|$ and

$$d\sigma = \sin\theta_2(\sin\theta_3)^2\dots(\sin\theta_{n-1})^{n-2}d\theta_1\dots d\theta_{n-1},$$

with $\theta_1 \in [0, 2\pi)$ and $\theta_j \in [0,\pi)$ for $j = 2,\dots,n-1$. Notice that

$$\begin{aligned} \mathcal{H}^k(\mathbb{S}^k) &= \int_0^{2\pi} d\theta_1 \int_0^\pi \sin\theta_2\,d\theta_2\dots\int_0^\pi(\sin\theta_{k-1})^{k-2}\,d\theta_{k-1} \\ &= \mathcal{H}^{k-1}(\mathbb{S}^{k-1})\int_0^\pi(\sin t)^{k-2}\,dt. \end{aligned}$$

Then we get

$$\begin{aligned} \int_{\mathbb{S}^{n-1}}|v\cdot e|\,d\sigma(v) &= \mathcal{H}^{n-2}(\mathbb{S}^{n-2})\int_0^\pi(\sin t)^{n-2}|\cos t|\,dt \\ &= \mathcal{H}^{n-2}(\mathbb{S}^{n-2})\Big(\int_0^{\frac{\pi}{2}}(\sin t)^{n-2}\cos t\,dt - \int_{\frac{\pi}{2}}^\pi(\sin t)^{n-2}\cos t\,dt\Big) \\ &= \frac{\mathcal{H}^{n-2}(\mathbb{S}^{n-2})}{n-1}\Big(\int_0^{\frac{\pi}{2}}\frac{d}{dt}(\sin t)^{n-1}\,dt - \int_{\frac{\pi}{2}}^\pi\frac{d}{dt}(\sin t)^{n-1}\,dt\Big) \\ &= \frac{2\mathcal{H}^{n-2}(\mathbb{S}^{n-2})}{n-1}. \end{aligned}$$

Therefore

$$n\omega_n K_{1,n} = 2\frac{\mathcal{H}^{n-2}(\mathbb{S}^{n-2})}{n-1} = 2\mathcal{L}^{n-1}(B_1(0)) = 2\omega_{n-1},$$

and hence (1.13) becomes

$$\lim_{s\to 1}(1-s)[u]_{W^{s,1}(\Omega)} = 2\omega_{n-1}|Du|(\Omega),$$

for any $u \in BV(\Omega)$.

1.2.2.2. *Estimating the nonlocal part of the s-perimeter.* The aim of this subsection consists in proving that if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with Lipschitz boundary and $E \subseteq \mathbb{R}^n$ has finite perimeter in Ω_β, for some $\beta \in (0, r_0)$ and r_0 as in Proposition B.1.1, then

$$\limsup_{s\to 1}(1-s)\operatorname{Per}_s^{NL}(E,\Omega) \le 2\omega_{n-1}\lim_{\varrho\to 0^+}\operatorname{Per}(E, N_\varrho(\partial\Omega)). \tag{1.14}$$

Actually, we prove something slightly more general than (1.14). Namely, that to estimate the nonlocal part of the s-perimeter we do not necessarily need to use the sets Ω_ϱ: any "regular" approximation of Ω will do.

More precisely, let A_k, $D_k \subseteq \mathbb{R}^n$ be two sequences of bounded open sets with Lipschitz boundary strictly approximating Ω respectively from the inside and from the outside, that is

(i) $A_k \subseteq A_{k+1} \Subset \Omega$ and $A_k \nearrow \Omega$, i.e. $\bigcup_k A_k = \Omega$,
(ii) $\Omega \Subset D_{k+1} \subseteq D_k$ and $D_k \searrow \overline{\Omega}$, i.e. $\bigcap_k D_k = \overline{\Omega}$.

We define for every k

$$\Omega_k^+ := D_k \setminus \overline{\Omega}, \qquad \Omega_k^- := \Omega \setminus \overline{A_k} \qquad T_k := \Omega_k^+ \cup \partial\Omega \cup \Omega_k^-,$$
$$d_k := \min\{d(A_k, \partial\Omega),\, d(D_k, \partial\Omega)\} > 0.$$

In particular, we observe that we can consider Ω_ϱ with $\varrho < 0$ in place of A_k and with $\varrho > 0$ in place of D_k. Then T_k would be $N_\varrho(\partial\Omega)$ and $d_k = \varrho$.

Proposition 1.2.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $E \subseteq \mathbb{R}^n$ be a set having finite perimeter in D_1. Then*

$$\limsup_{s\to 1}(1-s)\operatorname{Per}_s^{NL}(E,\Omega) \le 2\omega_{n-1}\lim_{k\to\infty}\operatorname{Per}(E,T_k).$$

In particular, if $\operatorname{Per}(E,\partial\Omega)=0$*, then*

$$\lim_{s\to 1}(1-s)\operatorname{Per}_s(E,\Omega) = \omega_{n-1}\operatorname{Per}(E,\Omega).$$

Proof. Since Ω is regular and $\operatorname{Per}(E,\Omega)<\infty$, we already know that

$$\lim_{s\to 1}(1-s)\operatorname{Per}_s^L(E,\Omega) = \omega_{n-1}\operatorname{Per}(E,\Omega).$$

Notice that, since $|D\chi_E|$ is a finite Radon measure on D_1 and $T_k \searrow \partial\Omega$ as $k \nearrow \infty$, we have that

$$\exists \lim_{k\to\infty}\operatorname{Per}(E,T_k) = \operatorname{Per}(E,\partial\Omega).$$

Consider the nonlocal part of the fractional perimeter,

$$\operatorname{Per}_s^{NL}(E,\Omega) = \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\setminus\Omega) + \mathcal{L}_s(\mathcal{C}E\cap\Omega, E\setminus\Omega),$$

and take any k. Then

$$\begin{aligned}
\mathcal{L}_s(E\cap\Omega, \mathcal{C}E\setminus\Omega) &= \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\cap\Omega_k^+) + \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\cap(\mathcal{C}\Omega\setminus D_k))\\
&\le \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\cap\Omega_k^+) + \frac{n\omega_n}{s}|\Omega|\frac{1}{d_k^s}\\
&\le \mathcal{L}_s(E\cap\Omega_k^-, \mathcal{C}E\cap\Omega_k^+) + 2\frac{n\omega_n}{s}|\Omega|\frac{1}{d_k^s}\\
&\le \mathcal{L}_s(E\cap(\Omega_k^-\cup\Omega_k^+), \mathcal{C}E\cap(\Omega_k^-\cup\Omega_k^+)) + 2\frac{n\omega_n}{s}|\Omega|\frac{1}{d_k^s}\\
&= \operatorname{Per}_s^L(E,T_k) + 2\frac{n\omega_n}{s}|\Omega|\frac{1}{d_k^s}.
\end{aligned}$$

Since we can bound the other term in the same way, we get

$$\operatorname{Per}_s^{NL}(E,\Omega) \le 2\operatorname{Per}_s^L(E,T_k) + 4\frac{n\omega_n}{s}|\Omega|\frac{1}{d_k^s}.$$

By hypothesis we know that T_k is a bounded open set with Lipschitz boundary

$$\partial T_k = \partial A_k \cup \partial D_k.$$

Therefore using (1.7) we have

$$\lim_{s\to 1}(1-s)\operatorname{Per}_s^L(E,T_k) = \omega_{n-1}\operatorname{Per}(E,T_k),$$

and hence

$$\limsup_{s\to 1}(1-s)\operatorname{Per}_s^{NL}(E,\Omega) \le 2\omega_{n-1}\operatorname{Per}(E,T_k).$$

Since this holds true for any k, we get the claim. □

1.2.2.3. *Convergence in almost every* Ω_ϱ. Having a "continuous" approximating sequence (the Ω_ϱ) rather than numerable ones allows us to improve Proposition 1.2.5.

We first recall that if E has finite perimeter, then De Giorgi's structure Theorem (see, e.g., [79, Theorem 15.9]) guarantees in particular that

$$|D\chi_E| = \mathcal{H}^{n-1} \llcorner \partial^* E$$

and hence

$$\mathrm{Per}(E, B) = \mathcal{H}^{n-1}(\partial^* E \cap B) \qquad \text{for every Borel set } B \subseteq \mathbb{R}^n,$$

where $\partial^* E$ is the reduced boundary of E.

Corollary 1.2.6. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let r_0 be as in Proposition B.1.1. Let $E \subseteq \mathbb{R}^n$ be a set having finite perimeter in Ω_β, for some $\beta \in (0, r_0)$, and define*

$$S := \{\delta \in (-r_0, \beta) \,|\, \mathrm{Per}(E, \partial\Omega_\delta) > 0\}.$$

Then the set S is at most countable. Moreover

$$\lim_{s \to 1}(1-s)\,\mathrm{Per}_s(E, \Omega_\delta) = \omega_{n-1}\,\mathrm{Per}(E, \Omega_\delta), \tag{1.15}$$

for every $\delta \in (-r_0, \beta) \setminus S$.

Proof. We observe that

$$\mathrm{Per}(E, \partial\Omega_\delta) = \mathcal{H}^{n-1}(\partial^* E \cap \{\bar{d}_\Omega = \delta\}),$$

for every $\delta \in (-r_0, \beta)$, and

$$M := \mathcal{H}^{n-1}(\partial^* E \cap (\Omega_\beta \setminus \overline{\Omega_{-r_0}})) \leq \mathrm{Per}(E, \Omega_\beta) < \infty. \tag{1.16}$$

Then we define the sets

$$S_k := \Big\{\delta \in (-r_0, \beta) \,|\, \mathcal{H}^{n-1}(\partial^* E \cap \{\bar{d}_\Omega = \delta\}) > \frac{1}{k}\Big\},$$

for every $k \in \mathbb{N}$ and we remark that

$$S = \bigcup_{k \in \mathbb{N}} S_k.$$

Since by (1.16) we have

$$\mathcal{H}^{n-1}\Big(\bigcup_{-r_0 < \delta < \beta} (\partial^* E \cap \{\bar{d}_\Omega = \delta\})\Big) = M,$$

the number of elements in each S_k is at most

$$\sharp S_k \leq M\,k.$$

As a consequence the set S is at most countable, as claimed.

Finally, since Ω_δ is a bounded open set with Lipschitz boundary for every $\delta \in (-r_0, r_0)$ (see Proposition B.1.1), we obtain (1.15) by Proposition 1.2.5. □

1.2.2.4. *Conclusion.* We are now ready to prove Theorem 1.1.7.

Proof of Theorem 1.1.7. We begin by observing that if $E \subseteq \mathbb{R}^n$ and we have two open sets $\mathcal{O}_1 \subseteq \mathcal{O}_2$, then

$$\operatorname{Per}_s(E, \mathcal{O}_1) \leq \operatorname{Per}_s(E, \mathcal{O}_2).$$

More precisely, we have

$$\begin{aligned}\operatorname{Per}_s(E, \mathcal{O}_2) = \operatorname{Per}_s(E, \mathcal{O}_1) &+ \mathcal{L}_s\big(E \cap (\mathcal{O}_2 \setminus \mathcal{O}_1), \mathcal{C}E \cap (\mathcal{O}_2 \setminus \mathcal{O}_1)\big)\\ &+ \mathcal{L}_s\big(E \cap (\mathcal{O}_2 \setminus \mathcal{O}_1), \mathcal{C}E \setminus \mathcal{O}_2\big) + \mathcal{L}_s\big(\mathcal{C}E \cap (\mathcal{O}_2 \setminus \mathcal{O}_1), E \setminus \mathcal{O}_2\big).\end{aligned} \tag{1.17}$$

Moreover, we also have

$$\operatorname{Per}_s^L(E, \mathcal{O}_1) \leq \operatorname{Per}_s(E, \mathcal{O}_2) \quad \text{and} \quad \operatorname{Per}(E, \mathcal{O}_1) \leq \operatorname{Per}(E, \mathcal{O}_2).$$

Now suppose that E has locally finite perimeter in Ω and let $\Omega' \Subset \Omega$. Notice that we can find a bounded open set $\mathcal{O}$ with Lipschitz boundary, such that

$$\Omega' \Subset \mathcal{O} \Subset \Omega.$$

Since E has finite perimeter in $\mathcal{O}$, by point (iv) of Corollary 1.2.2, we know that E has finite s-perimeter in $\mathcal{O}$ (and hence also in $\Omega' \Subset \mathcal{O}$) for every $s \in (0,1)$. Moreover, by Theorem 1.1.4 we obtain

$$\liminf_{s \to 1} (1-s) \operatorname{Per}_s^L(E, \Omega') \leq \liminf_{s \to 1} (1-s) \operatorname{Per}_s^L(E, \mathcal{O}) < \infty.$$

The converse implication is proved similarly.

Now suppose that E has locally finite perimeter in Ω and let $\mathcal{O} \Subset \Omega$ have Lipschitz boundary. Let $r_0 = r_0(\mathcal{O}) > 0$ be as in Proposition B.1.1. Since $\mathcal{O} \Subset \Omega$, we can find $\beta \in (0, r_0)$ small enough such that $\mathcal{O}_\beta \Subset \Omega$. Moreover, since E has locally finite perimeter in Ω, E has finite perimeter in $\mathcal{O}_\beta$.

Then, by Corollary 1.2.6, we can find $\delta \in (0, \beta)$ such that $\operatorname{Per}(E, \partial \mathcal{O}_\delta) = 0$ and we have

$$\lim_{s \to 1} (1-s) \operatorname{Per}_s(E, \mathcal{O}_\delta) = \omega_{n-1} \operatorname{Per}(E, \mathcal{O}_\delta). \tag{1.18}$$

We also remark that, since $|\partial \mathcal{O}| = 0$, we can rewrite (1.17) as

$$\begin{aligned}\operatorname{Per}_s(E, \mathcal{O}_\delta) = \operatorname{Per}_s(E, \mathcal{O}) &+ \operatorname{Per}_s^L(E, \mathcal{O}_\delta \setminus \overline{\mathcal{O}})\\ &+ \mathcal{L}_s\big(E \cap (\mathcal{O}_\delta \setminus \overline{\mathcal{O}}), \mathcal{C}E \setminus \mathcal{O}_\delta\big) + \mathcal{L}_s\big(\mathcal{C}E \cap (\mathcal{O}_\delta \setminus \overline{\mathcal{O}}), E \setminus \mathcal{O}_\delta\big).\end{aligned} \tag{1.19}$$

Let

$$I_s := \mathcal{L}_s\big(E \cap (\mathcal{O}_\delta \setminus \overline{\mathcal{O}}), \mathcal{C}E \setminus \mathcal{O}_\delta\big) + \mathcal{L}_s\big(\mathcal{C}E \cap (\mathcal{O}_\delta \setminus \overline{\mathcal{O}}), E \setminus \mathcal{O}_\delta\big)$$

and notice that

$$I_s \leq \operatorname{Per}_s^{NL}(E, \mathcal{O}_\delta). \tag{1.20}$$

Hence, since $\operatorname{Per}(E, \partial \mathcal{O}_\delta) = 0$, by (1.20) and Proposition 1.2.5 we obtain

$$\lim_{s \to 1} (1-s) I_s = 0. \tag{1.21}$$

Furthermore, since E has finite perimeter in $\mathcal{O}_\delta \setminus \overline{\mathcal{O}}$, which is a bounded open set with Lipschitz boundary, by (1.7) of Theorem 1.1.4, we find

$$\lim_{s \to 1} (1-s) \operatorname{Per}_s^L(E, \mathcal{O}_\delta \setminus \overline{\mathcal{O}}) = \omega_{n-1} \operatorname{Per}(E, \mathcal{O}_\delta \setminus \overline{\mathcal{O}}). \tag{1.22}$$

Therefore, by (1.19), (1.18), (1.21) and (1.22), and exploiting the fact that $\mathrm{Per}(E,\cdot)$ is a measure, we get

$$\begin{aligned}\lim_{s\to1}(1-s)\,\mathrm{Per}(E,\mathcal{O}) &= \omega_{n-1}\big(\mathrm{Per}(E,\mathcal{O}_\delta)-\mathrm{Per}(E,\mathcal{O}_\delta\setminus\overline{\mathcal{O}})\big)\\ &= \omega_{n-1}\,\mathrm{Per}(E,\overline{\mathcal{O}}).\end{aligned} \tag{1.23}$$

Finally, since by (1.7) we know that

$$\lim_{s\to1}(1-s)\,\mathrm{Per}_s^L(E,\mathcal{O}) = \omega_{n-1}\,\mathrm{Per}(E,\mathcal{O}), \tag{1.24}$$

by (1.23) and (1.24) we obtain

$$\lim_{s\to1}(1-s)\,\mathrm{Per}_s^{NL}(E,\mathcal{O}) = \omega_{n-1}\,\mathrm{Per}(E,\partial\mathcal{O}),$$

concluding the proof of the Theorem. □

1.3. Irregularity of the boundary

1.3.1. The measure theoretic boundary as "support" of the local part of the s-perimeter. First of all we show that the (local part of the) s-perimeter does indeed measure a quantity related to the measure theoretic boundary.

LEMMA 1.3.1. *Let $E\subseteq\mathbb{R}^n$ be a set of locally finite s-perimeter. Then*

$$\partial^-E=\{x\in\mathbb{R}^n\,|\,\mathrm{Per}_s^L(E,B_r(x))>0 \textit{ for every } r>0\}.$$

PROOF. The claim follows from the following observation. Let A, $B\subseteq\mathbb{R}^n$ such that $A\cap B=\varnothing$; then

$$\mathcal{L}_s(A,B)=0 \quad\Longleftrightarrow\quad |A|=0 \quad\text{or}\quad |B|=0.$$

Therefore

$$\begin{aligned} x\in\partial^-E \quad&\Longleftrightarrow\quad |E\cap B_r(x)|>0 \text{ and } |\mathcal{C}E\cap B_r(x)|>0 \quad \forall\, r>0\\ &\Longleftrightarrow\quad \mathcal{L}_s(E\cap B_r(x),\mathcal{C}E\cap B_r(x))>0 \quad \forall\, r>0,\end{aligned}$$

concluding the proof □

This characterization of ∂^-E can be thought of as a fractional analogue of (A.7). However we can not really think of ∂^-E as the support of

$$\mathrm{Per}_s^L(E,\cdot\,):\Omega\longmapsto\mathrm{Per}_s^L(E,\Omega),$$

in the sense that, in general

$$\partial^-E\cap\Omega=\varnothing \quad\not\Rightarrow\quad \mathrm{Per}_s^L(E,\Omega)=0.$$

For example, consider $E:=\{x_n\le0\}\subseteq\mathbb{R}^n$ and notice that $\partial^-E=\{x_n=0\}$. Let $\Omega:=B_1(2e_n)\cup B_1(-2e_n)$. Then $\partial^-E\cap\Omega=\varnothing$, but

$$\mathrm{Per}_s^L(E,\Omega)=\mathcal{L}_s(B_1(2e_n),B_1(-2e_n))>0.$$

On the other hand, the only obstacle is the non connectedness of the set Ω and indeed we obtain the following

PROPOSITION 1.3.2. *Let $E\subseteq\mathbb{R}^n$ be a set of locally finite s-perimeter and let $\Omega\subseteq\mathbb{R}^n$ be an open set. Then*

$$\partial^-E\cap\Omega\neq\varnothing \quad\Longrightarrow\quad \mathrm{Per}_s^L(E,\Omega)>0.$$

Moreover, if Ω is connected

$$\partial^-E\cap\Omega=\varnothing \quad\Longrightarrow\quad \mathrm{Per}_s^L(E,\Omega)=0.$$

Therefore, if $\widehat{\mathcal{O}}(\mathbb{R}^n)$ *denotes the family of bounded and connected open sets, then* $\partial^- E$ *can be considered as the "support" of*

$$\begin{aligned}\mathrm{Per}_s^L(E,\,\cdot\,):\widehat{\mathcal{O}}(\mathbb{R}^n)&\longrightarrow[0,\infty)\\ \Omega&\longmapsto\mathrm{Per}_s^L(E,\Omega),\end{aligned}$$

in the sense that, if $\Omega\in\widehat{\mathcal{O}}(\mathbb{R}^n)$, *then*

$$\mathrm{Per}_s^L(E,\Omega)>0\quad\Longleftrightarrow\quad\partial^-E\cap\Omega\neq\varnothing.$$

Proof. Let $x\in\partial^-E\cap\Omega$. Since Ω is open, we have $B_r(x)\subseteq\Omega$ for some $r>0$ and hence

$$\mathrm{Per}_s^L(E,\Omega)\geq\mathrm{Per}_s^L(E,B_r(x))>0.$$

Let Ω be connected and suppose $\partial^-E\cap\Omega=\varnothing$. Notice that we have the partition of $\mathbb{R}^n$ as $\mathbb{R}^n=E_{ext}\cup\partial^-E\cup E_{int}$ (see Appendix A). Thus we can write Ω as the disjoint union

$$\Omega=(E_{ext}\cap\Omega)\cup(E_{int}\cap\Omega).$$

However, since Ω is connected and both E_{ext} and E_{int} are open, we must have $E_{ext}\cap\Omega=\varnothing$ or $E_{int}\cap\Omega=\varnothing$. Now, if $E_{ext}\cap\Omega=\varnothing$ (the other case is analogous), then $\Omega\subseteq E_{int}$ and hence $|\mathcal{C}E\cap\Omega|=0$. Thus

$$\mathrm{Per}_s^L(E,\Omega)=\mathcal{L}_s(E\cap\Omega,\mathcal{C}E\cap\Omega)=0,$$

concluding the proof. □

1.3.2. A notion of fractal dimension. Let $\Omega\subseteq\mathbb{R}^n$ be an open set. Then

$$t>s\qquad\Longrightarrow\qquad W^{t,1}(\Omega)\hookrightarrow W^{s,1}(\Omega),$$

(see, e.g., [38, Proposition 2.1]). As a consequence, for every $u\in L^1(\Omega)$ there exists a unique $R(u)\in[0,1]$ such that

$$[u]_{W^{s,1}(\Omega)}\quad\begin{cases}<\infty, & \forall\,s\in(0,R(u))\\ =\infty, & \forall\,s\in(R(u),1)\end{cases}$$

that is

$$\begin{aligned}R(u)&=\sup\left\{s\in(0,1)\,\middle|\,[u]_{W^{s,1}(\Omega)}<\infty\right\}\\ &=\inf\left\{s\in(0,1)\,\middle|\,[u]_{W^{s,1}(\Omega)}=\infty\right\}.\end{aligned}\tag{1.25}$$

In particular, exploiting this result for characteristic functions, in [99] the author suggested the following definition of fractal dimension.

Definition 1.3.3. *Let* $\Omega\subseteq\mathbb{R}^n$ *be an open set and let* $E\subseteq\mathbb{R}^n$ *such that* $|E\cap\Omega|<\infty$. *If* $\partial^-E\cap\Omega\neq\varnothing$, *we define*

$$\mathrm{Dim}_F(\partial^-E,\Omega):=n-R(\chi_E),$$

the fractal dimension of ∂^-E *in* Ω, *relative to the fractional perimeter. If* $\Omega=\mathbb{R}^n$, *we drop it in the formulas.*

Notice that in the case of sets (1.25) becomes

$$\begin{aligned}R(\chi_E)&=\sup\left\{s\in(0,1)\,\middle|\,\mathrm{Per}_s^L(E,\Omega)<\infty\right\}\\ &=\inf\left\{s\in(0,1)\,\middle|\,\mathrm{Per}_s^L(E,\Omega)=\infty\right\}.\end{aligned}\tag{1.26}$$

We observe that, since $\mathrm{Per}_s^L(\mathcal{C}E,\Omega)=\mathrm{Per}_s^L(E,\Omega)$, in order to define the fractal dimension of ∂^-E in Ω, it is actually enough to require that either $|E\cap\Omega|<\infty$ or $|\mathcal{C}E\cap\Omega|<\infty$. Clearly, when the open set Ω is bounded, such assumptions are trivially satisfied.

In particular we can consider Ω to be the whole of $\mathbb{R}^n$, or a bounded open set with Lipschitz boundary. In the first case the local part of the fractional perimeter coincides with the whole fractional perimeter, while in the second case we know that we can bound the nonlocal part with $2\,\mathrm{Per}_s(\Omega) < \infty$ for every $s \in (0,1)$. Therefore, in both cases in (1.26) we can as well take the whole fractional perimeter $\mathrm{Per}_s(E,\Omega)$ instead of just the local part.

Now we recall the definition of Minkowski dimension, given in terms of the Minkowski contents. For equivalent definitions of the Minkowski dimension and for the main properties, we refer to [80] and [51] and the references cited therein.

For simplicity, given $\Gamma \subseteq \mathbb{R}^n$ we set

$$\bar{N}^{\Omega}_{\varrho}(\Gamma) := \overline{N_{\varrho}(\Gamma)} \cap \Omega = \{x \in \Omega \,|\, d(x,\Gamma) \leq \varrho\},$$

for any $\varrho > 0$.

Definition 1.3.4. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. For any $\Gamma \subseteq \mathbb{R}^n$ and $r \in [0,n]$ we define the inferior and superior r-dimensional Minkowski contents of Γ relative to the set Ω as, respectively*

$$\underline{\mathcal{M}}^r(\Gamma,\Omega) := \liminf_{\varrho\to 0} \frac{|\bar{N}^{\Omega}_{\varrho}(\Gamma)|}{\varrho^{n-r}}, \qquad \overline{\mathcal{M}}^r(\Gamma,\Omega) := \limsup_{\varrho\to 0} \frac{|\bar{N}^{\Omega}_{\varrho}(\Gamma)|}{\varrho^{n-r}}.$$

Then we define the lower and upper Minkowski dimensions of Γ in Ω as

$$\begin{aligned} \underline{\mathrm{Dim}}_{\mathcal{M}}(\Gamma,\Omega) &:= \inf\big\{r \in [0,n] \,|\, \underline{\mathcal{M}}^r(\Gamma,\Omega) = 0\big\} \\ &= n - \sup\big\{r \in [0,n] \,|\, \underline{\mathcal{M}}^{n-r}(\Gamma,\Omega) = 0\big\}, \\ \overline{\mathrm{Dim}}_{\mathcal{M}}(\Gamma,\Omega) &:= \sup\big\{r \in [0,n] \,|\, \overline{\mathcal{M}}^r(\Gamma,\Omega) = \infty\big\} \\ &= n - \inf\big\{r \in [0,n] \,|\, \overline{\mathcal{M}}^{n-r}(\Gamma,\Omega) = \infty\big\}. \end{aligned}$$

If they agree, we write

$$\mathrm{Dim}_{\mathcal{M}}(\Gamma,\Omega)$$

for the common value and call it the Minkowski dimension of Γ in Ω. If $\Omega = \mathbb{R}^n$ or $\Gamma \Subset \Omega$, we drop the Ω in the formulas.

Remark 1.3.5. Let $\mathrm{Dim}_{\mathcal{H}}$ denote the Hausdorff dimension. In general one has

$$\mathrm{Dim}_{\mathcal{H}}(\Gamma) \leq \underline{\mathrm{Dim}}_{\mathcal{M}}(\Gamma) \leq \overline{\mathrm{Dim}}_{\mathcal{M}}(\Gamma),$$

and all the inequalities might be strict (for some examples, see, e.g., [80, Section 5.3]). However for some sets, like self-similar sets which satisfy appropriate symmetric and regularity conditions, they are all equal (see, e.g., [80, Corollary 5.8]).

Now we give a proof of the relation (1.1) (obtained in [99]). For related results, see also [94] and [52].

Proposition 1.3.6. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Then for every $E \subseteq \mathbb{R}^n$ such that $\partial^- E \cap \Omega \neq \varnothing$ and $\overline{\mathrm{Dim}}_{\mathcal{M}}(\partial^- E,\Omega) \geq n-1$ we have*

$$\mathrm{Dim}_F(\partial^- E,\Omega) \leq \overline{\mathrm{Dim}}_{\mathcal{M}}(\partial^- E,\Omega).$$

Proof. By hypothesis we have

$$\overline{\mathrm{Dim}}_{\mathcal{M}}(\partial^- E,\Omega) = n - \inf\big\{r \in (0,1) \,|\, \overline{\mathcal{M}}^{n-r}(\partial^- E,\Omega) = \infty\big\},$$

and we need to show that

$$\inf\big\{r \in (0,1) \,|\, \overline{\mathcal{M}}^{n-r}(\partial^- E,\Omega) = \infty\big\} \leq \sup\{s \in (0,1) \,|\, \mathrm{Per}^L_s(E,\Omega) < \infty\}.$$

Up to modifying E on a set of Lebesgue measure zero we can suppose that $\partial E = \partial^- E$, as in Remark MTA. Notice that this does not affect the s-perimeter.

Now for any $s \in (0,1)$

$$\begin{aligned} 2\,\mathrm{Per}_s^L(E,\Omega) &= \int_\Omega dx \int_\Omega \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\,dy \\ &= \int_\Omega dx \int_0^\infty d\varrho \int_{\partial B_\varrho(x)\cap\Omega} \frac{|\chi_E(x)-\chi_E(y)|}{|x-y|^{n+s}}\,d\mathcal{H}^{n-1}(y) \\ &= \int_\Omega dx \int_0^\infty \frac{d\varrho}{\varrho^{n+s}} \int_{\partial B_\varrho(x)\cap\Omega} |\chi_E(x)-\chi_E(y)|\,d\mathcal{H}^{n-1}(y). \end{aligned}$$

Notice that

$$d(x,\partial E) > \varrho \quad \Longrightarrow \quad \chi_E(y) = \chi_E(x), \quad \forall\, y \in \overline{B_\varrho(x)},$$

and hence

$$\begin{aligned} \int_{\partial B_\varrho(x)\cap\Omega} |\chi_E(x)-\chi_E(y)|\,d\mathcal{H}^{n-1}(y) &\le \int_{\partial B_\varrho(x)\cap\Omega} \chi_{\bar N_\varrho(\partial E)}(x)\,d\mathcal{H}^{n-1}(y) \\ &\le n\omega_n \varrho^{n-1}\chi_{\bar N_\varrho(\partial E)}(x). \end{aligned}$$

Therefore

$$2\,\mathrm{Per}_s^L(E,\Omega) \le n\omega_n \int_0^\infty \frac{d\varrho}{\varrho^{1+s}} \int_\Omega \chi_{\bar N_\varrho(\partial E)}(x) = n\omega_n \int_0^\infty \frac{|\bar N_\varrho^\Omega(\partial E)|}{\varrho^{1+s}}\,d\varrho.$$

We claim that

$$\overline{\mathcal{M}}^{n-r}(\partial E,\Omega) < \infty \quad \Longrightarrow \quad \mathrm{Per}_s^L(E,\Omega) < \infty, \quad \forall\, s \in (0,r). \tag{1.27}$$

Indeed

$$\limsup_{\varrho\to 0} \frac{|\bar N_\varrho^\Omega(\partial E)|}{\varrho^r} < \infty \quad \Longrightarrow \quad \exists\, C > 0 \text{ s.t. } \sup_{\varrho\in(0,C]} \frac{|\bar N_\varrho^\Omega(\partial E)|}{\varrho^r} \le M < \infty.$$

Hence

$$\begin{aligned} 2\,\mathrm{Per}_s^L(E,\Omega) &\le n\omega_n \Big\{ \int_0^C \frac{|\bar N_\varrho^\Omega(\partial E)|}{\varrho^{1-(r-s)+r}}\,d\varrho + \int_C^\infty \frac{|\bar N_\varrho^\Omega(\partial E)|}{\varrho^{1+s}}\,d\varrho \Big\} \\ &\le n\omega_n \Big\{ M \int_0^C \frac{1}{\varrho^{1-(r-s)}}\,d\varrho + |\Omega| \int_C^\infty \frac{1}{\varrho^{1+s}}\,d\varrho \Big\} \\ &= n\omega_n \Big\{ \frac{M}{r-s} C^{r-s} + \frac{|\Omega|}{sC^s} \Big\} < \infty, \end{aligned}$$

proving (1.27). This implies that

$$r \le \sup\{s \in (0,1) \,|\, \mathrm{Per}_s^L(E,\Omega) < \infty\},$$

for every $r \in (0,1)$ such that $\overline{\mathcal{M}}^{n-r}(\partial E,\Omega) < \infty$.

Thus, for $\varepsilon > 0$ very small, we have

$$\inf\big\{r \in (0,1) \,|\, \overline{\mathcal{M}}^{n-r}(\partial^- E,\Omega) = \infty\big\} - \varepsilon \le \sup\{s \in (0,1) \,|\, \mathrm{Per}_s^L(E,\Omega) < \infty\}.$$

Letting ε tend to zero, we conclude the proof. □

In particular, if Ω has Lipschitz boundary we obtain:

COROLLARY 1.3.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $E \subseteq \mathbb{R}^n$ such that $\partial^- E \cap \Omega \neq \varnothing$ and $\overline{\mathrm{Dim}}_{\mathcal{M}}(\partial^- E,\Omega) \in [n-1,n)$. Then*

$$\mathrm{Per}_s(E,\Omega) < \infty \qquad \textit{for every } s \in \big(0, n - \overline{\mathrm{Dim}}_{\mathcal{M}}(\partial^- E,\Omega)\big).$$

Remark 1.3.8. Actually, Proposition 1.3.6 and Corollary 1.3.7 still remain true when $\Omega = \mathbb{R}^n$, provided the set E we are considering is bounded. Indeed, if E is bounded, we can apply the previous results with $\Omega = B_R$ such that $E \Subset \Omega$. Moreover, since Ω has a regular boundary, as remarked above we can take the whole s-perimeter in (1.26), instead of just the local part. But then, since $\mathrm{Per}_s(E, \Omega) = \mathrm{Per}_s(E)$, we see that

$$\mathrm{Dim}_F(\partial^- E, \Omega) = \mathrm{Dim}_F(\partial^- E, \mathbb{R}^n).$$

1.3.2.1. *Remarks about the Minkowski content of $\partial^- E$.* In the beginning of the proof of Proposition 1.3.6 we chose a particular representative for the class of E in order to have $\partial E = \partial^- E$. This can be done since it does not affect the s-perimeter and we are already considering the Minkowski dimension of $\partial^- E$.

On the other hand, if we consider a set F such that $|E \Delta F| = 0$, we can use the same proof to obtain the inequality

$$\mathrm{Dim}_F(\partial^- E, \Omega) \leq \overline{\mathrm{Dim}}_{\mathcal{M}}(\partial F, \Omega).$$

It is then natural to ask whether we can find a "better" representative F, whose (topological) boundary ∂F has Minkowski dimension strictly smaller than that of $\partial^- E$.

First of all, we remark that the Minkowski content can be influenced by changes in sets of measure zero. Roughly speaking, this is because the Minkowski content is not a purely measure theoretic notion, but rather a combination of metric and measure.

For example, let $\Gamma \subseteq \mathbb{R}^n$ and define $\Gamma' := \Gamma \cup \mathbb{Q}^n$. Then $|\Gamma \Delta \Gamma'| = 0$, but $N_\delta(\Gamma') = \mathbb{R}^n$ for every $\delta > 0$.

In particular, considering different representatives for E we will get different topological boundaries and hence different Minkowski dimensions.

However, since the measure theoretic boundary minimizes the size of the topological boundary, that is

$$\partial^- E = \bigcap_{|F \Delta E| = 0} \partial F,$$

(see Appendix A), it minimizes also the Minkowski dimension.
Indeed, for every F such that $|F \Delta E| = 0$ we have

$$\begin{aligned} \partial^- E \subseteq \partial F \quad &\Longrightarrow \quad \bar{N}_\varrho^\Omega(\partial^- E) \subseteq \bar{N}_\varrho^\Omega(\partial F) \\ &\Longrightarrow \quad \overline{\mathcal{M}}^r(\partial^- E, \Omega) \leq \overline{\mathcal{M}}^r(\partial F, \Omega) \\ &\Longrightarrow \quad \overline{\mathrm{Dim}}_{\mathcal{M}}(\partial^- E, \Omega) \leq \overline{\mathrm{Dim}}_{\mathcal{M}}(\partial F, \Omega). \end{aligned}$$

1.3.3. Fractal dimension of the von Koch snowflake. The von Koch snowflake $S \subseteq \mathbb{R}^2$ is an example of a bounded open set with fractal boundary, for which the Minkowski dimension and the fractal dimension introduced above coincide.

Moreover its boundary is "nowhere rectifiable", in the sense that $\partial S \cap B_r(p)$ is not $(n-1)$-rectifiable for any $r > 0$ and $p \in \partial S$.

First of all we recall how to construct the von Koch curve. Then the snowflake is made of three von Koch curves.

Let Γ_0 be a line segment of unit length. The set Γ_1 consists of the four segments obtained by removing the middle third of Γ_0 and replacing it by the other two sides of the equilateral triangle based on the removed segment.
We construct Γ_2 by applying the same procedure to each of the segments in Γ_1 and so on. Thus Γ_k comes from replacing the middle third of each straight line segment of Γ_{k-1} by the other two sides of an equilateral triangle.

As k tends to infinity, the sequence of polygonal curves Γ_k approaches a limiting curve Γ, called the von Koch curve.

If we start with an equilateral triangle with unit length side and perform the same construction on all three sides, we obtain the von Koch snowflake Σ (see Figure 1). Let S be the bounded region enclosed by Σ, so that S is open and $\partial S = \Sigma$. We still call S the von Koch snowflake.

It can be shown (see, e.g., [51]) that the Hausdorff dimension of the von Koch snowflake is equal to its Minkowski dimension and

$$\mathrm{Dim}_{\mathcal{H}}(\Sigma) = \mathrm{Dim}_{\mathcal{M}}(\Sigma) = \frac{\log 4}{\log 3}$$

Now we explain how to construct S in a recursive way and we observe that

$$\partial^- S = \partial S = \Sigma.$$

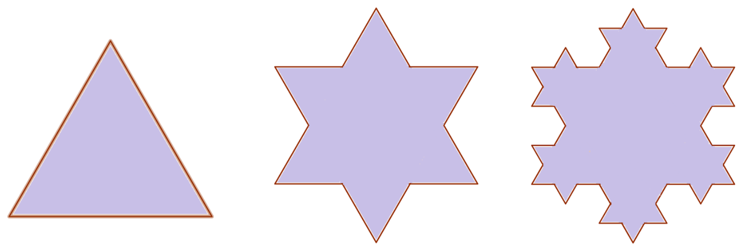


FIGURE 1. *The first three steps of the construction of the von Koch snowflake*

As starting point for the snowflake take the equilateral triangle T of side 1, with barycenter in the origin and a vertex on the y-axis, $P = (0, t)$ with $t > 0$.
Then T_1 is made of three triangles of side $1/3$, T_2 of $3 \cdot 4$ triangles of side $1/3^2$ and so on. In general T_k is made of $3 \cdot 4^{k-1}$ triangles of side $1/3^k$, call them $T_k^1, \dots, T_k^{3\cdot 4^{k-1}}$. Let x_k^i be the baricenter of T_k^i and Per_k^i the vertex which does not touch T_{k-1}.

Then $S = T \cup \bigcup T_k$. Also notice that T_k and T_{k-1} touch only on a set of measure zero.

For each triangle T_k^i there exists a rotation $\mathcal{R}_k^i \in SO(n)$ such that

$$T_k^i = F_k^i(T) := \mathcal{R}_k^i\Big(\frac{1}{3^k}T\Big) + x_k^i.$$

We choose the rotations so that $F_k^i(P) = \mathrm{Per}_k^i$.

Notice that for each triangle T_k^i we can find a small ball which is contained in the complementary of the snowflake, $B_k^i \subseteq \mathcal{C}S$, and touches the triangle in the vertex Per_k^i. Actually these balls can be obtained as the images of the affine transformations F_k^i of a fixed ball B.

To be more precise, fix a small ball contained in the complementary of T, which has the center on the y-axis and touches T in the vertex P, say $B := B_{1/1000}(0, t + 1/1000)$. Then

$$B_k^i := F_k^i(B) \subseteq \mathcal{C}S \tag{1.28}$$

for every i, k. To see this, imagine constructing the snowflake S using the same affine transformations F_k^i but starting with $T \cup B$ in place of T.

We know that $\partial^- S \subseteq \partial S$ (see Appendix A).
On the other hand, let $p \in \partial S$. Then every ball $B_\delta(p)$ contains at least a triangle $T_k^i \subseteq S$ and its corresponding ball $B_k^i \subseteq \mathcal{C}S$ (and actually infinitely many). Therefore

$$0 < |B_\delta(p) \cap S| < \omega_n \delta^n$$

for every $\delta > 0$ and hence $p \in \partial^- S$.

PROOF OF THEOREM 1.1.1. Since S is bounded, its boundary is $\partial^- S = \Sigma$, and $\mathrm{Dim}_{\mathcal{M}}(\Sigma) = \frac{\log 4}{\log 3}$, we obtain (1.2) from Corollary 1.3.7 and Remark 1.3.8.

Exploiting the construction of S given above and (1.28) we prove (1.3).
We have

$$\begin{aligned}
\mathrm{Per}_s(S) &= \mathcal{L}_s(S, \mathcal{C}S) = \mathcal{L}_s(T, \mathcal{C}S) + \sum_{k=1}^{\infty} \mathcal{L}_s(T_k, \mathcal{C}S) \\
&= \mathcal{L}_s(T, \mathcal{C}S) + \sum_{k=1}^{\infty} \sum_{i=1}^{3\cdot 4^{k-1}} \mathcal{L}_s(T_k^i, \mathcal{C}S) \geq \sum_{k=1}^{\infty} \sum_{i=1}^{3\cdot 4^{k-1}} \mathcal{L}_s(T_k^i, \mathcal{C}S) \\
&\geq \sum_{k=1}^{\infty} \sum_{i=1}^{3\cdot 4^{k-1}} \mathcal{L}_s(T_k^i, B_k^i) \qquad \text{(by (1.28))} \\
&= \sum_{k=1}^{\infty} \sum_{i=1}^{3\cdot 4^{k-1}} \mathcal{L}_s(F_k^i(T), F_k^i(B)) \\
&= \sum_{k=1}^{\infty} \sum_{i=1}^{3\cdot 4^{k-1}} \left(\frac{1}{3^k}\right)^{2-s} \mathcal{L}_s(T, B) \qquad \text{(by Proposition 1.3.12)} \\
&= \frac{3}{3^{2-s}} \mathcal{L}_s(T, B) \sum_{k=0}^{\infty} \left(\frac{4}{3^{2-s}}\right)^k.
\end{aligned}$$

We remark that

$$\mathcal{L}_s(T, B) \leq \mathcal{L}_s(T, \mathcal{C}T) = \mathrm{Per}_s(T) < \infty,$$

for every $s \in (0, 1)$.

To conclude, notice that the last series is divergent if $s \geq 2 - \frac{\log 4}{\log 3}$. □

Exploiting the self-similarity of the von Koch curve, we show that the fractal dimension of S is the same in every open set which contains a point of ∂S.

COROLLARY 1.3.9. *Let $S \subseteq \mathbb{R}^2$ be the von Koch snowflake. Then*

$$\mathrm{Dim}_F(\partial S, \Omega) = \frac{\log 4}{\log 3}$$

for every open set Ω such that $\partial S \cap \Omega \neq \varnothing$.

PROOF. Since $\mathrm{Per}_s(S, \Omega) \leq \mathrm{Per}_s(S)$, we have

$$\mathrm{Per}_s(S, \Omega) < \infty, \qquad \forall\, s \in \left(0, 2 - \frac{\log 4}{\log 3}\right).$$

On the other hand, if $p \in \partial S \cap \Omega$, then $B_r(p) \subseteq \Omega$ for some $r > 0$. Now notice that $B_r(p)$ contains a rescaled version of the von Koch curve, including all the triangles T_k^i which constitute it and the relative balls B_k^i. We can thus repeat the argument above to obtain

$$\mathrm{Per}_s(S, \Omega) \geq \mathrm{Per}_s(S, B_r(p)) = \infty, \qquad \forall\, s \in \left[2 - \frac{\log 4}{\log 3}, 1\right),$$

concluding the proof. □

1.3.4. Self-similar fractal boundaries.

Proof of Theorem 1.1.2. Arguing as we did with the von Koch snowflake, we show that $\mathrm{Per}_s(T)$ is bounded both from above and from below by the series

$$\sum_{k=0}^{\infty}\Big(\frac{b}{\lambda^{n-s}}\Big)^k,$$

which converges if and only if $s<n-\frac{\log b}{\log\lambda}$.

Indeed

$$\begin{aligned}\mathrm{Per}_s(T)&=\mathcal{L}_s(T,\mathcal{C}T)=\sum_{k=1}^{\infty}\sum_{i=1}^{ab^{k-1}}\mathcal{L}_s(T_k^i,\mathcal{C}T)\\&\leq\sum_{k=1}^{\infty}\sum_{i=1}^{ab^{k-1}}\mathcal{L}_s(T_k^i,\mathcal{C}T_k^i)=\sum_{k=1}^{\infty}\sum_{i=1}^{ab^{k-1}}\mathcal{L}_s(F_k^i(T_0),F_k^i(\mathcal{C}T_0))\\&=\frac{a}{\lambda^{n-s}}\mathcal{L}_s(T_0,\mathcal{C}T_0)\sum_{k=0}^{\infty}\Big(\frac{b}{\lambda^{n-s}}\Big)^k,\end{aligned}$$

and

$$\begin{aligned}\mathrm{Per}_s(T)&=\mathcal{L}_s(T,\mathcal{C}T)=\sum_{k=1}^{\infty}\sum_{i=1}^{ab^{k-1}}\mathcal{L}_s(T_k^i,\mathcal{C}T)\\&\geq\sum_{k=1}^{\infty}\sum_{i=1}^{ab^{k-1}}\mathcal{L}_s(T_k^i,S_k^i)=\sum_{k=1}^{\infty}\sum_{i=1}^{ab^{k-1}}\mathcal{L}_s(F_k^i(T_0),F_k^i(S_0))\\&=\frac{a}{\lambda^{n-s}}\mathcal{L}_s(T_0,S_0)\sum_{k=0}^{\infty}\Big(\frac{b}{\lambda^{n-s}}\Big)^k.\end{aligned}$$

Also notice that, since $\mathrm{Per}(T_0)<\infty$, we have

$$\mathcal{L}_s(T_0,S_0)\leq\mathcal{L}_s(T_0,\mathcal{C}T_0)=\mathrm{Per}_s(T_0)<\infty,$$

for every $s\in(0,1)$. □

Now suppose that T does not satisfy (1.5). Then we can obtain a set T' which does, simply by removing a portion S_0 from the building block T_0.
To be more precise, let $S_0\subseteq T_0$ be such that

$$|S_0|>0,\quad |T_0\setminus S_0|>0\quad\text{and}\quad \mathrm{Per}(T_0\setminus S_0)<\infty.$$

Then define a new building block $T_0':=T_0\setminus S_0$ and the set

$$T':=\bigcup_{k=1}^{\infty}\bigcup_{i=1}^{ab^{k-1}}F_k^i(T_0').$$

This new set has exactly the same structure of T, since we are using the same collection $\{F_k^i\}$ of affine maps.

Notice that

$$S_0\subseteq T_0\quad\Longrightarrow\quad F_k^i(S_0)\subseteq F_k^i(T_0),$$

and

$$F_k^i(T_0')=F_k^i(T_0)\setminus F_k^i(S_0),$$

for every k, i. Thus

$$T' = T \setminus \Big(\bigcup_{k=1}^{\infty}\bigcup_{i=1}^{ab^{k-1}} F_k^i(S_0)\Big)$$

satisfies (1.5).

Remark 1.3.10. Roughly speaking, what matters in order to obtain a set which satisfies the hypothesis of Theorem 1.1.2 is that there exists a bounded open set T_0 such that

$$|F_k^i(T_0)\cap F_h^j(T_0)| = 0, \qquad \text{if } i\neq j \text{ or } k\neq h.$$

This can be thought of as a compatibility criterion for the family of affine maps $\{F_k^i\}$. We also need to ask that the ratio of the logarithms of the growth factor and the scaling factor is $\frac{\log b}{\log \lambda}\in(n-1,n)$. Then we are free to choose as building block any set $T_0'\subseteq T_0$ such that

$$|T_0'|>0, \qquad |T_0\setminus T_0'|>0 \qquad \text{and } \operatorname{Per}(T_0')<\infty,$$

and the set

$$T' := \bigcup_{k=1}^{\infty}\bigcup_{i=1}^{ab^{k-1}} F_k^i(T_0').$$

satisfies the hypothesis of Theorem 1.1.2.

Therefore, even if the Sierpinski triangle and the Menger sponge do not satisfy (1.5), we can exploit their structure to construct new sets which do.

However, we remark that the new boundary $\partial^- T'$ will look very different from the original fractal. Actually, in general it will be a mix of unrectifiable pieces and smooth pieces. In particular, we can not hope to get an analogue of Corollary 1.3.9. Still, the following Remark shows that the new (measure theoretic) boundary retains at least some of the "fractal nature" of the original set.

Remark 1.3.11. If the set T of Theorem 1.1.2 is bounded, exploiting Proposition 1.3.6 and Remark 1.3.8 we obtain

$$\overline{\operatorname{Dim}}_{\mathcal{M}}(\partial^- T)\geq\frac{\log b}{\log\lambda}>n-1.$$

Moreover, notice that if Ω is a bounded open set with Lipschitz boundary, then

$$\operatorname{Per}(E,\Omega)<\infty \quad\Longrightarrow\quad \operatorname{Dim}_F(E,\Omega)=n-1.$$

Therefore, if $T\Subset B_R$, then

$$\operatorname{Per}(T)=\operatorname{Per}(T,B_R)=\infty,$$

even if T is bounded (and hence $\partial^- T$ is compact).

1.3.4.1. *Sponge-like sets.* The simplest way to construct the set T' consists in simply removing a small ball $S_0 := B\Subset T_0$ from T_0.

In particular, suppose that $|T_0\Delta T|=0$, as with the Sierpinski triangle.
Define

$$S := \bigcup_{k=1}^{\infty}\bigcup_{i=1}^{ab^{k-1}} F_k^i(B) \quad\text{and}\quad T' := \bigcup_{k=1}^{\infty}\bigcup_{i=1}^{ab^{k-1}} F_k^i(T_0\setminus B)=T\setminus S.$$

Then

$$|T_0\Delta T|=0 \quad\Longrightarrow\quad |T'\Delta(T_0\setminus S)|=0. \tag{1.29}$$

Now the set $E := T_0 \setminus S$ looks like a sponge, in the sense that it is a bounded open set with an infinite number of holes (each one at a positive, but non-fixed distance from the others).

From (1.29) we get $\mathrm{Per}_s(E) = \mathrm{Per}_s(T')$. Thus, since T' satisfies the hypothesis of Theorem 1.1.2, we obtain

$$\mathrm{Dim}_F(\partial^- E) = \frac{\log b}{\log \lambda}.$$

1.3.4.2. *Dendrite-like sets.* Depending on the form of the set T_0 and on the affine maps $\{F_k^i\}$, we can define more intricate sets T'.

As an example we consider the Sierpinski triangle $E \subseteq \mathbb{R}^2$.
It is of the form $E = T_0 \setminus T$, where the building block T_0 is an equilateral triangle, say with side length one, a vertex on the y-axis and baricenter in 0. The pieces T_k^i are obtained with a scaling factor $\lambda = 2$ and the growth factor is $b = 3$ (see, e.g., [51] for the construction). As usual, we consider the set

$$T = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{3^{k-1}} T_k^i.$$

However, as remarked above, we have $|T \Delta T_0| = 0$.

Starting from $k = 2$ each triangle T_k^i touches with (at least) a vertex (at least) another triangle T_h^j. Moreover, each triangle T_k^i gets touched in the middle point of each side (and actually it gets touched in infinitely many points).

Exploiting this situation, we can remove from T_0 six smaller triangles, so that the new building block T_0' is a star polygon centered in 0, with six vertices, one in each vertex of T_0 and one in each middle point of the sides of T_0.

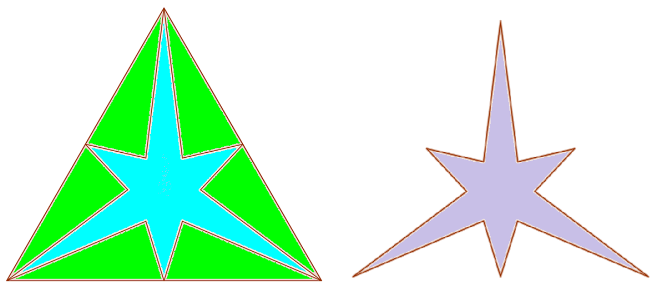


FIGURE 2. *Removing the six triangles (in green) to obtain the new "building block" T_0' (on the right)*

The resulting set

$$T' = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{3^{k-1}} F_k^i(T_0')$$

will have an infinite number of ramifications.

Since T' satisfies the hypothesis of Theorem 1.1.2, we obtain

$$\mathrm{Dim}_F(\partial^- T') = \frac{\log 3}{\log 2}.$$

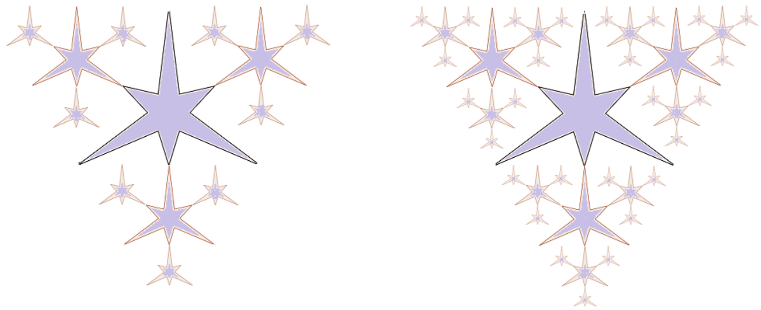


FIGURE 3. *The third and fourth steps of the iterative construction of the set T'*

1.3.4.3. *"Exploded" fractals.* In all the previous examples, the sets T_k^i are accumulated in a bounded region.

On the other hand, imagine making a fractal like the von Koch snowflake or the Sierpinski triangle "explode" and then rearrange the pieces T_k^i in such a way that $d(T_k^i, T_h^j) \geq d$, for some fixed $d > 0$.

Since the shape of the building block is not important, we can consider $T_0 := B_{1/4}(0) \subseteq \mathbb{R}^n$, with $n \geq 2$. Moreover, since the parameter a does not influence the dimension, we can fix $a = 1$.

Then we rearrange the pieces obtaining

$$E := \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{b^{k-1}} B_{\frac{1}{4\lambda^k}}(k, 0, \dots, 0, i). \tag{1.30}$$

Define for simplicity

$$B_k^i := B_{\frac{1}{4\lambda^k}}(k, 0, \dots, 0, i) \quad \text{and} \quad x_k^i := k\, e_1 + i\, e_n,$$

and notice that

$$B_k^i = \lambda^{-k} B_{\frac{1}{4}}(0) + x_k^i.$$

Since for every k, h and every $i \neq j$ we have

$$d(B_k^i, B_h^j) \geq \frac{1}{2},$$

the boundary of the set E is the disjoint union of $(n-1)$-dimensional spheres

$$\partial^- E = \partial E = \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{b^{k-1}} \partial B_k^i,$$

and in particular is smooth.

The (global) perimeter of E is

$$\text{Per}(E) = \sum_{k=1}^{\infty} \sum_{i=1}^{b^{k-1}} \text{Per}(B_k^i) = \frac{1}{\lambda} \text{Per}(B_{1/4}(0)) \sum_{k=0}^{\infty} \Big(\frac{b}{\lambda^{n-1}} \Big)^k = +\infty,$$

since $\frac{\log b}{\log \lambda} > n - 1$.

However E has locally finite perimeter, since its boundary is smooth and every ball B_R intersects only finitely many B_k^i's,

$$\text{Per}(E, B_R) < \infty, \qquad \forall\, R > 0.$$

Therefore it also has locally finite s-perimeter for every $s \in (0,1)$

$$\mathrm{Per}_s(E, B_R) < \infty, \qquad \forall\, R > 0, \qquad \forall\, s \in (0,1).$$

What is interesting is that the set E satisfies the hypothesis of Theorem 1.1.2 and hence it also has finite global s-perimeter for every $s < \sigma_0 := n - \frac{\log b}{\log \lambda}$,

$$\mathrm{Per}_s(E) < \infty \qquad \forall\, s \in (0,\sigma_0) \quad \text{and} \quad \mathrm{Per}_s(E) = \infty \qquad \forall\, s \in [\sigma_0, 1).$$

Thus we obtain Proposition 1.1.3.

Proof of Proposition 1.1.3. It is enough to choose a natural number $b \geq 2$ and take $\lambda := b^{\frac{1}{n-\sigma}}$. Notice that $\lambda > 1$ and

$$\frac{\log b}{\log \lambda} = n - \sigma \in (n-1, n).$$

Then we can define E as in (1.30) and we are done. □

1.3.5. Elementary properties of the s-perimeter. In the following Proposition we collect some elementary but useful properties of the fractional perimeter which we have exploited throughout the chapter.

Proposition 1.3.12. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set.*

(i) (Subadditivity) Let E, $F \subseteq \mathbb{R}^n$ be such that $|E \cap F| = 0$. Then

$$\mathrm{Per}_s(E \cup F, \Omega) \leq \mathrm{Per}_s(E, \Omega) + \mathrm{Per}_s(F, \Omega).$$

(ii) (Translation invariance) Let $E \subseteq \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Then

$$\mathrm{Per}_s(E + x, \Omega + x) = \mathrm{Per}_s(E, \Omega).$$

(iii) (Rotation invariance) Let $E \subseteq \mathbb{R}^n$ and $\mathcal{R} \in SO(n)$ a rotation. Then

$$\mathrm{Per}_s(\mathcal{R}E, \mathcal{R}\Omega) = \mathrm{Per}_s(E, \Omega).$$

(iv) (Scaling) Let $E \subseteq \mathbb{R}^n$ and $\lambda > 0$. Then

$$\mathrm{Per}_s(\lambda E, \lambda \Omega) = \lambda^{n-s}\, \mathrm{Per}_s(E, \Omega).$$

Proof. (i) follows from the following observations. Let A_1, A_2, $B \subseteq \mathbb{R}^n$. If $|A_1 \cap A_2| = 0$, then

$$\mathcal{L}_s(A_1 \cup A_2, B) = \mathcal{L}_s(A_1, B) + \mathcal{L}_s(A_2, B).$$

Moreover

$$A_1 \subseteq A_2 \quad \Longrightarrow \quad \mathcal{L}_s(A_1, B) \leq \mathcal{L}_s(A_2, B),$$

and

$$\mathcal{L}_s(A, B) = \mathcal{L}_s(B, A).$$

Therefore

$$\begin{aligned}
\mathrm{Per}_s(E \cup F, \Omega) &= \mathcal{L}_s((E \cup F) \cap \Omega, \mathcal{C}(E \cup F)) + \mathcal{L}_s((E \cup F) \setminus \Omega, \mathcal{C}(E \cup F) \cap \Omega) \\
&= \mathcal{L}_s(E \cap \Omega, \mathcal{C}(E \cup F)) + \mathcal{L}_s(F \cap \Omega, \mathcal{C}(E \cup F)) \\
&\qquad + \mathcal{L}_s(E \setminus \Omega, \mathcal{C}(E \cup F) \cap \Omega) + \mathcal{L}_s(F \setminus \Omega, \mathcal{C}(E \cup F) \cap \Omega) \\
&\leq \mathcal{L}_s(E \cap \Omega, \mathcal{C}E) + \mathcal{L}_s(F \cap \Omega, \mathcal{C}F) \\
&\qquad + \mathcal{L}_s(E \setminus \Omega, \mathcal{C}E \cap \Omega) + \mathcal{L}_s(F \setminus \Omega, \mathcal{C}F \cap \Omega) \\
&= \mathrm{Per}_s(E, \Omega) + \mathrm{Per}_s(F, \Omega).
\end{aligned}$$

(ii), (iii) and (iv) follow simply by changing variables in $\mathcal{L}_s$ and the following observations:

$$\begin{aligned}&(x+A_1)\cap(x+A_2)=x+A_1\cap A_2, \qquad x+\mathcal{C}A=\mathcal{C}(x+A),\\ &\mathcal{R}A_1\cap\mathcal{R}A_2=\mathcal{R}(A_1\cap A_2), \qquad \mathcal{R}(\mathcal{C}A)=\mathcal{C}(\mathcal{R}A),\\ &(\lambda A_1)\cap(\lambda A_2)=\lambda(A_1\cap A_2), \qquad \lambda(\mathcal{C}A)=\mathcal{C}(\lambda A).\end{aligned}$$

For example, for claim (iv) we have

$$\begin{aligned}\mathcal{L}_s(\lambda A,\lambda B)&=\int_{\lambda A}\int_{\lambda B}\frac{dx\,dy}{|x-y|^{n+s}}=\int_A\lambda^n\,dx\int_B\frac{\lambda^n\,dy}{\lambda^{n+s}|x-y|^{n+s}}\\ &=\lambda^{n-s}\mathcal{L}_s(A,B).\end{aligned}$$

Then

$$\begin{aligned}\mathrm{Per}_s(\lambda E,\lambda\Omega)&=\mathcal{L}_s(\lambda E\cap\lambda\Omega,\mathcal{C}(\lambda E))+\mathcal{L}_s(\lambda E\cap\mathcal{C}(\lambda\Omega),\mathcal{C}(\lambda E)\cap\lambda\Omega)\\ &=\mathcal{L}_s(\lambda(E\cap\Omega),\lambda\mathcal{C}E)+\mathcal{L}_s(\lambda(E\setminus\Omega),\lambda(\mathcal{C}E\cap\Omega))\\ &=\lambda^{n-s}\left(\mathcal{L}_s(E\cap\Omega,\mathcal{C}E)+\mathcal{L}_s(E\setminus\Omega,\mathcal{C}E\cap\Omega)\right)\\ &=\lambda^{n-s}\,\mathrm{Per}_s(E,\Omega).\end{aligned}$$

This concludes the proof of the Proposition. □

1.4. Proof of Example 1.1.1

Note that $E\subseteq(0,a^2]$. Let $\Omega:=(-1,1)\subseteq\mathbb{R}$. Then $E\Subset\Omega$ and $\mathrm{dist}(E,\partial\Omega)=1-a^2=:d>0$. Now

$$\mathrm{Per}_s(E)=\int_E\int_{\mathcal{C}E\cap\Omega}\frac{dxdy}{|x-y|^{1+s}}+\int_E\int_{\mathcal{C}\Omega}\frac{dxdy}{|x-y|^{1+s}}.$$

As for the second term, we have

$$\int_E\int_{\mathcal{C}\Omega}\frac{dxdy}{|x-y|^{1+s}}\le\frac{2|E|}{sd^s}<\infty.$$

We split the first term into three pieces

$$\begin{aligned}&\int_E\int_{\mathcal{C}E\cap\Omega}\frac{dxdy}{|x-y|^{1+s}}\\ &=\int_E\int_{-1}^0\frac{dxdy}{|x-y|^{1+s}}+\int_E\int_{\mathcal{C}E\cap(0,a)}\frac{dxdy}{|x-y|^{1+s}}+\int_E\int_a^1\frac{dxdy}{|x-y|^{1+s}}\\ &=\mathcal{I}_1+\mathcal{I}_2+\mathcal{I}_3.\end{aligned}$$

Note that $\mathcal{C}E\cap(0,a)=\bigcup_{k\in\mathbb{N}}I_{2k-1}=\bigcup_{k\in\mathbb{N}}(a^{2k},a^{2k-1})$.
A simple calculation shows that, if $a<b\le c<d$, then

$$\begin{aligned}&\int_a^b\int_c^d\frac{dxdy}{|x-y|^{1+s}}=\\ &\frac{1}{s(1-s)}\big[(c-a)^{1-s}+(d-b)^{1-s}-(c-b)^{1-s}-(d-a)^{1-s}\big].\end{aligned}\tag{1.31}$$

Also note that, if $n > m \geq 1$, then

$$\begin{aligned}(1-a^n)^{1-s}-(1-a^m)^{1-s} &= \int_m^n \frac{d}{dt}(1-a^t)^{1-s}\,dt \\ &= (s-1)\log a \int_m^n \frac{a^t}{(1-a^t)^s}\,dt \\ &\leq a^m(s-1)\log a \int_m^n \frac{1}{(1-a^t)^s}\,dt \\ &\leq (n-m)a^m \frac{(s-1)\log a}{(1-a)^s}.\end{aligned} \tag{1.32}$$

Now consider the first term

$$\mathcal{I}_1 = \sum_{k=1}^{\infty} \int_{a^{2k+1}}^{a^{2k}} \int_{-1}^{0} \frac{dxdy}{|x-y|^{1+s}}.$$

Use (1.31) and notice that $(c-a)^{1-s} - (d-a)^{1-s} \leq 0$ to get

$$\int_{-1}^{0} \int_{a^{2k+1}}^{a^{2k}} \frac{dxdy}{|x-y|^{1+s}} \leq \frac{1}{s(1-s)}\left[(a^{2k})^{1-s} - (a^{2k+1})^{1-s}\right] \leq \frac{1}{s(1-s)}(a^{2(1-s)})^k.$$

Then, as $a^{2(1-s)} < 1$ we get

$$\mathcal{I}_1 \leq \frac{1}{s(1-s)} \sum_{k=1}^{\infty} (a^{2(1-s)})^k < \infty.$$

As for the last term

$$\mathcal{I}_3 = \sum_{k=1}^{\infty} \int_{a^{2k+1}}^{a^{2k}} \int_{a}^{1} \frac{dxdy}{|x-y|^{1+s}},$$

use (1.31) and notice that $(d-b)^{1-s} - (d-a)^{1-s} \leq 0$ to get

$$\begin{aligned}\int_{a^{2k+1}}^{a^{2k}} \int_{a}^{1} \frac{dxdy}{|x-y|^{1+s}} &\leq \frac{1}{s(1-s)}\left[(1-a^{2k+1})^{1-s} - (1-a^{2k})^{1-s}\right] \\ &\leq \frac{-\log a}{s(1-a)^s} a^{2k} \quad \text{by (1.32)}.\end{aligned}$$

Thus

$$\mathcal{I}_3 \leq \frac{-\log a}{s(1-a)^s} \sum_{k=1}^{\infty} (a^2)^k < \infty.$$

Finally we split the second term

$$\mathcal{I}_2 = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} \int_{a^{2k+1}}^{a^{2k}} \int_{a^{2j}}^{a^{2j-1}} \frac{dxdy}{|x-y|^{1+s}}$$

into three pieces according to the cases $j > k$, $j = k$ and $j < k$.

If $j = k$, using (1.31) we get

$$\int_{a^{2k+1}}^{a^{2k}} \int_{a^{2k}}^{a^{2k-1}} \frac{dxdy}{|x-y|^{1+s}} =$$
$$= \frac{1}{s(1-s)} \left[(a^{2k} - a^{2k+1})^{1-s} + (a^{2k-1} - a^{2k})^{1-s} - (a^{2k-1} - a^{2k+1})^{1-s} \right]$$
$$= \frac{1}{s(1-s)} \left[a^{2k(1-s)}(1-a)^{1-s} + a^{(2k-1)(1-s)}(1-a)^{1-s} \right.$$
$$\left. - a^{(2k-1)(1-s)}(1-a^2)^{1-s} \right]$$
$$= \frac{1}{s(1-s)} (a^{2(1-s)})^k \left[(1-a)^{1-s} + \frac{(1-a)^{1-s}}{a^{1-s}} - \frac{(1-a^2)^{1-s}}{a^{1-s}} \right].$$

Summing over $k \in \mathbb{N}$ we get

$$\sum_{k=1}^{\infty} \int_{a^{2k+1}}^{a^{2k}} \int_{a^{2k}}^{a^{2k-1}} \frac{dxdy}{|x-y|^{1+s}} =$$
$$= \frac{1}{s(1-s)} \frac{a^{2(1-s)}}{1-a^{2(1-s)}} \left[(1-a)^{1-s} + \frac{(1-a)^{1-s}}{a^{1-s}} - \frac{(1-a^2)^{1-s}}{a^{1-s}} \right] < \infty.$$

In particular note that

$$(1-s)\operatorname{Per}_s(E) \geq (1-s)\mathcal{I}_2$$
$$\geq \frac{1}{s(1-a^{2(1-s)})} \left[a^{2(1-s)}(1-a)^{1-s} + a^{1-s}(1-a)^{1-s} - a^{1-s}(1-a^2)^{1-s} \right],$$

which tends to $+\infty$ when $s \to 1$. This shows that E cannot have finite perimeter.

To conclude let $j > k$, the case $j < k$ being similar, and consider

$$\sum_{k=1}^{\infty} \sum_{j=k+1}^{\infty} \int_{a^{2j}}^{a^{2j-1}} \int_{a^{2k+1}}^{a^{2k}} \frac{dxdy}{|x-y|^{1+s}}.$$

Again, using (1.31) and $(d-b)^{1-s} - (d-a)^{1-s} \leq 0$, we get

$$\int_{a^{2j}}^{a^{2j-1}} \int_{a^{2k+1}}^{a^{2k}} \frac{dxdy}{|x-y|^{1+s}}$$
$$\leq \frac{1}{s(1-s)} \left[(a^{2k+1} - a^{2j})^{1-s} - (a^{2k+1} - a^{2j-1})^{1-s} \right]$$
$$= \frac{a^{1-s}}{s(1-s)} (a^{2(1-s)})^k \left[(1 - a^{2(j-k)-1})^{1-s} - (1 - a^{2(j-k)-2})^{1-s} \right]$$
$$\leq \frac{a^{1-s}}{s(1-s)} (a^{2(1-s)})^k \frac{(s-1)\log a}{(1-a)^s} a^{2(j-k)-2} \qquad \text{by (1.32)}$$
$$= \frac{-\log a}{s(1-a^s)a^{s+1}} (a^{2(1-s)})^k (a^2)^{j-k},$$

for $j \geq k+2$. Then

$$\sum_{k=1}^{\infty} \sum_{j=k+2}^{\infty} \int_{a^{2j}}^{a^{2j-1}} \int_{a^{2k+1}}^{a^{2k}} \frac{dxdy}{|x-y|^{1+s}}$$
$$\leq \frac{-\log a}{s(1-a^s)a^{s+1}} \sum_{k=1}^{\infty} (a^{2(1-s)})^k \sum_{h=2}^{\infty} (a^2)^h < \infty.$$

If $j = k + 1$ we get

$$\sum_{k=1}^{\infty} \int_{a^{2k+2}}^{a^{2k+1}} \int_{a^{2k+1}}^{a^{2k}} \frac{dxdy}{|x-y|^{1+s}} \leq \frac{1}{s(1-s)} \sum_{k=1}^{\infty} (a^{2k+1} - a^{2k+2})^{1-s}$$

$$= \frac{a^{1-s}(1-a)^{1-s}}{s(1-s)} \sum_{k=1}^{\infty} (a^{2(1-s)})^{k} < \infty.$$

This shows that also $\mathcal{I}_2 < \infty$, so that $\mathrm{Per}_s(E) < \infty$ for every $s \in (0,1)$ as claimed.

CHAPTER 2

Approximation of sets of finite fractional perimeter by smooth sets and comparison of local and global s-minimal surfaces

2.1. Introduction and main results

This chapter is divided in two parts.

In the first part we prove that a set has (locally) finite fractional perimeter if and only if it can be approximated (in an appropriate way) by smooth open sets. To be more precise, we show that a set E has locally finite s-perimeter if and only if we can find a sequence of smooth open sets which converge in measure to E, whose boundaries converge to that of E in a uniform sense, and whose s-perimeters converge to that of E in every bounded open set.

The second part of this chapter is concerned with sets minimizing the fractional perimeter.

We recall that, given a set A and an open set Ω, we will write $A \Subset \Omega$ to mean that the closure $\overline{A}$ of A is compact and $\overline{A} \subseteq \Omega$. In particular, notice that if $A \Subset \Omega$, then A must be bounded.

We consider sets which are locally s-minimal in an open set $\Omega \subseteq \mathbb{R}^n$, namely sets which minimize the s-perimeter in every open subset $\Omega' \Subset \Omega$, and we prove existence and compactness results which extend those of [21].

We also compare this definition of local s-minimal set with the definition of s-minimal set introduced in [21], proving that they coincide when the domain Ω is a bounded open set with Lipschitz boundary (see Theorem 2.1.7).

In particular, the following existence results are proven:

- if Ω is an open set and E_0 is a fixed set, then there exists a set E which is locally s-minimal in Ω and such that $E \setminus \Omega = E_0 \setminus \Omega$;

- there exist minimizers in the class of subgraphs, namely nonlocal nonparametric minimal surfaces (see Theorem 2.1.16 for a precise statement);
- if Ω is an open set which has finite s-perimeter, then for every fixed set E_0 there exists a set E which is s-minimal in Ω and such that $E \setminus \Omega = E_0 \setminus \Omega$.

On the other hand, we show that when the domain Ω is unbounded the nonlocal part of the s-perimeter can be infinite, thus preventing the existence of competitors having finite s-perimeter in Ω and hence also of "global" s-minimal sets. In particular, we study this situation in a cylinder $\Omega^\infty := \Omega \times \mathbb{R} \subseteq \mathbb{R}^{n+1}$, considering as exterior data the subgraph of a (locally) bounded function.

In the following subsections we present the precise statements of the main results of this chapter.

2.1.1. Sets having (locally) finite s-perimeter. We recall that we implicitly assume that all the sets we consider contain their measure theoretic interior, do not intersect their measure theoretic exterior, and are such that their topological boundary coincides with their measure theoretic boundary—see Remark MTA and Appendix A for the details.

We recall that we say that a set $E \subseteq \mathbb{R}^n$ has locally finite s-perimeter in an open set $\Omega \subseteq \mathbb{R}^n$ if

$$\operatorname{Per}_s(E, \Omega') < \infty \qquad \text{for every open set } \Omega' \Subset \Omega.$$

We remark that the family of sets having finite s-perimeter in Ω need not coincide with the family of sets of locally finite s-perimeter in Ω, not even when Ω is "nice" (say bounded and with Lipschitz boundary). To be more precise, since

$$\operatorname{Per}_s(E, \Omega) = \sup_{\Omega' \Subset \Omega} \operatorname{Per}_s(E, \Omega'), \tag{2.1}$$

(see Proposition 2.2.9 and Remark 2.2.10), a set which has finite s-perimeter in Ω has also locally finite s-perimeter. However the converse, in general, is false.
When Ω is not bounded it is clear that also for sets of locally finite s-perimeter the sup in (2.1) may be infinite (consider, e.g., $\Omega = \mathbb{R}^n$ and $E = \{x_n \leq 0\}$).

Actually, as shown in Remark 2.2.11, this may happen even when Ω is bounded and has Lipschitz boundary. Roughly speaking, this is because the set E might oscillate more and more as it approaches the boundary $\partial\Omega$.

2.1.2. Approximation by smooth open sets. We denote by $N_\varrho(\Gamma)$ the ϱ-neighborhood of a set $\Gamma \subseteq \mathbb{R}^n$, that is

$$N_\varrho(\Gamma) := \{x \in \mathbb{R}^n \,|\, d(x, \Gamma) < \varrho\}.$$

The main approximation result is the following. In particular it shows that open sets with smooth boundary are dense in the family of sets of locally finite s-perimeter.

THEOREM 2.1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. A set $E \subseteq \mathbb{R}^n$ has locally finite s-perimeter in Ω if and only if there exists a sequence $E_h \subseteq \mathbb{R}^n$ of open sets with smooth boundary and $\varepsilon_h \longrightarrow 0^+$ such that*

$$(i) \quad E_h \xrightarrow{loc} E, \qquad \sup_{h \in \mathbb{N}} \operatorname{Per}_s(E_h, \Omega') < \infty \quad \textit{for every } \Omega' \Subset \Omega,$$

$$(ii) \quad \lim_{h \to \infty} \operatorname{Per}_s(E_h, \Omega') = \operatorname{Per}_s(E, \Omega') \quad \textit{for every } \Omega' \Subset \Omega,$$

$$(iii) \quad \partial E_h \subseteq N_{\varepsilon_h}(\partial E).$$

Moreover, if $\Omega = \mathbb{R}^n$ and the set E is such that $|E| < \infty$ and $\operatorname{Per}_s(E) < \infty$, then

$$E_h \longrightarrow E, \qquad \lim_{h \to \infty} \operatorname{Per}_s(E_h) = \operatorname{Per}_s(E), \tag{2.2}$$

and we can require each set E_h to be bounded (instead of asking (iii)).

We recall that, as we have observed in Section 0.2.1.3, such a result is well known for Caccioppoli sets (see, e.g., [79]) and indeed this density property can be used to define the (classical) perimeter functional as the relaxation (with respect to L^1_{loc} convergence) of the $\mathcal{H}^{n-1}$ measure of boundaries of smooth open sets, that is

$$\operatorname{Per}(E,\Omega)=\inf\Big\{\liminf_{k\to\infty}\mathcal{H}^{n-1}(\partial E_h\cap\Omega)\,\Big|\,E_h\subseteq\mathbb{R}^n \text{ open with smooth boundary, s.t. } E_h\xrightarrow{loc}E\Big\}. \tag{2.3}$$

The scheme of the proof of Theorem 2.1.1 is the following.

First of all, in Section 2.3.1 we prove appropriate approximation results for the functional

$$\mathcal{E}(u,\Omega)=\frac{1}{2}\iint_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2}\frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy,$$

which we believe might be interesting on their own.

Then we exploit the generalized coarea formula

$$\mathcal{E}(u,\Omega)=\int_{-\infty}^{\infty}\operatorname{Per}_s(\{u>t\},\Omega)\,dt,$$

and Sard's Theorem to obtain the approximation of the set E by superlevel sets of smooth functions which approximate χ_E.

Finally, a diagonal argument guarantees the convergence of the s-perimeters in every open set $\Omega'\Subset\Omega$.

REMARK 2.1.2. Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set with Lipschitz boundary and consider a set E which has finite s-perimeter in Ω. Notice that if we apply Theorem 2.1.1, in point (ii) we do not get the convergence of the s-perimeters in Ω, but only in every $\Omega'\Subset\Omega$. On the other hand, if we can find an open set $\mathcal{O}$ such that $\Omega\Subset\mathcal{O}$ and

$$\operatorname{Per}_s(E,\mathcal{O})<\infty,$$

then we can apply Theorem 2.1.1 in $\mathcal{O}$. In particular, since $\Omega\Subset\mathcal{O}$, by point (ii) we obtain

$$\lim_{h\to\infty}\operatorname{Per}_s(E_h,\Omega)=\operatorname{Per}_s(E,\Omega). \tag{2.4}$$

Still, when Ω is a bounded open set with Lipschitz boundary, we can always obtain the convergence (2.4) at the cost of weakening a little our request on the uniform convergence of the boundaries.

THEOREM 2.1.3. *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set with Lipschitz boundary. A set $E\subseteq\mathbb{R}^n$ has finite s-perimeter in Ω if and only if there exists a sequence $\{E_h\}$ of open sets with smooth boundary and $\varepsilon_h\longrightarrow 0^+$ such that*

$$\begin{aligned}
&(i)\quad E_h\xrightarrow{loc}E,\qquad \sup_{h\in\mathbb{N}}\operatorname{Per}_s(E_h,\Omega)<\infty,\\
&(ii)\quad \lim_{h\to\infty}\operatorname{Per}_s(E_h,\Omega)=\operatorname{Per}_s(E,\Omega),\\
&(iii)\quad \partial E_h\setminus N_{\varepsilon_h}(\partial\Omega)\subseteq N_{\varepsilon_h}(\partial E).
\end{aligned}$$

Notice that in point (iii) we do not ask the convergence of the boundaries in the whole of $\mathbb{R}^n$ but only in $\mathbb{R}^n\setminus N_\delta(\partial\Omega)$ (for any fixed $\delta>0$). Since $N_{\varepsilon_h}(\partial\Omega)\searrow\partial\Omega$, roughly speaking, the convergence holds in $\mathbb{R}^n$ "in the limit".

Moreover, we remark that point (ii) in Theorem 2.1.3 guarantees the convergence of the s-perimeters also in every $\Omega'\Subset\Omega$ (see Remark 2.3.6).

Finally, from the lower semicontinuity of the s-perimeter and Theorem 2.1.3, we obtain

COROLLARY 2.1.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $E \subseteq \mathbb{R}^n$. Then*

$$\mathrm{Per}_s(E,\Omega) = \inf\Big\{ \liminf_{h\to\infty} \mathrm{Per}_s(E_h,\Omega) \,\Big|\, E_h \subseteq \mathbb{R}^n \text{ open with smooth boundary, s.t. } E_h \xrightarrow{loc} E \Big\}.$$

For similar approximation results see also [28] and [30].

It is interesting to observe that in [47] the authors have proved, by exploiting the divergence Theorem, that if $E \subseteq \mathbb{R}^n$ is a bounded open set with smooth boundary, then

$$\mathrm{Per}_s(E) = c_{n,s} \int_{\partial E}\int_{\partial E} \frac{2 - |\nu_E(x) - \nu_E(y)|^2}{|x-y|^{n+s-2}} d\mathcal{H}^{n-1}_x d\mathcal{H}^{n-1}_y, \tag{2.5}$$

where ν_E denotes the external normal of E and

$$c_{n,s} := \frac{1}{2s(n+s-2)}.$$

Notice that in order to consider the right hand side of (2.5), we need the boundary of the set E to be at least locally $(n-1)$-rectifiable, so that the Hausdorff dimension of ∂E is $n-1$ and E has a well defined normal vector at $\mathcal{H}^{n-1}$-a.e. $x \in \partial E$. Therefore, the equality (2.5) cannot hold true for a generic set E having finite s-perimeter, since, as remarked in the beginning of the Introduction, such a set could have a nowhere rectifiable boundary.

Nevertheless, as a consequence of the equality (2.5), of the lower semicontinuity of the s-perimeter and of Theorem 2.1.1, we obtain the following Corollary, which can be thought of as an analogue of (2.3) in the fractional setting.

COROLLARY 2.1.5. *Let $E \subseteq \mathbb{R}^n$ be such that $|E| < \infty$. Then*

$$\mathrm{Per}_s(E) = \inf\Big\{ \liminf_{h\to\infty} c_{n,s} \int_{\partial E_h}\int_{\partial E_h} \frac{2 - |\nu_{E_h}(x) - \nu_{E_h}(y)|^2}{|x-y|^{n+s-2}} d\mathcal{H}^{n-1}_x d\mathcal{H}^{n-1}_y \,\Big|\, E_h \subseteq \mathbb{R}^n \text{ bounded open set with smooth boundary, s.t. } E_h \xrightarrow{loc} E \Big\}.$$

2.1.3. Nonlocal minimal surfaces. First of all we recall the definition of (locally) s-minimal sets.

DEFINITION 2.1.6. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $s \in (0,1)$. We say that a set $E \subseteq \mathbb{R}^n$ is* s-minimal *in Ω if $\mathrm{Per}_s(E,\Omega) < \infty$ and*

$$F \setminus \Omega = E \setminus \Omega \quad \Longrightarrow \quad \mathrm{Per}_s(E,\Omega) \le \mathrm{Per}_s(F,\Omega).$$

We say that a set $E \subseteq \mathbb{R}^n$ is locally s-minimal *in Ω if it is s-minimal in every open subset $\Omega' \Subset \Omega$.*

When the open set $\Omega \subseteq \mathbb{R}^n$ is bounded and has Lipschitz boundary, the notions of s-minimal set and locally s-minimal set coincide.

THEOREM 2.1.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $E \subseteq \mathbb{R}^n$. The following are equivalent:*

(i) E is s-minimal in Ω;

(ii) $\mathrm{Per}_s(E,\Omega) < \infty$ and

$$\mathrm{Per}_s(E,\Omega) \le \mathrm{Per}_s(F,\Omega) \qquad \text{for every } F \subseteq \mathbb{R}^n \quad \text{s.t.} \quad E\Delta F \Subset \Omega;$$

(iii) E is locally s-minimal in Ω.

We remark that a set as in (ii) is called a local minimizer for $\mathrm{Per}_s(-,\Omega)$ in [5] and a "nonlocal area minimizing surface" in Ω in [36].

REMARK 2.1.8. The implications $(i) \implies (ii) \implies (iii)$ actually hold in any open set $\Omega \subseteq \mathbb{R}^n$.

In [21] the authors proved that if Ω is a bounded open set with Lipschitz boundary, then given any fixed set $E_0 \subseteq \mathbb{R}^n$ we can find a set E which is s-minimal in Ω and such that $E \setminus \Omega = E_0 \setminus \Omega$.

This is because

$$\mathrm{Per}_s(E_0 \setminus \Omega, \Omega) \le \mathrm{Per}_s(\Omega) < \infty,$$

so the exterior datum $E_0 \setminus \Omega$ is itself an admissible competitor with finite s-perimeter in Ω and we can use the direct method of the Calculus of Variations to obtain a minimizer.

In Section 2.2.3 we prove a compactness property which we use in Section 2.4.3 to prove the following existence results, which extend that of [21].

THEOREM 2.1.9. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $E_0 \subseteq \mathbb{R}^n$. Then there exists a set $E \subseteq \mathbb{R}^n$ s-minimal in Ω, with $E \setminus \Omega = E_0 \setminus \Omega$, if and only if there exists a set $F \subseteq \mathbb{R}^n$, with $F \setminus \Omega = E_0 \setminus \Omega$ and such that $\mathrm{Per}_s(F,\Omega) < \infty$.*

An immediate consequence of this Theorem is the existence of s-minimal sets in open sets having finite s-perimeter.

COROLLARY 2.1.10. *Let $s \in (0,1)$ and let $\Omega \subseteq \mathbb{R}^n$ be an open set such that*

$$\mathrm{Per}_s(\Omega) < \infty.$$

Then for every $E_0 \subseteq \mathbb{R}^n$ there exists a set $E \subseteq \mathbb{R}^n$ s-minimal in Ω, with $E \setminus \Omega = E_0 \setminus \Omega$.

Even if we cannot find a competitor with finite s-perimeter, we can always find a locally s-minimal set.

COROLLARY 2.1.11. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $E_0 \subseteq \mathbb{R}^n$. Then there exists a set $E \subseteq \mathbb{R}^n$ locally s-minimal in Ω, with $E \setminus \Omega = E_0 \setminus \Omega$.*

In Section 2.4.2 we also prove compactness results for (locally) s-minimal sets (by slightly modifying the proof of [21, Theorem 3.3], which proved compactness for s-minimal sets in a ball). Namely, we prove that every limit set of a sequence of (locally) s-minimal sets is itself (locally) s-minimal.

THEOREM 2.1.12. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $\{E_k\}$ be a sequence of s-minimal sets in Ω, with $E_k \xrightarrow{loc} E$. Then E is s-minimal in Ω and*

$$\mathrm{Per}_s(E,\Omega) = \lim_{k\to\infty} \mathrm{Per}_s(E_k,\Omega). \tag{2.6}$$

COROLLARY 2.1.13. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $\{E_h\}$ be a sequence of sets locally s-minimal in Ω, with $E_h \xrightarrow{loc} E$. Then E is locally s-minimal in Ω and*

$$\mathrm{Per}_s(E,\Omega') = \lim_{h\to\infty} \mathrm{Per}_s(E_h,\Omega'), \qquad \text{for every } \Omega' \Subset \Omega. \tag{2.7}$$

2.1.3.1. *Minimal sets in cylinders.* We have seen in Corollary 2.1.11 that a locally s-minimal set always exists, no matter what the domain Ω or the exterior data $E_0 \setminus \Omega$ are.

On the other hand, by Theorem 2.1.9 we know that the only requirement needed for the existence of an s-minimal set is the existence of a competitor with finite s-perimeter. We show that even in the case of a regular domain, like the cylinder $\Omega^\infty := \Omega \times \mathbb{R}$, with $\Omega \subseteq \mathbb{R}^n$ bounded with $C^{1,1}$ boundary, such a competitor might not exist. Roughly speaking, this is a consequence of the unboundedness of the domain Ω^∞, which forces the nonlocal part of the s-perimeter to be infinite.

In Section 2.5 we study (locally) s-minimal sets in Ω^∞, with respect to the exterior data given by the subgraph of a function v, that is

$$\mathcal{S}g(v) := \big\{(x,t) \in \mathbb{R}^{n+1} \,|\, t < v(x)\big\}.$$

In particular, we consider sets which are s-minimal in the "truncated" cylinders

$$\Omega^k := \Omega \times (-k,k),$$

showing that if the function v is locally bounded, then these s-minimal sets cannot "oscillate" too much. Namely their boundaries are constrained in a cylinder $\Omega \times (-M,M)$ independently on k. As a consequence, we can find k_0 big enough such that a set E is locally s-minimal in Ω^∞ if and only if it is s-minimal in Ω^{k_0} (see Lemma 2.5.2 and Proposition 2.5.3 for the precise statements).

However, in general a set s-minimal in Ω^∞ does not exist. As an example we prove that there cannot exist an s-minimal set having as exterior data the subgraph of a bounded function.

Frst of all, we recall that we can write the fractional perimeter as the sum

$$\operatorname{Per}_s(E,\Omega) = \operatorname{Per}_s^L(E,\Omega) + \operatorname{Per}_s^{NL}(E,\Omega),$$

where

$$\begin{aligned}\operatorname{Per}_s^L(E,\Omega) &:= \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\cap\Omega) = \frac{1}{2}[\chi_E]_{W^{s,1}(\Omega)},\\ \operatorname{Per}_s^{NL}(E,\Omega) &:= \mathcal{L}_s(E\cap\Omega, \mathcal{C}E\setminus\Omega) + \mathcal{L}_s(E\setminus\Omega, \mathcal{C}E\cap\Omega).\end{aligned}$$

We can think of $\operatorname{Per}_s^L(E,\Omega)$ as the local part of the fractional perimeter, in the sense that if $|(E\Delta F)\cap\Omega| = 0$, then $\operatorname{Per}_s^L(F,\Omega) = \operatorname{Per}_s^L(E,\Omega)$.

The main result of Section 2.5 is the following:

THEOREM 2.1.14. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Let $E \subseteq \mathbb{R}^{n+1}$ be such that*

$$\Omega \times (-\infty,-k] \subseteq E \cap \Omega^\infty \subseteq \Omega \times (-\infty,k], \tag{2.8}$$

for some $k \in \mathbb{N}$, and suppose that $\operatorname{Per}_s(E,\Omega^{k+1}) < \infty$. Then

$$\operatorname{Per}_s^L(E,\Omega^\infty) < \infty.$$

On the other hand, if

$$\{x_{n+1} \le -k\} \subseteq E \subseteq \{x_{n+1} \le k\}, \tag{2.9}$$

then

$$\operatorname{Per}_s^{NL}(E,\Omega^\infty) = \infty.$$

In particular, if Ω has $C^{1,1}$ boundary and $v \in L^\infty(\mathbb{R}^n)$, there cannot exist an s-minimal set in Ω^∞ with exterior data

$$\mathcal{S}g(v) \setminus \Omega^\infty = \{(x,t) \in \mathbb{R}^{n+1} \,|\, x \in \mathcal{C}\Omega, \quad t < v(x)\}.$$

REMARK 2.1.15. From Theorem 2.1.9 we see that if $v \in L^\infty(\mathbb{R}^n)$, there cannot exist a set $E \subseteq \mathbb{R}^{n+1}$ such that $E \setminus \Omega^\infty = \mathcal{S}g(v) \setminus \Omega^\infty$ and $\operatorname{Per}_s(E,\Omega^\infty) < \infty$.

As a consequence of the computations developed in the proof of Theorem 2.1.14, in the end of Section 2.5 we also show that we cannot define a "naive" fractional nonlocal version of the area functional as

$$\mathscr{A}_s(u,\Omega) := \operatorname{Per}_s(\mathcal{S}g(u),\Omega^\infty),$$

since this would be infinite even for very regular functions.

To conclude, we remark that as an immediate consequence of Corollary 2.1.11 and [43, Theorem 1.1], we obtain an existence result for the Plateau's problem in the class of subgraphs.

THEOREM 2.1.16. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with $C^{1,1}$ boundary. For every function $v \in C(\mathbb{R}^n)$ there exists a function $u \in C(\overline{\Omega})$ such that, if*

$$\tilde{u} := \chi_\Omega u + (1 - \chi_\Omega)v,$$

then $\mathcal{S}g(\tilde{u})$ is locally s-minimal in Ω^∞.

Notice that, as remarked in [43], the function $\tilde{u}$ need not be continuous. Indeed, because of boundary stickiness effects of s-minimal surfaces (see, e.g., [45]), in general we might have

$$u_{|\partial\Omega} \neq v_{|\partial\Omega}.$$

2.2. Tools

We collect here some auxiliary results that we will exploit in the following sections. We begin by pointing out the following easy but useful result.

PROPOSITION 2.2.1. *Let $\Omega' \subseteq \Omega \subseteq \mathbb{R}^n$ be open sets and let $E \subseteq \mathbb{R}^n$. Then*

$$\begin{aligned}\mathrm{Per}_s(E,\Omega) = \mathrm{Per}_s(E,\Omega') &+ \mathcal{L}_s\big(E\cap(\Omega\setminus\Omega'),\mathcal{C}E\setminus\Omega\big) + \mathcal{L}_s\big(E\setminus\Omega,\mathcal{C}E\cap(\Omega\setminus\Omega')\big)\\ &+ \mathcal{L}_s\big(E\cap(\Omega\setminus\Omega'),\mathcal{C}E\cap(\Omega\setminus\Omega')\big).\end{aligned}$$

As a consequence,

(i) if $E \subseteq \Omega$, then

$$\mathrm{Per}_s(E,\Omega) = \mathrm{Per}_s(E),$$

(ii) if E, $F \subseteq \mathbb{R}^n$ have finite s-perimeter in Ω and $E\Delta F \subseteq \Omega' \subseteq \Omega$, then

$$\mathrm{Per}_s(E,\Omega) - \mathrm{Per}_s(F,\Omega) = \mathrm{Per}_s(E,\Omega') - \mathrm{Per}_s(F,\Omega').$$

REMARK 2.2.2. In particular, if E has finite s-perimeter in Ω, then it has finite s-perimeter also in every open set $\Omega' \subseteq \Omega$.

2.2.1. Bounded open sets with Lipschitz boundary. It is convenient to recall here some notation and results concerning the signed distance function, since we will make extensive use of such results in the subsequent sections.

Given a set $E \subseteq \mathbb{R}^n$, with $E \neq \varnothing$, the distance function from E is defined as

$$d_E(x) = d(x,E) := \inf_{y\in E}|x-y|, \qquad \text{for } x \in \mathbb{R}^n.$$

The signed distance function from ∂E, negative inside E, is then defined as

$$\bar{d}_E(x) = \bar{d}(x,E) := d(x,E) - d(x,\mathcal{C}E).$$

We also define for every $r \in \mathbb{R}$ the sets

$$E_r := \{x \in \mathbb{R}^n \,|\, \bar{d}_E(x) < r\}.$$

Notice that if $\varrho > 0$, then

$$N_\varrho(\partial\Omega) = \{|\bar{d}_\Omega| < \varrho\} = \Omega_\varrho \setminus \overline{\Omega_{-\varrho}}$$

is the ϱ-tubular neighborhood of $\partial\Omega$.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. It is well known (see, e.g., [48, Theorem 4.1]) that also the bounded open sets Ω_r have Lipschitz boundary, when r is small enough, say $|r| < r_0$. Also notice that

$$\partial\Omega_r = \{\bar{d}_\Omega = r\}.$$

Moreover the perimeter of Ω_r can be bounded uniformly in $r \in (-r_0, r_0)$ (see also Appendix B.1 for a more detailed discussion)

PROPOSITION 2.2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then there exists $r_0 > 0$ such that Ω_r is a bounded open set with Lipschitz boundary for every $r \in (-r_0, r_0)$ and*

$$\sup_{|r|<r_0} \mathcal{H}^{n-1}(\{\bar{d}_\Omega = r\}) < \infty. \tag{2.10}$$

As a consequence, exploiting the embedding $BV(\mathbb{R}^n) \hookrightarrow W^{s,1}(\mathbb{R}^n)$ we obtain a uniform bound for the (global) s-perimeters of the sets Ω_r (see Corollary 1.2.2).

COROLLARY 2.2.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then there exists $r_0 > 0$ such that*

$$\sup_{|r|<r_0} \mathrm{Per}_s(\Omega_r) < \infty. \tag{2.11}$$

2.2.1.1. *Increasing sequences.* In particular, Proposition 2.2.3 shows that if Ω is a bounded open set with Lpschitz boundary, then we can approximate it strictly from the inside with a sequence of bounded open sets $\Omega_k := \Omega_{-1/k} \Subset \Omega$. Moreover, (2.10) gives a uniform bound on the measure of the boundaries of the approximating sets.

Now we prove that any open set $\Omega \neq \varnothing$ can be approximated strictly from the inside with a sequence of bounded open sets with smooth boundaries.

PROPOSITION 2.2.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. For every $\varepsilon > 0$ there exists a bounded open set $\mathcal{O}_\varepsilon \subseteq \mathbb{R}^n$ with smooth boundary, such that*

$$\mathcal{O}_\varepsilon \Subset \Omega \qquad \textit{and} \qquad \partial\mathcal{O}_\varepsilon \subseteq N_\varepsilon(\partial\Omega).$$

PROOF. We show that we can approximate the set $\Omega_{-\varepsilon/2}$ with a bounded open set $\mathcal{O}_\varepsilon$ with smooth boundary such that $\partial\mathcal{O}_\varepsilon \subseteq N_{\varepsilon/4}(\partial\Omega_{-\varepsilon/2})$.
In general $\mathcal{O}_\varepsilon \not\subseteq \Omega_{-\varepsilon/2}$. However

$$\mathcal{O}_\varepsilon \subseteq N_{\varepsilon/4}(\Omega_{-\varepsilon/2}) \Subset \Omega \quad \text{and indeed} \quad \Omega_{-3\varepsilon/4} \subseteq \mathcal{O}_\varepsilon \subseteq \Omega_{-\varepsilon/4}, \tag{2.12}$$

proving the claim.

Let $u := \chi_{\Omega_{-\varepsilon/2}}$ and consider the regularized function

$$v := u_{\varepsilon/4} = u * \eta_{\varepsilon/4}$$

(see Section 2.3.1 for the details about the mollifier η_ε). Since $v \in C^\infty(\mathbb{R}^n)$, we know from Sard's Theorem that the superlevel set $\{v > t\}$ is an open set with smooth boundary for a.e. $t \in (0, 1)$. Moreover notice that $0 \le v \le 1$, with

$$\operatorname{supp} v \subseteq N_{\varepsilon/4}(\operatorname{supp} u) = N_{\varepsilon/4}(\Omega_{-\varepsilon/2}) \subseteq \Omega_{-\varepsilon/4},$$

and

$$v(x) = 1 \qquad \text{for every } x \in \left\{ y \in \Omega_{-\varepsilon/2} \,\middle|\, d(y, \partial\Omega_{-\varepsilon/2}) > \frac{\varepsilon}{4} \right\} \supseteq \Omega_{-\frac{3}{4}\varepsilon}.$$

This shows that $\mathcal{O}_\varepsilon := \{v > t\}$ (for any "regular" t) satisfies (2.12). □

COROLLARY 2.2.6. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Then there exists a sequence $\{\Omega_k\}$ of bounded open sets with smooth boundary such that $\Omega_k \nearrow \Omega$ strictly, i.e.*

$$\Omega_k \Subset \Omega_{k+1} \Subset \Omega \qquad \textit{and} \qquad \bigcup_{k\in\mathbb{N}} \Omega_k = \Omega.$$

In particular $\Omega_k \xrightarrow{loc} \Omega$.

Proof. It is enough to notice that we can approximate Ω strictly from the inside with bounded open sets $\mathcal{O}_k \subseteq \mathbb{R}^n$, that is

$$\mathcal{O}_k \Subset \mathcal{O}_{k+1} \Subset \Omega \qquad \text{and} \qquad \bigcup_{k\in\mathbb{N}} \mathcal{O}_k = \Omega.$$

Then we can exploit Proposition 2.2.5, and in particular (2.12), to find bounded open sets $\Omega_k \subseteq \mathbb{R}^n$ with smooth boundary such that

$$\mathcal{O}_k \Subset \Omega_k \Subset \mathcal{O}_{k+1}.$$

Indeed we can take as Ω_k a set $\mathcal{O}_\varepsilon$ corresponding to $\mathcal{O}_{k+1}$, with ε small enough to guarantee $\mathcal{O}_k \Subset \mathcal{O}_\varepsilon$.
As for the sets $\mathcal{O}_k$, if Ω is bounded we can simply take $\mathcal{O}_k := \Omega_{-2^{-k}}$. If Ω is not bounded, we can consider the sets $\Omega \cap B_{2^k}$ and define

$$\mathcal{O}_k := \big\{x \in \Omega \cap B_{2^k} \,|\, d\big(x, \partial(\Omega \cap B_{2^k})\big) > 2^{-k}\big\}.$$

To conclude, notice that we have $\chi_{\Omega_k} \longrightarrow \chi_\Omega$ pointwise everywhere in $\mathbb{R}^n$, which implies the convergence in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. □

2.2.1.2. *Some uniform estimates for ϱ-neighborhoods.* The uniform bound (2.10) on the perimeters of the sets Ω_δ allows us to obtain the following estimates, which will be used in the sequel.

Lemma 2.2.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $\delta \in (0, r_0)$. Then*

$$\begin{aligned} &(i) \quad \mathcal{L}_s(\Omega_{-\delta}, \Omega \setminus \Omega_{-\delta}) \le C\,\delta^{1-s}, \\ &(ii) \quad \mathcal{L}_s(\Omega, \Omega_\delta \setminus \Omega) \le C\,\delta^{1-s} \quad \textit{and} \quad \mathcal{L}_s(\Omega \setminus \Omega_{-\delta}, \mathcal{C}\Omega) \le C\,\delta^{1-s}, \end{aligned} \tag{2.13}$$

where the constant C is

$$C := \frac{n\omega_n}{s(1-s)} \sup_{|r|<r_0} \mathcal{H}^{n-1}(\{\bar{d}_\Omega = r\}).$$

Proof. By using the coarea formula for $\bar{d}_\Omega$ and exploiting (2.10), we get

$$\begin{aligned} \mathcal{L}_s(\Omega_{-\delta}, \Omega \setminus \Omega_{-\delta}) &= \int_{-\delta}^0 \Big(\int_{\{\bar{d}_\Omega = \varrho\}} \Big(\int_{\Omega_{-\delta}} \frac{dx}{|x-y|^{n+s}} \Big) d\mathcal{H}^{n-1}_y \Big) d\varrho \\ &\le \int_{-\delta}^0 \Big(\int_{\{\bar{d}_\Omega = \varrho\}} \Big(\int_{\mathcal{C}B_{\varrho+\delta}(y)} \frac{dx}{|x-y|^{n+s}} \Big) d\mathcal{H}^{n-1}_y \Big) d\varrho \\ &= \frac{n\omega_n}{s} \int_{-\delta}^0 \frac{\mathcal{H}^{n-1}(\{\bar{d}_\Omega = \varrho\})}{(\varrho+\delta)^s}\, d\varrho \\ &\le M \frac{n\omega_n}{s(1-s)} \int_{-\delta}^0 \frac{d}{d\varrho}(\varrho+\delta)^{1-s}\, d\varrho = M \frac{n\omega_n}{s(1-s)}\, \delta^{1-s}. \end{aligned}$$

In the same way we obtain point (ii),

$$\begin{aligned}\mathcal{L}_s(\Omega_\delta\setminus\Omega,\Omega)&=\int_0^\delta\Big(\int_{\{\bar{d}_\Omega=\varrho\}}\Big(\int_\Omega\frac{dx}{|x-y|^{n+s}}\Big)d\mathcal{H}^{n-1}_y\Big)d\varrho\\&\le\int_0^\delta\Big(\int_{\{\bar{d}_\Omega=\varrho\}}\Big(\int_{\mathcal{C}B_\varrho(y)}\frac{dx}{|x-y|^{n+s}}\Big)d\mathcal{H}^{n-1}_y\Big)d\varrho\\&=\frac{n\omega_n}{s}\int_0^\delta\frac{\mathcal{H}^{n-1}(\{\bar{d}_\Omega=\varrho\})}{\varrho^s}\,d\varrho\\&\le M\frac{n\omega_n}{s(1-s)}\int_0^\delta\frac{d}{d\varrho}\varrho^{1-s}\,d\varrho=M\frac{n\omega_n}{s(1-s)}\,\delta^{1-s},\end{aligned}$$

(the other estimate in point (ii) is analogous). □

2.2.2. (Semi)continuity of the s-perimeter. As shown in [21, Theorem 3.1], Fatou's Lemma gives the lower semicontinuity of the functional $\mathcal{L}_s$.

PROPOSITION 2.2.8. *Suppose*

$$A_k\xrightarrow{loc}A\qquad\text{and}\qquad B_k\xrightarrow{loc}B.$$

Then

$$\mathcal{L}_s(A,B)\le\liminf_{k\to\infty}\mathcal{L}_s(A_k,B_k). \tag{2.14}$$

In particular, if

$$E_k\xrightarrow{loc}E\qquad\text{and}\qquad\Omega_k\xrightarrow{loc}\Omega,$$

then

$$\operatorname{Per}_s(E,\Omega)\le\liminf_{k\to\infty}\operatorname{Per}_s(E_k,\Omega_k).$$

PROOF. If the right hand side of (2.14) is infinite, we have nothing to prove, so we can suppose that it is finite. By definition of the liminf, we can find $k_i\nearrow\infty$ such that

$$\lim_{i\to\infty}\mathcal{L}_s(A_{k_i},B_{k_i})=\liminf_{k\to\infty}\mathcal{L}_s(A_k,B_k)=:I.$$

Since $\chi_{A_{k_i}}\to\chi_A$ and $\chi_{B_{k_i}}\to\chi_B$ in $L^1_{\text{loc}}(\mathbb{R}^n)$, up to passing to a subsequence we can suppose that

$$\chi_{A_{k_i}}\longrightarrow\chi_A\qquad\text{and}\qquad\chi_{B_{k_i}}\longrightarrow\chi_B\qquad\text{a.e. in }\mathbb{R}^n.$$

Then, since

$$\mathcal{L}_s(A_{k_i},B_{k_i})=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n+s}}\chi_{A_{k_i}}(x)\chi_{B_{k_i}}(y)\,dx\,dy,$$

Fatou's Lemma gives

$$\mathcal{L}_s(A,B)\le\liminf_{i\to\infty}\mathcal{L}_s(A_{k_i},B_{k_i})=I,$$

proving (2.14).

The second inequality follows just by summing the contributions defining the fractional perimeter. □

Keeping Ω fixed we obtain [21, Theorem 3.1].

On the other hand, if we keep the set E fixed and approximate the open set Ω with a sequence of open subsets $\Omega_k\subseteq\Omega$, we get a continuity property.

PROPOSITION 2.2.9. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $\{\Omega_k\}$ be any sequence of open sets such that $\Omega_k \xrightarrow{loc} \Omega$. Then for every set $E \subseteq \mathbb{R}^n$*

$$\mathrm{Per}_s(E, \Omega) \le \liminf_{k\to\infty} \mathrm{Per}_s(E, \Omega_k).$$

Moreover, if $\Omega_k \subseteq \Omega$ for every k, then

$$\mathrm{Per}_s(E, \Omega) = \lim_{k\to\infty} \mathrm{Per}_s(E, \Omega_k), \tag{2.15}$$

(whether it is finite or not).

PROOF. Since $\Omega_k \xrightarrow{loc} \Omega$, Proposition 2.2.8 gives the first statement. Now notice that if $\Omega_k \subseteq \Omega$, Proposition 2.2.1 implies

$$\mathrm{Per}_s(E, \Omega_k) \le \mathrm{Per}_s(E, \Omega),$$

and hence

$$\limsup_{k\to\infty} \mathrm{Per}_s(E, \Omega_k) \le \mathrm{Per}_s(E, \Omega),$$

concluding the proof. □

REMARK 2.2.10. As a consequence, exploiting Corollary 2.2.6, we get

$$\mathrm{Per}_s(E, \Omega) = \sup_{\Omega' \subsetneq \Omega} \mathrm{Per}_s(E, \Omega') = \sup_{\Omega' \Subset \Omega} \mathrm{Per}_s(E, \Omega').$$

REMARK 2.2.11. Consider the set $E \subseteq \mathbb{R}$ constructed in the proof of [40, Example 2.10]. That is, let $\beta_k > 0$ be a decreasing sequence such that

$$M := \sum_{k=1}^{\infty} \beta_k < \infty \quad \text{and} \quad \sum_{k=1}^{\infty} \beta_{2k}^{1-s} = \infty, \quad \forall\, s \in (0,1).$$

Then define

$$\sigma_m := \sum_{k=1}^{m} \beta_k, \qquad I_m := (\sigma_m, \sigma_{m+1}), \qquad E := \bigcup_{j=1}^{\infty} I_{2j},$$

and let $\Omega := (0, M)$. As shown in [40],

$$\mathrm{Per}_s(E, \Omega) = \infty, \qquad \forall\, s \in (0,1).$$

On the other hand

$$\mathrm{Per}(E, \Omega') < \infty, \qquad \forall\, \Omega' \Subset \Omega,$$

hence E has locally finite s-perimeter in Ω, for every $s \in (0,1)$.

Indeed, notice that the intervals I_{2j} accumulate near M. Thus, for every $\varepsilon > 0$, all but a finite number of the intervals I_{2j}'s fall outside of the open set $\mathcal{O}_\varepsilon := (\varepsilon, M - \varepsilon)$. Therefore $\mathrm{Per}(E, \mathcal{O}_\varepsilon) < \infty$ and hence

$$\mathrm{Per}_s(E, \mathcal{O}_\varepsilon) < \infty, \quad \forall\, s \in (0,1).$$

Since $\mathcal{O}_\varepsilon \nearrow \Omega$ as $\varepsilon \to 0^+$, the set E has locally finite s-perimeter in Ω for every $s \in (0,1)$.

PROPOSITION 2.2.12. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $\{E_h\}$ be a sequence of sets such that*

$$E_h \xrightarrow{loc} E \qquad \textit{and} \qquad \lim_{h\to\infty} \mathrm{Per}_s(E_h, \Omega) = \mathrm{Per}_s(E, \Omega) < \infty.$$

Then

$$\lim_{h\to\infty} \mathrm{Per}_s(E_h, \Omega') = \mathrm{Per}_s(E, \Omega') \qquad \textit{for every open set } \Omega' \subseteq \Omega.$$

PROOF. The claim follows from classical properties of limits of sequences. Indeed, let

$$\begin{aligned} a_h &:= \mathrm{Per}_s(E_h, \Omega'), \\ b_h &:= \mathcal{L}_s\big(E_h \cap (\Omega \setminus \Omega'), \mathcal{C}E_h \setminus \Omega\big) + \mathcal{L}_s\big(E_h \setminus \Omega, \mathcal{C}E_h \cap (\Omega \setminus \Omega')\big) \\ &\quad + \mathcal{L}_s\big(E_h \cap (\Omega \setminus \Omega'), \mathcal{C}E_h \cap (\Omega \setminus \Omega')\big), \end{aligned}$$

and let a and b be the corresponding terms for E.
Notice that, by Proposition 2.2.1, we have

$$\mathrm{Per}_s(E_h, \Omega) = a_h + b_h \qquad \text{and} \qquad \mathrm{Per}_s(E, \Omega) = a + b.$$

From Proposition 2.2.8 we have

$$a \le \liminf_{h\to\infty} a_h \qquad \text{and} \qquad b \le \liminf_{h\to\infty} b_h,$$

and by hypothesis we know that

$$\lim_{h\to\infty} (a_h + b_h) = a + b.$$

Therefore

$$a + b \le \liminf_{h\to\infty} a_h + \liminf_{h\to\infty} b_h \le \liminf_{h\to\infty} (a_h + b_h) = a + b,$$

and hence

$$0 \le \liminf_{h\to\infty} b_h - b = a - \liminf_{h\to\infty} a_h \le 0,$$

so that

$$a = \liminf_{h\to\infty} a_h \qquad \text{and} \qquad b = \liminf_{h\to\infty} b_h.$$

Then, since

$$\limsup_{h\to\infty} a_h + \liminf_{h\to\infty} b_h \le \limsup_{h\to\infty} (a_h + b_h) = a + b,$$

we obtain

$$a = \liminf_{h\to\infty} a_h \le \limsup_{h\to\infty} a_h \le a,$$

concluding the proof. □

2.2.3. Compactness.

PROPOSITION 2.2.13 (Compactness). *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. If $\{E_h\}$ is a sequence of sets such that*

$$\limsup_{h\to\infty} \mathrm{Per}_s^L(E_h, \Omega') \le c(\Omega') < \infty, \quad \forall\, \Omega' \Subset \Omega, \tag{2.16}$$

then there exists a subsequence $\{E_{h_i}\}$ and $E \subseteq \mathbb{R}^n$ such that

$$E_{h_i} \cap \Omega \xrightarrow{loc} E \cap \Omega.$$

PROOF. We want to use a compact Sobolev embedding (see, e.g., [38, Corollary 7.2]) to construct a limit set via a diagonal argument.

Thanks to Corollary 2.2.6 we know that we can find an increasing sequence of bounded open sets $\{\Omega_k\}$ with smooth boundary such that

$$\Omega_k \Subset \Omega_{k+1} \Subset \Omega \qquad \text{and} \qquad \bigcup_{k\in\mathbb{N}} \Omega_k = \Omega.$$

Moreover, hypothesis (2.16) guarantees that

$$\forall k \quad \exists h(k) \text{ s.t.} \quad \mathrm{Per}_s^L(E_h, \Omega_k) \le c_k < \infty, \quad \forall h \ge h(k). \tag{2.17}$$

Clearly

$$\|\chi_{E_h}\|_{L^1(\Omega_k)} \leq |\Omega_k| < \infty,$$

and hence, since $[\chi_{E_h}]_{W^{s,1}(\Omega_k)} = 2\,\mathrm{Per}_s^L(E_h,\Omega_k)$, we have

$$\|\chi_{E_h}\|_{W^{s,1}(\Omega_k)} \leq c'_k, \quad \forall h \geq h(k).$$

Therefore [38, Corollary 7.2] (notice that each Ω_k is an extension domain) guarantees for every fixed k the existence of a subsequence $h_i \nearrow \infty$ (with $h_1 \geq h(k)$) such that

$$E_{h_i} \cap \Omega_k \xrightarrow{i\to\infty} E^k$$

in measure, for some set $E^k \subseteq \Omega_k$.

Applying this argument for $k=1$ we get a subsequence $\{h_i^1\}$ with

$$E_{h_i^1} \cap \Omega_1 \xrightarrow{i\to\infty} E^1.$$

Applying again this argument in Ω_2, with $\{E_{h_i^1}\}$ in place of $\{E_h\}$, we get a subsequence $\{h_i^2\}$ of $\{h_i^1\}$ with

$$E_{h_i^2} \cap \Omega_2 \xrightarrow{i\to\infty} E^2.$$

Notice that, since $\Omega_1 \subseteq \Omega_2$, we must have $E^2 \cap \Omega_1 = E^1$ in measure (by the uniqueness of the limit in Ω_1). We can also suppose that $h_1^2 > h_1^1$.
Proceeding inductively in this way we get an increasing subsequence $\{h_1^k\}$ such that

$$E_{h_1^i} \cap \Omega_k \xrightarrow{i\to\infty} E^k, \qquad \text{for every } k \in \mathbb{N},$$

with $E^{k+1} \cap \Omega_k = E^k$. Therefore if we define $E := \bigcup_k E^k$, since $\bigcup_k \Omega_k = \Omega$, we get

$$E_{h_1^i} \cap \Omega \xrightarrow{loc} E,$$

concluding the proof. □

REMARK 2.2.14. If E_h is s-minimal in Ω_k for every $h \geq h(k)$, then by minimality we get

$$\mathrm{Per}_s^L(E_h,\Omega_k) \leq \mathrm{Per}_s(E_h,\Omega_k) \leq \mathrm{Per}_s(E_h \setminus \Omega_k,\Omega_k) \leq \mathrm{Per}_s(\Omega_k) =: c_k < \infty,$$

since Ω_k is bounded and has Lipschitz boundary. Therefore $\{E_h\}$ satisfies the hypothesis of Proposition 2.2.13 and we can find a convergent subsequence.

2.3. Generalized coarea and approximation by smooth sets

We begin by showing that the s-perimeter satisfies a generalized coarea formula (see also [99] and [5, Lemma 10]). In the end of this section we will exploit this formula to prove that a set E of locally finite s-perimeter can be approximated by smooth sets whose s-perimeter converges to that of E.

Let $\Omega \subseteq \mathbb{R}^n$ be an open set. Given a function $u : \mathbb{R}^n \longrightarrow \mathbb{R}$, we define the functional

$$\mathcal{E}(u,\Omega) := \frac{1}{2}\int_\Omega\int_\Omega \frac{|u(x)-u(y)|}{|x-y|^{n+s}}dx\,dy + \int_\Omega\int_{\mathcal{C}\Omega} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}dx\,dy, \tag{2.18}$$

that is, half the "Ω-contribution" to the $W^{s,1}$-seminorm of u.
Notice that

$$\mathcal{E}(\chi_E,\Omega) = \mathrm{Per}_s(E,\Omega)$$

and, clearly

$$\mathcal{E}(u,\mathbb{R}^n) = \frac{1}{2}[u]_{W^{s,1}(\mathbb{R}^n)}.$$

PROPOSITION 2.3.1 (Coarea). *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $u : \mathbb{R}^n \longrightarrow \mathbb{R}$. Then*

$$\mathcal{E}(u,\Omega) = \int_{-\infty}^{\infty} \mathrm{Per}_s(\{u>t\},\Omega)\, dt. \tag{2.19}$$

In particular

$$\frac{1}{2}[u]_{W^{s,1}(\Omega)} = \int_{-\infty}^{\infty} \mathrm{Per}_s^L(\{u>t\},\Omega)\, dt.$$

PROOF. Notice that for every x, $y \in \mathbb{R}^n$ we have

$$|u(x)-u(y)| = \int_{-\infty}^{\infty} |\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)|\, dt. \tag{2.20}$$

Indeed, the function $t \longmapsto |\chi_{\{u>t\}}(x) - \chi_{\{u>t\}}(y)|$ takes only the values $\{0,1\}$ and it is different from 0 precisely in the interval having $u(x)$ and $u(y)$ as extremes. Therefore, if we plug (2.20) into (2.18) and use Fubini's Theorem, we get

$$\mathcal{E}(u,\Omega) = \int_{-\infty}^{\infty} \mathcal{E}(\chi_{\{u>t\}},\Omega)\, dt = \int_{-\infty}^{\infty} \mathrm{Per}_s(\{u>t\},\Omega)\, dt,$$

as wanted. □

2.3.1. Approximation results for the functional $\mathcal{E}$. In this section we prove the approximation properties for the functional $\mathcal{E}$ which we need for the proofs of Theorem 2.1.1 and Theorem 2.1.3. To this end we consider a (symmetric) smooth function η such that

$$\eta \in C_c^\infty(\mathbb{R}^n), \quad \mathrm{supp}\,\eta \subseteq B_1, \quad \eta \geq 0, \quad \eta(-x) = \eta(x), \quad \int_{\mathbb{R}^n} \eta\, dx = 1,$$

and we define the mollifier

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n}\eta\Big(\frac{x}{\varepsilon}\Big),$$

for every $\varepsilon \in (0,1)$. Notice that $\mathrm{supp}\,\eta_\varepsilon \subseteq B_\varepsilon$ and $\int_{\mathbb{R}^n} \eta_\varepsilon = 1$.

Given $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we define the ε-regularization of u as the convolution

$$u_\varepsilon(x) := (u * \eta_\varepsilon)(x) = \int_{\mathbb{R}^n} u(x-\xi)\eta_\varepsilon(\xi)\, d\xi, \quad \text{for every } x \in \mathbb{R}^n.$$

It is well known that $u_\varepsilon \in C^\infty(\mathbb{R}^n)$ and

$$u_\varepsilon \longrightarrow u \qquad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

Moreover, if $u = \chi_E$, then

$$0 \leq u_\varepsilon \leq 1 \qquad \text{and} \quad u_\varepsilon(x) = \begin{cases} 1, & \text{if } |B_\varepsilon(x) \setminus E| = 0 \\ 0, & \text{if } |B_\varepsilon(x) \cap E| = 0 \end{cases}, \tag{2.21}$$

(see, e.g., [79, Section 12.3]).

LEMMA 2.3.2. (*i*) *Let $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and let $\Omega \subseteq \mathbb{R}^n$ be an open set. Then*

$$\mathcal{E}(u,\Omega) < \infty \quad \Longrightarrow \quad \lim_{\varepsilon\to 0^+} \mathcal{E}(u_\varepsilon,\Omega') = \mathcal{E}(u,\Omega') \qquad \forall\, \Omega' \Subset \Omega. \tag{2.22}$$

(*ii*) *Let $u \in W^{s,1}(\mathbb{R}^n)$. Then*

$$\lim_{\varepsilon\to 0}[u_\varepsilon]_{W^{s,1}(\mathbb{R}^n)} = [u]_{W^{s,1}(\mathbb{R}^n)}.$$

(*iii*) *Let $u \in W^{s,1}(\mathbb{R}^n)$. Then there exists $\{u_k\} \subseteq C_c^\infty(\mathbb{R}^n)$ such that*

$$\|u-u_k\|_{L^1(\mathbb{R}^n)} \longrightarrow 0 \quad \text{and} \quad \lim_{k\to\infty}[u_k]_{W^{s,1}(\mathbb{R}^n)} = [u]_{W^{s,1}(\mathbb{R}^n)}.$$

Moreover, if $u = \chi_E$, then $0 \leq u_k \leq 1$.

PROOF. (i) Given $\mathcal{O}\subseteq\mathbb{R}^n$, let $Q(\mathcal{O}):=\mathbb{R}^{2n}\setminus(\mathcal{C}\mathcal{O})^2$, so that

$$\mathcal{E}(u,\mathcal{O})=\frac{1}{2}\iint_{Q(\mathcal{O})}\frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy.$$

Notice that if $\mathcal{O}\subseteq\Omega$, then $Q(\mathcal{O})\subseteq Q(\Omega)$ and hence

$$\mathcal{E}(u,\mathcal{O})\leq\mathcal{E}(u,\Omega). \tag{2.23}$$

Now let $\Omega'\Subset\Omega$ and notice that for ε small enough we have

$$Q(\Omega'-\varepsilon\xi)\subseteq Q(\Omega)\qquad\text{for every }\xi\in B_1. \tag{2.24}$$

As a consequence

$$\mathcal{E}(u_\varepsilon,\Omega')\leq\int_{B_1}\mathcal{E}(u,\Omega'-\varepsilon\xi)\eta(\xi)\,d\xi\leq\mathcal{E}(u,\Omega). \tag{2.25}$$

The second inequality follows from (2.24), (2.23) and $\int_{B_1}\eta=1$.
As for the first inequality, we have

$$\begin{aligned}
&\iint_{Q(\Omega')}\frac{|u_\varepsilon(x)-u_\varepsilon(y)|}{|x-y|^{n+s}}dx\,dy\\
&\quad=\iint_{Q(\Omega')}\Big|\int_{\mathbb{R}^n}\big(u(x-\xi)-u(y-\xi)\big)\frac{1}{\varepsilon^n}\eta\Big(\frac{\xi}{\varepsilon}\Big)\,d\xi\Big|\frac{dx\,dy}{|x-y|^{n+s}}\\
&\quad=\iint_{Q(\Omega')}\Big|\int_{B_1}\big(u(x-\varepsilon\xi)-u(y-\varepsilon\xi)\big)\eta(\xi)\,d\xi\Big|\frac{dx\,dy}{|x-y|^{n+s}}\\
&\quad\leq\int_{B_1}\Big(\iint_{Q(\Omega')}\frac{|u(x-\varepsilon\xi)-u(y-\varepsilon\xi)|}{|x-y|^{n+s}}dx\,dy\Big)\eta(\xi)\,d\xi\\
&\quad=\int_{B_1}\Big(\iint_{Q(\Omega'-\varepsilon\xi)}\frac{|u(x)-u(y)|}{|x-y|^{n+s}}dx\,dy\Big)\eta(\xi)\,d\xi.
\end{aligned}$$

We prove something stronger than the claim, that is

$$\lim_{\varepsilon\to0^+}\mathcal{E}(u_\varepsilon-u,\Omega')=0. \tag{2.26}$$

Indeed, notice that

$$|\mathcal{E}(u_\varepsilon,\Omega')-\mathcal{E}(u,\Omega')|\leq\mathcal{E}(u_\varepsilon-u,\Omega').$$

Let $\psi:\mathbb{R}^{2n}\longrightarrow\mathbb{R}$ be defined as

$$\psi(x,y):=\frac{u(x)-u(y)}{|x-y|^{n+s}}.$$

Moreover, for every $\varepsilon>0$ and $\xi\in B_1$, we consider the left translation by $\varepsilon(\xi,\xi)$ in $\mathbb{R}^{2n}$, that is

$$(L_{\varepsilon\xi}f)(x,y):=f(x-\varepsilon\xi,y-\varepsilon\xi),$$

for every $f:\mathbb{R}^{2n}\longrightarrow\mathbb{R}$.
Since $\psi\in L^1(Q(\Omega))$, for every $\delta>0$ there exists $\Psi\in C^1_c(Q(\Omega))$ such that

$$\|\psi-\Psi\|_{L^1(Q(\Omega))}\leq\frac{\delta}{2}.$$

We have

$$\begin{aligned}\mathcal{E}(u_\varepsilon - u, \Omega') &= \iint_{Q(\Omega')} \frac{|u_\varepsilon(x) - u_\varepsilon(y) - u(x) + u(y)|}{|x-y|^{n+s}} dx\,dy \\ &\le \int_{B_1} \Big(\iint_{Q(\Omega')} \frac{|u(x-\varepsilon\xi) - u(y-\varepsilon\xi) - u(x) + u(y)|}{|x-y|^{n+s}} dx\,dy \Big) \eta(\xi)\,d\xi \\ &= \int_{B_1} \|L_{\varepsilon\xi}\psi - \psi\|_{L^1(Q(\Omega'))} \eta(\xi)\,d\xi \\ &\le \int_{B_1} \Big(\|L_{\varepsilon\xi}\psi - L_{\varepsilon\xi}\Psi\|_{L^1(Q(\Omega'))} + \|L_{\varepsilon\xi}\Psi - \Psi\|_{L^1(Q(\Omega'))} \\ &\qquad\qquad + \|\Psi - \psi\|_{L^1(Q(\Omega'))} \Big) \eta(\xi)\,d\xi.\end{aligned}$$

Notice that

$$\|L_{\varepsilon\xi}\psi - L_{\varepsilon\xi}\Psi\|_{L^1(Q(\Omega'))} = \|\psi - \Psi\|_{L^1(Q(\Omega'-\varepsilon\xi))} \le \|\psi - \Psi\|_{L^1(Q(\Omega))}$$

and hence

$$\mathcal{E}(u_\varepsilon - u, \Omega') \le \delta + \int_{B_1} \|L_{\varepsilon\xi}\Psi - \Psi\|_{L^1(Q(\Omega'))} \eta(\xi)\,d\xi.$$

For $\varepsilon > 0$ small enough we have

$$\operatorname{supp}(L_{\varepsilon\xi}\Psi - \Psi) \subseteq N_1(\operatorname{supp}\Psi) =: K \Subset \mathbb{R}^{2n},$$

and

$$|\Psi(x-\varepsilon\xi, y-\varepsilon\xi) - \Psi(x,y)| \le 2 \max_{\operatorname{supp}\Psi} |\nabla\Psi|\,\varepsilon.$$

Thus

$$\int_{B_1} \|L_{\varepsilon\xi}\Psi - \Psi\|_{L^1(Q(\Omega'))} \eta(\xi)\,d\xi \le 2|K| \max_{\operatorname{supp}\Psi} |\nabla\Psi|\,\varepsilon.$$

Passing to the limit as $\varepsilon \to 0^+$ then gives

$$\limsup_{\varepsilon\to 0^+} \mathcal{E}(u_\varepsilon - u, \Omega') \le \delta.$$

Since δ is arbitrary, we get (2.26).

(ii) Reasoning as above we obtain

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} &\frac{|u_\varepsilon(x) - u_\varepsilon(y)|}{|x-y|^{n+s}} dx\,dy \\ &\le \int_{B_1} \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x-\varepsilon\xi) - u(y-\varepsilon\xi)|}{|x-y|^{n+s}} dx\,dy \Big) \eta(\xi)\,d\xi \\ &= \int_{B_1} \Big(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x) - u(y)|}{|x-y|^{n+s}} dx\,dy \Big) \eta(\xi)\,d\xi \\ &= [u]_{W^{s,1}(\mathbb{R}^n)} \int_{B_1} \eta(\xi)\,d\xi,\end{aligned}$$

that is

$$[u_\varepsilon]_{W^{s,1}(\mathbb{R}^n)} \le [u]_{W^{s,1}(\mathbb{R}^n)}.$$

This and Fatou's Lemma give

$$[u]_{W^{s,1}(\mathbb{R}^n)} \le \liminf_{\varepsilon\to 0} [u_\varepsilon]_{W^{s,1}(\mathbb{R}^n)} \le \limsup_{\varepsilon\to 0} [u_\varepsilon]_{W^{s,1}(\mathbb{R}^n)} \le [u]_{W^{s,1}(\mathbb{R}^n)},$$

concluding the proof.

(iii) The proof is a classical cut-off argument. We consider a sequence of cut-off functions $\psi_k \in C_c^\infty(\mathbb{R}^n)$ such that

$$0 \le \psi_k \le 1, \quad \text{supp}\ \psi_k \subseteq B_{k+1} \quad \text{and} \quad \psi_k \equiv 1 \quad \text{in } B_k.$$

We can also assume that

$$\sup_{k\in\mathbb{N}} |\nabla \psi_k| \le M_0 < \infty.$$

It is enough to show that

$$\lim_{k\to\infty} \|u - \psi_k u\|_{L^1(\mathbb{R}^n)} = 0 \quad \text{and} \quad \lim_{k\to\infty} [\psi_k u]_{W^{s,1}(\mathbb{R}^n)} = [u]_{W^{s,1}(\mathbb{R}^n)}. \tag{2.27}$$

Indeed then we can use *(ii)* to approximate each $\psi_k u$ with a smooth function $u_k := (u\psi_k) * \eta_{\varepsilon_k}$, for ε_k small enough to have

$$\|\psi_k u - u_k\|_{L^1(\mathbb{R}^n)} < 2^{-k} \quad \text{and} \quad |[\psi_k u]_{W^{s,1}(\mathbb{R}^n)} - [u_k]_{W^{s,1}(\mathbb{R}^n)}| < 2^{-k}.$$

Therefore

$$\|u - u_k\|_{L^1(\mathbb{R}^n)} \le \|u - \psi_k u\|_{L^1(\mathbb{R}^n)} + 2^{-k} \longrightarrow 0$$

and

$$|[u]_{W^{s,1}(\mathbb{R}^n)} - [u_k]_{W^{s,1}(\mathbb{R}^n)}| \le |[u]_{W^{s,1}(\mathbb{R}^n)} - [\psi_k u]_{W^{s,1}(\mathbb{R}^n)}| + 2^{-k} \longrightarrow 0.$$

Also notice that

$$\text{supp}\ u_k \subseteq N_{\varepsilon_k}(\text{supp}\ \psi_k u) \subseteq B_{k+2}$$

so that $u_k \in C_c^\infty(\mathbb{R}^n)$ for every k. Moreover, from the definition of u_k it follows that if $u = \chi_E$, then $0 \le u_k \le 1$.

For a proof of (2.27) see, e.g., [60, Lemma 12]. □

Now we show that if Ω is a bounded open set with Lipschitz boundary and if $u = \chi_E$, then we can find smooth functions u_h such that

$$\mathcal{E}(u_h, \Omega) \longrightarrow \mathcal{E}(u, \Omega).$$

We first need the following two results.

Lemma 2.3.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $u \in L^\infty(\mathbb{R}^n)$ be such that $\mathcal{E}(u,\Omega) < \infty$. For every $\delta \in (0, r_0)$ let*

$$\varphi_\delta := 1 - \chi_{\{|\bar{d}_\Omega| < \delta\}}.$$

Then

$$u\varphi_\delta \xrightarrow{\delta\to 0} u \quad \text{in } L^1(\mathbb{R}^n), \tag{2.28}$$

and

$$\lim_{\delta \searrow 0^+} \mathcal{E}(u\varphi_\delta, \Omega) = \mathcal{E}(u, \Omega).$$

Proof. First of all, notice that

$$\int_{\mathbb{R}^n} |u\varphi_\delta - u|\,dx = \int_{\{|\bar{d}_\Omega| < \delta\}} |u|\,dx \le \|u\|_{L^\infty(\mathbb{R}^n)}\,|\{|\bar{d}_\Omega| < \delta\}| \xrightarrow{\delta\to 0} 0.$$

Now

$$\int_\Omega \int_\Omega \frac{|(u\varphi_\delta)(x) - (u\varphi_\delta)(y)|}{|x-y|^{n+s}}\,dx\,dy$$
$$= \int_{\Omega_{-\delta}} \int_{\Omega_{-\delta}} \frac{|u(x) - u(y)|}{|x-y|^{n+s}}\,dx\,dy + 2\int_{\Omega_{-\delta}} \Big(\int_{\Omega\setminus\Omega_{-\delta}} \frac{|u(x)|}{|x-y|^{n+s}}\,dy \Big) dx.$$

Since $\Omega_{-\delta} \subseteq \Omega$, we have

$$\int_{\Omega_{-\delta}} \int_{\Omega_{-\delta}} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy \leq \int_{\Omega} \int_{\Omega} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy.$$

On the other hand, since $|\Omega \setminus \Omega_{-\delta}| \longrightarrow 0$, we get

$$\frac{|u(x)-u(y)|}{|x-y|^{n+s}}\chi_{\Omega_{-\delta}}(x)\chi_{\Omega_{-\delta}}(y) \xrightarrow{\delta\to 0} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\chi_{\Omega}(x)\chi_{\Omega}(y),$$

for a.e. $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Therefore, by Fatou's Lemma we obtain

$$[u]_{W^{s,1}(\Omega)} \leq \liminf_{\delta\searrow 0} [u]_{W^{s,1}(\Omega_{-\delta})} \leq \limsup_{\delta\searrow 0} [u]_{W^{s,1}(\Omega_{-\delta})} \leq [u]_{W^{s,1}(\Omega)}. \tag{2.29}$$

Moreover, by point (i) of (2.13) we get

$$\begin{aligned} 2\int_{\Omega_{-\delta}} \Big(\int_{\Omega\setminus\Omega_{-\delta}} \frac{|u(x)|}{|x-y|^{n+s}}\,dy\Big)dx &\leq 2\|u\|_{L^\infty(\mathbb{R}^n)}\mathcal{L}_s(\Omega_{-\delta}, \Omega\setminus\Omega_{-\delta}) \\ &\leq 2C\|u\|_{L^\infty(\mathbb{R}^n)}\,\delta^{1-s}. \end{aligned}$$

Therefore we find

$$\lim_{\delta\searrow 0}[u\varphi_\delta]_{W^{s,1}(\Omega)} = [u]_{W^{s,1}(\Omega)}.$$

Now

$$\begin{aligned} &\int_{\Omega}\int_{\mathcal{C}\Omega} \frac{|(u\varphi_\delta)(x)-(u\varphi_\delta)(y)|}{|x-y|^{n+s}}\,dx\,dy \\ &\quad = \int_{\Omega_{-\delta}}\int_{\mathcal{C}\Omega_\delta} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy + \int_{\Omega_{-\delta}}\Big(\int_{\Omega_\delta\setminus\Omega} \frac{|u(x)|}{|x-y|^{n+s}}\,dy\Big)dx \\ &\qquad + \int_{\Omega\setminus\Omega_{-\delta}}\Big(\int_{\mathcal{C}\Omega_\delta} \frac{|u(x)|}{|x-y|^{n+s}}\,dy\Big)dx. \end{aligned}$$

Since $\Omega_{-\delta} \subseteq \Omega$ and $\mathcal{C}\Omega_\delta \subseteq \mathcal{C}\Omega$, we have

$$\int_{\Omega_{-\delta}}\int_{\mathcal{C}\Omega_\delta} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy \leq \int_{\Omega}\int_{\mathcal{C}\Omega} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy.$$

Moreover, since both $|\Omega\setminus\Omega_{-\delta}| \longrightarrow 0$ and $|\mathcal{C}\Omega\setminus\mathcal{C}\Omega_\delta| \longrightarrow 0$, we have

$$\frac{|u(x)-u(y)|}{|x-y|^{n+s}}\chi_{\Omega_{-\delta}}(x)\chi_{\mathcal{C}\Omega_\delta}(y) \xrightarrow{\delta\to 0} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\chi_{\Omega}(x)\chi_{\mathcal{C}\Omega}(y),$$

for a.e. $(x,y) \in \mathbb{R}^n \times \mathbb{R}^n$.

Therefore, again by Fatou's Lemma we obtain

$$\lim_{\delta\searrow 0}\int_{\Omega_{-\delta}}\int_{\mathcal{C}\Omega_\delta} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy = \int_{\Omega}\int_{\mathcal{C}\Omega} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy.$$

Furthermore, by point (ii) of (2.13) we get

$$\begin{aligned} \int_{\Omega_{-\delta}}\Big(\int_{\Omega_\delta\setminus\Omega} \frac{|u(x)|}{|x-y|^{n+s}}\,dy\Big)dx &\leq \|u\|_{L^\infty(\mathbb{R}^n)}\mathcal{L}_s(\Omega_{-\delta}, \Omega_\delta\setminus\Omega) \\ &\leq \|u\|_{L^\infty(\mathbb{R}^n)}\mathcal{L}_s(\Omega, \Omega_\delta\setminus\Omega) \leq C\|u\|_{L^\infty(\mathbb{R}^n)}\delta^{1-s} \end{aligned}$$

and also

$$\int_{\Omega\setminus\Omega_{-\delta}}\Big(\int_{\mathcal{C}\Omega_\delta} \frac{|u(x)|}{|x-y|^{n+s}}\,dy\Big)dx \leq C\|u\|_{L^\infty(\mathbb{R}^n)}\delta^{1-s}.$$

Thus
$$\lim_{\delta\searrow 0}\int_\Omega\int_{\mathcal{C}\Omega}\frac{|(u\varphi_\delta)(x)-(u\varphi_\delta)(y)|}{|x-y|^{n+s}}\,dx\,dy=\int_\Omega\int_{\mathcal{C}\Omega}\frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy,$$
concluding the proof. □

Lemma 2.3.4. *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $v\in L^\infty(\mathbb{R}^n)$ be such that $\mathcal{E}(v,\Omega)<\infty$ and*
$$v\equiv 0\quad in\ \{|\bar{d}_\Omega|<\delta/2\},$$
for some $\delta\in(0,r_0)$. Then
$$\big|\mathcal{E}(v,\Omega)-\mathcal{E}(v,\Omega_{-\delta/2})\big|\le C\|v\|_{L^\infty(\mathbb{R}^n)}\delta^{1-s},$$
where $C=C(n,s,\Omega)>0$ does not depend on v.

Proof. Since
$$v\equiv 0\quad\text{in }\{|\bar{d}_\Omega|<\delta/2\},$$
we have
$$\mathcal{E}(v,\Omega)=\mathcal{E}(v,\Omega_{-\delta/2})+2\int_{\Omega\setminus\Omega_{-\delta/2}}\Big(\int_{\mathcal{C}\Omega_{\delta/2}}\frac{|v(y)|}{|x-y|^{n+s}}\,dy\Big)dx.$$
Now, by point (ii) of (2.13) we have
$$\begin{aligned}\int_{\Omega\setminus\Omega_{-\delta/2}}\Big(\int_{\mathcal{C}\Omega_{\delta/2}}\frac{|v(y)|}{|x-y|^{n+s}}\,dy\Big)&\le\|v\|_{L^\infty(\mathbb{R}^n)}\mathcal{L}_s(\Omega\setminus\Omega_{-\delta/2},\mathcal{C}\Omega)\\&\le 2^{s-1}C\|v\|_{L^\infty(\mathbb{R}^n)}\,\delta^{1-s}.\end{aligned}$$
□

Proposition 2.3.5. *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $u\in L^\infty(\mathbb{R}^n)$ be such that $\mathcal{E}(u,\Omega)<\infty$. Then there exists a sequence $\{u_h\}\subseteq C^\infty(\mathbb{R}^n)$ such that*

(i) $\|u_h\|_{L^\infty(\mathbb{R}^n)}\le\|u\|_{L^\infty(\mathbb{R}^n)}$, *and* $0\le u_h\le 1$ *if* $0\le u\le 1$,

(ii) $u_h\xrightarrow{h\to\infty}u$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

(iii) $\lim_{h\to\infty}\mathcal{E}(u_h,\Omega)=\mathcal{E}(u,\Omega)$.

Proof. By Lemma 2.3.3 we know that for every $h\in\mathbb{N}$ we can find δ_h small enough such that

$$\|u-u\varphi_{\delta_h}\|_{L^1(\mathbb{R}^n)}<2^{-h}\quad\text{and}\quad\big|\mathcal{E}(u,\Omega)-\mathcal{E}(u\varphi_{\delta_h},\Omega)\big|<2^{-h}.\tag{2.30}$$

We can assume that $\delta_h\searrow 0$.

By point (i) of Lemma 2.3.2 we know that for every h we can find ε_h small enough such that

$$\|(u\varphi_{\delta_h})*\eta_{\varepsilon_h}-u\varphi_{\delta_h}\|_{L^1(B_h)}<2^{-h}\tag{2.31}$$

and

$$\big|\mathcal{E}(u\varphi_{\delta_h},\Omega_{-\delta_h/2})-\mathcal{E}((u\varphi_{\delta_h})*\eta_{\varepsilon_h},\Omega_{-\delta_h/2})\big|<2^{-h}.\tag{2.32}$$

Taking ε_h small enough, we can also assume that

$$(u\varphi_{\delta_h})*\eta_{\varepsilon_h}\equiv 0\qquad\text{in }\{|\bar{d}_\Omega|<\delta_h/2\},\tag{2.33}$$

since the ε-convolution enlarges the support at most to an ε-neighborhood of the original support.

Let $u_h := (u\varphi_{\delta_h}) * \eta_{\varepsilon_h}$. Since we are taking the ε_h-regularization of the function $u\varphi_{\delta_h}$, which is just the product of u with a characteristic function, point (i) of our claim is immediate.

By (2.31) and the first part of (2.30) we get point (ii).

As for point (iii), exploiting (2.33) and Lemma 2.3.4, we obtain

$$\begin{aligned}
&\big|\mathcal{E}(u,\Omega) - \mathcal{E}(u_h,\Omega)\big| \\
&\qquad \leq \big|\mathcal{E}(u,\Omega) - \mathcal{E}(u\varphi_{\delta_h},\Omega)\big| + \big|\mathcal{E}(u\varphi_{\delta_h},\Omega) - \mathcal{E}(u\varphi_{\delta_h},\Omega_{-\delta_h/2})\big| \\
&\qquad\qquad + \big|\mathcal{E}(u\varphi_{\delta_h},\Omega_{-\delta_h/2}) - \mathcal{E}(u_h,\Omega_{-\delta_h/2})\big| \\
&\qquad\qquad\qquad + \big|\mathcal{E}(u_h,\Omega_{-\delta_h/2}) - \mathcal{E}(u_h,\Omega)\big| \\
&\qquad \leq 2^{-h} + 2^s C\|u\|_{L^\infty(\mathbb{R}^n)}\delta_h^{1-s} + 2^{-h},
\end{aligned}$$

which goes to 0 as $h \longrightarrow \infty$. □

2.3.2. Proofs of Theorem 2.1.1 and Theorem 2.1.3. Exploiting Lemma 2.3.2 and the coarea formula, we can now prove Theorem 2.1.1.

Proof of Theorem 2.1.1. The "if part" is trivial. Indeed, just from point (i) and the lower semicontinuity of the s-perimeter we get

$$\operatorname{Per}_s(E,\Omega') \leq \liminf_{h\to\infty} \operatorname{Per}_s(E_h,\Omega') < \infty,$$

for every $\Omega' \Subset \Omega$.

Now suppose that E has locally finite s-perimeter in Ω.

The scheme of the proof is similar to that of the classical case (see, e.g., the proof of [79, Theorem 13.8]).

Given a sequence $\varepsilon_h \searrow 0^+$ we consider the ε_h-regularization of $u := \chi_E$ and define the sets

$$E_h^t := \{u_{\varepsilon_h} > t\} \quad \text{with } t \in (0,1).$$

Sard's Theorem guarantees that for a.e. $t \in (0,1)$ the sequence $\{E_h^t\}_h$ is made of open sets with smooth boundary. We will get our sets E_h by opportunely choosing t.

Since $u_{\varepsilon_h} \longrightarrow \chi_E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, it is readily seen that for a.e. $t \in (0,1)$

$$E_h^t \xrightarrow{loc} E,$$

and hence the lower semicontinuity of the s-perimeter gives

$$\operatorname{Per}_s(E,\mathcal{O}) \leq \liminf_{h\to\infty} \operatorname{Per}_s(E_h^t,\mathcal{O}), \tag{2.34}$$

for every open set $\mathcal{O} \subseteq \mathbb{R}^n$.

Moreover from (2.21) we have

$$\{0 < u_\varepsilon < 1\} \subseteq N_\varepsilon(\partial E) \qquad \forall\, \varepsilon > 0,$$

and hence, since $\partial E_h^t \subseteq \{u_{\varepsilon_h} = t\}$, we obtain

$$\partial E_h^t \subseteq N_{\varepsilon_h}(\partial E), \tag{2.35}$$

which will give (iii) once we choose our t.

We improve (2.34) by showing that, if $\Omega' \Subset \Omega$ is a fixed bounded open set, then for a.e. $t \in (0,1)$ (with the set of exceptional values of t possibly depending on Ω'),

$$\operatorname{Per}_s(E,\Omega') = \liminf_{h\to\infty} \operatorname{Per}_s(E_h^t,\Omega'). \tag{2.36}$$

By (2.34) and Fatou's Lemma, we have

$$\operatorname{Per}_s(E,\Omega') \leq \int_0^1 \liminf_{h\to\infty} \operatorname{Per}_s(E_h^t,\Omega')\,dt \leq \liminf_{h\to\infty} \int_0^1 \operatorname{Per}_s(E_h^t,\Omega')\,dt. \tag{2.37}$$

Let $\mathcal{O}$ be a bounded open set such that $\Omega' \Subset \mathcal{O} \Subset \Omega$. Since E has locally finite s-perimeter in Ω, we have $\mathrm{Per}_s(E,\mathcal{O}) < \infty$. Then, since $\Omega' \Subset \mathcal{O}$, point (i) of Lemma 2.3.2 (with $\mathcal{O}$ in the place of Ω) implies

$$\lim_{h\to\infty} \mathcal{E}(u_{\varepsilon_h},\Omega') = \mathcal{E}(\chi_E,\Omega') = \mathrm{Per}_s(E,\Omega'). \tag{2.38}$$

Since $0 \le u_{\varepsilon_h} \le 1$, we have $E_h^t = \mathbb{R}^n$ if $t<0$ and $E_h^t = \varnothing$ if $t>1$, and hence rewriting (2.38) exploiting the coarea formula,

$$\lim_{h\to\infty} \int_0^1 \mathrm{Per}_s(E_h^t,\Omega')\,dt = \mathrm{Per}_s(E,\Omega').$$

This and (2.37) give

$$\int_0^1 \liminf_{h\to\infty} \mathrm{Per}_s(E_h^t,\Omega')\,dt = \mathrm{Per}_s(E,\Omega') = \int_0^1 \mathrm{Per}_s(E,\Omega')\,dt,$$

which implies

$$\mathrm{Per}_s(E,\Omega') = \liminf_{h\to\infty} \mathrm{Per}_s(E_h^t,\Omega'), \quad \text{for a.e. } t\in(0,1), \tag{2.39}$$

as claimed.

Now let the sets $\Omega_k \Subset \Omega$ be as in Corollary 2.2.6. From (2.39) we deduce that for a.e. $t\in(0,1)$ we have

$$\mathrm{Per}_s(E,\Omega_k) = \liminf_{h\to\infty} \mathrm{Per}_s(E_h^t,\Omega_k), \qquad \forall\, k\in\mathbb{N}. \tag{2.40}$$

Therefore, combining all we wrote so far, we find that for a.e. $t\in(0,1)$ the sequence $\{E_h^t\}_h$ is made of open sets with smooth boundary such that $E_h^t \xrightarrow{loc} E$ and both (2.35) and (2.40) hold true.

To conclude, by a diagonal argument we can find $t_0\in(0,1)$ and $h_i \nearrow \infty$ such that, if we define $E_i := E_{h_i}^{t_0}$, then $\{E_i\}$ is a sequence of open sets with smooth boundary such that $E_i \xrightarrow{loc} E$, with $\partial E_i \subseteq N_{\varepsilon_{h_i}}(\partial E)$, and

$$\mathrm{Per}_s(E,\Omega_k) = \lim_{i\to\infty} \mathrm{Per}_s(E_i,\Omega_k), \qquad \forall\, k\in\mathbb{N}. \tag{2.41}$$

Now notice that if $\Omega' \Subset \Omega$, then there exists a k such that $\Omega' \Subset \Omega_k$. Therefore by (2.41) and Proposition 2.2.12 we get (ii).

This concludes the proof of the first part of the claim.

Now suppose that $\Omega = \mathbb{R}^n$ and $|E|$, $\mathrm{Per}_s(E) < \infty$.

Since $|E| < \infty$, we know that $u_\varepsilon \longrightarrow \chi_E$ in $L^1(\mathbb{R}^n)$. Therefore we obtain $E_h^t \longrightarrow E$ for a.e. $t\in(0,1)$.
Moreover, from point (ii) of Lemma 2.3.2 we know that

$$\mathcal{E}(u,\mathbb{R}^n) < \infty \qquad \Longrightarrow \qquad \lim_{\varepsilon\to 0} \mathcal{E}(u_\varepsilon,\mathbb{R}^n) = \mathcal{E}(u,\mathbb{R}^n).$$

We can thus repeat the proof above and obtain

$$\mathrm{Per}_s(E) = \liminf_{h\to\infty} \mathrm{Per}_s(E_h^t),$$

for a.e. $t\in(0,1)$. For any fixed "good" $t_0\in(0,1)$ this directly implies, with no need of a diagonal argument, the existence of a subsequence $h_i \nearrow \infty$ such that

$$\mathrm{Per}_s(E) = \lim_{i\to\infty} \mathrm{Per}_s(E_{h_i}^{t_0}).$$

We are left to show that in this case we can take the sets E_h to be bounded.

To this end, it is enough to replace the functions u_{ε_k} with the functions u_k obtained in point (iii) of Lemma 2.3.2.
Indeed, since u_k has compact support, for each $t \in (0,1)$ the set

$$E_k^t := \{u_k > t\}$$

is bounded. Since $u_k \longrightarrow u$ in $L^1(\mathbb{R}^n)$ we still find

$$E_k^t \xrightarrow{loc} E \quad \text{for a.e. } t \in (0,1),$$

and, since $0 \leq u_k \leq 1$ and

$$\lim_{k\to\infty} \mathcal{E}(u_k, \mathbb{R}^n) = \mathrm{Per}_s(E),$$

we can use again the coarea formula to conclude as above. □

Proof of Theorem 2.1.3. Exploiting the approximating sequence obtained in Proposition 2.3.5, we can now prove Theorem 2.1.3 exactly as above.
As for point (iii), recall that the functions u_h of Proposition 2.3.5 are defined as

$$u_h = (\chi_E \varphi_{\delta_h}) * \eta_{\varepsilon_h}.$$

Notice that, since we can suppose that $\varepsilon_h < \delta_h/2$, we have

$$u_h = \chi_E * \eta_{\varepsilon_h}, \qquad \text{in } \mathbb{R}^n \setminus N_{2\delta_h}(\partial\Omega).$$

Therefore, for every $t \in (0,1)$ we find

$$\partial\{u_h > t\} \subseteq N_{\varepsilon_h}(\partial E) \subseteq N_{2\delta_h}(\partial E), \qquad \text{in } \mathbb{R}^n \setminus N_{2\delta_h}(\partial\Omega).$$

This gives point (iii) once we choose an appropriate t, as in the proof of Theorem 2.1.1. □

Remark 2.3.6. We remark that by Proposition 2.2.12 we have also

$$\lim_{h\to\infty} \mathrm{Per}_s(E_h, \Omega') = \mathrm{Per}_s(E, \Omega'), \qquad \text{for every } \Omega' \Subset \Omega.$$

2.4. Existence and compactness of (locally) s-minimal sets

2.4.1. Proof of Theorem 2.1.7.

Proof of Theorem 2.1.7. $(i) \Longrightarrow (ii)$ is obvious.
$(ii) \Longrightarrow (iii)$ Let $\Omega' \Subset \Omega$ and let $F \subseteq \mathbb{R}^n$ be such that $F \setminus \Omega' = E \setminus \Omega'$.
Since $E \Delta F \subseteq \Omega' \Subset \Omega$, we have

$$\mathrm{Per}_s(E, \Omega) \leq \mathrm{Per}_s(F, \Omega).$$

Then, since $F \setminus \Omega' = E \setminus \Omega'$, by Proposition 2.2.1 we get

$$\mathrm{Per}_s(E, \Omega') \leq \mathrm{Per}_s(F, \Omega').$$

$(iii) \Longrightarrow (i)$ Let E be locally s-minimal in Ω.
First of all we prove that $\mathrm{Per}_s(E, \Omega) < \infty$.
Indeed, since E is locally s-minimal in Ω, in particular it is s-minimal in every Ω_r, with $r \in (-r_0, 0)$. Thus, by minimality and (2.11), we get

$$\mathrm{Per}_s(E, \Omega_r) \leq \mathrm{Per}_s(E \setminus \Omega_r, \Omega_r) \leq \mathrm{Per}_s(\Omega_r) \leq M < \infty,$$

for every $r \in (-r_0, 0)$. Therefore by (2.15) we obtain $\mathrm{Per}_s(E, \Omega) \leq M$.
Now let $F \subseteq \mathbb{R}^n$ be such that $F \setminus \Omega = E \setminus \Omega$. Take a sequence $\{r_k\} \subseteq (-r_0, 0)$ such that $r_k \nearrow 0$, let $\Omega_k := \Omega_{r_k}$, and define

$$F_k := (F \cap \Omega_k) \mathscr{H} \, \mathrm{Per}(E \setminus \Omega_k).$$

The local minimality of E gives

$$\mathrm{Per}_s(E,\Omega_k)\le \mathrm{Per}_s(F_k,\Omega_k),\qquad \text{for every } k\in\mathbb{N},$$

and by (2.15) we know that

$$\mathrm{Per}_s(E,\Omega)=\lim_{k\to\infty}\mathrm{Per}_s(E,\Omega_k).$$

Since $F_k=F$ outside $\Omega\setminus\Omega_k$, and $F_k=E$ in $\Omega\setminus\Omega_k$, we obtain

$$\begin{aligned}\mathrm{Per}_s(F,\Omega_k)-\mathrm{Per}_s(F_k,\Omega_k)&=\mathcal{L}_s(F\cap\Omega_k,\mathcal{C}F\cap(\Omega\setminus\Omega_k))\\&+\mathcal{L}_s(\mathcal{C}F\cap\Omega_k,F\cap(\Omega\setminus\Omega_k))-\mathcal{L}_s(F\cap\Omega_k,\mathcal{C}E\cap(\Omega\setminus\Omega_k))\\&-\mathcal{L}_s(\mathcal{C}F\cap\Omega_k,E\cap(\Omega\setminus\Omega_k)).\end{aligned}$$

Notice that each of the four terms in the right hand side is less or equal than $\mathcal{L}_s(\Omega_k,\Omega\setminus\Omega_k)$. Thus

$$a_k:=|\,\mathrm{Per}_s(F,\Omega_k)-\mathrm{Per}_s(F_k,\Omega_k)|\le 4\,\mathcal{L}_s(\Omega_k,\Omega\setminus\Omega_k).$$

Notice that from point (i) of (2.13) we have $a_k\longrightarrow 0$.

Now

$$\mathrm{Per}_s(F,\Omega)+a_k\ge \mathrm{Per}_s(F,\Omega_k)+a_k\ge \mathrm{Per}_s(F_k,\Omega_k)\ge \mathrm{Per}_s(E,\Omega_k),$$

and hence, passing to the limit $k\to\infty$, we get

$$\mathrm{Per}_s(F,\Omega)\ge \mathrm{Per}_s(E,\Omega).$$

Since F was an arbitrary competitor for E, we see that E is s-minimal in Ω. □

2.4.2. Proofs of Theorem 2.1.12 and Corollary 2.1.13. We slightly modify the proof of [21, Theorem 3.3] to show that the conclusion remains true in any bounded open set Ω with Lipschitz boundary.

Proof of Theorem 2.1.12. Assume $F=E$ outside Ω and let

$$F_k:=(F\cap\Omega)\mathscr{H}\,\mathrm{Per}(E_k\setminus\Omega).$$

Since $F_k=E_k$ outside Ω and E_k is s-minimal in Ω, we have

$$\mathrm{Per}_s(F_k,\Omega)\ge \mathrm{Per}_s(E_k,\Omega).$$

On the other hand, since $F_k=F$ inside Ω, we have

$$|\,\mathrm{Per}_s(F_k,\Omega)-\mathrm{Per}_s(F,\Omega)|\le \mathcal{L}_s(\Omega,(F_k\Delta F)\setminus\Omega)=\mathcal{L}_s(\Omega,(E_k\Delta E)\setminus\Omega)=:b_k.$$

Thus

$$\mathrm{Per}_s(F,\Omega)+b_k\ge \mathrm{Per}_s(F_k,\Omega)\ge \mathrm{Per}_s(E_k,\Omega).$$

If we prove that $b_k\longrightarrow 0$, then by lower semicontinuty of the fractional perimeter

$$\mathrm{Per}_s(F,\Omega)\ge\limsup_{k\to\infty}\mathrm{Per}_s(E_k,\Omega)\ge\liminf_{k\to\infty}\mathrm{Per}_s(E_k,\Omega)\ge \mathrm{Per}_s(E,\Omega).\tag{2.42}$$

This shows that E is s-minimal in Ω. Moreover, (2.6) follows from (2.42) by taking $F=E$.

We are left to show $b_k\longrightarrow 0$.

Let r_0 be as in Proposition 2.2.3 and let $R>r_0$. In the end we will let $R\longrightarrow\infty$. Define

$$a_k(r):=\mathcal{H}^{n-1}\big((E_k\Delta E)\cap\{\bar d_\Omega=r\}\big)$$

for every $r\in[0,r_0)$.

We split b_k as the sum

$$\begin{aligned}b_k=\mathcal{L}_s\big(\Omega,(E_k\Delta E)\cap(\Omega_{r_0}\setminus\Omega)\big)+\mathcal{L}_s\big(\Omega,(E_k\Delta E)\cap(\Omega_R\setminus\Omega_{r_0})\big)\\+\mathcal{L}_s\big(\Omega,(E_k\Delta E)\setminus\Omega_R\big).\end{aligned}$$

Notice that if $x \in \Omega$ and $y \in (\Omega_R \setminus \Omega_{r_0})$, then $|x-y| \geq r_0$, and hence

$$\begin{aligned}\mathcal{L}_s\big(\Omega, (E_k \Delta E) \cap (\Omega_R \setminus \Omega_{r_0})\big) &= \int_{\Omega_R \setminus \Omega_{r_0}} \chi_{E_k \Delta E}(y)\, dy \int_\Omega \frac{1}{|x-y|^{n+s}} dx \\ &\leq \frac{|\Omega|}{r_0^{n+s}} |(E_k \Delta E) \cap (\Omega_R \setminus \Omega_{r_0})|.\end{aligned}$$

Since $E_k \xrightarrow{loc} E$ and $\Omega_R \setminus \Omega_{r_0}$ is bounded, for every fixed R we find

$$\lim_{k \to \infty} \mathcal{L}_s\big(\Omega, (E_k \Delta E) \cap (\Omega_R \setminus \Omega_{r_0})\big) = 0.$$

As for the last term, we have

$$\mathcal{L}_s\big(\Omega, (E_k \Delta E) \setminus \Omega_R\big) \leq \mathcal{L}_s(\Omega, \mathcal{C}\Omega_R) \leq \int_\Omega dx \int_{\mathcal{C}B_R(x)} \frac{dy}{|x-y|^{n+s}} = \frac{n\omega_n}{s\, R^s} |\Omega|.$$

We are left to estimate the first term. By using the coarea formula, we obtain

$$\begin{aligned}&\mathcal{L}_s\big(\Omega, (E_k \Delta E) \cap (\Omega_{r_0} \setminus \Omega)\big) \\ &\qquad = \int_0^{r_0} \Big(\int_{\{\bar{d}_\Omega = r\}} \chi_{E_k \Delta E}(y) \Big(\int_\Omega \frac{dx}{|x-y|^{n+s}} \Big) d\mathcal{H}^{n-1}_y \Big) dr \\ &\qquad \leq \int_0^{r_0} \Big(\int_{\{\bar{d}_\Omega = r\}} \chi_{E_k \Delta E}(y) \Big(\int_{\mathcal{C}B_r(y)} \frac{dx}{|x-y|^{n+s}} \Big) d\mathcal{H}^{n-1}_y \Big) dr \\ &\qquad = \frac{n\omega_n}{s} \int_0^{r_0} \frac{a_k(r)}{r^s}\, dr.\end{aligned}$$

Notice that

$$\int_0^{r_0} a_k(r)\, dr = |(E_k \Delta E) \cap (\Omega_{r_0} \setminus \Omega)| \xrightarrow{k \to \infty} 0,$$

so that

$$a_k(r) \xrightarrow{k \to \infty} 0 \qquad \text{for a.e. } r \in [0, r_0).$$

Moreover, exploiting (2.10) we get

$$\int_0^{r_0} \frac{a_k(r)}{r^s}\, dr \leq M \int_0^{r_0} \frac{1}{r^s}\, dr = \frac{M}{1-s} r_0^{1-s},$$

and hence, by dominated convergence, we obtain

$$\lim_{k \to \infty} \int_0^{r_0} \frac{a_k(r)}{r^s}\, dr = 0.$$

Therefore

$$\limsup_{k \to \infty} b_k \leq \frac{n\omega_n}{s} |\Omega|\, R^{-s}.$$

Letting $R \longrightarrow \infty$, we obtain $b_k \longrightarrow 0$, concluding the proof. ◻

Proof of Corollary 2.1.13. Let the sets $\Omega_k \Subset \Omega$ be as in Corollary 2.2.6. By Theorem 2.1.12 we see that E is s-minimal in each Ω_k. Moreover (2.6) gives

$$\operatorname{Per}_s(E, \Omega_k) = \lim_{h \to \infty} \operatorname{Per}_s(E_h, \Omega_k),$$

for every k. Now if $\Omega' \Subset \Omega$, then $\Omega' \subseteq \Omega_k$ for some k. Thus E is s-minimal in Ω' and we obtain (2.7) by Proposition 2.2.12. ◻

2.4.3. Proofs of Theorem 2.1.9 and Corollary 2.1.11. We can exploit Proposition 2.2.13 to extend the existence result [21, Theorem 3.2] to any open set Ω, provided a competitor with finite fractional perimeter exists.

Proof of Theorem 2.1.9. The "only if" part is trivial. Now suppose there exists a competitor for E_0 with finite s-perimeter in Ω. Then

$$\inf\{\mathrm{Per}_s(E,\Omega) \,|\, E\setminus\Omega = E_0\setminus\Omega\} < \infty$$

and we can find a minimizing sequence, that is $\{E_h\}$ with $E_h\setminus\Omega = E_0\setminus\Omega$ and

$$\lim_{h\to\infty}\mathrm{Per}_s(E_h,\Omega) = \inf\{\mathrm{Per}_s(E,\Omega) \,|\, E\setminus\Omega = E_0\setminus\Omega\}.$$

Let $\Omega' \Subset \Omega$. Since, for every $h\in\mathbb{N}$ we have

$$\mathrm{Per}_s(E_h,\Omega') \leq \mathrm{Per}_s(E_h,\Omega) \leq M < \infty,$$

we can use Proposition 2.2.13 to find a set $E' \subseteq \Omega$ such that

$$E_h\cap\Omega \xrightarrow{loc} E'$$

(up to subsequence). Since $E_h\setminus\Omega = E_0\setminus\Omega$ for every h, if we set $E := E'\mathscr{H}\,\mathrm{Per}(E_0\setminus\Omega)$, then

$$E_h \xrightarrow{loc} E.$$

The semicontinuity of the fractional perimeter concludes the proof. □

Remark 2.4.1. In particular, if Ω is a bounded open set with Lipschitz boundary, then (as already proved in [21]) we can always find an s-minimal set for every $s\in(0,1)$, no matter what the external data $E_0\setminus\Omega$ is. Indeed in this case

$$\mathrm{Per}_s(E_0\setminus\Omega,\Omega) \leq \mathrm{Per}_s(\Omega) < \infty.$$

Actually, in order to have the existence of s-minimal sets for some fixed $s\in(0,1)$, the open set Ω need not be bounded nor have a regular boundary. It is enough to have

$$\mathrm{Per}_s(\Omega) < \infty.$$

Then $E_0\setminus\Omega$ has finite s-perimeter in Ω and we can apply Theorem 2.1.9.

Now we prove that a locally s-minimal set always exists, without having to assume the existence of a competitor having finite fractional perimeter.

Proof of Corollary 2.1.11. Let the sets Ω_k be as in Corollary 2.2.6.
From Theorem 2.1.9 and Remark 2.4.1 we know that for every k we can find a set E_k which is s-minimal in Ω_k and such that $E_k\setminus\Omega_k = E_0\setminus\Omega_k$.
Notice that, since the sequence Ω_k is increasing, the set E_h is s-minimal in Ω_k for every $h\geq k$.
This gives us a sequence $\{E_h\}$ satisfying the hypothesis of Proposition 2.2.13 (see Remark2.2.14), and hence (up to a subsequence)

$$E_h\cap\Omega \xrightarrow{loc} F,$$

for some $F\subseteq\Omega$. Since $E_h\setminus\Omega = E_0\setminus\Omega$ for every h, if we set $E := F\mathscr{H}\,\mathrm{Per}(E_0\setminus\Omega)$, we obtain

$$E_h \xrightarrow{loc} E.$$

Theorem 2.1.12 guarantees that E is s-minimal in every Ω_k and hence also locally s-minimal in Ω. Indeed, if $\Omega'\Subset\Omega$, then for some k big enough we have $\Omega'\subseteq\Omega_k$. Now, since E is s-minimal in Ω_k, it is s-minimal also in Ω'. □

2.5. Locally s-minimal sets in cylinders

Given a bounded open set $\Omega \subseteq \mathbb{R}^n$, we consider the cylinders

$$\Omega^k := \Omega \times (-k, k), \qquad \Omega^\infty := \Omega \times \mathbb{R}.$$

We recall that, given any set $E_0 \subseteq \mathbb{R}^{n+1}$, by Corollary 2.1.11 we can find a set $E \subseteq \mathbb{R}^{n+1}$ which is locally s-minimal in Ω^∞ and such that $E \setminus \Omega^\infty = E_0 \setminus \Omega^\infty$.

Remark 2.5.1. Actually, if Ω has Lipschitz boundary then E is s-minimal in every cylinder $\mathcal{O} = \Omega \times (a, b)$ of finite height (notice that $\mathcal{O}$ is not compactly contained in Ω^∞). Indeed, $\mathcal{O}$ is a bounded open set with Lipschitz boundary and E is locally s-minimal in $\mathcal{O}$. Thus, by Theorem 2.1.7, E is s-minimal in $\mathcal{O}$.
As a consequence, E is s-minimal in every bounded open subset $\Omega' \subseteq \Omega^\infty$.

We are going to consider as exterior data the subgraph

$$E_0 = \mathcal{S}g(v) := \{(x, t) \in \mathbb{R}^{n+1} \,|\, t < v(x)\},$$

of a function $v : \mathbb{R}^n \longrightarrow \mathbb{R}$, which is locally bounded, i.e.

$$M_r := \sup_{|x| \le r} |v(x)| < \infty, \qquad \text{for every } r > 0. \tag{2.43}$$

The following result is an immediate consequence of (the proof of) [43, Lemma 3.3].

Lemma 2.5.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with $C^{1,1}$ boundary and let $v : \mathbb{R}^n \longrightarrow \mathbb{R}$ be locally bounded. There exists a constant $M = M(n, s, \Omega, v) > 0$ such that if $E \subseteq \mathbb{R}^{n+1}$ is locally s-minimal in Ω^∞, with $E \setminus \Omega^\infty = \mathcal{S}g(v) \setminus \Omega^\infty$, then*

$$\Omega \times (-\infty, -M] \subseteq E \cap \Omega^\infty \subseteq \Omega \times (-\infty, M].$$

As a consequence

$$E \setminus \big(\Omega \times [-M, M]\big) = \mathcal{S}g(v) \setminus \big(\Omega \times [-M, M]\big). \tag{2.44}$$

Proof. By Remark 2.5.1, the set E is s-minimal in Ω^∞ in the sense considered in [43]. Thus, [43, Lemma 3.3] guarantees that

$$E \cap \Omega^\infty \subseteq \Omega \times (-\infty, M].$$

Moreover, the same argument used in the proof shows also that

$$\mathcal{C}E \cap \Omega^\infty \subseteq \Omega \times [-M, \infty),$$

(up to considering a bigger M).
Since $M > M_{R_0}$, where R_0 is such that $\Omega \Subset B_{R_0}$, we get (2.44), concluding the proof. □

Roughly speaking, Lemma 2.5.2 gives an a priori bound on the variation of ∂E in the "vertical" direction. In particular, from (2.44) we see that it is enough to look for a locally s-minimal set among sets which coincide with $\mathcal{S}g(v)$ out of $\Omega \times [-M, M]$.
As a consequence, we can prove that a set is locally s-minimal in Ω^∞ if and only if it is s-minimal in $\Omega \times [-M, M]$.

Proposition 2.5.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with $C^{1,1}$ boundary and let $v : \mathbb{R}^n \longrightarrow \mathbb{R}$ be locally bounded. Let M be as in Lemma 2.5.2 and let k_0 be the smallest integer $k_0 > M$. Let $F \subseteq \mathbb{R}^{n+1}$ be s-minimal in Ω^{k_0}, with respect to the exterior data*

$$F \setminus \Omega^{k_0} = \mathcal{S}g(v) \setminus \Omega^{k_0}. \tag{2.45}$$

Then F is s-minimal in Ω^k for every $k \ge k_0$, hence is locally s-minimal in Ω^∞.

PROOF. Let $E \subseteq \mathbb{R}^{n+1}$ be locally s-minimal in Ω^∞, with respect to the exterior data

$$E \setminus \Omega^\infty = \mathcal{S}g(v) \setminus \Omega^\infty.$$

Recall that by Remark 2.5.1 the set E is s-minimal in Ω^k for every k. In particular

$$\mathrm{Per}_s(E, \Omega^k) < \infty \qquad \forall\, k \in \mathbb{N}.$$

To prove the Proposition, it is enough to show that

$$\mathrm{Per}_s(F, \Omega^k) = \mathrm{Per}_s(E, \Omega^k), \qquad \text{for every } k \geq k_0. \tag{2.46}$$

Indeed, notice that by (2.45) and (2.44) we have

$$F \setminus \Omega^{k_0} = \mathcal{S}g(v) \setminus \Omega^{k_0} = E \setminus \Omega^{k_0}, \tag{2.47}$$

hence, clearly,

$$F \setminus \Omega^k = E \setminus \Omega^k, \qquad \forall\, k \geq k_0.$$

Then, since E is s-minimal in Ω^k, from (2.46) we conclude that also F is s-minimal in Ω^k, for every $k \geq k_0$. In turn, this implies that F is locally s-minimal in Ω^∞.

Exploiting Proposition 2.2.1, by (2.47) we obtain that for every $k \geq k_0$

$$\mathrm{Per}_s(F, \Omega^k) = \mathrm{Per}_s(F, \Omega^{k_0}) + c_k, \qquad \mathrm{Per}_s(E, \Omega^k) = \mathrm{Per}_s(E, \Omega^{k_0}) + c_k, \tag{2.48}$$

where

$$\begin{aligned} c_k = \mathcal{L}_s\big(\mathcal{S}g(v) \cap (\Omega^k \setminus \Omega^{k_0}), \mathcal{C}\mathcal{S}g(v) \setminus \Omega^k\big) + \mathcal{L}_s\big(\mathcal{S}g(v) \setminus \Omega^k, \mathcal{C}\mathcal{S}g(v) \cap (\Omega^k \setminus \Omega^{k_0})\big) \\ + \mathcal{L}_s\big(\mathcal{S}g(v) \cap (\Omega^k \setminus \Omega^{k_0}), \mathcal{C}\mathcal{S}g(v) \cap (\Omega^k \setminus \Omega^{k_0})\big), \end{aligned}$$

which is finite and does not depend on E nor F. To see that c_k is finite, simply notice that

$$c_k \leq \mathrm{Per}_s(E, \Omega^k) < \infty.$$

Now, by (2.47) and the minimality of F we have

$$\mathrm{Per}_s(F, \Omega^{k_0}) \leq \mathrm{Per}_s(E, \Omega^{k_0}).$$

On the other hand, since also the set E is s-minimal in Ω^{k_0}, again by (2.47) we get

$$\mathrm{Per}_s(E, \Omega^{k_0}) \leq \mathrm{Per}_s(F, \Omega^{k_0}).$$

This and (2.48) give

$$\mathrm{Per}_s(F, \Omega^k) = \mathrm{Per}_s(F, \Omega^{k_0}) + c_k = \mathrm{Per}_s(E, \Omega^k),$$

proving (2.46) and concluding the proof. □

It is now natural to wonder whether the set F is actually s-minimal in Ω^∞. The answer, in general, is no. Indeed, Theorem 2.1.14 shows that in general we cannot hope to find an s-minimal set in Ω^∞.

PROOF OF THEOREM 2.1.14. Notice that by (2.8) we have

$$\begin{aligned} E \cap (\Omega^\infty \setminus \Omega^{k+1}) = \Omega \times (-\infty, -k-1), \\ \mathcal{C}E \cap (\Omega^\infty \setminus \Omega^{k+1}) = \Omega \times (k+1, \infty), \end{aligned}$$

and

$$E \cap \Omega^{k+1} \subseteq \Omega \times (-k-1, k), \qquad \mathcal{C}E \cap \Omega^{k+1} \subseteq \Omega \times (-k, k+1).$$

Thus

$$\begin{aligned}\mathrm{Per}_s^L(E,\Omega^\infty) &= \mathrm{Per}_s^L(E,\Omega^{k+1}) + \mathcal{L}_s(E\cap(\Omega^\infty\setminus\Omega^{k+1}),\mathcal{C}E\cap\Omega^{k+1})\\ &\qquad + \mathcal{L}_s(\mathcal{C}E\cap(\Omega^\infty\setminus\Omega^{k+1}),E\cap\Omega^{k+1}) + \mathrm{Per}_s^L(E,\Omega^\infty\setminus\Omega^{k+1})\\ &\le \mathrm{Per}_s^L(E,\Omega^{k+1}) + 2\mathcal{L}_s(\Omega\times(-\infty,-k-1),\Omega\times(-k,k+1))\\ &\qquad + \mathcal{L}_s(\Omega\times(-\infty,-k-1),\Omega\times(k+1,\infty)).\end{aligned}$$

Since $d(\Omega\times(-\infty,-k-1),\Omega\times(-k,k+1))=1$, we get

$$\begin{aligned}&\mathcal{L}_s(\Omega\times(-\infty,-k-1),\Omega\times(-k,k+1))\\ &\qquad\le \int_{\Omega\times(-k,k+1)}\Big(\int_{\mathcal{C}B_1(X)}\frac{dY}{|X-Y|^{n+1+s}}\Big)\,dX\\ &\qquad= \frac{(n+1)\omega_{n+1}}{s}(2k+1)|\Omega|.\end{aligned}$$

As for the last term, since $n+1\ge 2$, we have

$$\begin{aligned}&\mathcal{L}_s(\Omega\times(-\infty,-k-1),\Omega\times(k+1,\infty))\\ &\qquad= \int_\Omega\int_\Omega\Big(\int_{-\infty}^{-k-1}\int_{k+1}^\infty\frac{dt\,d\tau}{(|x-y|^2+(t-\tau)^2)^{\frac{n+1+s}{2}}}\Big)dx\,dy\\ &\qquad\le |\Omega|^2\int_{-\infty}^{-k-1}\Big(\int_{k+1}^\infty\frac{dt}{(t-\tau)^{n+1+s}}\Big)d\tau\\ &\qquad= \frac{|\Omega|^2}{n+s}\int_{-\infty}^{-k-1}\frac{d\tau}{(k+1-\tau)^{n+s}}\\ &\qquad= \frac{|\Omega|^2}{(n+s)(n-1+s)}\,\frac{1}{(2k+2)^{n-1+s}}.\end{aligned}$$

This shows that $\mathrm{Per}_s^L(E,\Omega^\infty)<\infty$.

Now suppose that $E\subseteq\mathbb{R}^{n+1}$ satisfies (2.9). Then

$$\mathrm{Per}_s^{NL}(E,\Omega^\infty)\ge 2\mathcal{L}_s(\Omega\times(-\infty,-k),\mathcal{C}\Omega\times(k,\infty)).$$

Since Ω is bounded, we can take $R>0$ big enough such that $\Omega\Subset B_R$. For every $T>T_0:=\max\{k,R\}$ we have

$$\Omega\times(-\infty,-T)\subseteq\Omega\times(-\infty,-k)\quad\text{and}\quad(B_T\setminus B_R)\times(T,\infty)\subseteq\mathcal{C}\Omega\times(k,\infty).$$

Thus for every $T>T_0$

$$\begin{aligned}\mathcal{L}_s(\Omega\times(-\infty,-k),\mathcal{C}\Omega\times(k,\infty)) &\ge \mathcal{L}_s(\Omega\times(-\infty,-T),(B_T\setminus B_R)\times(T,\infty))\\ &= \int_\Omega dx\int_{B_T\setminus B_R}dy\int_{-\infty}^{-T}dt\int_T^\infty\frac{d\tau}{(|x-y|^2+(\tau-t)^2)^{\frac{n+1+s}{2}}} =: a_T.\end{aligned}$$

Notice that for every $x\in\Omega$, $y\in B_T\setminus B_R$, $t\in(-\infty,-T)$ and $\tau\in(T,\infty)$, we have

$$|x-y|\le|x|+|y|\le R+T\le 2T\le\tau-t,$$

and hence

$$\begin{aligned}a_T &\ge \frac{1}{2^{\frac{n+1+s}{2}}}\int_\Omega dx\int_{B_T\setminus B_R}dy\int_{-\infty}^{-T}dt\int_T^\infty\frac{d\tau}{(\tau-t)^{n+1+s}}\\ &= \frac{|\Omega|}{2^{\frac{n+1+s}{2}}(n+s)(n-1+s)}\,\frac{|B_T\setminus B_R|}{(2T)^{n-1+s}}.\end{aligned}$$

Since $|B_T \setminus B_R| \sim T^n$ as $T \to \infty$, we get $a_T \longrightarrow \infty$. Therefore, since

$$\mathrm{Per}_s^{NL}(E, \Omega^\infty) \geq 2a_T \qquad \text{for every } T > T_0,$$

we obtain $\mathrm{Per}_s^{NL}(E, \Omega^\infty) = \infty$.

To conclude, let Ω be bounded, with $C^{1,1}$ boundary, and let $v \in L^\infty(\mathbb{R}^n)$. Suppose that there exists a set $E \subseteq \mathbb{R}^{n+1}$ which is s-minimal in Ω^∞ with respect to the exterior data $E \setminus \Omega^\infty = \mathcal{S}g(v) \setminus \Omega^\infty$.
Then, thanks to Lemma 2.5.2, we can find k big enough such that E satisfies (2.9). Since this implies $\mathrm{Per}_s(E, \Omega^\infty) = \infty$, we reach a contradiction concluding the proof. □

Corollary 2.5.4. *In particular*

$$u \in BV_{\mathrm{loc}}(\mathbb{R}^n) \cap L^\infty_{\mathrm{loc}}(\mathbb{R}^n) \quad \Longrightarrow \quad \mathrm{Per}_s^L(\mathcal{S}g(u), \Omega^\infty) < \infty, \tag{2.49}$$

and

$$u \in L^\infty(\mathbb{R}^n) \quad \Longrightarrow \quad \mathrm{Per}_s^{NL}(\mathcal{S}g(u), \Omega^\infty) = \infty, \tag{2.50}$$

for every bounded open set $\Omega \subseteq \mathbb{R}^n$.

Furthermore, if $|u| \leq M$ *in* Ω *and there exists* $\Sigma \subseteq \mathbb{S}^{n-1}$ *with* $\mathcal{H}^{n-1}(\Sigma) > 0$ *such that either*

$$u(r\omega) \leq M \quad \text{or} \quad u(r\omega) \geq -M \qquad \text{for every } \omega \in \Sigma \quad \text{and} \quad r \geq r_0,$$

then $\mathrm{Per}_s^{NL}(\mathcal{S}g(u), \Omega^\infty) = \infty$.

Proof. Both (2.49) and (2.50) are immediate from Theorem 2.1.14, so we only need to prove the last claim.

Since Ω is bounded, we can find $R > 0$ such that $\Omega \Subset B_R$.
For every $T > T_0 := \max\{M, R, r_0\}$ define

$$\mathcal{S}(T) := \{x = r\omega \in \mathbb{R}^n \,|\, r \in (T_0, T),\, \omega \in \Sigma\}.$$

Notice that $\mathcal{S}(T) \subseteq B_T$ and

$$\begin{aligned}|\mathcal{S}(T)| &= \int_{T_0}^T \Big(\int_{\partial B_r} \chi_{\mathcal{S}(T)}\, d\mathcal{H}^{n-1} \Big) dr = \int_{T_0}^T \mathcal{H}^{n-1}(r\Sigma)\, dr \\ &= \frac{\mathcal{H}^{n-1}(\Sigma)}{n}(T^n - T_0^n).\end{aligned}$$

Suppose that $u(r\omega) \leq M$ for every $r \geq r_0$ and $\omega \in \Sigma$. Then, arguing as in the second part of the proof of Theorem 2.1.14, we obtain

$$\begin{aligned}\mathrm{Per}_s^{NL}(\mathcal{S}g(u), \Omega^\infty) &\geq \mathcal{L}_s(\mathcal{S}g(u) \cap \Omega^\infty, \mathcal{C}\mathcal{S}g(u) \setminus \Omega^\infty) \\ &\geq \mathcal{L}_s(\Omega \times (-\infty, -T), \mathcal{S}(T) \times (T, \infty)) \\ &\geq \frac{|\Omega|}{2^{\frac{n+1+s}{2}}(n+s)(n-1+s)} \frac{|\mathcal{S}(T)|}{(2T)^{n-1+s}},\end{aligned}$$

for every $T > T_0$. Since

$$\frac{|\mathcal{S}(T)|}{(2T)^{n-1+s}} \sim T^{1-s},$$

which tends to ∞ as $T \to \infty$, we get our claim. □

In the classical framework, the area functional of a function $u \in C^{0,1}(\mathbb{R}^n)$ is defined as

$$\mathcal{A}(u, \Omega) := \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx = \mathcal{H}^n\big(\{(x, u(x)) \in \mathbb{R}^{n+1} \,|\, x \in \Omega\}\big),$$

for any bounded open set $\Omega \subseteq \mathbb{R}^n$. Exploiting the subgraph of u one then defines the relaxed area functional of a function $u \in BV_{\mathrm{loc}}(\mathbb{R}^n)$ as

$$\mathcal{A}(u,\Omega) := \mathrm{Per}(\mathcal{S}g(u),\Omega^\infty). \tag{2.51}$$

Notice that when u is Lipschitz the two definitions coincide.

One might then be tempted to define a nonlocal fractional version of the area functional by replacing the classical perimeter in (2.51) with the s-perimeter, that is

$$\mathcal{A}_s(u,\Omega) := \mathrm{Per}_s(\mathcal{S}g(u),\Omega^\infty).$$

However Corollary 2.5.4 shows that this definition is ill-posed even for regular functions u.

On the other hand, it is worth remarking that one could use just the local part of the s-perimeter, but then the resulting functional

$$\mathcal{A}_s^L(u,\Omega) := \mathrm{Per}_s^L(\mathcal{S}g(u),\Omega^\infty) = \frac{1}{2}[\chi_{\mathcal{S}g(u)}]_{W^{s,1}(\Omega^\infty)}$$

has a local nature.

Exploiting [35, Theorem 1], we obtain the following:

Lemma 2.5.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $u \in BV(\Omega) \cap L^\infty(\Omega)$. Then*

$$\lim_{s\to 1^-}(1-s)\mathcal{A}_s^L(u,\Omega) = \omega_n \mathcal{A}(u,\Omega).$$

Proof. Let k be such that $|u| \le k$. Then $E = \mathcal{S}g(u)$ satisfies (2.8) and hence, arguing as in the beginning of the proof of Theorem 2.1.14, we get

$$\mathcal{A}_s^L(u,\Omega) = \mathrm{Per}_s^L(\mathcal{S}g(u),\Omega^{k+1}) + O(1),$$

as $s \to 1$. Since $\mathcal{S}g(u)$ has finite perimeter in Ω^{k+1}, which is a bounded open set with Lipschitz boundary, we conclude using [35, Theorem 1] (see also Theorem 1.1.7 for the asymptotics as $s \to 1$ of the s-perimeter).

Indeed, notice that since $|u| \le k$, we have

$$\mathrm{Per}(\mathcal{S}g(u),\Omega^{k+1}) = \mathrm{Per}(\mathcal{S}g(u),\Omega^\infty) = \mathcal{A}(u,\Omega). \qquad \square$$

CHAPTER 3

Complete stickiness of nonlocal minimal surfaces for small values of the fractional parameter

3.1. Introduction and main results

In this chapter, we deal with the behavior of s-minimal sets when the fractional parameter $s \in (0,1)$ is small. In particular

- we give the asymptotic behavior of the fractional mean curvature as $s \to 0^+$,
- we classify the behavior of s-minimal surfaces, in dependence of the exterior data at infinity.

Moreover, we prove the continuity of the fractional mean curvature in all variables for $s \in [0,1]$.

It is convenient to recall the definition of the s-fractional mean curvature of a set E at a point $q \in \partial E$ (which is the fractional counterpart of the classical mean curvature). It is defined as the principal value integral

$$H_s[E](q) := \text{P.V.} \int_{\mathbb{R}^n} \frac{\chi_{\mathcal{C}E}(y) - \chi_E(y)}{|y-q|^{n+s}} \, dy,$$

that is

$$H_s[E](q) := \lim_{\varrho \to 0^+} H_s^\varrho[E](q), \qquad \text{where} \qquad H_s^\varrho[E](q) := \int_{\mathcal{C}B_\varrho(q)} \frac{\chi_{\mathcal{C}E}(y) - \chi_E(y)}{|y-q|^{n+s}} \, dy.$$

For the main properties of the fractional mean curvature, we refer, e.g., to [2].

Let us also recall here the notation for the area of the $(n-1)$-dimensional sphere as

$$\varpi_n := \mathcal{H}^{n-1}\left(\{x \in \mathbb{R}^n \,|\, |x| = 1\}\right),$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. The volume of the n-dimensional unit ball is then

$$\omega_n = |B_1| = \frac{\varpi_n}{n}.$$

Moreover, we set $\varpi_0 := 0$.

This chapter is organized as follows. We set some notations and recall some known results in the following Subsection 3.1.2. Also, we give some preliminary results on the contribution from infinity of sets in Section 3.2.

In Section 3.3, we consider exterior data "occupying at infinity" in measure, with respect to an appropriate weight, less than an half-space. To be precise

$$\alpha(E_0) < \frac{\varpi_n}{2}.$$

In this hypothesis:

- In Subsection 3.3.1 we give some asymptotic estimates of the density, in particular showing that when s is small enough s-minimal sets cannot fill their domain.
- In Subsection 3.3.2 we give some estimates on the fractional mean curvature. In particular we show that if a set E has an exterior tangent ball of radius δ at some point $p \in \partial E$, then the s-fractional mean curvature of E in p is strictly positive for every $s < s_\delta$.
- In Subsection 3.3.3 we prove that when the fractional parameter is small and the exterior data at infinity occupies (in measure, with respect to the weight) less than half the space, then s-minimal sets completely stick at the boundary (that is, they are empty inside the domain), or become "topologically dense" in their domain. A similar result, which says that s-minimal sets fill the domain or their complementaries become dense, can be obtained in the same way, when the exterior data occupies in the appropriate sense more than half the space (so this threshold is somehow optimal).
- Subsection 3.3.4 narrows the set of minimal sets that become dense in the domain for s small. As a matter of fact, if the exterior data does not completely surround the domain, s-minimal sets completely stick at the boundary.

In Section 3.4, we provide some examples in which we are able to explicitly compute the contribution from infinity of sets. Section 3.5 contains the continuity of the fractional mean curvature operator in all its variables for $s \in [0, 1]$. As a corollary, we show that for $s \to 0^+$ the fractional mean curvature at a regular point of the boundary of a set, takes into account only the behavior of that set at infinity. The continuity property implies that the mean curvature at a regular point on the boundary of a set may change sign, as s varies, depending on the signs of the two asymptotics as $s \to 1^-$ and $s \to 0^+$.

In Appendix B and Appendix C we collect some useful results that we use in the present chapter. Worth mentioning are Appendixes C.2 and C.3. The first of the two gathers some known results on the regularity of s-minimal surfaces, so as to state the Euler-Lagrange equation pointwisely in the interior of Ω. In the latter we prove that the Euler-Lagrange equation holds (at least as a inequality) at $\partial E \cap \partial \Omega$, as long as the two boundaries do not intersect "transversally".

3.1.1. Statements of the main results. We remark that the quantity α,

$$\alpha(E) = \lim_{s \to 0^+} s \int_{\mathcal{C}B_1} \frac{\chi_E(y)}{|y|^{n+s}}\, dy, \tag{3.1}$$

may not exist—see [40, Example 2.8 and 2.9]. For this reason, we define

$$\overline{\alpha}(E) := \limsup_{s \to 0^+} s \int_{\mathcal{C}B_1} \frac{\chi_E(y)}{|y|^{n+s}}\, dy, \qquad \underline{\alpha}(E) := \liminf_{s \to 0^+} s \int_{\mathcal{C}B_1} \frac{\chi_E(y)}{|y|^{n+s}}\, dy. \tag{3.2}$$

This set parameter plays an important role in describing the asymptotic behavior of the fractional mean curvature as $s \to 0^+$ for unbounded sets. As a matter of fact, the limit as $s \to 0^+$ of the fractional mean curvature for a *bounded* set is a positive, universal constant (independent of the set), see, e.g., [47, Appendix B]). On the other hand, this asymptotic behavior changes for *unbounded* sets, due to the set function $\alpha(E)$, as described explicitly in the following result:

THEOREM 3.1.1. *[Proof in Section 3.5] Let $E \subseteq \mathbb{R}^n$ and let $p \in \partial E$ be such that ∂E is $C^{1,\gamma}$ near p, for some $\gamma \in (0, 1]$. Then*

$$\liminf_{s \to 0^+} s\, H_s[E](p) = \varpi_n - 2\overline{\alpha}(E)$$
$$\limsup_{s \to 0^+} s\, H_s[E](p) = \varpi_n - 2\underline{\alpha}(E).$$

We notice that if E is bounded, then $\underline{\alpha}(E) = \overline{\alpha}(E) = \alpha(E) = 0$, hence Theorem 3.1.1 reduces in this case to formula (B.1) in [47]. Actually, we can estimate the fractional mean curvature from below (above) uniformly with respect to the radius of the exterior (interior) tangent ball to E. To be more precise, if there exists an exterior tangent ball at $p \in \partial E$ of radius $\delta > 0$, then for every $s < s_\delta$ we have

$$\liminf_{\varrho \to 0^+} s\, H_s^\varrho[E](p) \geq \frac{\varpi_n - 2\overline{\alpha}(E)}{4}.$$

More explicitly, we have the following result:

THEOREM 3.1.2. *[Proof in Section 3.3.2] Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Let $E_0 \subseteq \mathcal{C}\Omega$ be such that*

$$\overline{\alpha}(E_0) < \frac{\varpi_n}{2}, \tag{3.3}$$

and let

$$\beta = \beta(E_0) := \frac{\varpi_n - 2\overline{\alpha}(E_0)}{4}.$$

We define

$$\delta_s = \delta_s(E_0) := e^{-\frac{1}{s} \log \frac{\varpi_n + 2\beta}{\varpi_n + \beta}}, \tag{3.4}$$

for every $s \in (0, 1)$. Then, there exists $s_0 = s_0(E_0, \Omega) \in (0, \frac{1}{2}]$ such that, if $E \subseteq \mathbb{R}^n$ is such that $E \setminus \Omega = E_0$ and E has an exterior tangent ball of radius (at least) δ_σ, for some $\sigma \in (0, s_0)$, at some point $q \in \partial E \cap \overline{\Omega}$, then

$$\liminf_{\varrho \to 0^+} H_s^\varrho[E](q) \geq \frac{\beta}{s} > 0, \qquad \forall\, s \in (0, \sigma]. \tag{3.5}$$

Given an open set $\Omega \subseteq \mathbb{R}^n$ and $\delta \in \mathbb{R}$, we consider the open set

$$\Omega_\delta := \{x \in \mathbb{R}^n \,|\, \bar{d}_\Omega(x) < \delta\},$$

where $\bar{d}_\Omega$ denotes the signed distance function from $\partial\Omega$, negative inside Ω.

It is well known (see, e.g., [4, 66]) that if Ω is bounded and $\partial\Omega$ is of class C^2, then the distance function is also of class C^2 in a neighborhood of $\partial\Omega$. Namely, there exists $r_0 > 0$ such that

$$\bar{d}_\Omega \in C^2(N_{2r_0}(\partial\Omega)), \qquad \text{where} \qquad N_{2r_0}(\partial\Omega) := \{x \in \mathbb{R}^n \,|\, |\bar{d}_\Omega(x)| < 2r_0\}.$$

As a consequence, since $|\nabla \bar{d}_\Omega| = 1$, the open set Ω_δ has C^2 boundary for every $|\delta| < 2r_0$. For a more detailed discussion, see Appendix B.1.1 and the references cited therein.

The constant r_0 will have the above meaning throughout this whole chapter.

We give the next definition.

DEFINITION 3.1.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. We say that a set E is δ-dense in Ω for some fixed $\delta > 0$ if $|B_\delta(x) \cap E| > 0$ for any $x \in \Omega$ for which $B_\delta(x) \Subset \Omega$.*

Notice that if E is δ-dense then E cannot have an exterior tangent ball of radius greater or equal than δ at any point $p \in \partial E \cap \Omega_{-\delta}$.
We observe that the notion for a set of being δ-dense is a "topological" notion, rather than a measure theoretic one. Indeed, δ-dense sets need not be "irregular" nor "dense" in the measure theoretic sense (see Remark 3.3.4).

With this definition and using Theorem 3.1.2 we obtain the following classification.

THEOREM 3.1.4. *[Proof in Section 3.3.3] Let Ω be a bounded and connected open set with C^2 boundary. Let $E_0 \subseteq \mathcal{C}\Omega$ such that*

$$\overline{\alpha}(E_0) < \frac{\varpi_n}{2}.$$

Then the following two results hold.
A) Let s_0 and δ_s be as in Theorem 3.1.2. There exists $s_1 = s_1(E_0, \Omega) \in (0, s_0]$ such that if $s < s_1$ and E is an s-minimal set in Ω with exterior data E_0, then either

$$(A.1)\ E \cap \Omega = \varnothing \qquad or \qquad (A.2)\ E \text{ is } \delta_s - dense.$$

B) Either
(B.1) there exists $\tilde{s} = \tilde{s}(E_0, \Omega) \in (0,1)$ such that if E is an s-minimal set in Ω with exterior data E_0 and $s \in (0, \tilde{s})$, then

$$E \cap \Omega = \varnothing,$$

or
(B.2) there exist $\delta_k \searrow 0$, $s_k \searrow 0$ and a sequence of sets E_k such that each E_k is s_k-minimal in Ω with exterior data E_0 and for every k

$$\partial E_k \cap B_{\delta_k}(x) \neq \varnothing \quad \forall\ B_{\delta_k}(x) \Subset \Omega.$$

We remark here that Definition 3.1.3 allows the s-minimal set to completely fill Ω. The next theorem states that for s small enough (and $\overline{\alpha}(E) < \varpi_n/2$) we can exclude this possibility.

THEOREM 3.1.5. *[Proof in Section 3.3.1] Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set of finite classical perimeter and let $E_0 \subseteq \mathcal{C}\Omega$ be such that*

$$\overline{\alpha}(E_0) < \frac{\varpi_n}{2}.$$

For every $\delta > 0$ and every $\gamma \in (0,1)$ there exists $\sigma_{\delta,\gamma} = \sigma_{\delta,\gamma}(E_0, \Omega) \in (0, \frac{1}{2}]$ such that if $E \subseteq \mathbb{R}^n$ is s-minimal in Ω, with exterior data E_0 and $s < \sigma_{\delta,\gamma}$, then

$$\big|(\Omega \cap B_\delta(x)) \setminus E\big| \geq \gamma \,\frac{\varpi_n - 2\overline{\alpha}(E_0)}{\varpi_n - \overline{\alpha}(E_0)} \big|\Omega \cap B_\delta(x)\big|, \qquad \forall\, x \in \overline{\Omega}. \tag{3.6}$$

REMARK 3.1.6. Let Ω and E_0 be as in Theorem 3.1.5 and fix $\gamma = \frac{1}{2}$.

(1) Notice that we can find $\bar{\delta} > 0$ and $\bar{x} \in \Omega$ such that

$$B_{2\bar{\delta}}(\bar{x}) \subseteq \Omega.$$

Now if $s < \sigma_{\bar{\delta},\frac{1}{2}}$ and E is s-minimal in Ω with respect to E_0, (3.6) says that

$$|B_{\bar{\delta}}(\bar{x}) \cap \mathcal{C}E| > 0.$$

Then (since the ball is connected), either $B_{\bar{\delta}}(\bar{x}) \subseteq \mathcal{C}E$ or there exists a point

$$x_0 \in \partial E \cap \overline{B}_{\bar{\delta}}(\bar{x}).$$

In this case, since $d(x_0, \partial\Omega) \geq \bar{\delta}$, [21, Corollary 4.3] implies that

$$B_{\bar{\delta}c_s}(z) \subseteq \mathcal{C}E \cap B_{\bar{\delta}}(x_0) \subseteq \mathcal{C}E \cap \Omega$$

for some z, where $c_s \in (0,1]$ denotes the constant of the clean ball condition (as introduced in [21, Corollary 4.3]) and depends only on s (and n). In both cases, there exists a ball of radius $\bar{\delta} c_s$ contained in $\mathcal{C}E \cap \Omega$.

(2) If $s < \sigma_{\bar{\delta},\frac{1}{2}}$ and E is s-minimal and δ_s-dense, then we have that

$$\delta_s > c_s \bar{\delta}.$$

On the other hand, we have an explicit expression for δ_s, given in (3.4). Therefore, if one could prove that c_s goes to zero slower than δ_s, one could exclude the existence of s-minimal sets that are δ_s-dense (for all sufficiently small s).

An interesting result is related to s-minimal sets whose exterior data does not completely surround Ω. In this case, the s-minimal set, for small values of s, is always empty in Ω. More precisely:

THEOREM 3.1.7. *[Proof in Section 3.3.4] Let Ω be a bounded and connected open set with C^2 boundary. Let $E_0 \subseteq \mathcal{C}\Omega$ such that*

$$\overline{\alpha}(E_0) < \frac{\varpi_n}{2},$$

and let s_1 be as in Theorem 3.1.4. Suppose that there exists $R > 0$ and $x_0 \in \partial\Omega$ such that

$$B_R(x_0) \setminus \Omega \subseteq \mathcal{C}E_0.$$

Then, there exists $s_3 = s_3(E_0, \Omega) \in (0, s_1]$ such that if $s < s_3$ and E is an s-minimal set in Ω with exterior data E_0, then

$$E \cap \Omega = \varnothing.$$

We notice that Theorem 3.1.7 prevents the existence of s-minimal sets that are δ-dense (for any δ).

REMARK 3.1.8. The indexes s_1 and s_3 are defined as follows

$$s_1 := \sup\{s \in (0, s_0) \,|\, \delta_s < r_0\}$$

and

$$s_3 := \sup\Big\{s \in (0, s_0) \,\Big|\, \delta_s < \frac{1}{2}\min\{r_0, R\}\Big\}.$$

Clearly, $s_3 \le s_1 \le s_0$.

REMARK 3.1.9. We point out that condition (3.3) is somehow optimal. Indeed, when $\alpha(E_0)$ exists and

$$\alpha(E_0) = \frac{\varpi_n}{2},$$

several configurations may occur, depending on the position of Ω with respect to the exterior data $E_0 \setminus \Omega$. As an example, take

$$\mathfrak{P} = \{(x', x_n) \,\big|\, x_n > 0\}.$$

Then, for any $\Omega \subseteq \mathbb{R}^n$ a bounded open set with C^2 boundary, the only s-minimal set with exterior data given by $\mathfrak{P} \setminus \Omega$ is $\mathfrak{P}$ itself. So, if E is s-minimal with respect to $\mathfrak{P} \setminus \Omega$ then

$$\begin{aligned} \Omega \subseteq \mathfrak{P} \qquad &\Longrightarrow \qquad E \cap \Omega = \Omega \\ \Omega \subseteq \mathbb{R}^n \setminus \mathfrak{P} \qquad &\Longrightarrow \qquad E \cap \Omega = \varnothing. \end{aligned}$$

On the other hand, if one takes $\Omega = B_1$, then

$$E \cap B_1 = \mathfrak{P} \cap B_1.$$

As a further example, we consider the supergraph

$$E_0 := \{(x', x_n) \,\big|\, x_n > \tanh x_1\},$$

for which we have that (see Example 3.4.4)

$$\alpha(E_0) = \frac{\varpi_n}{2}.$$

Then for every s-minimal set in Ω with exterior data $E_0 \setminus \Omega$, we have that

$$\Omega \subseteq \{(x', x_n) \mid x_n > 1\} \qquad \Longrightarrow \qquad E \cap \Omega = \Omega$$
$$\Omega \subseteq \{(x', x_n) \mid x_n < -1\} \qquad \Longrightarrow \qquad E \cap \Omega = \varnothing.$$

Taking $\Omega = B_2$, we have by the maximum principle in Proposition C.4.2 that every set E which is s-minimal in B_2, with respect to $E_0 \setminus B_2$, satisfies

$$B_2 \cap \{(x', x_n) \mid x_n > 1\} \subseteq E, \qquad B_2 \cap \{(x', x_n) \mid x_n < -1\} \subseteq \mathcal{C}E.$$

On the other hand, we are not able to establish what happens in $B_2 \cap \{(x', x_n) \mid -1 < x_n < 1\}$.

Remark 3.1.10. We notice that when E is s-minimal in Ω with respect to E_0, then $\mathcal{C}E$ is s-minimal in Ω with respect to $\mathcal{C}E_0$. Moreover

$$\underline{\alpha}(E_0) > \frac{\varpi_n}{2} \qquad \Longrightarrow \qquad \overline{\alpha}(\mathcal{C}E_0) < \frac{\varpi_n}{2}.$$

So in this case we can apply Theorems 3.1.2, 3.1.4, 3.1.5 and 3.1.7 to $\mathcal{C}E$ with respect to the exterior data $\mathcal{C}E_0$. For instance, if E is s-minimal in Ω with exterior data E_0 with

$$\underline{\alpha}(E_0) > \frac{\varpi_n}{2},$$

and $s < s_1(\mathcal{C}E_0, \Omega)$, then either

$$E \cap \Omega = \Omega \qquad \text{or} \qquad \mathcal{C}E \ \text{ is } \ \delta_s(\mathcal{C}E_0) - \text{dense}.$$

The analogues of the just mentioned Theorems can be obtained similarly.

We point out that from our main results and the last two remarks, we have a complete classification of nonlocal minimal surfaces when s is small whenever

$$\alpha(E_0) \neq \frac{\varpi_n}{2}.$$

In the last section of the chapter, we prove the continuity of the fractional mean curvature in all variables (see Theorem 3.5.2 and Proposition 3.5.3). As a consequence, we have the following result.

Proposition 3.1.11. *Let $E \subseteq \mathbb{R}^n$ and let $p \in \partial E$ such that ∂E is $C^{1,\alpha}$ in $B_R(p)$ for some $R > 0$ and $\alpha \in (0, 1]$. Then the function*

$$H_{(\,\cdot\,)}[E](\,\cdot\,) : (0, \alpha) \times (\partial E \cap B_R(p)) \longrightarrow \mathbb{R}, \qquad (s, x) \longmapsto H_s[E](x)$$

is continuous.
Moreover, if $\partial E \cap B_R(p)$ is C^2 and for every $x \in \partial E \cap B_R(p)$ we define

$$\tilde{H}_s[E](x) := \begin{cases} s(1-s)H_s[E](x), & \textit{for } s \in (0,1) \\ \varpi_{n-1} H[E](x), & \textit{for } s = 1, \end{cases}$$

then the function

$$\tilde{H}_{(\,\cdot\,)}[E](\,\cdot\,) : (0, 1] \times (\partial E \cap B_R(p)) \longrightarrow \mathbb{R}, \qquad (s, x) \longmapsto \tilde{H}_s[E](x)$$

is continuous.
Finally, if $\partial E \cap B_R(p)$ is $C^{1,\alpha}$ and $\alpha(E)$ exists, and if for every $x \in \partial E \cap B_R(p)$ we denote

$$\tilde{H}_0[E](x) := \varpi_n - 2\alpha(E),$$

then the function

$$\tilde{H}_{(\,\cdot\,)}[E](\,\cdot\,):[0,\alpha)\times(\partial E\cap B_R(p))\longrightarrow\mathbb{R},\qquad (s,x)\longmapsto\tilde{H}_s[E](x)$$

is continuous.

As a consequence of the continuity of the fractional mean curvature and the asymptotic result in Theorem 3.1.1 we establish that, by varying the fractional parameter s, the nonlocal mean curvature may change sign at a point where the classical mean curvature is negative, as one can observe in Theorem 3.5.7.

3.1.2. Definitions, known facts and notations. We recall here some basic facts on s-minimal sets and surfaces, on the fractional mean curvature operator, and some notations, that we will use in the course of this chapter.

3.1.2.1. *Measure theoretic assumption.* We recall the following notations and measure theoretic assumptions, which are assumed throughout the chapter.

Let $E\subseteq\mathbb{R}^n$ be a measurable set. Up to modifications in sets of measure zero, we can assume (see Remark MTA and Appendix A) that E contains the measure theoretic interior

$$E_{int}:=\Big\{x\in\mathbb{R}^n\,|\,\exists\, r>0\text{ s.t. }|E\cap B_r(x)|=\frac{\varpi_n}{n}r^n\Big\}\subseteq E,$$

the complementary $\mathcal{C}E$ contains the measure theoretic exterior

$$E_{ext}:=\{x\in\mathbb{R}^n\,|\,\exists\, r>0\text{ s.t. }|E\cap B_r(x)|=0\}\subseteq\mathcal{C}E,$$

and the topological boundary of E coincides with its measure theoretic boundary, $\partial E=\partial^-E$, where

$$\begin{aligned}\partial^-E&:=\mathbb{R}^n\setminus(E_{int}\cup E_{ext})\\&=\Big\{x\in\mathbb{R}^n\,|\,0<|E\cap B_r(x)|<\frac{\varpi_n}{n}r^n\text{ for every }r>0\Big\}.\end{aligned}$$

In particular, we remark that both E_{int} and E_{ext} are open sets.

3.1.2.2. *Hölder continuous functions.* We will use the following notation for the class of Hölder continuous functions.

Let $\alpha\in(0,1]$, let $S\subseteq\mathbb{R}^n$ and let $v:S\longrightarrow\mathbb{R}^m$. The α-Hölder semi-norm of v in S is defined as

$$[v]_{C^{0,\alpha}(S,\mathbb{R}^m)}:=\sup_{x\neq y\in S}\frac{|v(x)-v(y)|}{|x-y|^\alpha}.$$

With a slight abuse of notation, we will omit the $\mathbb{R}^m$ in the formulas. We also define

$$\|v\|_{C^0(S)}:=\sup_{x\in S}|v(x)|\quad\text{and}\quad\|v\|_{C^{0,\alpha}(S)}:=\|v\|_{C^0(S)}+[v]_{C^{0,\alpha}(S)}.$$

Given an open set $\Omega\subseteq\mathbb{R}^n$, we define the space of uniformly Hölder continuous functions $C^{0,\alpha}(\overline{\Omega},\mathbb{R}^m)$ as

$$C^{0,\alpha}(\overline{\Omega},\mathbb{R}^m):=\{v\in C^0(\overline{\Omega},\mathbb{R}^m)\,|\,\|v\|_{C^{0,\alpha}(\overline{\Omega})}<\infty\}.$$

Recall that $C^1(\overline{\Omega})$ is the space of those functions $u:\overline{\Omega}\longrightarrow\mathbb{R}$ such that $u\in C^0(\overline{\Omega})\cap C^1(\Omega)$ and such that ∇u can be continuously extended to $\overline{\Omega}$. For every $S\subseteq\overline{\Omega}$ we write

$$\|u\|_{C^{1,\alpha}(S)}:=\|u\|_{C^0(S)}+\|\nabla u\|_{C^{0,\alpha}(S)},$$

and we define

$$C^{1,\alpha}(\overline{\Omega}):=\{u\in C^1(\overline{\Omega})\,|\,\|u\|_{C^{1,\alpha}(\overline{\Omega})}<\infty\}.$$

We will usually consider the local versions of the above spaces. Given an open set $\Omega \subseteq \mathbb{R}^n$, the space of locally Hölder continuous functions $C^{k,\alpha}(\Omega)$, with $k \in \{0,1\}$, is defined as

$$C^{k,\alpha}(\Omega) := \{u \in C^k(\Omega) \,|\, \|u\|_{C^{k,\alpha}(\mathcal{O})} < \infty \text{ for every } \mathcal{O} \Subset \Omega\}.$$

3.1.2.3. *The Euler-Lagrange equation.* We recall that the fractional mean curvature gives the Euler-Lagrange equation of an s-minimal set. To be more precise, if E is s-minimal in Ω, then

$$H_s[E] = 0, \qquad \text{on} \quad \partial E \cap \Omega,$$

in an appropriate viscosity sense (see [21, Theorem 5.1]).

Actually, by exploiting the interior regularity theory of s-minimal sets, the equation is satisfied in the classical sense in a neighborhood of every "viscosity point" (see Appendix C.2). That is, if E has at $p \in \partial E \cap \Omega$ a tangent ball (either interior or exterior), then ∂E is C^∞ in $B_r(p)$, for some $r > 0$ small enough, and

$$H_s[E](x) = 0, \qquad \forall\, x \in \partial E \cap B_r(p).$$

Moreover, if the boundary of Ω is of class $C^{1,1}$, then the Euler-Lagrange equation (at least as an inequality) holds also at a point $p \in \partial E \cap \partial\Omega$, provided that the boundary ∂E and the boundary $\partial\Omega$ do not intersect "transversally" in p (see Theorem C.3.1).

3.2. Contribution to the mean curvature coming from infinity

In this section, we study in detail the quantities $\alpha(E), \overline{\alpha}(E), \underline{\alpha}(E))$ as defined in (3.1), (3.2). As a first remark, notice that these definitions are independent on the radius of the ball (see [40, Observation 3 in Subsection 3.3]) so we have that for any $R > 0$

$$\overline{\alpha}(E) = \limsup_{s \to 0^+} s \int_{\mathcal{C}B_R} \frac{\chi_E(y)}{|y|^{n+s}}\,dy, \quad \underline{\alpha}(E) := \liminf_{s \to 0^+} s \int_{\mathcal{C}B_R} \frac{\chi_E(y)}{|y|^{n+s}}\,dy. \tag{3.7}$$

Notice that

$$\overline{\alpha}(E) = \varpi_n - \underline{\alpha}(\mathcal{C}E), \quad \underline{\alpha}(E) = \varpi_n - \overline{\alpha}(\mathcal{C}E).$$

We define

$$\alpha_s(q,r,E) := \int_{\mathcal{C}B_r(q)} \frac{\chi_E(y)}{|q-y|^{n+s}}\,dy.$$

Then, the quantity $\alpha_s(q,r,E)$ somehow "stabilizes" for small s independently on how large or where we take the ball, as rigorously given by the following result:

PROPOSITION 3.2.1. *Let $K \subseteq \mathbb{R}^n$ be a compact set and $[a,b] \subseteq \mathbb{R}$ be a closed interval, with $0 < a < b$. Then*

$$\lim_{s \to 0^+} s|\alpha_s(q,r,E) - \alpha_s(0,1,E)| = 0 \qquad \text{uniformly in } q \in K, r \in [a,b].$$

Moreover, for any bounded open set $\Omega \subseteq \mathbb{R}^n$ and any fixed $r > 0$, we have that

$$\limsup_{s \to 0^+} s \inf_{q \in \overline{\Omega}} \alpha_s(q,r,E) = \limsup_{s \to 0^+} s \sup_{q \in \overline{\Omega}} \alpha_s(q,r,E) = \overline{\alpha}(E). \tag{3.8}$$

PROOF. Let us fix $r \in [a,b]$ and $q \in K$, and $R > 0$ such that $K \subseteq \overline{B}_R$. Let also $\varepsilon \in (0,1)$ be a fixed positive small quantity (that we will take arbitrarily small further on), such that

$$R > (\varepsilon b)/(1-\varepsilon).$$

We notice that if $x \in B_r(q)$, we have that $|x| < r + |q| < R/\varepsilon$, hence $B_r(q) \subseteq B_{R/\varepsilon}$. We write that

$$\alpha_s(q,R,E) = \int_{\mathcal{C}B_r(q)} \frac{\chi_E(y)}{|q-y|^{n+s}}\,dy = \int_{\mathcal{C}B_{R/\varepsilon}} \frac{\chi_E(y)}{|q-y|^{n+s}}\,dy + \int_{B_{R/\varepsilon}\setminus B_r(q)} \frac{\chi_E(y)}{|q-y|^{n+s}}\,dy.$$

Now for $y \in \mathcal{C}B_{R/\varepsilon}$ we have that $|y-q| \geq |y| - |q| \geq (1-\varepsilon)|y|$, thus for any $q \in \overline{B}_R$

$$\int_{\mathcal{C}B_{R/\varepsilon}} \frac{\chi_E(y)}{|q-y|^{n+s}}\, dy \leq (1-\varepsilon)^{-n-s} \int_{\mathcal{C}B_{R/\varepsilon}} \frac{\chi_E(y)}{|y|^{n+s}}\, dy = (1-\varepsilon)^{-n-s}\alpha_s(0, R/\varepsilon, E). \tag{3.9}$$

Moreover

$$\begin{aligned} \int_{B_{R/\varepsilon}\setminus B_r(q)} \frac{\chi_E(y)}{|q-y|^{n+s}}\, dy &\leq \int_{B_{R/\varepsilon}\setminus B_r(q)} \frac{dy}{|q-y|^{n+s}} \leq \varpi_n \int_r^{R/\varepsilon+R} t^{-s-1}\, dt \\ &= \varpi_n \frac{r^{-s} - R^{-s}\varepsilon^s(1+\varepsilon)^{-s}}{s} \leq \varpi_n \frac{a^{-s} - R^{-s}\varepsilon^s(1+\varepsilon)^{-s}}{s}. \end{aligned} \tag{3.10}$$

Notice also that since $B_r(q) \subseteq B_{R/\varepsilon}$ and $|q-y| \leq |q| + |y| \leq (\varepsilon+1)|y|$ for any $y \in \mathcal{C}B_{R/\varepsilon}$, we obtain that

$$\int_{\mathcal{C}B_r(q)} \frac{\chi_E(y)}{|q-y|^{n+s}}\, dy \geq \int_{\mathcal{C}B_{R/\varepsilon}} \frac{\chi_E(y)}{|q-y|^{n+s}}\, dy \geq (1+\varepsilon)^{-n-s} \int_{\mathcal{C}B_{R/\varepsilon}} \frac{\chi_E(y)}{|y|^{n+s}}\, dy. \tag{3.11}$$

Putting(3.9), (3.10) and (3.11) together, we get that

$$\begin{aligned} 0 \leq \alpha_s(q,r,E) - (1+\varepsilon)^{-n-s}\alpha_s(0,R/\varepsilon,E) &\leq \alpha_s(0,R/\varepsilon,E)\left((1-\varepsilon)^{-n-s} - (1+\varepsilon)^{-n-s}\right) \\ &\quad + \varpi_n \frac{a^{-s} - R^{-s}\varepsilon^s(1+\varepsilon)^{-s}}{s}. \end{aligned}$$

Now we have that

$$|\alpha_s(0,R/\varepsilon,E) - \alpha_s(0,1,E)| \leq \left| \int_{B_{R/\varepsilon}\setminus B_1} \frac{dy}{|y|^{n+s}} \right| \leq \varpi_n \frac{|1 - R^{-s}\varepsilon^s|}{s}.$$

So by the triangle inequality we obtain

$$\begin{aligned} |\alpha_s(q,r,E) - (1+\varepsilon)^{-n-s}\alpha_s(0,1,E)| &\leq \alpha_s(0,R/\varepsilon,E)\left((1-\varepsilon)^{-n-s} - (1+\varepsilon)^{-n-s}\right) \\ &\quad + \frac{\varpi_n}{s}\left[a^{-s} - R^{-s}\varepsilon^s(1+\varepsilon)^{-s} + (1+\varepsilon)^{-n-s}|1 - R^{-s}\varepsilon^s|\right]. \end{aligned}$$

Hence, it holds that

$$\limsup_{s\to 0^+} s|\alpha_s(q,r,E) - (1+\varepsilon)^{-n}\alpha_s(0,1,E)| \leq \left((1-\varepsilon)^{-n} - (1+\varepsilon)^{-n}\right)\overline{\alpha}(E),$$

uniformly in $q \in K$ and in $r \in [a,b]$.
Letting $\varepsilon \to 0^+$, we conclude that

$$\lim_{s\to 0^+} s|\alpha_s(q,r,E) - \alpha_s(0,1,E)| = 0,$$

uniformly in $q \in K$ and in $r \in [a,b]$.

Now, we consider K such that $K = \overline{\Omega}$. Using the inequalities (3.9), (3.10) and (3.11) we have that for any $q \in \overline{\Omega}$

$$\begin{aligned} (1+\varepsilon)^{-n-s} \int_{\mathcal{C}B_{R/\varepsilon}} \frac{\chi_E(y)}{|y|^{n+s}}\, dy &\leq \int_{\mathcal{C}B_r(q)} \frac{\chi_E(y)}{|q-y|^{n+s}}\, dy \\ &\leq (1-\varepsilon)^{-n-s} \int_{\mathcal{C}B_{R/\varepsilon}} \frac{\chi_E(y)}{|y|^{n+s}}\, dy + \varpi_n \frac{a^{-s} - R^{-s}\varepsilon^s(1+\varepsilon)^{-s}}{s}. \end{aligned}$$

Passing to limsup it follows that

$$\begin{aligned} (1+\varepsilon)^{-n}\overline{\alpha}(E) &\leq \limsup_{s\to 0^+} s \inf_{q\in\overline{\Omega}} \int_{\mathcal{C}B_r(q)} \frac{\chi_E(y)}{|q-y|^{n+s}}\, dy \\ &\leq \limsup_{s\to 0^+} s \sup_{q\in\overline{\Omega}} \int_{\mathcal{C}B_r(q)} \frac{\chi_E(y)}{|q-y|^{n+s}}\, dy \leq (1-\varepsilon)^{-n}\overline{\alpha}(E). \end{aligned}$$

Sending $\varepsilon \to 0$ we obtain the conclusion. □

REMARK 3.2.2. Let $E \subseteq \mathbb{R}^n$ be such that $|E| < \infty$. Then

$$\alpha(E) = 0.$$

Indeed,

$$|\alpha_s(0, 1, E)| \le |E|,$$

hence

$$\limsup_{s \to 0} s|\alpha_s(0, 1, E)| = 0.$$

Now, we discuss some useful properties of $\overline{\alpha}$. Roughly speaking, the quantity $\overline{\alpha}$ takes into account the "largest possible asymptotic opening" of a set, and so it possesses nice geometric features such as monotonicity, additivity and geometric invariances. The detailed list of these properties is the following:

PROPOSITION 3.2.3.
(i) (Monotonicity) Let $E, F \subseteq \mathbb{R}^n$ be such that for some $r > 0$ and $q \in \mathbb{R}^n$

$$E \setminus B_r(q) \subseteq F \setminus B_r(q).$$

Then

$$\overline{\alpha}(E) \le \overline{\alpha}(F).$$

(ii) (Additivity) Let $E, F \subseteq \mathbb{R}^n$ be such that for some $r > 0$ and $q \in \mathbb{R}^n$

$$(E \cap F) \setminus B_r(q) = \varnothing.$$

Then

$$\overline{\alpha}(E \cup F) \le \overline{\alpha}(E) + \overline{\alpha}(F).$$

Moreover, if $\alpha(E), \alpha(F)$ exist, then $\alpha(E \cup F)$ exists and

$$\alpha(E \cup F) = \alpha(E) + \alpha(F).$$

(iii) (Invariance with respect to rigid motions) Let $E \subseteq \mathbb{R}^n$, $x \in \mathbb{R}^n$ and $\mathcal{R} \in \mathcal{SO}(n)$ be a rotation. Then

$$\overline{\alpha}(E + x) = \overline{\alpha}(E) \quad \text{and} \quad \overline{\alpha}(\mathcal{R}E) = \overline{\alpha}(E).$$

(iv) (Scaling) Let $E \subseteq \mathbb{R}^n$ and $\lambda > 0$. Then for some $r > 0$ and $q \in \mathbb{R}^n$

$$\alpha_s(q, r, \lambda E) = \lambda^{-s} \alpha_s\left(\frac{q}{\lambda}, \frac{r}{\lambda}, E\right) \quad \text{and} \quad \overline{\alpha}(\lambda E) = \overline{\alpha}(E).$$

(v) (Symmetric difference) Let $E, F \subseteq \mathbb{R}^n$. Then for every $r > 0$ and $q \in \mathbb{R}^n$

$$|\alpha_s(q, r, E) - \alpha_s(q, r, F)| \le \alpha_s(q, r, E \Delta F).$$

As a consequence, if $|E \Delta F| < \infty$ and $\alpha(E)$ exists, then $\alpha(F)$ exists and

$$\alpha(E) = \alpha(F).$$

PROOF. (i) It is enough to notice that for every $s \in (0, 1)$

$$\alpha_s(q, r, E) \le \alpha_s(q, r, F).$$

Then, passing to limsup and recalling (3.8) we conclude that

$$\overline{\alpha}(E) \le \overline{\alpha}(F).$$

(ii) We notice that for every $s \in (0, 1)$

$$\alpha_s(q, r, E \cup F) = \alpha_s(q, r, E) + \alpha_s(q, r, F)$$

and passing to limsup and liminf as $s \to 0^+$ we obtain the desired claim.
(iii) By a change of variables, we have that

$$\alpha_s(0,1,E+x) = \int_{\mathcal{C}B_1} \frac{\chi_{E+x}(y)}{|y|^{n+s}}\,dy = \int_{\mathcal{C}B_1(-x)} \frac{\chi_E(y)}{|x+y|^{n+s}}\,dy = \alpha_s(-x,1,E).$$

Accordingly, the invariance by translation follows after passing to limsup and using (3.8). In addition, the invariance by rotations is obvious, using a change of variables.
(iv) Changing the variable $y = \lambda x$ we deduce that

$$\begin{aligned}\alpha_s(q,r,\lambda E) &= \int_{\mathcal{C}B_r(q)} \frac{\chi_{\lambda E}(y)}{|q-y|^{n+s}}\,dy = \lambda^{-s} \int_{\mathcal{C}B_{\frac{r}{\lambda}}(\frac{q}{\lambda})} \frac{\chi_E(x)}{|\frac{q}{\lambda}-x|^{n+s}}\,dx \\ &= \lambda^{-s}\alpha_s\left(\frac{q}{\lambda},\frac{r}{\lambda},E\right).\end{aligned}$$

Hence, the claim follows by passing to limsup as $s \to 0^+$.
(v) We have that

$$\begin{aligned}|\alpha_s(q,r,E)-\alpha_s(q,r,F)| &\le \int_{\mathcal{C}B_r(q)} \frac{|\chi_E(y)-\chi_F(y)|}{|y-q|^{n+s}}\,dy = \int_{\mathcal{C}B_r(q)} \frac{\chi_{E\Delta F}(y)}{|y-q|^{n+s}}\,dy \\ &= \alpha_s(q,r,E\Delta F).\end{aligned}$$

The second part of the claim follows applying the Remark 3.2.2. □

We recall the definition (see (3.1) in [40])

$$\mu(E) := \lim_{s\to 0^+} s\,\mathrm{Per}_s(E,\Omega),$$

where Ω is a bounded open set with C^2 boundary. Moreover, we define

$$\overline{\mu}(E) = \limsup_{s\to 0^+} s\,\mathrm{Per}_s(E,\Omega)$$

and give the following result:

Proposition 3.2.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with finite classical perimeter and let $E_0 \subseteq \mathcal{C}\Omega$. Then*

$$\overline{\mu}(E_0) = \overline{\alpha}(E_0)|\Omega|.$$

Proof. Let $R > 0$ be fixed such that $\Omega \subseteq B_R$, $y \in \Omega$ be any fixed point and $\varepsilon \in (0,1)$ be small enough such that $R/\varepsilon > R+1$. This choice of ε assures that $B_1(y) \subseteq B_{R/\varepsilon}$. We have that

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\chi_{E_0}(x)}{|x-y|^{n+s}}\,dx = \int_{\mathcal{C}B_{R/\varepsilon}} \frac{\chi_{E_0}(x)}{|x-y|^{n+s}}\,dx + \int_{B_{R/\varepsilon}\setminus B_1(y)} \frac{\chi_{E_0}(x)}{|x-y|^{n+s}}\,dx \\ + \int_{B_1(y)} \frac{\chi_{E_0}(x)}{|x-y|^{n+s}}\,dx.\end{aligned}$$

Since $|x-y| \ge (1-\varepsilon)|x|$ whenever $x \in \mathcal{C}B_{R/\varepsilon}$, we get

$$\int_{\mathcal{C}B_{R/\varepsilon}} \frac{\chi_{E_0}(x)}{|x-y|^{n+s}}\,dx \le (1-\varepsilon)^{-n-s} \int_{\mathcal{C}B_{R/\varepsilon}} \frac{\chi_{E_0}(x)}{|x|^{n+s}}\,dx.$$

Also we have that

$$\int_{B_{R/\varepsilon}\setminus B_1(y)} \frac{\chi_{E_0}(x)}{|x-y|^{n+s}}\,dx \le \varpi_n \int_1^{R/\varepsilon+R} \varrho^{-s-1}\,d\varrho \le \varpi_n \frac{1-\left(\frac{R}{\varepsilon}+R\right)^{-s}}{s}.$$

Also, we can assume that $s < 1/2$ (since we are interested in what happens for $s \to 0$). In this way, if $|x-y|<1$ we have that $|x-y|^{-n-s} \le |x-y|^{-n-\frac{1}{2}}$, and so

$$\int_{B_1(y)} \frac{\chi_{E_0}(x)}{|x-y|^{n+s}}\,dx \le \int_{B_1(y)} \frac{\chi_{E_0}(x)}{|x-y|^{n+\frac{1}{2}}}\,dx.$$

Also, since $E_0 \subseteq \mathcal{C}\Omega$, we have that

$$\int_{B_1(y)} \frac{\chi_{E_0}(x)}{|x-y|^{n+\frac{1}{2}}}\,dx \le \int_{B_1(y)\setminus\Omega} \frac{dx}{|x-y|^{n+\frac{1}{2}}} \le \int_{\mathcal{C}\Omega} \frac{dx}{|x-y|^{n+\frac{1}{2}}}.$$

This means that

$$\int_\Omega \int_{B_1(y)} \frac{\chi_{E_0}(x)}{|x-y|^{n+s}}\,dx\,dy \le \int_\Omega \int_{\mathcal{C}\Omega} \frac{dx}{|x-y|^{n+\frac{1}{2}}} = \operatorname{Per}_{\frac{1}{2}}(\Omega) = c < \infty,$$

since Ω has a finite classical perimeter. In this way, it follows that

$$\begin{aligned} s\operatorname{Per}_s(E_0,\Omega) &= \int_\Omega \int_{\mathbb{R}^n} \frac{\chi_{E_0}(x)}{|x-y|^{n+s}} dx\,dy \\ &\le s(1-\varepsilon)^{-n-s}|\Omega| \int_{\mathcal{C}B_{R/\varepsilon}} \frac{\chi_{E_0}(x)}{|x|^{n+s}} dx + \varpi_n \left(1 - \left(\frac{R}{\varepsilon}+R\right)^{-s}\right)|\Omega| + sc. \end{aligned} \tag{3.12}$$

Furthermore, notice that if $x \in B_{R/\varepsilon}$ we have that $|x-y| \le (1+\varepsilon)|x|$, hence

$$\int_{\mathbb{R}^n} \frac{\chi_{E_0}(x)}{|x-y|^{n+s}}\,dx \ge \int_{\mathcal{C}B_{R/\varepsilon}} \frac{\chi_{E_0}(x)}{|x-y|^{n+s}}\,dx \ge (1+\varepsilon)^{-n-s} \int_{\mathcal{C}B_{R/\varepsilon}} \frac{\chi_{E_0}(x)}{|x|^{n+s}}\,dx.$$

Thus for any $\varepsilon > 0$

$$s\operatorname{Per}_s(E_0,\Omega) \ge s|\Omega|(1+\varepsilon)^{-n-s} \int_{\mathcal{C}B_{R/\varepsilon}} \frac{\chi_{E_0}(x)}{|x|^{n+s}}\,dx.$$

Passing to limsup as $s \to 0^+$ here above and in (3.12) it follows that

$$(1+\varepsilon)^{-n}\overline{\alpha}(E_0)\,|\Omega| \le \overline{\mu}(E_0) \le (1-\varepsilon)^{-n}\overline{\alpha}(E_0)\,|\Omega|.$$

Sending $\varepsilon \to 0$, we obtain the desired conclusion. □

3.3. Classification of nonlocal minimal surfaces for small s

3.3.1. Asymptotic estimates of the density (Theorem 3.1.5). The importance of Theorem 3.1.5 is threefold:

- first of all, it is an interesting result in itself, by stating (in the usual hypothesis in which the contribution from infinity of the exterior data E_0 is less than that of a half-space) that any ball of fixed radius, centered at some $x \in \overline{\Omega}$, contains at least a portion of the complement of an s-minimal set E, when s is small enough. We further observe that Theorem 3.1.5 actually provides a "uniform" measure theoretic estimate of how big this portion is, purely in terms of the fixed datum $\overline{\alpha}(E_0)$.
- Moreover, we point out that Definition 3.1.3 does not exlude apriori "full" sets, i.e. sets E such that $E \cap \Omega = \Omega$. Hence, in the situation of point (A) of Theorem 3.1.4, one may wonder whether an s-minimal set E, which is δ_s-dense, can actually completely cover Ω. The answer is no: Theorem 3.1.5 proves in particular that the contribution from infinity forces the domain Ω, for s small enough, to contain at least a non-trivial portion of the complement of E.
- Finally, the density estimate of Theorem 3.1.5 serves as an auxiliary result for the proof of part (B) of our main Theorem 3.1.4.

Proof of Theorem 3.1.5. We begin with two easy but useful preliminary remarks. We observe that, given a set $F \subseteq \mathbb{R}^n$ and two open sets $\Omega' \subseteq \Omega$, we have

$$\mathrm{Per}_s(F, \Omega') \leq \mathrm{Per}_s(F, \Omega). \tag{3.13}$$

Also, we point out that, given an open set $\mathcal{O} \subseteq \mathbb{R}^n$ and a set $F \subseteq \mathbb{R}^n$, then by the definition of the fractional perimeter, it holds

$$F \cap \Omega = \varnothing \quad \Longrightarrow \quad \mathrm{Per}_s(F, \mathcal{O}) = \int_F \int_{\mathcal{O}} \frac{dx\, dy}{|x-y|^{n+s}}. \tag{3.14}$$

With these observations at hand, we are ready to proceed with the proof of the Theorem. We argue by contradiction.

Suppose that there exists $\delta > 0$ and $\gamma \in (0,1)$ for which we can find a sequence $s_k \searrow 0$, a sequence of sets $\{E_k\}$ such that each E_k is s_k-minimal in Ω with exterior data E_0, and a sequence of points $\{x_k\} \subseteq \overline{\Omega}$ such that

$$\big|(\Omega \cap B_\delta(x_k)) \setminus E_k\big| < \gamma\, \frac{\varpi_n - 2\overline{\alpha}(E_0)}{\varpi_n - \overline{\alpha}(E_0)} \big|\Omega \cap B_\delta(x_k)\big|. \tag{3.15}$$

As a first step, we are going to exploit (3.15) in order to obtain a bound from below for the limit as $k \to \infty$ of $s_k\, \mathrm{Per}_{s_k}(E_k, \Omega \cap B_\delta(x_k))$ (see the forthcoming inequality (3.17)).

First of all we remark that, since $\overline{\Omega}$ is compact, up to passing to subsequences we can suppose that $x_k \longrightarrow x_0$, for some $x_0 \in \overline{\Omega}$. Now we observe that from (3.15) it follows that

$$\begin{aligned} |E_k \cap (\Omega \cap B_\delta(x_k))| &= |\Omega \cap B_\delta(x_k)| - \big|(\Omega \cap B_\delta(x_k)) \setminus E_k\big| \\ &> \frac{(1-\gamma)\varpi_n - (1-2\gamma)\overline{\alpha}(E_0)}{\varpi_n - \overline{\alpha}(E_0)}\, |\Omega \cap B_\delta(x_k)|, \end{aligned}$$

and hence, since $x_k \longrightarrow x_0$,

$$\liminf_{k\to\infty} |E_k \cap (\Omega \cap B_\delta(x_k))| \geq \frac{(1-\gamma)\varpi_n - (1-2\gamma)\overline{\alpha}(E_0)}{\varpi_n - \overline{\alpha}(E_0)}\, |\Omega \cap B_\delta(x_0)|. \tag{3.16}$$

Notice that, since Ω is bounded, we can find $R > 0$ such that $\Omega \Subset B_R(q)$ for every $q \in \overline{\Omega}$. Then we obtain that

$$\begin{aligned} \mathrm{Per}_{s_k}(E_k, \Omega \cap B_\delta(x_k)) &\geq \int_{E_k \cap (\Omega \cap B_\delta(x_k))} \Big(\int_{\mathcal{C}E_k \setminus (\Omega \cap B_\delta(x_k))} \frac{dz}{|y-z|^{n+s_k}} \Big) dy \\ &\geq \int_{E_k \cap (\Omega \cap B_\delta(x_k))} \Big(\int_{\mathcal{C}\Omega} \frac{\chi_{\mathcal{C}E_0}(z)}{|y-z|^{n+s_k}}\, dz \Big) dy \\ &\geq \int_{E_k \cap (\Omega \cap B_\delta(x_k))} \Big(\inf_{q \in \overline{\Omega}} \int_{\mathcal{C}\Omega} \frac{\chi_{\mathcal{C}E_0}(z)}{|q-z|^{n+s_k}}\, dz \Big) dy \\ &\geq \big|E_k \cap (\Omega \cap B_\delta(x_k))\big| \inf_{q \in \overline{\Omega}} \int_{\mathcal{C}B_R(q)} \frac{\chi_{\mathcal{C}E_0}(z)}{|q-z|^{n+s_k}}\, dz. \end{aligned}$$

So, thanks to Proposition 3.2.1 and recalling (3.16), we find

$$\begin{aligned} &\liminf_{k\to\infty} s_k\, \mathrm{Per}_{s_k}(E_k, \Omega \cap B_\delta(x_k)) \\ &\qquad \geq \Big(\liminf_{k\to\infty} |E_k \cap (\Omega \cap B_\delta(x_k))| \Big) \Big(\liminf_{k\to\infty} s_k \inf_{q \in \overline{\Omega}} \int_{\mathcal{C}B_R(q)} \frac{\chi_{\mathcal{C}E_0}(z)}{|q-z|^{n+s_k}}\, dz \Big) \\ &\qquad = \big(\varpi_n - \overline{\alpha}(E_0)\big) \Big(\liminf_{k\to\infty} |E_k \cap (\Omega \cap B_\delta(x_k))| \Big) \\ &\qquad \geq \big(\varpi_n - \overline{\alpha}(E_0)\big) \frac{(1-\gamma)\varpi_n - (1-2\gamma)\overline{\alpha}(E_0)}{\varpi_n - \overline{\alpha}(E_0)}\, |\Omega \cap B_\delta(x_0)|. \end{aligned} \tag{3.17}$$

On the other hand, as a second step we claim that

$$\limsup_{k\to\infty} s_k \operatorname{Per}_{s_k}(E_k, \Omega\cap B_\delta(x_k)) \le \overline{\alpha}(E_0)\big|\Omega\cap B_\delta(x_0)\big|. \tag{3.18}$$

We point out that obtaining the inequality (3.18) is a crucial step of the proof. Indeed, exploiting both (3.18) and (3.17), we obtain

$$\begin{aligned}\overline{\alpha}(E_0)\,|\Omega\cap B_\delta(x_0)| &\ge \liminf_{k\to\infty} s_k \operatorname{Per}_{s_k}(E_k, \Omega\cap B_\delta(x_k))\\ &\ge \big((1-\gamma)\varpi_n - (1-2\gamma)\overline{\alpha}(E_0)\big)|\Omega\cap B_\delta(x_0)|.\end{aligned} \tag{3.19}$$

Then, since $x_0\in\overline{\Omega}$ implies that

$$|\Omega\cap B_\delta(x_0)|>0,$$

by (3.19) we get

$$\overline{\alpha}(E_0)\ge(1-\gamma)\varpi_n-(1-2\gamma)\overline{\alpha}(E_0)\quad\text{ that is }\quad(1-\gamma)\overline{\alpha}(E_0)\ge(1-\gamma)\frac{\varpi_n}{2}.$$

Therefore, since $\gamma\in(0,1)$ and by hypothesis $\overline{\alpha}(E_0)<\frac{\varpi_n}{2}$, we reach a contradiction, concluding the proof.

We are left to prove (3.17). For this, we exploit the minimality of the sets E_k in order to compare the s_k-perimeter of E_k with the s_k-perimeter of appropriate competitors F_k.

We first remark that, since $x_k\longrightarrow x_0$, for every $\varepsilon>0$ there exists $\tilde{k}_\varepsilon$ such that

$$\Omega\cap B_\delta(x_k)\subseteq\Omega\cap B_{\delta+\varepsilon}(x_0),\qquad\forall\, k\ge\tilde{k}_\varepsilon. \tag{3.20}$$

We fix a small $\varepsilon>0$. We will let $\varepsilon\to0$ later on.

We also observe that, since E_k is s_k-minimal in Ω, it is s_k-minimal also in every $\Omega'\subseteq\Omega$, hence in particular in $\Omega\cap B_{\delta+\varepsilon}(x_0)$. Now we proceed to define the sets

$$F_k:=E_0\cup(E_k\cap(\Omega\setminus B_{\delta+\varepsilon}(x_0)))=E_k\setminus\big(\Omega\cap B_{\delta+\varepsilon}(x_0)\big). \tag{3.21}$$

Then, by (3.13), (3.20), (3.21) and by the minimality of E_k in $\Omega\cap B_{\delta+\varepsilon}(x_0)$, for every $k\ge\tilde{k}_\varepsilon$ we find that

$$\operatorname{Per}_{s_k}(E_k,\Omega\cap B_\delta(x_k))\le\operatorname{Per}_{s_k}(E_k,\Omega\cap B_{\delta+\varepsilon}(x_0))\le\operatorname{Per}_{s_k}(F_k,\Omega\cap B_{\delta+\varepsilon}(x_0)).$$

We observe that by the definition (3.21) we have that

$$F_k\cap\big(\Omega\cap B_{\delta+\varepsilon}(x_0)\big)=\varnothing.$$

Therefore, recalling (3.14) and the definition (3.21) of the sets F_k, we obtain that

$$\begin{aligned}\operatorname{Per}_{s_k}(F_k,\Omega\cap B_{\delta+\varepsilon}(x_0))&=\int_{E_0\cup(E_k\cap(\Omega\setminus B_{\delta+\varepsilon}(x_0)))}\int_{\Omega\cap B_{\delta+\varepsilon}(x_0)}\frac{dy\,dz}{|y-z|^{n+s_k}}\\ &=\int_{E_0}\int_{\Omega\cap B_{\delta+\varepsilon}(x_0)}\frac{dy\,dz}{|y-z|^{n+s_k}}+\int_{E_k\cap(\Omega\setminus B_{\delta+\varepsilon}(x_0))}\int_{\Omega\cap B_{\delta+\varepsilon}(x_0)}\frac{dy\,dz}{|y-z|^{n+s_k}}\\ &\le\int_{E_0}\int_{\Omega\cap B_{\delta+\varepsilon}(x_0)}\frac{dy\,dz}{|y-z|^{n+s_k}}+\int_{\Omega\setminus B_{\delta+\varepsilon}(x_0)}\int_{\Omega\cap B_{\delta+\varepsilon}(x_0)}\frac{dy\,dz}{|y-z|^{n+s_k}}\\ &=:I_k^1+I_k^2.\end{aligned}$$

Furthermore, again by (3.14), we have that

$$I_k^1=\operatorname{Per}_{s_k}(E_0,\Omega\cap B_{\delta+\varepsilon}(x_0))\quad\text{ and }\quad I_k^2=\operatorname{Per}_{s_k}(\Omega\setminus B_{\delta+\varepsilon}(x_0),\Omega\cap B_{\delta+\varepsilon}(x_0)). \tag{3.22}$$

We observe that the open set $\Omega\cap B_{\delta+\varepsilon}(x_0)$ has finite classical perimeter. Thus, we can exploit the equalities (3.22) and apply Proposition 3.2.4 twice, obtaining

$$\limsup_{k\to\infty} s_k I_k^1\le\overline{\alpha}(E_0)\big|\Omega\cap B_{\delta+\varepsilon}(x_0)\big|,$$

and

$$\limsup_{k\to\infty} s_k I_k^2 \le \overline{\alpha}(\Omega\setminus B_{\delta+\varepsilon}(x_0))\big|\Omega\cap B_{\delta+\varepsilon}(x_0)\big|, \tag{3.23}$$

for every $\varepsilon>0$. Also notice that, since Ω is bounded, by Remark 3.2.2 we have

$$\overline{\alpha}(\Omega\setminus B_{\delta+\varepsilon}(x_0)) = \alpha(\Omega\setminus B_{\delta+\varepsilon}(x_0)) = 0,$$

and hence, by (3.23),

$$\lim_{k\to\infty} s_k I_k^2 = 0.$$

Therefore, combining these computations we find that

$$\limsup_{k\to\infty} s_k \operatorname{Per}_{s_k}(E_k,\Omega\cap B_\delta(x_k)) \le \limsup_{k\to\infty} s_k I_k^1 \le \overline{\alpha}(E_0)\big|\Omega\cap B_{\delta+\varepsilon}(x_0)\big|,$$

for every $\varepsilon>0$ small. To conclude, we let $\varepsilon\to 0$ and we obtain (3.18). □

It is interesting to observe that, as a straightforward consequence of Theorem 3.1.5, when $\alpha(E_0)=0$ we know that any sequence of s-minimal sets is asymptotically empty inside Ω, as $s\to 0^+$. More precisely

Corollary 3.3.1. *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set of finite classical perimeter and let $E_0\subseteq\mathcal{C}\Omega$ be such that $\alpha(E_0)=0$. Let $s_k\in(0,1)$ be such that $s_k\searrow 0$ and let $\{E_k\}$ be a sequence of sets such that each E_k is s_k-minimal in Ω with exterior data E_0. Then*

$$\lim_{k\to\infty} |E_k\cap\Omega| = 0.$$

Proof. Fix $\delta>0$. Since $\overline{\Omega}$ is compact, we can find a finite number of points $x_1,\dots,x_m\in\overline{\Omega}$ such that

$$\overline{\Omega}\subseteq\bigcup_{i=1}^m B_\delta(x_i).$$

By Theorem 3.1.5 (by using the fact that $\alpha(E_0)=0$) we know that for every $\gamma\in(0,1)$ we can find a $k(\gamma)$ big enough such that

$$\big|(\Omega\cap B_\delta(x_i))\setminus E_k\big| \ge \gamma\,\big|\Omega\cap B_\delta(x_i)\big|.$$

Then,

$$\big|E_k\cap(\Omega\cap B_\delta(x_i))\big| = \big|\Omega\cap B_\delta(x_i)\big| - \big|(\Omega\cap B_\delta(x_i))\setminus E_k\big| \le (1-\gamma)|\Omega\cap B_\delta(x_i)|,$$

for every $i=1,\dots,m$ and every $k\ge k(\gamma)$. Thus

$$|E_k\cap\Omega| \le (1-\gamma)\sum_{i=1}^m |\Omega\cap B_\delta(x_i)|,$$

for every $k\ge k(\gamma)$, and hence

$$\limsup_{k\to\infty} |E_k\cap\Omega| \le (1-\gamma)\sum_{i=1}^m |\Omega\cap B_\delta(x_i)|,$$

for every $\gamma\in(0,1)$. Letting $\gamma\longrightarrow 1^-$ concludes the proof. □

We recall here that any set E_0 of finite measure has $\alpha(E_0)=0$ (check Remark 3.2.2).

3.3.2. Estimating the fractional mean curvature (Theorem 3.1.2). Thanks to the previous preliminary work, we are now in the position of completing the proof of Theorem 3.1.2.

Proof of Theorem 3.1.2. Let $R := 2\max\{1, \operatorname{diam}(\Omega)\}$. First of all, (3.8) implies that

$$\liminf_{s\to 0^+}\left(\varpi_n R^{-s} - 2s\sup_{q\in\overline{\Omega}}\int_{\mathcal{C}B_R(q)}\frac{\chi_E(y)}{|q-y|^{n+s}}\,dy\right) = \varpi_n - 2\overline{\alpha}(E_0) = 4\beta.$$

Notice that by (3.3), $\beta > 0$. Hence for every s small enough, say $s < s' \le \frac{1}{2}$ with $s' = s'(E_0,\Omega)$, we have that

$$\varpi_n R^{-s} - 2s\sup_{q\in\overline{\Omega}}\int_{\mathcal{C}B_R(q)}\frac{\chi_E(y)}{|q-y|^{n+s}}\,dy \ge \frac{7}{2}\beta. \tag{3.24}$$

Now, let $E\subseteq\mathbb{R}^n$ be such that $E\setminus\Omega = E_0$, suppose that E has an exterior tangent ball of radius $\delta < R/2$ at $q\in\partial E\cap\overline{\Omega}$, that is

$$B_\delta(p)\subseteq \mathcal{C}E \quad\text{and}\quad q\in\partial B_\delta(p),$$

and let $s < s'$. Then for ϱ small enough (say $\varrho < \delta/2$) we conclude that

$$H_s^\varrho[E](q) = \int_{B_R(q)\setminus B_\varrho(q)}\frac{\chi_{\mathcal{C}E}(y)-\chi_E(y)}{|q-y|^{n+s}}\,dy + \int_{\mathcal{C}B_R(q)}\frac{\chi_{\mathcal{C}E}(y)-\chi_E(y)}{|q-y|^{n+s}}\,dy.$$

Let $D_\delta = B_\delta(p)\cap B_\delta(p')$, where p' is the symmetric of p with respect to q, i.e. the ball $B_\delta(p')$ is the ball tangent to $B_\delta(p)$ in q. Let also K_δ be the convex hull of D_δ and let $\operatorname{Per}_\delta := K_\delta - D_\delta$. Notice that $B_\varrho(q)\subseteq K_\delta\subseteq B_R(q)$. Then

$$\begin{aligned}\int_{B_R(q)\setminus B_\varrho(q)}\frac{\chi_{\mathcal{C}E}(y)-\chi_E(y)}{|q-y|^{n+s}}\,dy &= \int_{D_\delta\setminus B_\varrho(q)}\frac{\chi_{\mathcal{C}E}(y)-\chi_E(y)}{|q-y|^{n+s}}\,dy\\ &\quad+\int_{\operatorname{Per}_\delta\setminus B_\varrho(q)}\frac{\chi_{\mathcal{C}E}(y)-\chi_E(y)}{|q-y|^{n+s}}\,dy + \int_{B_R(q)\setminus K_\delta}\frac{\chi_{\mathcal{C}E}(y)-\chi_E(y)}{|q-y|^{n+s}}\,dy.\end{aligned}$$

Since $B_\delta(p)\subseteq\mathcal{C}E$, by symmetry we obtain that

$$\begin{aligned}&\int_{D_\delta\setminus B_\varrho(q)}\frac{\chi_{\mathcal{C}E}(y)-\chi_E(y)}{|q-y|^{n+s}}\,dy\\ &\qquad=\int_{B_\delta(p)\setminus B_\varrho(q)}\frac{dy}{|q-y|^{n+s}} + \int_{B_\delta(p')\setminus B_\varrho(q)}\frac{\chi_{\mathcal{C}E}(y)-\chi_E(y)}{|q-y|^{n+s}}\,dy \ge 0.\end{aligned}$$

Moreover, from [43, Lemma 3.1] (here applied with $\lambda = 1$) we have that

$$\left|\int_{\operatorname{Per}_\delta\setminus B_\varrho(q)}\frac{\chi_{\mathcal{C}E}(y)-\chi_E(y)}{|q-y|^{n+s}}\,dy\right| \le \int_{\operatorname{Per}_\delta}\frac{dy}{|q-y|^{n+s}} \le \frac{C_0}{1-s}\delta^{-s},$$

with $C_0 = C_0(n) > 0$. Notice that $B_\delta(q)\subseteq K_\delta$ so

$$\left|\int_{B_R(q)\setminus K_\delta}\frac{\chi_{\mathcal{C}E}(y)-\chi_E(y)}{|q-y|^{n+s}}\,dy\right| \le \int_{B_R(q)\setminus B_\delta(q)}\frac{dy}{|q-y|^{n+s}} = \varpi_n\frac{\delta^{-s}-R^{-s}}{s}.$$

Therefore for every $\varrho < \delta/2$ one has that

$$\int_{B_R(q)\setminus B_\varrho(q)}\frac{\chi_{\mathcal{C}E}(y)-\chi_E(y)}{|q-y|^{n+s}}\,dy \ge -\frac{C_0}{1-s}\delta^{-s} - \frac{\varpi_n}{s}\delta^{-s} + \frac{\varpi_n}{s}R^{-s}.$$

Thus, using (3.24)

$$\begin{aligned}
H_s^\varrho[E](q) &= \int_{B_R(q)\setminus B_\varrho(q)} \frac{\chi_{\mathcal{C}E}(y)-\chi_E(y)}{|q-y|^{n+s}}\,dy + \int_{\mathcal{C}B_R(q)} \frac{\chi_{\mathcal{C}E}(y)-\chi_E(y)}{|q-y|^{n+s}}\,dy\\
&\geq -\frac{C_0}{1-s}\delta^{-s} - \frac{\varpi_n}{s}\delta^{-s} + \frac{\varpi_n}{s}R^{-s} + \int_{\mathcal{C}B_R(q)} \frac{dy}{|q-y|^{n+s}} - 2\int_{\mathcal{C}B_R(q)} \frac{\chi_E(y)}{|q-y|^{n+s}}\,dy\\
&\geq -\delta^{-s}\Big(\frac{C_0}{1-s}+\frac{\varpi_n}{s}\Big) + \frac{\varpi_n}{s}R^{-s} + \Big(\frac{\varpi_n}{s}R^{-s} - 2\sup_{q\in\overline{\Omega}}\int_{\mathcal{C}B_R(q)} \frac{\chi_E(y)}{|q-y|^{n+s}}\,dy\Big)\\
&\geq -\delta^{-s}\Big(\frac{C_0}{1-s}+\frac{\varpi_n}{s}\Big) + \frac{\varpi_n}{s}R^{-s} + \frac{7\beta}{2s}\\
&\geq -\delta^{-s}\Big(2C_0+\frac{\varpi_n}{s}\Big) + \frac{\varpi_n}{s}R^{-s} + \frac{7\beta}{2s},
\end{aligned}$$

where we also exploited that $s < s' \leq 1/2$. Since $R > 1$, we have

$$R^{-s} \to 1^-, \qquad \text{as } s \to 0^+.$$

Therefore we can find $s'' = s''(E_0,\Omega)$ small enough such that

$$\varpi_n R^{-s} \geq \varpi_n - \frac{\beta}{2}, \qquad \forall s < s''.$$

Now let

$$s_0 = s_0(E_0,\Omega) := \min\Big\{s', s'', \frac{\beta}{2C_0}\Big\}.$$

Then, for every $s < s_0$ we have

$$\begin{aligned}
H_s^\varrho[E](q) &\geq \frac{1}{s}\Big\{-\delta^{-s}\big((2C_0)s+\varpi_n\big) + \varpi_n R^{-s} + \frac{7}{2}\beta\Big\}\\
&\geq \frac{1}{s}\big\{-\delta^{-s}(\varpi_n+\beta) + \varpi_n + 3\beta\big\},
\end{aligned} \tag{3.25}$$

for every $\varrho \in (0,\delta/2)$.

Notice that if we fix $s \in (0,s_0)$, then for every

$$\delta \geq e^{-\frac{1}{s}\log\frac{\varpi_n+2\beta}{\varpi_n+\beta}} =: \delta_s(E_0),$$

we have that

$$-\delta^{-s}(\varpi_n+\beta) + \varpi_n + 3\beta \geq \beta > 0.$$

To conclude, we let $\sigma \in (0,s_0)$ and suppose that E has an exterior tangent ball of radius δ_σ at $q \in \partial E \cap \overline{\Omega}$. Notice that, since $\delta_\sigma < 1$, we have

$$-(\delta_\sigma)^{-s}(\varpi_n+\beta) + \varpi_n + 3\beta \geq -(\delta_\sigma)^{-\sigma}(\varpi_n+\beta) + \varpi_n + 3\beta = \beta, \qquad \forall\, s \in (0,\sigma].$$

Then (3.25) gives that

$$\liminf_{\varrho\to 0^+} H_s^\varrho[E](q) \geq \frac{\beta}{s} > 0, \qquad \forall\, s \in (0,\sigma],$$

which concludes the proof. □

Remark 3.3.2. We remark that

$$\log\frac{\varpi_n+2\beta}{\varpi_n+\beta} > 0,$$

thus

$$\delta_s \to 0^+ \qquad \text{as } s \to 0^+.$$

As a consequence of Theorem 3.1.2, we have that, as $s \to 0^+$, the s-minimal sets with small mass at infinity have small mass in Ω. The precise result goes as follows:

Corollary 3.3.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, let $E \subseteq \mathbb{R}^n$ be such that*

$$\overline{\alpha}(E) < \frac{\varpi_n}{2},$$

and suppose that ∂E is of class C^2 in Ω. Then, for every $\Omega' \Subset \Omega$ there exists $\tilde{s} = \tilde{s}(E \cap \overline{\Omega'}) \in (0, s_0)$ such that for every $s \in (0, \tilde{s}]$

$$H_s[E](q) \geq \frac{\varpi_n - 2\overline{\alpha}(E)}{4s} > 0, \qquad \forall\, q \in \partial E \cap \overline{\Omega'}. \tag{3.26}$$

Proof. Since ∂E is of class C^2 in Ω and $\Omega' \Subset \Omega$, the set E satisfies a uniform exterior ball condition of radius $\tilde{\delta} = \tilde{\delta}(E \cap \overline{\Omega'})$ in $\overline{\Omega'}$, meaning that E has an exterior tangent ball of radius at least $\tilde{\delta}$ at every point $q \in \partial E \cap \overline{\Omega'}$.

Now, since $\delta_s \to 0^+$ as $s \to 0^+$, we can find $\tilde{s} = \tilde{s}(E \cap \overline{\Omega'}) < s_0(E \setminus \Omega, \Omega)$, small enough such that $\delta_s < \tilde{\delta}$ for every $s \in (0, \tilde{s}]$. Then we can conclude by applying Theorem 3.1.2. □

3.3.3. Classification of s-minimal surfaces (Theorem 3.1.4). To classify the behavior of the s-minimal surfaces when s is small, we need to take into account the "worst case scenario", that is the one in which the set behaves very badly in terms of oscillations and lack of regularity. To this aim, we make an observation about δ-dense sets.

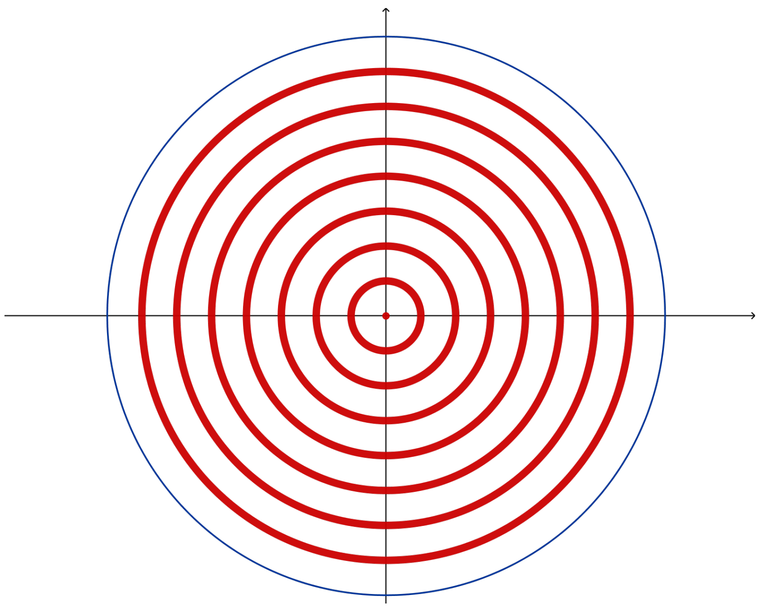


Figure 1. A δ-dense set of measure $< \varepsilon$

Remark 3.3.4. For every $k \geq 1$ and every $\varepsilon < 2^{-k-1}$, we define the sets

$$\Gamma_k^\varepsilon := B_\varepsilon \cup \bigcup_{i=1}^{2^k-1} \left\{ x \in \mathbb{R}^n \,\Big|\, \frac{i}{2^k} - \varepsilon < |x| < \frac{i}{2^k} + \varepsilon \right\} \quad \text{and} \quad \Gamma_k := \{0\} \cup \bigcup_{i=1}^{2^k-1} \partial B_{\frac{i}{2^k}}.$$

Notice that for every $\delta > 0$ there exists $\tilde{k} = \tilde{k}(\delta)$ such that for every $k \geq \tilde{k}$ we have

$$B_\delta(x) \cap \Gamma_k \neq \varnothing, \qquad \forall\, B_\delta(x) \subseteq B_1.$$

Thus, for every $k \geq \tilde{k}(\delta)$ and $\varepsilon < 2^{-k-1}$, the set Γ_k^ε is δ-dense in B_1. Moreover, notice that

$$\Gamma_k = \bigcap_{\varepsilon\in(0,2^{-k-1})} \Gamma_k^\varepsilon \quad \text{and} \quad \lim_{\varepsilon\to 0^+} |\Gamma_k^\varepsilon| = 0.$$

It is also worth remarking that the sets Γ_k^ε have smooth boundary. In particular, for every $\delta > 0$ and every $\varepsilon > 0$ small, we can find a set $E \subseteq B_1$ which is δ-dense in B_1 and whose measure is $|E| < \varepsilon$. This means that we can find an open set E with smooth boundary, whose measure is arbitrarily small and which is "topologically arbitrarily dense" in B_1.

We introduce the following useful geometric observation.

PROPOSITION 3.3.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and connected open set with C^2 boundary and let $\delta \in (0, r_0)$, for r_0 given in (B.1). If E is not δ-dense in Ω and $|E \cap \Omega| > 0$, then there exists a point $q \in \partial E \cap \Omega$ such that E has an exterior tangent ball at q of radius δ (contained in Ω), i.e. there exist $p \in \mathcal{C}E \cap \Omega$ such that*

$$B_\delta(p) \Subset \Omega, \qquad q \in \partial B_\delta(p) \cap \partial E \quad \textit{and} \quad B_\delta(p) \subseteq \mathcal{C}E.$$

PROOF. Using Definition 3.1.3, we have that there exists $x \in \Omega$ for which $B_\delta(x) \Subset \Omega$ and $|B_\delta(x) \cap E| = 0$, so $B_\delta(x) \subseteq E_{ext}$. If $B_\delta(x)$ is tangent to ∂E then we are done.

Notice that

$$B_\delta(x) \Subset \Omega \quad \Longrightarrow \quad d(x, \partial\Omega) > \delta,$$

and let

$$\delta' := \min\{r_0, d(x, \partial\Omega)\} \in (\delta, r_0].$$

Now we consider the open set $\Omega_{-\delta'} \subseteq \Omega$

$$\Omega_{-\delta'} := \{\bar{d}_\Omega < -\delta'\},$$

so $x \in \Omega_{-\delta'}$. According to Remark B.1.4 and Lemma B.1.5 we have that $\Omega_{-\delta'}$ has C^2 boundary and that

$$\Omega_{-\delta'} \text{ satisfies the uniform interior ball condition of radius at least } r_0. \tag{3.27}$$

We have two possibilities:

$$\begin{aligned} &\text{i)} \quad \overline{E} \cap \Omega_{-\delta'} \neq \varnothing \\ &\text{ii)} \quad \varnothing \neq \overline{E} \cap \Omega \subseteq \Omega \setminus \Omega_{-\delta'}. \end{aligned} \tag{3.28}$$

If i) happens, we pick any point $y \in \overline{E} \cap \Omega_{-\delta'}$. The set $\overline{\Omega_{-\delta'}}$ is path connected (see Proposition B.1.6), so there exists a path $c : [0,1] \longrightarrow \mathbb{R}^n$ that connects x to y and that stays inside $\overline{\Omega_{-\delta'}}$, that is

$$c(0) = x, \qquad c(1) = y \quad \text{and} \quad c(t) \in \overline{\Omega_{-\delta'}}, \quad \forall\, t \in [0,1].$$

Moreover, since $\delta < \delta'$, we have

$$B_\delta\big(c(t)\big) \Subset \Omega \qquad \forall\, t \in [0,1].$$

Hence, we can "slide the ball" $B_\delta(x)$ along the path and we obtain the desired claim thanks to Lemma B.2.1.

Now, if we are in the case ii) of (3.28), then $\Omega_{-\delta'} \subseteq E_{ext}$, so we dilate $\Omega_{-\delta'}$ until we first touch $\overline{E}$. That is, we consider

$$\tilde{\varrho} := \inf\{\varrho \in [0, \delta'] \,\big|\, \Omega_{-\varrho} \subseteq E_{ext}\}.$$

Notice that by hypothesis $\tilde{\varrho} > 0$. Then

$$\overline{\Omega_{-\tilde{\varrho}}} \subseteq \overline{E_{ext}} = E_{ext} \cup \partial E.$$

If
$$\partial\Omega_{-\tilde\varrho}\cap\partial E=\varnothing \quad \text{then} \quad \overline{\Omega_{-\tilde\varrho}}\subseteq E_{ext},$$
hence we have that
$$d=d\left(\overline{E}\cap\Omega\setminus\Omega_{-\delta'},\overline{\Omega_{-\tilde\varrho}}\right)\in(0,\tilde\varrho),$$
therefore
$$\Omega_{-\tilde\varrho}\subseteq\Omega_{-(\tilde\varrho-d)}\subseteq E_{ext}.$$
This is in contradiction with the definition of $\tilde\varrho$. Hence, there exists $q\in\partial\Omega_{-\tilde\varrho}\cap\partial E$.

Recall that, by definition of $\tilde\varrho$, we have $\Omega_{-\tilde\varrho}\subseteq\mathcal{C}E$. Thanks to (3.27), there exists a tangent ball at q interior to $\Omega_{-\tilde\varrho}$, hence a tangent ball at q exterior to E, of radius at least $r_0>\delta$. This concludes the proof of the lemma. □

We observe that part (A) of Theorem 3.1.4 is essentially a consequence of Theorem 3.1.2. Indeed, if an s-minimal set E is not δ_s-dense and it is not empty in Ω, then by Proposition 3.3.5 we can find a point $q\in\partial E\cap\Omega$ at which E has an exterior tangent ball of radius δ_s. Then Theorem 3.1.2 implies that the s-fractional mean curvature of E in q is strictly positive, contradicting the Euler-Lagrange equation.

On the other hand, part (B) of Theorem 3.1.4 follows from a careful asymptotic use of the density estimates provided by Theorem 3.1.5. For the reader's facility, we also recall that r_0 has the same meaning here and across the chapter, as clarified in Appendix B.1.1. We now proceed with the precise arguments of the proof.

Proof of Theorem 3.1.4. We begin by proving part (A).
First of all, since $\delta_s\to0^+$, we can find $s_1=s_1(E_0,\Omega)\in(0,s_0]$ such that $\delta_s<r_0$ for every $s\in(0,s_1)$.

Now let $s\in(0,s_1)$ and let E be s-minimal in Ω, with exterior data E_0.

We suppose that $E\cap\Omega\neq\varnothing$ and prove that E has to be δ_s-dense.

Suppose by contradiction that E is not δ_s-dense. Then, in view of Proposition 3.3.5, there exists $p\in\mathcal{C}E\cap\Omega$ such that
$$q\in\partial B_{\delta_s}(p)\cap(\partial E\cap\Omega) \quad \text{and} \quad B_{\delta_s}(p)\subseteq\mathcal{C}E.$$
Hence we use the Euler-Lagrange theorem at q, i.e.
$$H_s[E](q)\le0,$$
to obtain a contradiction with Theorem 3.1.2. This says that E is not δ_s-dense and concludes the proof of part (A) of Theorem 3.1.4.

Now we prove the part (B) of the Theorem.
Suppose that point $(B.1)$ does not hold true. Then we can find a sequence $s_k\searrow0$ and a sequence of sets E_k such that each E_k is s_k-minimal in Ω with exterior data E_0 and
$$E_k\cap\Omega\neq\varnothing.$$
We can assume that $s_k<s_1(E_0,\Omega)$ for every k. Then part (A) implies that each E_k is δ_{s_k}-dense, that is
$$|E_k\cap B_{\delta_{s_k}}(x)|>0 \quad \forall\, B_{\delta_{s_k}}(x)\Subset\Omega.$$
Fix $\gamma=\frac12$, take a sequence $\delta_h\searrow0$ and let $\sigma_{\delta_h,\frac12}$ be as in Theorem 3.1.5. Recall that $\delta_s\searrow0$ as $s\searrow0$. Thus for every h we can find k_h big enough such that
$$s_{k_h}<\sigma_{\delta_h,\frac12} \quad \text{and} \quad \delta_{s_{k_h}}<\delta_h. \tag{3.29}$$
In particular, this implies
$$|E_{k_h}\cap B_{\delta_h}(x)|\ge|E_k\cap B_{\delta_{s_{k_h}}}(x)|>0 \quad \forall\, B_{\delta_h}(x)\Subset\Omega, \tag{3.30}$$

for every h. On the other hand, by (3.29) and Theorem 3.1.5, we also have that

$$|\mathcal{C}E_{k_h} \cap B_{\delta_h}(x)| > 0 \quad \forall B_{\delta_h}(x) \Subset \Omega. \tag{3.31}$$

This concludes the proof of part (B). Indeed, notice that since $B_{\delta_h}(x)$ is connected, (3.30) and (3.31) together imply that

$$\partial E_{k_h} \cap B_{\delta_h}(x) \neq \varnothing \quad \forall B_{\delta_h}(x) \Subset \Omega.$$

□

3.3.4. Stickiness to the boundary is a typical behavior (Theorem 3.1.7). Now we show that the "typical behavior" of the nonlocal minimal surfaces is to stick at the boundary whenever they are allowed to do it, in the precise sense given by Theorem 3.1.7.

Proof of Theorem 3.1.7. Let

$$\delta := \frac{1}{2}\min\{r_0, R\},$$

and notice that (see Remark B.1.3)

$$B_\delta(x_0 + \delta\nu_\Omega(x_0)) \subseteq B_R(x_0) \setminus \Omega \subseteq \mathcal{C}E_0.$$

Since $\delta_s \to 0^+$, we can find $s_3 = s_3(E_0, \Omega) \in (0, s_0]$ such that $\delta_s < \delta$ for every $s \in (0, s_3)$.

Now let $s \in (0, s_3)$ and let E be s-minimal in Ω, with exterior data E_0.

We claim that

$$B_\delta(x_0 - r_0\nu_\Omega(x_0)) \subseteq E_{ext}. \tag{3.32}$$

We observe that this is indeed a crucial step to prove Theorem 3.1.7. Indeed, once this is established, by Remark B.1.3 we obtain that

$$B_\delta(x_0 - r_0\nu_\Omega(x_0)) \Subset \Omega.$$

Hence, since $\delta_s < \delta$, we deduce from (3.32) that E is not δ_s-dense. Thus, since $s < s_3 \leq s_1$, Theorem 3.1.4 implies that $E \cap \Omega = \varnothing$, which concludes the proof of Theorem 3.1.7.

This, we are left to prove (3.32). Suppose by contradiction that

$$\overline{E} \cap B_\delta(x_0 - r_0\nu_\Omega(x_0)) \neq \varnothing,$$

and consider the segment $c : [0,1] \longrightarrow \mathbb{R}^n$,

$$c(t) := x_0 + \big((1-t)\delta - t\, r_0\big)\nu_\Omega(x_0).$$

Notice that

$$B_\delta\big(c(0)\big) \subseteq E_{ext} \quad \text{and} \quad B_\delta\big(c(1)\big) \cap \overline{E} \neq \varnothing,$$

so

$$t_0 := \sup\Big\{\tau \in [0,1] \,\Big|\, \bigcup_{t\in[0,\tau]} B_\delta\big(c(t)\big) \subseteq E_{ext}\Big\} < 1.$$

Arguing as in Lemma B.2.1, we conclude that

$$B_\delta\big(c(t_0)\big) \subseteq E_{ext} \quad \text{and} \quad \exists\, q \in \partial B_\delta\big(c(t_0)\big) \cap \partial E.$$

By definition of c, we have that either $q \in \Omega$ or

$$q \in \partial\Omega \cap B_R(x_0).$$

In both cases (see [21, Theorem 5.1] and Theorem (C.3.1)) we have

$$H_s[E](q) \leq 0,$$

which gives a contradiction with Theorem 3.1.2 and concludes the proof. □

3.4. The contribution from infinity of some supergraphs

We compute in this Subsection the contribution from infinity of some particular supergraphs.

Example 3.4.1 (The cone). Let $S \subseteq \mathbb{S}^{n-1}$ be a portion of the unit sphere, $\mathfrak{o} := \mathcal{H}^{n-1}(S)$ and

$$C := \{t\sigma \mid t \geq 0,\ \sigma \in S)\}.$$

Then the contribution from infinity is given by the opening of the cone,

$$\alpha(C) = \mathfrak{o}. \tag{3.33}$$

Indeed,

$$\alpha_s(0,1,C) = \int_{\mathcal{C}B_1} \frac{\chi_C(y)}{|y|^{n+s}}\, dy = \mathcal{H}^{n-1}(S) \int_1^\infty t^{-s-1}\, dt = \frac{\mathfrak{o}}{s},$$

and we obtain the claim by passing to the limit. Notice that this says in particular that the contribution from infinity of a half-space is $\varpi_n/2$.

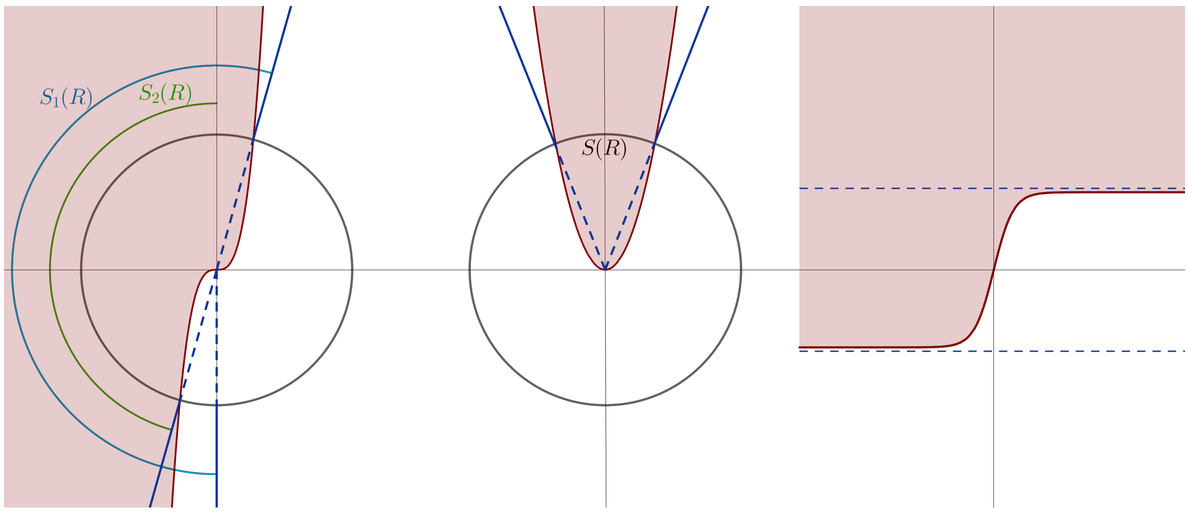


Figure 2. The contribution from infinity of x^3, x^2 and $\tanh x$

Example 3.4.2 (The parabola). We consider the supergraph

$$E := \{(x', x_n) \mid x_n \geq |x'|^2\},$$

and we show that, in this case,

$$\alpha(E) = 0.$$

In order to see this, we take any $R > 0$, intersect the ball B_R with the parabola and build a cone on this intersection (see the second picture in Figure 2), i.e. we define

$$S(R) := \partial B_R \cap E, \qquad C_R = \{t\sigma \mid t \geq 0,\ \sigma \in S(R)\}.$$

We can explicitly compute the opening of this cone, that is

$$\mathfrak{o}(R) = \left(\arcsin \frac{\left(\sqrt{4R^2+1} - 1\right)^{1/2}}{R\sqrt{2}} \right) \frac{\varpi_n}{\pi}.$$

Since $E \subseteq C_R$ outside of B_R, thanks to the monotonicity property in Proposition 3.2.3 and to (3.33), we have that

$$\overline{\alpha}(E) \leq \overline{\alpha}(C_R) = \mathfrak{o}(R).$$

Sending $R \to \infty$, we find that

$$\overline{\alpha}(E) = 0, \qquad \text{thus} \qquad \alpha(E) = 0.$$

More generally, if we consider for any given $c, \varepsilon > 0$ a function u such that

$$u(x') > c|x'|^{1+\varepsilon}, \qquad \text{for any } |x'| > R \text{ for some } R > 0$$

and

$$E := \{(x', x_n) \,\big|\, x_n \geq u(x')\},$$

then

$$\alpha(E) = 0.$$

On the other hand, if we consider a function that is not rotation invariant, things can go differently, as we see in the next example.

Example 3.4.3 (The supergraph of x^3). We consider the supergraph

$$E := \{(x, y) \,\big|\, y \geq x^3\}.$$

In this case, we show that

$$\alpha(E) = \pi.$$

For this, given $R > 0$, we intersect ∂B_R with E and denote by $S_1(R)$ and $S_2(R)$ the arcs on the circle as the first picture in Figure 2. We consider the cones

$$C^1_R := \{t\sigma \,\big|\, t \geq 0,\ \sigma \in S_1(R)\} \quad C^2_R := \{t\sigma \,\big|\, t \geq 0,\ \sigma \in S_2(R)\}$$

and notice that outside of B_R, it holds that $C^2_R \subseteq E \subseteq C^1_R$. Let $\overline{x}_R$ be the solution of

$$x^6 + x^2 = R^2,$$

that is the x-coordinate in absolute value of the intersection points $\partial B_R \cap \partial E$. Since $f(x) = x^6 + x^2$ is increasing on $(0, \infty)$ and $R^2 = f(\overline{x}_R) < f(R^{1/3})$, we have that $\overline{x}_R < R^{1/3}$. Hence

$$\mathfrak{o}^1(R) = \pi + \arcsin \frac{\overline{x}_R}{R} \leq \pi + \arcsin \frac{R^{1/3}}{R}, \quad \mathfrak{o}^2(R) \geq \pi - \arcsin \frac{R^{1/3}}{R}.$$

Thanks to the monotonicity property in Proposition 3.2.3 and to (3.33) we have that

$$\overline{\alpha}(E) \leq \alpha(C^1_R) = \mathfrak{o}^1(R), \quad \underline{\alpha}(E) \geq \alpha(C^2_R) = \mathfrak{o}^2(R)$$

and sending $R \to \infty$ we obtain that

$$\overline{\alpha}(E) \leq \pi, \quad \underline{\alpha}(E) \geq \pi.$$

Thus $\alpha(E)$ exists and we obtain the desired conclusion.

Example 3.4.4 (The supergraph of a bounded function). We consider the supergraph

$$E := \{(x', x_n) \,\big|\, x_n \geq u(x')\}, \qquad \text{with} \qquad \|u\|_{L^\infty(\mathbb{R}^n)} < M.$$

We show that, in this case,

$$\alpha(E) = \frac{\varpi_n}{2}.$$

To this aim, let

$$\mathfrak{P}_1 := \{(x', x_n) \,\big|\, x_n > M\}$$
$$\mathfrak{P}_2 := \{(x', x_n) \,\big|\, x_n < -M\}.$$

We have that

$$\mathfrak{P}_1 \subseteq E, \qquad \mathfrak{P}_2 \subseteq \mathcal{C}E.$$

Hence by Proposition 3.2.3

$$\underline{\alpha}(E) \geq \overline{\alpha}(\mathfrak{P}_1) = \frac{\varpi_n}{2}, \qquad \underline{\alpha}(\mathcal{C}E) \geq \overline{\alpha}(\mathfrak{P}_2) = \frac{\varpi_n}{2}.$$

Since $\underline{\alpha}(\mathcal{C}E) = \varpi_n - \overline{\alpha}(E)$ we find that

$$\overline{\alpha}(E) \leq \frac{\varpi_n}{2},$$

thus the conclusion. An example of this type is depicted in Figure 2 (more generally, the result holds for the supergraph in $\mathbb{R}^n$ $\{(x', x_n) \bigm| x_n \geq \tanh x_1\}$).

EXAMPLE 3.4.5 (The supergraph of a sublinear graph). More generally, we can take the supergraph of a function that grows sublinearly at infinity, i.e.

$$E := \{(x', x_n) \bigm| x_n > u(x')\}, \qquad \text{with} \qquad \lim_{|x'|\to+\infty} \frac{|u(x')|}{|x'|} = 0.$$

In this case, we show that

$$\alpha(E) = \frac{\varpi_n}{2}.$$

Indeed, for any $\varepsilon > 0$ we have that there exists $R = R(\varepsilon) > 0$ such that

$$|u(x')| < \varepsilon|x'|, \quad \forall\, |x'| > R.$$

We denote

$$S_1(R) := \partial B_R \cap \{(x', x_n) \bigm| x_n > \varepsilon|x'|\}, \qquad S_2(R) := \partial B_R \cap \{(x', x_n) \bigm| x_n < -\varepsilon|x'|\}$$

and

$$C_R^i = \{t\sigma \bigm| t \geq 0,\ \sigma \in S_i(R)\}, \quad \text{for} \ \ i = 1, 2.$$

We have that outside of B_R

$$C_R^1 \subseteq E, \qquad C_R^2 \subseteq \mathcal{C}E,$$

and

$$\alpha(C_R^1) = \alpha(C_R^2) = \frac{\varpi_n}{\pi}\left(\frac{\pi}{2} - \arctan\varepsilon\right).$$

We use Proposition 3.2.3, (i), and letting ε go to zero, we obtain that $\alpha(E)$ exists and

$$\alpha(E) = \frac{\varpi_n}{2}.$$

A particular example of this type is given by

$$E := \{(x', x_n) \bigm| x_n > c|x'|^{1-\varepsilon}\}, \quad \text{when } |x'| > R \ \text{ for some } \varepsilon \in (0, 1],\ c \in \mathbb{R},\ R > 0.$$

In particular using the additivity property in Proposition 3.2.3 we can compute α for sets that lie between two graphs.

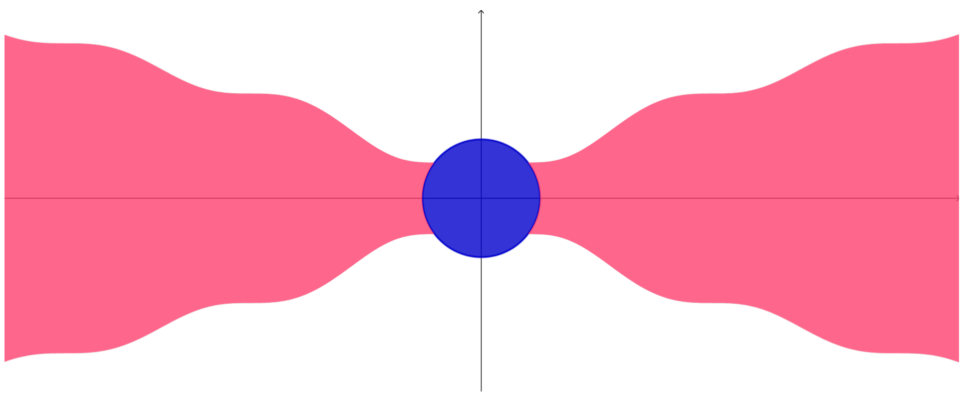


FIGURE 3. The “butterscotch hard candy” graph

Example 3.4.6 (The "butterscotch hard candy"). Let $E \subseteq \mathbb{R}^n$ be such that

$$E \cap \{|x'| > R\} \subseteq \{(x', x_n) \mid |x'| > R\,,\ |x_n| < c|x'|^{1-\varepsilon}\},$$

for some $\varepsilon \in (0,1]$, $c > 0$ and $R > 0$ (an example of such a set E is given in Figure 3). In this case, we have that

$$\alpha(E) = 0.$$

Indeed, we can write $E_1 := E \cap \{|x'| > R\}$ and $E_2 := E \cap \{|x'| \le R\}$. Then, using the computations in Example 3.4.5, we have by the monotonicity and the additivity properties in Proposition 3.2.3 that

$$\overline{\alpha}(E_1) \le \alpha\big(\{x_n > -c|x'|^{1-\varepsilon}\}\big) - \alpha\big(\{x_n > c|x'|^{1-\varepsilon}\}\big) = 0.$$

Moreover, E_2 lies inside $\{|x_1| \le R\}$. Hence, again by Proposition 3.2.3 and by Example 3.4.1, we find

$$\overline{\alpha}(E_2) \le \alpha\big(\{|x_1| \le R\}\big) = \alpha\big(\{x_1 \le R\}\big) - \alpha\big(\{x_1 < -R\}\big) = 0.$$

Consequently, using again the additivity property in Proposition 3.2.3, we obtain that

$$\overline{\alpha}(E) \le \overline{\alpha}(E_1) + \overline{\alpha}(E_2) = 0,$$

that is the desired result.

We can also compute α for sets that have different growth ratios in different directions. For this, we have the following example.

Example 3.4.7 (The supergraph of a superlinear function on a small cone). We consider a set lying in the half-space, deprived of a set that grows linearly at infinity. We denote by $\tilde{S}$ the portion of the sphere given by

$$\tilde{S} := \Big\{\sigma \in \mathbb{S}^{n-2} \,\Big|\, \sigma = (\cos\sigma_1, \sin\sigma_1\cos\sigma_2, \dots, \sin\sigma_1 \dots \sin\sigma_{n-2}),$$
$$\text{with } \sigma_i \in \Big(\frac{\pi}{2} - \bar{\varepsilon}, \frac{\pi}{2} + \bar{\varepsilon}\Big)\,,\ i = 1, \dots, n-2\Big\},$$

where $\bar{\varepsilon} \in (0, \pi/2)$. For $x_0 \in \mathbb{R}^n$ and $k > 0$ we define the supergraph $E \subseteq \mathbb{R}^n$ as

$$E := \big\{(x', x_n) \in \mathbb{R}^n \mid x_n \ge u(x')\big\} \quad \text{where} \quad u(x') = \begin{cases} k|x' - x_0'| & \text{for } x' \in X, \\ 0 & \text{for } x' \notin X, \end{cases}$$
$$X = \{x' \in \mathbb{R}^{n-1} \text{ s.t. } x' = t\sigma + x_0',\ \sigma \in \tilde{S}\}.$$

We remark that $X \subseteq \{x_n = 0\}$ is the cone "generated" by $\tilde{S}$ and centered at x_0. Then

$$\alpha(E) = \frac{\varpi_n}{2} - \mathcal{H}^{n-2}(\tilde{S}) \int_0^k \frac{dt}{(1+t^2)^{\frac{n}{2}}}. \tag{3.34}$$

Let

$$\mathfrak{P}_+ := \{(x', x_n) \mid x_n > 0\}, \qquad \mathfrak{P}_- := \{(x', x_n) \mid x_n < 0\}$$

and we consider the subgraph

$$F := \big\{(x', x_n) \mid 0 < x_n < u(x')\big\}.$$

Then

$$E \cup F = \mathfrak{P}_+, \qquad \mathfrak{P}_- \cup F = \mathcal{C}E.$$

Using the additivity property in Proposition 3.2.3, we see that

$$\overline{\alpha}(E) \ge \frac{\varpi_n}{2} - \overline{\alpha}(F), \qquad \varpi_n - \underline{\alpha}(E) = \overline{\alpha}(\mathcal{C}E) \le \frac{\varpi_n}{2} + \overline{\alpha}(F). \tag{3.35}$$

Let $R > 0$ be arbitrary. We get that

$$\alpha_s(x_0, R, F) \leq \int_{(B'_R(x'_0)\times\mathbb{R})\cap\mathcal{C}B_R(x_0)} \frac{\chi_F(y)}{|y-x_0|^{n+s}}\,dy + \int_{\mathcal{C}(B'_R(x'_0)\times\mathbb{R})} \frac{\chi_F(y)}{|y-x_0|^{n+s}}\,dy$$

so

$$\begin{aligned}
\alpha_s(x_0, R, F) \leq & \int_{B'_R(x'_0)} \frac{dy'}{|y'-x'_0|^{n-1+s}} \int_{\frac{\sqrt{R^2-|y'-x'_0|^2}}{|y'-x'_0|}}^{\infty} \frac{dt}{(1+t^2)^{\frac{n+s}{2}}} \\
& + \int_{\mathcal{C}B'_R(x'_0)\cap X} \frac{dy'}{|y'-x'_0|^{n-1+s}} \int_0^k \frac{dt}{(1+t^2)^{\frac{n+s}{2}}} \\
= & \, I_1 + I_2.
\end{aligned} \tag{3.36}$$

Using that $1 + t^2 \geq \max\{1, t^2\}$ and passing to polar coordinates, we obtain that

$$\begin{aligned}
I_1 = & \int_{B'_R(x'_0)} \frac{dy'}{|y'-x'_0|^{n-1+s}} \left(\int_{\frac{\sqrt{R^2-|y'-x'_0|^2}}{|y'-x'_0|}}^{\frac{R}{|y'-x'_0|}} \frac{dt}{(1+t^2)^{\frac{n+s}{2}}} + \int_{\frac{R}{|y'-x'_0|}}^{\infty} \frac{dt}{(1+t^2)^{\frac{n+s}{2}}} \right) \\
\leq & \, \varpi_{n-1} \left(\int_0^R \tau^{-s-2} \left(R - \sqrt{R^2-\varrho^2} \right) d\varrho + \frac{R^{-n-s+1}}{n+s-1} \int_0^R \varrho^{n-2}\, d\varrho \right) \\
= & \, \varpi_{n-1} \left(R^{-s} \int_0^1 \tau^{-s-2} \left(1 - \sqrt{1-\tau^2} \right) d\tau + \frac{R^{-s}}{(n+s-1)(n-1)} \right).
\end{aligned}$$

Also, for any $\tau \in (0, 1)$ we have that

$$1 - \sqrt{1-\tau^2} \leq c\tau^2,$$

for some positive constant c, independent on n, s. Therefore

$$I_1 \leq \frac{c\varpi_{n-1}R^{-s}}{1-s} + \frac{\varpi_{n-1}R^{-s}}{(n-1)(n+s-1)}.$$

Moreover,

$$I_2 = \mathcal{H}^{n-2}(\tilde{S}) \frac{R^{-s}}{s} \int_0^k \frac{dt}{(1+t^2)^{\frac{n+s}{2}}}.$$

So passing to limsup and liminf as $s \to 0^+$ in (3.36) and using Fatou's lemma we obtain that

$$\overline{\alpha}(F) \leq \mathcal{H}^{n-2}(\tilde{S}) \int_0^k \frac{dt}{(1+t^2)^{\frac{n}{2}}}, \qquad \underline{\alpha}(F) \geq \mathcal{H}^{n-2}(\tilde{S}) \int_0^k \frac{dt}{(1+t^2)^{\frac{n}{2}}}.$$

In particular $\alpha(F)$ exists, and from (3.35) we get that

$$\frac{\varpi_n}{2} - \alpha(F) \leq \underline{\alpha}(E) \leq \overline{\alpha}(E) \leq \frac{\varpi_n}{2} - \alpha(F).$$

Therefore, $\alpha(E)$ exists and

$$\alpha(E) = \frac{\varpi_n}{2} - \mathcal{H}^{n-2}(\tilde{S}) \int_0^k \frac{dt}{(1+t^2)^{\frac{n}{2}}}.$$

3.5. Continuity of the fractional mean curvature and a sign changing property of the nonlocal mean curvature

We use a formula proved in [25] to show that the s-fractional mean curvature is continuous with respect to $C^{1,\alpha}$ convergence of sets, for any $s < \alpha$ and with respect to C^2 convergence of sets, for s close to 1.

By $C^{1,\alpha}$ convergence of sets we mean that our sets locally converge in measure and can locally be described as the supergraphs of functions which converge in $C^{1,\alpha}$.

Definition 3.5.1. *Let $E \subseteq \mathbb{R}^n$ and let $q \in \partial E$ such that ∂E is $C^{1,\alpha}$ near q, for some $\alpha \in (0,1]$. We say that the sequence $E_k \subseteq \mathbb{R}^n$ converges to E in a $C^{1,\alpha}$ sense (and write $E_k \xrightarrow{C^{1,\alpha}} E$) in a neighborhood of q if:*
(i) the sets E_k locally converge in measure to E, i.e.

$$|(E_k \Delta E) \cap B_r| \xrightarrow{k\to\infty} 0 \qquad \text{for any } r > 0$$

and
(ii) the boundaries ∂E_k converge to ∂E in $C^{1,\alpha}$ sense in a neighborhood of q.
We define in a similar way the C^2 convergence of sets.

More precisely, we denote

$$Q_{r,h}(x) := B'_r(x') \times (x_n - h, x_n + h),$$

for $x \in \mathbb{R}^n$, $r, h > 0$. If $x = 0$, we drop it in formulas and simply write $Q_{r,h} := Q_{r,h}(0)$. Notice that up to a translation and a rotation, we can suppose that $q = 0$ and

$$E \cap Q_{2r,2h} = \{(x', x_n) \in \mathbb{R}^n \,|\, x' \in B'_{2r},\, u(x') < x_n < 2h\}, \tag{3.37}$$

for some $r, h > 0$ small enough and $u \in C^{1,\alpha}(\overline{B}'_{2r})$ such that $u(0) = 0$. Then, point (ii) means that we can write

$$E_k \cap Q_{2r,2h} = \{(x', x_n) \in \mathbb{R}^n \,|\, x' \in B'_{2r},\, u_k(x') < x_n < 2h\}, \tag{3.38}$$

for some functions $u_k \in C^{1,\alpha}(\overline{B}'_{2r})$ such that

$$\lim_{k\to\infty} \|u_k - u\|_{C^{1,\alpha}(\overline{B}'_{2r})} = 0. \tag{3.39}$$

We remark that, by the continuity of u, up to considering a smaller r, we can suppose that

$$|u(x')| < \frac{h}{2}, \qquad \forall\, x' \in B'_{2r}. \tag{3.40}$$

We have the following result.

Theorem 3.5.2. *Let $E_k \xrightarrow{C^{1,\alpha}} E$ in a neighborhood of $q \in \partial E$. Let $q_k \in \partial E_k$ be such that $q_k \longrightarrow q$ and let $s, s_k \in (0, \alpha)$ be such that $s_k \xrightarrow{k\to\infty} s$. Then*

$$\lim_{k\to\infty} H_{s_k}[E_k](q_k) = H_s[E](q).$$

Let $E_k \xrightarrow{C^2} E$ in a neighborhood of $q \in \partial E$. Let $q_k \in \partial E_k$ be such that $q_k \longrightarrow q$ and let $s_k \in (0,1)$ be such that $s_k \xrightarrow{k\to\infty} 1$. Then

$$\lim_{k\to\infty} (1 - s_k) H_{s_k}[E_k](q_k) = \varpi_{n-1} H[E](q).$$

A similar problem is studied also in [29], where the author estimates the difference between the fractional mean curvature of a set E with $C^{1,\alpha}$ boundary and that of the set $\Phi(E)$, where Φ is a $C^{1,\alpha}$ diffeomorphism of $\mathbb{R}^n$, in terms of the $C^{0,\alpha}$ norm of the Jacobian of the diffeomorphism Φ.

When $s \to 0^+$ we do not need the $C^{1,\alpha}$ convergence of sets, but only the uniform boundedness of the $C^{1,\alpha}$ norms of the functions defining the boundary of E_k in a neighborhood of the boundary points. However, we have to require that the measure of the symmetric difference is uniformly bounded. More precisely:

Proposition 3.5.3. *Let $E\subseteq\mathbb{R}^n$ be such that $\alpha(E)$ exists. Let $q\in\partial E$ be such that*
$$E\cap Q_{r,h}(q)=\{(x',x_n)\in\mathbb{R}^n\,|\,x'\in B'_r(q'),\,u(x')<x_n<h+q_n\},$$
for some $r,h>0$ small enough and $u\in C^{1,\alpha}(\overline{B}'_r(q'))$ such that $u(q')=q_n$. Let $E_k\subseteq\mathbb{R}^n$ be such that
$$|E_k\Delta E|<C_1$$
for some $C_1>0$. Let $q_k\in\partial E_k\cap B_d$, for some $d>0$, such that
$$E_k\cap Q_{r,h}(q_k)=\{(x',x_n)\in\mathbb{R}^n\,|\,x'\in B'_r(q'_k),\,u_k(x')<x_n<h+q_{k,n}\}$$
for some functions $u_k\in C^{1,\alpha}(\overline{B}'_r(q'_k))$ such that $u_k(q'_k)=q_{k,n}$ and
$$\|u_k\|_{C^{1,\alpha}(\overline{B}'_r(q'_k))}<C_2$$
for some $C_2>0$. Let $s_k\in(0,\alpha)$ be such that $s_k\xrightarrow{k\to\infty}0$. Then
$$\lim_{k\to\infty}s_kH_{s_k}[E_k](q_k)=\varpi_n-2\alpha(E).$$

In particular, fixing $E_k=E$ in Theorem 3.5.2 and Proposition 3.5.3 we obtain Proposition 3.1.11 stated in the Introduction.

To prove Theorem 3.5.2 we prove at first the following preliminary result.

Lemma 3.5.4. *Let $E_k\xrightarrow{C^{1,\alpha}}E$ in a neighborhood of $0\in\partial E$. Let $q_k\in\partial E_k$ be such that $q_k\longrightarrow0$. Then*
$$E_k-q_k\xrightarrow{C^{1,\beta}}E\qquad\text{in a neighborhood of }0,$$
for every $\beta\in(0,\alpha)$.
Moreover, if $E_k\xrightarrow{C^2}E$ in a neighborhood of $0\in\partial E$, $q_k\in\partial E_k$ are such that $q_k\longrightarrow0$ and $\mathcal{R}_k\in SO(n)$ are such that
$$\lim_{k\to\infty}|\mathcal{R}_k-Id|=0,$$
then
$$\mathcal{R}_k(E_k-q_k)\xrightarrow{C^2}E\qquad\text{in a neighborhood of }0\,.$$

Proof. First of all, notice that since $q_k\longrightarrow0$, for k big enough we have
$$|q'_k|<\frac{1}{2}r\qquad\text{and}\qquad|q_{k,n}|=|u_k(q'_k)|<\frac{1}{8}h.$$
By (3.40) and (3.39), we see that for k big enough
$$|u_k(x')|\le\frac{3}{4}h,\qquad\forall\,x'\in B'_{2r}.$$
Therefore
$$|u_k(x')-q_{k,n}|<\frac{7}{8}h<h,\qquad\forall\,x'\in B'_{2r}.$$
If we define
$$\tilde{u}_k(x'):=u_k(x'+q'_k),\qquad x'\in\overline{B}'_r,$$
for every k big enough we have
$$(E_k-q_k)\cap Q_{r,h}=\{(x',x_n)\in\mathbb{R}^n\,|\,x'\in B'_r,\,\tilde{u}_k(x')<x_n<h\}.\tag{3.41}$$
It is easy to check that the sequence E_k-q_k locally converges in measure to E. We claim that
$$\lim_{k\to\infty}\|\tilde{u}_k-u\|_{C^{1,\beta}(\overline{B}'_r)}=0.\tag{3.42}$$

Indeed, let
$$\tau_k u(x') := u(x' + q_k').$$
We have that
$$\|\tilde{u}_k - \tau_k u\|_{C^1(\overline{B}_r')} \leq \|u_k - u\|_{C^1\left(\overline{B}'_{\frac{3}{2}r}\right)}$$
and that
$$\|\tau_k u - u\|_{C^1(\overline{B}_r')} \leq \|\nabla u\|_{C^0\left(\overline{B}'_{\frac{3}{2}r}\right)} |q_k'| + \|u\|_{C^{1,\alpha}\left(\overline{B}'_{\frac{3r}{2}}\right)} |q_k'|^\alpha.$$
Thus by the triangular inequality
$$\lim_{k\to\infty} \|\tilde{u}_k - u\|_{C^1(\overline{B}_r')} = 0,$$
thanks to (3.39) and the fact that $q_k \to 0$.

Now, notice that $\nabla(\tilde{u}_k) = \tau_k(\nabla u_k)$, so
$$[\nabla\tilde{u}_k - \nabla u]_{C^{0,\beta}(\overline{B}_r')} \leq [\tau_k(\nabla u_k - \nabla u)]_{C^{0,\beta}(\overline{B}_r')} + [\tau_k(\nabla u) - \nabla u)]_{C^{0,\beta}(\overline{B}_r')}.$$
Therefore
$$[\tau_k(\nabla u_k - \nabla u)]_{C^{0,\beta}(\overline{B}_r')} \leq [\nabla u_k - \nabla u]_{C^{0,\beta}\left(\overline{B}'_{\frac{3r}{2}}\right)}$$
and for every $\delta > 0$ we obtain
$$[\tau_k(\nabla u) - \nabla u]_{C^{0,\beta}(\overline{B}_r')} \leq \frac{2}{\delta^\beta}\|\tau_k(\nabla u) - \nabla u\|_{C^0\left(\overline{B}'_{\frac{3r}{2}}\right)} + 2[\nabla u]_{C^{0,\alpha}(\overline{B}_r')}\delta^{\alpha-\beta}.$$
Sending $k \to \infty$ we find that
$$\limsup_{k\to\infty}[\tau_k(\nabla u) - \nabla u)]_{C^{0,\beta}(\overline{B}_r')} \leq 2[\nabla u]_{C^{0,\alpha}(\overline{B}_r')}\delta^{\alpha-\beta}$$
for every $\delta > 0$, hence
$$\lim_{k\to\infty} [\nabla\tilde{u}_k - \nabla u]_{C^{0,\beta}(\overline{B}_r')} = 0.$$
This concludes the proof of the first part of the Lemma.
As for the second part, the C^2 convergence of sets in a neighborhood of 0 can be proved similarly. Some care must be taken when considering rotations, since one needs to use the implicit function theorem. □

Proof of Theorem 3.5.2. Up to a translation and a rotation, we can suppose that $q = 0$ and $\nu_E(0) = 0$. Then we can find $r, h > 0$ small enough and $u \in C^{1,\alpha}(\overline{B}_r')$ such that we can write $E \cap Q_{2r,2h}$ as in (3.37).

Since $s_k \to s \in (0, \alpha)$ for k large enough we can suppose that $s_k, s \in [\sigma_0, \sigma_1]$ for $0 < \sigma_0 < \sigma_1 < \beta < \alpha$. Notice that there exists $\delta > 0$ such that

$$B_\delta \Subset Q_{r,h}. \tag{3.43}$$

We take an arbitrary $R > 1$ as large as we want and define the sets
$$F_k := (E_k \cap B_R) - q_k.$$
From Lemma 3.5.4 we have that in a neighborhood of 0
$$F_k \xrightarrow{C^{1,\beta}} E \cap B_R.$$
In other words,

$$\lim_{k\to\infty} |F_k \Delta (E \cap B_R)| = 0. \tag{3.44}$$

Moreover, if u_k is a function defining E_k as a supergraph in a neighborhood of 0 as in (3.38), denoting $\tilde{u}_k(x') = u_k(x' + q_k')$ we have that
$$F_k \cap Q_{r,h} = \{(x', x_n) \in \mathbb{R}^n \,|\, x' \in B_r', \, \tilde{u}_k(x') < x_n < h\}$$

and that

$$\lim_{k\to\infty}\|\tilde{u}_k-u\|_{C^{1,\beta}(\overline{B}'_r)}=0,\qquad \|\tilde{u}_k\|_{C^{1,\beta}(\overline{B}'_r)}\le M \;\text{ for some }\; M>0. \tag{3.45}$$

We also remark that, by (3.40) we can write

$$E\cap Q_{r,h}=\{(x',x_n)\in\mathbb{R}^n\,|\,x'\in B'_r,\,u(x')<x_n<h\}.$$

Exploiting (3.41) we can write the fractional mean curvature of F_k in 0 by using formula (C.1), that is

$$\begin{aligned}H_{s_k}[F_k](0)=2\int_{B'_r}&\Big\{G_{s_k}\Big(\frac{\tilde{u}_k(y')-\tilde{u}_k(0)}{|y'|}\Big)-G_{s_k}\Big(\nabla\tilde{u}_k(0)\cdot\frac{y'}{|y'|}\Big)\Big\}\frac{dy'}{|y'|^{n-1+s_k}}\\&+\int_{\mathbb{R}^n}\frac{\chi_{\mathcal{C}F_k}(y)-\chi_{F_k}(y)}{|y|^{n+s_k}}\chi_{\mathcal{C}Q_{r,h}}(y)\,dy.\end{aligned} \tag{3.46}$$

Now, we denote as in (C.2)

$$\mathcal{G}(s_k,\tilde{u}_k,y'):=\mathcal{G}(s_k,\tilde{u}_k,0,y')=G_{s_k}\Big(\frac{\tilde{u}_k(y')-\tilde{u}_k(0)}{|y'|}\Big)-G_{s_k}\Big(\nabla\tilde{u}_k(0)\cdot\frac{y'}{|y'|}\Big)$$

and we rewrite the identity in (3.46) as

$$H_{s_k}[F_k](0)=2\int_{B'_r}\mathcal{G}(s_k,\tilde{u}_k,y')\frac{dy'}{|y'|^{n-1+s_k}}+\int_{\mathbb{R}^n}\frac{\chi_{\mathcal{C}F_k}(y)-\chi_{F_k}(y)}{|y|^{n+s_k}}\chi_{\mathcal{C}Q_{r,h}}(y)\,dy.$$

Also, with this notation and by formula (C.1) we have for E

$$H_s[E\cap B_R](0)=2\int_{B'_r}\mathcal{G}(s,u,y')\frac{dy'}{|y'|^{n-1+s}}+\int_{\mathbb{R}^n}\frac{\chi_{\mathcal{C}(E\cap B_R)}(y)-\chi_{E\cap B_R}(y)}{|y|^{n+s}}\chi_{\mathcal{C}Q_{r,h}}(y)\,dy.$$

We can suppose that $r<1$. We begin by showing that for every $y'\in B'_r\setminus\{0\}$ we have

$$\lim_{k\to\infty}\mathcal{G}(s_k,\tilde{u}_k,y')=\mathcal{G}(s,u,y'). \tag{3.47}$$

First of all, we observe that

$$|\mathcal{G}(s_k,\tilde{u}_k,y')-\mathcal{G}(s,u,y')|\le|\mathcal{G}(s_k,\tilde{u}_k,y')-\mathcal{G}(s,\tilde{u}_k,y')|+|\mathcal{G}(s,\tilde{u}_k,y')-\mathcal{G}(s,u,y')|.$$

Then

$$\begin{aligned}|\mathcal{G}(s_k,\tilde{u}_k,y')-\mathcal{G}(s,\tilde{u}_k,y')|&=\Big|\int_{\nabla\tilde{u}_k(0)\cdot\frac{y'}{|y'|}}^{\frac{\tilde{u}_k(y')-\tilde{u}_k(0)}{|y'|}}(g_{s_k}(t)-g_s(t))\,dt\Big|\\&\le 2\int_0^{+\infty}|g_{s_k}(t)-g_s(t)|\,dt.\end{aligned}$$

Notice that for every $t\in\mathbb{R}$

$$\lim_{k\to\infty}|g_{s_k}(t)-g_s(t)|=0,\qquad\text{and}\qquad|g_{s_k}(t)-g_s(t)|\le 2g_{\sigma_0}(t),\quad\forall\,k\in\mathbb{N}.$$

Since $g_{\sigma_0}\in L^1(\mathbb{R})$, by the Dominated Convergence Theorem we obtain that

$$\lim_{k\to\infty}|\mathcal{G}(s_k,\tilde{u}_k,y')-\mathcal{G}(s,\tilde{u}_k,y')|=0.$$

We estimate

$$
\begin{aligned}
|\mathcal{G}(s,\tilde u_k,y')-\mathcal{G}(s,u,y')| &\le \Big|G_s\Big(\frac{\tilde u_k(y')-\tilde u_k(0)}{|y'|}\Big)-G_s\Big(\frac{u(y')-u(0)}{|y'|}\Big)\Big| \\
&\qquad + \Big|G_s\Big(\nabla\tilde u_k(0)\cdot\frac{y'}{|y'|}\Big)-G_s\Big(\nabla u(0)\cdot\frac{y'}{|y'|}\Big)\Big| \\
&\le \Big|\frac{\tilde u_k(y')-\tilde u_k(0)}{|y'|}-\frac{u(y')-u(0)}{|y'|}\Big| + |\nabla\tilde u_k(0)-\nabla u(0)| \\
&= \Big|\nabla(\tilde u_k-u)(\xi)\cdot\frac{y'}{|y'|}\Big| + |\nabla\tilde u_k(0)-\nabla u(0)| \\
&\le 2\|\nabla\tilde u_k-\nabla u\|_{C^0(\overline{B}'_r)},
\end{aligned}
$$

which, by (3.42), tends to 0 as $k\to\infty$. This proves the pointwise convergence claimed in (3.47).

Therefore, for every $y'\in B'_r\setminus\{0\}$,

$$
\lim_{k\to\infty}\frac{\mathcal{G}(s_k,\tilde u_k,y')}{|y'|^{n-1+s_k}}=\frac{\mathcal{G}(s,u,y')}{|y'|^{n-1+s}}.
$$

Thus, by (C.3) we obtain that

$$
\Big|\frac{\mathcal{G}(s_k,\tilde u_k,y')}{|y'|^{n-1+s_k}}\Big| \le \|\tilde u_k\|_{C^{1,\beta}(\overline{B}'_r)}\frac{1}{|y'|^{n-1-(\beta-s_k)}} \le \frac{M}{|y'|^{n-1-(\beta-\sigma_1)}}\in L^1_{\mathrm{loc}}(\mathbb{R}^{n-1}),
$$

given (3.45). The Dominated Convergence Theorem then implies that

$$
\lim_{k\to\infty}\int_{B'_r}\mathcal{G}(s_k,\tilde u_k,y')\frac{dy'}{|y'|^{n-1+s_k}}=\int_{B'_r}\mathcal{G}(s,u,y')\frac{dy'}{|y'|^{n-1+s}}. \tag{3.48}
$$

Now, we show that

$$
\lim_{k\to\infty}\int_{\mathbb{R}^n}\frac{\chi_{\mathcal{C}F_k}(y)-\chi_{F_k}(y)}{|y|^{n+s_k}}\chi_{\mathcal{C}Q_{r,h}}(y)\,dy=\int_{\mathbb{R}^n}\frac{\chi_{\mathcal{C}(E\cap B_R)}(y)-\chi_{E\cap B_R}(y)}{|y|^{n+s}}\chi_{\mathcal{C}Q_{r,h}}(y)\,dy. \tag{3.49}
$$

For this, we observe that

$$
\Big|\int_{\mathcal{C}Q_{r,h}}(\chi_{\mathcal{C}(E\cap B_R)}(y)-\chi_{E\cap B_R}(y))\Big(\frac{1}{|y|^{n+s_k}}-\frac{1}{|y|^{n+s}}\Big)dy\Big| \le \int_{\mathcal{C}B_\delta}\Big|\frac{1}{|y|^{n+s_k}}-\frac{1}{|y|^{n+s}}\Big|dy,
$$

where we have used (3.43) in the last inequality. For $y\in\mathcal{C}B_1$

$$
\Big|\frac{1}{|y|^{n+s_k}}-\frac{1}{|y|^{n+s}}\Big|\le\frac{2}{|y|^{n+\sigma_0}}\in L^1(\mathcal{C}B_1)
$$

and for $y\in B_1\setminus B_\delta$

$$
\Big|\frac{1}{|y|^{n+s_k}}-\frac{1}{|y|^{n+s}}\Big|\le\frac{2}{|y|^{n+\sigma_1}}\in L^1(B_1\setminus B_\delta).
$$

We use then the Dominated Convergence Theorem and get that

$$
\lim_{k\to\infty}\int_{\mathcal{C}Q_{r,h}}(\chi_{\mathcal{C}(E\cap B_R)}(y)-\chi_{E\cap B_R}(y))\Big(\frac{1}{|y|^{n+s_k}}-\frac{1}{|y|^{n+s}}\Big)dy=0.
$$

Now

$$
\begin{aligned}
\Big|\int_{\mathcal{C}Q_{r,h}}\frac{\chi_{\mathcal{C}F_k}(y)-\chi_{F}k(y)-\big(\chi_{\mathcal{C}(E\cap B_R)(y)}-\chi_{E\cap B_R}(y)\big)}{|y|^{n+s_k}}\,dy\Big| &= 2\int_{\mathcal{C}Q_{r,h}}\frac{\chi_{F_k\Delta(E\cap B_R)}(y)}{|y|^{n+s_k}}\,dy \\
&\le 2\frac{|F_k\Delta(E\cap B_R)|}{\delta^{n+\sigma_1}}\xrightarrow{k\to\infty}0,
\end{aligned}
$$

according to (3.44). The last two limits prove (3.49). Recalling (3.48), we obtain that

$$\lim_{k\to\infty} H_{s_k}[F_k](0) = H_s[E\cap B_R](0).$$

We have that $H_{s_k}[F_k](0) = H_{s_k}[E_k\cap B_R](q_k)$, so

$$\begin{aligned}|H_{s_k}[E_k](q_k) - H_s[E](0)| &\le |H_{s_k}[E_k](q_k) - H_{s_k}[E_k\cap B_R](q_k)| \\ &\quad + |H_{s_k}[F_k](0) - H_s[E\cap B_R](0)| + |H_s[E\cap B_R](0) - H_s[E](0)|.\end{aligned}$$

Since

$$|H_{s_k}[E_k](q_k) - H_{s_k}[E_k\cap B_R](q_k)| + |H_s[E](0) - H_s[E\cap B_R](0)| \le \frac{4\varpi_n}{\sigma_0} R^{-\sigma_0},$$

sending $R\to\infty$

$$\lim_{k\to\infty} H_{s_k}[E_k](q_k) = H_s[E](0).$$

This concludes the proof of the first part of the Theorem.

In order to prove the second part of Theorem 3.5.2, we fix $R>1$ and we denote

$$F_k := \mathcal{R}_k\big((E_k\cap B_R) - q_k\big),$$

where $\mathcal{R}_k \in SO(n)$ is a rotation such that

$$\mathcal{R}_k : \nu_{E_k}(0) \longmapsto \nu_E(0) = -e_n \quad \text{and} \quad \lim_{k\to\infty} |\mathcal{R}_k - \mathrm{Id}| = 0.$$

Thus, by Lemma 3.5.4 we know that $F_k \xrightarrow{C^2} E$ in a neighborhood of 0.
To be more precise,

$$\lim_{k\to\infty} |F_k\Delta(E\cap B_R)| = 0. \tag{3.50}$$

Moreover, there exist $r,h>0$ small enough and $v_k, u \in C^2(\overline{B}'_r)$ such that

$$\begin{aligned}F_k\cap Q_{r,h} &= \{(x',x_n)\in\mathbb{R}^n \,|\, x'\in B'_r,\ v_k(x') < x_n < h\},\\ E\cap Q_{r,h} &= \{(x',x_n)\in\mathbb{R}^n \,|\, x'\in B'_r,\ u(x') < x_n < h\}\end{aligned}$$

and that

$$\lim_{k\to\infty} \|v_k - u\|_{C^2(\overline{B}'_r)} = 0. \tag{3.51}$$

Notice that $0\in\partial F_k$ and $\nu_{F_k}(0) = e_n$ for every k, that is,

$$v_k(0) = u(0) = 0, \quad \nabla v_k(0) = \nabla u(0) = 0. \tag{3.52}$$

We claim that

$$\lim_{k\to\infty} (1-s_k)\big|H_{s_k}[F_k](0) - H_{s_k}[E\cap B_R](0)\big| = 0. \tag{3.53}$$

By (3.52) and formula (C.1) we have that

$$\begin{aligned}H_{s_k}[F_k](0) &= 2\int_{B'_r} \frac{dy'}{|y'|^{n+s_k-1}} \int_0^{\frac{v_k(y')}{|y'|}} \frac{dt}{(1+t^2)^{\frac{n+s_k}{2}}} + \int_{\mathcal{C}Q_{r,h}} \frac{\chi_{\mathcal{C}F_k}(y) - \chi_{F_k}(y)}{|y|^{n+s_k}}\,dy\\ &= H^{loc}_{s_k}[F_k](0) + \int_{\mathcal{C}Q_{r,h}} \frac{\chi_{\mathcal{C}F_k}(y) - \chi_{F_k}(y)}{|y|^{n+s_k}}\,dy.\end{aligned}$$

We use the same formula for $E\cap B_R$ and prove at first that

$$\begin{aligned}\left|\int_{\mathcal{C}Q_{r,h}} \frac{\chi_{\mathcal{C}F_k}(y) - \chi_{F_k}(y) - \chi_{\mathcal{C}(E\cap B_R)}(y) + \chi_{E\cap B_R}(y)}{|y|^{n+s_k}}\,dy\right| &\le \frac{|F_k\Delta(E\cap B_R)|}{\delta^{n+s_k}}\\ &\le \frac{|F_k\Delta(E\cap B_R)|}{\delta^{n+1}},\end{aligned}$$

(where we have used (3.43)), which tends to 0 as $k \to \infty$, by (3.50).

Moreover, notice that by the Mean Value Theorem and (3.52) we have

$$|(v_k - u)(y')| \leq \frac{1}{2}|D^2(v_k - u)(\xi')||y'|^2 \leq \frac{\|v_k - u\|_{C^2(\overline{B}'_r)}}{2}|y'|^2.$$

Thus

$$\left|H^{loc}_{s_k}[F_k](0) - H^{loc}_{s_k}[E \cap B_R](0)\right| \leq 2\int_{B'_r} \frac{dy'}{|y'|^{n+s_k-1}}\left|\int_{\frac{u(y')}{|y'|}}^{\frac{v_k(y')}{|y'|}} \frac{dt}{(1+t^2)^{\frac{n+s_k}{2}}}\right|$$

$$\leq 2\int_{B'_r} |y'|^{-n-s_k}|(v_k - u)(y')|\, dy' \leq \frac{\varpi_{n-1}\,\|v_k - u\|_{C^2(\overline{B}'_r)}}{1 - s_k} r^{1-s_k},$$

hence by (3.51) we obtain

$$\lim_{k\to\infty}(1 - s_k)\left|H^{loc}_{s_k}[F_k](0) - H^{loc}_{s_k}[E \cap B_R](0)\right| = 0. \tag{3.54}$$

This concludes the proof of claim (3.53).

Now we use the triangle inequality and have that

$$\left|(1 - s_k)H_{s_k}[E_k](q_k) - H[E](0)\right| \leq (1 - s_k)\left|H_{s_k}[E_k](q_k) - H_{s_k}[F_k](0)\right|$$
$$+ (1 - s_k)\left|H_{s_k}[F_k](0) - H_{s_k}[E \cap B_R](0)\right| + \left|(1 - s_k)H_{s_k}[E \cap B_R](0) - H[E](0)\right|.$$

The last term in the right hand side converges by [2, Theorem 12]. As for the first term, notice that

$$H_{s_k}[F_k](0) = H_{s_k}[E_k \cap B_R](q_k),$$

hence

$$\lim_{k\to\infty}(1 - s_k)\left|H_{s_k}[E_k \cap B_R](q_k) - H_{s_k}[E_k](q_k)\right| \leq \limsup_{k\to\infty}(1 - s_k)\frac{2\varpi_n}{s_k}R^{-s_k} = 0.$$

Sending $k \to \infty$ in the triangle inequality above, we conclude the proof of the second part of Theorem 3.5.2. □

REMARK 3.5.5. In relation to the second part of the proof, we point out that using the directional fractional mean curvature defined in [2, Definition 6, Theorem 8], we can write

$$H^{loc}_{s_k}[F_k](0) = 2\int_{\mathbb{S}^{n-2}} \left[\int_0^r \varrho^{n-2}\left(\int_0^{v_k(\varrho e)} \frac{dt}{(\varrho^2 + t^2)^{\frac{n+s_k}{2}}}\right) d\varrho\right] d\mathcal{H}^{n-2}_e$$
$$= 2\int_{\mathbb{S}^{n-2}} \overline{K}_{s_k,e}\, d\mathcal{H}^{n-2}_e.$$

One is then actually able to prove that

$$\lim_{k\to\infty}(1 - s_k)\overline{K}_{s_k,e}[E_k - q_k](0) = H_e[E](0),$$

uniformly in $e \in \mathbb{S}^{n-2}$, by using formula (3.54) and the first claim of [2, Theorem 12].

REMARK 3.5.6. The proof of Theorem 3.5.2, as well as the proof of the next Proposition 3.5.3, settles the case in which $n \geq 2$. For $n = 1$, the proof follows in the same way, after observing that the local contribution to the fractional mean curvature is equal to zero because of symmetry. As a matter of fact, the formula in (C.1) for the fractional mean curvature (which has no meaning for $n = 1$) is not required.
We remark also that in our notation $\varpi_0 = 0$. This gives consistency to the second claim of Theorem 3.5.2 also for $n = 1$.

We prove now the continuity of the fractional mean curvature as $s \to 0$.

Proof of Proposition 3.5.3. Up to a translation, we can take $q = 0$ and $u(0) = 0$. For $R > 2\max\{r,h\}$, we write

$$\begin{aligned} H_{s_k}[E_k](q_k) &= \text{P.V.} \int_{Q_{r,h}(q_k)} \frac{\chi_{\mathcal{C}E_k}(y) - \chi_{E_k}(y)}{|y-q_k|^{n+s_k}}\,dy + \int_{\mathcal{C}Q_{r,h}(q_k)} \frac{\chi_{\mathcal{C}E_k}(y) - \chi_{E_k}(y)}{|y-q_k|^{n+s_k}}\,dy \\ &= \text{P.V.} \int_{Q_{r,h}(q_k)} \frac{\chi_{\mathcal{C}E_k}(y) - \chi_{E_k}(y)}{|y-q_k|^{n+s_k}}\,dy + \int_{B_R(q_k)\setminus Q_{r,h}(q_k)} \frac{\chi_{\mathcal{C}E_k}(y) - \chi_{E_k}(y)}{|y-q_k|^{n+s_k}}\,dy \\ &\quad + \int_{\mathcal{C}B_R(q_k)} \frac{\chi_{\mathcal{C}E_k}(y) - \chi_{E_k}(y)}{|y-q_k|^{n+s_k}}\,dy \\ &= I_1(k) + I_2(k) + I_3(k). \end{aligned}$$

Now using (C.1), (C.2) and (C.3) we have that

$$\begin{aligned} |I_1(k)| &\le 2\int_{B'_r(q'_k)} \frac{|\mathcal{G}(s_k,u_k,q'_k,y')|}{|y'-q'_k|^{n+s_k-1}}\,dy' \le 2\|u_k\|_{C^{1,\alpha}(\overline{B}'_r(q'_k))} \int_{B'_r(q'_k)} \frac{|y'-q'_k|^{\alpha}}{|y'-q'_k|^{n+s_k-1}}\,dy' \\ &\le 2C_2\varpi_{n-1}\frac{r^{\alpha-s_k}}{\alpha-s_k}. \end{aligned}$$

Using (3.43) we also have that

$$|I_2(k)| \le \int_{B_R(q_k)\setminus B_\delta(q_k)} \frac{dy}{|y-q_k|^{n+s_k}} = \varpi_n \frac{\delta^{-s_k} - R^{-s_k}}{s_k}.$$

Thus

$$\lim_{k\to\infty} s_k\big(|I_1(k)| + |I_2(k)|\big) = 0. \tag{3.55}$$

Furthermore

$$\begin{aligned} &\big|s_kI_3(k) - \big(\varpi_n - 2s_k\alpha_{s_k}(0,R,E)\big)\big| \\ &\le \left| s_k\int_{\mathcal{C}B_R(q_k)} \frac{dy}{|y-q_k|^{n+s_k}} - 2s_k\int_{\mathcal{C}B_R(q_k)} \frac{\chi_{E_k}(y)}{|y-q_k|^{n+s_k}}\,dy - \varpi_n + 2s_k\alpha_{s_k}(q_k,R,E))\right| \\ &\quad + 2s_k|\alpha_{s_k}(q_k,R,E) - \alpha_{s_k}(0,R,E)| \\ &\le |\varpi_nR^{-s_k} - \varpi_n| + 2s_k\left| \int_{\mathcal{C}B_R(q_k)} \frac{\chi_{E_k}(y)}{|y-q_k|^{n+s_k}}\,dy - \int_{\mathcal{C}B_R(q_k)} \frac{\chi_E(y)}{|y-q_k|^{n+s_k}}\,dy\right| \\ &\quad + 2s_k|\alpha_{s_k}(q_k,R,E) - \alpha_{s_k}(0,R,E)| \\ &\le |\varpi_nR^{-s_k} - \varpi_n| + 2s_k\int_{\mathcal{C}B_R(q_k)} \frac{\chi_{E_k\Delta E}(y)}{|y-q_k|^{n+s_k}}\,dy + 2s_k|\alpha_{s_k}(q_k,R,E) - \alpha_{s_k}(0,R,E)| \\ &\le |\varpi_nR^{-s_k} - \varpi_n| + 2C_1s_kR^{-n-s_k} + 2s_k|\alpha_{s_k}(q_k,R,E) - \alpha_{s_k}(0,R,E)|, \end{aligned}$$

where we have used that $|E_k\Delta E| < C_1$.

Therefore, since $q_k \in B_d$ for every k, as a consequence of Proposition 3.2.1 it follows that

$$\lim_{k\to\infty} \big|s_kI_3(k) - \big(\varpi_n - 2s_k\alpha_{s_k}(0,R,E)\big)\big| = 0. \tag{3.56}$$

Hence, by (3.55) and (3.56), we get that

$$\lim_{k\to\infty} s_kH_{s_k}[E_k](q_k) = \varpi_n - 2\lim_{k\to\infty} s_k\alpha_{s_k}(0,R,E) = \varpi_n - 2\alpha(E),$$

concluding the proof. □

Proof of Theorem 3.1.1. Arguing as in the proof of Proposition 3.5.3, by keeping fixed $E_k = E$ and $q_k = p$, we obtain

$$\liminf_{s\to 0} s\, H_s[E](p) = \varpi_n - 2\limsup_{s\to 0} s\,\alpha_s(0,R,E) = \varpi_n - 2\overline{\alpha}(E),$$

and similarly for the limsup. □

As a corollary of Theorem 3.5.2 and Theorem 3.1.1, we have the following result.

Theorem 3.5.7. *Let $E \subseteq \mathbb{R}^n$ and let $p \in \partial E$ be such that $\partial E \cap B_r(p)$ is C^2 for some $r > 0$. Suppose that the classical mean curvature of E in p is $H(p) < 0$. Also assume that*

$$\overline{\alpha}(E) < \frac{\varpi_n}{2}.$$

Then there exist $\sigma_0 < \tilde{s} < \sigma_1$ in $(0,1)$ such that

(i) $H_s[E](p) > 0$ *for every $s \in (0, \sigma_0]$, and actually*

$$\liminf_{s\to 0^+} s\; H_s[E](p) = \varpi_n - 2\overline{\alpha}(E),$$

(ii) $H_{\tilde{s}}[E](p) = 0$,

(iii) $H_s[E](p) < 0$ *for every $s \in [\sigma_1, 1)$, and actually*

$$\lim_{s\to 1}(1-s)\; H_s[E](p) = \varpi_{n-1} H[E](p).$$

CHAPTER 4

On nonlocal minimal graphs

4.1. Introduction

The aim of this chapter consists in introducing a functional framework for studying minimizers of the fractional perimeter that can be globally written as subgraphs.

More precisely, we define a functional $\mathcal{F}_s$, which can be considered as a fractional and nonlocal version of the area functional, and we exploit it to study nonlocal minimal graphs.

One of the main difficulties in defining a fractional and nonlocal version of the classical area functional is that, as observed in Chapter 2,

$$\mathrm{Per}_s(\{x_{n+1} < u(x)\}, \Omega \times \mathbb{R}) = \infty,$$

independently of the regularity of u—see Theorem 2.1.14 and Corollary 2.5.4. Nevertheless, this problem can be avoided by working in the "truncated cylinders" $\Omega \times (-M, M)$. In the functional setting that we introduce, this leads us to consider a family of functionals $\mathcal{F}_s^M$, instead of only the global functional $\mathcal{F}_s$.

Exploiting these approximating functionals, we prove existence and uniqueness results for minimizers of the functional $\mathcal{F}_s$—and actually of more general functionals—for a large class of exterior data which includes locally bounded functions.

Moreover, one of the main contributions of this chapter consists in proving the equivalence of:

- minimizers of the functional $\mathcal{F}_s$,
- minimizers ot the fractional perimeter,
- weak solutions of the fractional mean curvature equation,
- viscosity solutions of the fractional mean curvature equation,
- smooth pointwise solutions of the fractional mean curvature equation,

(see Theorem 4.1.11).

We observe that the functional framework introduced in this chapter easily adapts to the obstacle problem. Hence we prove also existence and uniqueness results for the nonlocal Plateau problem with (eventually discontinuous) obstacles.

Now we proceed to give the definitions and the precise statements of the main results of the chapter.

4.1.1. Definitions and main results. Let $g:\mathbb{R}\to\mathbb{R}$ be a continuous function satisfying

$$g(t)=g(-t)\quad\text{for every } t\in\mathbb{R},\qquad 0<g\le 1\quad\text{in } \mathbb{R},$$

and

$$\lambda:=\int_0^{+\infty} g(t)t\,dt<\infty.$$

Then, we define

$$G(t):=\int_0^t g(\tau)\,d\tau\quad\text{and}\quad \mathcal{G}(t):=\int_0^t G(\tau)\,d\tau=\int_0^t\left(\int_0^\tau g(\sigma)\,d\sigma\right)d\tau.$$

Given any function $u:\mathbb{R}^n\to\mathbb{R}$, we also formally set

$$\mathcal{F}(u,\Omega):=\iint_{Q(\Omega)}\mathcal{G}\left(\frac{u(x)-u(y)}{|x-y|}\right)\frac{dx\,dy}{|x-y|^{n-1+s}},\tag{4.1}$$

where

$$Q(\Omega):=\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2.$$

A particularly important example of function g is given by

$$g_s(t):=\frac{1}{(1+t^2)^{\frac{n+1+s}{2}}}.\tag{4.2}$$

We indicate with G_s and $\mathcal{G}_s$ respectively the first and second integrals of g_s as in (4.13). Furthermore, $\mathcal{F}_s$ denotes the functional corresponding to $\mathcal{G}_s$ in light of definition 4.1.

We will consider the following space

$$\mathcal{W}^s(\Omega):=\{u:\mathbb{R}^n\to\mathbb{R}\,|\,u|_\Omega\in W^{s,1}(\Omega)\}.$$

Given a function $\varphi:\mathcal{C}\Omega\to\mathbb{R}$ we also define the space

$$\mathcal{W}^s_\varphi(\Omega):=\{v\in\mathcal{W}^s(\Omega)\,|\,v=\varphi\text{ a.e. in }\mathcal{C}\Omega\}.\tag{4.3}$$

Our aim will be that of minimizing the functional $\mathcal{F}$ in $\mathcal{W}^s_\varphi(\Omega)$, given a fixed function $\varphi:\mathcal{C}\Omega\to\mathbb{R}$ as exterior data.

However, we remark that the functional $\mathcal{F}$ is not well defined on functions $u\in\mathcal{W}^s_\varphi(\Omega)$, unless the function φ has a suitable growth at infinity, namely

$$\int_\Omega\Big(\int_{\mathcal{C}\Omega}\frac{|\varphi(y)|}{|x-y|^{n+s}}dy\Big)dx<\infty,\tag{4.4}$$

which is a quite restrictive condition.

Nevertheless, as ensured by Lemma 4.5.1—exploiting the fractional Hardy-type inequality of Theorem D.1.4—the following definition of minimizer is well posed:

DEFINITION 4.1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. A function $u \in \mathcal{W}^s(\Omega)$ is a* minimizer *of $\mathcal{F}$ in Ω if*

$$\iint_{Q(\Omega)} \left\{ \mathcal{G}\left(\frac{u(x)-u(y)}{|x-y|}\right) - \mathcal{G}\left(\frac{v(x)-v(y)}{|x-y|}\right) \right\} \frac{dx\,dy}{|x-y|^{n-1+s}} \leq 0$$

for every $v \in \mathcal{W}^s(\Omega)$ such that $v = u$ almost everywhere in $\mathcal{C}\Omega$.

Fixed $\varphi : \mathcal{C}\Omega \to \mathbb{R}$, we consider the problem of finding a function $u \in \mathcal{W}^s_\varphi(\Omega)$ which is a minimizer for the functional $\mathcal{F}$ in the sense of Definition 4.1.1.

One of the main difficulties of this chapter will be that of finding such a minimizer without imposing the global condition (4.1.1) on the exterior data. This will be done by asking a suitable weaker condition on the exterior data φ and by exploiting a "truncation procedure" for the functional $\mathcal{F}$.

DEFINITION 4.1.2. *Let Ω be a bounded open set and let $u : \mathcal{C}\Omega \to \mathbb{R}$. Given an open set $\mathcal{O} \subseteq \mathbb{R}^n$ such that $\Omega \subseteq \mathcal{O}$, we define the "truncated tail" of u at a point $x \in \Omega$ as*

$$\mathrm{Tail}_s(u, \mathcal{O} \setminus \Omega; x) := \int_{\mathcal{O}\setminus\Omega} \frac{|u(y)|}{|x-y|^{n+s}}\,dy.$$

It is convenient to recall that, given a set $F \subseteq \mathbb{R}^n$, we denote

$$F_r := \left\{ x \in \mathbb{R}^n \,|\, \bar{d}_F(x) < r \right\},$$

for any $r \in \mathbb{R}$, with $\bar{d}_F$ denoting the signed distance function from ∂F, negative inside F. In particular, if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set and $\varrho > 0$, then

$$\Omega_{-\varrho} \Subset \Omega \Subset \Omega_\varrho.$$

We will make extensive use of this notation in the present chapter.

One of the main results of this chapter consists in proving the existence and uniqueness of a minimizer $u \in \mathcal{W}^s_\varphi(\Omega)$ for exterior data φ whose tail is integrable in a large enough neighborhood of Ω.

THEOREM 4.1.3. *Let $n \geq 1$, $s \in (0,1)$, and $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then, there is a constant $\Theta > 1$, depending only on n and s and g, such that, given any function $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ with $\mathrm{Tail}_s(\varphi, \Omega_{\Theta\,\mathrm{diam}(\Omega)} \setminus \Omega; \,\cdot\,) \in L^1(\Omega)$, there exists a unique minimizer u of $\mathcal{F}$ within $\mathcal{W}^s_\varphi(\Omega)$. Moreover, u satisfies*

$$\|u\|_{W^{s,1}(\Omega)} \leq C \left(\left\| \mathrm{Tail}_s(\varphi, \Omega_{\Theta\,\mathrm{diam}(\Omega)} \setminus \Omega; \,\cdot\,) \right\|_{L^1(\Omega)} + 1 \right), \tag{4.5}$$

for some constant $C > 0$ depending only on n, s, g and Ω.

We remark that asking

$$\|\mathrm{Tail}_s(\varphi, \mathcal{O} \setminus \Omega; \,\cdot\,)\|_{L^1(\Omega)} = \int_\Omega \Big(\int_{\mathcal{O}\setminus\Omega} \frac{|\varphi(y)|}{|x-y|^{n+s}} dy \Big) dx < \infty,$$

is a much weaker requirement than asking (4.1.1), since we impose no conditions on φ in $\mathcal{C}\mathcal{O}$.

The proof of Theorem 4.1.3 is the content of Section 4.5.2. The argument exploits the minimizers of appropriate truncated functionals $\mathcal{F}^M(\,\cdot\,, \Omega)$, considered within their natural domain, and an apriori bound on the $W^{s,1}(\Omega)$ norm, which gives (4.5). These topics are studied in Section 4.5.1.

See also Section 4.2.2 for the definition of the functionals $\mathcal{F}^M(\,\cdot\,,\Omega)$ and for their main functional properties, and Section 4.2.3 for the relationship existing between $\mathcal{F}^M(\,\cdot\,,\Omega)$ and the s-perimeter—in the geometric case $g = g_s$.

We also observe that if φ is bounded in $\mathcal{O}\setminus\Omega$, then, since Ω is bounded and has Lipschitz boundary, we have $\mathrm{Tail}_s(\varphi,\mathcal{O}\setminus\Omega;\,\cdot\,)\in L^1(\Omega)$—for more details about the integrability of the truncated tail, we refer to Lemma 4.5.10.

Hence, the boundedness of φ in a large enough neighborhood of Ω is enough to guarantee the existence of a unique minimizer of $\mathcal{F}$. Furthermore, in this case we prove that the minimizer is bounded also in Ω. More precisely:

THEOREM 4.1.4. *Let $n\geq 1$, $s\in(0,1)$, $\Omega\subseteq\mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and $R_0>0$ be such that $\Omega\subseteq B_{R_0}$. There exists a large constant $\Theta>1$, depending only on n, s and g, such that if $u\in\mathcal{W}^s(\Omega)$ is a minimizer of $\mathcal{F}$ in Ω, bounded in $B_{\Theta R_0}\setminus\Omega$, then u is also bounded in Ω and*

$$\|u\|_{L^\infty(\Omega)}\leq R_0+\|u\|_{L^\infty(B_{\Theta R_0}\setminus\Omega)}.$$

We observe that, even when the exterior data φ is not bounded in a neighborhood of Ω, we are nevertheless able to prove that the minimizer of $\mathcal{F}$ in $\mathcal{W}^s_\varphi(\Omega)$, if it exists, is locally bounded inside Ω (see Proposition 4.5.12).

Moreover, we point out that in order to obtain the global boundedness of the minimizer $u\in\mathcal{W}^s_\varphi(\Omega)$ inside Ω, it is actually enough to require the function φ to be bounded only in a neighborhood $\Omega_r\setminus\Omega$, with $r>0$ as small as we want. However, we remark that in this case the apriori L^∞ bound is not as clean as the one of Theorem 4.1.4 (see Theorem 4.5.14 for the precise statement).

Let us also mention that in Section 4.6 we will partially extend the above results to the obstacle problem. More precisely, we will prove the existence and uniqueness of a minimizer, in the case of locally bounded exterior data only, and we will establish an apriori bound on the $L^\infty(\Omega)$ of the minimizer.

The Euler-Lagrange operator associated to the minimization of $\mathcal{F}$ is

$$\mathscr{H}u(x):=2\,\mathrm{P.V.}\int_{\mathbb{R}^n}G\Big(\frac{u(x)-u(y)}{|x-y|}\Big)\frac{dy}{|x-y|^{n+s}}.$$

We remark that in order for $\mathscr{H}u(x)$ to be well defined, the function u must be regular enough (e.g. $C^{1,\alpha}$ for some $\alpha>s$) in a neighborhood of the point x.

On the other hand, we can always define $\mathscr{H}u$ in the distributional sense, for any measurable $u:\mathbb{R}^n\to\mathbb{R}$, as the linear functional

$$\langle\mathscr{H}u,v\rangle:=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}G\Big(\frac{u(x)-u(y)}{|x-y|}\Big)\big(v(x)-v(y)\big)\frac{dx\,dy}{|x-y|^{n+s}},$$

for every $v\in W^{s,1}(\mathbb{R}^n)$.

This observation prompts us to give the following definition of weak solution:

DEFINITION 4.1.5. *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set and let $f\in C(\overline{\Omega})$. We say that a function $u:\mathbb{R}^n\to\mathbb{R}$ is a weak solution of $\mathscr{H}u=f$ in Ω if*

$$\langle\mathscr{H}u,v\rangle=\int_\Omega f\,v\,dx,\qquad\forall\,v\in C^\infty_c(\Omega).$$

Some elementary properties of the operator $\mathscr{H}$ are studied in Section 4.2.4.

Exploiting the convexity of the functional $\mathcal{F}$, it is easy to verify—see Lemma 4.5.4—that if we add the requirement that $u\in\mathcal{W}^s(\Omega)$, then

$$u\text{ is a minimizer of }\mathcal{F}\text{ in }\Omega\quad\Longleftrightarrow\quad u\text{ is a weak solution of }\mathscr{H}u=0\text{ in }\Omega.$$

Besides distributional solutions, another natural notion of solutions to consider for the problem

$$\begin{cases} \mathscr{H}u = f & \text{in } \Omega, \\ u = \varphi & \text{in } \mathcal{C}\Omega. \end{cases}$$

is that of viscosity solutions. We will use $C^{1,1}$ functions as test functions.

DEFINITION 4.1.6. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let $f \in C(\overline{\Omega})$. We say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is a (viscosity) subsolution of $\mathscr{H}u = f$ in Ω, and we write*

$$\mathscr{H}u \le f \qquad \text{in } \Omega,$$

if u is upper semicontinuous in Ω and whenever the following happens:

(i) $x_0 \in \Omega$,
(ii) $v \in C^{1,1}(B_r(x_0))$, for some $r < d(x_0, \partial\Omega)$,
(iii) $v(x_0) = u(x_0)$ and $v(y) \ge u(y)$ for every $y \in B_r(x_0)$,

then if we define

$$\tilde{v}(x) := \begin{cases} v(x) & \text{if } x \in B_r(x_0), \\ u(x) & \text{if } x \in \mathbb{R}^n \setminus B_r(x_0), \end{cases}$$

we have

$$\mathscr{H}\tilde{v}(x_0) \le f(x_0).$$

A supersolution is defined similarly. A (viscosity) solution is a function $u : \mathbb{R}^n \to \mathbb{R}$ which is continuous in Ω and which is both a subsolution and a supersolution.

We remark that in the definition of a viscosity subsolution we do not ask u to be upper semicontinuous in $\overline{\Omega}$ but only in Ω. Furthermore, we do not ask u to belong to the functional space $\mathcal{W}^s(\Omega)$.

Another important result of this chapter consists in proving that viscosity (sub)solutions are also weak (sub)solutions.

THEOREM 4.1.7. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let $f \in C(\overline{\Omega})$. Let $u : \mathbb{R}^n \to \mathbb{R}$ be such that u is locally integrable in $\mathbb{R}^n$ and u is locally bounded in Ω. If u is a viscosity subsolution,*

$$\mathscr{H}u \le f \quad \text{in } \Omega,$$

then u is a weak subsolution,

$$\langle \mathscr{H}u, v\rangle \le \int_\Omega f v\,dx, \qquad \forall\, v \in C_c^\infty(\Omega) \text{ s.t. } v \ge 0.$$

It is worth to mention also a global version, for viscosity solutions, of this Theorem. Given a continuous function $f \in C(\mathbb{R}^n)$, we say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is a viscosity solution of $\mathscr{H}u = f$ in $\mathbb{R}^n$ if $u \in C(\mathbb{R}^n)$ and u is a viscosity solution in every bounded open set $\Omega \subseteq \mathbb{R}^n$.

COROLLARY 4.1.8. *Let $f \in C(\mathbb{R}^n)$ and let $u : \mathbb{R}^n \to \mathbb{R}$. If u is a viscosity solution of $\mathscr{H}u = f$ in $\mathbb{R}^n$, then u is a weak solution,*

$$\langle \mathscr{H}u, v\rangle = \int_{\mathbb{R}^n} f v\,dx, \qquad \forall\, v \in C_c^\infty(\mathbb{R}^n).$$

The study of viscosity (sub)solutions and the proof of Theorem 4.1.7 are carried out in Section 4.3.

4.1.1.1. *Geometric case.* The case in which $g = g_s$ is particularly important, because it is connected with the nonlocal minimal surfaces. In particular,

$$\mathscr{H}_s u(x) = H_s[\mathcal{S}g(u)](x, u(x)),$$

is the s-fractional mean curvature of the subgraph of u,

$$\mathcal{S}g(u) := \{X = (x, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x_{n+1} < u(x)\},$$

at the point $(x, u(x)) \in \partial\mathcal{S}g(u)$ (provided u is regular enough near x).

Therefore, the equation

$$\mathscr{H}_s u = 0$$

is, at least formally, the Euler-Lagrange equation of an s-minimal set which can be globally written as a subgraph.

Before going on, we recall that the s-fractional perimeter of a set $E \subseteq \mathbb{R}^{n+1}$ in an open set $\mathcal{O} \subseteq \mathbb{R}^{n+1}$ is defined as

$$\operatorname{Per}_s(E, \mathcal{O}) = \mathcal{L}_s(E \cap \mathcal{O}, \mathcal{C}E \cap \mathcal{O}) + \mathcal{L}_s(E \cap \mathcal{O}, \mathcal{C}E \setminus \mathcal{O}) + \mathcal{L}_s(E \setminus \mathcal{O}, \mathcal{C}E \cap \mathcal{O}),$$

where

$$\mathcal{L}_s(A, B) := \int_A \int_B \frac{dX\, dY}{|X - Y|^{n+1+s}},$$

for every couple of disjoint sets $A, B \subseteq \mathbb{R}^{n+1}$. We also observe that we can rewrite the s-perimeter as

$$\operatorname{Per}_s(E, \mathcal{O}) = \frac{1}{2} \iint_{\mathbb{R}^{2(n+1)} \setminus (\mathcal{C}\mathcal{O})^2} \frac{|\chi_E(X) - \chi_E(Y)|}{|X - Y|^{n+1+s}} dX\, dY.$$

The s-fractional mean curvature of E at $X \in \partial E$ is the principal value integral

$$H_s[E](X) := \text{P.V.} \int_{\mathbb{R}^{n+1}} \frac{\chi_{\mathcal{C}E}(Y) - \chi_E(Y)}{|X - Y|^{n+1+s}} dy.$$

Definition 4.1.9. *Let $\mathcal{O} \subseteq \mathbb{R}^{n+1}$ be an open set and let $E \subseteq \mathbb{R}^{n+1}$. We say that E is s-minimal in $\mathcal{O}$ if $\operatorname{Per}_s(E, \mathcal{O}) < \infty$ and*

$$F \setminus \mathcal{O} = E \setminus \mathcal{O} \qquad \Longrightarrow \qquad \operatorname{Per}_s(E, \mathcal{O}) \le \operatorname{Per}_s(F, \mathcal{O}).$$

We say that E is locally s-minimal in $\mathcal{O}$ if it is s-minimal in every $\mathcal{O}' \Subset \mathcal{O}$.

In this chapter we are interested in the case where the domain is a cylinder, $\mathcal{O} = \Omega \times \mathbb{R}$. For simplicity, we introduce the following notation:

$$\Omega^M := \Omega \times (-M, M), \quad \forall\, M \ge 0 \qquad \text{and} \qquad \Omega^\infty := \Omega \times \mathbb{R}.$$

We remark that when Ω is bounded and has Lipschitz boundary, then a set E is locally s-minimal in Ω^∞ if and only if it is s-minimal in Ω^M, for every $M > 0$—see Remark 2.5.1.

We show that appropriately rearranging a set E in the vertical direction we decrease the s-perimeter.

More precisely, given a set $E \subseteq \mathbb{R}^{n+1}$, we consider the function $w_E : \mathbb{R}^n \to \mathbb{R}$ defined by

$$w_E(x) := \lim_{R \to +\infty} \left(\int_{-R}^{R} \chi_E(x, t)\, dt - R \right) \tag{4.6}$$

for every $x \in \mathbb{R}^n$, together with its subgraph $E_\star := \mathcal{S}g(w_E)$.

Then we have the following result.

THEOREM 4.1.10. *Let $n \geq 1$, $s \in (0,1)$, and $\Omega \subseteq \mathbb{R}^n$ be an open set with Lipschitz boundary. Let $E \subseteq \mathbb{R}^{n+1}$ be such that $E \setminus \Omega^\infty$ is a subgraph and*

$$\Omega \times (-\infty, -M) \subseteq E \cap \Omega^\infty \subseteq \Omega \times (-\infty, M), \tag{4.7}$$

for some $M > 0$. Then,

$$\mathrm{Per}_s(E_\star, \Omega^M) \leq \mathrm{Per}_s(E, \Omega^M). \tag{4.8}$$

The inequality is strict unless $E_\star = E$.

The proof of Theorem 4.1.10 can be found in Section 4.4 and is based on a rearrangement inequality that we establish for rather general 1-dimensional integral set functions.

Combining the main results of this chapter and exploiting the interior regularity proved in [19], we obtain the following Theorem—whose proof is in Section 4.5.4.

THEOREM 4.1.11. *Let $n \geq 1$, $s \in (0,1)$, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and let $u \in \mathcal{W}^s(\Omega)$. Then, the following are equivalent:*

(i) u is a weak solution of $\mathscr{H}_s u = 0$ in Ω,
(ii) u is a minimizer of $\mathcal{F}_s$ in Ω,
(iii) $u \in L^\infty_{\mathrm{loc}}(\Omega)$ and $\mathcal{S}g(u)$ is locally s-minimal in $\Omega \times \mathbb{R}$,
(iv) $u \in C^\infty(\Omega)$ and u is a pointwise solution of $\mathscr{H}_s u = 0$ in Ω.

Moreover, if $u \in \mathcal{W}^s(\Omega) \cap L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then all of the above are equivalent to:

(v) u is a viscosity solution of $\mathscr{H}_s u = 0$ in Ω.

We also point out the following global version of Theorem 4.1.11:

COROLLARY 4.1.12. *Let $u \in W^{s,1}_{\mathrm{loc}}(\mathbb{R}^n)$. Then, the following are equivalent:*

(i) u is a viscosity solution of $\mathscr{H}_s u = 0$ in $\mathbb{R}^n$,
(ii) u is a weak solution of $\mathscr{H}_s u = 0$ in $\mathbb{R}^n$,
(iii) u is a minimizer of $\mathcal{F}_s$ in Ω, for every bounded open set $\Omega \subseteq \mathbb{R}^n$,
(iv) $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathcal{S}g(u)$ is locally s-minimal in $\mathbb{R}^{n+1}$,
(v) $u \in C^\infty(\mathbb{R}^n)$ and u is a pointwise solution of $\mathscr{H}_s u = 0$ in $\mathbb{R}^n$.

In [43] the authors observed that if a set E is locally s-minimal in Ω^∞, with $\Omega \subseteq \mathbb{R}^n$ a bounded open set with C^2 boundary, and $E = \mathcal{S}g(\varphi)$ in $\mathcal{C}\Omega^\infty$, with $\varphi \in L^\infty(B_{\tilde{R}} \setminus \Omega)$ for some $\tilde{R} = \tilde{R}(n, s, \Omega) > 0$ big enough, then

$$\Omega \times (-\infty, -M_0) \subseteq E \cap \Omega^\infty \subseteq \Omega \times (-\infty, M_0),$$

for some $M_0 = M_0(n, s, \Omega, \varphi) > 0$. Roughly speaking, this is an a priori bound on the "vertical variation" of the nonlocal minimal surface ∂E in terms of the exterior data φ and can be thought of as the geometric counterpart of Theorem 4.1.4.

Exploiting this observation, Theorem 4.1.10 and Theorem 4.1.11, we conclude that there exists a unique locally s-minimal set $E \subseteq \mathbb{R}^{n+1}$ in Ω^∞ having as exterior data the subgraph of φ and E is the subgraph of the function $u \in \mathcal{W}^s_\varphi(\Omega)$ that minimizes $\mathcal{F}_s$.

THEOREM 4.1.13. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^2 boundary and let $\tilde{R}(n, s, \Omega)$ be as defined above. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be such that $\varphi \in L^\infty(B_{\tilde{R}} \setminus \Omega)$. If $E \subseteq \mathbb{R}^{n+1}$ is locally s-minimal in Ω^∞ and $E \setminus \Omega^\infty = \mathcal{S}g(\varphi) \setminus \Omega^\infty$, then $E = \mathcal{S}g(u)$, for some $u \in \mathfrak{B}_{M_0} \mathcal{W}^s_\varphi(\Omega)$, with $M_0(n, s, \Omega, \varphi) > 0$ defined as above. Moreover, u is the unique minimizer of $\mathcal{F}_s$ in $\mathcal{W}^s_\varphi(\Omega)$.*

We point out that the existence of a locally s-minimal set as in Theorem 4.1.13 is ensured by Corollary 2.1.11.

In particular, Theorem 4.1.13 extends the result obtained in [43] to a much wider family of exterior data φ. Moreover, it is interesting to observe that, to the best of the authors' knowledge, this also provides the only uniqueness result available for (locally) s-minimal sets, besides the trivial case where the exterior data is an half-space.

We conclude the Introduction with some observations concerning the regularity of the minimizers of the functional $\mathcal{F}_s$.

Thanks to the interior regularity results proven in [19] and the fact that the subgraph of a minimizer u of $\mathcal{F}_s$ is locally s-minimal, we know that $u \in C^\infty(\Omega)$.

On the other hand, we point out that a minimizer u of $\mathcal{F}_s$ need not be continuous across the boundary of Ω and indeed, in general the subgraph $\mathcal{S}g(u)$ sticks to the boundary of the cylinder Ω^∞. For examples of this typically nonlocal phenomenon, we refer in particular to [45, Theorems 1.2 and 1.4]. Indeed, the exterior data considered in [45, Theorem 1.2] is the subgraph of the function $\varphi : \mathbb{R} \setminus (-1,1) \to \mathbb{R}$ defined as

$$\varphi(t) := -M \quad \text{if } t \leq -1 \quad \text{and} \quad \varphi(t) := M \quad \text{if } t \geq 1.$$

Hence, by Theorem 4.1.13, we know that there exists a unique locally s-minimal set E with exterior data $\mathcal{S}g(\varphi)$, which is given by $E = \mathcal{S}g(u)$, where $u \in \mathcal{W}^s_\varphi(-1,1)$ is the minimizer of $\mathcal{F}_s$. Then [45, Theorem 1.2] says that u does not attain the exterior data φ, which is smooth and globally bounded, in a continuous way, but rather "sticks" to the boundary of the cylinder $(-1,1) \times \mathbb{R}$.

The same behavior is observed in [45, Theorem 1.4] where the exterior data can be chosen to be a small, smooth and compactly supported bump function. Again, by Theorem 4.1.13, we know that the locally s-minimal set is given by the subgraph of the function u which minimizes $\mathcal{F}_s$. Furthermore, we remark that in this case the exterior data can be taken to be "arbitrarily close" to the constant function 0, so in some sense this kind of phenomenon is the typical boundary behavior of minimizers of $\mathcal{F}_s$.

Furthermore, we mention the forthcoming paper [15], where this behavior is investigated in the case where the fractional parameter s is small, also in the presence of obstacles.

Nevertheless, even if in general the minimizer of $\mathcal{F}_s$ is not continuous across the boundary of the domain, not even when the exterior data is smooth and globally bounded, we point out that no gap phenomenon occurs, as shown by Proposition 4.7.5.

Finally, we mention that in Section 4.7 we prove some approximation results for subgraphs having (locally) finite fractional perimeter. In particular, exploiting the surprising density result established in [44], we show that s-minimal subgraphs can be appropriately approximated by subgraphs of σ-harmonic functions, for any fixed $\sigma \in (0,1)$—see Theorem 4.7.4.

4.2. Preliminary results

4.2.1. Elementary properties of the functions g, G, and $\mathcal{G}$. We begin by recalling the following definitions given in the introduction. We consider a continuous function $g : \mathbb{R} \to \mathbb{R}$ satisfying

$$g(t) = g(-t) \quad \text{for every } t \in \mathbb{R}, \tag{4.9}$$

$$0 < g \leq 1 \quad \text{in } \mathbb{R}, \tag{4.10}$$

and

$$\lambda := \int_0^{+\infty} t g(t)\, dt < \infty. \tag{4.11}$$

We also observe that

$$\Lambda := \int_{\mathbb{R}} g(t)\, dt \leq 2(\lambda + 1) < \infty. \tag{4.12}$$

As remarked in the Introduction, it is easily seen that the function g_s defined in (4.2) satisfies these assumptions. When considering g_s, we will denote

$$\Lambda_{n,s} := \int_{\mathbb{R}} g_s(t)\, dt.$$

Then, we define

$$G(t) := \int_0^t g(\tau)\, d\tau, \qquad \mathcal{G}(t) := \int_0^t G(\tau)\, d\tau = \int_0^t \left(\int_0^\tau g(\sigma)\, d\sigma \right) d\tau \tag{4.13}$$

and

$$\overline{G}(t) := \int_{-\infty}^t g(\tau)\, d\tau = \int_{-t}^{+\infty} g(\tau)\, d\tau, \tag{4.14}$$

for every $t \in \mathbb{R}$. Notice that

$$\overline{G}(t) = \frac{\Lambda}{2} + G(t) \quad \text{for every } t \in \mathbb{R}. \tag{4.15}$$

The following lemma collects the main properties of these functions that will be used in the forthcoming sections.

LEMMA 4.2.1. *The functions G and $\mathcal{G}$ are respectively of class C^1 and C^2. Furthermore, the following facts hold true.*

(a) The function G is odd, increasing, satisfies $G(0) = 0$ and

$$c_\star \min\{1, |t|\} \le |G(t)| \le \min\left\{ \frac{\Lambda}{2}, |t| \right\} \quad \textit{for every } t \in \mathbb{R}, \tag{4.16}$$

where

$$c_\star = c_\star(g) := \inf_{t \in [0,1]} g(t) > 0. \tag{4.17}$$

Moreover,

$$|G(t) - G(\tau)| \le |t - \tau| \quad \textit{for every } t, \tau \in \mathbb{R}. \tag{4.18}$$

(b) The function $\mathcal{G}$ is even, increasing on $[0, \infty)$, strictly convex and such that $\mathcal{G}(0) = 0$. It satisfies

$$\frac{c_\star}{2} \min\left\{|t|, t^2\right\} \le \mathcal{G}(t) \le \frac{t^2}{2}, \tag{4.19}$$

$$\frac{\Lambda}{2}|t| - \lambda \le \mathcal{G}(t) \le \frac{\Lambda}{2}|t|, \tag{4.20}$$

for every $t \in \mathbb{R}$, and

$$|\mathcal{G}(t) - \mathcal{G}(\tau)| \le \frac{\Lambda}{2}\, |t - \tau| \quad \textit{for every } t, \tau \in \mathbb{R}. \tag{4.21}$$

PROOF. Almost all the statements follow immediately from definitions (4.13) and (4.14). The only properties that require an explicit proof are the lower bounds on $|G|$ and $\mathcal{G}$

To obtain the left-hand inequality in (4.16) we assume without loss of generality that $t \ge 0$ and distinguish between the cases $t > 1$ and $t \in [0, 1]$. In the first situation, the claim simply follows by (4.13) along with the monotonicity of G and (4.17), as indeed

$$G(t) \ge G(1) = \int_0^1 g(t)\, dt \ge c_\star.$$

Conversely, when $t \in [0, 1]$ we have

$$G(t) = \int_0^t g(\tau)\, d\tau \ge c_\star\, t,$$

thanks again to (4.17).

To get the lower bound in (4.19), we first notice that we can restrict ourselves to $t \geq 1$, since the case $t \in [0,1]$ can be deduced straight-away from (4.16) and the definition of $\mathcal{G}$. For $t \geq 1$ we apply (4.16) to compute

$$\mathcal{G}(t) = \int_0^1 G(\tau)\, d\tau + \int_1^t G(\tau)\, d\tau \geq c_\star \left(\int_0^1 \tau\, d\tau + \int_1^t d\tau \right) = \frac{c_\star}{2}\big(1 + 2(t-1)\big) \geq \frac{c_\star}{2}\, t.$$

Finally, to establish the first inequality in (4.20), we recall definitions (4.11)-(4.13) and compute, for $t \geq 0$,

$$\begin{aligned} \mathcal{G}(t) - \frac{\Lambda}{2} t &= \int_0^t \left(\int_0^\tau g(\sigma)\, d\sigma \right) d\tau - \left(\int_0^{+\infty} g(\sigma)\, d\sigma \right) t = - \int_0^t \left(\int_\tau^{+\infty} g(\sigma)\, d\sigma \right) d\tau \\ &= - \int_0^t \left(\int_0^\sigma g(\sigma)\, d\tau \right) d\sigma - \int_t^{+\infty} \left(\int_0^t g(\sigma)\, d\tau \right) d\sigma \\ &= - \int_0^t \sigma g(\sigma)\, d\sigma - t \int_t^{+\infty} g(\sigma)\, d\sigma = -\lambda + \int_t^{+\infty} (\sigma - t) g(\sigma)\, d\sigma \geq -\lambda. \end{aligned}$$

Note that the third identity follows by Fubini's theorem. The proof of the lemma is thus complete. □

We stress that hypothesis (4.11) has only been used to deduce the left-hand inequality in (4.20). If one drops it, the weaker lower bound

$$\mathcal{G}(t) \geq \frac{c_\star}{2}|t| - \frac{c_\star}{2} \quad \text{for every } t \in \mathbb{R}$$

can still be easily deduced from (4.19). This estimate is indeed sufficient for most of the applications presented in the remainder of this chapter. However, we will make crucial use of the finer bound (4.20) at some point in the proof of Proposition 4.5.12. Therefore, such result and all those that rely on it need assumption (4.11) to hold.

Note that the function $g(t) = 1/(1+t^2)$ fulfills hypotheses (4.9), (4.10), (4.12), but not (4.11). Also, the corresponding second antiderivative $\mathcal{G}$ does not satisfies the lower bound in (4.20) or any bound of the form $\mathcal{G}(t) \geq \Lambda |t|/2 - C$ for some constant $C > 0$.

4.2.2. Functional theoretic properties of the fractional area functionals. In this subsection we introduce the area-type functionals $\mathcal{F}^M$ and determine some basic properties of the local part $\mathcal{A}$ and nonlocal part $\mathcal{N}^M$.

First of all, we observe that we can split the functional $\mathcal{F}$ defined in (4.1) into the two components

$$\mathcal{F}(u, \Omega) = \mathcal{A}(u, \Omega) + \mathcal{N}(u, \Omega),$$

with

$$\mathcal{A}(u, \Omega) := \int_\Omega \int_\Omega \mathcal{G}\left(\frac{u(x) - u(y)}{|x-y|} \right) \frac{dx\, dy}{|x-y|^{n-1+s}} \tag{4.22}$$

and

$$\mathcal{N}(u, \Omega) := 2 \int_\Omega \int_{\mathcal{C}\Omega} \mathcal{G}\left(\frac{u(x) - u(y)}{|x-y|} \right) \frac{dx\, dy}{|x-y|^{n-1+s}}.$$

As shown in Lemma 4.2.2, in order for the local part $\mathcal{A}(u, \Omega)$ to be well defined, it is necessary and sufficient that $u \in W^{s,1}(\Omega)$. On the other hand, for the nonlocal part $\mathcal{N}(u, \Omega)$ to be well defined, we would have to impose some restrictive condition on the behavior of u in the whole $\mathbb{R}^n$—namely (4.4).

For this reason, given any real number $M \geq 0$ we define for a function $u : \mathbb{R}^n \to \mathbb{R}$ the "truncated" nonlocal part

(4.23)
$$\mathcal{N}^M(u,\Omega) := \int_\Omega \left\{ \int_{\mathcal{C}\Omega} \left[\int_{\frac{-M-u(y)}{|x-y|}}^{\frac{u(x)-u(y)}{|x-y|}} \overline{G}(t)\,dt + \int_{\frac{u(x)-u(y)}{|x-y|}}^{\frac{M-u(y)}{|x-y|}} \overline{G}(-t)\,dt \right] \frac{dy}{|x-y|^{n-1+s}} \right\} dx,$$

and we introduce the functional

$$\mathcal{F}^M(u,\Omega) := \mathcal{A}(u,\Omega) + \mathcal{N}^M(u,\Omega). \tag{4.24}$$

When $g = g_s$ we will add the subscript s to the functionals, that is, we will write $\mathcal{A}_s$, $\mathcal{N}_s^M$ and $\mathcal{F}_s^M$.

As a motivation for introducing the functionals $\mathcal{N}^M$, we observe that in the geometric situation—that is, when $g = g_s$—considering the functional $\mathcal{N}_s^M$ in place of $\mathcal{N}_s$ amounts, roughly speaking, to considering the nonlocal part of the fractional perimeter of the subgraph $\mathcal{S}g(u)$ in the "truncated" cylinder $\Omega^M = \Omega \times (-M, M)$ instead of considering the nonlocal part of the fractional perimeter in the whole cylinder $\Omega \times \mathbb{R}$—which would be infinite. This relationship with the fractional perimeter will be made precise in the forthcoming Subsection 4.2.3.

From the functional point of view, as proved in Lemma 4.2.3, the advantage of considering $\mathcal{N}^M$ instead of $\mathcal{N}$ consists in that we do not need to impose any condition on the function u outside of the domain Ω for $\mathcal{N}^M(u,\Omega)$ to be well defined.

We now proceed to establish the natural domain of definition of the local part $\mathcal{A}$. Notice that for the integral defining it to be meaningful (albeit possibly infinite) one only needs u to be defined in Ω.

LEMMA 4.2.2. *Let $n \geq 1$, $s \in (0,1)$, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $u : \Omega \to \mathbb{R}$ be a measurable function. Then*

$$\frac{c_\star}{2}\Big([u]_{W^{s,1}(\Omega)} - c_s(\Omega) \Big) \leq \mathcal{A}(u,\Omega) \leq \frac{\Lambda}{2}\, [u]_{W^{s,1}(\Omega)}, \tag{4.25}$$

where $c_\star > 0$ is the constant defined in (4.17) and

$$c_s(\Omega) := \frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{1-s}\, |\Omega| \operatorname{diam}(\Omega)^{1-s}. \tag{4.26}$$

Therefore,

$$u \in W^{s,1}(\Omega) \quad \textit{if and only if} \quad \mathcal{A}(u,\Omega) < \infty.$$

PROOF. The upper bound in (4.25) immediately follows by observing that $\mathcal{G}(t) \leq \Lambda|t|/2$ for every $t \in \mathbb{R}$, thanks to the right-hand inequality in formula (4.19) of Lemma 4.2.1. To get the lower bound, we recall the left-hand side of (4.19) and compute

$$\mathcal{A}(u,\Omega) \geq \frac{c_\star}{2} \left(\int_\Omega \int_\Omega \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy - \int_\Omega \int_\Omega \frac{dx\,dy}{|x-y|^{n-1+s}} \right).$$

The conclusion follows now by Lemma D.1.1 in Appendix D.1. Finally, we observe that if u is a measurable function such that $[u]_{W^{s,1}(\Omega)} < \infty$, then $u \in L^1(\Omega)$ by Lemma D.1.2. □

In the following result we present an equivalent representation for $\mathcal{N}^M(u,\Omega)$, given in terms of the function $\mathcal{G}$. We also establish its finiteness when the restriction of u to Ω belongs to the space $W^{s,1}(\Omega)$. Interestingly, no assumption on the behavior of u outside of Ω is needed.

LEMMA 4.2.3. *Let $n \geq 1$, $s \in (0,1)$, $M \geq 0$, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $u : \mathbb{R}^n \to \mathbb{R}$ be a measurable function. Then,*

$$|\mathcal{N}^M(u,\Omega)| \leq C\,\Lambda\big(\|u\|_{W^{s,1}(\Omega)} + M\big), \tag{4.27}$$

where Λ is the positive constant defined in (4.12) and $C > 0$ is a constant depending only on n, s and Ω. Hence,

$$|\mathcal{N}^M(u,\Omega)| < \infty \quad \textit{if} \quad u|_\Omega \in W^{s,1}(\Omega).$$

Furthermore, we have the identity

$$\begin{aligned} \mathcal{N}^M(u,\Omega) = \int_\Omega \Bigg\{ \int_{\mathcal{C}\Omega} \Bigg[2\,\mathcal{G}\left(\frac{u(x)-u(y)}{|x-y|}\right) - \mathcal{G}\left(\frac{M+u(y)}{|x-y|}\right) \\ -\mathcal{G}\left(\frac{M-u(y)}{|x-y|}\right)\Bigg] \frac{dy}{|x-y|^{n-1+s}} \Bigg\} dx + M\Lambda \int_\Omega \int_{\mathcal{C}\Omega} \frac{dx\,dy}{|x-y|^{n+s}}. \end{aligned} \tag{4.28}$$

PROOF. We can assume that $u|_\Omega \in W^{s,1}(\Omega)$, otherwise (4.27) is trivially satisfied. Taking advantage of (4.15) and of the right-hand inequality in (4.16), we get that

$$|\mathcal{N}^M(u,\Omega)| \leq 2\Lambda \left[\int_\Omega \left(|u(x)| \int_{\mathcal{C}\Omega} \frac{dy}{|x-y|^{n+s}} \right) dx + M \int_\Omega \int_{\mathcal{C}\Omega} \frac{dx\,dy}{|x-y|^{n+s}} \right].$$

We remark that the last double integral in the previous formula is the s-fractional perimeter of Ω in $\mathbb{R}^n$, which is finite, since Ω is bounded and has Lipschitz boundary. Then (4.27) follows by Corollary D.1.5.

On the other hand, identity (4.28) is a simple consequence of definition (4.23), formula (4.15) and the symmetry properties of G and $\mathcal{G}$. □

We stress that, in order to have $\mathcal{N}^M(u,\Omega)$ finite, the requirement $u|_\Omega \in W^{s,1}(\Omega)$ is far from being optimal. In fact, as the previous proof showed, it suffices that $u|_\Omega$ lies in a suitable weighted L^1 space over Ω—that contains for instance $L^\infty(\Omega)$. Nevertheless, such a requirement does not limit our analysis, since it is needed to have $\mathcal{A}(u,\Omega)$ finite, according to Lemma 4.2.2. We inform the interested reader that a more precise result on the natural domain of definition of $\mathcal{N}_s^M$ will be provided by Lemma 4.2.7 in the forthcoming Subsection 4.2.3.

Furthermore, we observe that if $u : \mathbb{R}^n \to \mathbb{R}$ is such that $u \in L^\infty(\Omega)$ and $M \geq \|u\|_{L^\infty(\Omega)}$, then $\mathcal{N}^M(u,\Omega) \geq 0$—since the integrand inside the square brackets in (4.23) is non-negative. On the other hand, we remark that in general the nonlocal part $\mathcal{N}^M(\,\cdot\,,\Omega)$ can assume also negative values, as proved in the following Example 4.2.1.

EXAMPLE 4.2.1. Let $n \geq 1$, $s \in (0,1)$, $M \geq 0$, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. There exists a positive constant $C = C(n,s,\Omega,g,M) > 0$ big enough such that, if $u : \mathbb{R}^n \to \mathbb{R}$ is the constant function $u \equiv T$, for some $T \geq C$, then

$$\mathcal{F}^M(u,\Omega) = \mathcal{N}^M(u,\Omega) < 0.$$

Proof. Let us fix $R > 0$ such that $\Omega \Subset B_R$. By identity (4.28) and recalling that $\mathcal{G} \geq 0$, we obtain

$$\begin{aligned}\mathcal{N}^M(u,\Omega) &= -\int_\Omega \left\{ \int_{\mathcal{C}\Omega} \left[\mathcal{G}\left(\frac{M+T}{|x-y|}\right) + \mathcal{G}\left(\frac{M-T}{|x-y|}\right) \right] \frac{dy}{|x-y|^{n-1+s}} \right\} dx \\ &\qquad + M\Lambda \int_\Omega \int_{\mathcal{C}\Omega} \frac{dx\,dy}{|x-y|^{n+s}} \\ &\leq -\int_\Omega \int_{\mathcal{C}\Omega} \mathcal{G}\left(\frac{M+T}{|x-y|}\right) \frac{dx\,dy}{|x-y|^{n-1+s}} + M\Lambda \int_\Omega \int_{\mathcal{C}\Omega} \frac{dx\,dy}{|x-y|^{n+s}} \\ &\leq -\int_\Omega \int_{B_R\setminus\Omega} \mathcal{G}\left(\frac{M+T}{|x-y|}\right) \frac{dx\,dy}{|x-y|^{n-1+s}} + M\Lambda \int_\Omega \int_{\mathcal{C}\Omega} \frac{dx\,dy}{|x-y|^{n+s}}.\end{aligned}$$

By exploiting (4.19), the fact that Ω is bounded and has Lipschitz boundary—hence it has finite s-perimeter—and Lemma D.1.1, we find that

$$\begin{aligned}\int_\Omega \int_{B_R\setminus\Omega} \mathcal{G}\left(\frac{M+T}{|x-y|}\right) \frac{dx\,dy}{|x-y|^{n-1+s}} &\geq \frac{c_\star}{2} \int_\Omega \int_{B_R\setminus\Omega} \left[\frac{M+T}{|x-y|} - 1\right] \frac{dx\,dy}{|x-y|^{n-1+s}} \\ &= C_1(M+T) - C_2,\end{aligned}$$

with C_1, $C_2 > 0$ depending only on n, s, Ω and g. Therefore,

$$\mathcal{N}^M(u,\Omega) \leq -C_1(M+T) + C_2 + M\Lambda \int_\Omega \int_{\mathcal{C}\Omega} \frac{dx\,dy}{|x-y|^{n+s}},$$

which is negative, provided we take $T > 0$ big enough. This concludes the proof. □

We collect the results of Lemmas 4.2.2 and 4.2.3 in the following unifying statement.

Lemma 4.2.4. *Let $n \geq 1$, $s \in (0,1)$, $M \geq 0$, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $u \in \mathcal{W}^s(\Omega)$. Then, $\mathcal{F}^M(u,\Omega)$ is finite and it holds*

$$\left|\mathcal{F}^M(u,\Omega)\right| \leq C\,\Lambda \left(\|u\|_{W^{s,1}(\Omega)} + M\right),$$

for some constant $C > 0$ depending only on n, s and Ω.

We conclude this subsection by specifying the convexity properties enjoyed by the functionals $\mathcal{A}$, $\mathcal{N}^M$, and $\mathcal{F}^M$.

Lemma 4.2.5. *Let $s \in (0,1)$ and let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. The following facts hold true:*

(i) The functional $\mathcal{A}(\,\cdot\,,\Omega)$ is convex on $W^{s,1}(\Omega)$.
(ii) Given any $M \geq 0$ and measurable function $\varphi : \mathcal{C}\Omega \to \mathbb{R}$, the functionals $\mathcal{N}^M(\,\cdot\,,\Omega)$ and $\mathcal{F}^M(\,\cdot\,,\Omega)$ are strictly convex on the space $\mathcal{W}^s_\varphi(\Omega)$ defined in (4.3).

Proof. The convexity of the functionals is an immediate consequence of the (strict) convexity of $\mathcal{G}$ warranted by Lemma 4.2.1. We point out that the convexity of $\mathcal{N}^M(\,\cdot\,,\Omega)$ is due also to the fact that the second and third summands appearing inside square brackets in the representation (4.28) are constant on $\mathcal{W}^s_\varphi(\Omega)$. Indeed, given u, $v \in \mathcal{W}^s_\varphi(\Omega)$ and $t \in (0,1)$, we have the identity

(4.29)
$$\begin{aligned}&\mathcal{N}^M(tu + (1-t)v,\Omega) - t\,\mathcal{N}^M(u,\Omega) - (1-t)\mathcal{N}^M(v,\Omega) \\ &\quad = 2\int_\Omega \left\{ \int_{\mathcal{C}\Omega} \left[\mathcal{G}\left(t\frac{u(x)-\varphi(y)}{|x-y|} + (1-t)\frac{v(x)-\varphi(y)}{|x-y|} \right) - t\,\mathcal{G}\left(\frac{u(x)-\varphi(y)}{|x-y|}\right) \right.\right. \\ &\qquad\qquad \left.\left. - (1-t)\mathcal{G}\left(\frac{v(x)-\varphi(y)}{|x-y|}\right) \right] \frac{dy}{|x-y|^{n-1+s}} \right\} dx,\end{aligned}$$

and the convexity of $\mathcal{G}$ guarantees that the integrand in the double integral above is nonpositive. Furthermore, the strict convexity of $\mathcal{G}$ implies that the quantity in (4.29) is equal to zero if and only if

$$\frac{u(x)-\varphi(y)}{|x-y|}=\frac{v(x)-\varphi(y)}{|x-y|}\quad\text{for a.e. }(x,y)\in\Omega\times\mathcal{C}\Omega,$$

i.e. if and only if $u=v$ almost everywhere in Ω—and hence in $\mathbb{R}^n$. □

4.2.3. Geometric properties of the fractional area functionals. This subsection is devoted to the description of a few geometric properties enjoyed by $\mathcal{A}_s$, $\mathcal{N}_s^M$ and $\mathcal{F}_s^M$. More specifically, in this subsection we consider the case $g=g_s$ and we show the connection existing between the fractional perimeter Per_s and these functionals, that ultimately motivates their introduction.

First of all, we remark that we can split the s-perimeter into its local and nonlocal parts, as

$$\mathrm{Per}_s(E,\mathcal{O})=\mathrm{Per}_s^L(E,\mathcal{O})+\mathrm{Per}_s^{NL}(E,\mathcal{O}),$$

with

$$\mathrm{Per}_s^L(E,\mathcal{O}):=\mathcal{L}_s(E\cap\mathcal{O},\mathcal{C}E\cap\mathcal{O})=\frac{1}{2}[\chi_E]_{W^{s,1}(\mathcal{O})}.$$

We begin with a result that deals with the local part $\mathcal{A}_s$.

Before going on, we recall that by Lemma D.1.2 we know that a function having finite $W^{s,1}(\Omega)$-seminorm also belongs to $L^1(\Omega)$.

LEMMA 4.2.6. *Let $s\in(0,1)$, $\Omega\subseteq\mathbb{R}^n$ be a bounded open set and let $u:\Omega\to\mathbb{R}$ be a measurable function. Then,*

$$u\in W^{s,1}(\Omega)\quad\textit{if and only if}\quad\mathrm{Per}_s^L\left(\mathcal{S}g(u),\Omega^\infty\right)<\infty.$$

In particular, it holds

$$\mathrm{Per}_s^L\left(\mathcal{S}g(u),\Omega^\infty\right)=\mathcal{A}_s(u,\Omega)+\mathrm{Per}_s^L\left(\{x_{n+1}<0\},\Omega^\infty\right).\tag{4.30}$$

PROOF. Using Lebesgue's monotone convergence theorem, we write

$$\mathrm{Per}_s^L\left(\mathcal{S}g(u),\Omega^\infty\right)=\lim_{\delta\to0^+}\iint_{\Omega^2\cap\{|x-y|>\delta\}}dx\,dy\int_{-\infty}^{u(x)}dx_{n+1}\int_{u(y)}^{+\infty}\frac{dy_{n+1}}{|X-Y|^{n+1+s}}.$$

Fix any small $\delta>0$ and let $(x,y)\in\Omega^2\cap\{|x-y|>\delta\}$. Shifting variables, we see that

$$\int_{-\infty}^{u(x)}dx_{n+1}\int_{u(y)}^{+\infty}\frac{dy_{n+1}}{|X-Y|^{n+1+s}}=\int_{-\infty}^{u(x)-u(y)}dx_{n+1}\int_0^{+\infty}\frac{dy_{n+1}}{|X-Y|^{n+1+s}},$$

so that

$$\begin{aligned}\mathrm{Per}_s^L\left(\mathcal{S}g(u),\Omega^\infty\right)=\lim_{\delta\to0^+}&\iint_{\Omega^2\cap\{|x-y|>\delta\}}dx\,dy\int_0^{u(x)-u(y)}dx_{n+1}\int_0^{+\infty}\frac{dy_{n+1}}{|X-Y|^{n+1+s}}\\&+\mathrm{Per}_s^L\left(\{x_{n+1}<0\},\Omega^\infty\right).\end{aligned}$$

After a renormalization of both variables x_{n+1} and y_{n+1}, we have

$$\int_0^{u(x)-u(y)}dx_{n+1}\int_0^{+\infty}\frac{dy_{n+1}}{|X-Y|^{n+1+s}}=\frac{1}{|x-y|^{n-1+s}}\int_0^{\frac{u(x)-u(y)}{|x-y|}}dt\int_0^{+\infty}\frac{dr}{[1+(r-t)^2]^{\frac{n+1+s}{2}}}.$$

Changing coordinates once again and recalling definition (4.2), we obtain that

$$\int_0^{u(x)-u(y)} dx_{n+1} \int_0^{+\infty} \frac{dy_{n+1}}{|X-Y|^{n+1+s}} = \frac{1}{|x-y|^{n-1+s}} \int_0^{\frac{u(x)-u(y)}{|x-y|}} \left(\int_{-t}^{+\infty} \frac{d\tau'}{[1+(\tau')^2]^{\frac{n+1+s}{2}}} \right) dt$$
$$= \frac{1}{|x-y|^{n-1+s}} \int_0^{\frac{u(x)-u(y)}{|x-y|}} \left(\int_{-\infty}^{t} g_s(\tau)\, d\tau \right) dt.$$

By (4.13) and (4.12), we get

$$\int_0^{\frac{u(x)-u(y)}{|x-y|}} \left(\int_{-\infty}^{t} g_s(\tau)\, d\tau \right) dt = \frac{\Lambda_{n,s}}{2} \frac{u(x)-u(y)}{|x-y|} + \mathcal{G}_s \left(\frac{u(x)-u(y)}{|x-y|} \right).$$

Since, by symmetry,

$$\iint_{\Omega^2 \cap \{|x-y|>\delta\}} \frac{u(x)-u(y)}{|x-y|} \frac{dx\, dy}{|x-y|^{n-1+s}} = 0,$$

we conclude that

$$\mathrm{Per}_s^L(\mathcal{S}g(u), \Omega^\infty) = \lim_{\delta \to 0^+} \iint_{\Omega^2 \cap \{|x-y|>\delta\}} \mathcal{G}_s \left(\frac{u(x)-u(y)}{|x-y|} \right) dx\, dy + \mathrm{Per}_s^L(\{x_{n+1} < 0\}, \Omega^\infty).$$

The claim of the lemma now follows by taking advantage once again of Lebesgue's monotone convergence theorem and recalling definition (4.22). □

LEMMA 4.2.7. *Let $s \in (0,1)$, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $u : \mathbb{R}^n \to \mathbb{R}$ be such that $u|_\Omega \in L^\infty(\Omega)$. Then, for any $M \geq \|u\|_{L^\infty(\Omega)}$, the quantity $\mathcal{N}_s^M(u,\Omega)$ is finite and it holds*

$$\mathcal{N}_s^M(u,\Omega) = \mathcal{L}_s \left(\mathcal{S}g(u) \cap \Omega^M, \mathcal{C}\mathcal{S}g(u) \setminus \Omega^\infty \right) + \mathcal{L}_s \left(\mathcal{C}\mathcal{S}g(u) \cap \Omega^M, \mathcal{S}g(u) \setminus \Omega^\infty \right). \tag{4.31}$$

PROOF. Thanks to the fact that $M \geq \|u\|_{L^\infty(\Omega)}$, we write

$$\mathcal{L}_s \left(\mathcal{S}g(u) \cap \Omega^M, \mathcal{C}\mathcal{S}g(u) \setminus \Omega^\infty \right) = \int_\Omega dx \int_{\mathcal{C}\Omega} dy \int_{-M}^{u(x)} dx_{n+1} \int_{u(y)}^{+\infty} \frac{dy_{n+1}}{|X-Y|^{n+1+s}},$$
$$\mathcal{L}_s \left(\mathcal{C}\mathcal{S}g(u) \cap \Omega^M, \mathcal{S}g(u) \setminus \Omega^\infty \right) = \int_\Omega dx \int_{\mathcal{C}\Omega} dy \int_{u(x)}^{M} dx_{n+1} \int_{-\infty}^{u(y)} \frac{dy_{n+1}}{|X-Y|^{n+1+s}}.$$

By arguing as in the proof of Lemma 4.2.6 and recalling definitions (4.2), (4.13) and (4.14), we then have

$$\int_{-M}^{u(x)} dx_{n+1} \int_{u(y)}^{+\infty} \frac{dy_{n+1}}{|X-Y|^{n+1+s}} = \int_{-M-u(y)}^{u(x)-u(y)} dx_{n+1} \int_0^{+\infty} \frac{dy_{n+1}}{|X-Y|^{n+1+s}}$$
$$= \frac{1}{|x-y|^{n-1+s}} \int_{\frac{-M-u(y)}{|x-y|}}^{\frac{u(x)-u(y)}{|x-y|}} dx_{n+1} \int_{-x_{n+1}}^{+\infty} \frac{d\tau}{(1+\tau^2)^{\frac{n+1+s}{2}}}$$
$$= \frac{1}{|x-y|^{n-1+s}} \int_{\frac{-M-u(y)}{|x-y|}}^{\frac{u(x)-u(y)}{|x-y|}} \overline{G}_s(t)\, dt$$

for every $x \in \Omega$ and $y \in \mathcal{C}\Omega$. Hence,

$$\mathcal{L}_s \left(\mathcal{S}g(u) \cap \Omega^M, \mathcal{C}\mathcal{S}g(u) \setminus \Omega^\infty \right) = \int_\Omega dx \int_{\mathcal{C}\Omega} dy \left(\frac{1}{|x-y|^{n-1+s}} \int_{\frac{-M-u(y)}{|x-y|}}^{\frac{u(x)-u(y)}{|x-y|}} \overline{G}_s(t)\, dt \right).$$

Similarly,

$$\mathcal{L}_s\left(\mathcal{C}\mathcal{S}g(u)\cap\Omega^M,\mathcal{S}g(u)\setminus\Omega^\infty\right)=\int_\Omega dx\int_{\mathcal{C}\Omega}dy\left(\frac{1}{|x-y|^{n+1+s}}\int_{\frac{u(x)-u(y)}{|x-y|}}^{\frac{M-u(y)}{|x-y|}}\overline{G}_s(-t)\,dt\right).$$

By combining the last two identities and recalling definition (4.23), we are led to (4.31). ☐

PROPOSITION 4.2.8. *Let $s\in(0,1)$ and $\Omega\subseteq\mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $u:\mathbb{R}^n\to\mathbb{R}$ be such that $u|_\Omega\in L^\infty(\Omega)$ and take $M\geq\|u\|_{L^\infty(\Omega)}$. Then,*

$$u|_\Omega\in W^{s,1}(\Omega)\quad\text{if and only if}\quad \mathrm{Per}_s\left(\mathcal{S}g(u),\Omega^M\right)<\infty. \tag{4.32}$$

In particular, it holds

$$\mathrm{Per}_s\left(\mathcal{S}g(u),\Omega^M\right)=\mathcal{F}_s^M(u,\Omega)+\kappa_{\Omega,M}, \tag{4.33}$$

where $\kappa_{\Omega,M}$ is the non-negative constant given by

$$\kappa_{\Omega,M}:=\mathrm{Per}_s^L\left(\{x_{n+1}<0\},\Omega^\infty\right)-\mathrm{Per}_s^L\left(\{x_{n+1}<0\},\Omega^\infty\setminus\Omega^M\right). \tag{4.34}$$

PROOF. The proposition is an almost immediate consequence of Lemmas 4.2.6 and 4.2.7. First, we observe that the following identities are true:

$$\begin{aligned}
\mathcal{L}_s\left(\mathcal{S}g(u)\cap\Omega^M,\Omega^M\setminus\mathcal{S}g(u)\right)&=\int_\Omega dx\int_\Omega dy\int_{-M}^{u(x)}dx_{n+1}\int_{u(y)}^M\frac{dy_{n+1}}{|X-Y|^{n+1+s}},\\
\mathcal{L}_s\left(\mathcal{S}g(u)\cap\Omega^M,\mathcal{C}\mathcal{S}g(u)\setminus\Omega^M\right)&=\int_\Omega dx\int_\Omega dy\int_{-M}^{u(x)}dx_{n+1}\int_M^{+\infty}\frac{dy_{n+1}}{|X-Y|^{n+1+s}}\\
&\quad+\int_\Omega dx\int_{\mathcal{C}\Omega}dy\int_{-M}^{u(x)}dx_{n+1}\int_{u(y)}^{+\infty}\frac{dy_{n+1}}{|X-Y|^{n+1+s}},\\
\mathcal{L}_s\left(\Omega^M\setminus\mathcal{S}g(u),\mathcal{S}g(u)\setminus\Omega^M\right)&=\int_\Omega dx\int_\Omega dy\int_{u(x)}^M dx_{n+1}\int_{-\infty}^{-M}\frac{dy_{n+1}}{|X-Y|^{n+1+s}}\\
&\quad+\int_\Omega dx\int_{\mathcal{C}\Omega}dy\int_{u(x)}^M dx_{n+1}\int_{-\infty}^{u(y)}\frac{dy_{n+1}}{|X-Y|^{n+1+s}}.
\end{aligned}$$

Note that we took advantage of the fact that $M\geq\|u\|_{L^\infty(\Omega)}$ in order to obtain the above formulas. In light of this, it is not hard to see that

$$\begin{aligned}
\mathrm{Per}_s\left(\mathcal{S}g(u),\Omega^M\right)&=\mathrm{Per}_s^L\left(\mathcal{S}g(u),\Omega^\infty\right)-\mathrm{Per}_s^L\left(\{x_{n+1}<0\},\Omega^\infty\setminus\Omega^M\right)\\
&\quad+\mathcal{L}_s\left(\mathcal{S}g(u)\cap\Omega^M,\mathcal{C}\mathcal{S}g(u)\setminus\Omega^\infty\right)+\mathcal{L}_s\left(\mathcal{C}\mathcal{S}g(u)\cap\Omega^M,\mathcal{S}g(u)\setminus\Omega^\infty\right).
\end{aligned}$$

Identity (4.33) follows by recalling definition (4.24) and applying (4.30) and (4.31). ☐

4.2.4. Some facts about the Euler-Lagrange operator. We collect here some observations about the nonlocal integrodifferential operator $\mathscr{H}$, which is formally defined on a function $u:\mathbb{R}^n\to\mathbb{R}$ at a point $x\in\mathbb{R}^n$ by

$$\mathscr{H}u(x):=2\,\mathrm{P.V.}\int_{\mathbb{R}^n}G\left(\frac{u(x)-u(y)}{|x-y|}\right)\frac{dy}{|x-y|^{n+s}}.$$

We begin by introducing the following useful notation

$$\delta_g(u,x;\xi):=G\Big(\frac{u(x)-u(x+\xi)}{|\xi|}\Big)-G\Big(\frac{u(x-\xi)-u(x)}{|\xi|}\Big),$$

and we observe that by symmetry we can write

$$\mathscr{H}u(x) = \text{P.V.}\int_{\mathbb{R}^n} \frac{\delta_g(u,x;\xi)}{|\xi|^{n+s}}\,d\xi. \tag{4.35}$$

From now on, unless otherwise stated, we will always consider $\mathscr{H}u(x)$ as written in (4.35).

We remark that when $g = g_s$, we will write $\mathscr{H}_s$ for the corresponding nonlocal operator. From a geometric standpoint, the quantity $\mathscr{H}_s u$ describes the s-mean curvature of the subgraph of u. Indeed, it holds

$$H_s[\mathcal{S}g(u)](x,u(x)) = \mathscr{H}_s u(x) \tag{4.36}$$

for every $x \in \mathbb{R}^n$ at which u is of class $C^{1,\alpha}$, for some $\alpha > s$—see [16, Appendix B.1] for the details of this computation.

We also define

$$\mathscr{H}^{\geq r}u(x) := \int_{\mathbb{R}^n\setminus B_r} \frac{\delta_g(u,x;\xi)}{|\xi|^{n+s}}\,d\xi, \qquad \forall\, r>0,$$

so that

$$\mathscr{H}u(x) = \lim_{r\to 0^+} \mathscr{H}^{\geq r}u(x).$$

Remark 4.2.9. Let $u:\mathbb{R}^n \to \mathbb{R}$. Then $\mathscr{H}^{\geq r}u(x)$ is finite for every $x \in \mathbb{R}^n$ and $r>0$. Indeed, exploiting the boundedness of G we find

$$\Big|\frac{\delta_g(u,x;\xi)}{|\xi|^{n+s}}\Big| \leq \frac{\Lambda}{|\xi|^{n+s}},$$

which is summable in $\mathbb{R}^n \setminus B_r$. In particular

$$|\mathscr{H}^{\geq r}u(x)| \leq \frac{n\omega_n}{s}\Lambda\, r^{-s}.$$

One of the main advantages of writing the nonlocal operator $\mathscr{H}u(x)$ as in (4.35) is that the integral is well defined in the classical sense, provided u is regular enough around x.

Lemma 4.2.10. *Let $s \in (0,1)$ and let $u:\mathbb{R}^n \to \mathbb{R}$ such that $u \in C^{1,\gamma}(B_r(x))$, for some $x \in \mathbb{R}^n$, $r > 0$ and $\gamma \in (s,1]$. Then*

$$\mathscr{H}^{<\varrho}u(x) := \int_{B_\varrho} \frac{\delta_g(u,x;\xi)}{|\xi|^{n+s}}\,d\xi$$

is well defined for every $\varrho > 0$ and

$$\mathscr{H}u(x) = \mathscr{H}^{<\varrho}u(x) + \mathscr{H}^{\geq\varrho}u(x) = \int_{\mathbb{R}^n} \frac{\delta_g(u,x;\xi)}{|\xi|^{n+s}}\,d\xi. \tag{4.37}$$

Proof. We begin by proving that

$$\Big|\frac{u(x+\xi)+u(x-\xi)-2u(x)}{|\xi|}\Big| \leq 2^\gamma \|u\|_{C^{1,\gamma}(B_r(x))}|\xi|^\gamma, \quad \forall\,\xi \in B_r\setminus\{0\}. \tag{4.38}$$

Indeed, by the Mean Value Theorem we have

$$u(x+\xi)-u(x) = \nabla u(x+t\xi)\cdot\xi \quad\text{and}\quad u(x-\xi)-u(x) = \nabla u(x-\tau\xi)\cdot(-\xi),$$

for some t, $\tau \in [0,1]$. Thus

$$\begin{aligned}\Big|\frac{u(x+\xi)+u(x-\xi)-2u(x)}{|\xi|}\Big| &= \Big|\frac{\nabla u(x+t\xi)\cdot\xi - \nabla u(x-\tau\xi)\cdot\xi}{|\xi|}\Big|\\ &\leq |\nabla u(x+t\xi)-\nabla u(x-\tau\xi)| \leq [\nabla u]_{C^\gamma(B_r(x))}|(t+\tau)\xi|^\gamma\\ &\leq 2^\gamma\|u\|_{C^{1,\gamma}(B_r(x))}|\xi|^\gamma,\end{aligned}$$

as claimed.

Now we remark that, thanks to Remark 4.2.9, in order to prove the lemma it is enough to show that $\delta_g(u,x;\xi)|\xi|^{-n-s}$ is summable in B_ϱ, for $\varrho>0$ small enough. For this, just notice that by (4.18) we have

$$|\delta_g(u,x;\xi)| \le \left|\frac{u(x+\xi)+u(x-\xi)-2u(x)}{|\xi|}\right|.$$

Then the conclusion follows from (4.38). □

We stress that the right hand side of (4.37) is defined in the classical sense, not as a principal value. Also notice that, thanks to Remark 4.2.9, we need not ask any growth condition for u at infinity.

When u is not regular enough around x, the quantity $\mathscr{H}u(x)$ is in general not well-defined, due to the lack of cancellation required for the principal value to converge. Nevertheless, as already observed in the Introduction, we can understand the operator $\mathscr{H}$ as defined in the following weak (distributional) sense. Given a function $u:\mathbb{R}^n\to\mathbb{R}$, we set

$$\langle \mathscr{H}u, v\rangle := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G\left(\frac{u(x)-u(y)}{|x-y|}\right)\big(v(x)-v(y)\big)\frac{dx\,dy}{|x-y|^{n+s}} \tag{4.39}$$

for every $v\in C_c^\infty(\mathbb{R}^n)$. More generally, it is immediate to see that (4.39) is well-defined for every $v:\mathbb{R}^n\to\mathbb{R}$ such that $[v]_{W^{s,1}(\mathbb{R}^n)}<\infty$. Indeed, taking advantage of the boundedness of G, one has that

$$|\langle \mathscr{H}u, v\rangle| \le \frac{\Lambda}{2}\,[v]_{W^{s,1}(\mathbb{R}^n)}, \tag{4.40}$$

with Λ as in (4.12). Hence, $\mathscr{H}u$ induces a continuous linear functional on $W^{s,1}(\mathbb{R}^n)$, that is

$$\langle \mathscr{H}u, -\rangle \in \big(W^{s,1}(\mathbb{R}^n)\big)^*.$$

Remarkably, this holds for every measurable function $u:\mathbb{R}^n\to\mathbb{R}$, regardless of its regularity.

Estimate (4.40) says that the pairing $(u,v)\mapsto\langle\mathscr{H}u,v\rangle$ is continuous in the second component v, with respect to the $W^{s,1}(\mathbb{R}^n)$ topology. The next lemma shows that we also have continuity in u with respect to convergence a.e. in $\mathbb{R}^n$.

LEMMA 4.2.11. *Let u_k, $u:\mathbb{R}^n\to\mathbb{R}$ be such that $u_k\to u$ almost everywhere in $\mathbb{R}^n$ and let $v\in W^{s,1}(\mathbb{R}^n)$. Then*

$$\lim_{k\to\infty}\langle\mathscr{H}u_k,v\rangle = \langle\mathscr{H}u,v\rangle.$$

Lemma 4.2.11 is a simple consequence of Lebesgue's dominated convergence theorem, thanks to the fact that $\|G\|_{L^\infty(\mathbb{R})}=\Lambda/2$.

The next result shows that the nonlocal mean curvature operator $\mathscr{H}$ naturally arises when computing the Euler-Lagrange equation associated to the fractional area functional.

LEMMA 4.2.12. *Let $n\ge1$, $s\in(0,1)$, $M\ge0$, $\Omega\subseteq\mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and $u\in\mathcal{W}^s(\Omega)$. Then,*

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0}\mathcal{F}^M(u+\varepsilon v,\Omega)=\langle\mathscr{H}u,v\rangle\quad \textit{for every } v\in\mathcal{W}^s_0(\Omega). \tag{4.41}$$

PROOF. First, notice that $u+\varepsilon v\in\mathcal{W}^s(\Omega)$ for every $\varepsilon\in\mathbb{R}$. Hence, by Lemma 4.2.4, both $\mathcal{F}^M(u,\Omega)$ and $\mathcal{F}^M(u+\varepsilon v,\Omega)$ are finite. Now, by Lagrange's mean value theorem, there exists a function $\tilde\tau_\varepsilon:\mathbb{R}\times\mathbb{R}\to[-|\varepsilon|,|\varepsilon|]$ such that

$$\mathcal{G}(A+\varepsilon B)-\mathcal{G}(A)=\varepsilon\, G(A+\tilde\tau_\varepsilon(A,B)B)\,B$$

for every $A, B \in \mathbb{R}$. As $v = 0$ in $\mathcal{C}\Omega$, calling

$$\tau_\varepsilon(x,y) := \tilde{\tau}_\varepsilon\left(\frac{u(x)-u(y)}{|x-y|}, \frac{v(x)-v(y)}{|x-y|}\right) \quad \text{for every } x, y \in \mathbb{R}^n,$$

we have

$$\begin{aligned}&\mathcal{F}^M(u+\varepsilon v, \Omega) - \mathcal{F}^M(u,\Omega)\\ &\quad = \varepsilon \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G\left(\frac{u(x)-u(y)}{|x-y|} + \tau_\varepsilon(x,y)\frac{v(x)-v(y)}{|x-y|}\right)\big(v(x)-v(y)\big)\frac{dx\,dy}{|x-y|^{n+s}}.\end{aligned}$$

Since G is bounded, $v \in \mathcal{W}^s_0(\Omega)$, and $|\tau_\varepsilon| \leq \varepsilon$, we may conclude the proof using Lebesgue's dominated convergence theorem. □

For more details about the Euler-Lagrange equation of minimizers, we refer to Lemma 4.5.4.

4.3. Viscosity implies weak

4.3.1. Viscosity (sub)solutions. We are interested in viscosity solutions of the equation

$$\begin{cases} \mathscr{H} u = f & \text{in } \Omega,\\ u = \varphi & \text{in } \mathbb{R}^n \setminus \Omega.\end{cases}$$

We will use $C^{1,1}$ functions as test functions. First we point out the following easy remark.

Remark 4.3.1. Let $u,\ v : \mathbb{R}^n \to \mathbb{R}$ be such that

$$u(x_0) = v(x_0) \qquad \text{and} \qquad u(x) \leq v(x) \quad \forall\, x \in \mathbb{R}^n.$$

Then

$$\delta_g(u, x_0; \xi) \geq \delta_g(v, x_0; \xi) \qquad \forall\, \xi \in \mathbb{R}^n,$$

hence also

$$\mathscr{H} u(x_0) \geq \mathscr{H} v(x_0).$$

Indeed, it is enough to notice that

$$\delta_g(u, x_0; \xi) = G\Big(\frac{u(x_0) - u(x_0+\xi)}{|\xi|}\Big) + G\Big(\frac{u(x_0)-u(x_0-\xi)}{|\xi|}\Big),$$

and recall that G is increasing.

Definition 4.3.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let $f \in C(\overline{\Omega})$. We say that a function $u : \mathbb{R}^n \to \mathbb{R}$ is a (viscosity) subsolution of $\mathscr{H} u = f$ in Ω, and we write*

$$\mathscr{H} u \leq f \qquad \text{in } \Omega,$$

if u is upper semicontinuous in Ω and whenever the following happens:

(i) $x_0 \in \Omega$,
(ii) $v \in C^{1,1}(B_r(x_0))$, for some $r < d(x_0, \partial\Omega)$,
(iii) $v(x_0) = u(x_0)$ and $v(y) \geq u(y)$ for every $y \in B_r(x_0)$,

then if we define

$$\tilde{v}(x) := \begin{cases} v(x) & \text{if } x \in B_r(x_0),\\ u(x) & \text{if } x \in \mathbb{R}^n \setminus B_r(x_0),\end{cases}$$

we have

$$\mathscr{H}\tilde{v}(x_0) \leq f(x_0).$$

A supersolution is defined similarly. A (viscosity) solution is a function $u : \mathbb{R}^n \to \mathbb{R}$ which is continuous in Ω and which is both a subsolution and a supersolution.

From now on, we will concentrate on viscosity subsolutions, the corresponding statements for supersolutions being obtained by considering $-u$ in place of u.
Unless otherwise stated, f will always be supposed to be continuous in the closure of Ω.

A crucial observation is the following.
Roughly speaking, for a function u to be touched from above at some point x_0 by a $C^{1,1}$ function means that u is $C^{1,1}$ "from above" at x_0. From the geometric point of view, the subgraph of u has an exterior tangent paraboloid at the point $(x_0, u(x_0))$.

This regularity of u "from above" at a point x_0, coupled with the property of being a viscosity subsolution, is enough to guarantee that $\mathscr{H}u(x_0)$ is well defined.
As a consequence, a viscosity subsolution is a classical subsolution in every "viscosity point", i.e. in every point which can be touched from above by a $C^{1,1}$ function. More precisely:

PROPOSITION 4.3.3. *Let*

$$\mathscr{H}u \leq f \qquad \textit{in } \Omega,$$

and let $x_0 \in \Omega$. Suppose that there exists a function $v \in C^{1,1}(B_r(x_0))$ that touches u from above at x_0, that is

$$v(x_0) = u(x_0) \qquad \textit{and} \qquad v(y) \geq u(y) \quad \forall\, y \in B_r(x_0).$$

Then $\mathscr{H}u(x_0)$ is defined in the classical sense and

$$\mathscr{H}u(x_0) \leq f(x_0).$$

PROOF. We begin by showing that $\delta_g(u, x_0; \xi)|\xi|^{-n-s}$ is integrable in $\mathbb{R}^n$, so that $\mathscr{H}u(x_0)$ is well defined in the classical sense.

For the argument we follow [74, Proposition 1]. We consider the functions

$$v_\varrho(y) := \begin{cases} v(y) & \text{if } y \in B_\varrho(x_0), \\ u(y) & \text{if } y \in \mathbb{R}^n \setminus B_\varrho(x_0), \end{cases}$$

for every $\varrho \in (0, r]$ and we denote

$$\delta_g^+(v_\varrho, x_0; \xi) := \max\{\delta_g(v_\varrho, x_0; \xi), 0\} \qquad \text{and} \qquad \delta_g^-(v_\varrho, x_0; \xi) := \max\{-\delta_g(v_\varrho, x_0; \xi), 0\}.$$

We remark that, since $v \in C^{1,1}(B_r(x_0))$, the function $\delta_g(v_\varrho, x_0; \xi)|\xi|^{-n-s}$ is integrable in $\mathbb{R}^n$, that is

$$\int_{\mathbb{R}^n} \frac{\delta_g^+(v_\varrho, x_0; \xi) + \delta_g^-(v_\varrho, x_0; \xi)}{|\xi|^{n+s}}\, d\xi = \int_{\mathbb{R}^n} \frac{|\delta_g(v_\varrho, x_0; \xi)|}{|\xi|^{n+s}}\, d\xi < +\infty.$$

Moreover, notice that

$$\delta_g(u, x_0; \xi) \geq \delta_g(v_{\varrho_1}, x_0; \xi) \geq \delta_g(v_{\varrho_2}, x_0; \xi), \qquad \text{for every } 0 < \varrho_1 \leq \varrho_2 \leq r.$$

Therefore, in particular

$$\int_{\mathbb{R}^n} \frac{\delta_g^-(u, x_0; \xi)}{|\xi|^{n+s}}\, d\xi \leq \int_{\mathbb{R}^n} \frac{|\delta_g(v_r, x_0; \xi)|}{|\xi|^{n+s}}\, d\xi < +\infty. \tag{4.42}$$

Now, since u is a subsolution, we have

$$\int_{\mathbb{R}^n} \frac{\delta_g(v_\varrho, x_0; \xi)}{|\xi|^{n+s}}\, d\xi \leq f(x_0),$$

that is

$$\int_{\mathbb{R}^n} \frac{\delta_g^+(v_\varrho, x_0; \xi)}{|\xi|^{n+s}}\, d\xi \leq \int_{\mathbb{R}^n} \frac{\delta_g^-(v_\varrho, x_0; \xi)}{|\xi|^{n+s}}\, d\xi + f(x_0).$$

Since
$$\delta_g^+(v_\varrho, x_0; \xi) \nearrow \delta_g^+(u, x_0; \xi) \qquad \text{as } \varrho \searrow 0,$$
the monotone convergence Theorem gives
$$\lim_{\varrho \to 0^+} \int_{\mathbb{R}^n} \frac{\delta_g^+(v_\varrho, x_0; \xi)}{|\xi|^{n+s}}\, d\xi = \int_{\mathbb{R}^n} \frac{\delta_g^+(u, x_0; \xi)}{|\xi|^{n+s}}\, d\xi.$$
Moreover
$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\delta_g^+(v_{\varrho_1}, x_0; \xi)}{|\xi|^{n+s}}\, d\xi &\le \int_{\mathbb{R}^n} \frac{\delta_g^-(v_{\varrho_1}, x_0; \xi)}{|\xi|^{n+s}}\, d\xi + f(x_0) \\ &\le \int_{\mathbb{R}^n} \frac{\delta_g^-(v_{\varrho_2}, x_0; \xi)}{|\xi|^{n+s}}\, d\xi + f(x_0) < +\infty,\end{aligned}$$
for every $0 < \varrho_1 \le \varrho_2 \le r$. Thus
$$\int_{\mathbb{R}^n} \frac{\delta_g^+(u, x_0; \xi)}{|\xi|^{n+s}}\, d\xi \le \int_{\mathbb{R}^n} \frac{\delta_g^-(v_\varrho, x_0; \xi)}{|\xi|^{n+s}}\, d\xi + f(x_0) < +\infty, \tag{4.43}$$
for every $\varrho \in (0, r]$. By (4.42) and (4.43), we see that $\delta_g(u, x_0; \xi)|\xi|^{-n-s}$ is integrable in $\mathbb{R}^n$ and hence $\mathscr{H} u(x_0)$ is well defined.

Finally, since for every $\varrho \in (0, r]$ we have
$$\frac{\delta_g^-(v_\varrho, x_0; \xi)}{|\xi|^{n+s}} \le \frac{\delta_g^-(v_r, x_0; \xi)}{|\xi|^{n+s}},$$
which is integrable in $\mathbb{R}^n$, by Lebesgue's dominated convergence Theorem we can pass to the limit $\varrho \to 0$ in the right hand side of (4.43), obtaining
$$\int_{\mathbb{R}^n} \frac{\delta_g^+(u, x_0; \xi)}{|\xi|^{n+s}}\, d\xi \le \int_{\mathbb{R}^n} \frac{\delta_g^-(u, x_0; \xi)}{|\xi|^{n+s}}\, d\xi + f(x_0),$$
that is
$$\mathscr{H} u(x_0) \le f(x_0),$$
as claimed. □

For later use, it is convenient to introduce the following definition.

Definition 4.3.4. *Let $u : \mathbb{R}^n \to \mathbb{R}$ and let $x_0 \in \mathbb{R}^n$. The function u is $C^{1,1}$ at x_0, and we write $u \in C^{1,1}(x_0)$, if there exist $\ell \in \mathbb{R}^n$ and M, $r > 0$ such that*
$$|u(x_0 + \xi) - u(x_0) - \ell \cdot \xi| \le M|\xi|^2, \qquad \forall\, \xi \in B_r. \tag{4.44}$$

We remark that we clearly have
$$u \in C^{1,1}(B_R(x_0)) \quad \Longrightarrow \quad u \in C^{1,1}(x_0).$$
Roughly speaking, being $C^{1,1}$ at x_0 means that there exist both an interior and an exterior tangent paraboloid to the subgraph of u at the point $(x_0, u(x_0))$.

As a consequence of Proposition 4.3.3, we obtain the following Corollary:

Corollary 4.3.5. *Let*
$$\mathscr{H} u \le f \qquad \text{in } \Omega,$$
and let $x_0 \in \Omega$. If $u \in C^{1,1}(x_0)$, then $\mathscr{H} u(x_0)$ is well defined and
$$\mathscr{H} u(x_0) \le f(x_0).$$

Proof. Consider the paraboloid
$$q(x) := u(x_0) + \ell \cdot (x - x_0) + M|x - x_0|^2, \qquad \forall\, x \in B_r(x_0),$$
with ℓ, M and r as in Definition 4.3.4. Then $q \in C^{1,1}(B_r(x_0))$ and by (4.44) we know that q touches u from above at x_0. Thus the conclusion follows from Proposition 4.3.3. □

4.3.2. Sup-convolutions. In this subsection we introduce and study the sup-convolutions u^ε of a viscosity subsolution u. These provide a sequence of subsolutions which converge to u and which enjoy nice regularity properties, since they are semiconvex functions.

We will consider only globally bounded subsolutions.

DEFINITION 4.3.6. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded function. We define the sup-convolution u^ε of u as*

$$u^\varepsilon(x) := \sup_{y\in\mathbb{R}^n} \left\{ u(y) - \frac{1}{\varepsilon}|y-x|^2 \right\} \qquad \forall\, x \in \mathbb{R}^n,$$

for every $\varepsilon > 0$.

Now we point out some easy properties of sup-convolutions.
We begin by remarking that, by definition,

$$u^\varepsilon \geq u \quad \text{in } \mathbb{R}^n.$$

Moreover, notice that if we denote

$$\sup_{\mathbb{R}^n} |u| =: M < +\infty,$$

then

$$u^\varepsilon(x) = \sup_{|y-x|\leq\sqrt{2M\varepsilon}} \left\{ u(y) - \frac{1}{\varepsilon}|y-x|^2 \right\} \qquad \forall\, x \in \mathbb{R}^n. \tag{4.45}$$

Indeed, if $|y-x| > \sqrt{2M\varepsilon}$, then

$$u(y) - \frac{1}{\varepsilon}|y-x|^2 < -M \leq u(x),$$

but we know that

$$u^\varepsilon(x) \geq u(x).$$

REMARK 4.3.7. Given an open set $\Omega \subseteq \mathbb{R}^n$, we denote

$$\Omega^\varepsilon := \left\{ x \in \Omega \,|\, d(x, \partial\Omega) > 2\sqrt{2M\varepsilon} \right\}. \tag{4.46}$$

If u is upper semicontinuous in an open set Ω, then for every $x \in \Omega^\varepsilon$ there exists $y_0 \in \Omega$ such that

$$u^\varepsilon(x) = u(y_0) - \frac{1}{\varepsilon}|y_0 - x|^2 = \max_{|y-x|\leq\sqrt{2M\varepsilon}} \left\{ u(y) - \frac{1}{\varepsilon}|y-x|^2 \right\}.$$

This follows straightforwardly from (4.45) and the upper semicontinuity of u.

In the following Theorem we collect some important properties of sup-convolutions which can be found in [4, Chapter 1].
We first recall the definition of semiconvex functions.

DEFINITION 4.3.8. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $u : \Omega \to \mathbb{R}$. We say that u is semiconvex in Ω if there exists a constant $c \geq 0$ such that*

$$x \mapsto u(x) + \frac{c}{2}|x|^2$$

is convex in any ball $B \subseteq \Omega$. The smallest constant $c \geq 0$ for which this happens is called the semiconvexity constant of u and denoted $sc(u, \Omega)$.

THEOREM 4.3.9. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded function. Then u^ε is semiconvex in $\mathbb{R}^n$, with semiconvexity constant*

$$sc(u^\varepsilon, \mathbb{R}^n) \leq \frac{2}{\varepsilon}.$$

Therefore $u^\varepsilon \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ and $\nabla u^\varepsilon \in BV_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$. Moreover $u^\varepsilon \in C^{1,1}(x)$ for almost every $x \in \mathbb{R}^n$.

If u is upper semicontinuous in an open set $\Omega \subseteq \mathbb{R}^n$, then

$$u^\varepsilon(x) \searrow u(x) \quad \text{as } \varepsilon \searrow 0, \qquad \forall\, x \in \Omega.$$

The convergence is locally uniform if u is continuous in Ω.

PROOF. The semiconvexity of u^ε follows by [4, Proposition 4, (i)]. Then, by [4, Theorem 15] this implies that $u^\varepsilon \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^n)$ and by [4, Theorem 16] that $\nabla u^\varepsilon \in BV_{\mathrm{loc}}(\mathbb{R}^n, \mathbb{R}^n)$. That $u^\varepsilon \in C^{1,1}(x)$ for almost every $x \in \mathbb{R}^n$ follows from the Taylor expansion in point [4, Theorem 16, (ii)]. Finally, the convergence of u^ε to u is obtained by arguing as in the proof of [4, Proposition 4, (ii)]. □

One of the most important features of sup-convolutions consists in preserving the viscosity subsolution property (eventually up to a small error). More precisely:

PROPOSITION 4.3.10. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, let $f \in C(\overline{\Omega})$ and let $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded function,*

$$M := \sup_{\mathbb{R}^n} |u| < +\infty,$$

such that

$$\mathscr{H} u \leq f \quad \text{in } \Omega.$$

Then

$$\mathscr{H} u^\varepsilon \leq f + c_\varepsilon \quad \text{in } \Omega^\varepsilon,$$

where Ω^ε is defined in (4.46) and the constant $c_\varepsilon \geq 0$ depends only on ε, M and the modulus of continuity of f. More precisely,

$$c_\varepsilon := \sup_{\substack{x,\, y \in \overline{\Omega} \\ |x-y| \leq \sqrt{2M\varepsilon}}} |f(x) - f(y)|.$$

In particular

$$c_\varepsilon \searrow 0 \quad \text{as } \varepsilon \searrow 0 \qquad \text{and} \qquad c_\varepsilon = 0 \quad \text{if} \quad f \text{ is constant.} \tag{4.47}$$

PROOF. Let $x_0 \in \Omega^\varepsilon$ and suppose that there exists $v \in C^{1,1}(B_r(x_0))$ such that

$$v(x_0) = u^\varepsilon(x_0) \quad \text{and} \quad v(x) \geq u^\varepsilon(x), \quad \forall\, x \in B_r(x_0).$$

We need to show that

$$\mathscr{H} \tilde{v}(x_0) \leq f(x_0) + c_\varepsilon.$$

By Remark 4.3.7 we know that we can find $y_0 \in \Omega$ such that $|y_0 - x_0| \leq \sqrt{2M\varepsilon}$ and

$$u^\varepsilon(x_0) = u(y_0) - \frac{1}{\varepsilon}|y_0 - x_0|^2,$$

Then we define

$$\psi(x) := v(x + x_0 - y_0) + \frac{1}{\varepsilon}|y_0 - x_0|^2, \qquad \forall\, x \in B_r(y_0),$$

and we remark that clearly $\psi \in C^{1,1}(B_r(y_0))$. Moreover

$$\psi(y_0) = v(x_0) + \frac{1}{\varepsilon}|y_0 - x_0|^2 = u^\varepsilon(x_0) + \frac{1}{\varepsilon}|y_0 - x_0|^2 = u(y_0).$$

Then notice that, since $v \geq u^\varepsilon$ in $B_r(x_0)$, by definition of u^ε we obtain

$$u(y) - \frac{1}{\varepsilon}|y-x|^2 \leq u^\varepsilon(x) \leq v(x), \qquad \forall\, y \in \mathbb{R}^n \text{ and } x \in B_r(x_0).$$

Taking $y \in B_r(y_0)$ and $x := y + x_0 - y_0$ gives

$$u(y) \leq \psi(y), \qquad \forall\, y \in B_r(y_0).$$

Thus ψ touches u from above at y_0 and hence

$$\mathscr{H}\tilde{\psi}(y_0) \leq f(y_0).$$

Now notice that by changing variables we find

$$\begin{aligned}
\mathscr{H}^{<r}\tilde{\psi}(y_0) &= 2\text{P.V.}\int_{B_r(y_0)} G\Big(\frac{\psi(y_0)-\psi(y)}{|y-y_0|}\Big)\frac{dy}{|y-y_0|^{n+s}} \\
&= 2\text{P.V.}\int_{B_r(x_0)} G\Big(\frac{v(x_0)-v(x)}{|x-x_0|}\Big)\frac{dx}{|x-x_0|^{n+s}} \\
&= \mathscr{H}^{<r}\tilde{v}(x_0).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathscr{H}^{\geq r}\tilde{\psi}(y_0) &= 2\int_{\mathbb{R}^n\setminus B_r(y_0)} G\Big(\frac{u^\varepsilon(x_0)+\frac{1}{\varepsilon}|y_0-x_0|^2 - u(y)}{|y-y_0|}\Big)\frac{dy}{|y-y_0|^{n+s}} \\
&= 2\int_{\mathbb{R}^n\setminus B_r(x_0)} G\Big(\frac{u^\varepsilon(x_0)+\frac{1}{\varepsilon}|y_0-x_0|^2 - u(x+y_0-x_0)}{|x-x_0|}\Big)\frac{dx}{|x-x_0|^{n+s}}.
\end{aligned}$$

We remark that by taking $y := x + y_0 - x_0$ in the definition of $u^\varepsilon(x)$ as a sup, we get

$$\frac{1}{\varepsilon}|y_0-x_0|^2 - u(x+y_0-x_0) \geq -u^\varepsilon(x), \qquad \forall\, x \in \mathbb{R}^n \setminus B_r(x_0).$$

Hence

$$\mathscr{H}^{\geq r}\tilde{\psi}(y_0) \geq 2\int_{\mathbb{R}^n\setminus B_r(x_0)} G\Big(\frac{u^\varepsilon(x_0)-u^\varepsilon(x)}{|x-x_0|}\Big)\frac{dx}{|x-x_0|^{n+s}} = \mathscr{H}^{\geq r}\tilde{v}(x_0).$$

This implies that

$$\mathscr{H}\tilde{v}(x_0) \leq \mathscr{H}\tilde{\psi}(y_0) \leq f(y_0) \leq f(x_0) + c_\varepsilon,$$

as claimed. To conclude the proof, notice that (4.47) follows from the definition of c_ε and the uniform continuity of f. □

As a consequence, exploiting the regularity of u^ε we find that u^ε is a classical subsolution almost everywhere in Ω^ε.

Corollary 4.3.11. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, let $f \in C(\overline{\Omega})$ and let $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded function such that*

$$\mathscr{H}u \leq f \quad \text{in } \Omega.$$

Then for almost every $x \in \Omega^\varepsilon$ we have that $\mathscr{H}u^\varepsilon(x)$ is well defined and

$$\mathscr{H}u^\varepsilon(x) \leq f(x) + c_\varepsilon.$$

Proof. By Theorem 4.3.9 we know that $u^\varepsilon \in C^{1,1}(x)$ for almost every $x \in \mathbb{R}^n$. Then the conclusion follows from Proposition 4.3.10 and Corollary 4.3.5. □

4.3.3. Weak (sub)solutions. Given a function $u:\mathbb{R}^n\to\mathbb{R}$, we define

$$\langle \mathscr{H} u, v\rangle := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} G\Big(\frac{u(x)-u(y)}{|x-y|}\Big)\big(v(x)-v(y)\big)\frac{dx\,dy}{|x-y|^{n+s}},$$

for every $v\in W^{s,1}(\mathbb{R}^n)$. In particular, this is well defined for every $v\in C_c^\infty(\Omega)$, where we understand that v is extended by zero outside Ω.

DEFINITION 4.3.12. *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set and let $f\in C(\overline{\Omega})$. We say that a function $u:\mathbb{R}^n\to\mathbb{R}$ is a weak subsolution in Ω if*

$$\langle \mathscr{H} u, v\rangle \le \int_\Omega f v\,dx, \qquad \forall\, v\in C_c^\infty(\Omega)\ s.t.\ v\ge 0.$$

We want to pass from a function u which is a subsolution almost everywhere to a weak subsolution. In order to do this, it is enough to ask u to have a gradient with bounded variation. More precisely, we introduce the space

$$\begin{aligned} BH(\Omega) &:= \big\{u\in W^{1,1}(\Omega)\,|\,\nabla u\in BV(\Omega,\mathbb{R}^n)\big\}\\ &= \big\{u\in W^{1,1}(\Omega)\,|\,\partial_j u\in BV(\Omega),\ \forall\, j=1,\dots,n\big\},\end{aligned}$$

endowed with the norm

$$\|u\|_{BH(\Omega)} := \|u\|_{W^{1,1}(\Omega)} + |D^2u|(\Omega).$$

For the properties of the space $BH(\Omega)$ of functions of bounded Hessian, we refer the interested reader to [37]. We remark that sometimes the notation $BV^2(\Omega)=BH(\Omega)$ is also used in the literature.

We only recall the following "density" property, [37, Proposition 1.4]:

PROPOSITION 4.3.13. *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set with C^2 boundary and let $u\in BH(\Omega)$. Then there exist $u_k\in C^2(\Omega)\cap W^{2,1}(\Omega)$ such that*

$$\lim_{k\to\infty}\big\{\|u-u_k\|_{W^{1,1}(\Omega)} + \big||D^2u|(\Omega)-|D^2u_k|(\Omega)\big|\big\}=0.$$

Exploiting this density property, we can prove the following:

LEMMA 4.3.14. *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set and let $u\in BH(\Omega)$. Let $\Omega'\Subset\Omega$ and let $d:=dist(\Omega',\partial\Omega)/2$. Then*

$$\int_{\Omega'}|u(x+\xi)+u(x-\xi)-2u(x)|\,dx\le 2|\xi|^2|D^2u|(\Omega), \qquad \forall\,\xi\in B_d. \tag{4.48}$$

PROOF. Let $\mathcal{O}\subseteq\Omega$ be a bounded open set with C^2 boundary such that

$$\Omega'\Subset\mathcal{O}, \qquad \text{with} \quad dist(\Omega',\partial\mathcal{O})>d. \tag{4.49}$$

By Proposition 4.3.13 we can find $u_k\in C^2(\mathcal{O})\cap W^{2,1}(\mathcal{O})$ such that

$$\lim_{k\to\infty}\big\{\|u-u_k\|_{W^{1,1}(\mathcal{O})} + \big||D^2u|(\mathcal{O})-|D^2u_k|(\mathcal{O})\big|\big\}=0. \tag{4.50}$$

Now notice that

$$\begin{aligned}|u_k(x+\xi)+u_k(x-\xi)-2u_k(x)| \le{}& |u_k(x+\xi)-u_k(x)-\nabla u_k(x)\cdot\xi|\\ &+|u_k(x-\xi)-u_k(x)-\nabla u_k(x)\cdot(-\xi)|.\end{aligned}$$

Then by Taylor's formula with integral remainder we have

$$|u_k(x+\xi)-u_k(x)-\nabla u_k(x)\cdot\xi|\le|\xi|^2\int_0^1|D^2u_k(x+t\xi)|\,dt,$$

and similarly for $-\xi$. Integrating in x over Ω' and switching the order of integration gives

$$\begin{aligned}\int_{\Omega'} |u_k(x+\xi)+u_k(x-\xi)-2u_k(x)|\,dx &\le |\xi|^2\int_{\Omega'}\Big(\int_{-1}^1 |D^2u_k(x+t\xi)|\,dt\Big)dx\\ &= |\xi|^2\int_{-1}^1\Big(\int_{\Omega'}|D^2u_k(x+t\xi)|\,dx\Big)dt \le 2|\xi|^2|D^2u_k|(\mathcal{O}),\end{aligned}$$

since $|\xi|<d$ and $\mathcal{O}$ satisfies (4.49).

Then by Fatou's Lemma and (4.50) we obtain

$$\begin{aligned}\int_{\Omega'}|u(x+\xi)+u(x-\xi)-2u(x)|\,dx &\le 2|\xi|^2\liminf_{k\to\infty}|D^2u_k|(\mathcal{O}) = 2|\xi|^2|D^2u|(\mathcal{O})\\ &\le 2|\xi|^2|D^2u|(\Omega),\end{aligned}$$

proving (4.48) and concluding the proof of the Lemma. ☐

Proposition 4.3.15. *Let $\Omega\subseteq\mathbb{R}^n$ be a bounded open set and let $u\in BH(\Omega)$. Then*

$$\mathscr{H}u\in L^1_{\mathrm{loc}}(\Omega),$$

and

$$\langle\mathscr{H}u,v\rangle=\int_\Omega \mathscr{H}u(x)v(x)\,dx,\qquad \forall\, v\in C_c^\infty(\Omega).$$

Proof. Let $\Omega'\Subset\Omega$ and let $d:=dist(\Omega',\partial\Omega)/2$. We recall that

$$|\delta_g(u,x;\xi)|\le\frac{|u(x+\xi)+u(x-\xi)-2u(x)|}{|\xi|}.$$

Therefore, by Remark 4.2.9 and (4.48) we obtain

$$\begin{aligned}\int_{\Omega'}|\mathscr{H}u(x)|\,dx &\le \int_{\Omega'}dx\int_{\mathbb{R}^n}\frac{|\delta_g(u,x;\xi)|}{|\xi|^{n+s}}\,d\xi\\ &\le |\Omega'|\frac{n\omega_n}{s}\Lambda\, d^{-s}+\int_{B_d}\Big(\int_{\Omega'}|u(x+\xi)+u(x-\xi)-2u(x)|\,dx\Big)\frac{d\xi}{|\xi|^{n+1+s}}\\ &\le |\Omega'|\frac{n\omega_n}{s}\Lambda\,d^{-s}+2|D^2u|(\Omega)\frac{n\omega_n}{1-s}d^{1-s}<+\infty.\end{aligned}\tag{4.51}$$

This proves that $\mathscr{H}u\in L^1_{\mathrm{loc}}(\Omega)$.

As a consequence of (4.51), since

$$|\mathscr{H}^{\ge\varrho}u(x)|\le\int_{\mathbb{R}^n}\frac{|\delta_g(u,x;\xi)|}{|\xi|^{n+s}}\,d\xi,\qquad\forall\,\varrho>0,$$

given $v\in C_c^\infty(\Omega)$ we can apply Lebesgue's dominated convergence Theorem to obtain

$$\lim_{\varrho\to0^+}\int_{\mathbb{R}^n}\mathscr{H}^{\ge\varrho}u(x)v(x)\,dx=\int_{\mathbb{R}^n}\mathscr{H}u(x)v(x)\,dx.$$

Now notice that by symmetry

$$\int_{\mathbb{R}^n}\mathscr{H}^{\ge\varrho}u(x)v(x)\,dx=\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}G\Big(\frac{u(x)-u(y)}{|x-y|}\Big)\big(v(x)-v(y)\big)\big(1-\chi_{B_\varrho}(x-y)\big)\frac{dx\,dy}{|x-y|^{n+s}}.$$

Finally, since $v\in C_c^\infty(\Omega)\subseteq W^{s,1}(\mathbb{R}^n)$, we can apply again Lebesgue's dominated convergence Theorem to obtain

$$\lim_{\varrho\to0^+}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}G\Big(\frac{u(x)-u(y)}{|x-y|}\Big)\big(v(x)-v(y)\big)\big(1-\chi_{B_\varrho}(x-y)\big)\frac{dx\,dy}{|x-y|^{n+s}}=\langle\mathscr{H}u,v\rangle,$$

concluding the proof. ☐

Then exploiting Proposition 4.3.15, Theorem 4.3.9 and Corollary 4.3.11 we immediately obtain the following:

Corollary 4.3.16. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, let $f \in C(\overline{\Omega})$ and let $u : \mathbb{R}^n \to \mathbb{R}$ be a bounded function such that*

$$\mathscr{H}u \leq f \quad \text{in } \Omega.$$

Then

$$\langle \mathscr{H}u^\varepsilon, v\rangle \leq \int_\Omega \big(f(x) + c_\varepsilon\big)v(x)\,dx, \qquad \forall\, v \in C_c^\infty(\Omega^\varepsilon) \text{ s.t. } v \geq 0. \tag{4.52}$$

4.3.4. Viscosity implies weak. As a consequence of Lemma 4.2.11, exploiting the fact that supconvolutions are weak subsolutions we obtain the following:

Theorem 4.3.17. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, let $f \in C(\overline{\Omega})$ and let $u : \mathbb{R}^n \to \mathbb{R}$ be a viscosity subsolution,*

$$\mathscr{H}u \leq f \quad \text{in } \Omega.$$

Suppose that u is bounded and assume also that there exists a closed set $S \subseteq \mathbb{R}^n \setminus \Omega$ such that $|S| = 0$ and u is upper semicontinuous in $\mathbb{R}^n \setminus S$. Then u is a weak subsolution in Ω,

$$\langle \mathscr{H}u, v\rangle \leq \int_\Omega fv\,dx, \qquad \forall\, v \in C_c^\infty(\Omega) \text{ s.t. } v \geq 0.$$

Proof. The hypothesis on u and Theorem 4.3.9 imply that

$$u^\varepsilon(x) \to u(x), \qquad \forall\, x \in \mathbb{R}^n \setminus S,$$

and hence almost everywhere in $\mathbb{R}^n$. Let $v \in C_c^\infty(\Omega)$ be such that $v \geq 0$. Notice that

$$\operatorname{supp} v \subseteq \Omega^\varepsilon,$$

for every ε small enough. Thus, by (4.52) and recalling (4.47), we obtain

$$\langle \mathscr{H}u, v\rangle = \lim_{\varepsilon\to 0^+}\langle \mathscr{H}u^\varepsilon, v\rangle \leq \lim_{\varepsilon\to 0^+}\int_\Omega (f + c_\varepsilon)v\,dx = \int_\Omega fv\,dx,$$

concluding the proof. □

In particular, if $|\partial\Omega| = 0$, we allow for $\partial\Omega \subseteq S$, so we are not asking u to be continuous across $\partial\Omega$.

We are now going to use an approximation procedure to extend Theorem 4.3.17 to the case of arbitrary exterior data.

The crucial point consists in the following observation, that follows essentially from the fact that $\mathscr{H}^{\geq d}u(x)$ can be bounded in terms of only d, independently both of u or x (see Remark 4.2.9).

Theorem 4.3.18. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let $f \in C(\overline{\Omega})$. Let $u : \mathbb{R}^n \to \mathbb{R}$ be locally integrable in $\mathbb{R}^n$ and suppose that*

$$\mathscr{H}u \leq f \quad \text{in } \Omega.$$

Let $u_k : \mathbb{R}^n \to \mathbb{R}$ be such that $u_k \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Given two open sets

$$\Omega' \Subset \mathcal{O} \subseteq \Omega,$$

we define

$$\bar{u}_k(x) := \begin{cases} u(x) & \text{if } x \in \mathcal{O}, \\ u_k(x) & \text{if } x \in \mathbb{R}^n \setminus \mathcal{O}. \end{cases}$$

Then for every $k \in \mathbb{N}$ there exists a constant $e_k \geq 0$ such that $e_k \to 0$ and

$$\mathscr{H}\bar{u}_k \leq f + e_k \quad \text{in } \Omega'.$$

PROOF. We denote $d := dist(\Omega', \partial\mathcal{O}) > 0$ and we remark that for every $x \in \Omega'$ we have

$$\delta_g(\bar{u}_k, x; \xi) = \delta_g(u, x; \xi), \qquad \forall\, \xi \in B_d. \tag{4.53}$$

On the other hand, let

$$\omega_k(x) := \mathscr{H}^{\geq d}\bar{u}_k(x) - \mathscr{H}^{\geq d}u(x), \qquad \forall\, x \in \Omega',$$

and let $R_0 > 0$ be such that $\Omega \subseteq B_{R_0}$. Then for every $x \in \Omega'$ and $R \geq R_0 + d$ we have

$$\begin{aligned} |\omega_k(x)| &\leq 2 \int_{\mathbb{R}^n \setminus B_d(x)} \left| G\Big(\frac{u(x) - \bar{u}_k(y)}{|x-y|}\Big) - G\Big(\frac{u(x) - u(y)}{|x-y|}\Big) \right| \frac{dy}{|x-y|^{n+s}} \\ &\leq 2 \int_{B_R(x) \setminus B_d(x)} \frac{|\bar{u}_k(y) - u(y)|}{|x-y|^{n+1+s}}\, dy + 2\Lambda \int_{\mathbb{R}^n \setminus B_R(x)} \frac{dy}{|x-y|^{n+s}} \\ &\leq \frac{2}{d^{n+1+s}} \|u_k - u\|_{L^1(B_{R+R_0})} + \frac{2\Lambda n \omega_n}{s} R^{-s}. \end{aligned}$$

Hence for every $k \in \mathbb{N}$ we obtain

$$\sup_{x \in \Omega'} |\omega_k(x)| \leq \frac{2}{d^{n+1+s}} \|u_k - u\|_{L^1(B_{R+R_0})} + \frac{2\Lambda n \omega_n}{s} R^{-s}, \qquad \forall\, R \geq R_0 + d.$$

Thus, if we define

$$e_k := \inf_{R \geq R_0 + d} \left(\frac{2}{d^{n+1+s}} \|u_k - u\|_{L^1(B_{R+R_0})} + \frac{2\Lambda n \omega_n}{s} R^{-s} \right),$$

we get

$$\sup_{x \in \Omega'} |\omega_k(x)| \leq e_k, \qquad \forall\, k \in \mathbb{N}. \tag{4.54}$$

Now notice that, since $u_k \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we have

$$\limsup_{k \to \infty} e_k \leq \limsup_{k \to \infty} \left(\frac{2}{d^{n+1+s}} \|u_k - u\|_{L^1(B_{R+R_0})} + \frac{2\Lambda n \omega_n}{s} R^{-s} \right) = \frac{2\Lambda n \omega_n}{s} R^{-s},$$

for every $R \geq R_0 + d$. Letting $R \nearrow +\infty$ proves that $e_k \to 0$.

Now let $x_0 \in \Omega'$ be such that there exists $v \in C^{1,1}(B_r(x_0))$ with $r < dist(x_0, \Omega')$ and

$$v(x_0) = \bar{u}_k(x_0) = u(x_0) \quad \text{and} \quad v(x) \geq \bar{u}_k(x) = u(x) \quad \forall\, x \in B_r(x_0).$$

By Proposition 4.3.3 we obtain

$$\mathscr{H}u(x_0) \leq f(x_0).$$

Hence, by (4.53) and (4.54) we get

$$\begin{aligned} \mathscr{H}\bar{u}_k(x_0) &= \mathscr{H}^{<d}u(x_0) + \mathscr{H}^{\geq d}\bar{u}_k(x_0) = \mathscr{H}u(x_0) + \omega_k(x_0) \\ &\leq f(x_0) + |\omega_k(x_0)| \leq f(x_0) + e_k. \end{aligned}$$

Finally, notice that if we set

$$\tilde{v}_k(x) := \begin{cases} v(x) & \text{if } x \in B_r(x_0), \\ \bar{u}_k(x) & \text{if } x \in \mathbb{R}^n \setminus B_r(x_0), \end{cases}$$

then by Remark 4.3.1 we obtain

$$\mathscr{H}\tilde{v}_k(x_0) \leq \mathscr{H}\bar{u}_k(x_0) \leq f(x_0) + e_k,$$

concluding the proof. □

With this fundamental approximation tool at hand, we are ready to prove the general "viscosity implies weak" Theorem.

Proof of Theorem 4.1.7. Let $v \in C_c^\infty(\Omega)$ such that $v \geq 0$. Then we can find two open sets such that
$$supp\, v \Subset \Omega' \Subset \mathcal{O} \Subset \Omega,$$
and such that $|\partial\mathcal{O}| = 0$.

Since u is locally integrable in $\mathbb{R}^n$, we can find a sequence of functions $u_k \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ such that
$$u_k \to u \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \quad \text{and} \quad \text{a.e. in } \mathbb{R}^n.$$

Now let $\bar{u}_k$ and e_k be as defined in Theorem 4.3.18. Notice that since u is locally bounded in Ω, it is bounded in $\mathcal{O}$, and hence the functions $\bar{u}_k$ are bounded in $\mathbb{R}^n$.

Moreover, since u is upper semicontinuous in Ω and u_k is continuous in $\mathbb{R}^n$, the functions $\bar{u}_k$ are upper semicontinuous in $\mathbb{R}^n \setminus \partial\mathcal{O}$.

We can thus apply Theorem 4.3.18 and Theorem 4.3.17 to obtain
$$\langle \mathscr{H}\bar{u}_k, v\rangle \leq \int_{\Omega'} (f + e_k)v\, dx, \qquad \forall\, k \in \mathbb{N}.$$
Then, since $\bar{u}_k \to u$ almost everywhere in $\mathbb{R}^n$ and $e_k \to 0$, by Lemma 4.2.11 we get
$$\langle \mathscr{H}u, v\rangle = \lim_{k\to\infty} \langle \mathscr{H}\bar{u}_k, v\rangle \leq \lim_{k\to\infty} \int_{\Omega'} (f + e_k)v\, dx = \int_\Omega fv\, dx.$$
This concludes the proof of the Theorem. □

4.4. Minimizers of $\mathcal{F}_s^M$ versus minimizers of Per_s

Here, we bring forward our analysis of the geometric properties enjoyed the functional $\mathcal{F}_s^M$, and in particular of its relation with the s-perimeter.

We will show that the s-perimeter of a set E, which is a subgraph outside Ω^∞ and whose boundary is trapped inside a strip of finite height inside Ω^∞, decreases under a vertical rearrangement that transforms E into a global subgraph. This fact will be a consequence of a new rearrangement inequality for rather general 1-dimensional integral set functions, that we establish in the following subsection.

4.4.1. A one-dimensional rearrangement inequality. Let $K : \mathbb{R} \to \mathbb{R}$ be a non-negative function. Given two sets $A, B \subseteq \mathbb{R}$, we define
$$H_K(A, B) := \int_A \int_B d\mu, \quad \text{where } d\mu = d\mu_K(x - y) := K(x - y)\, dx\, dy, \tag{4.55}$$
whenever this quantity is finite.

Fix two real numbers α, β and consider two sets A, B satisfying
$$(-\infty, \alpha) \subseteq A \quad \text{and} \quad (\beta, +\infty) \subseteq B. \tag{4.56}$$
We define the *decreasing rearrangement* A_* of A as
$$A_* := (-\infty, a_*), \quad \text{with } a_* := \lim_{R\to+\infty} \left(\int_{-R}^R \chi_A(t)\, dt - R \right). \tag{4.57}$$
Similarly, we define the *increasing rearrangement* B^* of B as
$$B^* := (b^*, +\infty), \quad \text{with } b^* := \lim_{R\to+\infty} \left(R - \int_{-R}^R \chi_B(t)\, dt \right). \tag{4.58}$$
Notice that, up to a set of vanishing measure—actually, a point—it holds
$$B^* = \mathcal{C}(\mathcal{C}B)_*. \tag{4.59}$$

The next result shows that the value of H_K decreases when their arguments are appropriately rearranged.

Proposition 4.4.1. *Let $A, B \subseteq \mathbb{R}$ be two sets satisfying*

$$(-\infty, \underline{\alpha}) \subseteq A \subseteq (-\infty, \bar{\alpha}) \quad \text{and} \quad (\bar{\beta}, +\infty) \subseteq B \subseteq (\underline{\beta}, +\infty),$$

for some real numbers $\underline{\alpha} < \bar{\alpha}$ and $\underline{\beta} < \bar{\beta}$. Let $K : \mathbb{R} \to \mathbb{R}$ be a non-negative function and suppose that

$$H_K\big((-\infty, \bar{\alpha}), (\underline{\beta}, +\infty)\big) < \infty. \tag{4.60}$$

Then,

$$H_K(A_*, B^*) \leq H_K(A, B). \tag{4.61}$$

Proof. First of all, we observe that we can restrict ourselves to assume that A and B are both open sets. Indeed, if A and B are merely measurable, by the outer regularity of the Lebesgue measure there exist two sequences of open sets $\{A_k\}$, $\{B_k\}$ with $A \subseteq A_k \subseteq (-\infty, \bar{\alpha})$ and $B \subseteq B_k \subseteq (\underline{\beta}, +\infty)$ for any $k \in \mathbb{N}$, and such that $|A_k \setminus A|, |B_k \setminus B| \to 0$ as $k \to +\infty$. Suppose now that (4.61) holds with A_k and B_k respectively in place of A and B. By this and the fact that, by definitions (4.57)-(4.58), it clearly holds $A_* \subseteq (A_k)_*$ and $B^* \subseteq (B_k)^*$ for any k, we deduce that

$$H_K(A_*, B^*) \leq \lim_{k \to +\infty} H_K((A_k)_*, (B_k)^*) \leq \lim_{k \to +\infty} H_K(A_k, B_k) = H_K(A, B).$$

The last identity follows from Lebesgue's dominated convergence theorem, which can be used thanks to (4.60). In light of this, it suffices to prove (4.61) when A and B are open sets.

Next, we recall that each open subset of the real line can be written as the union of countably many disjoint open intervals. In our setting, we have

$$A = \bigcup_{k=0}^{+\infty} A^{(k)}, \text{ with } A^{(k)} := \bigcup_{i=0}^{k} A_i,$$

and

$$B = \bigcup_{k=0}^{+\infty} B^{(k)}, \text{ with } B^{(k)} := \bigcup_{j=0}^{k} B_j,$$

for two sequences $\{A_i\}$, $\{B_j\}$ of open intervals satisfying $A_{i_1} \cap A_{i_2} = \varnothing$ for every $i_1 \neq i_2$ and $B_{j_1} \cap B_{j_2} = \varnothing$ for every $j_1 \neq j_2$, and such that $(-\infty, \underline{\alpha}) \subseteq A_0$ and $(\bar{\beta}, +\infty) \subseteq B_0$, Suppose now that (4.61) holds when A and B are the unions of finitely many disjoint open intervals. In particular, (4.61) is true with $A^{(k)}$ and $B^{(k)}$ in place of A and B, respectively. Hence,

$$H_K((A^{(k)})_*, (B^{(k)})^*) \leq H_K(A^{(k)}, B^{(k)}) \leq H_K(A, B) \tag{4.62}$$

for every $k \in \mathbb{N}$. On the other hand, it is easy to see that

$$(-\infty, \underline{\alpha}) \subseteq (A^{(k-1)})_* \subseteq (A^{(k)})_* \subseteq A_* \quad \text{and} \quad (\bar{\beta}, +\infty) \subseteq (B^{(k-1)})^* \subseteq (B^{(k)})^* \subseteq B^*$$

for every $k \in \mathbb{N}$. Therefore, both $|A_* \setminus (A^{(k)})_*|$ and $|B^* \setminus (B^{(k)})^*|$ go to 0 as $k \to +\infty$. Lebesgue's monotone convergence theorem then yields that

$$H_K(A_*, B^*) = \lim_{k \to +\infty} H_K((A^{(k)})_*, (B^{(k)})^*).$$

The combination of this and (4.62) gives (4.61).

In light of the considerations that we just made, we are left to prove (4.61) when A and B are unions of finitely many disjoint open intervals. Thus, we fix $M, N \in \mathbb{N} \cup \{0\}$ and assume that

$$A = \bigcup_{i=0}^{M} A_i \quad \text{and} \quad B = \bigcup_{j=0}^{N} B_j,$$

with

$$\begin{aligned} A_0 &:= (-\infty, a_0) \quad \text{and} \quad A_i := (a_{2i-1}, a_{2i}) \quad \text{for } i = 1, \dots, M, \\ B_0 &:= (b_0, +\infty) \quad \text{and} \quad B_j := (b_{2j}, b_{2j-1}) \quad \text{for } j = 1, \dots, N, \end{aligned}$$

where $\{a_i\}_{i=0}^{2M}, \{b_j\}_{j=0}^{2N} \subseteq \mathbb{R}$ are two sets of points satisfying $a_{i-1} < a_i$ and $b_{j-1} < b_j$, for every $i = 1, \dots, 2M$ and $j = 1, \dots, 2N$. In this framework, inequality (4.61) takes the form

$$\sum_{\substack{i=0,\dots,M \\ j=0,\dots,N}} \int_{A_i} \int_{B_j} d\mu \geq \int_{A_*} \int_{B^*} d\mu. \tag{4.63}$$

Clearly, when $M = N = 0$ there is nothing to prove, as it holds $A_* = A$ and $B^* = B$. In case either $M = 0$ or $N = 0$, the verification of (4.63) is also simple. Indeed, suppose for instance that $N = 0$ and $M \geq 1$. Then, $B^* = B = (b_0, +\infty)$ and $A_* = (-\infty, a_*)$ for some $a_* \in \mathbb{R}$. Up to a set of measure zero we may write A_* as the union of the $M+1$ disjoint intervals $\{C_i\}_{i=0}^{M}$ given by $C_i = A_i - \bar{a}_i$, with $\bar{a}_i \geq 0$ for every i. Accordingly,

$$\int_{A_*} \int_{B^*} d\mu = \sum_{i=0}^{M} \int_{C_i} \int_{b_0}^{+\infty} d\mu = \sum_{i=0}^{M} \int_{A_i} \int_{b_0 + \bar{a}_i}^{+\infty} d\mu \leq \sum_{i=0}^{M} \int_{A_i} \int_{b_0}^{+\infty} d\mu = \int_{A} \int_{B} d\mu,$$

that is (4.63). Note that the second identity follows by adding to both of the variables of the double integral the same quantity $\bar{a}_i$. That is, we applied the change of coordinates $x = w - \bar{a}_i$, $y = z - \bar{a}_i$ and got

$$\int_{C_i} \int_{b_0}^{+\infty} d\mu = \int_{C_i} \int_{b_0}^{+\infty} K(x-y)\, dx\, dy = \int_{A_i} \int_{b_0 + \bar{a}_i}^{+\infty} K(w-z)\, dw dz = \int_{A_i} \int_{b_0 + \bar{a}_i}^{+\infty} d\mu,$$

exploiting the fact that K is translation-invariant.

As the case $M = 0$, $N \geq 1$ is completely analogous, we can now address the validity of (4.63) when $M, N \geq 1$. Recalling definitions (4.57)-(4.58), it is immediate to see that

$$\begin{aligned} A_* &= (-\infty, a_*), \quad \text{with } a_* = a_0 + \sum_{\ell=1}^{M} |A_\ell| = a_0 + \sum_{\ell=1}^{M} (a_{2\ell} - a_{2\ell-1}), \\ B^* &= (b^*, +\infty), \quad \text{with } b^* = b_0 - \sum_{\ell=1}^{N} |B_j| = b_0 - \sum_{\ell=1}^{N} (b_{2\ell-1} - b_{2\ell}). \end{aligned}$$

Set

$$C_i := A_i - \bar{a}_i, \quad \text{with } \bar{a}_i := \sum_{\ell=0}^{i-1} (a_{2\ell+1} - a_{2\ell}) \text{ for } i = 1, \dots, M \text{ and } \bar{a}_0 := 0, \tag{4.64}$$

$$D_j := B_j + \bar{b}_j, \quad \text{with } \bar{b}_j := \sum_{\ell=0}^{j-1} (b_{2\ell} - b_{2\ell+1}) \text{ for } j = 1, \dots, N \text{ and } \bar{b}_0 := 0. \tag{4.65}$$

The families $\{C_i\}_{i=0}^M$ and $\{D_j\}_{j=0}^N$ are both made up of consecutive open intervals. Moreover, up to sets of measure zero, we have

$$A_* = \bigcup_{i=0}^{M} C_i \quad \text{and} \quad B^* = \bigcup_{j=0}^{N} D_j. \tag{4.66}$$

Consequently, we can equivalently express (4.63) as

$$\sum_{\substack{i=0,\dots,M \\ j=0,\dots,N}} \int_{A_i}\int_{B_j} d\mu \geq \sum_{\substack{i=0,\dots,M \\ j=0,\dots,N}} \int_{C_i}\int_{D_j} d\mu. \tag{4.67}$$

Fix any $j = 1, \dots, N$. We compute

$$\int_{A_0}\int_{B_j} d\mu = \int_{C_0}\int_{D_j - \bar{b}_j} d\mu = \int_{C_0+\bar{b}_j}\int_{D_j} d\mu = \int_{(C_0+\bar{b}_j)\setminus C_0}\int_{D_j} d\mu + \int_{C_0}\int_{D_j} d\mu.$$

Notice that the first identity follows from definitions (4.64)-(4.65), the second by applying to both variables of the double integral a shift of length $\bar{b}_j$, and the third since $C_0 \subseteq C_0+\bar{b}_j$. Similarly,

$$\int_{A_i}\int_{B_0} d\mu = \int_{C_i}\int_{(D_0-\bar{a}_i)\setminus D_0} d\mu + \int_{C_i}\int_{D_0} d\mu.$$

for every $i = 1, \dots, M$. Furthermore, by a translation of size $\bar{b}_j - \bar{a}_i$, we may also write

$$\int_{A_i}\int_{B_j} d\mu = \int_{C_i+\bar{a}_i}\int_{D_j-\bar{b}_j} d\mu = \int_{C_i+\bar{b}_j}\int_{D_j-\bar{a}_i} d\mu$$

for every $i = 1, \dots, M$ and $j = 1, \dots, N$. Finally, again by (4.64)-(4.65)—with $i = j = 0$—we have

$$\int_{A_0}\int_{B_0} d\mu = \int_{C_0}\int_{D_0} d\mu.$$

Applying the last four identities together with (4.66), formula (4.67) becomes

$$\sum_{\substack{i=0,\dots,M \\ j=0,\dots,N}} \int_{E_{i;j}}\int_{F_{j;i}} d\mu \geq \int_{a_0}^{a_*}\int_{b^*}^{b_0} d\mu, \tag{4.68}$$

where we put

(4.69)
$$\begin{aligned}
&E_{0;0} := \{a_0\}, && F_{0;0} := \{b_0\}, && \\
&E_{i;0} := C_i, && F_{0;i} := (D_0 - \bar{a}_i) \setminus D_0, && \text{for } i = 1, \dots, M, \\
&E_{0;j} := \big(C_0 + \bar{b}_j\big) \setminus C_0, && F_{j;0} := D_j, && \text{for } j = 1, \dots, N, \\
&E_{i;j} := C_i + \bar{b}_j, && F_{j;i} := D_j - \bar{a}_i, && \text{for } i = 1, \dots, M,\ j = 1, \dots, N.
\end{aligned}$$

We now claim that

$$[a_0, a_*] \times [b^*, b_0] \subseteq \bigcup_{\substack{i=0,\dots,M \\ j=0,\dots,N}} \overline{E_{i;j}} \times \overline{F_{j;i}}. \tag{4.70}$$

Observe that (4.70) is stronger than (4.68), and therefore that its validity would lead us to the conclusion of the proof.

Before showing that (4.70) is true, we make some considerations on the intervals $E_{i;j}$'s and $F_{j;i}$'s. Given a bounded non-empty interval $I \subseteq \mathbb{R}$, we indicate with $\ell(I)$ and $r(I)$ its left and right endpoint, respectively. We have that

$$r(E_{i-1;j}) = \ell(E_{i;j}), \qquad \text{for } i = 1, \ldots, M,\ j = 0, \ldots, N, \tag{4.71}$$

$$r(F_{j;i}) = \ell(F_{j-1;i}), \qquad \text{for } i = 0, \ldots, M,\ j = 1, \ldots, N, \tag{4.72}$$

$$r(E_{M;j}) \geq a_*, \qquad \text{for } j = 0, \ldots, N, \tag{4.73}$$

$$\ell(F_{N;i}) \leq b^*, \qquad \text{for } i = 0, \ldots, M. \tag{4.74}$$

To check (4.71), we recall definitions (4.69), (4.64), (4.65), and notice that

$$\begin{aligned} r(E_{i-1;j}) &= r(A_{i-1}) - \bar{a}_{i-1} + \bar{b}_j = a_{2i-2} - \bar{a}_i + (a_{2i-1} - a_{2i-2}) + \bar{b}_j \\ &= \ell(A_i) - \bar{a}_i + \bar{b}_j = \ell(E_{i;j}) \end{aligned}$$

for every $i = 1, \ldots, M$ and $j = 0, \ldots, N$. On the other hand, it holds

$$\begin{aligned} r(E_{M;j}) = r(A_M) - \bar{a}_M + \bar{b}_j &= a_{2M} - \sum_{\ell=0}^{M-1} (a_{2\ell+1} - a_{2\ell}) + \bar{b}_j \\ &= a_0 + \sum_{\ell=1}^{M} (a_{2\ell} - a_{2\ell-1}) + \bar{b}_j \geq a_*, \end{aligned}$$

which gives (4.73). Items (4.72) and (4.74) follow analogously.

In view of formulas (4.71), (4.73) and (4.72), (4.74), we immediately deduce that

$$[a_0, a_*] \subseteq \bigcup_{i=0}^{M} \overline{E_{i;j}} \quad \text{for any } j = 0, \ldots, N \tag{4.75}$$

and

$$[b^*, b_0] \subseteq \bigcup_{j=0}^{N} \overline{F_{j;i}} \quad \text{for any } i = 0, \ldots, M, \tag{4.76}$$

respectively—recall that $\ell(E_{0;j}) = a_0$ and $r(F_{0;i}) = b_0$ for any such j and i.

On top of the previous facts, we also claim that

$$\ell(E_{i;j}) > \ell(E_{i;j-1}) \quad \text{for every } i = 1, \ldots, M,\ j = 1, \ldots, N \tag{4.77}$$

and

$$r(F_{j;i}) < r(F_{j;i-1}) \quad \text{for every } i = 1, \ldots, M,\ j = 1, \ldots, N. \tag{4.78}$$

Indeed, for $i = 1, \ldots, M$ and $j = 1, \ldots, N$ we have

$$r(F_{j;i}) = r(D_j) - \bar{a}_i = r(D_j) - \bar{a}_{i-1} - (a_{2i-1} - a_{2i-2}) < r(D_j) - \bar{a}_{i-1} = r(F_{j;i}).$$

This proves (4.78), while (4.77) can be checked in a similar fashion.

Thanks to the previous remarks, we can now address the proof of (4.70). Let

$$p = (x, y) \in [a_0, a_*] \times [b^*, b_0] \tag{4.79}$$

and suppose by contradiction that p does not belong to the right-hand side of (4.70). I.e.,

$$p \notin \overline{E_{i;j}} \times \overline{F_{j;i}} \quad \text{for every } i = 0, \ldots, M \text{ and } j = 0, \ldots, N. \tag{4.80}$$

In light of (4.75), we know that in correspondence to every $j = 0, \ldots, N$ we can pick an $i_j \in \{0, \ldots, M\}$ in such a way that

$$x \in \overline{E_{i_j;j}}. \tag{4.81}$$

We claim that

$$\{i_j\}_{j=0}^N \text{ is non-increasing.} \tag{4.82}$$

Indeed, suppose that we have constructed the (finite) sequence $\{i_\ell\}$ up to the index $\ell = j-1$, with $j \in \{1, \dots, N\}$. Of course, when $i_{j-1} = M$ we necessarily have $i_j \le i_{j-1}$. On the other hand, if $i_{j-1} \le M-1$, using (4.77) and (4.71), we infer that

$$\ell(E_{i_{j-1}+1;j}) > \ell(E_{i_{j-1}+1;j-1}) = r(E_{i_{j-1};j-1}) \ge x.$$

Hence, also in this case i_j falls within the set $\{0, \dots, i_{j-1}\}$ and (4.82) is established.

Next, by comparing (4.81) and (4.80), we notice that $y \notin \overline{F_{j;i_j}}$. This amounts to say that, for every index $j = 0, \dots, N$,

$$\text{either } \; y < \ell(F_{j;i_j}) \; \text{ or } \; y > r(F_{j;i_j}). \tag{4.83}$$

We now claim that the latter possibility cannot occur, i.e., that

$$y < \ell(F_{j;i_j}) \tag{4.84}$$

for every $j = 0, \dots, N$. Note that (4.84) would lead us to a contradiction. Indeed, by using it with $j = N$ and in combination with (4.79) and (4.74), we would get

$$b^* \le y < \ell(F_{N;i_N}) \le b^*,$$

which is clearly impossible. Therefore, to finish the proof we are only left to show that (4.84) holds true for every $j = 0, \dots, N$. To achieve this, we argue inductively. First, we check that (4.84) is verified for $j = 0$. Indeed, by (4.79) and (4.69),

$$y \le b_0 = r(F_{0;i_0}),$$

and thus (4.83) yields that $y < \ell(F_{0;i_0})$. Secondly, we pick any $j \in \{1, \dots, N\}$ and assume that (4.84) is valid with $j-1$ in place of j. Then, recalling (4.72), (4.82) and possibly (4.78) (applied iteratively), we get that

$$y < \ell(F_{j-1;i_{j-1}}) = r(F_{j;i_{j-1}}) \le r(F_{j;i_j}).$$

By comparing this with (4.83), we finally deduce that claim (4.84) holds true. Thus, the proof is complete. □

4.4.2. Vertical rearrangements and the s-perimeter. We now take advantage of Proposition 4.4.1 to show that Per_s decreases under vertical rearrangements. Given a set $E \subseteq \mathbb{R}^{n+1}$, we consider the function $w_E : \mathbb{R}^n \to \mathbb{R}$ defined by

$$w_E(x) := \lim_{R \to +\infty} \left(\int_{-R}^{R} \chi_E(x,t)\, dt - R \right)$$

for every $x \in \mathbb{R}^n$, together with its subgraph $E_\star := \mathcal{S}g(w_E)$. We have the following result.

Proof of Theorem 4.1.10. Denote with G either the set E or its rearrangement $E_\star$. Observe that E and $E_\star$ coincide outside of Ω^∞, and are both given by the subgraph of the same function $v : \mathcal{C}\Omega \to \mathbb{R}$. Hence,

$$G \setminus \Omega^\infty = \Big\{ (x,t) \in (\mathcal{C}\Omega) \times \mathbb{R} \,|\, t < v(x) \Big\}. \tag{4.85}$$

It is also clear that $E_\star$ satisfies (4.7). Accordingly,

$$\Omega \times (-\infty, -M) \subseteq G \cap \Omega^\infty \subseteq \Omega \times (-\infty, M). \tag{4.86}$$

We compute

$$\begin{aligned}
\operatorname{Per}_s(G,\Omega^M) &= \mathcal{L}_s(G\cap\Omega^M,\mathcal{C}G\cap\Omega^M)+\mathcal{L}_s(G\cap\Omega^M,\mathcal{C}G\setminus\Omega^M)+\mathcal{L}_s(G\setminus\Omega^M,\mathcal{C}G\cap\Omega^M)\\
&= \mathcal{L}_s(G\cap\Omega^M,\mathcal{C}G\cap\Omega^M)\\
&\quad+\mathcal{L}_s(G\cap\Omega^M,\mathcal{C}G\cap(\Omega^\infty\setminus\Omega^M))+\mathcal{L}_s(G\cap\Omega^M,\mathcal{C}G\setminus\Omega^\infty)\\
&\quad+\mathcal{L}_s(G\cap(\Omega^\infty\setminus\Omega^M),\mathcal{C}G\cap\Omega^M)+\mathcal{L}_s(G\setminus\Omega^\infty,\mathcal{C}G\cap\Omega^M)\\
&= \mathcal{L}_s(G\cap\Omega^\infty,\mathcal{C}G\cap\Omega^\infty)-\mathcal{L}_s(G\cap(\Omega^\infty\setminus\Omega^M),\mathcal{C}G\cap(\Omega^\infty\setminus\Omega^M))\\
&\quad+\mathcal{L}_s(G\cap\Omega^M,\mathcal{C}G\setminus\Omega^\infty)+\mathcal{L}_s(G\setminus\Omega^\infty,\mathcal{C}G\cap\Omega^M).
\end{aligned}$$

Thanks to (4.86), we may write

$$\mathcal{L}_s(G\cap(\Omega^\infty\setminus\Omega^M),\mathcal{C}G\cap(\Omega^\infty\setminus\Omega^M))=\mathcal{L}_s(\Omega\times(-\infty,-M),\Omega\times(M,+\infty))=:C_M.$$

Note that C_M is a constant depending only on n, s, Ω, and M. Moreover, using (4.85) and again (4.86), we have

$$\mathcal{L}_s(G\cap\Omega^M,\mathcal{C}G\setminus\Omega^\infty)=\mathcal{L}_s(G\cap\Omega^\infty,\mathcal{C}G\setminus\Omega^\infty)-D_M^{(1)}$$

and

$$\mathcal{L}_s(G\setminus\Omega^\infty,\mathcal{C}G\cap\Omega^M)=\mathcal{L}_s(G\setminus\Omega^\infty,\mathcal{C}G\cap\Omega^\infty)-D_M^{(2)},$$

where $D_M^{(1)}:=\mathcal{L}_s(\Omega\times(-\infty,-M),\mathcal{C}\mathcal{S}g(v)\setminus\Omega^\infty)$ and $D_M^{(2)}:=\mathcal{L}_s(\mathcal{S}g(v)\setminus\Omega^\infty,\Omega\times(M,+\infty))$ are constants depending only on n, s, Ω, M, and v. Putting together the last four identities, we find that

$$\begin{aligned}
\operatorname{Per}_s(G,\Omega^M) &= \mathcal{L}_s(G\cap\Omega^\infty,\mathcal{C}G\cap\Omega^\infty)+\mathcal{L}_s(G\cap\Omega^\infty,\mathcal{C}G\setminus\Omega^\infty)+\mathcal{L}_s(G\setminus\Omega^\infty,\mathcal{C}G\cap\Omega^\infty)\\
&\quad-C_M-D_M^{(1)}-D_M^{(2)}.
\end{aligned}$$

In particular, inequality (4.1.10) will be verified if we prove that

$$\begin{aligned}
&\mathcal{L}_s(E_\star\cap\Omega^\infty,\mathcal{C}E_\star\cap\Omega^\infty)\le\mathcal{L}_s(E\cap\Omega^\infty,\mathcal{C}E\cap\Omega^\infty),\\
&\mathcal{L}_s(E_\star\cap\Omega^\infty,\mathcal{C}E_\star\setminus\Omega^\infty)\le\mathcal{L}_s(E\cap\Omega^\infty,\mathcal{C}E\setminus\Omega^\infty),\\
&\mathcal{L}_s(E_\star\setminus\Omega^\infty,\mathcal{C}E_\star\cap\Omega^\infty)\le\mathcal{L}_s(E\setminus\Omega^\infty,\mathcal{C}E\cap\Omega^\infty).
\end{aligned}\tag{4.87}$$

Set

$$G(x):=\Big\{t\in\mathbb{R}\,|\,(x,t)\in G\Big\}\quad\text{for }x\in\mathbb{R}^n$$

and

$$K_a(t):=\frac{1}{(a^2+t^2)^{\frac{n+1+s}{2}}}\quad\text{for }a,t\in\mathbb{R}.$$

Using the notation of (4.55) and Fubini's theorem, we may write

$$\begin{aligned}
\mathcal{L}_s(G\cap\Omega^\infty,\mathcal{C}G\cap\Omega^\infty)&=\int_\Omega\int_\Omega H_{K_{|x-y|}}(G(x),\mathcal{C}G(y))\,dx\,dy,\\
\mathcal{L}_s(G\cap\Omega^\infty,\mathcal{C}G\setminus\Omega^\infty)&=\int_\Omega\int_{\mathcal{C}\Omega} H_{K_{|x-y|}}(G(x),\mathcal{C}G(y))\,dx\,dy,\\
\mathcal{L}_s(G\setminus\Omega^\infty,\mathcal{C}G\cap\Omega^\infty)&=\int_{\mathcal{C}\Omega}\int_\Omega H_{K_{|x-y|}}(G(x),\mathcal{C}G(y))\,dx\,dy.
\end{aligned}\tag{4.88}$$

Recalling the definition of decreasing rearrangement of a subset of the real line introduced in (4.57), we observe that $E(x)_*=(-\infty,w_E(x))=E_\star(x)$ for every $x\in\mathbb{R}^n$. Also notice that $H_{K_a}((-\infty,\alpha),(\beta,+\infty))<\infty$ for every $\alpha,\beta\in\mathbb{R}$ and $a\neq0$. By this, we are allowed to apply Proposition 4.4.1 and deduce that

$$H_{K_{|x-y|}}(E_\star(x),\mathcal{C}E_\star(y))\le H_{K_{|x-y|}}(E(x),\mathcal{C}E(y))\quad\text{for a.e. }x,y\in\mathbb{R}^n,$$

where we also took advantage of property (4.59). In view of (4.88), this last inequality ensures the validity of (4.87). The proof is thus finished. □

4.5. Minimizers

This section is devoted to the study of the minimizers of $\mathcal{F}$. As observed in the Introduction, we will prove the existence of minimizers with the aid of an appropriate approximation procedure, which makes use of the "truncated functionals" $\mathcal{F}^M$ and of their own minimizers. For this reason, we introduce straight away the following auxiliary functional spaces. Given a bounded open set $\Omega \subseteq \mathbb{R}^n$, $s \in (0,1)$ and $M \geq 0$, we define

$$\mathfrak{B}\mathcal{W}^s(\Omega) := \{u \in \mathcal{W}^s(\Omega) \,|\, u|_\Omega \in L^\infty(\Omega)\}$$
$$\text{and} \quad \mathfrak{B}_M\mathcal{W}^s(\Omega) := \{u \in \mathfrak{B}\mathcal{W}^s(\Omega) \,|\, \|u\|_{L^\infty(\Omega)} \leq M\}.$$

Moreover, given a function $\varphi : \mathcal{C}\Omega \to \mathbb{R}$, we define

$$\mathfrak{B}\mathcal{W}^s_\varphi(\Omega) := \{u \in \mathfrak{B}\mathcal{W}^s(\Omega) \,|\, u = \varphi \text{ a.e. in } \mathcal{C}\Omega\}$$
$$\text{and} \quad \mathfrak{B}_M\mathcal{W}^s_\varphi(\Omega) := \{u \in \mathfrak{B}_M\mathcal{W}^s(\Omega) \,|\, u = \varphi \text{ a.e. in } \mathcal{C}\Omega\}.$$

We begin by recalling the definition of minimizer in the context of the Dirichlet problem. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, $s \in (0,1)$ and let $\varphi : \mathcal{C}\Omega \to \mathbb{R}$. We say that a function $u \in \mathcal{W}^s_\varphi$ is a minimizer of $\mathcal{F}$ in $\mathcal{W}^s_\varphi(\Omega)$ if

$$\iint_{Q(\Omega)} \left\{ \mathcal{G}\left(\frac{u(x)-u(y)}{|x-y|}\right) - \mathcal{G}\left(\frac{v(x)-v(y)}{|x-y|}\right) \right\} \frac{dx\,dy}{|x-y|^{n-1+s}} \leq 0,$$

for every $v \in \mathcal{W}^s_\varphi(\Omega)$.

It is now convenient to point out the following useful result, which is easily obtained by arguing as in the proof of Lemma 4.2.3, exploiting formula (4.28) and the global Lipschitzianity of $\mathcal{G}$—see (4.21).

Lemma 4.5.1. *Let $n \geq 1$, $s \in (0,1)$, $M \geq 0$, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $\varphi : \mathcal{C}\Omega \to \mathbb{R}$. There exists a constant $C > 0$, depending only on n, s and Ω, such that*

$$\iint_{Q(\Omega)} \left| \mathcal{G}\left(\frac{u(x)-u(y)}{|x-y|}\right) - \mathcal{G}\left(\frac{v(x)-v(y)}{|x-y|}\right) \right| \frac{dx\,dy}{|x-y|^{n-1+s}} \leq C\,\Lambda \|u-v\|_{W^{s,1}(\Omega)},$$

for every u, $v \in \mathcal{W}^s_\varphi(\Omega)$, with Λ as defined in (4.12). Moreover, we have the identity

(4.89)
$$\mathcal{F}^M(u,\Omega) - \mathcal{F}^M(v,\Omega) = \iint_{Q(\Omega)} \left\{ \mathcal{G}\left(\frac{u(x)-u(y)}{|x-y|}\right) - \mathcal{G}\left(\frac{v(x)-v(y)}{|x-y|}\right) \right\} \frac{dx\,dy}{|x-y|^{n-1+s}}.$$

As a consequence, if $u, u_k \in \mathcal{W}^s_\varphi(\Omega)$ are such that $\|u-u_k\|_{W^{s,1}(\Omega)} \to 0$ as $k \to \infty$, then

$$\lim_{k\to\infty} \mathcal{F}^M(u_k,\Omega) = \mathcal{F}^M(u,\Omega).$$

Remark 4.5.2. In this Remark we collect the following straightforward but important consequences of Lemma 4.5.1:

(i) it guarantees that the definition of minimizer is well posed;
(ii) it provides an equivalent characterization of a minimizer of $\mathcal{F}$ in $\mathcal{W}^s_\varphi(\Omega)$ as a function $u \in \mathcal{W}^s_\varphi(\Omega)$ that minimizes $\mathcal{F}^M(\,\cdot\,,\Omega)$, i.e. such that
$$\mathcal{F}^M(u,\Omega) = \inf\left\{\mathcal{F}^M(v,\Omega) \,|\, v \in \mathcal{W}^s_\varphi(\Omega)\right\};$$
(iii) by point (ii) and by the strict convexity of $\mathcal{F}^M$—see point (ii) of Lemma 4.2.5—we obtain that a minimizer of $\mathcal{F}$ in $\mathcal{W}^s_\varphi(\Omega)$, if it exists, is unique;

(iv) as a consequence of the density of the spaces $C_c^\infty(\Omega)$ and $W^{s,1}(\Omega) \cap L^\infty(\Omega)$ in the fractional Sobolev space $W^{s,1}(\Omega)$—see, e.g., Appendix D.2—Lemma 4.5.1 implies that to verify the minimality of $u \in \mathcal{W}^s_\varphi(\Omega)$ we can limit ourselves to consider either competitors $v \in \mathcal{W}^s_\varphi(\Omega)$ such that $v|_\Omega \in C_c^\infty(\Omega)$, or $v \in \mathfrak{B}\mathcal{W}^s_\varphi(\Omega)$.

In light of point (ii) of Remark 4.5.2, we could have considered as definition of minimizer just that of a function $u \in \mathcal{W}^s_\varphi(\Omega)$ that minimizes the functional $\mathcal{F}^0$—or the functional $\mathcal{F}^M$, for some fixed $M > 0$—in $\mathcal{W}^s_\varphi(\Omega)$. However, we remark that such a definition is not very helpful when trying to prove existence results and indeed it presents some difficulties, first of all the fact that the functional $\mathcal{F}^M$ in general is not non-negative in the space $\mathcal{W}^s_\varphi(\Omega)$ and may indeed change sign—see Example 4.2.1. Hence, lower semicontinuity and compactness properties are not straightforward.

Now we turn our attention to the Euler-Lagrange equation satisfied by minimizers.

We recall that, given a bounded open set $\Omega \subseteq \mathbb{R}^n$ and $s \in (0,1)$, we say that a measurable function $u : \mathbb{R}^n \to \mathbb{R}$ is a weak solution of $\mathscr{H} u = 0$ in Ω if

$$\langle \mathscr{H} u, v \rangle = 0 \quad \text{for every } v \in C_c^\infty(\Omega). \tag{4.90}$$

Remark 4.5.3. Notice that, if Ω has Lipschitz boundary, then, by density, in (4.90) we can as well consider $v \in \mathcal{W}^s_0(\Omega)$ as test function. Indeed, in light of Corollary D.1.5 we have that

$$\|v\|_{W^{s,1}(\mathbb{R}^n)} \le C(n,s,\Omega)\|v\|_{W^{s,1}(\Omega)} \quad \text{for every } v \in \mathcal{W}^s_0(\Omega).$$

Hence, since $\langle \mathscr{H} u, \cdot \rangle \in \big(W^{s,1}(\mathbb{R}^n)\big)^*$, by the density of $C_c^\infty(\Omega)$ in $W^{s,1}(\Omega)$ we find that (4.90) implies that

$$\langle \mathscr{H} u, v \rangle = 0 \quad \text{for every } v \in \mathcal{W}^s_0(\Omega).$$

Exploiting the convexity of the functionals $\mathcal{F}^M$, we can prove the equivalence between weak solutions (with "finite energy") and minimizers.

Lemma 4.5.4. *Let $n \ge 1$, $s \in (0,1)$, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and $u \in \mathcal{W}^s(\Omega)$. Then, u is a weak solution of $\mathscr{H} u = 0$ in Ω if and only if u is a minimizer of $\mathcal{F}$ in Ω.*

Proof. Suppose that u is a weak solution, let $v \in \mathcal{W}^s_u(\Omega)$ and define $w := v - u$. Notice that, since $w \in \mathcal{W}^s_0(\Omega)$, by Remark 4.5.3 we have

$$\langle \mathscr{H} u, w \rangle = 0.$$

Now we observe that the convexity of $\mathcal{G}$ implies that

$$\mathcal{G}(t) - \mathcal{G}(\tau) \ge G(\tau)(t-\tau) \quad \text{for every } t, \tau \in \mathbb{R}.$$

Thus, by (4.89) we obtain

$$\mathcal{F}^M(v,\Omega) - \mathcal{F}^M(u,\Omega) \ge \langle \mathscr{H} u, w \rangle = 0.$$

Since $v \in \mathcal{W}^s_u(\Omega)$ is arbitrary, the function u minimizes $\mathcal{F}^M(\,\cdot\,,\Omega)$ in $\mathcal{W}^s_u(\Omega)$, and hence—by point (ii) of Remark 4.5.2—u is a minimizer of $\mathcal{F}$ in Ω, in the sense of Definition 4.1.1.

To conclude the proof of the Lemma, the converse implication follows by point (ii) of Remark 4.5.2 and Lemma 4.2.12. □

It is interesting to observe that Lemmas 4.2.11 and 4.5.4 imply straight away that the set of minimizers of $\mathcal{F}$ is closed in $\mathcal{W}^s(\Omega)$, with respect to almost everywhere convergence.

Proposition 4.5.5. *Let $n \ge 1$, $s \in (0,1)$ and let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $\{u_k\} \subseteq \mathcal{W}^s(\Omega)$ be such that each u_k is a minimizer of $\mathcal{F}$ in Ω. If $u_k \to u$ a.e. in $\mathbb{R}^n$, for some function $u \in \mathcal{W}^s(\Omega)$, then u is a minimizer of $\mathcal{F}$ in Ω.*

Before going on, we briefly explain why we consider condition (4.4) to be too restrictive in our framework—even if at first glance it seems to be necessary, since it is required in order to guarantee that $\mathcal{F}$ is well defined on $\mathcal{W}^s_\varphi(\Omega)$—and why it makes sense to expect the existence of minimizers even when the exterior data $\varphi : \mathbb{R}^n \to \mathbb{R}$ does not satisfy (4.4).

First of all, we observe that if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with Lipschitz boundary and $\varphi : \mathbb{R}^n \to \mathbb{R}$ is bounded in a neighborhood of Ω, then it is readily seen that φ satisfies (4.4) if and only if

$$\int_{\mathbb{R}^n} \frac{|\varphi(y)|}{1 + |y|^{n+s}}\, dy < \infty. \tag{4.91}$$

We remark in particular that (4.91) forces φ to grow sublinearly at infinity.

Let now $u : \mathbb{R}^n \to \mathbb{R}$ be such that $u = \varphi$ almost everywhere in $\mathcal{C}\Omega$ and suppose that $u \in C^2(B_r(x))$, for some $x \in \Omega$ and $r > 0$. Then, the condition (4.91) is the same condition needed in order to guarantee the well definiteness of the fractional Laplacian

$$(-\Delta)^{\frac{s}{2}} u(x) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) + u(x-y)}{|y|^{n+s}}\, dy.$$

On the other hand, as observed in Lemma 4.2.10, the operator $\mathscr{H} u$ is well defined at x just thanks to the local regularity of u, with no need of assumptions about the growth of u at infinity. We further mention that condition (4.91) is needed in order to define the fractional $\frac{s}{2}$-Laplacian of a function as a tempered distribution. Contrarily, we can always define the operator $\mathscr{H} u$ in the distributional sense of (4.39), without having to make any assumption on the function u, besides measurability.

Also, we recall that we have a definition of minimizer of $\mathcal{F}$, namely Definition 4.1.1, which—as ensured by Lemma 4.5.1—makes sense without having to impose any restriction on the exterior data.

Thus, differently to what happens in the context of the fractional Laplacian, where condition (4.91) is totally natural, in our framework it seems to be unnecessarily restrictive.

Let us now switch our attention to the geometric situation, which corresponds to the choice $g = g_s$. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^2 boundary. We consider as exterior data a continuous function $\varphi \in C(\mathbb{R}^n)$, but we make no assumption on the behavior of φ at infinity. Then, we know that there exists a function $u : \mathbb{R}^n \to \mathbb{R}$ such that $u \in C^\infty(\Omega) \cap C(\overline{\Omega})$ and $u = \varphi$ in $\mathbb{R}^n \setminus \overline{\Omega}$, whose subgraph $\mathcal{S}g(u)$ is locally s-minimal in the cylinder Ω^∞. The existence follows from [**43**, Theorem 1.1] and Theorem 2.1.16, while the interior smoothness is guaranteed by [**19**, Theorem 1.1]. Thus, we know that in this case the "geometric problem" of (locally) minimizing the s-perimeter in Ω^∞ with respect to the exterior data $\mathcal{S}g(\varphi) \setminus \Omega^\infty$ has a solution, which is given by the subgraph of a function u, even if φ does not satisfy (4.4). To go one step further, we now observe that the function u is actually the minimizer of $\mathcal{F}_s$ in $\mathcal{W}^s_\varphi(\Omega)$. Indeed, thanks to the smoothness of u and the minimality of $\mathcal{S}g(u)$—see [**21**, Theorem 5.1]—we have that

$$\mathscr{H}_s u(x) = H_s[\mathcal{S}g(u)](x, u(x)) = 0 \qquad \text{for every } x \in \Omega,$$

and hence, by Proposition 4.3.15,

$$\langle \mathscr{H}_s u, v \rangle = 0 \qquad \text{for every } v \in C^\infty_c(\Omega).$$

Then, by Lemma 4.5.4, we conclude that u minimizes $\mathcal{F}_s$ in $\mathcal{W}^s_\varphi(\Omega)$.

For a more detailed discussion about the equivalence between stationary functions, minimizers of $\mathcal{F}$ and "geometric minimizers", in a more general situation, we refer to the forthcoming proof of Theorem 4.1.11 in Section 4.5.4.

4.5.1. Minimizers of the truncated functionals $\mathcal{F}^M$. As we have just anticipated, we are going to prove the existence of minimizers of $\mathcal{F}$ by making use of the minimizers of the truncated functionals $\mathcal{F}^M$. In order to motivate why we should expect this strategy to work, let us indulge a little longer in the discussion about the geometric situation.

Again, we consider a bounded open set $\Omega \subseteq \mathbb{R}^n$ with C^2 boundary and we fix as exterior data a continuous function $\varphi \in C(\mathbb{R}^n)$. As a first step, we observe that [43, Theorem 1.1] says that if $E \subseteq \mathbb{R}^{n+1}$ is a set which is locally s-minimal in the cylinder Ω^∞ and $E \setminus \Omega^\infty = \mathcal{S}g(\varphi) \setminus \Omega^\infty$, then E is globally a subgraph, that is, $E = \mathcal{S}g(u)$, for some function $u : \mathbb{R}^n \to \mathbb{R}$ such that $u \in C(\overline{\Omega})$ and $u = \varphi$ in $\mathbb{R}^n \setminus \overline{\Omega}$.

Therefore, we are reduced to prove the existence of a set E which is locally s-minimal in Ω^∞, with exterior data $\mathcal{S}g(\varphi) \setminus \Omega^\infty$.

We recall that, in order to do this, the argument exploited in the proof of Corollary 2.1.11 is the following. We first consider the minimization problem in the truncated cylinders Ω^k, that is, we take a set $E_k \subseteq \mathbb{R}^{n+1}$ which is s-minimal in Ω^k and such that $E_k \setminus \Omega^k = \mathcal{S}g(\varphi) \setminus \Omega^k$. The existence of such sets is guaranteed by [21, Theorem 3.2], since Ω^k is a bounded open set with Lipschitz boundary. Then, a compactness argument which exploits uniform perimeter estimates for s-minimal sets guarantees the existence of a set E such that $\chi_{E_k} \to \chi_E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$, up to subsequences. Notice that we have $E \setminus \Omega^\infty = \mathcal{S}g(\varphi) \setminus \Omega^\infty$. Finally, the s-minimality of the approximating sets E_k implies that the limit set E is locally s-minimal in Ω^∞—we refer the interested reader to Chapter 2 for the rigorous details of the argument.

Now we recall that, when restricted to the functional space $\mathfrak{B}_M\mathcal{W}^s(\Omega)$, the functional $\mathcal{F}_s^M$ corresponds to the s-fractional perimeter in the truncated cylinder Ω^M—by Proposition 4.2.8. Hence, the problem of finding a set $E_k \subseteq \mathbb{R}^{n+1}$ which is s-minimal in Ω^k, with respect to the exterior data $\mathcal{S}g(\varphi) \setminus \Omega^k$ corresponds, when $k \geq \|\varphi\|_{L^\infty(\Omega)}$, to the functional problem of minimizing $\mathcal{F}_s^k$ in the space $\mathfrak{B}_k\mathcal{W}^s_\varphi(\Omega)$—see Proposition 4.5.11. As we are going to prove in a moment, by making use of the direct method of the Calculus of Variations and exploiting the convexity of $\mathcal{F}_s^k$, this minimizing problem has a unique solution u_k. Then, if we want to follow the same strategy exploited in the geometric situation, we should aim to prove that $u_k \to u$ almost everywhere in $\mathbb{R}^n$, up to subsequences. This step is quite simple when working with sets, thanks to universal perimeter estimates. On the other hand, in the functional setting the situation is a little trickier and the existence of a limit function u is ensured by the uniform estimates provided by Proposition 4.5.9. Finally, we can exploit the minimality of the functions u_k in the space $\mathfrak{B}_k\mathcal{W}^s_\varphi(\Omega)$ to obtain the minimality of u in $\mathcal{W}^s_\varphi(\Omega)$.

Let us now get to the proofs of the aforementioned results.

We begin by observing that $\mathcal{F}^M$ is lower semicontinuous in $\mathfrak{B}_M\mathcal{W}^s(\Omega)$ with respect to pointwise convergence almost everywhere.

Lemma 4.5.6 (**Semicontinuity**). *Let $n \geq 1$, $s \in (0,1)$, $M > 0$, and $\Omega \subseteq \mathbb{R}^n$ be an open set. Let $\{u_k\}_{k\in\mathbb{N}} \subseteq \mathfrak{B}_M\mathcal{W}^s(\Omega)$ be a sequence of functions converging to some $u : \mathbb{R}^n \to \mathbb{R}$ a.e. in $\mathbb{R}^n$. Then,*

$$\mathcal{F}^M(u, \Omega) \leq \liminf_{k\to\infty} \mathcal{F}^M(u_k, \Omega).$$

Proof. The proof is a consequence of Fatou's lemma, applied separately to the functionals $\mathcal{A}$ and $\mathcal{N}^M$. Notice that, in order to use this result with $\mathcal{N}^M$, the uniform bound

$$\|u_k\|_{L^\infty(\Omega)} \leq M$$

is fundamental to guarantee that the quantity inside square brackets in (4.23) is non-negative—recall that $\overline{G} \geq 0$ by definition (4.14). ☐

Next is a compactness result for sequences uniformly bounded with respect to $\mathcal{A}$.

LEMMA 4.5.7 (**Compactness**). *Let $n \geq 1$, $s \in (0,1)$, and $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $\{u_k\}_{k\in\mathbb{N}}$ be a sequence functions $u_k : \Omega \to \mathbb{R}$ satisfying*

$$\sup_{k\in\mathbb{N}} \big(\|u_k\|_{L^1(\Omega)} + \mathcal{A}(u_k, \Omega) \big) < \infty.$$

Then, there exists a function $u \in W^{s,1}(\Omega)$ such that $\{u_k\}$ converges to u a.e. in Ω, up to a subsequence.

Lemma 4.5.7 follows at once from the compact embedding $W^{s,1}(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$—see, e.g., [38, Theorem 7.1]—and Lemma 4.2.2.

By combining the last two results, we easily obtain the existence of a (unique) minimizer u_M of $\mathcal{F}^M(\,\cdot\,, \Omega)$ among all functions in $\mathfrak{B}_M\mathcal{W}^s(\Omega)$ with fixed values outside of Ω.

PROPOSITION 4.5.8. *Let $n \geq 1$, $s \in (0,1)$, $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary, and $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ be a given function. For every $M > 0$, there exists a unique minimizer u_M of $\mathcal{F}^M(\,\cdot\,, \Omega)$ in $\mathfrak{B}_M\mathcal{W}^s_\varphi(\Omega)$, i.e., there exists a unique $u_M \in \mathfrak{B}_M\mathcal{W}^s_\varphi(\Omega)$ for which*

$$\mathcal{F}^M(u_M, \Omega) = \inf \Big\{ \mathcal{F}^M(v, \Omega) \,|\, v \in \mathfrak{B}_M\mathcal{W}^s_\varphi(\Omega) \Big\}. \tag{4.92}$$

PROOF. Since $\mathfrak{B}_M\mathcal{W}^s_\varphi(\Omega)$ is a convex subset of $\mathcal{W}^s_\varphi(\Omega)$, the uniqueness of the minimizer of $\mathcal{F}^M(\,\cdot\,, \Omega)$ within $\mathfrak{B}_M\mathcal{W}^s_\varphi(\Omega)$ is a consequence of the strict convexity of $\mathcal{F}^M(\,\cdot\,, \Omega)$—see point (ii) of Lemma 4.2.5. Therefore, we are only left to establish its existence.

Let $\{u^{(k)}\} \subseteq \mathfrak{B}_M\mathcal{W}^s_\varphi(\Omega)$ be a minimizing sequence, that is

$$\lim_{k\to\infty} \mathcal{F}^M(u^{(k)}, \Omega) = \inf \Big\{ \mathcal{F}^M(v, \Omega) \,|\, v \in \mathfrak{B}_M\mathcal{W}^s_\varphi(\Omega) \Big\} =: m.$$

Clearly, $\mathcal{F}^M(u^{(k)}, \Omega) \leq 2m$ for k large enough. Now, since $\|u^{(k)}\|_{L^\infty(\Omega)} \leq M$, we know that $\mathcal{N}^M(u^{(k)}, \Omega) \geq 0$—recall definitions (4.23) and (4.14)—and therefore $\mathcal{A}(u^{(k)}, \Omega) \leq 2m$ for k large. In light of Lemma 4.5.7, we then deduce that $\{u^{(k)}\}$ converges (up to a subsequence) to a function $u_M \in \mathfrak{B}_M\mathcal{W}^s_\varphi(\Omega)$ a.e. in $\mathbb{R}^n$. Identity (4.92) follows by applying Lemma 4.5.6. □

We briefly mention here that if for some $M_0 > 0$ we have $\|u_{M_0}\|_{L^\infty(\Omega)} < M_0$, then—as a consequence of the strict convexity of $\mathcal{F}^M$—we obtain that $u_M = u_{M_0}$ for every $M \geq M_0$. It is readily seen that this implies that the function u_{M_0} minimizes $\mathcal{F}$ in $\mathcal{W}^s_\varphi(\Omega)$. Therefore, in order to guarantee the existence of a minimizer, it is enough to prove an a priori L^∞ estimate. Depending on the exterior data, this is indeed possible—see Theorem 4.1.4 and Section 4.5.3.

We will not pursue this strategy here, but we will exploit it to prove the existence of a solution to the obstacle problem. For more details we thus refer to the proof of Theorem 4.6.1.

Instead, we now prove the following a priori estimate on the $W^{s,1}$ norm.

PROPOSITION 4.5.9. *Let $n \geq 1$, $s \in (0,1)$, $M \geq 0$, and $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ with $\mathrm{Tail}_s(\varphi, \Omega_{\Theta\,\mathrm{diam}(\Omega)} \setminus \Omega; \,\cdot\,) \in L^1(\Omega)$. If $u \in \mathcal{W}^s_\varphi(\Omega)$ is such that*

$$\mathcal{F}^M(u, \Omega) \leq \mathcal{F}^M(v, \Omega) \quad \textit{for every } v \in \mathfrak{B}_M\mathcal{W}^s_\varphi(\Omega),$$

then

$$\mathrm{diam}(\Omega)^{-s}\|u\|_{L^1(\Omega)} + [u]_{W^{s,1}(\Omega)} \leq C \left(\big\|\mathrm{Tail}_s(\varphi, \Omega_{\Theta\,\mathrm{diam}(\Omega)} \setminus \Omega; \,\cdot\,)\big\|_{L^1(\Omega)} + \mathrm{diam}(\Omega)^{1-s}|\Omega| \right),$$

for two constants $\Theta, C > 1$, depending only on n, s and g.

We observe that Proposition 4.5.9 applies in particular to the minimizers u_M, but we stress that, in general, in the hypothesis we are not assuming u to be bounded.

Proof of Proposition 4.5.9. We use the function $v := \chi_{\mathcal{C}\Omega} u \in \mathfrak{B}_M \mathcal{W}^s_\varphi(\Omega)$ as a competitor for u. We get

$$0 \leq \mathcal{F}^M(v,\Omega) - \mathcal{F}^M(u,\Omega) = -\mathcal{A}(u,\Omega) + 2\int_\Omega \int_{\mathcal{C}\Omega} \frac{H(x,y)}{|x-y|^{n-1+s}}\,dx\,dy, \tag{4.93}$$

with

$$H(x,y) := \mathcal{G}\left(\frac{v(x)-v(y)}{|x-y|}\right) - \mathcal{G}\left(\frac{u(x)-u(y)}{|x-y|}\right).$$

Write $d := \operatorname{diam}(\Omega)$. On the one hand, by Lemma 4.2.2,

$$\mathcal{A}(u,\Omega) \geq \frac{c_\star}{2}[u]_{W^{s,1}(\Omega)} - \frac{c_\star \mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{2(1-s)}|\Omega| d^{1-s}, \tag{4.94}$$

with $c_\star > 0$ as defined in (4.17). On the other hand, let $R := \Theta d$, with $\Theta \geq 1$ to be chosen later. Recalling the definition of v and taking advantage of point (b) of Lemma 4.2.1, we obtain

$$H(x,y) \leq \frac{\Lambda}{2}\frac{|u(y)|}{|x-y|} + \frac{c_\star}{2} - \frac{c_\star}{2}\frac{|u(x)-u(y)|}{|x-y|} \quad \text{for every } x \in \Omega,\ y \in \Omega_R \setminus \Omega$$

and

$$H(x,y) \leq \frac{\Lambda}{2}\frac{|u(x)|}{|x-y|} \quad \text{for every } x \in \Omega,\ y \in \mathcal{C}\Omega_R.$$

Hence, exploiting Lemma D.1.1 and observing that $c_\star \leq \Lambda$, we get

$$\begin{aligned}
2\int_\Omega \int_{\mathcal{C}\Omega} \frac{H(x,y)}{|x-y|^{n-1+s}}\,dx\,dy &\leq \int_\Omega \left(\int_{\Omega_R\setminus\Omega} \frac{\Lambda|u(y)| - c_\star|u(x)-u(y)|}{|x-y|^{n+s}}\,dy\right) dx \\
&\quad + c_\star \int_\Omega \int_{\Omega_R\setminus\Omega} \frac{dx\,dy}{|x-y|^{n-1+s}} \\
&\quad + \Lambda \int_\Omega |u(x)| \left(\int_{\mathcal{C}\Omega_R} \frac{dy}{|x-y|^{n+s}}\right) dx \\
&\leq \Lambda \bigg(\|\mathrm{Tail}_s(u, \Omega_{\Theta d}\setminus\Omega;\,\cdot\,)\|_{L^1(\Omega)} + \frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{1-s}\Theta^{1-s} d^{1-s}|\Omega| \\
&\quad + \Theta^{-s} d^{-s}\|u\|_{L^1(\Omega)} \bigg) - c_\star \int_\Omega \int_{\Omega_{\Theta d}\setminus\Omega} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy.
\end{aligned}$$

Putting together this estimate with (4.93) and (4.94), and recalling that $\Theta \geq 1$, we find that

$$\begin{aligned}
\int_\Omega \int_{\Omega_{\Theta d}} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\,dx\,dy &\leq \frac{\Lambda}{c_\star} \bigg(\|\mathrm{Tail}_s(u,\Omega_{\Theta d}\setminus\Omega;\,\cdot\,)\|_{L^1(\Omega)} \\
&\quad + 2\frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{1-s}\Theta^{1-s} d^{1-s}|\Omega| + \Theta^{-s} d^{-s}\|u\|_{L^1(\Omega)} \bigg).
\end{aligned} \tag{4.95}$$

Now we observe that

$$\operatorname{diam}(\Omega_d) = 3d \qquad \text{and} \qquad |\Omega_d \setminus \Omega| \geq c_n d^n, \tag{4.96}$$

for some dimensional constant $c_n > 0$ depending only on n. Indeed, the equality is an immediate consequence of the definition of Ω_d, while the measure estimate follows by observing that if we take a point $x_0 \in \partial\Omega_{d/2}$, then $B_{d/2}(x_0) \subseteq \Omega_d \setminus \Omega$, and hence

$$|\Omega_d \setminus \Omega| \geq |B_{d/2}(x_0)| = \frac{|B_1|}{2^n}\, d^n.$$

Since $v = 0$ in Ω and $v = u$ outside of Ω, using Lemma D.1.6 and exploiting (4.96), we may now estimate

$$\begin{aligned}\|u\|_{L^1(\Omega)} = \|u - v\|_{L^1(\Omega)} &\leq \frac{\operatorname{diam}(\Omega_d)^{n+s}}{|\Omega_d \setminus \Omega|} \int_\Omega |u(x)| \left(\int_{\Omega_d\setminus\Omega} \frac{dy}{|x-y|^{n+s}} \right) dx \\ &\leq C d^s \left(\int_\Omega \int_{\Omega_d\setminus\Omega} \frac{|u(x)-u(y)|}{|x-y|^{n+s}}\, dx\, dy + \|\operatorname{Tail}_s(u, \Omega_{\Theta d} \setminus \Omega; \cdot\,)\|_{L^1(\Omega)} \right),\end{aligned}$$

with $C > 0$ depending only on n and s. Using this estimate together with (4.95) and recalling that $\Theta \geq 1$, we get

$$\|u\|_{L^1(\Omega)} \leq C \left(d^s \|\operatorname{Tail}_s(u, \Omega_{\Theta d} \setminus \Omega; \cdot\,)\|_{L^1(\Omega)} + \Theta^{1-s} d|\Omega| + \Theta^{-s} \|u\|_{L^1(\Omega)} \right),$$

with $C > 0$ depending only on n, s and g. By taking Θ sufficiently large (in dependence of n, s and g only), we can reabsorb the L^1 norm of u on the left-hand side and obtain that

$$\|u\|_{L^1(\Omega)} \leq C \left(d^s \|\operatorname{Tail}_s(u, \Omega_{\Theta d} \setminus \Omega; \cdot\,)\|_{L^1(\Omega)} + d|\Omega| \right).$$

The conclusion follows by combining this estimate with (4.95). □

As shown in the following Lemma, the integrability of the truncated tail is equivalent to L^1 integrability plus weighted integrability arbitrarily close to the boundary of the domain.

Lemma 4.5.10. *Let $n \geq 1$, $s \in (0,1)$, $\Omega \Subset \mathcal{O} \subseteq \mathbb{R}^n$ two bounded open sets, such that Ω has Lipschitz boundary, and $\varphi : \mathcal{C}\Omega \to \mathbb{R}$. Then, $\operatorname{Tail}_s(\varphi, \mathcal{O} \setminus \Omega; \cdot\,) \in L^1(\Omega)$ if and only if $\varphi \in L^1(\mathcal{O} \setminus \Omega)$ and $\operatorname{Tail}_s(\varphi, \Omega_r \setminus \Omega; \cdot\,) \in L^1(\Omega \setminus \Omega_{-r})$, for some small $r > 0$.*

Moreover, suppose that $\varphi \in L^1(\mathcal{O} \setminus \Omega)$ and let $r > 0$ be small. Then:

(i) if $\varphi \in W^{s,1}(\Omega_r \setminus \Omega)$, then $\operatorname{Tail}_s(\varphi, \mathcal{O} \setminus \Omega; \cdot\,) \in L^1(\Omega)$;
(ii) if $\varphi \in L^\infty(\Omega_r \setminus \Omega)$, then $\operatorname{Tail}_\sigma(\varphi, \mathcal{O} \setminus \Omega; \cdot\,) \in L^1(\Omega)$, for every $\sigma \in (0,1)$.

Proof. To begin, let $d := \operatorname{diam}(\mathcal{O})$ and notice that

$$\frac{1}{|x-y|^{n+s}} \geq \frac{1}{d^{n+s}} \quad \text{for every } x \in \Omega \text{ and } y \in \mathcal{O} \setminus \Omega.$$

Hence

$$\|\varphi\|_{L^1(\mathcal{O}\setminus\Omega)} \leq \frac{d^{n+s}}{|\Omega|} \|\operatorname{Tail}_s(\varphi, \mathcal{O} \setminus \Omega; \cdot\,)\|_{L^1(\Omega)}.$$

Moreover, we clearly have

$$\|\operatorname{Tail}_s(\varphi, \Omega_r \setminus \Omega; \cdot\,)\|_{L^1(\Omega\setminus\Omega_{-r})} \leq \|\operatorname{Tail}_s(\varphi, \mathcal{O} \setminus \Omega; \cdot\,)\|_{L^1(\Omega)},$$

for every small $r > 0$.

Now suppose that $\varphi \in L^1(\mathcal{O} \setminus \Omega)$ and let $r > 0$ be small.

If $\operatorname{Tail}_s(\varphi, \Omega_r \setminus \Omega; \cdot\,) \in L^1(\Omega \setminus \Omega_{-r})$, then $\operatorname{Tail}_s(\varphi, \mathcal{O} \setminus \Omega; \cdot\,) \in L^1(\Omega)$. Indeed, since

$$|x-y| \geq r \quad \text{for every } x \in \Omega \text{ and } y \in \mathcal{O} \setminus \Omega_r,$$

we have

$$\|\operatorname{Tail}_s(\varphi, \mathcal{O} \setminus \Omega_r; \cdot\,)\|_{L^1(\Omega)} \leq \frac{|\Omega|}{r^{n+s}} \|\varphi\|_{L^1(\mathcal{O}\setminus\Omega_r)}. \tag{4.97}$$

Similarly,

$$\|\mathrm{Tail}_s(\varphi,\Omega_r\setminus\Omega;\,\cdot\,)\|_{L^1(\Omega_{-r})}\le\frac{|\Omega_{-r}|}{r^{n+s}}\|\varphi\|_{L^1(\Omega_r\setminus\Omega)}\le\frac{|\Omega|}{r^{n+s}}\|\varphi\|_{L^1(\Omega_r\setminus\Omega)}.$$

Therefore,

$$\begin{aligned}\|\mathrm{Tail}_s(\varphi,\mathcal{O}\setminus\Omega;\,\cdot\,)\|_{L^1(\Omega)}&=\|\mathrm{Tail}_s(\varphi,\mathcal{O}\setminus\Omega_r;\,\cdot\,)\|_{L^1(\Omega)}+\|\mathrm{Tail}_s(\varphi,\Omega_r\setminus\Omega;\,\cdot\,)\|_{L^1(\Omega_{-r})}\\&\quad+\|\mathrm{Tail}_s(\varphi,\Omega_r\setminus\Omega;\,\cdot\,)\|_{L^1(\Omega\setminus\Omega_{-r})}\\&\le\frac{|\Omega|}{r^{n+s}}\|\varphi\|_{L^1(\mathcal{O}\setminus\Omega)}+\|\mathrm{Tail}_s(\varphi,\Omega_r\setminus\Omega;\,\cdot\,)\|_{L^1(\Omega\setminus\Omega_{-r})}.\end{aligned}$$

If $\varphi\in W^{s,1}(\Omega_r\setminus\Omega)$, then—since for small $r>0$ the open set Ω_r has Lipschitz boundary—by Corollary D.1.5 we obtain

$$\|\mathrm{Tail}_s(\varphi,\Omega_r\setminus\Omega;\,\cdot\,)\|_{L^1(\Omega)}\le C(n,\Omega_r\setminus\Omega,s)\|\varphi\|_{W^{s,1}(\Omega_r\setminus\Omega)}.$$

Hence, recalling (4.97), we have

$$\|\mathrm{Tail}_s(\varphi,\mathcal{O}\setminus\Omega;\,\cdot\,)\|_{L^1(\Omega)}\le\frac{|\Omega|}{r^{n+s}}\|\varphi\|_{L^1(\mathcal{O}\setminus\Omega_r)}+C\|\varphi\|_{W^{s,1}(\Omega_r\setminus\Omega)}.$$

If $\varphi\in L^\infty(\Omega_r\setminus\Omega)$, then

$$\|\mathrm{Tail}_\sigma(\varphi,\Omega_r\setminus\Omega;\,\cdot\,)\|_{L^1(\Omega)}\le\|\varphi\|_{L^\infty(\Omega_r\setminus\Omega)}\,\mathrm{Per}_\sigma(\Omega),$$

for every $\sigma\in(0,1)$. Thus, we obtain point (ii) by exploiting (4.97) again. This concludes the proof of the Lemma. □

We conclude this Section by getting back to the geometric framework $g=g_s$. We exploit Theorem 4.1.10 in order to prove that the unique s-minimal set in Ω^M with respect to the exterior data $\mathcal{S}g(\varphi)\setminus\Omega^M$ is the subgraph of the minimizer u_M.

PROPOSITION 4.5.11. *Let $n\ge1$, $s\in(0,1)$, $M>0$, and $\Omega\subseteq\mathbb{R}^n$ a bounded open set with Lipschitz boundary. Let $\varphi:\mathbb{R}^n\to\mathbb{R}$, such that $\varphi=0$ a.e. in Ω and let u_M be the minimizer of $\mathcal{F}_s^M(\,\cdot\,,\Omega)$ within $\mathfrak{B}_M\mathcal{W}^s_\varphi(\Omega)$. Then, $\mathcal{S}g(u_M)$ is the unique set which is s-minimal in Ω^M with respect to the exterior data $\mathcal{S}g(\varphi)\setminus\Omega^M$.*

PROOF. Let $E\subseteq\mathbb{R}^{n+1}$ be s-minimal in Ω^M, with respect to the exterior data $\mathcal{S}g(\varphi)\setminus\Omega^M$—we know that such a set exists by [21, Theorem 3.2]. Let w_E be the function defined in (4.6) and notice that the set E satisfies the hypothesis of Theorem 4.1.10. Hence,

$$\mathrm{Per}_s(\mathcal{S}g(w_E),\Omega^M)\le\mathrm{Per}_s(E,\Omega^M).\tag{4.98}$$

As a consequence, we conclude that $E=\mathcal{S}g(w_E)$, since otherwise the inequality (4.98) would be strict, thus contradicting the minimality of E. Recalling (4.32), we have in particular that $w_E\in\mathfrak{B}_M\mathcal{W}^s_\varphi(\Omega)$. Then, by identity (4.33) and exploiting both the minimality of u_M and of E, we find that

$$0\ge\mathcal{F}_s^M(u_M,\Omega^M)-\mathcal{F}_s^M(w_E,\Omega^M)=\mathrm{Per}_s(\mathcal{S}g(u_M),\Omega^M)-\mathrm{Per}_s(\mathcal{S}g(w_E),\Omega^M)\ge0.$$

Since u_M is the unique minimizer of $\mathcal{F}_s^M(\,\cdot\,,\Omega)$ within $\mathfrak{B}_M\mathcal{W}^s_\varphi(\Omega)$, this implies that $u_M=w_E$, concluding the proof. □

4.5.2. Proof of Theorem 4.1.3. Proposition 4.5.8 shows that, for each $M > 0$, there exists a unique minimizer u_M of $\mathcal{F}^M(\,\cdot\,,\Omega)$ within the space $\mathfrak{B}_M\mathcal{W}^s_\varphi(\Omega)$. To establish the existence of a minimizer of $\mathcal{F}$, we now need u_M to converge as $M \to \infty$. This is achieved through the uniform $W^{s,1}$ estimate of Proposition 4.5.9, at the price of assuming some (weighted) integrability on the exterior datum in a sufficiently large neighborhood of Ω. The minimality of the limit function u is then obtained as a consequence of the minimality of the functions u_M.

Proof of Theorem 4.1.3. Let $\Theta > 1$ be the constant given by Proposition 4.5.9. For any $M > 0$, the minimizer u_M satisfies the hypotheses of Proposition 4.5.9. Therefore,

$$\|u_M\|_{W^{s,1}(\Omega)} \leq C\left(\left\|\mathrm{Tail}_s(\varphi, \Omega_{\Theta\,\mathrm{diam}(\Omega)} \setminus \Omega;\,\cdot\,)\right\|_{L^1(\Omega)} + 1\right), \tag{4.99}$$

for some constant $C > 0$ depending only on n, s, g and Ω, and, in particular, independent of M. By the compact fractional Sobolev embedding (see, e.g., [38, Theorem 7.1]), we conclude that there exists a function $u \in \mathcal{W}^s_\varphi(\Omega)$ to which $\{u_{M_j}\}$ converges in $L^1(\Omega)$ and a.e. in Ω, for some diverging sequence $\{M_j\}_{j\in\mathbb{N}}$. Letting $M = M_j \to +\infty$ in (4.99), by Fatou's Lemma we see that u satisfies (4.5). We are therefore left to show that u is a minimizer for $\mathcal{F}$ in $\mathcal{W}^s_\varphi(\Omega)$.

Take $v \in \mathfrak{B}\mathcal{W}^s_\varphi(\Omega)$. Then, for j large enough we have $M_j \geq \|v\|_{L^\infty(\Omega)}$, and hence, by the minimality of u_{M_j} we get $\mathcal{F}^{M_j}(u_{M_j},\Omega) \leq \mathcal{F}^{M_j}(v,\Omega)$. That is,

$$\begin{aligned}
0 \geq \mathcal{A}(u_{M_j}) &+ 2\int_\Omega \left\{\int_{\Omega_R\setminus\Omega} \mathcal{G}\left(\frac{u_{M_j}(x)-\varphi(y)}{|x-y|}\right)\frac{dy}{|x-y|^{n-1+s}}\right\}dx \\
&- \mathcal{A}(v) - 2\int_\Omega \left\{\int_{\Omega_R\setminus\Omega} \mathcal{G}\left(\frac{v(x)-\varphi(y)}{|x-y|}\right)\frac{dy}{|x-y|^{n-1+s}}\right\}dx \\
&+ 2\int_\Omega \left\{\int_{\mathcal{C}\Omega_R} \left[\mathcal{G}\left(\frac{u_{M_j}(x)-\varphi(y)}{|x-y|}\right) - \mathcal{G}\left(\frac{v(x)-\varphi(y)}{|x-y|}\right)\right]\frac{dy}{|x-y|^{n-1+s}}\right\}dx,
\end{aligned}$$

for any fixed $R \in (0, \Theta\,\mathrm{diam}(\Omega)]$. We now claim that letting $j \to +\infty$ in the above formula, we obtain the same inequality with u_{M_j} replaced by u.

Indeed, the quantities on the first line can be dealt with by using Fatou's lemma. Moreover, the Lipschitz character of $\mathcal{G}$—see (4.21)—and the fact that $u_{M_j} \to u$ in $L^1(\Omega)$ ensure that

$$\begin{aligned}
\int_\Omega \left\{\int_{\mathcal{C}\Omega_R} \left|\mathcal{G}\left(\frac{u_{M_j}(x)-\varphi(y)}{|x-y|}\right) - \mathcal{G}\left(\frac{u(x)-\varphi(y)}{|x-y|}\right)\right|\frac{dy}{|x-y|^{n-1+s}}\right\}dx& \\
\leq \frac{\Lambda}{2}\int_\Omega |u_{M_j}(x)-u(x)|\left\{\int_{\mathcal{C}\Omega_R}\frac{dy}{|x-y|^{n+s}}\right\}dx& \\
\leq CR^{-s}\|u_{M_j}-u\|_{L^1(\Omega)} \xrightarrow{j\to\infty} 0.&
\end{aligned}$$

Hence, the third line passes to the limit as well. All in all, we have proved that u minimizes $\mathcal{F}$ in $\mathfrak{B}\mathcal{W}^s_\varphi(\Omega)$. The minimality of u within the larger class $\mathcal{W}^s_\varphi(\Omega)$ follows from the density of $L^\infty(\Omega)\cap W^{s,1}(\Omega)$ in $W^{s,1}(\Omega)$ and Lemma 4.5.1—see point *(iv)* of Remark 4.5.2. To conclude, the uniqueness of the minimizer follows by point *(iii)* of Remark 4.5.2. □

4.5.3. Boundedness results. The purpose of this section consists in proving that minimizers of $\mathcal{F}$ are always locally bounded and that they are globally bounded if the exterior data is bounded near the boundary of the domain Ω.

More precisely, by exploiting a Stampacchia-type argument, we prove the following result:

PROPOSITION 4.5.12. *Let $n \geq 1$, $s \in (0,1)$, $R > 0$, and $u \in \mathcal{W}^s(B_{2R})$ be a minimizer of $\mathcal{F}$ in B_{2R}. Then,*

$$\sup_{B_R} u \leq C \left(R + \fint_{B_{2R}} u_+(x)\, dx \right),$$

for some constant $C > 0$ depending only on n, s and g.

Clearly, Proposition 4.5.12 implies that if $u \in \mathcal{W}^s(\Omega)$ is a minimizer of $\mathcal{F}$ in Ω, then $u \in L^\infty_{\mathrm{loc}}(\Omega)$. Since the proof is rather lenghty and technical, we postpone it to Section 4.5.3.1.

Moreover, we prove that a minimizer u of $\mathcal{F}$ in Ω belongs to $L^\infty(\Omega)$, provided it is bounded, outside Ω, in a sufficiently large neighborhood of Ω. Furthermore, we obtain an apriori estimate on the $L^\infty(\Omega)$ norm of u purely in terms of the exterior data. That is, we show the validity of Theorem 4.1.4 of the Introduction.

We establish this result by showing that, given any function $u : \mathbb{R}^n \to \mathbb{R}$, bounded in $B_R \setminus \Omega$ for some large $R > 0$, the value $\mathcal{F}^M(u, \Omega)$ decreases when u is truncated at a high enough level. This last statement can be made precise as follows.

For $N \geq 0$, we define

$$u^{(N)} := \begin{cases} \min\{u, N\} & \text{in } \Omega, \\ u & \text{in } \mathcal{C}\Omega. \end{cases}$$

Then, we have the following result.

PROPOSITION 4.5.13. *Let $n \geq 1$, $s \in (0,1)$, and $M \geq 0$. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $R_0 > 0$ be such that $\Omega \subseteq B_{R_0}$. Then, there exists a large constant $\Theta > 1$, depending only on n, s and g, such that for every function $u : \mathbb{R}^n \to \mathbb{R}$ bounded from above in $B_{\Theta R_0} \setminus \Omega$, it holds*

$$\mathcal{A}(u^{(N)}, \Omega) \leq \mathcal{A}(u, \Omega) \quad \text{and} \quad \mathcal{N}^M(u^{(N)}, \Omega) \leq \mathcal{N}^M(u, \Omega) \tag{4.100}$$

for every

$$N \geq R_0 + \sup_{B_{\Theta R_0} \setminus \Omega} u. \tag{4.101}$$

In particular,

$$\mathcal{F}^M(u^{(N)}, \Omega) \leq \mathcal{F}^M(u, \Omega)$$

for every N satisfying (4.101).

We observe that Proposition 4.5.13 directly implies Theorem 4.1.4, thanks to the uniqueness of the minimizer, which is a consequence of the strict convexity of $\mathcal{F}^M$—see point *(iii)* of Remark 4.5.2.

By exploiting the interior local boundedness and by appropriately modifying the proof of Proposition 4.5.13, we are able to prove that, in order to ensure the global boundedness of a minimizer of $\mathcal{F}$, it is actually enough that u be bounded outside the domain Ω in an arbitrarily small neighborhood of the boundary. However, we remark that in this case, in general, we do not have a clean a priori bound on the L^∞ norm.

We first recall that if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set with C^2 boundary, then there exists $r_0(\Omega) > 0$ such that Ω satisfies a uniform strict interior and strict exterior ball condition of radius $2r_0$. Then, if $\bar{d}_\Omega$ denotes the signed distance function from $\partial\Omega$, negative inside Ω, we have that $\bar{d}_\Omega \in C^2\big(N_{2r_0}(\partial\Omega)\big)$, with

$$N_\varrho(\partial\Omega) := \{x \in \mathbb{R}^n \,|\, d(x, \partial\Omega) < \varrho\} = \big\{|\bar{d}_\Omega| < \varrho\big\} \quad \forall\, \varrho > 0.$$

For the details, we refer to Appendix B.1.1—see in particular Remark B.1.3.

The precise result is the following:

THEOREM 4.5.14. *Let $n \geq 1$, $s \in (0,1)$ and $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^2 boundary. If $u \in \mathcal{W}^s(\Omega)$ is a minimizer of $\mathcal{F}$ in Ω and $u \in L^\infty(\Omega_d \setminus \Omega)$, for some $d \in (0, r_0)$, then $u \in L^\infty(\Omega)$, with*

$$\|u\|_{L^\infty(\Omega\setminus\Omega_{-\theta d})} \leq d + \max\left\{\|u\|_{L^\infty(\Omega_{-\theta d})}, \|u\|_{L^\infty(\Omega_d\setminus\Omega)}\right\},$$

where $\theta = \theta(n, s, g) \in (0,1)$ is a small positive constant.

We observe that if we further assume that $\mathrm{Tail}_s(u, \Omega_{\Theta\,\mathrm{diam}(\Omega)} \setminus \Omega; \,\cdot\,) \in L^1(\Omega)$, then, by exploiting both the apriori L^1 estimate of Proposition 4.5.9, the estimate on $\|u\|_{L^\infty(\Omega_{-\theta d})}$ given by Proposition 4.5.12—together with a covering argument—and the estimate provided by Theorem 4.5.14, we can obtain an apriori estimate on $\|u\|_{L^\infty(\Omega)}$ purely in terms of the exterior data and of the geometry of Ω.

We now proceed with the proofs of the aforementioned results.

To prove Proposition 4.5.13, we will make use of a couple of simple lemmas. First, we have the following elementary result on convex functions.

LEMMA 4.5.15. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be a convex function. Then, for every $A, B, C, D \in \mathbb{R}$ satisfying $\min\{C, D\} \leq A, B \leq \max\{C, D\}$ and $A + B = C + D$, it holds*

$$\phi(A) + \phi(B) \leq \phi(C) + \phi(D).$$

PROOF. Without loss of generality, we may suppose that $A \leq B$ and $C \leq D$. Since we have that $C \leq A \leq B \leq D$, there exist two values $\lambda, \mu \in [0,1]$ such that

$$A = \lambda C + (1-\lambda)D \quad \text{and} \quad B = \mu C + (1-\mu)D.$$

In view of the convexity of ϕ, it holds

$$\begin{aligned} \phi(A) + \phi(B) &= \phi(\lambda C + (1-\lambda)D) + \phi(\mu C + (1-\mu)D) \\ &\leq \lambda\phi(C) + (1-\lambda)\phi(D) + \mu\phi(C) + (1-\mu)\phi(D) \\ &= (\lambda+\mu)\,\phi(C) + (2-\lambda-\mu)\,\phi(D). \end{aligned} \tag{4.102}$$

By taking advantage of the fact that $A + B = C + D$, we now observe that

$$\lambda C + (1-\lambda)D + \mu C + (1-\mu)D = C + D,$$

or, equivalently,

$$(1-\lambda-\mu)(C-D) = 0.$$

Consequently, either $C = D$ or $\lambda + \mu = 1$ (or both). In any case, we conclude that the right-hand side of (4.102) is equal to $\phi(C) + \phi(D)$, and from this the thesis follows. □

We use Lemma 4.5.15 to obtain the following inequality for rather general convex functionals. In our later applications, we will simply take $F(U; x, y) = \mathcal{G}(U/|x-y|)$.

LEMMA 4.5.16. *Let $F : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be a measurable function, convex with respect to the first variable, i.e. satisfying*

$$F(\lambda u + (1-\lambda)v; x, y) \leq \lambda F(u; x, y) + (1-\lambda)F(v; x, y) \tag{4.103}$$

for every $\lambda \in (0,1)$, $u, v \in \mathbb{R}$, and for a.e. $x, y \in \mathbb{R}^n$. Given a measurable set $\mathcal{U} \subseteq \mathbb{R}^n \times \mathbb{R}^n$, consider the functional $\mathcal{F}$ defined by

$$\mathcal{F}(w) := \iint_{\mathcal{U}} F(u(x) - u(y); x, y)\, dx\, dy$$

for every $w : \mathbb{R}^n \to \mathbb{R}$. Then, for every $u, v : \mathbb{R}^n \to \mathbb{R}$, it holds

$$\mathcal{F}(\min\{u, v\}) + \mathcal{F}(\max\{u, v\}) \leq \mathcal{F}(u) + \mathcal{F}(v). \tag{4.104}$$

Proof. For fixed $(x, y) \in \mathcal{U}$, we write

$$A := m(x) - m(y), \quad B := M(x) - M(y), \quad C := u(x) - u(y), \quad D := v(x) - v(y),$$

and

$$\phi(t) = \phi_{x,y}(t) := F(t; x, y) \quad \text{for every } t \in \mathbb{R}.$$

Thanks to (4.103), the function φ is convex. Also, we claim that

$$\min\{C, D\} \le A, B \le \max\{C, D\} \tag{4.105}$$

and

$$A + B = C + D. \tag{4.106}$$

Indeed, identity (4.106) is immediate since $m+M \equiv u+v$. The inequalities in (4.105) are also obvious if $u(x) \le v(x)$ and $u(y) \le v(y)$ or if $u(x) > v(x)$ and $u(y) > v(y)$. On the other hand, when for example $u(x) \le v(x)$ and $u(y) > v(y)$, we have

$$A = u(x) - v(y) \quad \text{and} \quad B = v(x) - u(y).$$

Accordingly,

$$C = u(x) - u(y) < u(x) - v(y) = A = u(x) - v(y) \le v(x) - v(y) = D$$

and

$$C = u(x) - u(y) \le v(x) - u(y) = B = v(x) - u(y) < v(x) - v(y) = D.$$

Hence, (4.105) is proved in this case. Arguing analogously, one can check that (4.105) also holds when $u(x) > v(x)$ and $u(y) \le v(y)$.

Thanks to (4.105) and (4.106), we may apply Lemma 4.5.15 and deduce that

$$\phi(A) + \phi(B) \le \phi(C) + \phi(D).$$

That is,

$$\begin{aligned} F(m(x) - m(y); x, y) &+ F(M(x) - M(y); x, y) \\ &\le F(u(x) - u(y); x, y) + F(v(x) - v(y); x, y). \end{aligned}$$

Inequality (4.104) then plainly follows by integrating the last formula in x and y. □

With the aid of this last result, we can proceed to check the validity of Proposition 4.5.13.

Proof of Proposition 4.5.13. Write $v := u^{(N)}$ and $R := \Theta R_0$, with $\Theta \ge 2$ to be chosen later sufficiently large, in dependence of n, s and g only. From Lemma 4.5.16, it clearly follows that $\mathcal{A}(v, \Omega) \le \mathcal{A}(u, \Omega)$. Hence, we can focus on the inequality for the nonlocal part $\mathcal{N}^M$.

Thanks to representation (4.28), we have

$$\mathcal{N}^M(v, \Omega) - \mathcal{N}^M(u, \Omega) = 2 \int_\Omega \int_{\mathcal{C}\Omega} \left[\mathcal{G}\left(\frac{v(x) - v(y)}{|x-y|} \right) - \mathcal{G}\left(\frac{u(x) - u(y)}{|x-y|} \right) \right] \frac{dx\, dy}{|x-y|^{n-1+s}}.$$

Setting $\Omega_+ := \{x \in \Omega \,|\, u(x) > N\}$ and writing $\mathcal{C}\Omega = A_1 \cup A_2$, with $A_1 := B_R \setminus \Omega$ and $A_2 := \mathcal{C}B_R$, we infer from the above identity that the second inequality in (4.100) is equivalent to

$$\alpha_1 + \alpha_2 \le 0, \tag{4.107}$$

where we set

$$\alpha_i := \int_{\Omega_+} \left\{ \int_{A_i} \left[\mathcal{G}\left(\frac{N - u(y)}{|x-y|} \right) - \mathcal{G}\left(\frac{u(x) - u(y)}{|x-y|} \right) \right] \frac{dy}{|x-y|^{n-1+s}} \right\} dx,$$

for $i = 1, 2$.

First, we establish a (negative) upper bound for α_1. Let $x \in \Omega_+$ and $y \in A_1$. Since, by hypothesis (4.101), $u(y) \leq N < u(x)$ and G is increasing, we have

$$\mathcal{G}\left(\frac{N-u(y)}{|x-y|}\right) - \mathcal{G}\left(\frac{u(x)-u(y)}{|x-y|}\right) = \int_{\frac{u(x)-u(y)}{|x-y|}}^{\frac{N-u(y)}{|x-y|}} G(t)\,dt \leq -G\left(\frac{N-u(y)}{|x-y|}\right)\frac{u(x)-N}{|x-y|},$$

and consequently

$$\alpha_1 \leq -\int_\Omega (u(x)-N)_+ \left[\int_{A_1} G\left(\frac{N-u(y)}{|x-y|}\right)\frac{dy}{|x-y|^{n+s}}\right] dx.$$

In view of the fact that $B_{2R_0} \setminus B_{R_0} \subseteq A_1$ (as $R \geq 2R_0$) and, again, (4.101) and the monotonicity of G, we estimate

$$\int_{A_1} G\left(\frac{N-u(y)}{|x-y|}\right)\frac{dy}{|x-y|^{n+s}} \geq G\left(\frac{R_0}{R_0+R_0}\right)\frac{|B_{2R_0}\setminus B_{R_0}|}{(R_0+R_0)^{n+s}} \geq \frac{c_1}{R_0^s}$$

for every $x \in \Omega$ and for some constant $c_1 > 0$ depending only on n, s and g. Accordingly,

$$\alpha_1 \leq -\frac{c_1}{R_0^s}\int_\Omega (u(x)-N)_+\,dx. \tag{4.108}$$

On the other hand, to control α_2 we simply use that $\mathcal{G}$ is a globally Lipschitz function—see (4.21)—and compute

$$\begin{aligned}
\alpha_2 &\leq \frac{\Lambda}{2}\int_\Omega (u(x)-N)_+\left(\int_{\mathbb{R}^n\setminus B_R}\frac{dy}{|x-y|^{n+s}}\right)dx\\
&\leq \frac{\Lambda}{2}\int_\Omega (u(x)-N)_+\left(\int_{\mathbb{R}^n\setminus B_{R/2}}\frac{dz}{|z|^{n+s}}\right)dx \leq \frac{C_2}{R^s}\int_\Omega (u(x)-N)_+\,dx,
\end{aligned}$$

for some constant $C_2 > 0$ depending only on n, s and g. Notice that to get the second inequality we changed variables and took advantage of the inclusion $B_{R/2}(x) \subseteq B_R$, which holds for all $x \in \Omega \subseteq B_{R_0}$ since $R \geq 2R_0$. Combining this last estimate with (4.108), we obtain

$$\alpha_1 + \alpha_2 \leq -\left(\frac{c_1}{R_0^s} - \frac{C_2}{R^s}\right)\int_\Omega (u(x)-N)_+\,dx,$$

and (4.107) follows provided we take $R \geq (C_2/c_1)^{1/s}R_0$. □

A suitable modification of the proof of Proposition 4.5.13 allows us to obtain Theorem 4.5.14.

Proof of Theorem 4.5.14. We recall that

$$u^{(N)} := \chi_\Omega \min\{u, N\} + (1-\chi_\Omega)u.$$

We consider $u^{(N)}$ with

$$N \geq d + \max\left\{\sup_{\Omega_{-\theta d}} u, \sup_{\Omega_d\setminus\Omega} u\right\}, \tag{4.109}$$

where $\theta \leq 1/4$ will be chosen suitably small later. We fix $M \geq 0$ and we prove that

$$\mathcal{F}^M(u^{(N)}, \Omega) \leq \mathcal{F}^M(u, \Omega).$$

We remark that the analogous estimate holds true when we cut u from below, inside Ω. Hence, by the minimality of u and the uniqueness of the minimizer—see points (ii) and (iii) of Remark 4.5.2—this implies the claim of the Theorem.

By arguing as in the proof of Proposition 4.5.13, we are left to prove that

$$I := \int_{\Omega_+} \left\{ \int_{\mathcal{C}\Omega} \left[\mathcal{G}\left(\frac{N-u(y)}{|x-y|}\right) - \mathcal{G}\left(\frac{u(x)-u(y)}{|x-y|}\right)\right] \frac{dy}{|x-y|^{n-1+s}} \right\} dx \le 0,$$

where $\Omega_+ := \{\xi \in \Omega \,|\, u(\xi) > N\}$. It is important to observe that by (4.109) we have

$$\Omega_+ \subseteq \Omega \setminus \Omega_{-\theta d} = \{\xi \in \Omega \,|\, d(\xi, \partial\Omega) \le \theta d\}.$$

As a consequence, since $\theta \le 1/4$, for every $x \in \Omega_+$ we can find a point $z_x \in \Omega_d \setminus \Omega$ such that

$$B_{\frac{\theta d}{2}}(z_x) \subseteq B_{3\theta d}(x) \setminus \Omega \subseteq B_d(x) \setminus \Omega \subseteq \Omega_d \setminus \Omega. \tag{4.110}$$

This is a consequence of the uniform interior and exterior ball conditions satisfied by Ω. More precisely, we observe that

$$p := x - \bar{d}_\Omega(x) \nabla \bar{d}_\Omega(x) \in \partial\Omega,$$

is the unique closest point to x. That is, p is the unique point on $\partial\Omega$ such that $|x-p| = d(x, \partial\Omega)$. Then, Ω has an exterior tangent ball of radius r_0 at p. Notice that the center of the ball is obtained by moving in direction $\nu_\Omega(p) = \nabla \bar{d}_\Omega(x)$ of a distance r_0. Hence, if we move only of a distance $\theta d/2$, we obtain the desired ball. All in all, we can write explicitely

$$z_x := x + \left(\frac{\theta d}{2} - \bar{d}_\Omega(x)\right) \nabla \bar{d}_\Omega(x).$$

For the details about this kind of geometric considerations concerning the signed distance function, see Appendix B.1.1.

Now we split $I = I_1 + I_2$, with

$$I_1 := \int_{\Omega_+} \left\{ \int_{B_d(x)\setminus\Omega} \left[\mathcal{G}\left(\frac{N-u(y)}{|x-y|}\right) - \mathcal{G}\left(\frac{u(x)-u(y)}{|x-y|}\right)\right] \frac{dy}{|x-y|^{n-1+s}} \right\} dx,$$

and

$$I_2 := \int_{\Omega_+} \left\{ \int_{\mathcal{C}B_d(x)\setminus\Omega} \left[\mathcal{G}\left(\frac{N-u(y)}{|x-y|}\right) - \mathcal{G}\left(\frac{u(x)-u(y)}{|x-y|}\right)\right] \frac{dy}{|x-y|^{n-1+s}} \right\} dx.$$

Since $\mathcal{G}$ is globally Lipschitz—see (4.21)—we have

$$I_2 \le \frac{\Lambda}{2} \int_{\Omega_+} \big(u(x) - N\big) \left(\int_{\mathcal{C}B_d(x)} \frac{dy}{|x-y|^{n+s}} \right) dx = \frac{\Lambda \mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{2s} d^{-s}. \tag{4.111}$$

As for I_1, let $x \in \Omega_+$ and $y \in B_d(x) \setminus \Omega$. Since by (4.109) we have $u(y) \le N < u(x)$ and G is increasing, we obtain

$$I_1 \le -\int_{\Omega_+} \big(u(x) - N\big) \left[\int_{B_d(x)\setminus\Omega} G\left(\frac{N-u(y)}{|x-y|}\right) \frac{dy}{|x-y|^{n+s}} \right] dx.$$

Exploiting (4.109) and the monotonicity of G, we see that

$$G\left(\frac{N-u(y)}{|x-y|}\right) \ge G\left(\frac{d}{d}\right) = G(1) > 0,$$

for every $x \in \Omega_+$ and $y \in B_d(x) \setminus \Omega$. Recalling (4.110) we thus obtain

$$\begin{aligned}\int_{\Omega_+} \big(u(x) - N\big) \left[\int_{B_d(x)\setminus\Omega} G\left(\frac{N-u(y)}{|x-y|}\right) \frac{dy}{|x-y|^{n+s}} \right] dx \\ \geq G(1) \int_{\Omega_+} \big(u(x) - N\big) \left[\int_{B_{\frac{\theta d}{2}}(z_x)} \frac{dy}{|x-y|^{n+s}} \right] dx \\ \geq G(1) \frac{\big|B_{\frac{\theta d}{2}}(z_x)\big|}{(3\theta d)^{n+s}} \int_{\Omega_+} \big(u(x) - N\big)\, dx \\ = \frac{G(1)|B_1|}{2^n 3^{n+s}} (\theta d)^{-s} \int_{\Omega_+} \big(u(x) - N\big)\, dx.\end{aligned}$$

Therefore, using also (4.111) we get

$$I \leq -\left(\frac{G(1)|B_1|}{2^n 3^{n+s}} \theta^{-s} - \frac{\Lambda \mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{2s} \right) d^{-s} \int_{\Omega_+} \big(u(x) - N\big)\, dx,$$

which is negative, provided we take θ small enough. This concludes the proof. $\square$

4.5.3.1. *Proof of the interior local boundedness.* We get now to the proof of Proposition 4.5.12.

Proof of Proposition 4.5.12. Let $0 < \varrho < \tau \leq 2R$ and $\eta \in C^\infty(\mathbb{R}^n)$ be a cutoff function satisfying $0 \leq \eta \leq 1$ in $\mathbb{R}^n$, $\mathrm{supp}(\eta) \Subset B_\tau$, $\eta = 1$ in B_ϱ and $|\nabla\eta| \leq 2/(\tau - \varrho)$ in $\mathbb{R}^n$. For $k \geq 0$, we consider the functions $w = w_k := (u-k)_+$ and $v := u - \eta w$. Clearly, $v = u$ in $\mathcal{C}B_\tau$ and therefore

$$\iint_{Q(B_\tau)} \frac{H(x,y)}{|x-y|^{n-1+s}}\, dx\, dy \geq 0, \tag{4.112}$$

with

$$H(x,y) := \mathcal{G}\left(\frac{v(x)-v(y)}{|x-y|}\right) - \mathcal{G}\left(\frac{u(x)-u(y)}{|x-y|}\right).$$

We consider the sets $A(k) := \{x \in \mathbb{R}^n \,|\, u > k\}$ and $A(k,t) := B_t \cap A(k)$, for $t > 0$. First of all, we claim that

$$H(x,y) \leq -\frac{\Lambda}{2} \frac{|w(x)-w(y)|}{|x-y|} + \lambda\, \chi_{B_\varrho^2 \setminus (B_\varrho \setminus A(k,\varrho))^2}(x,y) \quad \text{for } x, y \in B_\varrho, \tag{4.113}$$

with λ and Λ as defined in (4.11) and (4.12) respectively. Clearly, (4.113) holds for every $x, y \in \mathcal{C}A(k)$, since $H(x,y) = 0$ for these points and $w = 0$ in $\mathcal{C}A(k)$. Furthermore, it is also valid for $x, y \in A(k,\varrho)$, as indeed, by (4.20) we have

$$H(x,y) = -\mathcal{G}\left(\frac{u(x)-u(y)}{|x-y|}\right) = -\mathcal{G}\left(\frac{w(x)-w(y)}{|x-y|}\right) \leq -\frac{\Lambda}{2} \frac{|w(x)-w(y)|}{|x-y|} + \lambda.$$

By symmetry, we are left to check (4.113) for $x \in A(k,\varrho)$ and $y \in B_\varrho \setminus A(k,\varrho)$. In this case, using $u(x) > k \geq u(y)$ along with (4.20), we get

$$\begin{aligned} H(x,y) &= \mathcal{G}\left(\frac{k-u(y)}{|x-y|}\right) - \mathcal{G}\left(\frac{u(x)-u(y)}{|x-y|}\right) \leq \frac{\Lambda}{2}\left(\frac{k-u(y)}{|x-y|} - \frac{u(x)-u(y)}{|x-y|}\right) + \lambda \\ &= -\frac{\Lambda}{2} \frac{u(x)-k}{|x-y|} + \lambda = -\frac{\Lambda}{2} \frac{|w(x)-w(y)|}{|x-y|} + \lambda. \end{aligned}$$

Hence, (4.113) is verified.

We now claim that

(4.114)
$$H(x,y) \leq \frac{\Lambda}{2}\left(\chi_{B_\tau^2}\frac{|w(x)-w(y)|}{|x-y|} + \frac{w(x)}{\max\{\tau-\varrho, |x-y|\}}\right) \quad \text{for } x \in B_\tau,\ y \in \mathcal{C}B_\varrho.$$

We already observed that $H(x,y) = 0$ for every $x, y \in \mathcal{C}A(k)$. When $x \in B_\tau \setminus A(k,\tau)$ and $y \in A(k)$, then

$$\begin{aligned} u(y)-u(x) &\geq u(y)-u(x)-\eta(y)(u(y)-k) = (1-\eta(y))u(y)+k\eta(y)-u(x) \\ &\geq (1-\eta(y))u(y)+k\eta(y)-k = (1-\eta(y))(u(y)-k) \geq 0 \end{aligned}$$

and therefore

$$H(x,y) = \mathcal{G}\left(\frac{u(y)-u(x)-\eta(y)(u(y)-k)}{|x-y|}\right) - \mathcal{G}\left(\frac{u(y)-u(x)}{|x-y|}\right) \leq 0,$$

by the monotonicity properties of $\mathcal{G}$. We are thus left to deal with $x \in A(k,\tau)$ and $y \in \mathcal{C}B_\varrho$. In this case, by the Lipschitz character of $\mathcal{G}$ and the properties of η,

$$\begin{aligned} H(x,y) &\leq \frac{\Lambda}{2}\frac{|\eta(x)w(x)-\eta(y)w(y)|}{|x-y|} \leq \frac{\Lambda}{2}\frac{\eta(y)|w(x)-w(y)|+w(x)|\eta(x)-\eta(y)|}{|x-y|} \\ &\leq \frac{\Lambda}{2}\left(\chi_{B_\tau^2}(y)\frac{|w(x)-w(y)|}{|x-y|} + \min\left\{\frac{1}{\tau-\varrho}, \frac{1}{|x-y|}\right\}w(x)\right), \end{aligned}$$

and (4.114) follows.

By taking advantage of estimates (4.113) and (4.114) in (4.112), by symmetry we deduce that

$$\begin{aligned} \iint_{B_\varrho^2} \frac{|w(x)-w(y)|}{|x-y|^{n+s}}\,dx\,dy \leq C\Bigg\{&\iint_{B_\tau^2\setminus B_\varrho^2}\frac{|w(x)-w(y)|}{|x-y|^{n+s}}\,dx\,dy + \int_{A(k,\varrho)}\int_{B_\varrho}\frac{dx\,dy}{|x-y|^{n-1+s}} \\ &+ \int_{B_\tau} w(x)\left(\frac{1}{\tau-\varrho}\int_{B_{\tau-\varrho}}\frac{dz}{|z|^{n-1+s}} + \int_{\mathcal{C}B_{\tau-\varrho}}\frac{dz}{|z|^{n+s}}\right)dx\Bigg\} \\ \leq C\Bigg\{&\iint_{B_\tau^2\setminus B_\varrho^2}\frac{|w(x)-w(y)|}{|x-y|^{n+s}}\,dx\,dy + |A(k,\varrho)|\varrho^{1-s} + \frac{\|w\|_{L^1(B_\tau)}}{(\tau-\varrho)^s}\Bigg\}, \end{aligned}$$

where for the second inequality we also used Lemma D.1.1. Adding to both sides C times the left-hand side and dividing by $1+C$, we get that

$$[w]_{W^{s,1}(B_\varrho)} \leq \theta\left([w]_{W^{s,1}(B_\tau)} + |A(k,\tau)|\tau^{1-s} + \frac{\|w\|_{L^1(B_\tau)}}{(\tau-\varrho)^s}\right)$$

for every $0 < \varrho < \tau \leq 2R$ and for some constant $\theta \in (0,1)$ depending only on n, s and g. Applying, e.g, [64, Lemma 1.1], we infer that

$$[w]_{W^{s,1}(B_{(\varrho+\tau)/2})} \leq C\left(|A(k,\tau)|\tau^{1-s} + \frac{\|w\|_{L^1(B_\tau)}}{(\tau-\varrho)^s}\right).$$

Let η be a cutoff acting between the balls B_ϱ and $B_{(3\varrho+\tau)/4}$. Then, by the fractional Sobolev inequality (see, e.g., [81, Theorem 1] or [38, Theorem 6.5]) and computations similar to other made previously, we have that

$$\begin{aligned} \|w\|_{L^{\frac{n}{n-s}}(B_\varrho)} \leq \|\eta w\|_{L^{\frac{n}{n-s}}(\mathbb{R}^n)} &\leq C\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|\eta(x)w(x)-\eta(y)w(y)|}{|x-y|^{n+s}}\,dx\,dy \\ &\leq C\left([w]_{W^{s,1}(B_{(\varrho+\tau)/2})} + \frac{\|w\|_{L^1(B_\tau)}}{(\tau-\varrho)^s}\right). \end{aligned}$$

Combining the last two inequalities and recalling that $w = w_k$, we arrive at

$$\|w_k\|_{L^{\frac{n}{n-s}}(B_\varrho)} \leq C\left(|A(k,\tau)|\tau^{1-s} + \frac{\|w_k\|_{L^1(B_\tau)}}{(\tau-\varrho)^s}\right) \tag{4.115}$$

for every $0 < r < \tau \leq 2R$ and $k \geq 0$.

Take now $k > h \geq 0$. We have

$$\|w_h\|_{L^1(B_\tau)} \geq \int_{A(k,\tau)} (u(x) - h)\, dx \geq (k-h)|A(k,\tau)|$$

and

$$\|w_h\|_{L^1(B_\tau)} \geq \int_{A(k,\tau)} (u(x) - h)\, dx \geq \int_{A(k,\tau)} (u(x) - k)\, dx = \|w_k\|_{L^1(B_\tau)}.$$

Thanks to these relations, (4.115), and Hölder's inequality, it is easy to see that

$$\varphi(k,\varrho) \leq \frac{C}{(k-h)^{s/n}}\left(\frac{\tau^{1-s}}{k-h} + \frac{1}{(\tau-\varrho)^s}\right)\varphi(h,\tau)^{1+\frac{s}{n}}, \tag{4.116}$$

where we set $\varphi(\ell, t) := \|w_\ell\|_{L^1(B_t)}$.

Consider two sequences $\{k_j\}$ and $\{r_j\}$ defined by $k_j := M(1-2^{-j})$ and $r_j := R(1+2^{-j})$ for every non-negative integer j, where $M > 0$ will be chosen later. By applying (4.116) with $k = k_{j+1}$, $\varrho = r_{j+1}$, $h = k_j$, and $\tau = r_j$, setting $\varphi_j := \varphi(k_j, r_j)$, and taking $M \geq R$, we find

$$\varphi_{j+1} \leq \frac{C(2^j \varphi_j)^{1+\frac{s}{n}}}{(R\sqrt[n]{M})^s}.$$

Applying now, e.g., [69, Lemma 7.1], we conclude that φ_j converges to 0—i.e., $u \leq M$ in B_R—, provided we choose M in such a way that

$$\|u_+\|_{L^1(B_{2R})} = \varphi_0 \leq c_\sharp R^n M,$$

for some constant $c_\sharp > 0$ depending only on n, s and g. This concludes the proof. □

4.5.4. Geometric minimizers. This section is concerned with the minimizers of the geometric situation, which corresponds to the choice $g = g_s$. Here, we provide the proofs of Theorems 4.1.11 and 4.1.13, which we have stated in the Introduction.

We begin by proving the equivalence between funtional minimizers, geometric minimizers, and the various notions of solutions to the equation $\mathscr{H}_s u = 0$ in Ω. This result is the consequence of the main theorems proved in this chapter, together with the interior regularity ensured by [19].

Proof of Theorem 4.1.11. $(i) \implies (ii)$ follows by Lemma 4.5.4.

As for the implication $(ii) \implies (iii)$, by Proposition 4.5.12 we know that $u \in L^\infty_{\mathrm{loc}}(\Omega)$. Then, let $\Omega_k \Subset \Omega$ be a sequence of bounded open sets with Lipschitz boundary, such that

$$\Omega_k \Subset \Omega_{k+1} \quad \text{and} \quad \bigcup_{k\in\mathbb{N}} \Omega_k = \Omega,$$

let M_k be a diverging sequence such that

$$M_k \geq \|u\|_{L^\infty(\Omega_k)}, \tag{4.117}$$

and consider the cylinders $\mathcal{O}^k := \Omega_k \times (-M_k, M_k)$. We prove that $\mathcal{S}g(u)$ is s-minimal in every $\mathcal{O}^k$. Since $\mathcal{O}^k \nearrow \Omega^\infty$, this readily implies that $\mathcal{S}g(u)$ is locally s-minimal in Ω^∞, as wanted.

Let $E \subseteq \mathbb{R}^{n+1}$ be such that $E \setminus \mathcal{O}^k = \mathcal{S}g(u) \setminus \mathcal{O}^k$ and let w_E be the function defined in (4.6). We can suppose that $\mathrm{Per}_s(E, \mathcal{O}^k) < \infty$, otherwise there is nothing to prove.

Then, by (4.117), we know that the set E satisfies (4.7) and hence Theorem 4.1.10 implies that

$$\mathrm{Per}_s(\mathcal{S}g(w_E),\mathcal{O}^k) \le \mathrm{Per}_s(E,\mathcal{O}^k). \tag{4.118}$$

Notice that, by Proposition 4.2.8, we have $w_E \in \mathfrak{B}_{M_k}\mathcal{W}^s_u(\Omega_k)$. Thus, since also $u \in \mathfrak{B}_{M_k}\mathcal{W}^s(\Omega_k)$, by identity (4.33), by the minimality of u and recalling (4.118), we obtain

$$\mathrm{Per}_s(\mathcal{S}g(u),\mathcal{O}^k) \le \mathrm{Per}_s(\mathcal{S}g(w_E),\mathcal{O}^k) \le \mathrm{Per}_s(E,\mathcal{O}^k).$$

The arbitrariness of the set E implies that $\mathcal{S}g(u)$ is s-minimal in $\mathcal{O}^k$, as claimed.

Now we prove that $(iii) \implies (iv)$. First of all, we observe that [19, Theorem 1.1] guarantees that $u \in C^\infty(\Omega)$. Therefore, given any $x \in \Omega$, we can find both an interior and an exterior tangent ball to $\mathcal{S}g(u)$ at the boundary point $(x,u(x)) \in \partial\mathcal{S}g(u) \cap \Omega^\infty$. The Euler-Lagrange equation satisfied by s-minimal sets—see [21, Theorem 5.1]—and identity (4.36) then imply that

$$\mathscr{H}_s u(x) = H_s[\mathcal{S}g(u)](x,u(x)) = 0.$$

The implication $(iv) \implies (i)$ follows from Proposition 4.3.15. Indeed, given $v \in C^\infty_c(\Omega)$, we can find a bounded open set Ω' such that

$$supp\, v \Subset \Omega' \Subset \Omega.$$

Then, since u is smooth in Ω, we have $u \in BH(\Omega')$ and hence Proposition 4.3.15 implies that

$$\langle \mathscr{H}_s u, v\rangle = \int_{\Omega'} \mathscr{H}_s u(x) v(x)\, dx = 0.$$

We observe that $(iv) \implies (v)$ always holds true, thanks to Remark 4.3.1. Finally, if we assume that $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then we have implication $(v) \implies (i)$ by Theorem 4.1.7. This concludes the proof of the Theorem. □

We observe that Corollary 4.1.12 is a straightforward consequence of Theorem 4.1.11.

We pass to the proof of the uniqueness of the locally s-minimal set with exterior data given by the subgraph of a function that is bounded in a big enough neighborhood of Ω.

Proof of Theorem 4.1.13. By [43, Lemma 3.3] we know that

$$\Omega \times (-\infty,-M_0) \subseteq E \cap \Omega^\infty \subseteq \Omega \times (-\infty, M_0).$$

We observe that, since E is locally s-minimal in Ω^∞ and Ω has regular boundary, by Theorem 2.1.7 and Remark 2.5.1 we know that E is s-minimal in Ω^{M_0}. In particular, we have $\mathrm{Per}_s(E,\Omega^{M_0}) < \infty$. Moreover, since E satisfies the hypothesis of Theorem 4.1.10, we get

$$\mathrm{Per}_s(\mathcal{S}g(w_E),\Omega^{M_0}) \le \mathrm{Per}_s(E,\Omega^{M_0}),$$

and $u := w_E \in \mathfrak{B}_{M_0}\mathcal{W}^s_\varphi(\Omega)$. The s-minimality of E implies that $E = \mathcal{S}g(u)$, since otherwise the inequality would be strict. Thus, $\mathcal{S}g(u)$ is locally s-minimal in Ω^∞ and, by Theorem 4.1.11, u minimizes $\mathcal{F}_s$ in $\mathcal{W}^s_\varphi(\Omega)$. The conclusion then follows from the uniqueness of such minimizer. □

4.6. Nonparametric Plateau problem with obstacles

In this section we consider the Plateau problem with (eventually discontinuous) obstacles. Namely, besides imposing the exterior data condition

$$u = \varphi \quad \text{a.e. in } \mathcal{C}\Omega,$$

we constrain the functions to lie above a fixed function which acts as an obstacle, that is

$$u \ge \psi \quad \text{a.e. in } A,$$

where $A \subseteq \Omega$ is a fixed open set.

We stress that the purpose of the present section is only that of showing that the functional setting introduced in the previous sections can be easily adapted to study the obstacle problem, so we do not aim at full generality in the statements nor in the proofs of our results. In particular, we limit ourselves to consider bounded obstacles and we prove the existence of a solution only in the case where the exterior data is bounded in a big enough neighborhood of the domain Ω. Furthermore, we will not investigate the regularity properties of such a solution and of the free boundary.

We begin by introducing appropriate functional spaces. Given a bounded open set $\Omega \subseteq \mathbb{R}^n$ with Lipschitz boundary, $s \in (0,1)$, an open set $A \subseteq \Omega$, an obstacle function $\psi \in L^\infty(A)$, the exterior data $\varphi : \mathcal{C}\Omega \to \mathbb{R}$, and $M \geq \|\psi\|_{L^\infty(A)}$, we define the spaces

$$\begin{aligned}
\mathcal{K}^s(\Omega,\varphi,A,\psi) &:= \{u \in \mathcal{W}^s_\varphi(\Omega) \,|\, u \geq \psi \text{ a.e. in } A\},\\
\mathfrak{B}\mathcal{K}^s(\Omega,\varphi,A,\psi) &:= \{u \in \mathfrak{B}\mathcal{W}^s_\varphi(\Omega) \,|\, u \geq \psi \text{ a.e. in } A\},\\
\mathfrak{B}_M\mathcal{K}^s(\Omega,\varphi,A,\psi) &:= \mathcal{K}^s(\Omega,\varphi,A,\psi) \cap \mathfrak{B}_M\mathcal{W}^s(\Omega).
\end{aligned}$$

We say that a function $u \in \mathcal{K}^s(\Omega,\varphi,A,\psi)$ solves the obstacle problem if u minimizes $\mathcal{F}$ in $\mathcal{K}^s(\Omega,\varphi,A,\psi)$, i.e. if

$$\iint_{Q(\Omega)} \left\{ \mathcal{G}\left(\frac{u(x)-u(y)}{|x-y|}\right) - \mathcal{G}\left(\frac{v(x)-v(y)}{|x-y|}\right)\right\} \frac{dx\,dy}{|x-y|^{n-1+s}} \leq 0,$$

for every $v \in \mathcal{K}^s(\Omega,\varphi,A,\psi)$. We remark that this definition is well posed, thanks to Lemma 4.5.1.

The main result of this section is the following existence and uniqueness Theorem.

THEOREM 4.6.1. *Let $n \geq 1$, $s \in (0,1)$, $\Omega \subseteq \mathbb{R}^n$ a bounded open set with Lipschitz boundary, $R_0 > 1$ be such that $\Omega \subseteq B_{R_0}$ and let $\Theta = \Theta(n,s,g) > 1$ be as in Theorem 4.1.4. Let $A \subseteq \Omega$ be an open set and let $\psi \in L^\infty(A)$. For every $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ such that $\varphi \in L^\infty(B_{\Theta R_0} \setminus \Omega)$, there exists a unique function $u \in \mathcal{K}^s(\Omega,\varphi,A,\psi)$ that solves the obstacle problem. Moreover*

$$\|u\|_{L^\infty(\Omega)} \leq R_0 + \max\big\{\|\varphi\|_{L^\infty(B_{\Theta R_0}\setminus\Omega)}, \|\psi\|_{L^\infty(A)}\big\}.$$

The proof of this Theorem is the content of Section 4.6.1. It is interesting to observe that a solution exists without having to impose regularity assumptions on the domain A where the obstacle is defined, nor on the obstacle function ψ—besides boundedness.

Before going on, we mention that in Section 4.6.2 we consider the geometric case corresponding to the choice $g = g_s$ and we show the connection between solutions of the functional obstacle problem and of the geometric obstacle problem.

Now we point out that a solution of the obstacle problem is a supersolution of the equation $\mathscr{H}u = 0$ in the whole domain Ω and a solution away from the contact set, that is, formally:

$$\mathscr{H}u \geq 0 \quad \text{in } \Omega \quad \text{and} \quad \mathscr{H}u = 0 \quad \text{in } \Omega \setminus \{u = \psi\}.$$

More precisely, we have the following result:

PROPOSITION 4.6.2. *Let $n \geq 1$, $s \in (0,1)$, $\Omega \subseteq \mathbb{R}^n$ a bounded open set with Lipschitz boundary, $\varphi : \mathcal{C}\Omega \to \mathbb{R}$, $A \subseteq \Omega$ an open set and $\psi \in L^\infty(A)$. Suppose that there exists a function $u \in \mathcal{K}^s(\Omega,\varphi,A,\psi)$ that solves the obstacle problem. Then*

$$\langle \mathscr{H}u, v\rangle \geq 0 \quad \forall\, v \in C^\infty_c(\Omega) \quad s.t.\ v \geq 0.$$

Furthermore, if $\mathcal{O} \subseteq \Omega$ is an open set such that

$$\inf_{\mathcal{O}\cap A}(u-\psi) \geq \delta,$$

for some $\delta > 0$*, then*

$$\langle \mathscr{H} u, v\rangle = 0 \quad \forall\, v \in C_c^\infty(\mathcal{O}).$$

In particular, if $\mathcal{O}$ *has Lipschitz boundary, then* u *minimizes* $\mathcal{F}$ *in* $\mathcal{W}_u^s(\mathcal{O})$.

PROOF. First of all, notice that if $v \in C_c^\infty(\Omega)$ is such that $v \geq 0$, then $u + \varepsilon v \in \mathcal{K}^s(\Omega, \varphi, A, \psi)$ for every $\varepsilon > 0$. Thus, by the minimality of u and recalling identity (4.89) in Lemma 4.5.1, we have

$$\mathcal{F}^0(u + \varepsilon v, \Omega) - \mathcal{F}^0(u, \Omega) \geq 0.$$

Passing to the limit $\varepsilon \to 0^+$ and recalling Lemma 4.2.12, we find $\langle \mathscr{H} u, v\rangle \geq 0$, as claimed.

In order to prove that u is a solution away from the contact set, let $v \in C_c^\infty(\mathcal{O})$ and observe that for every $|\varepsilon| \leq \delta/\|v\|_{L^\infty(\mathcal{O})}$ we have $u + \varepsilon v \in \mathcal{K}^s(\Omega, \varphi, A, \psi)$. Roughly speaking, since we are away from the contact set, we are allowed to deform the function u both from above and from below. Hence, again by the minimality of u and exploiting Lemma 4.2.12, we obtain $\langle \mathscr{H} u, v\rangle = 0$.

Finally, if $\mathcal{O}$ has Lipschitz boundary, then we conclude that u minimizes $\mathcal{F}$ in $\mathcal{W}_u^s(\mathcal{O})$ by Lemma 4.5.4. ☐

In particular, we observe that if $A \Subset \Omega$ has Lipschitz boundary, then u minimizes $\mathcal{F}$ in $\mathcal{W}_u^s(\Omega \setminus \overline{A})$.

4.6.1. Proof of Theorem 4.6.1. The argument is essentially the same one that we already employed to prove the existence of minimizers of $\mathcal{F}$ in $\mathcal{W}_\varphi^s(\Omega)$. We begin by considering the functions u_M that minimize $\mathcal{F}^M(\,\cdot\,, \Omega)$ in $\mathfrak{B}_M\mathcal{K}^s(\Omega, \varphi, A, \psi)$, then we show that they stabilize, by exploiting Proposition 4.5.13.

Step 1. First of all, we observe that

$$\mathcal{K}^s(\Omega, \varphi, A, \psi) \subseteq \mathcal{W}_\varphi^s(\Omega), \qquad \mathfrak{B}\mathcal{K}^s(\Omega, \varphi, A, \psi) \subseteq \mathfrak{B}\mathcal{W}_\varphi^s(\Omega)$$
$$\text{and} \quad \mathfrak{B}_M\mathcal{K}^s(\Omega, \varphi, A, \psi) \subseteq \mathfrak{B}_M\mathcal{W}_\varphi^s(\Omega)$$

are closed convex subsets. As a consequence, by arguing as in the proof of Proposition 4.5.8 and exploiting the convexity of $\mathcal{F}^M$ ensured by Lemma 4.2.5, we find that for every $M \geq \|\psi\|_{L^\infty(A)}$ there exists a unique $u_M \in \mathfrak{B}_M\mathcal{K}^s(\Omega, \varphi, A, \psi)$ such that

$$\mathcal{F}^M(u_M, \Omega) = \inf\left\{\mathcal{F}^M(v, \Omega) \,|\, v \in \mathfrak{B}_M\mathcal{K}^s(\Omega, \varphi, A, \psi)\right\}.$$

Step 2. Now we remark that, since the obstacle ψ is bounded, we can apply Proposition 4.5.13 to obtain an a priori bound on the L^∞ norm of the minimizers u_M, provided $M > 0$ is big enough. Let indeed

$$u^{(N)} := \chi_\Omega \min\{u, N\} + (1 - \chi_\Omega) u,$$

and notice that, if $u \in \mathcal{K}^s(\Omega, \varphi, A, \psi)$ and $N \geq \sup_A \psi$, then we clearly have $u^{(N)} \in \mathcal{K}^s(\Omega, \varphi, A, \psi)$. Therefore, if we consider

$$N := R_0 + \max\left\{\sup_{B_{\Theta R_0}\setminus\Omega} \varphi, \sup_A \psi\right\},$$

then by Proposition 4.5.13 and by the uniqueness of the minimizer of the functional $\mathcal{F}^M(\,\cdot\,, \Omega)$ in $\mathfrak{B}_M\mathcal{K}^s(\Omega, \varphi, A, \psi)$, we obtain

$$\sup_\Omega u_M \leq R_0 + \max\left\{\sup_{B_{\Theta R_0}\setminus\Omega} \varphi, \sup_A \psi\right\},$$

for every $M \geq N$. Since we can argue in the same way by truncating the functions from below, we find that

$$\|u_M\|_{L^\infty(\Omega)} \leq R_0 + \max\left\{\|\varphi\|_{L^\infty(B_{\Theta R_0}\setminus\Omega)}, \|\psi\|_{L^\infty(A)}\right\} =: N_0, \tag{4.119}$$

for every $M \geq N_0$.

Step 3. Fix $M_0 := N_0 + 1$ and observe that (4.119) ensures that

$$\|u_{M_0}\|_{L^\infty(\Omega)} \leq N_0 < M_0. \tag{4.120}$$

We claim that this implies that the function $u := u_{M_0}$ solves the obstacle problem. In order to prove this, let us consider $v \in \mathfrak{B}\mathcal{K}^s(\Omega, \varphi, A, \psi)$ and notice that by (4.120) we have

$$w := tv + (1-t)u \in \mathfrak{B}_{M_0}\mathcal{K}^s(\Omega, \varphi, A, \psi),$$

provided $t \in (0, 1)$ is small enough. Thus, by the minimality of u and exploiting the convexity of $\mathcal{F}^{M_0}$, we find

$$\mathcal{F}^{M_0}(u, \Omega) \leq \mathcal{F}^{M_0}(w, \Omega) \leq t\mathcal{F}^{M_0}(v, \Omega) + (1-t)\mathcal{F}^{M_0}(u, \Omega),$$

that is

$$\mathcal{F}^{M_0}(u, \Omega) \leq \mathcal{F}^{M_0}(v, \Omega).$$

This shows that u minimizes $\mathcal{F}^{M_0}(\,\cdot\,, \Omega)$ in $\mathfrak{B}\mathcal{K}^s(\Omega, \varphi, A, \psi)$. Thanks to Lemma 4.5.1, this implies that u minimizes $\mathcal{F}$ in the larger space $\mathcal{K}^s(\Omega, \varphi, A, \psi)$ and hence solves the obstacle problem. Finally, the strict convexity of $\mathcal{F}^M$ guarantees the uniqueness of such a solution—see also point *(iii)* of Remark 4.5.2—concluding the proof of the Theorem.

4.6.2. Geometric obstacle problem. In this section we study the obstacle problem for the fractional perimeter in the unbounded domain Ω^∞. This problem has been recently considered—in the case of bounded domains—in [20], where the authors proved a regularity result for the solution. Our aim consists in showing that, also in the presence of obstacles, minimizers of the functional problem are minimizers for the geometric problem.

Again, we stress that we do not aim at full generality. In particular, we limit ourselves to give the definition of a geometric minimizer in the setting that interests us, by considering only as domain a bounded open set $\Omega \subseteq \mathbb{R}^n$ with Lipschitz boundary and bounded obstacles.

Let $n \geq 1$, $s \in (0, 1)$, $\Omega \subseteq \mathbb{R}^n$ a bounded open set with Lipschitz boundary, $\varphi : \mathbb{R}^n \to \mathbb{R}$, $A \subseteq \Omega$ an open set and $\psi \in L^\infty(A)$. We define

$$\mathcal{O} := \left\{(x, x_{n+1}) \in \mathbb{R}^{n+1} \,|\, x \in A \text{ and } x_{n+1} < \psi(x)\right\}.$$

We say that a set $E \subseteq \mathbb{R}^{n+1}$ such that $E \setminus \Omega^\infty = \mathcal{S}g(\varphi) \setminus \Omega^\infty$ and $\mathcal{O} \subseteq E$ solves the geometric obstacle problem if for every $M \geq \|\psi\|_{L^\infty(A)}$ it holds $\mathrm{Per}_s(E, \Omega^M) < \infty$, and

$$\mathrm{Per}_s(E, \Omega^M) \leq \mathrm{Per}_s(F, \Omega^M),$$

for every $F \subseteq \mathbb{R}^{n+1}$ such that $F \setminus \Omega^M = E \setminus \Omega^M$ and $\mathcal{O} \subseteq F$.

REMARK 4.6.3. We observe that if $E \subseteq \mathbb{R}^{n+1}$ solves the geometric obstacle problem, then it is locally s-minimal in the open set $\Omega^\infty \setminus \overline{\mathcal{O}}$.

By exploiting Theorem 4.1.10 and Proposition 4.2.8, it is readily seen that if u solves the obstacle problem—with $g = g_s$—then its subgraph solves the geometric obstacle problem.

PROPOSITION 4.6.4. *Let $n \geq 1$, $s \in (0,1)$, $g = g_s$, $\Omega \subseteq \mathbb{R}^n$ a bounded open set with Lipschitz boundary, $R_0 > 1$ be such that $\Omega \subseteq B_{R_0}$ and let $\Theta = \Theta(n,s,g_s) > 1$ be as in Theorem 4.1.4. Let $A \subseteq \Omega$ be an open set, $\psi \in L^\infty(A)$ and let $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ such that $\varphi \in L^\infty(B_{\Theta R_0} \setminus \Omega)$. Let $u \in \mathfrak{BK}^s(\Omega, \varphi, A, \psi)$ be the unique solution of the obstacle problem, as in Theorem 4.6.1. Then, $\mathcal{S}g(u)$ solves the geometric obstacle problem.*

We conclude this section by proving that the subgraph of u is actually the unique solution to the geometric obstacle problem.

In order to do this, we consider $\Omega \subseteq \mathbb{R}^n$ to be a bounded open set with C^2 boundary and the domain of definition of the obstacle $A \subseteq \Omega$ to be either $A = \Omega$ or $A \Subset \Omega$ with C^2 boundary. Since we are considering a bounded obstacle $\psi \in L^\infty(A)$—and thanks to Remark 4.6.3—it is easy to check that the argument of the proof of [43, Lemma 3.3] works also in this situation.

Therefore, there exists $\tilde{R}(n,s,\Omega) > 0$ such that, if $\varphi \in L^\infty(B_{\tilde{R}} \setminus \Omega)$, and $E \subseteq \mathbb{R}^{n+1}$ solves the geometric obstacle problem, then

$$\Omega \times (-\infty, -M_0) \subseteq E \cap \Omega^\infty \subseteq \Omega \times (-\infty, M_0),$$

for some $M_0(n,s,\Omega,\varphi,A,\psi) > 0$. Let us now define $R_s := \max\{\Theta R_0, \tilde{R}\}$. Then, we have the following uniqueness result:

PROPOSITION 4.6.5. *Let $n \geq 1$, $s \in (0,1)$, $g = g_s$, $\Omega \subseteq \mathbb{R}^n$ a bounded open set with C^2 boundary. Let $A \subseteq \Omega$ be an open set such that either $A = \Omega$ or $A \Subset \Omega$ with C^2 boundary, $\psi \in L^\infty(A)$ and let $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ such that $\varphi \in L^\infty(B_{R_s} \setminus \Omega)$, with R_s as defined above. Let $u \in \mathfrak{BK}^s(\Omega, \varphi, A, \psi)$ be the unique solution of the obstacle problem, as in Theorem 4.6.1. Then, $\mathcal{S}g(u)$ is the unique solution of the geometric obstacle problem.*

The proof follows by arguing as in the proof of Theorem 4.1.13 and exploiting the uniqueness of the solution of the (functional) obstacle problem.

4.7. Approximation results

In this section we collect some approximating results for the functionals $\mathcal{F}^M(\cdot, \Omega)$. These results are interesting for various reasons. First of all, they are meaningful in themselves and they somehow complement the results proven in Section 2.3. More precisely, in Proposition 4.7.3 we show that a subgraph having finite s-perimeter can be approximated with smooth subgraphs, and not just with arbitrary smooth open sets as in Theorem 2.1.1.

Secondarily, we point out that the subgraphs of σ-harmonic functions are somehow less rigid than nonlocal minimal graphs. Indeed, thanks to the surprising result proved in [44], it is always possible to approximate a nonlocal minimal graph with σ-harmonic functions—see Theorem 4.7.4. On the other hand—as observed in [44]—the converse is not possible, because by exploiting [44, Theorem 1.1] it is possible to construct σ-harmonic functions that oscillate wildly, while nonlocal minimal graphs must satisfy uniform density estimates at the boundary points—see [21, Theorem 4.1].

Finally, we prove that there is no gap phenomenon when we minimize $\mathcal{F}$ with respect to regular exterior data—see Proposition 4.7.5. Indeed, as we have remarked in the introduction, even when the exterior data is a smooth and compactly supported function, the minimizer of $\mathcal{F}$, in general, is not continuous across the boundary of the domain, because of stickiness effects which are typically nonlocal.

Thus, it is natural to wonder whether the minimization of $\mathcal{F}$ among functions which are smooth in the whole of $\mathbb{R}^n$ leads to a value which is strictly bigger than that obtained by minimizing $\mathcal{F}$ in the larger space $\mathcal{W}^s_\varphi(\Omega)$. Roughly speaking, given $\varphi \in C^{0,1}(\mathbb{R}^n)$, we wonder whether the inequality

$$\inf\left\{\mathcal{F}^0(v,\Omega)\,|\,v\in C^{0,1}(\mathbb{R}^n)\text{ s.t. }v=\varphi\text{ in }\mathcal{C}\Omega\right\}\geq\inf_{v\in\mathcal{W}^s_\varphi(\Omega)}\mathcal{F}^0(v,\Omega)$$

can be strict. The answer is no. As shown by Proposition 4.7.5, this inequality is actually always an equality.

First of all, we remark that when we keep the exterior data fixed, then the approximation result follows from Lemma 4.5.1 and the density of $C^\infty_c(\Omega)$ in $W^{s,1}(\Omega)$. We have alreday exploited this fact in the proof of the existence of minimizers—see also point (iv) of Remark 4.5.2.

On the other hand, when we approximate also the exterior data we have the following useful result.

Proposition 4.7.1. *Let $n\geq 1$, $s\in(0,1)$, $\Omega\subseteq\mathbb{R}^n$ a bounded open set with Lipschitz boundary. Let u, $u_k\in L^1_{\mathrm{loc}}(\mathbb{R}^n)\cap W^{s,1}(\Omega_d)$, for some $d>0$, and suppose that $u_k\to u$ both in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and in $W^{s,1}(\Omega_d)$. Then*

$$\lim_{k\to\infty}\mathcal{F}^M(u_k,\Omega)=\mathcal{F}^M(u,\Omega),$$

for every $M\geq 0$.

Before getting to the proof of Proposition 4.7.1, we state some of its consequences.

If we consider a symmetric mollifier $\eta\in C^\infty_c(\mathbb{R}^n)$ as in (D.11) and we define the mollified functions $u_\varepsilon:=u*\eta_\varepsilon$, then we obtain the following corollary of Proposition 4.7.1.

Corollary 4.7.2. *Let $n\geq 1$, $s\in(0,1)$, $\Omega\subseteq\mathbb{R}^n$ a bounded open set with Lipschitz boundary and let $u\in L^1_{\mathrm{loc}}(\mathbb{R}^n)\cap W^{s,1}(\Omega_d)$, for some $d>0$. Then*

$$\lim_{\varepsilon\to 0}\mathcal{F}^M(u_\varepsilon,\Omega)=\mathcal{F}^M(u,\Omega),$$

for every $M\geq 0$.

Proof. It is well known that $u_\varepsilon\to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. On the other hand, since $u\in W^{s,1}(\Omega_d)$, we have also $u_\varepsilon\to u$ in $W^{s,1}(\Omega_{d/2})$—see, e.g., Lemma D.2.3. Hence, the conclusion follows from Proposition 4.7.1. □

When we consider subgraphs of locally bounded functions, Proposition 4.2.8 and Corollary 4.7.2 straightforwardly imply the desired approximation result which complements Theorem 2.1.1.

Proposition 4.7.3. *Let $n\geq 1$, $s\in(0,1)$, $\Omega\subseteq\mathbb{R}^n$ a bounded open set with Lipschitz boundary and let $u\in L^1_{\mathrm{loc}}(\mathbb{R}^n)\cap W^{s,1}(\Omega)\cap L^\infty_{\mathrm{loc}}(\Omega)$. Then, for every open set $\mathcal{O}\Subset\Omega$ with Lipschitz boundary and every $M\geq\|u\|_{L^\infty(\mathcal{O}_r)}$, with $r:=d(\mathcal{O},\partial\Omega)/2$, it holds*

$$\lim_{\varepsilon\to 0}\mathrm{Per}_s(\mathcal{S}g(u_\varepsilon),\mathcal{O}^M)=\mathrm{Per}_s(\mathcal{S}g(u),\mathcal{O}^M).\tag{4.121}$$

Moreover, if $u\in C(\mathcal{U})$, for some open set $\mathcal{U}\subseteq\mathbb{R}^n$, then for every compact set $K\Subset\mathcal{U}$ and every $\delta>0$ we have

$$\partial\mathcal{S}g(u_\varepsilon)\cap K^\infty\subseteq N_\delta\big(\mathcal{S}g(u)\big)\cap K^\infty,$$

for every $\varepsilon>0$ small enough.

Proof. It is enough to notice that for every $\varepsilon>0$ small enough we have

$$\|u_\varepsilon\|_{L^\infty(\mathcal{O})}\leq\|u\|_{L^\infty(\mathcal{O}_r)}\leq M.$$

Then, (4.121) follows by making use of Corollary 4.7.2 and of identity (4.33). To conclude, notice that $u \in C(\mathcal{U})$ implies that

$$\partial \mathcal{S}g(u) \cap \mathcal{U}^\infty = \{(x, u(x)) \in \mathbb{R}^{n+1} \,|\, x \in \mathcal{U}\},$$

and similarly for u_ε. Thus, the uniform convergence of the boundaries follows from the fact that $u_\varepsilon \to u$ locally uniformly in $\mathcal{U}$. $\square$

By exploiting [44, Theorem 1.1] to approximate the mollified functions u_ε, it is immediate to see that we can find a sequence of σ-harmonic functions u_k such that $\mathcal{F}^M(u_k, \Omega) \to \mathcal{F}^M(u, \Omega)$. In particular, if the function u is bounded in Ω and we take $g = g_s$, then we can approximate the s-perimeter of the subgraph of u with the s-perimeter of the subgraphs of the functions u_k.

We give a precise statement of this fact only in the case of nonlocal minimal graphs.

Theorem 4.7.4. *Let $n \geq 1$, $s \in (0,1)$, $\Omega \subseteq \mathbb{R}^n$ a bounded open set with Lipschitz boundary, let $\varphi : \mathcal{C}\Omega \to \mathbb{R}$ be such that*

$$\varphi \in L^1_{\mathrm{loc}}(\mathbb{R}^n \setminus \Omega) \cap W^{s,1}(\Omega_d \setminus \Omega) \cap L^\infty(\Omega_d \setminus \Omega),$$

for some $d > 0$ small, and let $R_0 > 0$ such that $\Omega_d \Subset B_{R_0}$. Let $u \in \mathfrak{B}\mathcal{W}^s_\varphi(\Omega)$ be the unique minimizer of $\mathcal{F}_s$ in $\mathcal{W}^s_\varphi(\Omega)$. Then, for every fixed $\sigma \in (0,1)$ and $\ell \in \mathbb{N}$, there exists a sequence of compactly supported functions $u_k \in H^\sigma(\mathbb{R}^n) \cap C^\sigma(\mathbb{R}^n)$ such that

(i) $(-\Delta)^\sigma u_k = 0$ *in* B_{k+R_0}

(ii) $u_k \to u$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *and in* $W^{s,1}(\Omega_{d/2})$

(iii) $\lim_{k\to\infty} \|u_k - u\|_{C^\ell(\Omega')} = 0$ *for every* $\Omega' \Subset \Omega$,

(iv) $\|u_k\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega_d)} + 1$,

(v) $\lim_{k\to\infty} \mathrm{Per}_s(\mathcal{S}g(u_k), \Omega^M) = \mathrm{Per}_s(\mathcal{S}g(u), \Omega^M)$, *for every* $M \geq \|u\|_{L^\infty(\Omega_d)} + 1$.

Moreover, for every compact set $K \Subset \Omega$ and every $\delta > 0$ it holds

$$\partial \mathcal{S}g(u_k) \cap K^\infty \subseteq N_\delta\big(\mathcal{S}g(u)\big) \cap K^\infty,$$

for every k big enough.

Proof. We begin by observing that, recalling Lemma 4.5.10 and exploiting Theorem 4.1.3, we know that there exists a unique function $u \in \mathcal{W}^s_\varphi(\Omega)$ that minimizes $\mathcal{F}$. Moreover, since φ is bounded near $\partial\Omega$, by Theorem 4.5.14 we know that $u \in L^\infty(\Omega)$. Finally, by [19, Theorem 1.1] we have $u \in C^\infty(\Omega)$. We also remark that, since $\varphi \in W^{s,1}(\Omega_d \setminus \Omega)$, it is readily seen that $u \in W^{s,1}(\Omega_d)$. As a consequence, we have that

$$\begin{aligned} & u_\varepsilon \to u && \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ and in } W^{s,1}(\Omega_{d/2}) \\ & \lim_{\varepsilon\to 0} \|u_\varepsilon - u\|_{C^\ell(\Omega')} = 0 && \text{for every } \Omega' \Subset \Omega, \\ & \|u_\varepsilon\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega_d)} && \text{for every } \varepsilon > 0 \text{ small enough.} \end{aligned} \tag{4.122}$$

Then, the claim follows by using [44, Theorem 1.1] to approximate the mollified functions u_ε and a diagonal argument. Indeed, fix $\varepsilon > 0$ and notice that, since u_ε is smooth in $\mathbb{R}^n$, by [44, Theorem 1.1] we can find for every $k \in \mathbb{N}$ a compactly supported function $u_k \in H^\sigma(\mathbb{R}^n) \cap C^\sigma(\mathbb{R}^n)$ such that

$$\begin{aligned} & (-\Delta)^\sigma u_k = 0 \qquad \text{in } B_{k+R_0} \\ & \|u_k - u_\varepsilon\|_{C^\ell(B_{k+R_0})} < \frac{1}{e^k}. \end{aligned}$$

In particular, this implies that

$$\begin{aligned}u_k \to u_\varepsilon \qquad &\text{in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ and in } W^{s,1}(\Omega_d),\\ \|u_k\|_{L^\infty(\Omega)} \leq \|u_\varepsilon\|_{L^\infty(\Omega)} &+ 1 \leq \|u\|_{L^\infty(\Omega_d)} + 1.\end{aligned}$$

Therefore, after a diagonal argument and recalling (4.122), we obtain a sequence of compactly supported functions $u_k \in H^\sigma(\mathbb{R}^n) \cap C^\sigma(\mathbb{R}^n)$ that satisfies points (i), (ii), (iii) and (iv). Then, point (v) follows by points (ii) and (iv), Proposition 4.7.1 and identity (4.33).

To conclude, notice that the locally uniform convergence of the boundaries follows from point (iii)—used just for the C^0 norm, as in the proof of Proposition 4.7.3.. ☐

Now we provide the proof of Proposition 4.7.1.

Proof of Proposition 4.7.1. First of all, we observe that by the Lipschitzianity of $\mathcal{G}$—see (4.21)—we have

$$\big|\mathcal{A}(u_k,\Omega) - \mathcal{A}(u,\Omega)\big| \leq \frac{\Lambda}{2}\|u_k - u\|_{W^{s,1}(\Omega)}.$$

As for the nonlocal part, we will exploit identity (4.28) and, again, the Lipschitzianity of $\mathcal{G}$. We have

$$\begin{aligned}\big|\mathcal{N}^M(u_k,\Omega) - \mathcal{N}^M(u,\Omega)\big| \leq \int_\Omega \bigg\{ \int_{\mathcal{C}\Omega} \bigg| &2\,\mathcal{G}\left(\frac{u_k(x)-u_k(y)}{|x-y|}\right) - \mathcal{G}\left(\frac{M+u_k(y)}{|x-y|}\right)\\ &-\mathcal{G}\left(\frac{M-u_k(y)}{|x-y|}\right) - 2\,\mathcal{G}\left(\frac{u(x)-u(y)}{|x-y|}\right) + \mathcal{G}\left(\frac{M+u(y)}{|x-y|}\right)\\ &+\mathcal{G}\left(\frac{M-u(y)}{|x-y|}\right)\bigg| \frac{dy}{|x-y|^{n-1+s}} \bigg\}\,dx.\end{aligned}$$

We split the domain $\mathcal{C}\Omega = \big(\Omega_r \setminus \Omega\big) \cup \big(B_R \setminus \Omega_r\big) \cup \mathcal{C}B_R$, with $r \in (0,d)$ small enough such that Ω_r has Lipschitz boundary, and $R > 0$ big—we will let $R \to \infty$ in the end—and we treat the three cases differently.

We begin by observing that—by appropriately regrouping the terms, using the triangle inequality, the Lipschitzianity of $\mathcal{G}$ and exploiting also Corollary D.1.5—the double integral over $\Omega \times \big(\Omega_r \setminus \Omega\big)$ can be estimated by

$$\Lambda \int_\Omega \bigg\{ \int_{\Omega_r\setminus\Omega} \frac{|u_k(x)-u_k(y)-u(x)-u(y)| + |u_k(y)-u(y)|}{|x-y|^{n+s}}\,dy \bigg\}\,dx \leq C\|u_k - u\|_{W^{s,1}(\Omega_r)}.$$

Similarly, the double integral over $\Omega \times \big(B_R \setminus \Omega_r\big)$ can be estimated by

$$\begin{aligned}&\Lambda \int_\Omega \bigg\{ \int_{B_R\setminus\Omega_r} \frac{|u_k(x)-u_k(y)-u(x)-u(y)| + |u_k(y)-u(y)|}{|x-y|^{n+s}}\,dy \bigg\}\,dx\\ &\leq \Lambda \bigg\{ \int_\Omega |u_k(x)-u(x)| \left(\int_{B_R\setminus\Omega_r} \frac{dy}{|x-y|^{n+s}} \right) dx\\ &\qquad + 2\int_{B_R\setminus\Omega_r} |u_k(y)-u(y)| \left(\int_\Omega \frac{dx}{|x-y|^{n+s}} \right) dy \bigg\}\\ &\leq \frac{3\,\Lambda\,\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{s\,r^s}\|u_k - u\|_{L^1(B_R)}.\end{aligned}$$

Now we observe that, since $u_k \to u$ in $L^1(\Omega)$, we have $\|u_k\|_{L^1(\Omega)} \leq 2\|u\|_{L^1(\Omega)}$ for all k big enough. Moreover, we take $R_0 > 0$ such that $\Omega \Subset B_{R_0}$ and $R > R_0$. Then, by regrouping the terms in a different way, we estimate the double integral over $\Omega \times \mathcal{C}B_R$ with

$$\begin{aligned}
&\frac{\Lambda}{2}\int_\Omega \{|u_k(x)-M|+|u_k(x)+M|+|u(x)-M|+|u(x)+M|\}\left(\int_{\mathcal{C}B_R}\frac{dy}{|x-y|^{n+s}}\right)dx\\
&\qquad\le \Lambda\int_\Omega \{|u_k(x)|+|u(x)|+2M\}\left(\int_{\mathcal{C}B_{R-R_0}(x)}\frac{dy}{|x-y|^{n+s}}\right)dx\\
&\qquad\le C\,\frac{\|u\|_{L^1(\Omega)}+M\,|\Omega|}{(R-R_0)^s}.
\end{aligned}$$

All in all, we have proved that

$$\left|\mathcal{F}^M(u_k,\Omega)-\mathcal{F}^M(u,\Omega)\right|\le C\left(\|u_k-u\|_{W^{s,1}(\Omega_r)}+\|u_k-u\|_{L^1(B_R)}+\frac{\|u\|_{L^1(\Omega)}+M\,|\Omega|}{(R-R_0)^s}\right).$$

Passing first to the limit $k\to\infty$, then to the limit $R\to\infty$, concludes the proof of the Proposition. □

We conclude this section by proving that there is no gap phenomenon in the minimization of $\mathcal{F}$. This is a simple consequence of the density of $C_c^\infty(\Omega)$ in $W^{s,1}(\Omega)$, which sostantially means that functions in $W^{s,1}(\Omega)$ do not have a well defined trace. Roughly speaking, this implies that we can approximate any function $u\in W^{s,1}(\Omega)$ with smooth functions that have a fixed boundary value.

Proposition 4.7.5. *Let $n\ge 1$, $s\in(0,1)$, $\Omega\subseteq\mathbb{R}^n$ a bounded open set with Lipschitz boundary and let $\varphi:\mathbb{R}^n\to\mathbb{R}$ be such that $\varphi\in C^{0,1}(\Omega_d)$, for some $d>0$, and $\varphi\in L^1(\Omega_{\Theta\,\mathrm{diam}(\Omega)})$, with $\Theta>1$ as in Theorem 4.1.3. Then,*

$$\inf\left\{\mathcal{F}^0(v,\Omega)\,|\,v\in C^{0,1}(\Omega_d)\ \text{s.t.}\ v=\varphi\ \text{in}\ \Omega_d\setminus\Omega\ \text{and a.e. in}\ \mathcal{C}\Omega_d\right\}=\min_{v\in\mathcal{W}^s_\varphi(\Omega)}\mathcal{F}^0(v,\Omega).$$

Proof. Notice that $\varphi\in C^{0,1}(\Omega_d)$ implies that $\varphi\in W^{s,1}(\Omega_d)$, since

$$\int_{\Omega_d}\int_{\Omega_d}\frac{|\varphi(x)-\varphi(y)|}{|x-y|^{n+s}}\,dx\,dy\le[\varphi]_{C^{0,1}(\Omega_d)}\int_{\Omega_d}\int_{\Omega_d}\frac{dx\,dy}{|x-y|^{n-1+s}},$$

which is finite, thanks to Lemma D.1.1. Then, by recalling point (i) of Lemma 4.5.10 and exploiting Theorem 4.1.3, we know that there exists a unique function $u\in\mathcal{W}^s_\varphi(\Omega)$ that minimizes $\mathcal{F}$. By point (ii) of Remark 4.5.2, this means that

$$\mathcal{F}^0(u,\Omega)=\inf_{v\in\mathcal{W}^s_\varphi(\Omega)}\mathcal{F}^0(v,\Omega).$$

Moreover, since $w:=u-\varphi\in W^{s,1}(\Omega)$, by the density of $C_c^\infty(\Omega)$ in $W^{s,1}(\Omega)$, we can find a sequence $\{w_k\}\subseteq C_c^\infty(\Omega)$ such that

$$\lim_{k\to\infty}\|w_k-w\|_{W^{s,1}(\Omega)}=0.$$

If we extend the functions w_k by zero outside Ω, this means that the functions $v_k:=\varphi+w_k\in C^{0,1}(\Omega_d)$ converge to u in $W^{s,1}(\Omega)$. By Lemma 4.5.1, this implies

$$\lim_{k\to\infty}\mathcal{F}^0(v_k,\Omega)=\mathcal{F}^0(u,\Omega),$$

concluding the proof of the Proposition. □

CHAPTER 5

Bernstein-Moser-type results for nonlocal minimal graphs

5.1. Introduction and main results

For simplicity, in this chapter sets that minimize Per_s in all bounded open subsets of $\mathbb{R}^{n+1}$ will be simply called *s-minimal* and their boundaries *s-minimal surfaces*.

In this brief chapter we are mostly interested in s-minimal sets $E \subseteq \mathbb{R}^{n+1}$ that are subgraphs of a measurable function $u : \mathbb{R}^n \to \mathbb{R}$, i.e., that satisfy

$$E = \{x = (x', x_{n+1}) \in \mathbb{R}^n \times \mathbb{R} \,|\, x_{n+1} < u(x')\}\,. \tag{5.1}$$

We will call the boundaries of such extremal sets *s-minimal graphs*.

We observe that, differently from the previous chapters, we will use here the notation $x = (x', x_{n+1}) \in \mathbb{R}^{n+1}$.

We recall that, if $u : \mathbb{R}^n \to \mathbb{R}$ is a function of class $C^{1,1}$ in a neighborhood of a point $x' \in \mathbb{R}^n$, and $E := \mathcal{S}g(u)$ as in (5.1), then

$$H_s[E](x', u(x')) = \mathscr{H}_s u(x'),$$

with

$$\mathscr{H}_s u(x') := 2\,\mathrm{P.V.} \int_{\mathbb{R}^n} G_s\left(\frac{u(x') - u(y')}{|x' - y'|}\right) \frac{dy'}{|x' - y'|^{n+s}}$$

and

$$G_s(t) := \int_0^t \frac{d\tau}{(1+\tau^2)^{\frac{n+1+s}{2}}} \quad \text{for } t \in \mathbb{R}. \tag{5.2}$$

Taking advantage of the convexity of the energy functional associated to $\mathscr{H}_s$ and of a suitable rearrangement inequality, we have shown in Chapter 4 that a set E given by (5.1) for some function $u : \mathbb{R}^n \to \mathbb{R}$ is s-minimal if and only if u is a solution of

$$\mathscr{H}_s u = 0 \quad \text{in } \mathbb{R}^n. \tag{5.3}$$

There are several notions of solutions of (5.3), such as smooth solutions, viscosity solutions, and weak solutions. However, all such definitions are equivalent under mild assumptions on u—see Corollary 4.1.12 for more details. In what follows, a solution of (5.3) will always indicate a function $u \in C^\infty(\mathbb{R}^n)$ that satisfies identity (5.3) pointwise. We stress that no growth assumptions at infinity are made on u.

The main contribution of this chapter is the following result.

THEOREM 5.1.1. *Let $n \geq \ell \geq 1$ be integers, $s \in (0,1)$, and suppose that*

$$(P_{s,\ell}) \qquad \text{there exist no singular } s\text{-minimal cones in } \mathbb{R}^\ell.$$

Let u be a solution of (5.3) having $n - \ell$ partial derivatives bounded on one side.
Then, u is an affine function.

We point out that throughout the chapter a *cone* is any subset $\mathscr{C}$ of the Euclidean space for which $\lambda x \in \mathscr{C}$ for every $x \in \mathscr{C}$ and $\lambda > 0$. In addition, a *singular* cone is a cone whose boundary is not smooth at the origin or, equivalently, any nontrivial cone that is not a half-space.

Characterizing the values of s and ℓ for which $(P_{s,\ell})$ is satisfied represents a challenging open problem, whose solution would lead to fundamental advances in the understanding of the regularity properties enjoyed by nonlocal minimal surfaces. Currently, property $(P_{s,\ell})$ is known to hold in the following cases:

- when $\ell = 1$ or $\ell = 2$, for every $s \in (0,1)$;
- when $3 \leq \ell \leq 7$ and $s \in (1 - \varepsilon_0, 1)$ for some small $\varepsilon_0 \in (0,1]$ depending only on ℓ.

Case $\ell = 1$ holds by definition, while $\ell = 2$ is the content of [**92**, Theorem 1]. On the other hand, case $3 \leq \ell \leq 7$ has been established in [**25**, Theorem 2]—see also [**18**] for a different approach yielding an explicit value for ε_0 when $\ell = 3$.

As a consequence of Theorem 5.1.1 and the last remarks, we immediately obtain the following result.

COROLLARY 5.1.2. *Let $n \geq \ell \geq 1$ be integers and $s \in (0,1)$. Assume that either*

- *$\ell \in \{1,2\}$, or*
- *$3 \leq \ell \leq 7$ and $s \in (1 - \varepsilon_0, 1)$, with $\varepsilon_0 = \varepsilon_0(\ell) > 0$ as in [**25**, Theorem 2].*

Let u be a solution of (5.3) having $n - \ell$ partial derivatives bounded on one side.
Then, u is an affine function.

We observe that Theorem 5.1.1 gives a new flatness result for s-minimal graphs, under the assumption that $(P_{s,\ell})$ holds true. It can be seen as a generalization of the fractional De Giorgi-type lemma contained in [**58**, Theorem 1.2], which is recovered here taking $\ell = n$. In this case, we indeed provide an alternative proof of said result.

On the other hand, the choice $\ell = 2$ gives an improvement of [**55**, Theorem 4], when specialized to s-minimal graphs. In light of these observations, Theorem 5.1.1 and Corollary 5.1.2 can be seen as a bridge between Bernstein-type theorems (flatness results in low dimensions) and Moser-type theorems (flatness results under global gradient bounds).

For classical minimal graphs—formally corresponding to the case $s = 1$ here (see, e.g., [**5**, **25**])—the counterpart of Corollary 5.1.2 has been recently obtained by A. Farina in [**54**]. In that case, the result is sharp and holds with $\ell = \min\{n, 7\}$. See also [**53**] by the same author for a previous result established for $\ell = 1$ and through a different argument.

Using the same ideas that lead to Theorem 5.1.1, we can prove the following rigidity result for entire s-minimal graphs that lie above a cone.

THEOREM 5.1.3. *Let $n \geq 1$ be an integer and $s \in (0,1)$. Let u be a solution of (5.3) and assume that there exists a constant $C > 0$ for which*

$$u(x') \geq -C(1 + |x'|) \quad \text{for every } x' \in \mathbb{R}^n. \tag{5.4}$$

Then, u is an affine function.

Of course, the same conclusion can be drawn if (5.4) is replaced by the specular

$$u(x') \leq C(1 + |x'|) \quad \text{for every } x' \in \mathbb{R}^n.$$

For classical minimal graphs, the corresponding version of Theorem 5.1.3 follows at once from the gradient estimate of Bombieri, De Giorgi & Miranda [**13**] and Moser's version of Bernstein's theorem [**84**]. See for instance [**68**, Theorem 17.6] for a clean statement and the details of its proof.

In the nonlocal scenario, a gradient bound for s-minimal graphs has been recently established in [**19**]. However, this result is partly weaker than the one of [**13**], since it provides a bound for the gradient of a solution of (5.3) in terms of its oscillation, and not just of its supremum (or infimum) as in [**13**]. Consequently, in [**19**] a rigidity result analogous to Theorem 5.1.3 is deduced, but with (5.4) replaced by the stronger, two-sided assumption: $|u(x')| \leq C(1 + |x'|)$ for every $x' \in \mathbb{R}^n$. Theorem 5.1.3 thus improves [**19**, Theorem 1.5] directly. Moreover, our proof is different, as it relies on geometric considerations rather than uniform regularity estimates.

Theorem 5.1.3 says in particular that there exist no non-flat s-minimal subgraphs that contain a half-space. Actually, a more general result is true for s-minimal sets that are not necessarily subgraphs, as shown by the following theorem.

THEOREM 5.1.4. *Let $n \geq 1$ be an integer and $s \in (0, 1)$. If E is an s-minimal set in $\mathbb{R}^{n+1}$ that contains a half-space, then E is a half-space.*

Interestingly, Theorem 5.1.4 can be used to obtain a stronger version of Theorem 5.1.3, where the bound in (5.4) is required to only hold at all points x' that lie in a half-space of $\mathbb{R}^n$. See Remark 5.6.1 at the end of Section 5.6.

The proof of Theorem 5.1.1 is based on the extension to the fractional framework of a strategy devised by A. Farina for classical minimal graphs and previously unpublished. As a result, the ideas contained in the following sections can be used to obtain a different, easier proof of [**54**, Theorem 1.1]—since, by Simons' theorem (see, e.g., [**79**, Theorem 28.10]), no singular classical minimal cones exist in dimension lower or equal to 7. Similarly, the same argument that we employ for Theorem 5.1.3 can be successfully applied to classical minimal graphs, giving a different, more geometric, proof of [**68**, Theorem 17.6].

The argument leading to Theorem 5.1.1 relies on a general splitting result for blow-downs of s-minimal graphs. Since it may have an interest on its own, we provide its statement here below.

THEOREM 5.1.5. *Let $n \geq 1$ be an integer and $s \in (0, 1)$. Let u be a solution of (5.3) and E as in (5.1). Assume that u is not affine and that, for some $k \in \{1, \ldots, n-1\}$, the partial derivative $\frac{\partial u}{\partial x_i}$ is bounded from below in $\mathbb{R}^n$ for every $i = 1, \ldots, k$.*

Then, every blow-down limit $\mathscr{C} \subseteq \mathbb{R}^{n+1}$ of E is a cylinder of the form

$$\mathscr{C} = \mathbb{R}^k \times P \times \mathbb{R},$$

for some singular s-minimal cone $P \subseteq \mathbb{R}^{n-k}$.

The notion of blow-down limit will be made precise in Section 5.2.

REMARK 5.1.6. As revealed by a simple inspection of its proof, Theorem 5.1.5 still holds if we require any k directional derivatives $\partial_{\nu_1} u, \ldots, \partial_{\nu_k} u$ (not necessarily the partial derivatives) to be bounded from below, provided that the directions $\nu_1, \ldots, \nu_k$ are linearly independent. Consequently, one can similarly modify the statements of Theorem 5.1.1 and Corollary 5.1.2 without affecting their validity.

The remainder of the chapter is structured as follows. In Section 5.2 we gather some known facts about sets with finite perimeter, the regularity of s-minimal surfaces, and their blow-downs. Section 5.3 is devoted to the proof of Theorem 5.1.5, while in Section 5.4 we show how Theorem 5.1.1 follows from it. Sections 5.5 and 5.6 contain the proofs of Theorems 5.1.4 and 5.1.3, respectively. The chapter is closed by Section 5.7, which includes the extension of a result due to Chern [**26**] to the framework of graphs having constant s-mean curvature.

5.2. Some remarks on nonlocal minimal surfaces and blow-down cones

As in the previous chapters, we implicitly assume that all the sets we consider contain their measure theoretic interior, do not intersect their measure theoretic exterior, and are such that their topological boundary coincides with their measure theoretic boundary—see Remark MTA and Appendix A for the details.

We now recall some known results about the regularity of s-minimal surfaces, which will be often used without mention in the subsequent sections.

Let $E \subseteq \mathbb{R}^{n+1}$ be an s-minimal set. Then, its boundary ∂E is n-rectifiable. Actually, by [**21**, Theorem 2.4], [**92**, Corollary 2], and [**58**, Theorem 1.1], ∂E is locally of class C^∞, except possibly for a set of singular points $\Sigma_E \subseteq \partial E$ satisfying

$$\mathcal{H}^d(\Sigma_E) = 0 \quad \text{for every } d > n-2.$$

In particular, the set E has locally finite (classical) perimeter in $\mathbb{R}^{n+1}$ and thus it makes sense to consider its reduced boundary $\partial^* E$.

Furthermore, thanks to the blow-up analysis developed in [**21**]—see in particular [**21**, Theorem 9.4]—and the tangential properties of the reduced boundary of a set of locally finite perimeter—see, e.g., [**79**, Theorem 15.5]—we have that $\partial^* E$ is smooth and the singular set is given by

$$\Sigma_E = \partial E \setminus \partial^* E.$$

Given a measurable set $E \subseteq \mathbb{R}^{n+1}$, a point $x \in \mathbb{R}^{n+1}$, and a real number $r > 0$, we write

$$E_{x,r} := \frac{E - x}{r}.$$

We call any L^1_{loc}-limit $E_{x,\infty}$ of E_{x,r_j} along a diverging sequence $\{r_j\}$ a *blow-down limit* of E at x.

Observe that doing a blow-down of a set E corresponds to the operation of looking at E from further and further away. As a result, in the limit one loses track of the point at which the blow-down was centered. That is, blow-down limits may depend on the chosen diverging sequence $\{r_j\}$ but not on the point of application x. This fact is certainly well-known to the experts. Nevertheless, we include in the following Remark a brief justification of it for the convenience of the less experienced reader.

REMARK 5.2.1. Let $x, y \in \mathbb{R}^{n+1}$ and $E \subseteq \mathbb{R}^{n+1}$ be a measurable set. Assume that there exists a set $F \subseteq \mathbb{R}^{n+1}$ such that $E_{x,r_j} \to F$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ as $j \to +\infty$, along a diverging sequence $\{r_j\}$. We claim that also

$$E_{y,r_j} \to F \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}) \text{ as } j \to +\infty. \tag{5.5}$$

To verify this assertion, let $R > 0$ be fixed and write $f_j := \chi_{E_{x,r_j}}$ and $f := \chi_F$. Notice that $\chi_{E_{y,r_j}} = \tau_{v_j} f_j := f_j(\cdot - v_j)$, with $v_j := (x-y)/r_j$. Since $v_j \to 0$ as $j \to 0$, we have

$$\begin{aligned} \big|(E_{y,r_j} \Delta F) \cap B_R\big| &= \|\chi_{E_{y,r_j}} - \chi_F\|_{L^1(B_R)} = \|\tau_{v_j} f_j - f\|_{L^1(B_R)} \\ &\le \|\tau_{v_j} f_j - \tau_{v_j} f\|_{L^1(B_R)} + \|\tau_{v_j} f - f\|_{L^1(B_R)} \\ &\le \|f_j - f\|_{L^1(B_{R+1})} + \|\tau_{v_j} f - f\|_{L^1(B_R)}, \end{aligned}$$

provided j is sufficiently large. Claim (5.5) follows since, by assumption, $f_j \to f$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ and $R > 0$ is arbitrary.

In light of this remark, we can assume blow-downs to be always centered at the origin. For simplicity of notation, we will write $E_r := E_{0,r} = E/r$ and use E_∞ to indicate any blow-down limit.

The next lemma collects some known facts about blow-downs of s-minimal sets.

LEMMA 5.2.2. *Let $E \subseteq \mathbb{R}^{n+1}$ be a nontrivial s-minimal set. Then, for every diverging sequence $\{r_j\}$, there exists a subsequence $\{r_{j_k}\}$ of $\{r_j\}$ and a set $E_\infty \subseteq \mathbb{R}^{n+1}$ such that $E_{r_{j_k}} \to E_\infty$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{n+1})$ as $k \to +\infty$. The set E_∞ is a nontrivial s-minimal cone. Furthermore, E_∞ is a half-space if and only if E is a half-space.*

PROOF. The existence of a limit of E_{r_j} (up to a subsequence) is a consequence of the fact that E_r is an s-minimal set and of Proposition 2.2.13 and Remark 2.2.14.

The fact that E_∞ is s-minimal is a consequence of the s-minimality of the sets $E_{r_{j_k}}$ and their L^1_{loc} convergence to E_∞—see Corollary 2.1.13.

Next we observe that, since E is nontrivial, we can find a point $x \in \partial E$. Thanks to Remark 5.2.1, we then have that

$$E_{x,r_{j_k}} \to E_\infty \text{ in } L^1_{\mathrm{loc}}(\mathbb{R}^{n+1}) \text{ as } k \to \infty.$$

Since $0 \in \partial E_{x,r_{j_k}}$ for every $k \in \mathbb{N}$, we can conclude that E_∞ is a cone by arguing as in [21, Theorem 9.2].

The nontriviality of E_∞ can be established, for instance, by using the uniform density estimates of [21]. Indeed, $0 \in \partial E_{x,r_{j_k}}$ for every $k \in \mathbb{N}$ and hence [21, Theorem 4.1] gives that $\min\{|E_{x,r_{j_k}} \cap B_1|, |B_1 \setminus E_{x,r_{j_k}}|\} \geq c$ for some constant $c > 0$ independent of k. As $E_{x,r_{j_k}} \to E_\infty$ in $L^1(B_1)$, it follows that both E_∞ and its complement have positive measure in B_1. Consequently, E_∞ is neither the empty set nor the whole $\mathbb{R}^{n+1}$.

Finally, if E_∞ is a half-space, one can deduce the flatness of ∂E from the ε-regularity theory of [21, Section 6] and the fact that $\partial E_{r_{j_k}} \to \partial E_\infty$ in the Hausdorff sense, thanks to the uniform density estimates. See, e.g., [58, Lemma 3.1] for more details on this argument. $\square$

5.3. Proof of Theorem 5.1.5

In this section we include a proof of the splitting result stated in the introduction, namely Theorem 5.1.5. The argument leading to it is based on the following classification result for nonlocal minimal cones that contain their translates. For classical minimal cones, it was proved in [70].

PROPOSITION 5.3.1. *Let $\mathscr{C} \subseteq \mathbb{R}^{n+1}$ be an s-minimal cone and assume that*

$$\mathscr{C} + v \subseteq \mathscr{C} \tag{5.6}$$

for some $v \in \mathbb{R}^{n+1} \setminus \{0\}$. Then, $\mathscr{C}$ is either a half-space or a cylinder in direction v.

PROOF. First of all, we notice that, since $\mathscr{C}$ is a cone and inclusion (5.6) holds true, the function $w := -\nu_{\mathscr{C}} \cdot v$ satisfies

$$w \geq 0 \quad \text{in } \partial^* \mathscr{C}. \tag{5.7}$$

To see this, let $x \in \partial^* \mathscr{C}$ and observe that, $\mathscr{C}$ being a cone, we have that $\mu x \in \overline{\mathscr{C}}$ for every $\mu > 0$. But then $\mu x + v \in \overline{\mathscr{C}} + v$ and, using (5.6), it follows that $\mu x + v \in \overline{\mathscr{C}}$. Consequently, $\mu \lambda x + \lambda v = \lambda(\mu x + v) \in \overline{\mathscr{C}}$ for every $\lambda, \mu > 0$. Choosing $\mu = 1/\lambda$ we get that $x + \lambda v \in \overline{\mathscr{C}}$ for every $\lambda > 0$, which gives that v points inside $\overline{\mathscr{C}}$. Recalling that the normal $\nu_{\mathscr{C}}$ points outside $\mathscr{C}$, we are immediately led to (5.7).

Now, by [19, Theorem 1.3(i)] we know that w solves

$$\mathscr{L}w + c^2 w = 0 \quad \text{in } \partial^*\mathscr{C}, \tag{5.8}$$

where

$$\mathscr{L}w(x) := \text{P.V.} \int_{\partial^*\mathscr{C}} \frac{w(y) - w(x)}{|x-y|^{n+1+s}} \, d\mathcal{H}^n_y,$$
$$c^2(x) := \frac{1}{2} \int_{\partial^*\mathscr{C}} \frac{|\nu_{\mathscr{C}}(x) - \nu_{\mathscr{C}}(y)|^2}{|x-y|^{n+1+s}} \, d\mathcal{H}^n_y,$$

for every $x \in \partial^*\mathscr{C}$. As $c^2 \geq 0$ in $\partial^* C$ and (5.7) holds true, we deduce from (5.8) that w is $\mathscr{L}$-superharmonic in $\partial^*\mathscr{C}$, i.e.,

$$-\mathscr{L}w \geq 0 \quad \text{in } \partial^*\mathscr{C}.$$

By [19, Corollary 6.8] (and the lower perimeter bound reported in [19, Theorem 3.1]), we then infer that, for every point $x \in \partial^*\mathscr{C}$ and radius $R > 0$, the function w satisfies

$$\inf_{B_R(x)\cap\partial^*\mathscr{C}} w \geq c_\star R^{1+s} \int_{\partial^*\mathscr{C}} \frac{w(y)}{(R + |y-x|)^{n+1+s}} \, d\mathcal{H}^n_y,$$

for some constant $c_\star \in (0,1]$ depending only on n and s.

Accordingly, either $w = 0$ in the whole $\partial^*\mathscr{C}$ or $\inf_{B_R(x)\cap\partial^*\mathscr{C}} w \geq c_{x,R}$ for some constant $c_{x,R} > 0$ and for every $x \in \partial^*\mathscr{C}$ and $R > 0$. In the first case, it is easy to see that $\mathscr{C}$ must be a cylinder in direction v. If the second situation occurs, then $\partial\mathscr{C}$ is a locally Lipschitz graph with respect to the direction v (see, e.g., [83, Theorem 5.6]), and hence smooth, due to [58, Theorem 1.1]. It being a cone, we conclude that $\mathscr{C}$ must be a half-space. □

With this in hand, we may now proceed to prove the splitting result.

Proof of Theorem 5.1.5. Let E denote the subgraph of u, as defined by (5.1). We recall that, as observed right before the statement of Theorem 5.1.1, the set E is s-minimal.

Let $\mathscr{C}$ be a blow-down cone of E. By definition, there exists a diverging sequence r_j for which $E_{r_j} = E/r_j \to \mathscr{C}$ in $L^1_{\text{loc}}(\mathbb{R}^{n+1})$. As noticed in Lemma 5.2.2, $\mathscr{C}$ is a nontrivial s-minimal cone. Moreover, $\mathscr{C}$ is not an half-space, since, otherwise, E would be a half-space too (again, by Lemma 5.2.2), contradicting the hypothesis that E is the subgraph of a non-affine function. We also recall that this is equivalent to the cone $\mathscr{C}$ being singular.

As E is a subgraph, it follows that $E - te_{n+1} \subseteq E$ for every $t > 0$. This yields that $E_{r_j} - e_{n+1} \subseteq E_{r_j}$ for every j. Hence, by $L^1_{\text{loc}}(\mathbb{R}^{n+1})$ convergence, $\mathscr{C} - e_{n+1} \subseteq \mathscr{C}$. Since $\mathscr{C}$ is not a half-space, by Proposition 5.3.1 we conclude that $\mathscr{C}$ is a cylinder in direction e_{n+1}, that is

$$\mathscr{C} + \lambda e_{n+1} = \mathscr{C} \quad \text{for every } \lambda \in \mathbb{R}, \tag{5.9}$$

or, equivalently, $\mathscr{C} = \mathscr{C}' \times \mathbb{R}$, for some singular s-minimal cone $\mathscr{C}' \subseteq \mathbb{R}^n$. Observe that the s-minimality of $\mathscr{C}'$ is a consequence of [21, Theorem 10.1]. Also note that to obtain (5.9) we only took advantage of the fact that E is an s-minimal subgraph and not the hypotheses on the partial derivatives of u.

Let now $i = 1, \dots, k$ be fixed. By the bound from below on the partial derivative $\frac{\partial u}{\partial x_i}$ and the fundamental theorem of calculus, there exists a constant $\kappa > 0$ such that

$$u(z' + te_i) - u(z') = \int_0^t \frac{\partial u(z' + \tau e_i)}{\partial x_i} \, d\tau \geq -\kappa t$$

for every $z' \in \mathbb{R}^n$ and $t > 0$. Let now u_j be the function defining the blown-down set E_{r_j}. Clearly, $u_j(z') = u(r_j z')/r_j$ and hence

$$u_j(y' + e_i) - u_j(y') = \frac{u(r_j y' + r_j e_i) - u(r_j y')}{r_j} \geq -\kappa$$

for every $y' \in \mathbb{R}^n$ and $j \in \mathbb{N}$. This means that $E_j - \kappa e_{n+1} + e_i \subseteq E_j$ for every $j \geq 1$. Passing to the limit and using (5.9), we deduce that $\mathscr{C} + e_i = \mathscr{C} - \kappa e_{n+1} + e_i \subseteq \mathscr{C}$. Taking advantage once again of Proposition 5.3.1 and of the fact that $\mathscr{C}$ is not a half-space, we infer that $\mathscr{C}$ is a cylinder in direction e_i for every $i = 1, \ldots, k$. The conclusion of the theorem follows. □

5.4. Proof of Theorem 5.1.1

First of all, we may assume that the partial derivatives of u bounded on one side are the first $n - \ell$. Also, up to flipping the variable x_i, for some $i \in \{1, \ldots, n - \ell\}$, we may suppose that those partial derivatives are all bounded from below. All in all, we have that

$$\frac{\partial u}{\partial x_i} \geq -\kappa \quad \text{for every } i = 1, \ldots, n - \ell,$$

for some constant $\kappa \geq 0$.

If u were not affine, then, by applying Theorem 5.1.5 with $k = n - \ell$, we would have that every blow-down cone $\mathscr{C}$ of the set E defined by (5.1) is given by

$$\mathscr{C} = \mathbb{R}^k \times P \times \mathbb{R},$$

for some singular s-minimal cone $P \subseteq \mathbb{R}^{n-k} = \mathbb{R}^\ell$. As this contradicts assumption ($P_{s,\ell}$), we conclude that u must be affine.

5.5. Proof of Theorem 5.1.4

Let Π be a half-space contained in E. Without loss of generality, we may assume that $\Pi = \{x \in \mathbb{R}^n \,|\, x_{n+1} < 0\}$. Consider then a blow-down $\mathscr{C}$ of E, which is a nontrivial s-minimal cone, by Lemma 5.2.2. In particular, $\Pi \subseteq \mathscr{C}$ and $0 \in \partial\Pi \cap \partial\mathscr{C}$. Using, e.g., [21, Corollary 6.2], we infer that $\mathscr{C} = \Pi$ and therefore that E is half-space as well, thanks again to Lemma 5.2.2.

5.6. Proof of Theorem 5.1.3

Suppose by contradiction that the function u is not affine and denote with E its subgraph. Up to a translation of E in the vertical direction, hypothesis (5.4) yields that E contains the cone

$$\mathscr{D} := \left\{ x \in \mathbb{R}^{n+1} \,|\, x_{n+1} < -C|x'| \right\}.$$

Consider now a blow-down $\mathscr{C}$ of E. On the one hand, we clearly have that $\mathscr{D} \subseteq \mathscr{C}$. On the other hand, by arguing as in the beginning of the proof of Theorem 5.1.5, we have that $\mathscr{C}$ must be a nontrivial vertical cylinder. More precisely, $\mathscr{C} = \mathscr{C}' \times \mathbb{R}$, for some nontrivial singular s-minimal cone $\mathscr{C}' \subseteq \mathbb{R}^n$. These two facts imply that $\mathscr{C}' = \mathbb{R}^n$, contradicting its nontriviality. This concludes the proof.

Remark 5.6.1. By a refinement of this argument we can prove a stronger version of Theorem 5.1.3, where hypothesis (5.4) is replaced by

$$u(x') \geq -C(1 + |x'|) \quad \text{for every } x' \in \mathbb{R}^n \text{ such that } x_1 < 0. \tag{5.10}$$

Indeed, arguing by contradiction as before, we see that any blow-down of the subgraph of u is a cylinder of the form $\mathscr{C}' \times \mathbb{R}$. In light of (5.10), the cone $\mathscr{C}'$ contains a half-space of $\mathbb{R}^n$ and is thus flat, due to Theorem 5.1.4. This leads to a contradiction.

5.7. Subgraphs of constant fractional mean curvature

We recall—see Chapter 4—that, given a measurable function $u : \mathbb{R}^n \to \mathbb{R}$, we can understand $\mathscr{H}_s u$ as a linear form on the fractional Sobolev space $W^{s,1}(\mathbb{R}^n)$, setting

$$\langle \mathscr{H}_s u, v \rangle := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} G_s\left(\frac{u(x') - u(y')}{|x' - y'|} \right) (v(x') - v(y')) \, \frac{dx' dy'}{|x' - y'|^{n+s}}$$

for every $v \in W^{s,1}(\mathbb{R}^n)$. This definition is indeed well-posed since G_s is bounded.

Let h be a real number. We say that a measurable function $u : \mathbb{R}^n \to \mathbb{R}$ is a weak solution of $\mathscr{H}_s u = h$ in $\mathbb{R}^n$ if it holds

$$\langle \mathscr{H}_s u, v \rangle = h \int_{\mathbb{R}^n} v(x') \, dx' \quad \text{for every } v \in W^{s,1}(\mathbb{R}^n).$$

We remark that by the density of $C_c^\infty(\mathbb{R}^n)$ in $W^{s,1}(\mathbb{R}^n)$, it is equivalent to consider the test functions v to be smooth and compactly supported.

We now prove that if the s-mean curvature of a global subgraph is constant, then this constant must be zero. More precisely, we have the following statement.

PROPOSITION 5.7.1. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be a weak solution of $\mathscr{H}_s u = h$ in $\mathbb{R}^n$, for some constant $h \in \mathbb{R}$. Then $h = 0$.*

PROOF. Recalling (5.2), we notice that

$$|G_s(t)| \le \int_0^{+\infty} \frac{d\tau}{(1 + \tau^2)^{\frac{n+1+s}{2}}} = \frac{\Lambda_{n,s}}{2} < +\infty \quad \text{for every } t \in \mathbb{R}.$$

Suppose that $h \ge 0$—the case $h \le 0$ is analogous. Let $R > 0$ and consider the test function $v = \chi_{B'_R} \in W^{s,1}(\mathbb{R}^n)$. We have

$$|\langle \mathscr{H}_s u, \chi_{B'_R} \rangle| \le \Lambda_{n,s} \int_{B'_R} \int_{\mathbb{R}^n \setminus B'_R} \frac{dx' dy'}{|x' - y'|^{n+s}} = C R^{n-s},$$

for some constant $C > 0$ depending only on n an s. Since u weakly solves $\mathscr{H}_s = h$ in $\mathbb{R}^n$, we deduce that

$$h |B'_1| R^n = h \int_{\mathbb{R}^n} \chi_{B'_R}(x') \, dx' = \langle \mathscr{H}_s u, \chi_{B'_R} \rangle \le C R^{n-s}$$

for all $R > 0$, that is $0 \le h R^s \le C / |B'_1|$. Letting $R \to +\infty$ we conclude that $h = 0$. □

We point out that, as a consequence of Proposition 5.7.1 and the results of Corollary 4.1.12, if a function $u \in W^{s,1}_{\mathrm{loc}}(\mathbb{R}^n)$ is a weak solution of $\mathscr{H}_s u = h$ in $\mathbb{R}^n$, then the subgraph of u must be an s-minimal set—thus extending to the nonlocal framework a celebrated result of Chern, namely the Corollary of Theorem 1 in [26].

We further remark that other definitions for solutions of the equation $\mathscr{H}_s u = h$ could have been considered, namely smooth pointwise solutions and viscosity solutions (for a rigorous definition see Definition 4.3.2). However, it is readily seen that a smooth pointwise solution is also a viscosity solution. Moreover, Corollary 4.1.8 shows that a viscosity solution is also a weak solution. Consequently, Proposition 5.7.1 applies to these other two notions of solutions as well.

CHAPTER 6

A free boundary problem: superposition of nonlocal energy plus classical perimeter

In this chapter we study the minimizers of the functional

$$\mathcal{N}(u,\Omega) + \mathrm{Per}\big(\{u>0\},\Omega\big),$$

with $\mathcal{N}(u,\Omega)$ being, roughly speaking, the Ω-contribution to the H^s seminorm of a function $u:\mathbb{R}^n\to\mathbb{R}$

The main contributions of the present chapter consist in establishing a monotonicity formula for the minimizers, in exploiting it to investigate the properties of blow-up limits and in proving a dimension reduction result. Moreover, we show that, when $s<1/2$, the perimeter dominates—in some sense—over the nonlocal energy. As a consequence, we obtain a regularity result for the free boundary $\{u=0\}$.

6.1. Introduction: definitions and main results

Let us begin by giving the rigorous definition of the functional that we are going to study.

Given $s\in(0,1)$ and a bounded open set $\Omega\subseteq\mathbb{R}^n$ with Lipschitz boundary, we consider the functional

$$\mathcal{F}_\Omega(u,E) := \iint_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \mathrm{Per}(E,\Omega), \tag{6.1}$$

where E is the positivity set of the function $u:\mathbb{R}^n\to\mathbb{R}$, that is

$$u\geq 0 \quad \text{a.e. in } E \qquad \text{and} \qquad u\leq 0 \quad \text{a.e. in } \mathcal{C}E.$$

We call such a pair (u, E) an *admissible pair*. Here above $\mathcal{C}E$ denotes the complement of E and $\operatorname{Per}(E, \Omega)$ denotes the (classical) perimeter of E in Ω.

Furthermore, we write

$$
\begin{aligned}
\mathcal{N}(u,\Omega) &:= \iint_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy \\
&= \iint_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + 2\iint_{\Omega\times\mathcal{C}\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,
\end{aligned} \tag{6.2}
$$

for the nonlocal energy of u appearing in the definition of $\mathcal{F}_\Omega$. Roughly speaking, this is the Ω-contribution to the H^s seminorm of u.

We will consider the following definition of minimizing pair.

Definition 6.1.1. *Given an admissible pair (u, E), we say that a pair (v, F) is an admissible competitor (for $\mathcal{F}_\Omega$ with respect to the pair (u, E)) if*

$$
\begin{aligned}
&\mathit{supp}(v-u) \Subset \Omega, \qquad F\Delta E \Subset \Omega, \\
&v-u \in H^s(\mathbb{R}^n) \qquad \mathit{and} \qquad \operatorname{Per}(F,\Omega) < +\infty.
\end{aligned} \tag{6.3}
$$

We say that the admissible pair (u, E) is minimizing in Ω if $\mathcal{F}_\Omega(u, E) < +\infty$ and

$$\mathcal{F}_\Omega(u,E) \le \mathcal{F}_\Omega(v,F),$$

for every admissible competitor (v, F).

We observe that in Proposition 6.2.9 we will provide some equivalent characterizations of minimizing pairs.

In particular, we are interested in the following minimization problem, with respect to fixed "exterior data". Given an admissible pair (u_0, E_0) and a bounded open set $\mathcal{O} \subseteq \mathbb{R}^n$ with Lipschitz boundary, such that

$$\Omega \Subset \mathcal{O}, \qquad \mathcal{N}(u_0,\Omega) < +\infty \qquad \text{and} \qquad \operatorname{Per}(E_0,\mathcal{O}) < +\infty,$$

we want to find an admissible pair (u, E) attaining the following infimum

$$
\begin{aligned}
\inf \big\{ \mathcal{N}(v,\Omega) + \operatorname{Per}(F,\mathcal{O}) \,|\, (v,F) \text{ admissible s.t. } v = u_0 \text{ a.e. in } \mathcal{C}\Omega \\
\text{and } F\setminus\Omega = E_0\setminus\Omega \big\}.
\end{aligned} \tag{6.4}
$$

Roughly speaking, as customary when dealing with minimization problems involving the classical perimeter, we are considering a (fixed) neighborhood $\mathcal{O}$ of Ω (as small as we like) in order to "read" the boundary data $\partial E_0 \cap \partial\Omega$.

In Section 6.2 we prove the existence of pairs solving this Dirichlet problem. Moreover, we show that a pair (u, E) realizing the infimum in (6.4) is also a minimizing pair in the sense of Definition 6.1.1.

Concerning the minimizers of the functional $\mathcal{F}$, we also establish the following uniform energy estimates, which turn out to be important when proving the existence of blow-up limits.

Theorem 6.1.2. *Let (u, E) be a minimizing pair in B_2. Then*

$$\iint_{\mathbb{R}^{2n}\setminus(\mathcal{C}B_1)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \operatorname{Per}(E,B_1) \le C\left(1 + \int_{\mathbb{R}^n} \frac{|u(y)|^2}{1+|y|^{n+2s}}\,dy\right),$$

for some $C = C(n, s) > 0$.

In order to study blow-up sequences, we will need a "localized" version of $\mathcal{N}(\,\cdot\,,\Omega)$ which is obtained through an extension technique studied in [23]. To be more precise, given a function $u:\mathbb{R}^n\to\mathbb{R}$, we consider the function $\overline{u}:\mathbb{R}^{n+1}_+\to\mathbb{R}$, where

$$\mathbb{R}^{n+1}_+ := \{(x,z)\in\mathbb{R}^{n+1}\,|\,x\in\mathbb{R}^n,\ z>0\},$$

defined via the convolution with an appropriate Poisson kernel,

$$\overline{u}(\,\cdot\,,z) = u*\mathcal{K}_s(\,\cdot\,,z),\quad \text{where}\quad \mathcal{K}_s(x,z) := c_{n,s}\frac{z^{2s}}{(|x|^2+z^2)^{(n+2s)/2}}. \tag{6.5}$$

Here above, $c_{n,s}>0$ is an appropriate normalizing constant. We observe that the extended function $\overline{u}$ is well defined, provided the function u belongs to the weighted Lebesgue space

$$L_s(\mathbb{R}^n) := \left\{u:\mathbb{R}^n\to\mathbb{R}\,\Big|\,\int_{\mathbb{R}^n}\frac{|u(\xi)|}{1+|\xi|^{n+2s}}d\xi<+\infty\right\}.$$

For a proof of this fact and for a detailed introduction to the extension operator, we refer the interested reader to [75]. In light of Remark 6.2.1, we can thus consider the extended function of a minimizer.

We use capital letters, like $X=(x,z)$, to denote points in $\mathbb{R}^{n+1}$. Given a set $\Omega\subseteq\mathbb{R}^{n+1}$, we write

$$\Omega_+ := \Omega\cap\{z>0\}\qquad\text{and}\qquad \Omega_0 := \Omega\cap\{z=0\}.$$

Moreover we identify the hyperplane $\{z=0\}\simeq\mathbb{R}^n$ via the projection function.

In particular, we exploit the energy naturally associated to the extension problem to define an extended functional, which has a local behavior.

To be more precise, given a bounded open set $\Omega\subseteq\mathbb{R}^{n+1}$ with Lipschitz boundary, such that $\Omega_0\neq\varnothing$, we define

$$\mathfrak{F}_\Omega(\mathcal{V},F) := c'_{n,s}\int_{\Omega_+}|\nabla\mathcal{V}|^2z^{1-2s}\,dX + \mathrm{Per}(F,\Omega_0), \tag{6.6}$$

for $\mathcal{V}:\mathbb{R}^{n+1}_+\to\mathbb{R}$ and $F\subseteq\mathbb{R}^n\simeq\{z=0\}$ the positivity set of the trace of $\mathcal{V}$ on $\{z=0\}$, that is

$$\mathcal{V}\big|_{\{z=0\}}\geq 0\quad\text{a.e. in } F\quad\text{and}\quad \mathcal{V}\big|_{\{z=0\}}\leq 0\quad\text{a.e. in } \mathcal{C}F.$$

We call such a pair $(\mathcal{V},F)$ an *admissible pair* for the extended functional.

From now on, whenever considering the extended functional, unless otherwise stated we will implicitly assume that the open set $\Omega\subseteq\mathbb{R}^{n+1}$ is such that $\Omega_0\neq\varnothing$.

DEFINITION 6.1.3. *Given an admissible pair $(\mathcal{U},E)$ such that $\mathfrak{F}_\Omega(\mathcal{U},E)<+\infty$, we say that a pair $(\mathcal{V},F)$ is an admissible competitor (for $\mathfrak{F}_\Omega$ with respect to $(\mathcal{U},E)$) if $\mathfrak{F}_\Omega(\mathcal{V},F)<+\infty$ and*

$$\mathit{supp}\,(\mathcal{V}-\mathcal{U})\Subset\Omega\qquad\mathit{and}\qquad E\Delta F\Subset\Omega_0.$$

We say that an admssible pair $(\mathcal{U},E)$ is minimal in Ω if $\mathfrak{F}_\Omega(\mathcal{U},E)<+\infty$ and

$$\mathfrak{F}_\Omega(\mathcal{U},E)\leq\mathfrak{F}_\Omega(\mathcal{V},F),$$

for every admissible competitor $(\mathcal{V},F)$.

We will study this extended functional in Section 6.3. In particular, we relate minimizers of the extended functional $\mathfrak{F}$ with minimizers of the original functional $\mathcal{F}$, proving the following:

PROPOSITION 6.1.4. *Let (u, E) be an admissible pair for $\mathcal{F}$, according to Definition 6.1.1, such that $\mathcal{F}_{B_R}(u.E) < +\infty$. Then, the pair (u, E) is minimizing in B_R if and only if the pair $(\overline{u}, E)$ is minimizing for $\mathfrak{F}_\Omega$, for every bounded open set $\Omega \subseteq \mathbb{R}^{n+1}$ with Lipschitz boundary and such that $\Omega_0 \Subset B_R$.*

We now introduce the following notation

$$\mathcal{B}_r := \{(x, z) \in \mathbb{R}^{n+1} \,|\, |x|^2 + z^2 < r^2\}, \qquad \mathcal{B}_r^+ := \mathcal{B}_r \cap \{z > 0\},$$

and

$$(\partial\mathcal{B}_r)^+ := \partial\mathcal{B}_r \cap \{z > 0\} = \{(x, z) \in \mathbb{R}^{n+1}_+ \,|\, |x|^2 + z^2 = r^2\}.$$

The main reason for considering the extended functional consists in the fact that it allows us to obtain a Weiss-type monotonicity formula—by exploiting a scaled and "corrected" version of the functional $\mathfrak{F}_{\mathcal{B}_r}$. More precisely:

THEOREM 6.1.5 (Weiss-type Monotonicity Formula). *Let (u, E) be a minimizing pair for $\mathcal{F}$ in B_R and define the function $\Phi_u : (0, R) \to \mathbb{R}$ by*

$$\Phi_u(r) := r^{1-n}\left(c'_{n,s}\int_{\mathcal{B}_r^+} |\nabla\overline{u}|^2 z^{1-2s}\, dX + \operatorname{Per}(E, B_r)\right) - c'_{n,s}\Big(s - \frac{1}{2}\Big) r^{-n}\int_{(\partial\mathcal{B}_r)^+} \overline{u}^2 z^{1-2s}\, d\mathcal{H}^n.$$

Then, the function Φ_u is increasing in $(0, R)$. Moreover, Φ_u is constant in $(0, R)$ if and only if the extension $\overline{u}$ is homogeneous of degree $s - \frac{1}{2}$ in $\mathcal{B}_R^+$ and E is a cone in B_R.

In order to prove the monotonicity formula, we will need to construct appropriate competitors for the minimizing pair $(\overline{u}, E)$ of the extended functional. For this, we need to consider the cone $E(r)$ spanned by the "spherical slice" $E \cap \partial B_r$, namely

$$E(r) := \{\lambda y \,|\, \lambda > 0,\, y \in E \cap \partial B_r\}. \tag{6.7}$$

In Section 6.8, we show that this cone is indeed well defined for a.e. $r > 0$ and its perimeter in every ball B_ϱ can be computed by means of a simple formula (see Proposition 6.8.4). We mention that for the proof of Theorem 6.1.5—which is in Section 6.4—we will also need a result concerning the surface density of a Caccioppoli set, namely Corollary 6.9.2.

In order to study blow-up sequences, we prove a general convergence result for minimizing pairs under appropriate conditions. More precisely:

THEOREM 6.1.6 (Proof in Section 6.5.2). *Let $(\overline{u}_m, E_m)$ be a sequence of minimizing pairs in $\mathcal{B}_R^+$. Suppose that $\overline{u}_m$ is the extension of u_m, and*

$$u_m \to u \text{ in } L^\infty(B_R), \quad \overline{u}_m \to \overline{u} \text{ in } L^\infty(\mathcal{B}_R^+), \quad \text{and} \quad \big|(E_m \Delta E) \cap B_R\big| \to 0$$

as $m \to +\infty$, for some admissible pair $(\overline{u}, E)$, with $\overline{u}$ continuous in $\overline{\mathbb{R}^{n+1}_+}$, being $\overline{u}$ the extension function of u. Then, $(\overline{u}, E)$ is a minimizing pair in $\mathcal{B}_r^+$, for every $r \in (0, R)$. Furthemore,

$$\lim_{m\to+\infty}\int_{\mathcal{B}_r^+} |\nabla\overline{u}_m|^2 z^{1-2s}\, dX = \int_{\mathcal{B}_r^+} |\nabla\overline{u}|^2 z^{1-2s}\, dX, \quad \forall\, r \in (0, R), \tag{6.8}$$

and

$$D\chi_{E_m} \overset{*}{\rightharpoonup} D\chi_E \quad \text{and} \quad \big|D\chi_{E_m}\big| \overset{*}{\rightharpoonup} |D\chi_E|, \qquad \text{in } B_R. \tag{6.9}$$

In particular,

$$\lim_{m\to+\infty}\operatorname{Per}(E_m, B_r) = \operatorname{Per}(E, B_r), \tag{6.10}$$

for every $r \in (0, R)$ *such that*

$$\mathcal{H}^{n-1}(\partial^* E \cap \partial B_r) = 0.$$

Exploiting the results that we have mentioned so far, we are able to study blow-up limits. Let us first introduce some notation.

Given a function $u : \mathbb{R}^n \to \mathbb{R}$ and a set $E \subseteq \mathbb{R}^n$, we define

$$u_\lambda(x) := \lambda^{\frac{1}{2}-s} u(\lambda x) \qquad \text{and} \qquad E_\lambda := \frac{1}{\lambda} E, \tag{6.11}$$

for every $\lambda > 0$. We observe that the scaling introduced in (6.11) is consistent with the natural scaling of the functionals that we are considering—see Remark 6.4.3.

Given a minimizing pair (u, E), we are interested in the blow-up sequence, that is the sequence of pairs (u_r, E_r) for $r \to 0$. We observe that, as a consequence of the natural scaling of the functionals and of the monotonicity formula, blow-up limits possess homogeneity properties.

We thus introduce the following notion. We say that the admissible pair (u, E) is a *minimizing cone* if it is a minimizing pair in B_R, for every $R > 0$, and it is such that u is homogeneous of degree $s - \frac{1}{2}$ and E is a cone (that is, χ_E is homogeneous of degree 0).

With this, we can now state the following result:

THEOREM 6.1.7 (Proof in Section 6.5.3). *Let* $s > 1/2$ *and* (u, E) *be a minimizing pair in* B_1 *with* $0 \in \partial E$. *Let* (u_r, E_r) *be as in* (6.11). *Assume that* $u \in C^{s-\frac{1}{2}}(B_1)$. *Then, there exist a minimizing cone* (u_0, E_0) *and a sequence* $r_k \searrow 0$ *such that* $u_{r_k} \to u_0$ *in* $L^\infty_{\text{loc}}(\mathbb{R}^n)$ *and* $E_{r_k} \xrightarrow{\text{loc}} E_0$.

We point out that the assumption $u \in C^{s-\frac{1}{2}}(B_1)$ in Theorem 6.1.7 is clearly weaker than asking u to be $C^{s-\frac{1}{2}}$ in the whole of $\mathbb{R}^n$, which is the requirement of [**42**, Theorem 1.3]. In particular, in Theorem 6.1.7 we are not even requiring u to be continuous outside B_1.

In Section 6.6 we observe that in the case $s < 1/2$ the perimeter is, in some sense, the leading term of the functional $\mathcal{F}_\Omega$. As a consequence, we are able to prove the following regularity result for the free boundary ∂E:

THEOREM 6.1.8. *Let* $s \in (0, 1/2)$ *and let* (u, E) *be a minimizing pair in* Ω. *Suppose that* $u \in L^\infty_{\text{loc}}(\Omega)$. *Then* E *has almost minimal boundary in* Ω.

More precisely, if $x_0 \in \Omega$ *and* $d := d(x_0, \Omega)/3$, *then for every* $r \in (0, d]$ *it holds*

$$\operatorname{Per}(E, B_r(x_0)) \le \operatorname{Per}(F, B_r(x_0)) + C\, r^{n-2s}, \qquad \forall\, F \subseteq \mathbb{R}^n \text{ s.t. } E \Delta F \Subset B_r(x_0), \tag{6.12}$$

where

$$C = C\left(s, x_0, d, \|u\|_{L^\infty(B_{2d}(x_0))}, \int_{\mathbb{R}^n} \frac{|u(y)|}{1 + |y|^{n+2s}}\, dy\right) > 0.$$

Therefore

(i) $\partial^* E$ *is locally* $C^{1,\frac{1-2s}{2}}$ *in* Ω,

(ii) the singular set $\partial E \setminus \partial^* E$ *is such that*

$$\mathcal{H}^\sigma\big((\partial E \setminus \partial^* E) \cap \Omega\big) = 0, \qquad \textit{for every } \sigma > n - 8.$$

We conclude this Introduction by mentioning the following dimension reduction result for global minimizers.

Only in the following Theorem and in Section 6.7 we redefine

$$\mathcal{F}_\Omega(u, E) := (c'_{n,s})^{-1} \mathcal{N}(u, \Omega) + \operatorname{Per}(E, \Omega),$$

so that the corresponding extended functional is constant-free.

We say that an admissible pair (u, E) is minimizing in $\mathbb{R}^n$ if it minimizes $\mathcal{F}_\Omega$ in any bounded open subset $\Omega \subseteq \mathbb{R}^n$ (in the sense of Definition 6.1.1).

THEOREM 6.1.9. *Let (u, E) be an admissible pair and define*

$$u^\star(x, x_{n+1}) := u(x) \qquad \text{and} \qquad E^\star := E \times \mathbb{R}.$$

Then, the pair (u, E) is minimizing in $\mathbb{R}^n$ if and only if the pair $(u^\star, E^\star)$ is minimizing in $\mathbb{R}^{n+1}$.

6.1.1. Notation and assumptions. Throughout the chapter Ω will be a bounded open set with Lipschitz boundary, unless otherwise stated.

Like we did in the previous chapters, we will make the following assumption regarding the sets that we consider.

6.1.1.1. *Measure theoretic assumption.* Let $F \subseteq \mathbb{R}^n$. Up to modifications in sets of measure zero, we can assume that F coincides with the set $F^{(1)}$ of points of density 1, which is a "good representative" for F in its L^1_{loc} class. In particular, we can thus assume that F contains its measure theoretic interior

$$F_{int} := \{x \in \mathbb{R}^n \,|\, \exists\, r > 0 \text{ s.t. } |F \cap B_r(x)| = \omega_n r^n\} \subseteq F,$$

the complementary $\mathcal{C}F$ contains its measure theoretic interior,

$$F_{ext} := \{x \in \mathbb{R}^n \,|\, \exists\, r > 0 \text{ s.t. } |F \cap B_r(x)| = 0\} \subseteq \mathcal{C}F,$$

and the topological boundary of F coincides with the measure theoretic boundary, $\partial F = \partial^- F$, where

$$\partial^- F := \mathbb{R}^n \setminus \big(F_{int} \cup F_{ext}\big) = \{x \in \mathbb{R}^n \,|\, 0 < |F \cap B_r(x)| < \omega_n r^n \;\forall\, r > 0\}. \tag{6.13}$$

For the details, we refer to Appendix A and Section 6.8.

6.2. Preliminary results

In this section we will prove some basic properties, such as the existence of a minimizing pair (u, E) for the functional $\mathcal{F}$ (using the direct method of Calculus of Variations) and the s-harmonicity of the function u. We also establish a comparison principle for minimizers. Finally, we show that if (u, E) is minimizing in Ω, then it is minimizing in every $\Omega' \Subset \Omega$.

We first point out the following useful remarks about the "tail energies". Given $s \in (0, 1)$ we define the weighted Lebesgue space

$$L^2_s(\mathbb{R}^n) := \left\{ u : \mathbb{R}^n \to \mathbb{R} \,\Big|\, \int_{\mathbb{R}^n} \frac{|u(\xi)|^2}{1 + |\xi|^{n+2s}} d\xi < +\infty \right\}.$$

REMARK 6.2.1. We observe that we have the continuous embedding

$$L^2_s(\mathbb{R}^n) \subseteq L_s(\mathbb{R}^n).$$

Indeed, if $u \in L^2_s(\mathbb{R}^n)$, then by Holder's inequality we have

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{|u(y)|}{1 + |y|^{n+2s}}\, dy &= \int_{\mathbb{R}^n} \frac{|u(y)|}{\big(1 + |y|^{n+2s}\big)^{\frac{1}{2}}} \frac{dy}{\big(1 + |y|^{n+2s}\big)^{\frac{1}{2}}} \\ &\le \left(\int_{\mathbb{R}^n} \frac{|u(y)|^2}{1 + |y|^{n+2s}}\, dy \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^n} \frac{dy}{1 + |y|^{n+2s}} \right)^{\frac{1}{2}} < +\infty. \end{aligned}$$

Moreover, it trivially holds true that

$$L^2_s(\mathbb{R}^n) \subseteq L^2_{\text{loc}}(\mathbb{R}^n).$$

Finally, we point out that, if $\Omega \subseteq \mathbb{R}^n$ is a bounded open set and if $u : \mathbb{R}^n \to \mathbb{R}$ is a measurable function, then

$$\mathcal{N}(u, \Omega) < +\infty \qquad \Longrightarrow \qquad u \in L^2_s(\mathbb{R}^n).$$

For the proof of this observation we refer, e.g., to Lemma D.1.3.

6.2.1. Existence of a minimizing pair for the Dirichlet problem and s-harmonicity. We begin by observing that, even if the choice of the neighborhood $\mathcal{O} \Supset \Omega$ for the Dirichlet problem is arbitrary, it does not influence the minimization problem (provided that the positivity set of the exterior data is regular enough).

REMARK 6.2.2. Let $\Omega \Subset \mathcal{O}' \Subset \mathcal{O}$. Let $E_0 \subseteq \mathbb{R}^n$ be such that

$$\operatorname{Per}(E_0, \mathcal{O}) < +\infty.$$

Then

$$\operatorname{Per}(E, \mathcal{O}) = \operatorname{Per}(E, \mathcal{O}') + \operatorname{Per}(E_0, \mathcal{O} \setminus \mathcal{O}'), \qquad \forall\, E \subseteq \mathbb{R}^n \text{ s.t. } E \setminus \Omega = E_0 \setminus \Omega. \tag{6.14}$$

In particular, the minimization problem (6.4) "does not depend" on the choice of $\mathcal{O} \supset\supset \Omega$, in the sense that if the exterior data (u_0, E_0) is an admissible pair such that

$$\mathcal{N}(u_0, \Omega) < +\infty \qquad \text{and} \qquad \operatorname{Per}(E_0, \mathcal{O}) < +\infty,$$

then a pair (u, E) realizes the infimum

$$\inf \big\{ \mathcal{N}(v, \Omega) + \operatorname{Per}(F, \mathcal{O}) \,|\, (v, F) \text{ admissible s.t. } v = u_0 \text{ a.e. in } \mathcal{C}\Omega \\ \text{and } F \setminus \Omega = E_0 \setminus \Omega \big\}$$

if and only if it realizes the infimum

$$\inf \big\{ \mathcal{N}(v, \Omega) + \operatorname{Per}(F, \mathcal{O}') \,|\, (v, F) \text{ admissible s.t. } v = u_0 \text{ a.e. in } \mathcal{C}\Omega \\ \text{and } F \setminus \Omega = E_0 \setminus \Omega \big\},$$

for every $\Omega \Subset \mathcal{O}' \Subset \mathcal{O}$.

Given a fixed bounded open set $\mathcal{O} \subseteq \mathbb{R}^n$ with Lipschitz boundary such that $\Omega \Subset \mathcal{O}$, we denote

$$\overline{\mathcal{F}}_\Omega(u, E) := \mathcal{N}(u, \Omega) + \operatorname{Per}(E, \mathcal{O}). \tag{6.15}$$

We notice that $\overline{\mathcal{F}}_\Omega$ is the functional involved in the minimization of the Dirichlet problem (6.4).

Now we show that Definition 6.1.1 is compatible with the minimization of $\overline{\mathcal{F}}_\Omega$, as given by (6.4).

LEMMA 6.2.3. *A pair (u, E) realizing the infimum in (6.4) is a minimizing pair in the sense of Definition 6.1.1.*

PROOF. First of all, notice that

$$\operatorname{Per}(E, \mathcal{O}) < +\infty.$$

Now let (v, F) be an admissible competitor for (u, E), according to Definition 6.1.1. Then

$$F \setminus \Omega' = E \setminus \Omega',$$

for some $\Omega' \Subset \Omega$ (with Lipschitz boundary), thanks to (6.3). So, by (6.14), we have that

$$\operatorname{Per}(F, \mathcal{O}) = \operatorname{Per}(F, \Omega) + \operatorname{Per}(F, \mathcal{O} \setminus \Omega) = \operatorname{Per}(F, \Omega) + \operatorname{Per}(E, \mathcal{O} \setminus \Omega).$$

Therefore, recalling (6.1) and (6.15), we conclude that

$$\mathcal{F}_\Omega(v, F) - \mathcal{F}_\Omega(u, E) = \overline{\mathcal{F}}_\Omega(v, F) - \overline{\mathcal{F}}_\Omega(u, E) \geq 0,$$

which gives the desired result. □

Definition 6.2.4. *We will say that a pair (u, E) minimizing the Dirichlet problem in (6.4) is a minimizing pair for $\overline{\mathcal{F}}_\Omega$ (with respect to the exterior data (u_0, E_0)).*

In particular, Lemma 6.2.3 says that a minimizing pair according to Definition 6.2.4 is a minimizing pair according to Definition 6.1.1. Now we show that there exists a minimizer for $\overline{\mathcal{F}}_\Omega$, as given by Definition 6.2.4:

Lemma 6.2.5. *Let $\mathcal{O} \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary such that $\Omega \Subset \mathcal{O}$ and let (u_0, E_0) be an admissible pair for (6.4) such that*

$$\mathcal{N}(u_0, \Omega) < +\infty \quad \textit{and} \quad \operatorname{Per}(E_0, \mathcal{O}) < +\infty. \tag{6.16}$$

Then, there exists a minimizing pair (u, E) for $\overline{\mathcal{F}}_\Omega$ with respect to the exterior data (u_0, E_0).

Proof. Since (u_0, E_0) is an admissible competitor, we have that

$$\begin{aligned} \inf \big\{ \mathcal{N}(v, \Omega) + \operatorname{Per}(F, \mathcal{O}) \,|\, (v, F) \text{ admissible s.t. } v = u_0 \text{ a.e. in } \mathcal{C}\Omega \\ \text{and } F \setminus \Omega = E_0 \setminus \Omega \big\} \\ \leq \overline{\mathcal{F}}_\Omega(u_0, E_0) < +\infty, \end{aligned}$$

thanks to (6.16).

Now let (u_k, E_k) be a minimizing sequence and notice that

$$[u_k]^2_{H^s(\Omega)} + \operatorname{Per}(E_k, \mathcal{O}) \leq \overline{\mathcal{F}}_\Omega(u_k, E_k) \leq M \qquad \text{for every } k,$$

for some $M > 0$. Thus by compactness (see, e.g., [38, Theorem 7.1] and [68, Theorem 1.19]) we have that

$$\begin{aligned} &u_k \to u \quad \text{in } L^2(\Omega) \quad \text{and a.e. in } \Omega, \\ &\chi_{E_k} \to \chi_E \quad \text{in } L^1(\mathcal{O}) \quad \text{and a.e. in } \mathcal{O} \quad \text{and} \quad E_k \setminus \Omega = E_0 \setminus \Omega, \end{aligned}$$

as $k \to +\infty$, up to subsequences. Since the functions u_k are fixed outside Ω, we actually have that $u_k \to u$ a.e. in $\mathbb{R}^n$. Therefore, by Fatou's Lemma, we get

$$\mathcal{N}(u, \Omega) \leq \liminf_{k \to +\infty} \mathcal{N}(u_k, \Omega). \tag{6.17}$$

We remark that the perimeter functional $\operatorname{Per}(\,\cdot\,, \mathcal{O})$ is lower semicontinuous with respect to L^1_{loc} convergence of sets (see, e.g., [68, Theorem 1.9]). This and (6.17) imply that $\overline{\mathcal{F}}_\Omega(u, E)$ attains the desired minimum.

Hence, to complete the proof of Lemma 6.2.5, we only need to check that

$$u \geq 0 \quad \text{a.e. in } E \cap \Omega \quad \text{and} \quad u \leq 0 \quad \text{a.e. in } \mathcal{C}E \cap \Omega, \tag{6.18}$$

to guarantee that (u, E) is an admissible pair.

To prove (6.18), we observe that, for a.e. $x \in E \cap \Omega$,

$$u_k(x) \to u(x) \quad \text{and} \quad \chi_{E_k}(x) \to \chi_E(x) = 1,$$

and hence $\chi_{E_k}(x) = 1$ for every k large enough. Therefore, for a.e. such x, we have that $u_k(x) \geq 0$ for all k large enough, and so also $u(x) \geq 0$, which proves the first part of (6.18). A similar argument holds for $\mathcal{C}E \cap \Omega$, thus completing the proof of (6.18). □

Thanks to Lemmata 6.2.3 and 6.2.5, we obtain the existence of a minimizing pair in the sense of Definition 6.1.1. In the next result we state the s-harmonicity of the function u of a minimizing pair (u, E):

LEMMA 6.2.6. *Let (u, E) be a minimizing pair in Ω, according to Definition 6.1.1. If $\mathcal{O} \subseteq \Omega$ is an open set such that*

$$\inf_{\mathcal{O}} |u| \geq \delta,$$

for some $\delta > 0$, then

$$(-\Delta)^s u(x) = 0 \qquad \text{for any } x \in \mathcal{O}.$$

In particular, if $u \in C(\Omega)$, then $(-\Delta)^s u = 0$ in $\Omega \setminus \{u = 0\}$.

The proof of the s-harmonicity of u is the same as in [42, Lemma 3.2], so we omit the proof here. Roughly speaking, since the Euler-Lagrange functional associated to the functional $\mathcal{N}$ in (6.2) is the fractional s-Laplacian, the idea consists in considering small perturbating functions u_ε having as positivity set the positivity set E of u, so that when we look at the difference between the energies we get

$$0 \leq \mathcal{F}_\Omega(u_\varepsilon, E) - \mathcal{F}_\Omega(u, E) = \mathcal{N}(u_\varepsilon, \Omega) - \mathcal{N}(u, \Omega).$$

LEMMA 6.2.7. *Let (u, E) be a minimizing pair for $\overline{\mathcal{F}}_\Omega$, with respect to the exterior data (u_0, E_0) and let $\alpha \in \mathbb{R}$. If*

$$u_0 \geq \alpha \qquad \text{a.e. in } \mathcal{C}\Omega \quad (\text{respectively } u_0 \leq \alpha \text{ a.e. in } \mathcal{C}\Omega),$$

then

$$u \geq \alpha \qquad \text{a.e. in } \mathbb{R}^n \quad (\text{respectively } u \leq \alpha \text{ a.e. in } \mathbb{R}^n).$$

The proof of the comparison principle in Lemma 6.2.7 is the same as in [42, Lemma 3.3], so we omit it.

6.2.2. Equivalent characterizations of a minimizing pair. In this subsection, we give some equivalent definitions of the notion of minimizing pair.

First of all, notice that, if $\Omega' \subseteq \Omega$, then the functional in (6.2) can be written as

$$\mathcal{N}(v, \Omega) = \mathcal{N}(v, \Omega') + [v]^2_{H^s(\Omega \setminus \Omega')} + 2 \iint_{(\Omega \setminus \Omega') \times \mathcal{C}\Omega} \frac{|v(x) - v(y)|^2}{|x - y|^{n+2s}}\, dx\, dy. \tag{6.19}$$

In particular, if $v = u$ a.e. in $\mathcal{C}\Omega'$ and $\mathcal{N}(u, \Omega) < +\infty$, then from (6.19) we see that

$$\mathcal{N}(v, \Omega) < +\infty \quad \Longleftrightarrow \quad \mathcal{N}(v, \Omega') < +\infty, \tag{6.20}$$

and

$$\mathcal{N}(v, \Omega') - \mathcal{N}(u, \Omega') = \mathcal{N}(v, \Omega) - \mathcal{N}(u, \Omega). \tag{6.21}$$

We also point out the following trivial but useful remark, which explains why in the definition of an admissible competitor we ask $u - v \in H^s(\mathbb{R}^n)$. This is indeed equivalent to asking $\mathcal{N}(v, \Omega) < +\infty$.

REMARK 6.2.8. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let $u : \mathbb{R}^n \to \mathbb{R}$ be such that $\mathcal{N}(u, \Omega) < +\infty$. Let $v : \mathbb{R}^n \to \mathbb{R}$ be such that $v = u$ a.e. in $\mathcal{C}\Omega$. Then

$$\mathcal{N}(v, \Omega) < \infty \qquad \Longleftrightarrow \qquad u - v \in H^s(\mathbb{R}^n). \tag{6.22}$$

First of all, we remark that

$$[u - v]_{H^s(\mathbb{R}^n)} < +\infty \qquad \Longrightarrow \qquad \|u - v\|_{L^2(\mathbb{R}^n)} = \|u - v\|_{L^2(\Omega)} < +\infty.$$

This is a consequence of a fractional Poincaré type inequality—see, e.g., Proposition D.1.6—which we can apply to the function $w := v - u$ thanks to the assumption

$$v = u \quad \text{a.e. in } \mathcal{C}\Omega,$$

so it is enough to show that

$$\mathcal{N}(v, \Omega) < \infty \qquad \Longleftrightarrow \qquad [u - v]_{H^s(\mathbb{R}^n)} < +\infty.$$

This equivalence follows from the equality

$$[u-v]^2_{H^s(\mathbb{R}^n)} = \iint_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-v(x)-u(y)+v(y)|^2}{|x-y|^{n+2s}}\,dx\,dy = \mathcal{N}(u-v,\Omega)$$

and the "triangle inequality"

$$\mathcal{N}(u_1+u_2,\Omega) \le 2\big(\mathcal{N}(u_1,\Omega)+\mathcal{N}(u_2,\Omega)\big).$$

As a consequence of formulas (6.14) and (6.19), we obtain the following equivalent characterizations of minimizing pairs:

Proposition 6.2.9. *Let (u,E) be an admissible pair according to Definition 6.1.1 such that $\mathcal{F}_\Omega(u,E)<+\infty$. Then, the following statements are equivalent:*

(i) the pair (u,E) is minimizing in Ω, according to Definition 6.1.1,
(ii) for every open subset $\Omega' \Subset \Omega$ we have

$$\mathcal{N}(u,\Omega')+\operatorname{Per}(E,\Omega) = \inf\big\{\mathcal{N}(v,\Omega')+\operatorname{Per}(F,\Omega)\,|\,(v,F) \text{ admissible}$$
$$\text{s.t. } v=u \text{ a.e. in } \mathcal{C}\Omega' \text{ and } F\setminus\Omega' = E\setminus\Omega'\big\}.$$

(iii) the pair (u,E) is minimizing in every open set $\Omega'\Subset\Omega$,
(iv) the pair (u,E) is minimizing in every open set $\Omega'\subseteq\Omega$.

Proof. We begin with the implication $(i) \implies (ii)$.
Let (v,F) be an admissible pair such that

$$v=u \text{ a.e. in } \mathcal{C}\Omega' \text{ and } F\setminus\Omega' = E\setminus\Omega'.$$

We can suppose that

$$\mathcal{N}(v,\Omega')<+\infty \qquad \text{and} \qquad \operatorname{Per}(F,\Omega)<+\infty,$$

otherwise there is nothing to prove. In particular, thanks to (6.20) we have

$$\mathcal{N}(v,\Omega)<+\infty.$$

Thus (v,F) is an admissible competitor for (u,E) in Ω, according to Definition 6.1.1. By minimality of (u,E) and equality (6.21) we obtain

$$\mathcal{N}(u,\Omega')+\operatorname{Per}(E,\Omega)-\mathcal{N}(v,\Omega')-\operatorname{Per}(F,\Omega) = \mathcal{F}_\Omega(u,E)-\mathcal{F}_\Omega(v,F)\le 0,$$

as wanted.

As for the implication $(ii) \implies (iii)$, let (v,F) be an admissible competitor for (u,E) in Ω'.
Then we can find an open set $\mathcal{O}\Subset\Omega'$ such that $v=u$ a.e. in $\mathcal{C}\mathcal{O}$ and $F\Delta E\subseteq\mathcal{O}$. Exploiting both (6.14) and (6.19), we find

$$\mathcal{F}_{\Omega'}(v,E)-\mathcal{F}_{\Omega'}(u,E) = \mathcal{N}(v,\mathcal{O})+\operatorname{Per}(F,\Omega)-\mathcal{N}(u,\mathcal{O})-\operatorname{Per}(E,\Omega),$$

which is nonnegative by (ii).

The implication $(iii) \implies (iv)$ is proved in the same way. If (v,F) is an admissible competitor for (u,E) in Ω', then we can find $\mathcal{O}\Subset\Omega'$ such that $\operatorname{supp}(v-u)\Subset\mathcal{O}$ and $F\Delta E\Subset\mathcal{O}$.
Then (v,F) is an admissible competitor for (u,E) in $\mathcal{O}$. Exploiting the minimality assumed in (iii) and using again both (6.14) and (6.19), we thus obtain

$$\mathcal{F}_{\Omega'}(v,E)-\mathcal{F}_{\Omega'}(u,E) = \mathcal{F}_{\mathcal{O}}(v,F)-\mathcal{F}_{\mathcal{O}}(u,E)\ge 0.$$

The last implication $(iv) \implies (i)$ follows trivially by taking $\Omega'=\Omega$. □

Remark 6.2.10. Notice that point (ii) of Proposition 6.2.9 says that (u,E) is a minimizing pair for $\overline{\mathcal{F}}_{\Omega'}$ for every open subset $\Omega'\Subset\Omega$ (with respect to the exterior data (u,E)).

6.3. The extended functional

In this section we deal with the extended functional defined in the Introduction. For an introduction to the extension operator, we refer the interested reader to [75].

We recall that, given a function $u : \mathbb{R}^n \to \mathbb{R}$, we denote by $\overline{u} : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ the extended function defined in (6.5), that is

$$\overline{u}(x,z) := c_{n,s} z^{2s} \int_{\mathbb{R}^n} \frac{u(\xi)}{(|x-\xi|^2 + z^2)^{\frac{n+2s}{2}}} d\xi, \quad \text{for every } (x,z) \in \mathbb{R}^{n+1}_+.$$

We observe that for the extended function $\overline{u}$ to be well defined, it is enough that $u \in L_s(\mathbb{R}^n)$. Hence, in light of Remark 6.2.1, if $u : \mathbb{R}^n \to \mathbb{R}$ is a measurable function such that $\mathcal{N}(u,\Omega) < +\infty$, then the extended function $\overline{u}$ is well defined.

We start with some preliminary observations:

REMARK 6.3.1. If $\mathcal{N}(u, B_R) < +\infty$, then

$$\int_{\Omega_+} |\nabla \overline{u}|^2 z^{1-2s}\, dX < +\infty,$$

for every bounded open set $\Omega \subseteq \mathbb{R}^{n+1}$ with Lipschitz boundary and such that $\Omega_0 \Subset B_R$ (see [21, Proposition 7.1]).

In particular, if (u, E) is an admissible pair for $\mathcal{F}$ s.t. $\mathcal{F}_{B_R}(u,E) < +\infty$, then $(\overline{u}, E)$ is an admissible pair for the extended functional $\mathfrak{F}$, and $\mathfrak{F}_\Omega(\overline{u}, E) < +\infty$ for every bounded open set $\Omega \subseteq \mathbb{R}^{n+1}$ with Lipschitz boundary and such that $\Omega_0 \Subset B_R$.

REMARK 6.3.2. Let (u, E) be an admissible pair for $\mathcal{F}$ s.t. $\mathcal{F}_{B_R}(u,E) < +\infty$. Let $\Omega \subseteq \mathbb{R}^{n+1}$ be a bounded open set with Lipschitz boundary ans such that $\Omega_0 \Subset B_R$, and let $(\mathcal{V}, F)$ be an admissible competitor for $\mathfrak{F}_\Omega$, with respect to $(\overline{u}, E)$, according to Definition 6.1.3. Define $v := \mathcal{V}\big|_{\{z=0\}}$. Then (v, F) is an admissible competitor for $\mathcal{F}_{B_R}$, with respect to (u, E), according to Definition 6.1.1.

Indeed, let $\Omega' \subseteq \mathbb{R}^{n+1}$ be a bounded open set with Lipschitz boundary, such that $\Omega \Subset \Omega'$ and $\Omega'_0 \Subset B_R$. From Remark 6.3.1, we know that

$$\int_{\Omega'_+ \setminus \Omega_+} |\nabla \overline{u}|^2 z^{1-2s}\, dX < +\infty,$$

and hence, since $\mathfrak{F}(\mathcal{V}, \Omega) < +\infty$ and supp $(\mathcal{V} - \overline{u}) \Subset \Omega$, we get

$$\int_{\Omega'_+} |\nabla \mathcal{V}|^2 z^{1-2s}\, dX < +\infty.$$

It can be shown that this implies that $\mathcal{N}(v, \Omega) < +\infty$ (see e.g. the proof of [42, Proposition 4.1]). Now, since $v = u$ in $\mathcal{C}\Omega$ and $\mathcal{N}(u, B_R) < +\infty$, using (6.20) we get $\mathcal{N}(v, B_R) < +\infty$ and $u - v \in H^s(\mathbb{R}^n)$ as claimed.

6.3.1. An equivalent problem. Now we show that we can use the extended functional $\mathfrak{F}$, defined in (6.6), to obtain an equivalent formulation of the minimization problem for $\mathcal{F}$.

We remark that, differently from the proof of [42, Proposition 4.1], in our framework we only "localize" the energy $\mathcal{N}$.

PROOF OF PROPOSITION 6.1.4. Let $r \in (0, R)$. From [21, Lemma 7.2] we know that if $v : \mathbb{R}^n \to \mathbb{R}$ is such that

$$\mathcal{N}(v, B_r) < +\infty \quad \text{and} \quad \text{supp}(v - u) \Subset B_r, \tag{6.23}$$

then

$$\mathcal{N}(v, B_r) - \mathcal{N}(u, B_r) = c'_{n,s} \inf_{(\Omega,\mathcal{V})\in\mathfrak{J}_v} \int_{\Omega_+} \big(|\nabla \mathcal{V}|^2 - |\nabla \overline{u}|^2\big) z^{1-2s}\, dX, \tag{6.24}$$

where the set $\mathfrak{J}_v$ consists of all the couples $(\Omega, \mathcal{V})$, with $\Omega \subseteq \mathbb{R}^{n+1}$ a bounded open set with Lipschitz boundary such that $\Omega_0 \subseteq B_r$ and $\mathcal{V} : \mathbb{R}^{n+1} \to \mathbb{R}$ such that $\mathcal{V} - \overline{u}$ is compactly supported inside Ω and $\mathcal{V}\big|_{\{z=0\}} = v$.

Notice that for every such couple $(\Omega, \mathcal{V}) \in \mathfrak{J}_v$ we can prescribe without loss of generality that $\mathcal{V} = \overline{u}$ outside Ω.

$\Longrightarrow$) Let (u, E) be a minimizing pair for $\mathcal{F}$ in B_r, with $r \in (0, R)$. We show that $(\overline{u}, E)$ is minimizing for $\mathfrak{F}_\Omega$ for every bounded open set $\Omega \subseteq \mathbb{R}^{n+1}$ with Lipschitz boundary and $\Omega_0 \subseteq B_r$.
From Remark 6.3.1, we know that $(\overline{u}, E)$ is admissible for the extended functional and $\mathfrak{F}_\Omega(\overline{u}, E) < +\infty$.

Now let $(\mathcal{V}, F)$ be an admissible competitor and define $v := \mathcal{V}|_{\{z=0\}}$, so that $(\Omega, \mathcal{V}) \in \mathfrak{J}_v$. Since $v - u = \mathcal{V}|_{\{z=0\}} - \overline{u}|_{\{z=0\}}$ is compactly supported in $\Omega_0 \subseteq B_r$, from Remark 6.3.2 we see that (v, F) is an admissible competitor for $\mathcal{F}$ in B_r. Thus, using the minimality of (u, E) and (6.24), we obtain

$$\begin{aligned}
0 &\le \mathcal{F}_{B_r}(v, F) - \mathcal{F}_{B_r}(u, E)\\
&= \mathcal{N}(v, B_r) - \mathcal{N}(u, B_r) + \operatorname{Per}(F, B_r) - \operatorname{Per}(E, B_r)\\
&= c'_{n,s} \inf_{(\Omega,\mathcal{V})\in\mathfrak{J}_v} \int_{\Omega_+} \big(|\nabla \mathcal{V}|^2 - |\nabla \overline{u}|^2\big) z^{1-2s}\, dX + \operatorname{Per}(F, B_r) - \operatorname{Per}(E, B_r)\\
&\le \mathfrak{F}_\Omega(\mathcal{V}, F) - \mathfrak{F}_\Omega(\overline{u}, E).
\end{aligned}$$

Since this holds for every admissible competitor, this shows that $(\overline{u}, E)$ is minimizing for $\mathfrak{F}_\Omega$.

$\Longleftarrow$) Suppose that $(\overline{u}, E)$ is minimizing for $\mathfrak{F}_\Omega$, for every $\Omega \subseteq \mathbb{R}^{n+1}$ as in the statement of Proposition 6.1.4.

Let (v, F) be an admissible competitor for $\mathcal{F}$ in B_R. In particular, we have that

$$\operatorname{supp}(v - u) \Subset B_R \qquad \text{and} \qquad E\Delta F \Subset B_R,$$

hence we can suppose that

$$\operatorname{supp}(v - u) \Subset B_r \qquad \text{and} \qquad E\Delta F \Subset B_r, \tag{6.25}$$

for some $r \in (0, R)$.

Notice that v satisfies (6.23) and that if $(\Omega, \mathcal{V}) \in \mathfrak{J}_v$, then $(\mathcal{V}, F)$ is an admissible competitor for $\mathfrak{F}_\Omega$ with respect to $(\overline{u}, E)$ and $\Omega_0 \subseteq B_r \Subset B_R$.

Thus, if $(\Omega, \mathcal{V}) \in \mathfrak{J}_v$, since $(\overline{u}, E)$ is minimizing for $\mathfrak{F}_\Omega$, we get

$$\int_{\Omega_+} \big(|\nabla \mathcal{V}|^2 - |\nabla \overline{u}|^2\big) z^{1-2s}\, dX + \operatorname{Per}(F, B_r) - \operatorname{Per}(E, B_r) = \mathfrak{F}_\Omega(\mathcal{V}, F) - \mathfrak{F}_\Omega(\overline{u}, E) \ge 0.$$

Since this holds true for every $(\Omega, \mathcal{V}) \in \mathfrak{J}_v$ and (v, F) satisfies (6.25), we get from (6.24) that

$$\begin{aligned}
&\mathcal{F}_{B_R}(v, F) - \mathcal{F}_{B_R}(u, E) = \mathcal{F}_{B_r}(v, F) - \mathcal{F}_{B_r}(u, E)\\
&= c'_{n,s} \inf_{(\Omega,\mathcal{V})\in\mathfrak{J}_v} \int_{\Omega_+} \big(|\nabla \mathcal{V}|^2 - |\nabla \overline{u}|^2\big) z^{1-2s}\, dX + \operatorname{Per}(F, B_r) - \operatorname{Per}(E, B_r) \ge 0.
\end{aligned}$$

This shows that (u, E) is minimizing in B_R. $\square$

6.4. Monotonicity formula

In this subsection, we obtain a monotonicity formula in the spirit of [100]. The main feature here is that we need to consider the associated extension problem to prove that some energy is monotone. As usual in this type of problems, this will imply a homogeneity of the functions involved. Other papers in which this approach has been exploited are [22, 42].

We introduce now some notation. We say that a set $A \subseteq \mathbb{R}^n$ is a cone if $\lambda A = A$ for any $\lambda > 0$. Notice that this is the same as asking χ_A to be homogeneous of degree 0, that is $\chi_A(\lambda x) = \chi_A(x)$ for any $x \in \mathbb{R}^n$ and any $\lambda > 0$.

First of all we show that the functional $\mathfrak{F}$ possesses a natural scaling. For this, recall the definition of the rescaled pairs (u_λ, E_λ) given in (6.11).

We recall also the notation

$$\mathcal{B}_r := \{(x,z) \in \mathbb{R}^{n+1} \,|\, |x|^2 + z^2 < r^2\} \qquad \text{and} \qquad \mathcal{B}_r^+ := \mathcal{B}_r \cap \{z > 0\}.$$

We can now prove the following scaling result:

LEMMA 6.4.1. *Let (u, E) be a minimizing pair for $\mathcal{F}$ in B_R. Define*

$$\mathfrak{G}_u(r) := r^{1-n}\mathfrak{F}_{\mathcal{B}_r}(\overline{u}, E) = r^{1-n}\Big(c'_{n,s}\int_{\mathcal{B}_r^+} |\nabla\overline{u}|^2 z^{1-2s}\, dX + \operatorname{Per}(E, B_r)\Big) \tag{6.26}$$

for any $r \in (0, R)$, where $\mathfrak{F}$ has been introduced in (6.6). Then, for any $\lambda > 0$,

$$\mathfrak{G}_u(\lambda r) = \mathfrak{G}_{u_\lambda}(r). \tag{6.27}$$

PROOF. We know that the perimeter scales as

$$\operatorname{Per}(E_\lambda, \Omega_\lambda) = \lambda^{1-n} \operatorname{Per}(E, \Omega). \tag{6.28}$$

As for the energy of the extended functions, it is enough to notice that if $\overline{u}_\lambda$ denotes the extension of u_λ (as given by (6.5)), then

$$\overline{u}_\lambda(X) = \lambda^{\frac{1}{2}-s}\overline{u}(\lambda X). \tag{6.29}$$

Plugging (6.28) and (6.29) into (6.4.1), we obtain the desired formula in (6.27). ☐

Now we "correct" $\mathfrak{G}_u$ by adding an appropriate term,

$$\Phi_u(r) := \mathfrak{G}_u(r) - \mathfrak{C}_u(r),$$

where

$$\mathfrak{C}_u := c'_{n,s}\Big(s - \frac{1}{2}\Big) r^{-n} \int_{(\partial\mathcal{B}_r)^+} \overline{u}^2 z^{1-2s}\, d\mathcal{H}^n,$$

and we prove a monotonicity formula for Φ_u. Here above, we used the notation

$$(\partial\mathcal{B}_r)^+ := \partial\mathcal{B}_r \cap \{z > 0\} = \{(x,z) \in \mathbb{R}^{n+1}_+ \,|\, |x|^2 + z^2 = r^2\}.$$

REMARK 6.4.2. It is not difficult to see that Φ_u has the same scale invariance property of $\mathfrak{G}_u$, i.e.

$$\Phi_u(\lambda r) = \Phi_{u_\lambda}(r). \tag{6.30}$$

REMARK 6.4.3. Before proving the Monotonicity Formula, we point out that (u, E) is minimal in Ω if and only if (u_λ, E_λ) is minimal in Ω_λ, for every $\lambda > 0$. This is a consequence of the homogeneous scaling

$$\mathcal{F}_{\Omega_\lambda}(v_\lambda, F_\lambda) = \lambda^{1-n}\mathcal{F}_\Omega(v, F).$$

Indeed, it is enough to notice that (v, F) is an admissible competitor for (u_λ, E_λ) in Ω_λ if and only if $(v_{1/\lambda}, F_{1/\lambda})$ is an admissible competitor for (u, E) in Ω. Then, if (u, E) is minimal in Ω, we find

$$\mathcal{F}_{\Omega_\lambda}(v, F) = \lambda^{1-n}\mathcal{F}_\Omega(v_{1/\lambda}, F_{1/\lambda}) \geq \lambda^{1-n}\mathcal{F}_\Omega(u, E) = \mathcal{F}_{\Omega_\lambda}(u_\lambda, E_\lambda).$$

Proof of Theorem 6.1.5. First of all, notice that Φ_u is differentiable a.e. in $(0, R)$. We want to prove that there exists a subset $\mathcal{G} \subseteq (0, R)$ with $\mathcal{L}^1\big((0, R) \setminus \mathcal{G}\big) = 0$ and such that

$$\exists\, \frac{d}{dr}\Phi_u(r) \geq 0 \qquad \text{for every } r \in \mathcal{G}. \tag{6.31}$$

We remark that, even if the function Φ_u in general is not continuous, (6.31) is enough to prove that Φ_u is increasing in $(0, R)$, thanks to Lemma 6.9.1 and Corollary 6.9.2.

Indeed, let

$$\theta_E(r) := \frac{\operatorname{Per}(E, B_r)}{r^{n-1}} \qquad \text{and} \qquad f(r) := \Phi_u(r) - \theta_E(r),$$

and notice that, since f is continuous and differentiable a.e. in $(0, R)$, we can write

$$f(r_2) - f(r_1) = \int_{r_1}^{r_2} f'(\varrho)\, d\varrho, \qquad \text{for every } 0 < r_1 < r_2 < R. \tag{6.32}$$

Now suppose that (6.31) holds true and notice that

$$\Phi_u'(r) = f'(r) + \theta_E'(r) \qquad \text{for a.e. } r \in (0, R).$$

Then, exploiting (6.32) and formula (6.114) we obtain

$$\begin{aligned}\Phi_u(r_2) - \Phi_u(r_1) = f(r_2) - f(r_1) + \theta_E(r_2) - \theta_E(r_1) &\geq \int_{r_1}^{r_2} f'(\varrho)\, d\varrho + \int_{r_1}^{r_2} \theta_E'(\varrho)\, d\varrho \\ = \int_{r_1}^{r_2} \Phi_u'(\varrho)\, d\varrho = \int_{(r_1,r_2)\cap\mathcal{G}} \Phi_u'(\varrho)\, d\varrho \geq 0,\end{aligned}$$

for every $0 < r_1 < r_2 < R$, thus proving the monotonicity of Φ_u.

We also remark that, if we denote

$$\wp(r) := \operatorname{Per}(E, B_r),$$

then θ_E is differentiable at $r \in (0, R)$ if and only if $\wp$ is differentiable at r, and in this case

$$\theta_E'(r) = r^{1-n}\wp'(r) - (n-1)r^{-n}\wp(r).$$

Now we define the subset $\mathcal{G} \subseteq (0, R)$.

Notice that since (u, E) is an admissible pair, we have that $|\{u < 0\} \cap E| = 0$. Exploiting spherical coordinates, we see that for a.e. $r > 0$

$$u(x) \geq 0 \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in E \cap \partial B_r.$$

In the same way, for a.e. $r > 0$

$$u(x) \leq 0 \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \mathcal{C}E \cap \partial B_r.$$

All in all, we see that, for a.e. $r > 0$,

$$\begin{aligned} & u(x) \geq 0 \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in E \cap \partial B_r \\ \text{and} \quad & u(x) \leq 0 \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \mathcal{C}E \cap \partial B_r. \end{aligned} \tag{6.33}$$

The set $\mathcal{G}$ is defined as the set of all those $r \in (0, R)$ which satisfy all the following properties:

(i) (6.33) holds true,
(ii) the functions f and $\wp$ are differentiable at r
(iii) it holds

$$\mathcal{H}^{n-2}(\partial^* E \cap \partial B_r) < +\infty,$$

and r is a Lebesgue point for the function

$$(0,R) \ni \varrho \longmapsto \mathcal{H}^{n-2}(\partial^* E \cap \partial B_\varrho),$$

(iv) the cone $E(r)$ with vertex in 0 spanned by the spherical slice $E \cap \partial B_r$ (as defined in (6.7)) is a Caccioppoli set.

We remark that by Remark 6.8.2, Proposition 6.8.4 and Remark 6.8.5, points (iii) and (iv) hold true for a.e. $r \in (0,R)$. Hence $\mathcal{L}^1\big((0,R) \setminus \mathcal{G}\big) = 0$.

Now we prove claim (6.31).
First of all, notice that thanks to the scaling property (6.30) we can assume without loss of generality that $r = 1$. We have

$$\begin{aligned}\mathfrak{G}_u'(1) = c_{n,s}'\Big((1-n)\int_{\mathcal{B}_1^+} |\nabla \overline{u}|^2 z^{1-2s}\,dX + \int_{(\partial\mathcal{B}_1)^+} |\nabla \overline{u}|^2 z^{1-2s}\,d\mathcal{H}^n\Big) \\ + \frac{d}{dr}\frac{\mathrm{Per}(E,B_r)}{r^{n-1}}\Big|_{r=1}\end{aligned} \tag{6.34}$$

and

$$\mathfrak{C}_u'(1) = c_{n,s}'\Big(s-\frac{1}{2}\Big)\int_{(\partial\mathcal{B}_1)^+} \big(2\overline{u}\,\overline{u}_\nu + (1-2s)\overline{u}^2\big) z^{1-2s}\,d\mathcal{H}^n, \tag{6.35}$$

where $\overline{u}_\nu$ denotes the normal derivative of $\overline{u}$, so that the normal gradient is $\overline{u}_\nu(X)X$.
To prove (6.35) notice that changing variables $X = rY$, with $z = rw$ yields

$$\mathfrak{C}_u(r) = c_{n,s}'\Big(s-\frac{1}{2}\Big) r^{1-2s}\int_{(\partial\mathcal{B}_1)^+} \overline{u}^2(rY) w^{1-2s}\,d\mathcal{H}^n(Y).$$

Then take the derivative in r and set $r=1$.

To show that $\Phi_u'(1) \geq 0$ we construct appropriate competitors for $(\overline{u},E)$ and compare the energies.
Given a small $\varepsilon > 0$, we consider the admissible competitor $(\mathcal{U}^\varepsilon, E^\varepsilon)$ for $(\overline{u},E)$ defined as

$$\mathcal{U}^\varepsilon(X) := \begin{cases} (1-\varepsilon)^{s-\frac{1}{2}}\overline{u}\big(\frac{1}{1-\varepsilon}X\big) & \text{if } X \in \mathcal{B}_{1-\varepsilon}^+, \\ |X|^{s-\frac{1}{2}}\overline{u}\big(\frac{X}{|X|}\big) & \text{if } X \in \mathcal{B}_1^+ \setminus \mathcal{B}_{1-\varepsilon}^+, \\ \overline{u}(X) & \text{if } X \in \mathbb{R}_+^{n+1} \setminus \mathcal{B}_1^+, \end{cases}$$

and

$$\chi_{E^\varepsilon}(x) := \begin{cases} \chi_E\big(\frac{1}{1-\varepsilon}x\big) & \text{if } x \in B_{1-\varepsilon}, \\ \chi_E\big(\frac{x}{|x|}\big) & \text{if } x \in B_1 \setminus B_{1-\varepsilon}, \\ \chi_E(x) & \text{if } x \in \mathbb{R}^n \setminus B_1, \end{cases}$$

that is

$$E^\varepsilon := \big((1-\varepsilon)E \cap B_{1-\varepsilon}\big) \cup \big(E(1) \cap (B_1 \setminus B_{1-\varepsilon})\big) \cup (E \setminus B_1).$$

Let $u^\varepsilon := \mathcal{U}^\varepsilon\big|_{\{z=0\}}$ be the trace of $\mathcal{U}^\varepsilon$. It is clear that

$$u^\varepsilon \geq 0 \quad \text{a.e. in } E^\varepsilon \setminus (B_1 \setminus B_{1-\varepsilon}) \quad \text{and} \quad u^\varepsilon \leq 0 \quad \text{a.e. in } \mathcal{C}E^\varepsilon \setminus (B_1 \setminus B_{1-\varepsilon}).$$

Moreover condition (6.33) (with $r = 1$) guarantees that the same holds also a.e. in $B_1 \setminus B_{1-\varepsilon}$, so that $(\mathcal{U}^\varepsilon, E^\varepsilon)$ is an admissible pair for $\mathfrak{F}$.

By construction $(\mathcal{U}^\varepsilon, E^\varepsilon)$ is an admissible competitor for $(\overline{u}, E)$ in every bounded open set $\Omega \subseteq \mathbb{R}^{n+1}$ with Lipschitz boundary and such that $\mathcal{B}_1^+ \Subset \Omega$ and $\Omega_0 \Subset B_R$, since

$$\mathcal{U}^\varepsilon = \overline{u} \quad \text{in} \quad \mathbb{R}^{n+1}_+ \setminus \mathcal{B}_1^+ \qquad \text{and} \qquad E^\varepsilon = E \quad \text{in} \quad \mathbb{R}^n \setminus B_1. \tag{6.36}$$

In particular we can take $\Omega = \mathcal{B}_\varrho$ for some $\varrho \in (1, R)$ (recall that we are assuming $1 = r < R$).

Since

$$\mathcal{H}^{n-2}(\partial^* E \cap \partial B_1) < +\infty \quad \Longrightarrow \quad \mathcal{H}^{n-1}(\partial^* E \cap \partial B_1) = 0,$$

by the definitions of $E(1)$ and E^ε and formulas (6.97), thanks to [79, Theorem 16.16] we have

$$\begin{aligned} \operatorname{Per}(E^\varepsilon, B_\varrho) &= \operatorname{Per}(E^\varepsilon, B_1) + \operatorname{Per}(E, B_\varrho \setminus \overline{B_1}) \\ &= \operatorname{Per}(E_{1/(1-\varepsilon)}, B_{1-\varepsilon}) + \operatorname{Per}(E(1), B_1 \setminus \overline{B_{1-\varepsilon}}) + \operatorname{Per}(E, B_\varrho \setminus \overline{B_1}). \end{aligned} \tag{6.37}$$

Thus, using also (6.36), we get from Proposition 6.1.4

$$\mathfrak{F}_{\mathcal{B}_1}(\mathcal{U}^\varepsilon, E^\varepsilon) - \mathfrak{F}_{\mathcal{B}_1}(\overline{u}, E) = \mathfrak{F}_{\mathcal{B}_\varrho}(\mathcal{U}^\varepsilon, E^\varepsilon) - \mathfrak{F}_{\mathcal{B}_\varrho}(\overline{u}, E) \geq 0. \tag{6.38}$$

We compute $\mathfrak{F}_{\mathcal{B}_1}(\mathcal{U}^\varepsilon, E^\varepsilon)$ by splitting it in $\mathcal{B}_{1-\varepsilon}^+$ and $\mathcal{B}_1^+ \setminus \mathcal{B}_{1-\varepsilon}^+$.
Notice that in $\mathcal{B}_{1-\varepsilon}^+$ the pair $(\mathcal{U}^\varepsilon, E^\varepsilon)$ is just the rescaled pair $(\overline{u}_{1/(1-\varepsilon)}, E_{1/(1-\varepsilon)})$. Then

$$\begin{aligned} \mathfrak{F}_{\mathcal{B}_1}(\mathcal{U}^\varepsilon, E^\varepsilon) &= \mathfrak{F}_{\mathcal{B}_{1-\varepsilon}}(\overline{u}_{1/(1-\varepsilon)}, E_{1/(1-\varepsilon)}) + \mathfrak{F}_{\mathcal{B}_1 \setminus \mathcal{B}_{1-\varepsilon}}(\mathcal{U}^\varepsilon, E^\varepsilon) \\ &= (1-\varepsilon)^{n-1} \mathfrak{G}_{u_{1/(1-\varepsilon)}}(1-\varepsilon) + \mathfrak{F}_{\mathcal{B}_1 \setminus \mathcal{B}_{1-\varepsilon}}(\mathcal{U}^\varepsilon, E^\varepsilon) \\ &= (1-\varepsilon)^{n-1} \mathfrak{G}_u(1) + \mathfrak{F}_{\mathcal{B}_1 \setminus \mathcal{B}_{1-\varepsilon}}(\mathcal{U}^\varepsilon, E^\varepsilon) \end{aligned}$$

Now we compute $\mathfrak{F}_{\mathcal{B}_1 \setminus \mathcal{B}_{1-\varepsilon}}(\mathcal{U}^\varepsilon, E^\varepsilon)$.
As for the perimeter, (recalling (6.37)) by formula (6.99) we have

$$\begin{aligned} \operatorname{Per}(E^\varepsilon, B_1 \setminus \overline{B_{1-\varepsilon}}) &= \operatorname{Per}(E(1), B_1) - \operatorname{Per}(E(1), B_{1-\varepsilon}) \\ &= \frac{\mathcal{H}^{n-2}(\partial^* E \cap \partial B_1)}{n-1} \big(1 - (1-\varepsilon)^{n-1}\big) \\ &= \varepsilon\, \mathcal{H}^{n-2}(\partial^* E \cap \partial B_1) + o(\varepsilon). \end{aligned}$$

Notice that in $\mathcal{B}_1^+ \setminus \mathcal{B}_{1-\varepsilon}^+$ we have

$$\begin{aligned} \nabla \mathcal{U}^\varepsilon(X) &= \Big(s - \frac{1}{2}\Big)|X|^{s-\frac{3}{2}} \overline{u}\left(\frac{X}{|X|}\right) \frac{X}{|X|} + |X|^{s-\frac{1}{2}} \frac{1}{|X|} \left[\nabla \overline{u}\left(\frac{X}{|X|}\right) \right. \\ &\qquad \left. - \left(\nabla \overline{u}\left(\frac{X}{|X|}\right) \cdot \frac{X}{|X|} \right) \frac{X}{|X|} \right] \\ &= \Big(s - \frac{1}{2}\Big)|X|^{s-\frac{3}{2}} \overline{u}\left(\frac{X}{|X|}\right) \frac{X}{|X|} + |X|^{s-\frac{1}{2}} \frac{1}{|X|} \overline{u}_\tau\left(\frac{X}{|X|}\right), \end{aligned}$$

where $\overline{u}_\tau$ denotes the tangential gradient of $\overline{u}$ on $(\partial \mathcal{B}_1)^+$.
Since $\overline{u}_\tau \cdot \frac{X}{|X|} = 0$, this gives

$$|\nabla \mathcal{U}^\varepsilon(X)|^2 = |X|^{2s-3} \left\{ \Big(s - \frac{1}{2}\Big)^2 \overline{u}^2\left(\frac{X}{|X|}\right) + \left| \overline{u}_\tau\left(\frac{X}{|X|}\right) \right|^2 \right\}. \tag{6.39}$$

Therefore

$$\begin{aligned}\int_{\mathcal{B}_1^+\setminus\mathcal{B}_{1-\varepsilon}^+}|\nabla\mathcal{U}^\varepsilon|^2 z^{1-2s}\,dX &= \int_{1-\varepsilon}^1 dt\int_{(\partial\mathcal{B}_t)^+}|\nabla\mathcal{U}^\varepsilon|^2 z^{1-2s}\,d\mathcal{H}^n\\
&= \int_{1-\varepsilon}^1 t^{2s-3}\,dt\int_{(\partial\mathcal{B}_t)^+}\left\{\left(s-\frac12\right)^2\overline{u}^2\left(\frac{X}{|X|}\right)+\left|\overline{u}_\tau\left(\frac{X}{|X|}\right)\right|^2\right\}z^{1-2s}\,d\mathcal{H}^n\\
&= \int_{1-\varepsilon}^1 t^{-2}\,dt\int_{(\partial\mathcal{B}_1)^+}\left\{\left(s-\frac12\right)^2\overline{u}^2+|\overline{u}_\tau|^2\right\}z^{1-2s}\,d\mathcal{H}^n\\
&= \left(\frac{1}{1-\varepsilon}-1\right)\int_{(\partial\mathcal{B}_1)^+}\left\{\left(s-\frac12\right)^2\overline{u}^2+|\overline{u}_\tau|^2\right\}z^{1-2s}\,d\mathcal{H}^n\\
&= \varepsilon\int_{(\partial\mathcal{B}_1)^+}\left\{\left(s-\frac12\right)^2\overline{u}^2+|\overline{u}_\tau|^2\right\}z^{1-2s}\,d\mathcal{H}^n+o(\varepsilon).\end{aligned}$$

Exploiting these computations, we get from (6.38)

$$\begin{aligned}0 \le \big((1-\varepsilon)^{n-1}-1\big)\mathfrak{G}_u(1)+\varepsilon\mathcal{H}^{n-2}(\partial^*E\cap\partial B_1)\\
+\varepsilon c'_{n,s}\int_{(\partial\mathcal{B}_1)^+}\left\{\left(s-\frac12\right)^2\overline{u}^2+|\overline{u}_\tau|^2\right\}z^{1-2s}\,d\mathcal{H}^n+o(\varepsilon).\end{aligned}$$

Dividing by ε and passing to the limit $\varepsilon\to0$ yields

$$\begin{aligned}c'_{n,s}\Big\{(1-n)\int_{\mathcal{B}_1^+}|\nabla\overline{u}|^2 z^{1-2s}\,dX+\int_{(\partial\mathcal{B}_1)^+}\left\{\left(s-\frac12\right)^2\overline{u}^2+|\overline{u}_\tau|^2\right\}z^{1-2s}\,d\mathcal{H}^n\Big\}\\
+(1-n)\operatorname{Per}(E,B_1)+\mathcal{H}^{n-2}(\partial^*E\cap\partial B_1)\ge0.\end{aligned}\tag{6.40}$$

Notice that

$$\frac{d}{dr}\frac{\operatorname{Per}(E,B_r)}{r^{n-1}}\Big|_{r=1}-(1-n)\operatorname{Per}(E,B_1)=\frac{d}{dr}\operatorname{Per}(E,B_r)\Big|_{r=1}\ge0,$$

since $\operatorname{Per}(E,B_r)$ is increasing in r and it is deifferentiable at $r=1$ by hypothesis. Actually, by Proposition 6.8.6 we have

$$\frac{d}{dr}\operatorname{Per}(E,B_r)\Big|_{r=1}\ge\mathcal{H}^{n-2}(\partial^*E\cap\partial B_1).\tag{6.41}$$

Let I denote the first line in (6.40). Then we have

$$\begin{aligned}0&\le I+(1-n)\operatorname{Per}(E,B_1)+\mathcal{H}^{n-2}(\partial^*E\cap\partial B_1)\\
&=I+\frac{d}{dr}\frac{\operatorname{Per}(E,B_r)}{r^{n-1}}\Big|_{r=1}+\left(\mathcal{H}^{n-2}(\partial^*E\cap\partial B_1)-\frac{d}{dr}\operatorname{Per}(E,B_r)\Big|_{r=1}\right),\end{aligned}$$

and hence, by (6.41),

$$I+\frac{d}{dr}\frac{\operatorname{Per}(E,B_r)}{r^{n-1}}\Big|_{r=1}\ge\frac{d}{dr}\operatorname{Per}(E,B_r)\Big|_{r=1}-\mathcal{H}^{n-2}(\partial^*E\cap\partial B_1)\ge0.$$

Therefore

$$\begin{aligned}\mathfrak{G}'_u(1)&=\frac{d}{dr}\frac{\operatorname{Per}(E,B_r)}{r^{n-1}}\Big|_{r=1}+I+c'_{n,s}\int_{(\partial\mathcal{B}_1)^+}\left\{|\overline{u}_\nu|^2-\left(s-\frac12\right)^2\overline{u}^2\right\}z^{1-2s}\,d\mathcal{H}^n\\
&\ge c'_{n,s}\int_{(\partial\mathcal{B}_1)^+}\left\{|\overline{u}_\nu|^2-\left(s-\frac12\right)^2\overline{u}^2\right\}z^{1-2s}\,d\mathcal{H}^n\\
&\qquad+\left(\frac{d}{dr}\operatorname{Per}(E,B_r)\Big|_{r=1}-\mathcal{H}^{n-2}(\partial^*E\cap\partial B_1)\right)\end{aligned}$$

and

$$\begin{aligned}\Phi_u'(1) &= \mathfrak{G}_u'(1) - \mathfrak{C}_u'(1)\\ &\geq c'_{n,s}\int_{(\partial\mathcal{B}_1)^+}\left\{|\overline{u}_\nu|^2 - \left(s-\frac{1}{2}\right)^2\overline{u}^2 - 2\left(s-\frac{1}{2}\right)\overline{u}\,\overline{u}_\nu - \left(s-\frac{1}{2}\right)(1-2s)\overline{u}^2\right\}z^{1-2s}\,d\mathcal{H}^n\\ &\qquad + \left(\frac{d}{dr}\operatorname{Per}(E,B_r)\Big|_{r=1} - \mathcal{H}^{n-2}(\partial^*E\cap\partial B_1)\right).\end{aligned}$$

Since

$$|\overline{u}_\nu|^2 - \left(s-\frac{1}{2}\right)^2\overline{u}^2 - 2\left(s-\frac{1}{2}\right)\overline{u}\,\overline{u}_\nu - \left(s-\frac{1}{2}\right)(1-2s)\overline{u}^2 = \left(\overline{u}_\nu - \left(s-\frac{1}{2}\right)\overline{u}\right)^2,$$

we conclude

$$\begin{aligned}\Phi_u'(1) \geq c'_{n,s}\int_{(\partial\mathcal{B}_1)^+}&\left(\overline{u}_\nu - \left(s-\frac{1}{2}\right)\overline{u}\right)^2 z^{1-2s}\,d\mathcal{H}^n\\ &+ \left(\frac{d}{dr}\operatorname{Per}(E,B_r)\Big|_{r=1} - \mathcal{H}^{n-2}(\partial^*E\cap\partial B_1)\right) \geq 0.\end{aligned} \tag{6.42}$$

This proves (6.31), concluding the proof of the monotonicity of Φ_u.

We are left to prove that if Φ_u is constant, then $\overline{u}$ is homogeneous of degree $s-\frac{1}{2}$ in $\mathcal{B}_R^+$ and E is a cone in B_R (the converse is a trivial consequence of the scaling invariance of Φ_u).

First of all, notice that

$$\Phi_u \equiv c \quad \text{in } (0,R) \qquad \Longrightarrow \qquad \Phi_u' \equiv 0 \quad \text{in } (0,R),$$

hence from (6.42) we find that

$$\nabla\overline{u}(X)\cdot X = \left(s-\frac{1}{2}\right)\overline{u}(X) \qquad \text{for a.e. } X\in\mathcal{B}_R^+, \tag{6.43}$$

and

$$\frac{d}{dr}\operatorname{Per}(E,B_r) = \mathcal{H}^{n-2}(\partial^*E\cap\partial B_r) \qquad \text{for a.e. } r\in(0,R). \tag{6.44}$$

Equality (6.43) implies that $\overline{u}$ is homogeneous of degree $s-\frac{1}{2}$ in $\mathcal{B}_R^+$ (see, e.g., [42, Lemma 4.2]).

Therefore, if we denote

$$f(r) := \Phi_u(r) - \theta_E(r),$$

thanks to the scaling invariance properties we have

$$f \equiv c' \quad \text{in } (0,R),$$

and hence

$$\operatorname{Per}(E,B_r) = r^{n-1}\big(\Phi_u(r) - f(r)\big) = r^{n-1}(c-c') \quad \forall\, r\in(0,R),$$

so that the function $\wp(r) := \operatorname{Per}(E,B_r)$ is continuous in $(0,R)$. Thus, by (6.44) we obtain that E is a cone in B_R, thanks to Proposition 6.8.6.

This concludes the proof. □

6.5. Blow-up sequence and homogeneous minimizers

This section is devoted to the study of blow-up sequences. We begin by proving the uniform energy estimates of Theorem 6.1.2. Then we establish a convergence result for minimizing pairs—namely Theorem 6.1.6—and finally we study blow-up limits, that is, we prove Theorem 6.1.7 (exploiting also the monotonicity formula).

6.5.1. Uniform energy estimates. In this brief subsection, we provide the proof of the uniform energy estimates satisfied by minimizing pairs of the functional $\mathcal{F}$.

PROOF OF THEOREM 6.1.2. The argument of the proof is the same as in the proof of [42, Theorem 1.1], with a minor modification needed in order to replace the fractional perimeter considered there with the classical perimeter.

More precisely, we need to replace formula (7.7) of [42] with the corresponding formula for the classical perimeter. To this end, we consider the set

$$F := B_1 \cup (E \setminus B_1)$$

and notice that

$$\mathrm{Per}(F, B_{3/2}) = \mathcal{H}^{n-1}(\partial^* F \cap \partial B_1) + \mathrm{Per}(E, B_{3/2} \setminus \overline{B_1})$$

and

$$\mathrm{Per}(E, B_{3/2}) = \mathrm{Per}(E, B_1) + \mathcal{H}^{n-1}(\partial^* E \cap \partial B_1) + \mathrm{Per}(E, B_{3/2} \setminus \overline{B_1}).$$

Hence

(6.45)

$$\begin{aligned} \mathrm{Per}(F, B_{3/2}) - \mathrm{Per}(E, B_{3/2}) &= \mathcal{H}^{n-1}(\partial^* F \cap \partial B_1) - \mathrm{Per}(E, B_1) - \mathcal{H}^{n-1}(\partial^* E \cap \partial B_1) \\ &\leq \mathcal{H}^{n-1}(\partial^* F \cap \partial B_1) - \mathrm{Per}(E, B_1) \leq \mathcal{H}^{n-1}(\partial B_1) - \mathrm{Per}(E, B_1). \end{aligned}$$

Then we can conclude the proof by arguing as in the proof of [42, Theorem 1.1], substituting formula (7.7) there with formula (6.45), whenever needed. □

6.5.2. Convergence of minimizers. In this subsection we establish some conditions that ensure the convergence of minimizing pairs—namely we prove Theorem 6.1.6. We will exploit this result in the particularly important case of blow-up sequences.

In order to prove Theorem 6.1.6, we need the following glueing Lemma, which is a modification of [42, Lemma 6.2] taking into account the classical perimeter in place of (the extension of) the fractional perimeter.

LEMMA 6.5.1. *Let* (u_i, E_i) *be admissible pairs such that* $\mathcal{F}_{B_R}(u_i, E_i) < +\infty$, *for* $i = 1, 2$, *and let* $\overline{u}_i$ *be the corresponding extension functions. Let* $r \in (0, R)$ *be such that*

$$\mathcal{H}^{n-1}(\partial^* E_i \cap \partial B_r) = 0, \qquad \textit{for } i = 1, 2,$$

define

$$F := (E_1 \cap B_r) \cup (E_2 \setminus B_r),$$

and fix $\varrho \in (r, R)$.

Then, for every small $\varepsilon > 0$, *there exists a function* $\mathcal{V} : \mathbb{R}^{n+1}_+ \to \mathbb{R}$ *such that* $(\mathcal{V}, F)$ *is an admissible pair with*

$$\mathcal{V} = \overline{u}_2 \qquad \textit{in a neighborhood of } (\partial \mathcal{B}_\varrho)^+,$$

and such that

(6.46)

$$\begin{aligned} &\int_{\mathcal{B}_\varrho^+} \big(|\nabla \mathcal{V}|^2 - |\nabla \overline{u}_2|^2\big) z^{1-2s}\, dX \\ &\qquad \leq \int_{\mathcal{B}_r^+} \big(|\nabla \overline{u}_1|^2 - |\nabla \overline{u}_2|^2\big) z^{1-2s}\, dX + C\varepsilon^{-2} \int_{\mathcal{B}_{r+\varepsilon}^+ \setminus \mathcal{B}_{r-\varepsilon}^+} |\overline{u}_1 - \overline{u}_2|^2 z^{1-2s}\, dX \\ &\qquad\quad + C \int_{\mathcal{B}_{r+\varepsilon}^+ \setminus \mathcal{B}_{r-\varepsilon}^+} \big(|\nabla \overline{u}_1|^2 + |\nabla \overline{u}_2|^2\big) z^{1-2s}\, dX, \end{aligned}$$

for some constant $C > 0$*, and*

$$\operatorname{Per}(F, B_\varrho) - \operatorname{Per}(E_2, B_\varrho) = \operatorname{Per}(E_1, B_r) - \operatorname{Per}(E_2, B_r) + \mathcal{H}^{n-1}\big((E_1 \Delta E_2) \cap \partial B_r\big). \tag{6.47}$$

Proof. The construction of the function $\mathcal{V}$ and the proof of inequality (6.46) are the same as in the proof of [42, Lemma 6.2].

Equality (6.47) follows from [79, Theorem 16.16]. □

Remark 6.5.2. We remark that, if $(\overline{u}, E)$ is a minimizing pair for the extended functional $\mathfrak{F}$ in $\mathcal{B}_r^+$, then $(\overline{u}, E)$ is minimizing also in $\mathcal{B}_\varrho^+$, for every $\varrho \in (0, r]$.

Proof of Theorem 6.1.6. First of all, we remark that (6.8) follows by arguing as in [42, Lemma 8.3].

Now we prove that the pair $(\overline{u}, E)$ is minimizing in $\mathcal{B}_r^+$ for every $r \in (0, R)$.
For this, we first point out that, thanks to Remark 6.5.2, it is enough to prove that $(\overline{u}, E)$ is minimizing in $\mathcal{B}_r^+$ for a.e. $r \in (0, R)$.

Then we show that $(\overline{u}, E)$ is minimizing in $\mathcal{B}_r^+$ for every $r \in \mathcal{G}$, where $\mathcal{G}$ is defined as the set of all radii $r \in (0, R)$ such that

$$\begin{aligned} &\mathcal{H}^{n-1}(\partial^* E_m \cap \partial B_r) = 0 = \mathcal{H}^{n-1}(\partial^* E \cap \partial B_r) \\ &\text{and} \quad \lim_{m\to\infty} \mathcal{H}^{n-1}\big((E \Delta E_m) \cap \partial B_r\big) = 0. \end{aligned} \tag{6.48}$$

Notice that $\mathcal{L}^1\big((0, R) \setminus \mathcal{G}\big) = 0$. Indeed, the first condition in (6.48) holds true for a.e. $r \in (0, R)$ (see e.g. Remark 6.8.2). Moreover, notice that by hypothesis

$$0 = \lim_{m\to\infty} \big|(E \Delta E_m) \cap B_R\big| = \int_0^R \mathcal{H}^{n-1}\big((E \Delta E_m) \cap \partial B_r\big)\, dr,$$

so that also the second condition holds for a.e. $r \in (0, R)$.

Now fix a radius $r \in \mathcal{G}$ and let $(\mathcal{V}, F)$ be an admissible competitor for $(\overline{u}, E)$ in $\mathcal{B}_r^+$. In particular, notice that since $E \Delta F \Subset B_r$, by (6.48) we have

$$\mathcal{H}^{n-1}(\partial^* F \cap \partial B_r) = 0 \quad \text{and} \quad \lim_{m\to\infty} \mathcal{H}^{n-1}\big((F \Delta E_m) \cap \partial B_r\big) = 0. \tag{6.49}$$

Then we fix a radius $\varrho \in (r, R)$ and we consider the pairs $(\mathcal{V}_m, F_m)$ defined by using Lemma 6.5.1, (with $(\overline{u}_1, E_1) := (\mathcal{V}, F)$ and $(\overline{u}_2, E_2) := (\overline{u}_m, E_m)$).
Notice that, by construction, the pair $(\mathcal{V}_m, F_m)$ is an admissible competitor for the pair $(\overline{u}_m, E_m)$ in $\mathcal{B}_\varrho^+$. Hence the minimality of $(\overline{u}_m, E_m)$ (see Remark 6.5.2) implies

$$c'_{n,s} \int_{\mathcal{B}_\varrho^+} |\nabla \overline{u}_m|^2 z^{1-2s}\, dX + \operatorname{Per}(E_m, B_\varrho) \le c'_{n,s} \int_{\mathcal{B}_\varrho^+} |\nabla \mathcal{V}_m|^2 z^{1-2s}\, dX + \operatorname{Per}(F_m, B_\varrho). \tag{6.50}$$

Moreover, by Lemma 6.5.1, we have

(6.51)

$$\begin{aligned} c'_{n,s} \int_{\mathcal{B}_\varrho^+} |\nabla \mathcal{V}_m|^2 z^{1-2s}\, dX + \operatorname{Per}(F_m, B_\varrho) &\le c'_{n,s} \int_{\mathcal{B}_r^+} |\nabla \mathcal{V}|^2 z^{1-2s}\, dX + \operatorname{Per}(F, B_r) + c_m(\varepsilon) \\ &+ c'_{n,s} \int_{\mathcal{B}_\varrho^+} |\nabla \overline{u}_m|^2 z^{1-2s}\, dX + \operatorname{Per}(E_m, B_\varrho) - c'_{n,s} \int_{\mathcal{B}_r^+} |\nabla \overline{u}_m|^2 z^{1-2s}\, dX - \operatorname{Per}(E_m, B_r), \end{aligned}$$

with

$$c_m(\varepsilon) := c'_{n,s} C \left(\varepsilon^{-2} \int_{\mathcal{B}^+_{r+\varepsilon} \setminus \mathcal{B}^+_{r-\varepsilon}} |\mathcal{V} - \overline{u}_m|^2 z^{1-2s}\, dX \right.$$
$$\left. + \int_{\mathcal{B}^+_{r+\varepsilon} \setminus \mathcal{B}^+_{r-\varepsilon}} \big(|\nabla \mathcal{V}|^2 + |\nabla \overline{u}_m|^2\big) z^{1-2s}\, dX \right) + \mathcal{H}^{n-1}\big((F \Delta E_m) \cap \partial B_r\big)$$
$$= C' \left(\varepsilon^{-2} \int_{\mathcal{B}^+_{r+\varepsilon} \setminus \mathcal{B}^+_{r-\varepsilon}} |\overline{u} - \overline{u}_m|^2 z^{1-2s}\, dX + \int_{\mathcal{B}^+_{r+\varepsilon} \setminus \mathcal{B}^+_{r-\varepsilon}} \big(|\nabla \overline{u}|^2 + |\nabla \overline{u}_m|^2\big) z^{1-2s}\, dX \right)$$
$$+ \mathcal{H}^{n-1}\big((F \Delta E_m) \cap \partial B_r\big),$$

(where we have used that $\mathcal{V} = \overline{u}$ outside $\mathcal{B}_{1-\varepsilon}$, provided $\varepsilon < \tilde{\varepsilon}$, by definition of competitor).

Putting together (6.50) and (6.51), we find

$$c'_{n,s} \int_{\mathcal{B}^+_r} |\nabla \overline{u}_m|^2 z^{1-2s}\, dX + \mathrm{Per}(E_m, B_r)$$
$$\le c'_{n,s} \int_{\mathcal{B}^+_r} |\nabla \mathcal{V}|^2 z^{1-2s}\, dX + \mathrm{Per}(F, B_r) + c_m(\varepsilon). \tag{6.52}$$

We remark that, arguing as in the proof of [42, Theorem 1.2] and recalling (6.49), we obtain

$$\lim_{\varepsilon \to 0} \lim_{m \to \infty} c_m(\varepsilon) = 0.$$

Thus, exploiting (6.8) and the lower semicontinuity of the perimeter, we obtain

$$c'_{n,s} \int_{\mathcal{B}^+_r} |\nabla \overline{u}|^2 z^{1-2s}\, dX + \mathrm{Per}(E, B_r) \le c'_{n,s} \int_{\mathcal{B}^+_r} |\nabla \mathcal{V}|^2 z^{1-2s}\, dX + \mathrm{Per}(F, B_r).$$

The arbitrariness of the competitor $(\mathcal{V}, F)$ implies that $(\overline{u}, E)$ is minimizing in $\mathcal{B}^+_r$.

We are left to prove (6.9). Indeed, we point out that

$$\big|D\chi_{E_m}\big| \stackrel{*}{\rightharpoonup} \big|D\chi_E\big|, \qquad \text{in } B_R$$

implies (6.10) (see, e.g., [79, Remark 21.15]).

In order to prove (6.9), we argue as in the proof of [79, Theorem 21.14]. A key observation is that, thanks to (6.52), we have the (locally) uniform boundedness

$$\sup_{m \in \mathbb{N}} \big|D\chi_{E_m}\big|(B_r) = \sup_{m \in \mathbb{N}} \mathrm{Per}(E_m, B_r) \le C(r) < +\infty, \qquad \forall\, r \in (0, R). \tag{6.53}$$

From (6.53) and the convergence $\big|(E_m \Delta E) \cap B_R\big| \to 0$, we first get

$$D\chi_{E_m} \stackrel{*}{\rightharpoonup} D\chi_E \qquad \text{in } B_R,$$

by [79, Theorem 12.15].

Now notice that, in order to conclude the proof of (6.9), it is enough to show that every subsequence of $\big|D\chi_{E_m}\big|$ has a subsequence which weakly-star converges to $\big|D\chi_E\big|$, in B_R.

We begin by remarking that every subsequence of $\big|D\chi_{E_m}\big|$ admits a weakly-star convergent subsequence, in B_R. Indeed, given such a subsequence $\big|D\chi_{E_{m_h}}\big|$, thanks to (6.53), [79, Theorem 4.33] implies that we can find a subsequence m_{h_k} of m_h such that

$$\big|D\chi_{E_{m_{h_k}}}\big| \stackrel{*}{\rightharpoonup} \mu \qquad \text{in } B_R,$$

for some Radon measure μ (which, a priori, might depend on the subsequence).

Finally, we claim that if for some subsequence we have

$$\big|D\chi_{E_{m_h}}\big| \stackrel{*}{\rightharpoonup} \mu \qquad \text{in } B_R, \tag{6.54}$$

for some Radon measure μ, then

$$\mu = \big|D\chi_E\big| \qquad \text{in } B_R.$$

In order to prove this claim, we first point out that

$$\big|D\chi_E\big| \le \mu \qquad \text{in } B_R, \tag{6.55}$$

by [79, Proposition 4.30].

Next we show that for every $x \in B_R$ we have

$$\mu\big(B_r(x)\big) \le \big|D\chi_E\big|\big(B_r(x)\big) = \operatorname{Per}(E, B_r(x)), \qquad \text{for a.e. } r \in (0, R - |x|). \tag{6.56}$$

To prove (6.56), let $r < R - |x|$ be such that

$$\begin{aligned} &\mathcal{H}^{n-1}\big(\partial^* E_m \cap \partial B_r(x)\big) = 0 = \mathcal{H}^{n-1}\big(\partial^* E \cap \partial B_r(x)\big) \\ &\text{and} \quad \lim_{m\to\infty} \mathcal{H}^{n-1}\big((E\Delta E_m) \cap \partial B_r(x)\big) = 0. \end{aligned}$$

As before, these conditions hold true for a.e. $r < R - |x|$.
Given such an r, we fix $\varrho \in (r, R-|x|)$ and we consider the pair $(\mathcal{V}_h, F_h)$ defined by using Lemma 6.5.1, with $(\overline{u}_1, E_1) := (\overline{u}, E)$ and $(\overline{u}_2, E_2) := (\overline{u}_{m_h}, E_{m_h})$ (up to a traslation). In particular, by definition,

$$F_h := (E \cap B_r(x)) \cup (E_{m_h} \setminus B_r(x)).$$

Each pair $(\mathcal{V}_h, F_h)$ is an admissible competitor for the pair $(\overline{u}_{m_h}, E_{m_h})$ in $\mathcal{B}^+_\varrho(x, 0)$.

Then, arguing as in the first part of the proof, we obtain

$$\begin{aligned} c'_{n,s} \int_{\mathcal{B}^+_r(x,0)} &|\nabla \overline{u}_{m_h}|^2 z^{1-2s}\, dX + \operatorname{Per}(E_{m_h}, B_r(x)) \\ &\le c'_{n,s} \int_{\mathcal{B}^+_r(x,0)} |\nabla \overline{u}|^2 z^{1-2s}\, dX + \operatorname{Per}(E, B_r(x)) + c_h(\varepsilon), \end{aligned}$$

(in place of (6.52)), that is

$$\big|D\chi_{E_{m_h}}\big|\big(B_r(x)\big) = \operatorname{Per}(E_{m_h}, B_r(x)) \le \operatorname{Per}(E, B_r(x)) + \omega_h(\varepsilon), \tag{6.57}$$

where

$$\omega_h(\varepsilon) := c_h(\varepsilon) + c'_{n,s} \int_{\mathcal{B}^+_r(x,0)} \big(|\nabla \overline{u}|^2 - |\nabla \overline{u}_{m_h}|^2\big) z^{1-2s}\, dX$$

and $c_h(\varepsilon)$ is defined as before.

Arguing again as in Lemma 8.3 and the proof of Theorem 1.2 of [42], we obtain

$$\lim_{\varepsilon\to 0} \lim_{h\to\infty} \omega_h(\varepsilon) = 0.$$

Hence, by (6.57) and (6.54) (see [79, Proposition 4.26]) we obtain (6.56).
Therefore, if $x \in B_R$, then by (6.55) and (6.56) we find

$$\big|D\chi_E\big|\big(B_r(x)\big) = \mu\big(B_r(x)\big), \qquad \text{for a.e. } r \in (0, R - |x|). \tag{6.58}$$

By the Lebesgue-Besicovitch Theorem (see [79, Theorem 5.8]), this implies that

$$\big|D\chi_E\big| = \mu \qquad \text{in } B_R.$$

Indeed, by (6.58) we have

$$D_\mu \big|D\chi_E\big|(x) = \lim_{r\to 0} \frac{\big|D\chi_E\big|\big(B_r(x)\big)}{\mu\big(B_r(x)\big)} = 1, \qquad \text{for } \mu\text{-a.e. } x \in \operatorname{supp} \mu \cap B_R.$$

Thus, since by (6.55) we have $\big|D\chi_E\big| \ll \mu$, we get

$$\big|D\chi_E\big| = \big(D_\mu \big|D\chi_E\big|\big)\mu = \mu, \qquad \text{in } B_R.$$

This concludes the proof of the claim and hence of (6.9). □

6.5.3. Blow-up sequence. This subsection is concerned with the existence of a blow-up limit and is dedicated to the proof of Theorem 6.1.7. We wll employ Theorem 6.1.2, Theorem 6.1.6 and also Theorem 6.1.5.

In order to prove Theorem 6.1.7 under the assumption that $u \in C^{s-\frac{1}{2}}(B_1)$, we need the following estimate, wich improves the corresponding estimate in [42] (see the first formula in display on page 4595 there):

LEMMA 6.5.3. *Let $s \in (1/2, 1)$, and let $u : \mathbb{R}^n \to \mathbb{R}$ be such that $u \in L^2_s(\mathbb{R}^n)$, $u \in C^{s-\frac{1}{2}}(B_1)$ and $u(0) = 0$. Let also u_r be as in (6.11). Then, $u_r \in L^2_s(\mathbb{R}^n)$ for every $r \in (0, 1)$, and*

$$\int_{\mathbb{R}^n} \frac{|u_r(y)|^2}{1+|y|^{n+2s}}\,dy \le C\left(\|u\|^2_{C^{s-\frac{1}{2}}(B_r)} + (1-r)\|u\|^2_{C^{s-\frac{1}{2}}(B_1)} + r\int_{\mathbb{R}^n} \frac{|u(y)|^2}{1+|y|^{n+2s}}\,dy\right),$$

for some $C = C(n, s) > 0$.

PROOF. We write

$$I := \int_{\mathbb{R}^n} \frac{|u_r(y)|^2}{1+|y|^{n+2s}}\,dy = I_1 + I_2 + I_3, \tag{6.59}$$

where

$$\begin{aligned} I_1 &:= \int_{B_1} \frac{|u_r(y)|^2}{1+|y|^{n+2s}}\,dy,\\ I_2 &:= \int_{B_{1/r}\setminus B_1} \frac{|u_r(y)|^2}{1+|y|^{n+2s}}\,dy\\ \text{and}\qquad I_3 &:= \int_{\mathcal{C}B_{1/r}} \frac{|u_r(y)|^2}{1+|y|^{n+2s}}\,dy. \end{aligned}$$

We start by estimating I_1. For this, we notice that, for any x, $\tilde{x} \in B_1$,

$$|u_r(x) - u_r(\tilde{x})| = r^{\frac{1}{2}-s}|u(rx) - u(r\tilde{x})| \le \|u\|_{C^{s-\frac{1}{2}}(B_r)}|x-\tilde{x}|^{s-\frac{1}{2}}. \tag{6.60}$$

Moreover, since $u(0) = 0$, we have that $u_r(0) = 0$, and so (6.60) implies that

$$|u_r(x)| \le \|u\|_{C^{s-\frac{1}{2}}(B_r)}|x|^{s-\frac{1}{2}}, \tag{6.61}$$

for any $x \in B_1$.

As a consequence of (6.61),

$$\begin{aligned} I_1 = \int_{B_1} \frac{|u_r(y)|^2}{1+|y|^{n+2s}}\,dy &\le \|u\|^2_{C^{s-\frac{1}{2}}(B_r)} \int_{B_1} \frac{|y|^{2s-1}}{1+|y|^{n+2s}}\,dy\\ &\le \|u\|^2_{C^{s-\frac{1}{2}}(B_r)} \int_{B_1} |y|^{2s-1}\,dy \le C\|u\|^2_{C^{s-\frac{1}{2}}(B_r)}, \end{aligned} \tag{6.62}$$

for some $C > 0$, possibly depending on n and s.

To estimate I_2, we exploit the change of variable $x := ry$ and we obtain that

$$I_2 = \int_{B_{1/r}\setminus B_1} \frac{r^{1-2s}\,|u(ry)|^2}{1+|y|^{n+2s}}\,dy = \int_{B_1\setminus B_r} \frac{r\,|u(x)|^2}{r^{n+2s}+|x|^{n+2s}}\,dx. \tag{6.63}$$

Now, we use that $u \in C^{s-\frac{1}{2}}(B_1)$ and the fact that $0 \in \partial E$ to see that

$$|u(x)| \le \|u\|_{C^{s-\frac{1}{2}}(B_1)}|x|^{s-\frac{1}{2}},$$

for any $x \in B_1$. Plugging this information into (6.63), we conclude that

$$
\begin{aligned}
I_2 &\le r\|u\|^2_{C^{s-\frac12}(B_1)} \int_{B_1\setminus B_r} \frac{|x|^{2s-1}}{r^{n+2s}+|x|^{n+2s}}\,dx \\
&\le r\|u\|^2_{C^{s-\frac12}(B_1)} \int_{B_1\setminus B_r} \frac{|x|^{2s-1}}{|x|^{n+2s}}\,dx = \omega_n(1-r)\|u\|^2_{C^{s-\frac12}(B_1)}.
\end{aligned} \tag{6.64}
$$

It remains to estimate I_3. To this end, we make the change of variable $x := ry$ and we see that

$$
\begin{aligned}
I_3 &= \int_{\mathcal{C}B_{1/r}} \frac{r^{1-2s}\,|u(ry)|^2}{1+|y|^{n+2s}}\,dy = \int_{\mathcal{C}B_1} \frac{r\,|u(x)|^2}{r^{n+2s}+|x|^{n+2s}}\,dx \\
&\le r\int_{\mathcal{C}B_1} \frac{|u(x)|^2}{|x|^{n+2s}}\,dx \le Cr\int_{\mathcal{C}B_1} \frac{|u(x)|^2}{1+|x|^{n+2s}}\,dx \le Cr\int_{\mathbb{R}^n} \frac{|u(x)|^2}{1+|x|^{n+2s}}\,dx,
\end{aligned} \tag{6.65}
$$

for some $C > 0$, possibly depending on n and s.

Putting together (6.62), (6.64) and (6.65), and recalling (6.59), we obtain the desired estimate. □

We can now complete the proof of Theorem 6.1.7.

PROOF OF THEOREM 6.1.7. As a first step we claim that there exist a function $u_0 \in C^{s-\frac12}_{\rm loc}(\mathbb{R}^n)$ and a sequence $r_k \searrow 0$ such that u_{r_k} converges to u_0 locally uniformly in $\mathbb{R}^n$, that is

$$
\lim_{k\to\infty} \|u_{r_k} - u_0\|_{C^0(B_R)} = 0, \qquad \forall\, R > 0. \tag{6.66}
$$

Indeed, arguing as in (6.60) and (6.61), we see that if $r < 1/R$, then $u_r \in C^{s-\frac12}(B_R)$, with

$$
\sup_{r<1/R} \|u_r\|_{C^{s-\frac12}(B_R)} \le C_R < +\infty. \tag{6.67}
$$

More precisely, if $x, \tilde{x} \in B_R$ and $r < 1/R$, then

$$
|u_r(x) - u_r(\tilde{x})| \le \|u\|_{C^{s-\frac12}(B_1)} |x-\tilde{x}|^{s-\frac12}.
$$

Hence, since $u_r(0) = r^{\frac12 - s}u(0) = 0$,

$$
|u_r(x)| \le \|u\|_{C^{s-\frac12}(B_1)} |x|^{s-\frac12}, \qquad \forall\, x \in B_R, \tag{6.68}
$$

if $r < 1/R$. In particular,

$$
\sup_{B_R} |u_r| \le \|u\|_{C^{s-\frac12}(B_1)} R^{s-\frac12},
$$

for every $r < 1/R$, concluding the proof of (6.67).

Then we get the claim by Ascoli-Arzelá Theorem, via a diagonal argument.

We also point out that from (6.68) we obtain

$$
|u_0(x)| \le \|u\|_{C^{s-\frac12}(B_1)} |x|^{s-\frac12}, \qquad \forall\, x \in \mathbb{R}^n. \tag{6.69}
$$

The second step consists in showing the convergence of the positivity sets.

We begin by recalling that, as observed in Remark 6.2.1, $u \in L^2_s(\mathbb{R}^n)$. Thus, by Lemma 6.5.3 we have

$$
\sup_{r\in(0,1)} \int_{\mathbb{R}^n} \frac{|u_r(y)|^2}{1+|y|^{n+2s}}\,dy \le \Lambda < +\infty,
$$

with $\Lambda = \Lambda(n, s, u) > 0$. Next we recall that, thanks to Remark 6.4.3, the pair (u_r, E_r) is minimal in $B_{1/r}$ and hence also in B_{2R}, if $r < 1/2R$. Therefore, by Theorem 6.1.2 we obtain

$$\iint_{\mathbb{R}^{2n}\setminus(\mathcal{C}B_R)^2} \frac{|u_r(x) - u_r(y)|^2}{|x-y|^{n+2s}}\,dx\,dy + \mathrm{Per}(E_r, B_R) \\ \leq C_R\left(1 + \int_{\mathbb{R}^n} \frac{|u_r(y)|^2}{1+|y|^{n+2s}}\,dy\right) \leq C_R(1+\Lambda) < +\infty.$$

In particular, we have

$$\sup_{k\in\mathbb{N}} \mathrm{Per}(E_{r_k}, B_R) \leq C_R(1+\Lambda) < +\infty, \qquad \forall\, R > 0.$$

Thus by compactness (see, e.g., [79, Corollary 12.27]), up to a subsequence, we get

$$\chi_{E_{r_k}} \to \chi_{E_0}, \qquad \text{both in } L^1_{\mathrm{loc}}(\mathbb{R}^n) \text{ and a.e. in } \mathbb{R}^n,$$

for some set $E_0 \subseteq \mathbb{R}^n$ of locally finite perimeter. Arguing as in the end of the proof of Lemma 6.2.5, we see that (u_0, E_0) is an admissible pair.

As a third step, let $\overline{u}_r$ and $\overline{u}_0$ be the extension functions of u_r and u_0 respectively. We claim that

$$\lim_{k\to\infty} \|\overline{u}_{r_k} - \overline{u}_0\|_{L^\infty(\mathcal{Q}_R)} = 0, \qquad \forall\, R > 0. \tag{6.70}$$

We first remark that if $w_k := u_{r_k} - u_0$ and $\overline{w}_k$ is the extension function of w_k, then

$$\overline{w}_k = \overline{u}_{r_k} - \overline{u}_0.$$

Hence, by [46, Lemma 3.1] we find

$$\|\overline{u}_{r_k} - \overline{u}_0\|_{L^\infty(\mathcal{Q}_R)} = \|\overline{w}_k\|_{L^\infty(\mathcal{Q}_R)} \leq C_R\left(\|w_k\|_{L^\infty(B_{2R})} + \int_{\mathbb{R}^n\setminus B_{2R}} \frac{|w_k(y)|}{|y|^{n+2s}}\,dy\right)$$

By (6.66) we know that

$$\lim_{k\to\infty} \|w_k\|_{L^\infty(B_{2R})} = 0.$$

Hence, in order to prove (6.70) we only need to show that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n\setminus B_{2R}} \frac{|w_k(y)|}{|y|^{n+2s}}\,dy = 0. \tag{6.71}$$

First of all, we remark that by Lemma 6.5.3 and Fatou's Lemma we obtain

$$\int_{\mathbb{R}^n} \frac{|u_0(y)|^2}{1+|y|^{n+2s}}\,dy \leq \liminf_{k\to\infty} \int_{\mathbb{R}^n} \frac{|u_{r_k}(y)|^2}{1+|y|^{n+2s}}\,dy \leq \Lambda,$$

and hence

$$\int_{\mathbb{R}^n} \frac{|w_k(y)|^2}{1+|y|^{n+2s}}\,dy \leq 2\left(\int_{\mathbb{R}^n} \frac{|u_0(y)|^2}{1+|y|^{n+2s}}\,dy + \int_{\mathbb{R}^n} \frac{|u_{r_k}(y)|^2}{1+|y|^{n+2s}}\,dy\right) \leq 4\Lambda, \tag{6.72}$$

for every $k \in \mathbb{N}$. We also remark that

$$\frac{1}{|y|^{n+2s}} \leq C_R \frac{1}{1+|y|^{n+2s}}, \qquad \forall\, y \in \mathcal{C}B_{2R}. \tag{6.73}$$

Now let $\varrho > 2R$. Then, by Holder's inequality, (6.73) and (6.72), we obtain

$$\begin{aligned}\int_{\mathbb{R}^n\setminus B_{2R}} \frac{|w_k(y)|}{|y|^{n+2s}}\,dy &= \int_{B_\varrho\setminus B_{2R}} \frac{|w_k(y)|}{|y|^{n+2s}}\,dy + \int_{\mathcal{C}B_\varrho} \frac{|w_k(y)|}{|y|^{n+2s}}\,dy\\ &\le C_{R,\varrho}\|w_k\|_{L^\infty(B_\varrho)} + \left(\int_{\mathcal{C}B_\varrho} \frac{|w_k(y)|^2}{|y|^{n+2s}}\,dy\right)^{\frac12}\left(\int_{\mathcal{C}B_\varrho}\frac{dy}{|y|^{n+2s}}\right)^{\frac12}\\ &\le C_{R,\varrho}\|w_k\|_{L^\infty(B_\varrho)} + \left(C_R\int_{\mathcal{C}B_\varrho} \frac{|w_k(y)|^2}{1+|y|^{n+2s}}\,dy\right)^{\frac12}\left(\frac{\omega_n}{2s}\varrho^{-2s}\right)^{\frac12}\\ &\le C_{R,\varrho}\|w_k\|_{L^\infty(B_\varrho)} + 2\left(\frac{\omega_n C_R\Lambda}{2s}\right)^{\frac12}\varrho^{-s}.\end{aligned}$$

By (6.66), passing to the limit $k\to\infty$ yields

$$\limsup_{k\to\infty}\int_{\mathbb{R}^n\setminus B_{2R}} \frac{|w_k(y)|}{|y|^{n+2s}}\,dy \le 2\left(\frac{\omega_n C_R\Lambda}{2s}\right)^{\frac12}\varrho^{-s},$$

for every $\varrho > 2R$. Then, passing to the limit $\varrho\to\infty$ proves (6.71) and hence also (6.70).

The final step consists in showing that (u_0, E_0) is a minimizing cone.
We first remark that $\overline{u}_0$ is continuous in $\overline{\mathbb{R}^{n+1}_+}$. This can be proved by arguing as in the proof of [42, Theorem 1.3], by exploiting (6.66) and (6.69).

Now we can apply Theorem 6.1.6 to conclude that the pair $(\overline{u}_0, E_0)$ is minimizing in $\mathcal{B}_R^+$, for every $R > 0$ and hence, by Proposition 6.1.4, the pair (u_0, E_0) is minimizing in B_R, for every $R > 0$.

We are left to show that (u_0, E_0) is a cone. For this, we are going to use Theorem 6.1.5.
Since Φ_u is monotone in $(0, 1)$, there exists the limit

$$\lim_{r\to0}\Phi_u(r) =: \Phi\in\mathbb{R}. \tag{6.74}$$

Now, if $\varrho > 0$ is such that

$$\mathcal{H}^{n-1}(\partial^* E_0\cap\partial B_\varrho) = 0,$$

then by (6.70), (6.8) and (6.10) we obtain

$$\lim_{k\to\infty}\Phi_{u_{r_k}}(\varrho) = \Phi_{u_0}(\varrho).$$

Hence, by (6.74) and the scaling invariance (6.30), we get

$$\Phi_{u_0}(\varrho) = \lim_{k\to\infty}\Phi_{u_{r_k}}(\varrho) = \lim_{k\to\infty}\Phi_u(r_k\varrho) = \Phi,$$

that is

$$\Phi_{u_0}(\varrho) = \Phi, \qquad \text{for a.e. } \varrho > 0. \tag{6.75}$$

Since Φ_{u_0} is increasing in $(0, +\infty)$, (6.75) actually implies that

$$\Phi_{u_0}\equiv\Phi, \qquad \text{in } (0, +\infty).$$

Therefore, by Theorem 6.1.5 we have that u_0 is homogeneous of degree $s-\frac12$ in $\mathbb{R}^n$ and E_0 is a cone.

This concludes the proof of Theorem 6.1.7. □

6.6. Regularity of the free boundary when $s < 1/2$

We observe that in the case $s < 1/2$ the perimeter is, in some sense, the leading term of the functional $\mathcal{F}$. More precisely, by comparing the energy of a minimizing pair with the energy of a simple competitor, we obtain the following estimate.

THEOREM 6.6.1. *Let (u, E) be a minimizing pair in Ω, with $s \in (0, 1/2)$, and suppose that $u \in L^\infty_{\rm loc}(\Omega)$ Let $x_0 \in \Omega$ and let $d := d(x_0, \partial\Omega)/3$. Let $r \in (0, d]$ and define*

$$u_* := \begin{cases} 0 & \text{in } B_r(x_0), \\ u & \text{in } \mathbb{R}^n \setminus B_r(x_0). \end{cases} \tag{6.76}$$

Then

$$\begin{aligned} &\mathcal{N}(u, B_r(x_0)) \le \mathcal{N}(u_*, B_r(x_0)) \\ &\qquad \le 2\left(\mathrm{Per}_{2s}(B_1)\|u\|^2_{L^\infty(B_{2d}(x_0))} + r^{2s}|B_1| C_0 \int_{\mathbb{R}^n} \frac{|u(y)|^2}{1+|y|^{n+2s}}\, dy \right) r^{n-2s} \end{aligned} \tag{6.77}$$

where

$$C_0 = C_0(s, x_0, d) := \sup\left\{ \frac{1+|y|^{n+2s}}{|x-y|^{n+2s}} \;:\; x \in \overline{B_d(x_0)}, y \in \mathbb{R}^n \setminus B_{2d}(x_0) \right\}. \tag{6.78}$$

PROOF. First of all, notice that the function u_* defined in (6.76) is such that

$$u_* \ge 0 \quad \text{a.e. in } E \qquad \text{and} \qquad u_* \le 0 \quad \text{a.e. in } \mathcal{C}E,$$

hence (u_*, E) is an admissible pair. Moreover

$$\mathrm{supp}(u_* - u) \Subset \Omega$$

by definition of u_*, so that (u_*, E) is an admissible competitor for (u, E). Thus, since

$$u = u_* \quad \text{in } \mathcal{C}B_r(x_0),$$

by minimality of (u, E) we get

$$\mathcal{N}(u, B_r(x_0)) - \mathcal{N}(u_*, B_r(x_0)) = \mathcal{N}(u, \Omega) - \mathcal{N}(u_*, \Omega) = \mathcal{F}_\Omega(u, E) - \mathcal{F}_\Omega(u_*, E) \le 0.$$

We recall that

$$\int_{B_r(x_0)} \int_{\mathcal{C}B_r(x_0)} \frac{dx\, dy}{|x-y|^{n+2s}} = \mathrm{Per}_{2s}(B_r(x_0)) = r^{n-2s}\, \mathrm{Per}_{2s}(B_1).$$

Now we can estimate the energy of u_* as follows:

$$\begin{aligned} \mathcal{N}(u_*, B_r(x_0)) &= 2\int_{B_r(x_0)} dx \int_{\mathcal{C}B_r(x_0)} \frac{|u(y)|^2}{|x-y|^{n+2s}}\, dy \\ &= 2\int_{B_r(x_0)} \Big(\int_{B_{2d}(x_0)\setminus B_r(x_0)} \frac{|u(y)|^2}{|x-y|^{n+2s}}\, dy + \int_{\mathcal{C}B_{2d}(x_0)} \frac{|u(y)|^2}{|x-y|^{n+2s}}\, dy \Big) dx \\ &\le 2\int_{B_r(x_0)} \Big(\|u\|^2_{L^\infty(B_{2d}(x_0))} \int_{B_{2d}(x_0)\setminus B_r(x_0)} \frac{dy}{|x-y|^{n+2s}} + C_0 \int_{\mathcal{C}B_{2d}(x_0)} \frac{|u(y)|^2}{1+|y|^{n+2s}}\, dy \Big) dx \\ &\le 2\|u\|^2_{L^\infty(B_{2d}(x_0))}\, \mathrm{Per}_{2s}(B_r(x_0)) + 2C_0 |B_r(x_0)| \int_{\mathbb{R}^n} \frac{|u(y)|^2}{1+|y|^{n+2s}}\, dy, \end{aligned}$$

proving (6.77) and concluding the proof of the Theorem. □

Since the nonlocal energy $\mathcal{N}(u, B_r)$ of a minimizing pair (u, E) goes to zero at least as a power r^{n-2s}, we can prove that the set E is almost minimal—in the sense of [97]—and hence the free boundary ∂E enjoys some regularity properties.

Proof of Theorem 6.1.8. First of all, we can assume that F has finite perimeter in $B_r(x_0)$, otherwise there is nothing to prove. Now let u_* be the function defined in (6.76). Notice that, since

$$E\Delta F \Subset B_r(x_0),$$

then, by definition of u_*,

$$u_* \geq 0 \quad \text{a.e. in } F \qquad \text{and} \qquad u_* \leq 0 \quad \text{a.e. in } \mathcal{C}F,$$

so that (u_*, F) is an admissible pair and is actually an admissible competitor for (u, E).

Therefore, the minimality of (u, E) implies

$$0 \geq \mathcal{F}_\Omega(u, E) - \mathcal{F}_\Omega(u_*, F) = \mathcal{F}_{B_r(x_0)}(u, E) - \mathcal{F}_{B_r(x_0)}(u_*, F).$$

Hence

$$\begin{aligned}
\operatorname{Per}(E, B_r(x_0)) &\leq \operatorname{Per}(F, B_r(x_0)) + \mathcal{N}(u_*, B_r(x_0)) - \mathcal{N}(u, B_r(x_0)) \\
&\leq \operatorname{Per}(F, B_r(x_0)) + 2 \int_{B_r(x_0)} dx \int_{\mathcal{C}B_r(x_0)} \frac{|u(y)|^2 - |u(x) - u(y)|^2}{|x-y|^{n+2s}}\, dy \\
&\leq \operatorname{Per}(F, B_r(x_0)) + 2 \int_{B_r(x_0)} dx \int_{\mathcal{C}B_r(x_0)} \frac{2|u(x)|\,|u(y)|}{|x-y|^{n+2s}}\, dy \\
&\leq \operatorname{Per}(F, B_r(x_0)) + 4\|u\|_{L^\infty(B_d(x_0))} \int_{B_r(x_0)} dx \int_{\mathcal{C}B_r(x_0)} \frac{|u(y)|}{|x-y|^{n+2s}}\, dy.
\end{aligned}$$

Estimating the last double integral as in the proof of Theorem 6.6.1, we find

$$\int_{B_r(x_0)} dx \int_{\mathcal{C}B_r(x_0)} \frac{|u(y)|}{|x-y|^{n+2s}}\, dy \leq C r^{n-2s},$$

concluding the proof of (6.12).

The claims about the regularity of ∂E follow from classical properties of almost minimal sets—see, e.g., [97]. □

6.7. Dimensional reduction

In this Section we prove a dimensional reduction result in the style of Federer—namely Theorem 6.1.9. In order to do this, we need to slightly modify the functonal $\mathcal{F}$ by multiplying $\mathcal{N}$ with the dimensional constant $(c'_{n,s})^{-1}$, so that the corresponding extended functional is "constant-free".

More precisely, only in this Section we will redefine

$$\mathcal{F}_\Omega(u, E) := (c'_{n,s})^{-1}\mathcal{N}(u, \Omega) + \operatorname{Per}(E, \Omega).$$

We say that an admissible pair (u, E) is minimizing in $\mathbb{R}^n$ if it minimizes $\mathcal{F}_\Omega$ in any bounded open subset $\Omega \subseteq \mathbb{R}^n$ (in the sense of Definition 6.1.1).

The corresponding extended functional then becomes

$$\mathfrak{F}_\Omega(\mathcal{V}, F) := \int_{\Omega_+} |\nabla \mathcal{V}|^2 z^{1-2s}\, dX + \operatorname{Per}(F, \Omega_0),$$

for $\Omega \subseteq \mathbb{R}^{n+1}$.

Proof of Theorem 6.1.9. The proof is basically a combination of the proof of [42, Theorem 2.2] and [79, Lemma 28.13]. Before going into the details of the proof, we point out some notation which we use only here. We denote by $\mathcal{P}(F, \mathcal{O})$ the perimeter of a set $F \subseteq \mathbb{R}^{n+1}$ in an open set $\mathcal{O} \subseteq \mathbb{R}^{n+1}$.

We write $\mathcal{X} := (x, x_{n+1}, z)$ and, with a slight abuse of notation,

$$\mathcal{B}_R^+ \times (-a, a) := \{\mathcal{X} = (x, x_{n+1}, z) \in \mathbb{R}^{n+2} \,|\, X = (x, z) \in \mathcal{B}_R^+,\ |x_{n+1}| < a\},$$

"reversing" for notational simplicity the order of x_{n+1} and z in the domains.
If $\mathcal{V} : \mathbb{R}^{n+2} \to \mathbb{R}$, $\mathcal{X} \longmapsto \mathcal{V}(\mathcal{X})$, we write $\nabla_{\mathcal{X}}\mathcal{V}$ for the "full" gradient of $\mathcal{V}$ and

$$\nabla_X \mathcal{V} := (\partial_1 \mathcal{V}, \dots, \partial_n \mathcal{V}, \partial_z \mathcal{V}).$$

In particular, notice that for every fixed x_{n+1} we have

$$|\nabla_{\mathcal{X}}\mathcal{V}|^2 = \sum_{i=1}^{n+1} |\partial_i \mathcal{V}|^2 + |\partial_z \mathcal{V}|^2 \geq \sum_{i=1}^{n} |\partial_i \mathcal{V}|^2 + |\partial_z \mathcal{V}|^2 = |\nabla_X \mathcal{V}|^2. \tag{6.79}$$

We also remark that if $\overline{u}$ and $\overline{u}^\star$ denote the extension functions of u and $u^\star$ respectively, then

$$\overline{u}^\star(x, x_{n+1}, z) = \overline{u}(x, z). \tag{6.80}$$

We first prove, by slicing, that if (u, E) is minimizing in $\mathbb{R}^n$, then $(u^\star, E^\star)$ is minimizing in $\mathbb{R}^{n+1}$.
Fix $a, R > 0$ and let $(\mathcal{V}, F)$ be a competitor for $(\overline{u}^\star, E^\star)$ in $\mathcal{B}_R^+ \times (-a, a)$. For every $|t| < a$ we define the hyperplane slices

$$\mathcal{V}_t(x, z) := \mathcal{V}(x, t, z) \qquad \text{and} \qquad F_t := \{x \in \mathbb{R}^n \,|\, (x, t) \in F\}.$$

By [79, Theorem 18.11], the slice F_t has locally finite perimeter in $\mathbb{R}^n$ for a.e. $t \in (-a, a)$. Moreover, since F is the positivity set of $\mathcal{V}$, we have

$$\mathcal{V}_t\big|_{\{z=0\}} \geq 0 \quad \text{a.e. in } F_t \qquad \text{and} \qquad \mathcal{V}_t\big|_{\{z=0\}} \leq 0 \quad \text{a.e. in } \mathbb{R}^n \setminus F_t,$$

for a.e. $t \in (-a, a)$. Furthermore

$$\operatorname{supp}(\mathcal{V}_t - \overline{u}) \Subset \mathcal{B}_R \qquad \text{and} \qquad F_t \Delta E \Subset B_R,$$

for a.e. $t \in (-a, a)$. Hence $(\mathcal{V}_t, F_t)$ is an admissible competitor for $(\overline{u}, E)$ in $\mathcal{B}_R$, for a.e. $t \in (-a, a)$ and so the minimality of (u, E) implies that

$$\int_{\mathcal{B}_R^+} |\nabla_X \mathcal{V}_t|^2 z^{1-2s}\, dX + \mathcal{H}^{n-1}(\partial^* F_t \cap B_R) \geq \int_{\mathcal{B}_R^+} |\nabla_X \overline{u}|^2 z^{1-2s}\, dX + \mathcal{H}^{n-1}(\partial^* E \cap B_R), \tag{6.81}$$

for a.e. $t \in (-a, a)$. By formula (18.25) of [79], we have

$$\begin{aligned} \int_{-a}^{a} \mathcal{H}^{n-1}(\partial^* F_t \cap B_R)\, dt &= \int_{\partial^* F \cap \left(B_R \times (-a,a)\right)} |\pi \nu_E|\, d\mathcal{H}^n \leq \mathcal{H}^n\big(\partial^* F \cap B_R \times (-a, a)\big) \\ &= \mathcal{P}\big(F, B_R \times (-a, a)\big), \end{aligned} \tag{6.82}$$

where $\pi : \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $\pi(x, x_{n+1}) := x$.
By (6.79) and (6.82) we obtain

$$\begin{aligned} &\int_{\mathcal{B}_R^+ \times (-a,a)} |\nabla_{\mathcal{X}} \mathcal{V}|^2 z^{1-2s}\, d\mathcal{X} + \mathcal{P}\big(F, B_R \times (-a, a)\big) \\ &\qquad \geq \int_{-a}^{a} \left(\int_{\mathcal{B}_R^+} |\nabla_X \mathcal{V}_t|^2 z^{1-2s}\, dX + \mathcal{H}^{n-1}(\partial^* F_t \cap B_R) \right) dt. \end{aligned} \tag{6.83}$$

On the other hand, by (6.80) and formula (28.38) of [79], we have

$$\begin{aligned}\int_{\mathcal{B}_R^+\times(-a,a)} &|\nabla_{\mathcal{X}}\overline{u}^\star|^2 z^{1-2s}\,d\mathcal{X} + \mathcal{P}\big(E^\star, B_R\times(-a,a)\big)\\ &= \int_{-a}^{a}\left(\int_{\mathcal{B}_R^+}|\nabla_X\overline{u}|^2 z^{1-2s}\,dX + \mathcal{H}^{n-1}(\partial^*E\cap B_R)\right)dt\\ &= 2a\left(\int_{\mathcal{B}_R^+}|\nabla_X\overline{u}|^2 z^{1-2s}\,dX + \mathcal{H}^{n-1}(\partial^*E\cap B_R)\right).\end{aligned}\tag{6.84}$$

Exploiting (6.81), (6.83) and (6.84) we finally get

$$\begin{aligned}\int_{\mathcal{B}_R^+\times(-a,a)} &|\nabla_{\mathcal{X}}\mathcal{V}|^2 z^{1-2s}\,d\mathcal{X} + \mathcal{P}\big(F, B_R\times(-a,a)\big)\\ &\geq \int_{\mathcal{B}_R^+\times(-a,a)}|\nabla_{\mathcal{X}}\overline{u}^\star|^2 z^{1-2s}\,d\mathcal{X} + \mathcal{P}\big(E^\star, B_R\times(-a,a)\big).\end{aligned}$$

This proves that the pair $(\overline{u}^\star, E^\star)$ is minimizing in $\mathbb{R}^{n+2}_+$ and hence that $(u^\star, E^\star)$ is minimizing in $\mathbb{R}^{n+1}$, as claimed.

Now let $(u^\star, E^\star)$ be minimizing in $\mathbb{R}^{n+1}$ and suppose that (u, E) is not minimizing in $\mathbb{R}^n$.
Then we can find $R > 0$ and an admissible competitor $(\mathcal{V}, F)$ for $(\overline{u}, E)$ in $\mathcal{B}_R^+$, such that

$$\int_{\mathcal{B}_R^+}|\nabla\mathcal{V}|^2 z^{1-2s}\,dX + \mathrm{Per}(F, B_R) + \varepsilon \leq \int_{\mathcal{B}_R^+}|\nabla\overline{u}|^2 z^{1-2s}\,dX + \mathrm{Per}(E, B_R),\tag{6.85}$$

for some $\varepsilon > 0$. Now we exploit [42, Corollary 5.2] in order to construct a competitor for $(\overline{u}^\star, E^\star)$.

More precisely, fix $a > 0$ (which in the end of the argument will be taken arbitrarily large) and let $\mathcal{Z} : \mathcal{B}_R^+ \times (-a, a) \to \mathbb{R}$ be the function constructed in [42, Corollary 5.2], from $\mathcal{U} := \overline{u}$ and $\mathcal{V}$. Then define the set

$$G := \big(F\times(-a,a)\big)\cup\big(E\times(\mathbb{R}\setminus(-a,a))\big)\subseteq\mathbb{R}^{n+1}$$

and notice that thanks to (5.8) in [42], the pair $(\mathcal{Z}, G)$ is an admissible competitor for $(\overline{u}^\star, E^\star)$ in $\mathcal{B}_R^+\times(-a-1, a+1)$.

Arguing as in Step three of the proof of [79, Lemma 28.13], we find

$$\begin{aligned}\mathcal{P}\big(G, B_R\times(-a-1,a+1)\big) &- \mathcal{P}\big(E^\star, B_R\times(-a-1,a+1)\big)\\ &\leq 2a\,(\mathrm{Per}(F,B_R) - \mathrm{Per}(E,B_R)) + 2\mathcal{H}^n(B_R).\end{aligned}\tag{6.86}$$

Moreover, by (5.10) in [42] the energy of $\mathcal{Z}$ is

$$\begin{aligned}\int_{\mathcal{B}_R^+\times(-a-1,a+1)} &|\nabla_{\mathcal{X}}\mathcal{Z}|^2 z^{1-2s}\,d\mathcal{X}\\ &= 2\int_{\mathcal{B}_R^+\times(a-1,a+1)}|\nabla_{\mathcal{X}}\mathcal{Z}|^2 z^{1-2s}\,d\mathcal{X} + 2(a-1)\int_{\mathcal{B}_R^+}|\nabla\mathcal{V}|^2 z^{1-2s}\,dX,\end{aligned}\tag{6.87}$$

with

$$2\int_{\mathcal{B}_R^+\times(a-1,a+1)}|\nabla_{\mathcal{X}}\mathcal{Z}|^2 z^{1-2s}\,d\mathcal{X} =: C(\mathcal{Z})$$

independent of a by (5.9) in [42].

Therefore, from (6.84), (6.86) and (6.87) we obtain

(6.88)
$$\begin{aligned}&\int_{\mathcal{B}_R^+\times(-a-1,a+1)} |\nabla_{\mathcal{X}}\mathcal{Z}|^2 z^{1-2s}\,d\mathcal{X} + \mathcal{P}\big(G, B_R\times(-a-1,a+1)\big)\\ &\qquad - \int_{\mathcal{B}_R^+\times(-a-1,a+1)} |\nabla_{\mathcal{X}}\overline{u}^\star|^2 z^{1-2s}\,d\mathcal{X} - \mathcal{P}\big(E^\star, B_R\times(-a-1,a+1)\big)\\ &\le 2(a-1)\Big(\int_{\mathcal{B}_R^+}|\nabla\mathcal{V}|^2 z^{1-2s}\,dX + \mathrm{Per}(F,B_R) - \int_{\mathcal{B}_R^+}|\nabla\overline{u}|^2 z^{1-2s}\,dX - \mathrm{Per}(E,B_R)\Big) + C,\end{aligned}$$

where

$$C := C(\mathcal{Z}) + 2\big(\mathrm{Per}(F,B_R) - \mathrm{Per}(E,B_R) + |B_R|\big) + 4\int_{\mathcal{B}_R^+}|\nabla\overline{u}|^2 z^{1-2s}\,dX,$$

which is independent of a.

Finally, by (6.85) and (6.88) we get

$$\begin{aligned}&\int_{\mathcal{B}_R^+\times(-a-1,a+1)} |\nabla_{\mathcal{X}}\mathcal{Z}|^2 z^{1-2s}\,d\mathcal{X} + \mathcal{P}\big(G, B_R\times(-a-1,a+1)\big)\\ &\qquad - \int_{\mathcal{B}_R^+\times(-a-1,a+1)} |\nabla_{\mathcal{X}}\overline{u}^\star|^2 z^{1-2s}\,d\mathcal{X} - \mathcal{P}\big(E^\star, B_R\times(-a-1,a+1)\big)\\ &\le -2(a-1)\varepsilon + C < 0,\end{aligned}$$

provided we take a big enough.

This contradicts the minimality of $(\overline{u}^\star, E^\star)$, concluding the proof. □

6.8. Slicing the perimeter and cones

In this section we collect some (more or less known) results about Caccioppoli sets which we used throughout the chapter. In particular, we recall the coarea formula (see [79, Theorem 18.8]), which we then exploit to construct a cone starting from a "spherical slice" of a Caccioppoli set and to prove a useful formula to compute the perimeter of such a cone.

This construction is used in the proof of the monotonicity formula in Theorem 6.1.5.

Theorem 6.8.1 (Coarea formula). *If M is a locally $\mathcal{H}^{n-1}$-rectifiable set in $\mathbb{R}^n$ and $u:\mathbb{R}^n\to\mathbb{R}$ is a Lipschitz function, then*

$$\int_{\mathbb{R}}\mathcal{H}^{n-2}\big(M\cap\{u=t\}\big)\,dt = \int_M |\nabla^M u|\,d\mathcal{H}^{n-1}, \tag{6.89}$$

where

$$\nabla^M u(x) = \nabla u(x) - (\nabla u(x)\cdot\nu_M(x))\nu_M(x)$$

is the tangential gradient of u. In particular, if $g: M\to[-\infty,+\infty]$ is a Borel function such that $g\ge 0$, then

$$\int_{\mathbb{R}} dt\int_{M\cap\{u=t\}} g\,d\mathcal{H}^{n-2} = \int_M g|\nabla^M u|\,d\mathcal{H}^{n-1}. \tag{6.90}$$

Now we recall that, as remarked in Section 6.1.1.1, given a set $E\subseteq\mathbb{R}^n$ we can always find a set $\tilde{E}$ such that

$$|\tilde{E}\Delta E| = 0$$

and

$$E_{int}\subseteq\tilde{E},\qquad E_{ext}\subseteq\mathcal{C}\tilde{E}\qquad\text{and}\qquad\partial\tilde{E} = \partial^-E. \tag{6.91}$$

Such a set $\tilde{E}$ is given e.g. by the set of points of density 1 of E, that is

$$E^{(1)} := \left\{ x \in \mathbb{R}^n \,|\, \exists \lim_{r \to 0^+} \frac{|E \cap B_r(x)|}{\omega_n r^n} = 1 \right\}$$

(see, e.g., Appendix A). In [99] it is also shown that the measure theoretic boundary $\partial^- E$ has a nice characterization as the smallest topological boundary among the topological boundaries in the equivalence class of E in L^1_{loc}, that is

$$\partial^- E = \bigcap_{|F \Delta E| = 0} \partial F = \partial E^{(1)}. \tag{6.92}$$

If, furthermore, E is a Caccioppoli set, then $\partial^- E$ is the support of the Radon measure $D\chi_E$,

$$\partial^- E = \mathrm{supp}\ D\chi_E$$

(see, e.g., [79, Proposition 12.19]).
In this sense, the set $E^{(1)}$ is a "good representative" for E in its L^1_{loc} equivalence class.

Recall also that the reduced boundary of a Caccioppoli set $E \subseteq \mathbb{R}^n$

$$\partial^* E := \left\{ x \in \mathrm{supp}\ D\chi_E \text{ s.t. } \exists \lim_{\varrho \to 0^+} \frac{D\chi_E(B_\varrho(x))}{|D\chi_E|(B_\varrho(x))} =: \nu_E(x) \in \mathbb{S}^{n-1} \right\}$$

is locally $\mathcal{H}^{n-1}$-rectifiable by De Giorgi's structure Theorem (see, e.g., [79, Theorem 15.9]). The Borel function $\nu_E : \partial^* E \to \mathbb{S}^{n-1}$ is the (measure theoretic) outer unit normal to E. Also notice that by Lebesgue-Besicovitch differentiation Theorem we have

$$D\chi_E = \nu_E |D\chi_E| \llcorner \partial^* E.$$

Moreover De Giorgi's structure Theorem also says that

$$|D\chi_E| = \mathcal{H}^{n-1} \llcorner \partial^* E, \qquad D\chi_E = \nu_E \mathcal{H}^{n-1} \llcorner \partial^* E,$$

so that, in particular,

$$\mathrm{Per}(E, B) = |D\chi_E|(B) = \mathcal{H}^{n-1}(\partial^* E \cap B), \tag{6.93}$$

for any Borel set $B \subseteq \mathbb{R}^n$.

Remark 6.8.2. Let $E \subseteq \mathbb{R}^n$ be a set having finite perimeter in $B_{\tilde{R}}$ and let $R < \tilde{R}$. Using formula (6.90) for $M = \partial^* E$, with $u(x) = |x|$ and $g = \chi_{B_R}$, we obtain

$$\int_0^R \mathcal{H}^{n-2}(\partial^* E \cap \partial B_t)\, dt = \int_{\partial^* E \cap B_R} |\nabla^{\partial^* E} u| \, d\mathcal{H}^{n-1} \leq \mathrm{Per}(E, B_R) < +\infty.$$

As a consequence the function

$$h : r \longmapsto \mathcal{H}^{n-2}(\partial^* E \cap \partial B_r)$$

is such that $h \in L^1(0, R)$ and

$$\mathcal{H}^{n-2}(\partial^* E \cap \partial B_r) < +\infty, \tag{6.94}$$

for a.e. $r > 0$. Notice that for any r such that (6.94) holds true, we have

$$\mathcal{H}^{n-1}(\partial^* E \cap \partial B_r) = 0. \tag{6.95}$$

Hence (6.95) also holds true for a.e. $r > 0$.

Furthermore, we remark that since $h \in L^1(0, R)$, a.e. $r \in (0, R)$ is a Lebesgue point for h.

We now recall the following result (see e.g. Lemma 4.2.1 on page 102 of [3]):

LEMMA 6.8.3. *Let $x \in \Omega$ and let E be a set of finite perimeter in Ω. For a.e. $\varrho \in \big(0,\, d(x, \partial\Omega)\big)$ there exists a set E_ϱ which has finite perimeter in Ω, such that $E \Delta E_\varrho$ is contained in $B_\varrho(x)$ and*

$$P\big(E_\varrho,\, \overline{B_\varrho(x)}\big) \le \frac{\varrho}{n-1} \frac{d}{d\varrho} \operatorname{Per}(E,\, B_\varrho(x)). \tag{6.96}$$

As a matter of fact, taking $x := 0$ up to a translation, the set E_ϱ given in Lemma 6.8.3 is exactly the cone defined in (6.7) (inside $\overline{B_\varrho}$), see the formula in display after (2.8) on page 104 of [3].
More precisely, we recall that we always suppose that the "good representative" of a set is chosen, by taking the points of Lebesgue density 1. In this sense, formula (6.7) has to be interpreted as

$$E(r) := \{\lambda y \,|\, \lambda > 0,\ y \in E^{(1)} \cap \partial B_r\}.$$

Lemma 6.8.3 then guarantees that for a.e. $r \in \big(0, d(0, \Omega)\big)$ the cone $E(r)$ is a Caccioppoli set.

We also observe that the cone structure of $E(r)$, together with (6.92), implies that

$$\begin{aligned} \partial E(r) \cap \partial B_t &= \frac{t}{r}\big(\partial E^{(1)} \cap \partial B_r\big) = \frac{t}{r}\big(\partial^- E \cap \partial B_r\big) \\ \text{and}\ \ \partial^* E(r) \cap \partial B_t &= \frac{t}{r}\big(\partial^* E \cap \partial B_r\big). \end{aligned} \tag{6.97}$$

The cone structure of $E(r)$ also implies that

$$x \cdot \nu_{E(r)}(x) = 0 \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial^* E(r), \tag{6.98}$$

see, e.g., [79, Proposition 28.8].

With these pieces of information we obtain that:

PROPOSITION 6.8.4. *Let $E \subseteq \mathbb{R}^n$ be a Caccioppoli set. Then for a.e. $r > 0$ the cone $E(r)$ is a Caccioppoli set and*

$$\operatorname{Per}(E(r), B_\varrho) = \frac{\mathcal{H}^{n-2}(\partial^* E \cap \partial B_r)}{(n-1) r^{n-2}} \varrho^{n-1}, \tag{6.99}$$

for every $\varrho > 0$.

PROOF. The computation relies on (6.97) and (6.98) and uses the coarea formula with $M = \partial^* E(r)$ and $u(x) = |x|$, so that $\nabla u(x) = \frac{x}{|x|}$. Indeed,

$$\begin{aligned} \operatorname{Per}(E(r), B_\varrho) &= \int_{\partial^* E(r)} \chi_{B_\varrho}\, d\mathcal{H}^{n-1} = \int_{\partial^* E(r)} \chi_{B_\varrho} |\nabla^{\partial^* E(r)} u|\, d\mathcal{H}^{n-1} \\ &= \int_0^\varrho \mathcal{H}^{n-2}(\partial^* E(r) \cap \partial B_t)\, dt = \int_0^\varrho \Big(\frac{t}{r}\Big)^{n-2} \mathcal{H}^{n-2}(\partial^* E \cap \partial B_r)\, dt \\ &= \frac{\mathcal{H}^{n-2}(\partial^* E \cap \partial B_r)}{(n-1) r^{n-2}} \int_0^\varrho \frac{d}{dt} t^{n-1}\, dt, \end{aligned}$$

proving (6.99). □

REMARK 6.8.5. The same argument shows that if $E \subseteq \mathbb{R}^n$ has finite perimeter in B_R, then for a.e. $r \in (0, R)$ the cone $E(r)$ is a Caccioppoli set and satisfies formula (6.99).

We remark that, as a consequence of formulas (6.96) and (6.99), we obtain that

$$\mathcal{H}^{n-2}(\partial^* E \cap \partial B_r) \le \frac{d}{dr} \operatorname{Per}(E, B_r), \qquad \text{for a.e. } r > 0.$$

We now prove this inequality by exploiting the coarea formula.

Proposition 6.8.6. *Let $E\subseteq\mathbb{R}^n$ be a set having finite perimeter in B_R. Then*

$$\mathcal{H}^{n-2}(\partial^*E\cap\partial B_r)\le\frac{d}{dr}\operatorname{Per}(E,B_r),\tag{6.100}$$

for a.e. $r\in(0,R)$. Moreover, the following are equivalent:

(i) the set E is a cone in B_R, i.e. there exists a cone $C\subseteq\mathbb{R}^n$ such that

$$\big|(E\Delta C)\cap B_R\big|=0,$$

(ii) the function

$$(0,R)\ni r\longmapsto\operatorname{Per}(E,B_r)$$

is continuous and

$$\mathcal{H}^{n-2}(\partial^*E\cap\partial B_r)=\frac{d}{dr}\operatorname{Per}(E,B_r),\qquad \textit{for a.e. } r\in(0,R).\tag{6.101}$$

Proof. We define the functions

$$h(r):=\mathcal{H}^{n-2}(\partial^*E\cap\partial B_r)\qquad\text{and}\qquad\wp(r):=\operatorname{Per}(E,B_r).$$

Then $h\in L^1(0,R)$ (see Remark 6.8.2) and $\wp$ is differentiable almost everywhere in $(0,R)$, since it is monotone non-decreasing. Let

$$\mathcal{G}:=\{r\in(0,R)\,|\,r\text{ is a Lebesgue point of }h\text{ and }\exists\,\wp'(r)\},$$

and notice that $\mathcal{L}^1\big((0,R)\setminus\mathcal{G}\big)=0$. We also remark that

$$r\in\mathcal{G}\quad\Longrightarrow\quad\mathcal{H}^{n-1}(\partial^*E\cap\partial B_r)=0.$$

We prove that the inequality (6.100) holds true for every $r\in\mathcal{G}$. To this end, we use the coarea formula for ∂^*E, with $u(x):=|x|$. Notice that

$$|\nabla^{\partial^*E}u(x)|=\sqrt{1-\Big(\frac{x}{|x|}\cdot\nu_E(x)\Big)^2}\le1.$$

Thus

$$\begin{aligned}\operatorname{Per}(E,B_{r+\varepsilon})-\operatorname{Per}(E,B_r)&=\operatorname{Per}(E,B_{r+\varepsilon}\setminus\overline{B_r})=\int_{\partial^*E\cap(B_{r+\varepsilon}\setminus\overline{B_r})}d\mathcal{H}^{n-1}\\&\ge\int_{\partial^*E\cap(B_{r+\varepsilon}\setminus\overline{B_r})}\sqrt{1-\Big(\frac{x}{|x|}\cdot\nu_E(x)\Big)^2}\,d\mathcal{H}^{n-1}=\int_r^{r+\varepsilon}\mathcal{H}^{n-2}(\partial^*E\cap\partial B_t)\,dt,\end{aligned}$$

for every $\varepsilon>0$ small enough. Since $r\in\mathcal{G}$, dividing by ε and passing to the limit $\varepsilon\to0^+$ yields (6.100).

Now we prove that (i) implies (ii). First of all, notice that since $\lambda C=C$ for every $\lambda>0$, we have

$$\operatorname{Per}(E,B_\varrho)=\operatorname{Per}(C,B_\varrho)=P\Big(\frac{\varrho}{r}C,\frac{\varrho}{r}B_r\Big)=\Big(\frac{\varrho}{r}\Big)^{n-1}\operatorname{Per}(C,B_r)=\Big(\frac{\varrho}{r}\Big)^{n-1}\operatorname{Per}(E,B_r),$$

for every $r,\varrho\in(0,R)$. Hence

$$\lim_{\varrho\to r}\operatorname{Per}(E,B_\varrho)=\lim_{\varrho\to r}\Big(\frac{\varrho}{r}\Big)^{n-1}\operatorname{Per}(E,B_r)=\operatorname{Per}(E,B_r),$$

proving that $\wp$ is continuous in $(0,R)$.

Since E is a cone in B_R, we have by [79, Proposition 28.8] that

$$x\cdot\nu_E(x)=0\qquad\text{for }\mathcal{H}^{n-1}\text{-a.e. }\partial^*E\cap B_R.$$

Hence, if $u(x):=|x|$, then we find

$$|\nabla^{\partial^*E}u(x)|=1\qquad\text{for }\mathcal{H}^{n-1}\text{-a.e. }\partial^*E\cap B_R.$$

Therefore, the coarea formula implies that

$$\operatorname{Per}(E, B_{r+\varepsilon}) - \operatorname{Per}(E, B_r) = \int_r^{r+\varepsilon} \mathcal{H}^{n-2}(\partial^* E \cap \partial B_t)\, dt,$$

for every $r \in \mathcal{G}$ and $\varepsilon > 0$ small enough. Dividing by ε and passing to the limit $\varepsilon \to 0^+$ thus proves (6.101).

We are left to show that (ii) implies (i). To this end, first notice that since $\wp$ is continuous and differentiable a.e. in $(0, R)$, by the Fundamental Theorem of Calculus we have

$$\operatorname{Per}(E, B_r) - \operatorname{Per}(E, B_\varrho) = \int_\varrho^r \frac{d}{dt} \operatorname{Per}(E, B_t)\, dt, \tag{6.102}$$

for every $0 < \varrho < r < R$. Then, from (6.101) and (6.102) we get

$$\int_{\partial^* E \cap (B_r \setminus B_\varrho)} d\mathcal{H}^{n-1} = \operatorname{Per}(E, B_r) - \operatorname{Per}(E, B_\varrho) = \int_\varrho^r \mathcal{H}^{n-2}(\partial^* E \cap \partial B_t)\, dt. \tag{6.103}$$

Therefore, by exploiting the coarea formula, from (6.103) we obtain

$$\int_{\partial^* E \cap (B_r \setminus B_\varrho)} \sqrt{1 - \left(\frac{x}{|x|} \cdot \nu_E(x)\right)^2}\, d\mathcal{H}^{n-1} = \int_{\partial^* E \cap (B_r \setminus B_\varrho)} d\mathcal{H}^{n-1},$$

for every $0 < \varrho < r < R$. Thus

$$x \cdot \nu_E(x) = 0 \qquad \text{for } \mathcal{H}^{n-1}\text{-a.e. } x \in \partial^* E \cap B_R.$$

By [79, Proposition 28.8], this implies that $E^{(1)}$ is a cone in B_R, concluding the proof. □

6.9. The surface density of a Caccioppoli set

The following Lemma is a variation of [34, Lemma 5.1] and [3, Exercises 3.2.4 and 1.3.6].

Lemma 6.9.1. *Let $\wp : (0, R) \to \mathbb{R}$ be a monotone non-decreasing function and let $\beta \in C^1\big((0, R), (0, +\infty)\big)$. Then*

$$\beta(t_2)\wp(t_2) - \beta(t_1)\wp(t_1) = \int_{[t_1,t_2)} \beta(r)\, dD\wp(r) + \int_{[t_1,t_2)} \beta'(r)\wp(r)\, dr, \tag{6.104}$$

for every $0 < t_1 < t_2 < R$. Moreover $\wp$ is differentiable a.e. in $(0, R)$ and

$$\beta(t_2)\wp(t_2) - \beta(t_1)\wp(t_1) \geq \int_{t_1}^{t_2} \big[\beta(r)\wp'(r) + \beta'(r)\wp(r)\big]\, dr, \quad \text{for every } 0 < t_1 < t_2 < R. \tag{6.105}$$

Proof. We start by proving (6.104). For this, we define

$$\alpha := \beta\wp. \tag{6.106}$$

By construction, $\alpha \in BV(0, R)$. We also set $\alpha_\star(t) := D\alpha\big([0, t)\big)$ and we claim that

$$\text{the distributional derivative of } \alpha_\star \text{ is equal to } D\alpha. \tag{6.107}$$

To check this, we observe that, by Fubini's Theorem, for any $\phi \in C_c^\infty(0, R)$,

$$\begin{aligned} -\int_{[0,R)} \phi(\tau)\, dD\alpha(\tau) &= \int_{[0,R)} \big(\phi(R) - \phi(\tau)\big)\, dD\alpha(\tau) = \int_{[0,R)} \left(\int_{[\tau,R)} \phi'(t)\, dt\right) dD\alpha(\tau) \\ &= \int_{[0,R)} \left(\int_{[0,t)} \phi'(t)\, dD\alpha(\tau)\right) dt = \int_{[0,R)} \phi'(t)\, D\alpha\big([0, t)\big)\, dt = \int_{[0,R)} \phi'(t)\, \alpha_\star(t)\, dt. \end{aligned}$$

This proves (6.107).

Now we claim that there exists $c \in \mathbb{R}$ such that, a.e. $t \in (0, R)$,

$$\alpha(t) = c + D\alpha\big([0, t)\big). \tag{6.108}$$

To this end, we set $\gamma(t) := \alpha(t) - \alpha_\star(t)$. Since $\alpha_\star$ is monotone non-decreasing, we see that $\gamma \in BV(0, R)$. Also, by (6.107), we have that the distributional derivative of γ vanishes identically, hence $D\gamma = 0$ and therefore γ is constant. This implies (6.108), as desired.

Now, from (6.108), it follows that

$$\alpha(t_2) - \alpha(t_1) = D\alpha\big([0, t_2)\big) - D\alpha\big([0, t_1)\big) = D\alpha\big([t_1, t_2)\big) = \int_{[t_1,t_2)} dD\alpha(t). \tag{6.109}$$

From this and (6.106), we obtain (6.104). Now we prove (6.105). For this, we use the Lebesgue-Besicovitch Theorem (see, e.g., [79, Theorem 5.8]) to write

$$D\wp = \Psi\mathcal{L}^1 + D^s\wp, \tag{6.110}$$

with $D^s\wp$ is the singular part of $D\wp$, that is a measure supported in a set of zero Lebesgue measure, and (see, e.g., [79, Corollary 5.11])

$$\Psi(t) := \lim_{\varrho\to 0^+} \frac{D\wp\big((t - \varrho, t + \varrho)\big)}{2\varrho}.$$

We define

$$\mathcal{G} := \left\{ t \in (0, R) \text{ s.t. } t \text{ is a Lebesgue point of } \Psi \text{ and } \lim_{\varrho\to 0^+} \frac{D^s\wp\big((t - \varrho, t + \varrho)\big)}{\varrho} = 0 \right\}$$

and

$$\mathcal{B} := \left\{ t \in (0, R) \text{ s.t. } \lim_{\varrho\to 0^+} \frac{D^s\wp\big((t - \varrho, t + \varrho)\big)}{\varrho} \neq 0 \right\}.$$

Since $\wp$ is non-decreasing, we have that

$$\mathcal{B} = \left\{ t \in (0, R) \text{ s.t. } \lim_{\varrho\to 0^+} \frac{D^s\wp\big((t - \varrho, t + \varrho)\big)}{\varrho} > 0 \right\},$$

hence $\mathcal{B}$ is a subset of the support of $D^s\wp$, and so it has zero Lebesgue measure. Consequently,

$$\mathcal{G} \text{ has full Lebesgue measure in } (0, R). \tag{6.111}$$

Now we claim that

$$\text{for any } t \in \mathcal{G}, \text{ the function } \wp \text{ is differentiable at } t \text{ and } \wp'(t) = \Psi(t). \tag{6.112}$$

To check this, we exploit (6.108) (here, by choosing $\beta := 1$) and we write that

$$\wp(t) = c + D\wp\big([0, t)\big), \tag{6.113}$$

for some $c \in \mathbb{R}$. Then, by (6.110), we infer that

$$D\wp\big([0, t)\big) = \int_{[0,t)} \Psi(\tau)\, d\tau + D^s\wp\big([0, t)\big),$$

and therefore, by (6.113),

$$\wp(t) = c + \int_{[0,t)} \Psi(\tau)\, d\tau + D^s\wp\big([0, t)\big).$$

As a consequence, if $t \in \mathcal{G}$ we have that

$$\lim_{\varrho \to 0^+} \frac{\wp(t+\varrho) - \wp(t)}{\varrho} = \lim_{\varrho \to 0^+} \frac{1}{\varrho} \left[\int_{[t,t+\varrho)} \Psi(\tau)\, d\tau + D^s \wp\big([t, t+\varrho)\big) \right] = \Psi(t) + 0,$$

and this proves (6.112).

In view of (6.111) and (6.112), we obtain that

$$\text{the function } \wp \text{ is differentiable a.e. in } (0,R), \text{ with } \wp' = \Psi.$$

This and (6.110) give that

$$D\wp = \wp' \mathcal{L}^1 + D^s \wp.$$

Hence, by (6.106),

$$D\alpha = D\beta \wp + \beta D\wp = \beta' \wp \mathcal{L}^1 + \beta\big(\wp' \mathcal{L}^1 + D^s \wp\big) = (\beta' \wp + \beta \wp') \mathcal{L}^1 + \beta D^s \wp.$$

Accordingly, in view of (6.109), and using that $\beta \geq 0$,

$$\begin{aligned} \alpha(t_2) - \alpha(t_1) &= \int_{[t_1,t_2)} (\beta' \wp + \beta \wp')(t)\, dt + \int_{[t_1,t_2)} \beta(t)\, dD^s \wp(t) \\ &\geq \int_{[t_1,t_2)} (\beta' \wp + \beta \wp')(t)\, dt. \end{aligned}$$

This completes the proof of (6.105). □

In particular, by applying Lemma 6.9.1 to the "surface density" of $F \subseteq \mathbb{R}^n$ in 0,

$$\theta_F(r) := \frac{\operatorname{Per}(F, B_r)}{r^{n-1}},$$

we obtain the following result:

Corollary 6.9.2. *Let $F \subseteq \mathbb{R}^n$ be a set having finite perimeter in B_R and let*

$$\wp(r) := \operatorname{Per}(F, B_r), \qquad \theta_F(r) := \frac{\operatorname{Per}(F, B_r)}{r^{n-1}}.$$

Then the function θ_F is differentiable a.e. in $(0,R)$, with

$$\theta_F'(r) = r^{1-n} \wp'(r) - (n-1) r^{-1} \theta_F(r) \qquad \text{for a.e. } r \in (0,R).$$

Moreover

$$\theta_F(t_2) - \theta_F(t_1) \geq \int_{t_1}^{t_2} \theta_F'(r)\, dr, \quad \text{for every } 0 < t_1 < t_2 < R. \tag{6.114}$$

CHAPTER 7

The Phillip Island penguin parade (a mathematical treatment)

7.1. Introduction

The goal of this chapter is to provide a simple, but rigorous, mathematical model which describes the formation of groups of penguins on the shore at sunset.

The results that we obtain are the following. First of all, we provide the construction of a mathematical model to describe the formation of groups of penguins on the shore and their march towards their burrows; this model is based on systems of ordinary differential equations, with a number of degree of freedom which is variable in time (we show that the model admits a unique solution, which needs to be appropriately defined). Then, we give some rigorous mathematical results which provide sufficient conditions for a group of penguins to reach the burrows. In addition, we provide some numerical simulations which show that the mathematical model well predicts, at least at a qualitative level, the formation of clusters of penguins and their march towards the burrows; these simulations are easily implemented by images and videos.

It would be desirable to have empirical data about the formation of penguins clusters on the shore and their movements, in order to compare and adapt the model to experimental data and possibly give a quantitative description of concrete scenarios.

The methodology used is based on direct observations on site, strict interactions with experts in biology and penguin ecology, mathematical formulation of the problem and rigorous deductive arguments, and numerical simulations.

In this introduction, we will describe the elements which lead to the construction of the model, presenting its basic features and also its limitations. Given the interdisciplinary flavor of the subject, it is not possible to completely split the biological discussion from the mathematical formulation, but we can mention that the main mathematical equation is given in formula (7.1). Before (7.1), the main information coming from live observations are presented. After (7.1), the mathematical quantities involved in the equation are discussed and elucidated. The existence and uniqueness theory for equation (7.1) is presented in Section 7.2. Some rigorous mathematical results about equation (7.1) are given in Section 7.3. Roughly speaking, these are results which give sufficient conditions on the initial data of the system and on the external environment for the successful homecoming of the penguins, and their precise formulation requires the development of the mathematical framework in (7.1).

In Section 7.4 we present numerics, images and videos which support our intuition and set the mathematical model of (7.1) into a concrete framework that is easily comparable with the real-world phenomenon.

Prior to this, we think it is important to describe our experience of the penguins parade in Phillip Island, both to allow the reader who is not familiar with the event to concretely take part in it, and to describe some peculiar environmental aspects which are crucial to understand our description (for instance, the weather in Phillip Island is completely different from the Antarctic one, so many of our considerations are meant to be limited to this particular habitat) – also, our personal experience in this bio-mathematical adventure is a crucial point, in our opinion, to describe how scientific curiosity can trigger academic activities.

7.1.1. Description of the penguins parade. An extraordinary event in the state of Victoria, Australia, consists in the march of the little penguins (whose scientific name is *Eudyptula minor*) who live in Phillip Island. At sunset, when it gets too dark for the little penguins to hunt their food in the sea, they come out to return to their homes (which are small cavities in the terrain, that are located at some dozens of meters from the water edge). What follows is the mathematical description that came out of the observations on site at Phillip Island, enriched by the scientific discussions we later had with penguin ecologists.

By watching the penguins parade in Phillip Island, it seemed to us that some simple features appeared in the very unusual pattern followed by the little penguins. First of all, they have the strong tendency to gather together in a sufficiently large number before starting their march home. They have the tendency to march on a straight line, compactly arranged in a cluster, or group. To make this group, they move back and forth, waiting for other fellows or even going back to the sea if no other mate is around.

If a little penguin remains isolated, some parameters in the model proposed may lead to a complete stop of the individual. More precisely, in the model that we propose, there is a term which makes the velocity vanish. In practice, this interruption in the penguin's movement is not due to physical impediments, but rather to the fact that there is no other penguin in a sufficiently small neighbourhood: in this sense, at a mathematical level, a quantified version of the notion of "isolation" leads the penguin to stop.

Of course, from the point of view of ethology, it would be desirable to have further non-invasive tests to measure how the situation that we describe is felt by the penguin at an emotional level (at the moment, we are not aware of experiments like this in the literature). Also, it would be highly desirable to have some precise experiments to determine how many penguins do not manage to return to their burrows within a certain time after dusk and stay either in the water or in the vicinity of the shore.

On one hand, in our opinion, it is likely that rigorous experiments on site will demonstrate that the phenomenon for which an isolated penguin stops is rather uncommon, but not completely exceptional, in nature. On the other hand, our model is general enough to take into account the possibility that a penguin stops its march, and, at a quantitative level, we emphasized this feature in the pictures of Section 7.4 to make the situation visible.

The reader who does not want to take into account the stopping function in the model can just set this function to be identically equal to 1 (the mathematical formulation of this remark will be given after formula (7.9)). In this particular case, our model will still exhibit the formation of groups of penguins moving together.

Though no experimental test has been run on the emotive feelings of penguins during their homecoming, in the parade that we have seen live it indeed happened that one little penguin remained isolated from the others: even though (s)he was absolutely fit and no concrete obstacle was obstructing the motion, (s)he got completely stuck for half an hour and the staff of the Nature Park had to go and provide assistance. We stress again that the fact that the penguin stopped did not seem to be caused by any physical impediment (as confirmed to us by the Ranger on site), since no extreme environmental condition was occurring, the animal was not underweight, and was able to come out of the water and move effortlessly on the shore autonomously for about 15 meters, before suddenly stopping.

For a short video (courtesy of Phillip Island Nature Parks) of the little penguins parade, in which the formation of groups is rather evident, see e.g. the file `Penguins1.MOV`, available at the webpage
`https://youtu.be/x488k4n3ip8`

The simple features listed above are likely to be a consequence of the morphological structure of the little penguins and of the natural environment. As a matter of fact, little penguins are a marine-terrestrial species. They are highly efficient swimmers but possess a rather inefficient form of locomotion on land (indeed, flightless penguins, as the ones in Phillip Island, waddle, more than walk). At dusk, about 80 minutes after sunset according to the data collected in [**88**], little penguins return ashore after their fishing activity in the sea. Since their bipedal locomotion is slow and rather goofy (at least from the human subjective perception, but also in comparison with the velocity or agility that is well known to be typical of predators in nature), and the easily recognizable countershading of the penguins is likely to make them visible to predators, the transition between the marine and terrestrial environment may be particularly stressful for the penguins (see [**73**]) and this fact is probably related to the formation of penguins groups (see e.g. [**33**]). Thus, in our opinion, the rules that we have listed may be seen as the outcome of the difficulty of the little penguins to perform their transition from a more favourable environment to an habitat in which their morphology turns out to be suboptimal.

At the moment, there seems to be no complete experimental evidence measuring the subjective perceptions of the penguins with respect to the surrounding environments. Nevertheless, given the swimming ability of the penguins and the environmental conditions, one may well conjecture that an area of high potential danger for a penguin is the one adjacent to the shore-line, since this is a habitat which provides little or no shelter, and it is also in an area of reduced visibility. As a matter of fact, to protect the penguins in this critical area next to the water edge, the Rangers in Phillip Island implemented a control on the presence of the foxes in the proximity of the shore, with the aim of limiting the number of possible predators.

7.1.2. Comparison with the existing literature. We observe that, to the best of our knowledge, there is still no specific mathematical attempt to describe in a concise way the penguins parade. The mathematical literature of penguins has mostly focused on the description of the heat flow in the penguins feathers (see [49]), on the numerical analysis to mark animals for later identification (see [95]), on the statistics of the Magellanic penguins at sea (see [96]), on the hunting strategies of fishing penguins (see [63]), and on the isoperimetric arrangement of the Antarctic penguins to prevent the heat dispersion caused by the polar wind and on the crystal structures and solitary waves produced by such arrangements (see [62] and [86]). We remark that the climatic situation in Phillip Island is rather different from the Antarctic one and, given the very mild temperatures of the area, we do not think that heat considerations should affect too much the behaviour and the moving strategies of the Victorian little penguins and their tendency to cluster seems more likely to be a defensive strategy against possible predators.

Though no mathematical formulation of the little penguins parade has been given till now, a series of experimental analysis has been recently performed on the specific environment of Phillip Island. We recall, in particular, [33], in which the association of the little penguins in groups is described, by collecting data spanning over several years, [27], in which there is a description of the effect of fog on the orientation of the little penguins (which may actually not come back home in conditions of poor visibility), [78] and [87], which presents a data analysis to show the fractal structure in space and time for the foraging of the little penguins, also in relation to Lévy flights and fractional Brownian motions.

For an exhaustive list of publications focused on the behaviour of the little penguins of Phillip Island, we refer to the web page
`https://www.penguins.org.au/conservation/research/publications/`
This pages contains more than 160 publications related to the environment of Phillip Island, with special emphasis on the biology of little penguins.

We recall that there is also a wide literature from the point of view of biology and ethology focused on collective mathematical behaviours, also in terms of formation of groups and hierarchies (see e.g. [11] [82] and [56]).

The mathematical literature studying the collective behaviour of animal groups is also rather broad: we mention in particular [7], which studied the local rules of interaction of individual birds in airborne flocks, [32], which analyzed the self-organization from a microscopic to a macroscopic scale, [12], which took into account movements with a speed depending on an additional variable, and [71] for different models on opinion formation within an interacting group.

We remark that our model is specifically tailored on the Phillip Island penguins : for instance, other colonies of penguins, such as those in St Kilda, exhibit behaviours different from those in Phillip Island, due to the different environmental conditions, see e.g. the scientific report by [67] for additional information on the penguins colony on the St Kilda breakwater.

7.1.3. Mathematical formulation. In this section we provide a mathematical description of the penguins parade, which was described in Section 7.1.1. The idea for providing an equation for this parade is to prescribe that the velocity of a group of penguins which travels in line is influenced by the natural environment and by the position of the other visible groups. Anytime a group is formed, the equation needs to be modified to encode the formation of this new structure. The main mathematical notation is described in Table 1.

In further details, to translate into a mathematical framework the simple observations on the penguins behaviour that we listed in Subsection 7.1.1, we propose the following

$p_i(t)$	one-dimensional position of the ith group of penguins at time t
$w_i(t)$	number of penguins belonging to the ith group of penguins at time t
f	function describing the environment (sea, shore, presence of predators, etc.)
$\mathfrak{P}_i$	stopping function
ε	speed of a solitary penguin in a neutral condition (may be zero)
$\mathcal{V}_i$	strategic speed of the ith group of penguins (depending on the position of the penguins, on the size of the group and on time)
v	speed of "large" penguins groups
m_i	influence of the "visible" penguins ahead and behind on the speed of the ith group
$\mathfrak{s}$	eye-sight of the penguins

TABLE 1. Notation.

equation:

$$\dot{p}_i(t) = \mathfrak{P}_i\big(p(t), w(t); t\big)\,\Big(\varepsilon + \mathcal{V}_i\big(p(t), w(t); t\big)\Big) + f\big(p_i(t), t\big). \tag{7.1}$$

The variable $t \geq 0$ represents time and $p(t)$ is a vector valued function of time, that takes into account the positions of the different groups of penguins. Roughly speaking, at time t, there are $n(t)$ groups of penguins, therefore $p(t)$ is an array with $n(t)$ components, and so we will write

$$p(t) = \big(p_1(t), \dots, p_{n(t)}(t)\big). \tag{7.2}$$

We stress that $n(t)$ may vary in time (in fact, it will be taken to be piecewise constant), hence the spatial dimension of the image of p is also a function of time. For any $i \in \{1, \dots, n(t)\}$, the ith group of penguins contains a number of penguins denoted by $w_i(t)$ (thus, the number of penguins belonging to each group is also a function of time).

In further detail, the following notation is used. The function $n : [0, +\infty) \to \mathbb{N}_0$, where $\mathbb{N}_0 := \mathbb{N} \setminus \{0\}$, is piecewise constant and nonincreasing, namely there exist a (possibly finite) sequence $0 = t_0 < t_1 < \dots < t_j < \dots$ and integers $n_1 > \dots > n_j > \dots$ such that

$$n(t) = n_j \in \mathbb{N}_0 \text{ for any } t \in (t_{j-1}, t_j). \tag{7.3}$$

In this model, for simplicity, the spatial occupancy of a cluster of penguins coincide with that of a single penguin: of course, in reality, there is a small repulsion playing among the penguins, which cannot stay too close to one another. This additional complication may also be taken into account in our model, by enlarging the spatial size of the cluster in dependence of the numerousness of the penguins in the group. In any case, for practical purposes, we think it is not too inaccurate to identify a group of penguins with just a single element, since the scale at which the parade occurs (several dozens of meters) is much larger than the size of a single penguin (little penguins are only about 30 cm. tall).

We also consider the array $w(t) = \big(w_1(t), \dots, w_{n(t)}(t)\big)$. We assume that w_i is piecewise constant, namely, $w_i(t) = \bar{w}_{i,j}$ for any $t \in (t_{j-1}, t_j)$, for some $\bar{w}_{i,j} \in \mathbb{N}_0$, namely the number of little penguins in each group remains constant, till the next penguins join the group at time t_j (if, for the sake of simplicity, one wishes to think that initially all the little penguins are separated one from the other, one may also suppose that $w_i(t) = 1$ for all $i \in \{1, \dots, n_1\}$ and $t \in [0, t_1)$).

By possibly renaming the variables, we suppose that the initial position of the groups is increasing with respect to the index, namely

$$p_1(0) < \dots < p_{n_1}(0). \tag{7.4}$$

The parameter $\varepsilon \geq 0$ represents a drift velocity of the penguins towards their house, which is located at the point $H \in (0, +\infty)$. The parameter ε, from the biological point of view, represents the fact that each penguin, in a neutral situation, has a natural tendency to move towards its burrow. We can also allow $\varepsilon = 0$ in our treatment (namely, the existence and uniqueness theory in Section 7.2 remains unchanged if $\varepsilon = 0$ and the rigorous results in Section 7.3 present cases in which they still hold true when $\varepsilon = 0$, compare in particular with assumptions (7.17) and (7.19)).

For concreteness, if $p_i(T) = H$ for some $T \geq 0$, we can set $p_i(t) := H$ for all $t \geq T$ and remove p_i from the equation of motion – that is, the penguin has safely come back home.

For any $i \in \{1, \dots, n(t)\}$, the quantity $\mathcal{V}_i\big(p(t), w(t); t\big)$ represents the strategic velocity of the ith group of penguins and it can be considered as a function with domain varying in time

$$\mathcal{V}_i(\cdot, \cdot; t) : \mathbb{R}^{n(t)} \times \mathbb{N}^{n(t)} \to \mathbb{R},$$

i.e.

$$\mathcal{V}_i(\cdot, \cdot; t) : \mathbb{R}^{n_j} \times \mathbb{N}^{n_j} \to \mathbb{R} \qquad \text{for any } t \in (t_{j-1}, t_j),$$

and, for any $(p, w) = (p_1, \dots, p_{n(t)}, w_1, \dots, w_{n(t)}) \in \mathbb{R}^{n(t)} \times \mathbb{N}^{n(t)}$, it is of the form

$$\mathcal{V}_i\big(p, w; t\big) := \Big(1 - \mu\big(w_i\big)\Big)\, m_i\big(p, w; t\big) + v\mu\big(w_i\big). \tag{7.5}$$

In this setting, for any $(p, w) = (p_1, \dots, p_{n(t)}, w_1, \dots, w_{n(t)}) \in \mathbb{R}^{n(t)} \times \mathbb{N}^{n(t)}$, we have that

$$m_i\big(p, w; t\big) := \sum_{j \in \{1, \dots, n(t)\}} \operatorname{sign}(p_j - p_i)\, w_j\, \mathfrak{s}(|p_i - p_j|), \tag{7.6}$$

where $\mathfrak{s} \in \mathrm{Lip}([0, +\infty))$ is nonnegative and nonincreasing and, as usual, we denoted the "sign function" as

$$\mathbb{R} \ni r \mapsto \operatorname{sign}(r) := \begin{cases} 1 & \text{if } r > 0, \\ 0 & \text{if } r = 0, \\ -1 & \text{if } r < 0. \end{cases}$$

Also, for any $\ell \in \mathbb{N}$, we set

$$\mu(\ell) := \begin{cases} 1 & \text{if } \ell \geq \kappa, \\ 0 & \text{if } \ell \leq \kappa - 1, \end{cases} \tag{7.7}$$

for a fixed $\kappa \in \mathbb{N}$, with $\kappa \geq 2$, and $v > \varepsilon$.

In our framework, the meaning of the strategic velocity of the ith group of penguins is the following. When the group of penguins is too small (i.e. it contains less than κ little penguins), then the term involving μ vanishes, thus the strategic velocity reduces to the term given by m_i; this term, in turn, takes into account the position of the other groups of penguins. That is, each penguin is endowed with a "eye-sight" (i.e., the capacity of seeing the other penguins that are "sufficiently close" to them), which is modelled by the function $\mathfrak{s}$ (for instance, if $\mathfrak{s}$ is identically equal to 1, then the penguin has a "perfect eye-sight"; if $\mathfrak{s}(r) = e^{-r^2}$, then the penguin sees close objects much better than distant ones; if $\mathfrak{s}$ is compactly supported, then the penguin does not see too far objects, etc.). Based on the position of the other mates that (s)he sees, the penguin has the tendency to move either forward or backward (the more penguins (s)he sees ahead, the more (s)he is inclined to move forward, the more penguins (s)he sees behind, the more (s)he is inclined to move backward, and nearby penguins weight more than distant ones, due to the monotonicity of $\mathfrak{s}$). This strategic tension coming from the position of the other penguins is encoded by the function m_i.

The eye-sight function can be also considered as a modification of the interaction model based simply on metric distance. Another interesting feature which has been observed in several animal groups (see e.g. [7]), is the so-called "topological interaction" model, in which every agent interacts only with a fixed number of agents, among the ones which are closer. A modification of the function $\mathfrak{s}$ can also take into account this possibility. It is of course very interesting to investigate by direct observations how much topological, quantitative and metric considerations influence the formation and the movement of little penguin clusters.

When the group of penguins is sufficiently large (i.e. it contains at least κ little penguins), then the term involving μ is equal to 1; in this case, the strategic velocity is v (that is, when the group of penguins is sufficiently rich in population, its strategy is to move forward with cruising speed equal to v).

The function $\mathfrak{P}_i\big(p(t), w(t); t\big)$ describes the case of extreme isolation of the ith individual from the rest of the herd. Here, we take $\overline{d} > \underline{d} > 0$, a nonincreasing function $\varphi \in \mathrm{Lip}(\mathbb{R}, [0,1])$, with $\varphi(r) = 1$ if $r \le \underline{d}$ and $\varphi(r) = 0$ if $r \ge \overline{d}$, and, for any $\ell \in \mathbb{N}_0$,

$$\mathfrak{w}(\ell) := \begin{cases} 1 & \text{if } \ell \ge 2, \\ 0 & \text{if } \ell = 1, \end{cases} \tag{7.8}$$

and we take as stopping function the function with variable domain

$$\mathfrak{P}_i(\cdot, \cdot; t) : \mathbb{R}^{n(t)} \times \mathbb{N}^{n(t)} \to [0,1],$$

i.e.

$$\mathfrak{P}_i(\cdot, \cdot; t) : \mathbb{R}^{n_j} \times \mathbb{N}^{n_j} \to [0,1] \qquad \text{for any } t \in (t_{j-1}, t_j),$$

given, for any $(p, w) = (p_1, \dots, p_{n(t)}, w_1, \dots, w_{n(t)}) \in \mathbb{R}^{n(t)} \times \mathbb{N}^{n(t)}$, by

$$\mathfrak{P}_i\big(p, w; t\big) := \max\Big\{\mathfrak{w}(w_i), \max_{\substack{j \in \{1, \dots, n(t)\} \\ j \ne i}} \varphi\big(|p_i - p_j|\big)\Big\}. \tag{7.9}$$

Here the notation "Lip" stands for bounded and Lipschitz continuous functions.

The case of φ identically equal to 1 can be also comprised in our setting. In this case, also $\mathfrak{P}_i$ is identically one (which corresponds to the case in which the stopping function has no effect).

The stopping function describes the fact that the group may present the tendency to suddenly stop. This happens when the group contains only one element (i.e., $\mathfrak{w}_i = 0$) and the other groups are far apart (at distance larger than $\overline{d}$).

Conversely, if the group contains at least two little penguins, or if there is at least another group sufficiently close (say at distance smaller than $\underline{d}$), then the group is self-confident, namely the function $\mathfrak{P}_i\big(p(t), w(t); t\big)$ is equal to 1 and the total intentional velocity of the group coincides with the strategic velocity.

Interestingly, the stopping function $\mathfrak{P}_i$ may be independent of the eye-sight function $\mathfrak{s}$: namely a little penguin can stop if (s)he feels too much exposed, even if (s)he can see other little penguins (for instance, if $\mathfrak{s}$ is identically equal to 1, the little penguin always sees the other members of the herd, still (s)he can stop if they are too far apart).

The function $f \in \mathrm{Lip}(\mathbb{R} \times [0, +\infty))$ takes into account the environment. For a neutral environment, one has that this term vanishes (where neutral means here that the environment does not favour or penalize the homecoming of the penguins). In practice, it may take into account the ebb and flow of the sea on the foreshore (where the little penguins parade starts), the possible ruggedness of the terrain, the presence of predators, etc. (as a variation, one can consider also a stochastic version of this term). This environment function can take into account several characteristics at the same time. For example, a possible situation that we wish to model is that in which the sea occupies the spatial region $(-\infty, 0)$, producing waves that are periodic in time, with frequency ϖ and amplitude δ; suppose also that the shore is located in the spatial region $(-\infty, 0)$, presenting a steep hill in the region $(1, 2)$ which can slow down the motion of the penguins, whose burrows are located at the point 4. In this setting, a possible choice of the environment function f is

$$\mathbb{R} \times [0, +\infty) \ni (p, t) \longmapsto f(p, t) = \delta \sin(\varpi t + \phi)\, \chi_{(-\infty,0)}(p) - h\, \chi_{(1,2)}(p).$$

In this notation $h > 0$ is a constant that takes into account "how steep" the hill located in the region $(1, 2)$ is, $\phi \in \mathbb{R}$ is an initial phase of the wave in the sea, and χ_E is the characteristic function of a set E, namely

$$\chi_E(x) := \begin{cases} 1 & \text{if } x \in E, \\ 0 & \text{if } x \notin E. \end{cases}$$

Given the interpretations above, equation (7.1) tries to comprise the pattern that we described in words and to set the scheme of motion of the little penguins into a mathematical framework.

7.1.4. Preliminary presentation of the mathematical results. In this chapter, three main mathematical results will be presented. First of all, in Section 7.2, we provide an existence and uniqueness theory for the solutions of equation (7.1).

From the mathematical viewpoint, we remark that (7.1) does not fall into the classical framework of the standard Cauchy initial value problem for ordinary differential equations (compare e.g. with formula (2.3) and Theorem 2.1 in [8]), since the right hand side of the equation is not Lipschitz continuous (and, in fact, it is not even continuous). This mathematical complication is indeed the counterpart of the real motion of the little penguins in the parade, which have the tendency to change their speed rather abruptly to maintain contact with the other elements of the herd. That is, on view, it does not seem unreasonable to model, as a simplification, the speed of the penguin as a discontinuous function, to take into account the sudden modifications of the waddling according to the position of the other penguins, with the conclusive aim of gathering together a sufficient number of penguins in a group which eventually will march concurrently in the direction of their burrows.

Then, in Section 7.3 we provide two rigorous results which guarantee suitable conditions under which all the penguins, or some of them, safely return to their burrows. In Theorem 7.3.1 we establish that if the sum of the drift velocity and the environmental function is strictly positive and if there is a time (which can be the initial time or a subsequent one) for which the group at the end of the line consists of at least two penguins, then all the penguins reach their burrows in a finite time, which can be explicitly estimated.

Also, in Theorem 7.3.2 we prove that if the sum of the drift and cruise velocities and of the environmental function is strictly positive and if there is a time for which one of the penguins group is sufficiently numerous, then all the penguins of this group and of the groups ahead safely return home in a finite time, which can be explicitly estimated.

Rigorous statements and proofs will be given in Sections 7.2 and 7.3.

7.1.5. Detailed organization of the chapter. The mathematical treatment of equation (7.1) that we provide in this chapter is the following.

In Section 7.2, we provide a notion of solution for which (7.1) is uniquely solvable in the appropriate setting. This notion of solution will be obtained by a "stop-and-go" procedure, which is compatible with the idea that when two (or more) groups of penguins meet, they form a new, bigger group which will move coherently in the sequel of the march.

In Section 7.3, we discuss a couple of concrete examples in which the penguins are able to safely return home: namely, we show that there are "nice" conditions in which the strategy of the penguins allows a successful homecoming.

In Section 7.4, we present a series of numerical simulations to compare our mathematical model with the real-world experience. This part also contains some figures produced by the numerics.

Several possible structural generalizations of the model proposed are presented in Section 7.5. Furthermore, the model that we propose can be easily generalized to a multi-dimensional setting, as discussed in Section 7.6.

The conclusions of our work will be summarized in Section 7.7.

7.2. Existence and uniqueness theory for equation (7.1)

We stress that equation (7.1) does not lie within the setting of ordinary differential equations, since the right hand side is not Lipschitz continuous (due to the discontinuity of the functions w and m_i, and in fact the right hand side also involves functions with domain varying in time). As far as we know, the weak formulations of ordinary differential equations as the ones treated by [39] do not take into consideration the setting of equation (7.1), so we briefly discuss here a direct approach to the existence and uniqueness theory for such equation. To this end, and to clarify our direct approach, we present two illustrative examples (see e.g. [59]).

Example 7.2.1. Setting $x : [0, +\infty) \to \mathbb{R}$, the ordinary differential equation

$$\dot{x}(t) = \begin{cases} -1 & \text{if } x(t) \geq 0, \\ 1 & \text{if } x(t) < 0 \end{cases} \tag{7.10}$$

is not well posed. Indeed, taking an initial datum $x(0) < 0$, it will evolve with the formula $x(t) = t + x(0)$ for any $t \in [0, -x(0)]$ till it hits the zero value. At that point, equation (7.10) would prescribe a negative velocity, which becomes contradictory with the positive velocity prescribed to the negative coordinates.

Example 7.2.2. The ordinary differential equation

$$\dot{x}(t) = \begin{cases} -1 & \text{if } x(t) > 0, \\ 0 & \text{if } x(t) = 0, \\ 1 & \text{if } x(t) < 0 \end{cases} \tag{7.11}$$

is similar to the one in (7.10), in the sense that it does not fit into the standard theory of ordinary differential equations, due to the lack of continuity of the right hand side. But, differently from the one in (7.10), it can be set into an existence and uniqueness theory by a simple "reset" algorithm.

Namely, taking an initial datum $x(0) < 0$, the solution evolves with the formula $x(t) = t + x(0)$ for any $t \in [0, -x(0)]$ till it hits the zero value. At that point, equation (7.11) would prescribe a zero velocity, thus a natural way to continue the solution is to take $x(t) = 0$ for any $t \in [-x(0), +\infty)$ (similarly, in the case of positive initial datum $x(0) > 0$, a natural way to continue the solution is $x(t) = -t + x(0)$ for any $t \in [0, x(0)]$ and $x(t) = 0$ for any $t \in [x(0), +\infty)$). The basic idea for this continuation method is to flow the equation according to the standard Cauchy theory of ordinary differential equations for as long as possible, and then, when the classical theory breaks, "reset" the equation with respect of the datum at the break time (this method is not universal and indeed it does not work for (7.10), but it produces a natural global solution for (7.11)).

In the light of Example 7.2.2, we now present a framework in which equation (7.1) possesses a unique solution (in a suitable "reset" setting). To this aim, we first notice that the initial number of groups of penguins is fixed to be equal to n_1 and each group is given by a fixed number of little penguins packed together (that is, the number of little penguins in the ith initial group being equal to $\bar{w}_{i,1}$ and i ranges from 1 to n_1). So, we set $\bar{w}_1 := (\bar{w}_{1,1}, \dots, \bar{w}_{n_1,1})$ and $\bar{\mathfrak{w}}_{i,1} = \mathfrak{w}(\bar{w}_{i,1})$, where $\mathfrak{w}$ was defined in (7.8). For any $p = (p_1, \dots, p_{n_1}) \in \mathbb{R}^{n_1}$, let also

$$\mathfrak{P}_{i,1}(p) := \max \Big\{ \bar{\mathfrak{w}}_{i,1}, \max_{\substack{j \in \{1,\dots,n_1\} \\ j \neq i}} \varphi\big(|p_i - p_j|\big) \Big\}. \tag{7.12}$$

The reader may compare this definition with the one in (7.9). For any $i \in \{1, \dots, n_1\}$ we also set

$$\bar{\mu}_{i,1} := \mu(\bar{w}_{i,1}),$$

where μ is the function defined in (7.7), and, for any $p = (p_1, \dots, p_{n_1}) \in \mathbb{R}^{n_1}$,

$$\bar{m}_{i,1}(p) := \sum_{j \in \{1,\dots,n_1\}} \operatorname{sign}(p_j - p_i)\, \bar{w}_{j,1}\, \mathfrak{s}(|p_i - p_j|).$$

This definition has to be compared with (7.6). Recalling (7.4) we also set

$$\mathcal{D}_1 := \{p = (p_1, \dots, p_{n_1}) \in \mathbb{R}^{n_1} \text{ s.t. } p_1 < \dots < p_{n_1}\}.$$

We remark that if $p \in \mathcal{D}_1$ then

$$\bar{m}_{i,1}(p) = \sum_{j \in \{i+1,\dots,n_1\}} \bar{w}_{j,1}\, \mathfrak{s}(|p_i - p_j|) - \sum_{j \in \{1,\dots,i-1\}} \bar{w}_{j,1}\, \mathfrak{s}(|p_i - p_j|)$$

and therefore

$$\bar{m}_{i,1}(p) \text{ is bounded and Lipschitz for any } p \in \mathcal{D}_1. \tag{7.13}$$

Then, we set

$$\mathcal{V}_{i,1}(p) := (1 - \bar{\mu}_{i,1})\, \bar{m}_{i,1}(p) + v\bar{\mu}_{i,1}.$$

This definition has to be compared with the one in (7.5). Notice that, in view of (7.13), we have that

$$\mathcal{V}_{i,1}(p) \text{ is bounded and Lipschitz for any } p \in \mathcal{D}_1. \tag{7.14}$$

So, we set

$$G_{i,1}(p,t) := \mathfrak{P}_{i,1}(p)\,\big(\varepsilon + \mathcal{V}_{i,1}(p)\big) + f(p_i, t).$$

From (7.12) and (7.14), we have that $G_{i,1}$ is bounded and Lipschitz in $\mathcal{D}_1 \times [0, +\infty)$. Consequently, from the global existence and uniqueness of solutions of ordinary differential equations, we have that there exist $t_1 \in (0, +\infty]$ and a solution $p^{(1)}(t) = (p_1^{(1)}(t), \dots, p_{n_1}^{(1)}(t)) \in \mathcal{D}_1$ of the Cauchy problem

$$\begin{cases} \dot{p}_i^{(1)}(t) = G_{i,1}\big(p^{(1)}(t), t\big) & \text{for } t \in (0, t_1), \\ p^{(1)}(0) \quad \text{given in } \mathcal{D}_1 \end{cases}$$

and

$$p^{(1)}(t_1) \in \partial \mathcal{D}_1, \tag{7.15}$$

see e.g. Theorem 1.4.1 in the book [72].

Notice that, as customary in the mathematical literature, we denoted by ∂ the "topological boundary" of a set. In particular,

$$\begin{aligned} \partial \mathcal{D}_1 \;=\; & \{p = (p_1, \dots, p_{n_1}) \in \mathbb{R}^{n_1} \text{ s.t. } p_1 \le \dots \le p_{n_1} \\ & \quad \text{and there exists } i \in \{1, \dots, n_1 - 1\} \text{ s.t. } p_i = p_{i+1}\}. \end{aligned}$$

The idea for studying the Cauchy problem in our framework is thus that, as long as the trajectory of the system stays in the interior of the domain $\mathcal{D}_1$, the forcing term remains uniformly Lipschitz, thus the flow does not develop any singularity. Hence the trajectory exists and it is defined up to the time (if any) in which it meets the boundary of the domain $\mathcal{D}_1$, that, in the biological framework, corresponds to the situation in which two (or more) penguins meet (i.e., they occupy the same position at the same time). In this case, the standard flow procedure of the ordinary differential equation is stopped, we will merge the joint penguins into a common cluster, and then repeat the argument.

In further detail, the solution of (7.1) will be taken to be $p^{(1)}$ in $[0, t_1)$, that is, we set $p(t) := p^{(1)}(t)$ for any $t \in [0, t_1)$. We also set that $n(t) := n_1$ and $w(t) := (\bar{w}_{1,1}, \dots, \bar{w}_{n_1,1})$. With this setting, we have that p is a solution of equation (7.1) in the time range $t \in (0, t_1)$ with prescribed initial datum $p(0)$. Condition (7.15) allows us to perform our "stop-and-go" reset procedure as follows: we denote by n_2 the number of distinct points in the set $\{p_1^{(1)}(t_1), \dots, p_{n_1}^{(1)}(t_1)\}$. Notice that (7.15) says that if t_1 is finite then $n_2 \le n_1 - 1$ (namely, at least two penguins have reached the same position). In this way, the set of points $\{p_1^{(1)}(t_1), \dots, p_{n_1}^{(1)}(t_1)\}$ can be identified by the set of n_2 distinct points, that we denote by $\{p_1^{(2)}(t_1), \dots, p_{n_2}^{(2)}(t_1)\}$, with the convention that

$$p_1^{(2)}(t_1) < \dots < p_{n_2}^{(2)}(t_1).$$

For any $i \in \{1, \dots, n_2\}$, we also set

$$\bar{w}_{i,2} := \sum_{\substack{j \in \{1, \dots, n_1\} \\ p_j^{(1)}(t_1) = p_i^{(2)}(t_1)}} \bar{w}_{j,1}.$$

This says that the new group of penguins indexed by i contains all the penguins that have reached that position at time t_1.

Thus, having the "new number of groups", that is n_2, the "new number of little penguins in each group", that is $\bar{w}_2 = (\bar{w}_{1,2}, \dots, \bar{w}_{n_2,2})$, and the "new initial datum", that is $p^{(2)}(t_1) = \big(p_1^{(2)}(t_1), \dots, p_{n_2}^{(2)}(t_1)\big)$, we can solve a new differential equation with these new parameters, exactly in the same way as before, and keep iterating this process.

Indeed, recursively, we suppose that we have found $t_1 < t_2 < \dots < t_k$, $p^{(1)} : [0, t_1] \to \mathbb{R}^{n_1}$, ..., $p^{(k)} : [0, t_k] \to \mathbb{R}^{n_k}$ and $\bar{w}_1 \in \mathbb{N}_0^{n_1}$, ..., $\bar{w}_k \in \mathbb{N}_0^{n_k}$ such that, setting

$$\begin{aligned} & p(t) := p^{(j)}(t) \in \mathcal{D}_j, \qquad n(t) := n_j \\ \text{and} \qquad & w(t) := \bar{w}_j \qquad \text{for } t \in [t_{j-1}, t_j) \text{ and } j \in \{1, \dots, k\}, \end{aligned}$$

one has that p solves (7.1) in each interval (t_{j-1}, t_j) for $j \in \{1, \dots, k\}$, with the "stop condition"

$$p^{(j)}(t_j) \in \partial\mathcal{D}_j,$$

where

$$\mathcal{D}_j := \{p = (p_1, \dots, p_{n_j}) \in \mathbb{R}^{n_j} \text{ s.t. } p_1 < \dots < p_{n_j}\}.$$

Then, since $p^{(k)}(t_k) \in \partial\mathcal{D}_k$, if t_k is finite, we find $n_{k+1} \le n_k - 1$ such that the set of points $\{p_1^{(k)}(t_k), \dots, p_{n_k}^{(k)}(t_k)\}$ coincides with a set of n_{k+1} distinct points, that we denote by $\{p_1^{(k+1)}(t_k), \dots, p_{n_k}^{(k+1)}(t_k)\}$, with the convention that

$$p_1^{(k+1)}(t_k) < \dots < p_{n_k}^{(k+1)}(t_k).$$

For any $i \in \{1, \dots, n_{k+1}\}$, we set

$$\bar{w}_{i,k+1} := \sum_{\substack{j\in\{1,\dots,n_k\} \\ p_j^{(k)}(t_k)=p_i^{(k+1)}(t_k)}} \bar{w}_{j,k}. \tag{7.16}$$

It is useful to observe that, in light of (7.16),

$$\sum_{i\in\{1,\dots,n_{k+1}\}} \bar{w}_{i,k+1} = \sum_{i\in\{1,\dots,n_k\}} \bar{w}_{i,k},$$

which says that the total number of little penguins remains always the same (more precisely, the sum of all the little penguins in all groups is constant in time).

Let also $\bar{\mathfrak{w}}_{i,k+1} = \mathfrak{w}(\bar{w}_{i,k+1})$. Then, for any $i \in \{1, \dots, n_{k+1}\}$ and any $p = (p_1, \dots, p_{n_{k+1}}) \in \mathbb{R}^{n_{k+1}}$, we set

$$\mathfrak{P}_{i,k+1}(p) := \max\Big\{\bar{\mathfrak{w}}_{i,k+1}, \max_{\substack{j\in\{1,\dots,n_{k+1}\} \\ j\neq i}} \varphi\big(|p_i - p_j|\big)\Big\}.$$

For any $i \in \{1, \dots, n_{k+1}\}$ we also define

$$\bar{\mu}_{i,k+1} := \mu(\bar{w}_{i,k+1}),$$

where μ is the function defined in (7.7) and, for any $p \in \mathbb{R}^{n_{k+1}}$,

$$\bar{m}_{i,k+1}(p) := \sum_{j\in\{1,\dots,n_{k+1}\}} \operatorname{sign}(p_j - p_i)\, \bar{w}_{j,k+1}\, \mathfrak{s}(|p_i - p_j|).$$

We notice that $\bar{m}_{i,k+1}(p)$ is bounded and Lipschitz for any $p \in \mathcal{D}_{k+1} := \{p = (p_1, \dots, p_{n_{k+1}}) \in \mathbb{R}^{n_{k+1}}$ s.t. $p_1 < \dots < p_{n_{k+1}}\}$.

We also define

$$\mathcal{V}_{i,k+1}(p) := (1 - \bar{\mu}_{i,k+1})\, \bar{m}_{i,k+1}(p) + v\bar{\mu}_{i,k+1}$$

and

$$G_{i,k+1}(p,t) := \mathfrak{P}_{i,k+1}(p)\, \big(\varepsilon + \mathcal{V}_{i,k+1}(p)\big) + f(p_i, t).$$

In this way, we have that $G_{i,k+1}$ is bounded and Lipschitz in $\mathcal{D}_{k+1} \times [0, +\infty)$ and so we find the next solution $p^{(k+1)}(t) = (p_1^{(k+1)}(t), \dots, p_{n_{k+1}}^{(k+1)}(t)) \in \mathcal{D}_{k+1}$ in the interval (t_k, t_{k+1}), with $p^{(k+1)}(t_{k+1}) \in \partial\mathcal{D}_{k+1}$, by solving the ordinary differential equation

$$\dot{p}_i^{(k+1)}(t) = G_{i,k+1}\big(p^{(k+1)}(t), t\big).$$

This completes the iteration argument and provides the desired notion of solution for equation (7.1).

7.3. Examples of safe return home

Here, we provide some sufficient conditions for the penguins to reach their home, located at the point H, which is taken to be "far away with respect to the initial position of the penguins", namely we suppose that

$$H > \max_{i\in\{1,\dots,n(0)\}} p_i(0),$$

and ε has to be thought sufficiently small. Let us mention that, in the parade that we saw live, one little penguin remained stuck and did not manage to return home – so, giving a mathematical treatment of the case in which the strategy of the penguins turns out to be successful somehow reassured us on the fate of the species.

To give a mathematical framework of the notion of homecoming, we introduce the function

$$[0,+\infty) \ni t \mapsto \mathcal{N}(t) := \sum_{\substack{j\in\{1,\dots,n(t)\}\\ p_j(t)=H}} w_j(t).$$

In the setting of Subsection 7.1.3, the function $\mathcal{N}(t)$ represents the number of penguins that have safely returned home at time t.

For counting reasons, we also point out that the total number of penguins is constant and given by

$$\mathcal{M} := \sum_{j\in\{1,\dots,n(0)\}} w_j(0) = \sum_{j\in\{1,\dots,n(t)\}} w_j(t),$$

for any $t \geq 0$.

The first result that we present says that if at some time the group of penguins that stay further behind gathers into a group of at least two elements, then all the penguins will manage to eventually return home. The mathematical setting goes as follows:

THEOREM 7.3.1. *Let $t_o \geq 0$ and assume that*

$$\varepsilon + \inf_{(r,t)\in\mathbb{R}\times[t_o,+\infty)} f(r,t) \geq \iota \tag{7.17}$$

for some $\iota > 0$, and

$$w_1(t_o) \geq 2. \tag{7.18}$$

Then, there exists $T \in \left[t_o,\, t_o + \frac{H-p_1(t_o)}{\iota}\right]$ such that

$$\mathcal{N}(T) = \mathcal{M}.$$

PROOF. We observe that $w_1(t)$ is nondecreasing in t, by (7.16), and therefore (7.18) implies that $w_1(t) \geq 2$ for any $t \geq t_o$. Consequently, from (7.8), we obtain that $\mathfrak{w}(w_1(t)) = 1$ for any $t \geq t_o$. This and (7.9) give that $\mathfrak{P}_1\big(p, w(t); t\big) = 1$ for any $t \geq t_o$ and any $p \in \mathbb{R}^{n(t)}$. Accordingly, the equation of motions in (7.1) gives that, for any $t \geq t_o$,

$$\dot{p}_1(t) = \varepsilon + \mathcal{V}_1\big(p(t), w(t); t\big) + f\big(p_1(t), t\big) \geq \varepsilon + f\big(p_1(t), t\big) \geq \iota,$$

by (7.17). That is, for any $j \in \{1, \dots, n(t)\}$,

$$p_j(t) \geq p_1(t) \geq \min\{H,\ p_1(t_o) + \iota\,(t - t_o)\},$$

which gives the desired result. □

A simple variation of Theorem 7.3.1 says that if, at some time, a group of little penguins reaches a sufficiently large size, then all the penguins in this group (as well as the ones ahead) safely reach their home. The precise statement (whose proof is similar to the one of Theorem 7.3.1, up to technical modifications, and is therefore omitted) goes as follows:

THEOREM 7.3.2. *Let $t_o \geq 0$ and assume that*

$$\varepsilon + v + \inf_{(r,t)\in\mathbb{R}\times[t_o,+\infty)} f(r,t) \geq \iota \tag{7.19}$$

for some $\iota > 0$, and

$$w_{j_o}(t_o) \geq \kappa,$$

for some $j_o \in \{1, \dots, n(t_o)\}$, where κ is defined in (7.7).

Then, there exists $T \in \left[t_o,\, t_o + \frac{H-p_{j_o}(t_o)}{\iota}\right]$ such that

$$\mathcal{N}(T) \geq \sum_{j\in\{j_o,\dots,n(t_o)\}} w_j(t_o).$$

7.4. Pictures, videos and numerics

In this section, we present some simple numerical experiments to facilitate the intuition at the base of the model presented in (7.1). These simulations may actually show some of the typical treats of the little penguins parade, such as the oscillations and sudden change of direction, the gathering of the penguins into clusters and the possibility that some elements of the herd remain isolated, either on the land or in the sea.

The possibility that a penguin remains isolated also in the sea may actually occur in the real-world experience, as demonstrated by the last penguin in the herd on the video (courtesy of Phillip Island Nature Parks) named `Penguins2.MOV` available online at the webpage
`https://youtu.be/dVk1uYbH_Xc`

In our simulations, for the sake of simplicity, we considered 20 penguins returning to their burrows from the shore – some of the penguins may start their trip from the sea (that occupies the region below level 0 in the simulations) in which waves and currents may affect the movements of the animals. The pictures that we produce (see Section 7.9) have the time variable on the horizontal axis and the space variable on the vertical axis (with the burrow of the penguins community set at level 4 for definiteness). The pictures are, somehow, self-explanatory. For instance, in Figure 1, we present a case in which, fortunately, all the little penguins manage to safely return home, after having gathered into groups: as a matter of fact, in the first of these pictures all the penguins safely reach home together at the same time (after having rescued the first penguin, who stayed still for a long period due to isolation); on the other hand, the second of these pictures shows that a first group of penguins, which was originated by the animals that were on the land at the initial time, reaches home slightly before the second group of penguins, which was originated by the animals that were in the sea at the initial time (notice also that the motion of the penguins in the sea appears to be affected by waves and currents).

We also observe a different scenario depicted in Figure 4 (with two different functions to represent the currents in the sea): in this situation, a big group of 18 penguins gathers together (collecting also penguins who were initially in the water) and safely returns home. Two penguins remain isolated in the water, and they keep slowly moving towards their final destination (that they eventually reach after a longer time).

Similarly, in Figure 2, almost all the penguins gather into a single group and reach home, while two penguins get together in the sea, they come to the shore and slowly waddle towards their final destination, and one single penguin remains isolated in the water, moved by the currents.

The situation in Figure 3 is slightly different, since the last penguin at the beginning moves towards the others, but (s)he does not manage to join the forming group by the time the other penguins decide to move consistently towards their burrows – so, unfortunately this last penguin, in spite of the initial effort, finally remains in the water.

With simple modifications of the function f, one can also consider the case in which the waves of the sea change with time and their influence may become more (or less) relevant for the swimming of the little penguins: as an example of this feature, see Figures 5 and 6.

In Figures 8 and 7 we give some examples of what happens when varying the parameters that we used in the numerics of the other figures. For example we consider different values of κ, the parameter which encodes when a group of penguins is big enough to be self confident and waddle home without being influenced by the other groups of penguins in sight.

By considering small values of κ we can represent a strong preference of the penguins to go straight towards their homes, instead of first trying to form a large group. This situation is depicted in the second picture of Figure 8 where we see that after a few time the penguins form two distinct small groups and go towards home without trying to form a unique large group together.

On the contrary, considering a large value of κ represents the preference of the penguins to gather in a very large group before starting their march towards home, like in the first picture of Figure 8. This situation could represent for example the penguins being timorous because of the presence of predators.

We think that the case in which one penguin, or a small number of penguins, remain(s) in water even after the return of the main group is worth of further investigation also by means of concrete experiments. One possible scenario is that the penguins in the water will just wait long enough for other penguins to get close to the shore and join them to form a new group; on the other hand, if all the other penguins have already returned, the few ones remained in the water will have to accept the risk of returning home isolated from the other conspecifics and in an unprotected situation, and we think that interesting biological features could be detected in this case.

Finally, we recall that once a group of little penguins is created, then it moves consistently altogether. This is of course a simplifying assumption, and it might happen in reality that one or a few penguins leave a large group after its formation – perhaps because one penguin is slower than the other penguins of the group, perhaps because (s)he gets distracted by other events on the beach, or simply because (s)he feels too exposed being at the side of the group and may prefer to form a new group in which (s)he finds a more central and protected position. We plan to describe this case in detail in a forthcoming project (also possibly in light of morphological and social considerations and taking into account a possible randomness in the system).

The situation in which one little penguin seems to think about leaving an already formed group can be observed in the video (courtesy of Phillip Island Nature Parks) named `Penguins2.MOV` and available online at
`https://youtu.be/dVk1uYbH_Xc`
(see in particular the behaviour of the second penguin from the bottom, i.e. the last penguin of the already formed large cluster).

We point out that all these pictures have been easily obtained by short programs in MATLAB.

We describe here the algorithm of the basic program, with waves of constant size and standard behaviour of all the little penguins. The modified versions (periodic strong waves, tired little penguins and so on) can be easily inferred from it.

We take into account N little penguins, we set their house at $H = 4$ and the sea below the location 0. Strong waves can go beyond the location 0 in some cases, but in the standard program we just consider normal ones. We take a small ε to represent the natural predisposition to go home of the little penguins, and we define a constant $\delta = (N + 1)\varepsilon$ that we need to define the velocity of the little penguins. We define the waves as $\text{WAVE} = \delta \sin(T)$, where T is the array of times. The speed of the animals is related to the one of waves in such a way that it becomes the strongest just when the little penguins form a group that is big enough.

The program starts with a “for” loop that counts all the animals in a range near the chosen little penguin. This “for” loop gives us two values: the indicator of the parameter PAN (short for “panic”) and the function W, that represents the number of animals in the same position of the one we are considering. We needed this function since we have seen that when the little penguins form a group that is big enough, they proceed towards their home with a cruise speed that is bigger than it was before. We define this cruise speed as vc (short for "velocity") in the program.

Then we start computing the speed V of the little penguin. If PAN is equal to zero, the little penguin freezes. His velocity is zero if he is on the shore (namely his position is greater or equal than zero), or it is given by the waves if he is in the water. It is worth noting that at each value of time the "for" loop counts the value of PAN, hence a little penguin can leave the stopping condition if he sees some mates and start moving again.

If PAN is not zero we have mainly two cases, according to the fact that a big group is formed or not. If this has happened, namely $W > \frac{N}{2}$, then the little penguin we are considering is in the group, so he goes towards home with a cruise speed vc, possibly modified by the presence of waves. If the group is not formed yet, the animal we are considering is surrounded by some mates, but they are not enough to proceed straight home. His speed is positive or negative, namely he moves forward or backward, in dependence of the amount of little penguins that he has ahead of him or behind him. Its speed is given by:

$$V = \varepsilon + M$$

where M is the number of penguins ahead of him minus the number of animals behind him multiplied by $\frac{\delta}{N}$, and ε has been defined before. As in the other cases, the speed can be modified by the presence of waves if the position is less than zero.

Now that we have computed the speed of the animal, we can obtain his position P after a discrete time interval t by considering $P(k + 1) = P(k) + Vt$.

The last "for" loop is done in order to put in the same position two animals that are closed enough. Then we reset the counting variables PAN, W and M and we restart the loop.

For completeness, we made the source codes of all the programs available on the webpage
`https://www.dropbox.com/sh/odgic3a0ke5qp0q/AABIMaasAcTwZQ3qKRoB--xra?dl=0`

An example of the code is given in Section 7.8. The simplicity of these programs shows that the model in (7.1) is indeed very simple to implement numerically, still producing sufficiently "realistic" results in terms of cluster formation and cruising speed of the groups. The parameters in the code are chosen as examples, producing simulations that show some features similar to those observed on site and in the videos. From one picture to another, what is varying is the initial conditions and the environment function (minor modifications in the code would allow also to change the number of penguins, their eyesight, the drift and cruise velocities, the stopping function, and also to take into account multi-dimensional cases).

Also, these pictures can be easily translated into animations. Simple videos that we have obtained by these numerics are available from the webpage
`https://www.youtube.com/playlist?list=PLASZVsOA5ReZgEinpnJFat66lo2kIkWTS`

The source codes of the animations are available online at
`https://www.dropbox.com/s/l1z5riqtc8jzxbs/scatter.txt?dl=0`

7.5. Discussion on the model proposed: simplifications, generalizations and further directions of investigation

We stress that the model proposed in (7.1) is of course a dramatic simplification of "reality". As often happens in science indeed, several simplifications have been adopted in order to allow a rigorous mathematical treatment and handy numerical computations: nevertheless the model is already rich enough to detect some specific features of the little penguins parade, such as the formation of groups, the oscillatory waddling of the penguins and the possibility of isolated and exposed individuals. Moreover, our model is flexible enough to allow specific distinctions between the single penguins (for instance, with minor modifications, one can take into account the possibility that different penguins have a different eye-sight, that they have a different reaction to isolation, or that they exhibit some specific social behaviour that favours the formation of clusters selected by specific characteristics); similarly, the modeling of the habitat may also encode different possibilities (such as the burrows of the penguins being located in different places), and multi-dimensional models can be also constructed using similar ideas (see Section 7.6 for details).

We observe that one can replace the quantities $v, \mathfrak{s}, \mu, \kappa, \varphi$ with $v_i, \mathfrak{s}_i, \mu_i, \kappa_i, \varphi_i$ if one wants to customize these features for every group.

Furthermore, natural modifications lead to the possibility that one or a few penguins may leave an already formed group: for instance, rather than forming one single group, the model can still consider the penguins of the cluster as separate elements, each one with its own peculiar behaviour. At the moment, for simplicity, we considered here the basic model in which, once a cluster is made up, it keeps moving without losing any of its elements – we plan to address in a future project in detail the case of groups which may also decrease the number of components, possibly in dependence of random fluctuations or social considerations among the members of the group.

In addition, for simplicity, in this chapter we modelled each group to be located at a precise point: though this is not a completely unrealistic assumption (given that the scale of the individual penguin is much smaller than that of the beach), one can also easily modify this feature by locating a cluster in a region comparable to its size.

In future projects, we plan to introduce other more sophisticated models, also taking into account stochastic oscillations and optimization methods, and, in the long run, to use these models in a detailed experimental confrontation taking advantage of the automated monitoring systems which is under development in Phillip Island.

The model that we propose here is also flexible enough to allow quantitative modifications of all the parameters involved. This is quite important, since these parameters may vary due to different conditions of the environment. For instance, the eye-sight of the penguins can be reduced by the fog (see [**27**]), and by the effect of moonlight and artificial light (see [**88**]).

Similarly, the number of penguins in each group and the velocity of the herd may vary due to structural changes of the beach: roughly speaking, from the empirical data, penguins typically gather into groups of 5–10 individuals (but we have also observed much larger groups forming on the beach) within 40 second intervals, see [**33**], but the way these groups are built varies year by year and, for instance, the number of individuals which always gather into the same group changes year by year in strong dependence with the breeding success of the season, see again [**33**]. Also, tidal phenomena may change the number of little penguins in each group and the velocity of the group, since the change of the beach width alters the perception of the risk of the penguins. For instance, a low tide produces a larger beach, with higher potential risk of predators, thus making the penguins gather in groups of larger size, see [**73**].

7.6. Multi-dimensional models

It is interesting to remark that the model in (7.1) can be easily generalized to the multi-dimensional case. That is, for any $i \in \{1, \ldots, n(t)\}$ the ith coordinate p_i can be taken to have image in some $\mathbb{R}^d$. More generally, the dimension of the target space can also vary in time, by allowing for any $i \in \{1, \dots, n(t)\}$ the ith coordinate p_i to range in some $\mathbb{R}^{d_i(t)}$, with $d_i(t)$ piecewise constant, namely $d_i(t) = d_{i,j} \in \mathbb{N}_0$ for any $t \in (t_{j-1}, t_j)$ (compare with (7.3)).

This modification just causes a small notational complication in (7.2), since each $p_i(t)$ would now be a vector in $\mathbb{R}^{d_i(t)}$ and the array $p(t)$ would now be of dimension $d_1(t) + \cdots + d_{n(t)}$. While we do not indulge here in this generalization, we observe that such mathematical extension can be useful, in practice, to consider the specific location of the burrows and describe for instance the movements of the penguins on the beach (say, a two-dimensional surface) which, as time flows, gather together in a single queue and move in the end on a one-dimensional line.

Of course, the rigorous results in Section 7.3 need to be structurally modified in higher dimension, since several notions of "proximity" of groups, "direction of march" and "orientation of the eye-sight" can be considered.

7.7. Conclusions

As a result of our direct observation at Phillip Island and a series of scientific discussions with penguin ecologists, we provide a simple, but rigorous, mathematical model which aims to describe the formation of groups of penguins on the shore at sunset and the return to their burrows.

The model is proved to possess existence and uniqueness of solutions and quantitative results on the homecoming of the penguins are given.

The framework is general enough to show the formation of groups of penguins marching together – as well as the possibility that some penguins remain isolated from the rest of the herd.

The model is also numerically implemented in simple and explicit simulations.

We believe that the method proposed can be suitably compared with the real penguins parade, thus triggering a specific field work on this rather peculiar topic. Indeed, at the moment, a precise collection of data focused on the penguins parade seems to be still missing in the literature, and we think that a mathematical formulation provides the necessary setting for describing specific behaviours in ethology, such as the formation of groups and the possible isolation of penguins, in a rigorous and quantitative way.

Given the simple and quantitative mathematical setting that we introduced here, we also believe that our formulation can be easily modified and improved to capture possible additional details of the penguins march provided by the biological data which may be collected in future specialized field work.

We hope that this problem will also take advantage of statistically sound observations by ecologists, possibly taking into account the speed of the penguins in different environments, the formation of groups of different size, the velocity of each group depending on its size and the links between group formations motivated by homecoming and the social structures of the penguin population.

Due to the lack of available biological theories and precise experimental data, the form of some of the functions considered in this chapter should just be considered as an example. This applies in particular to the strategic velocity function, to the eye-sight function and to the stopping function, and it would be desirable to run experiments to provide a better quantification of these notions.

It would be also interesting to detect how changes in the environment, such as modified visibility or presence of predators, influence the formation of groups, their size and their speed. In general, we think that it would be very important to provide precise conditions for clustering and to explore these conditions systematically.

In addition, it would be interesting to adapt models of this type to social studies, politics and evolutionary biology, in order to describe and quantify the phenomenon of “front runners” which “wait for the formation of groups of considerable size” in order to “more safely proceed towards their goal”.

7.8. Example of a program list

```
H=4; % Position of the burrow of the penguins community
S=-2; % The sea lies in the region (-\infty,0]. For simplicity we assume
      that penguins start near the shore, that is, the initial position
      of each penguin is at least S
eps=0.005; % Drift velocity of the penguins
vc=0.05; % Cruising speed of a big enough raft of penguins
N=20; % Number of penguins
delta=(N+1)*eps; % This parameter is used to compute the strategic
                                velocity of a penguin.
% These parameters define the time interval
TMAX=(H-S)/(2*eps);
t=0.01;
T=(0:t:TMAX);
TG=T(1:1,1:12000);
P=zeros(N,length(T));
% The following is the array of the initial positions of the N penguins
P(:,1)=[-1.95 -1.5 -1.05 -0.6 -0.55 -0.4 -0.2 0.1 0.2 0.4 0.8 0.85 0.9
                               1 1.1 1.15 1.2 1.65 3 3.4];
s=(H-S)/3; % The parameter encoding the eye-sight of the penguins
pgot=(H-S)/12; % The parameter representing the stopping function
M=zeros(1,N);
V=M;
PAN=-1;
W=0;

WAVE=sin(T)*delta;  % The "environment function". In this case only
                    waves are taken into account

for k=1 : length(T)-1
  for i=1 : N
    if P(i,k)<H
      for j=1: N  % This cycle checks if the ith penguin is in panic
        if -pgot<P(i,k)-P(j,k) & P(i,k)-P(j,k)<pgot
          PAN=PAN+1;
          if P(i,k)==P(j,k)
            W=W+1;  % This counts the number of penguins in the same
                    position of the ith penguin, that is the dimension
                    of the raft
          end
        end
      end
      if PAN==0  % The ith penguin is stuck because of panic
        if -3.5<P(i,k) & P(i,k)<0
          V(i)=-WAVE(k);
          P(i,k+1)=P(i,k)+V(i)*t;
        else
          P(i,k+1)=P(i,k);
```

```
        end
      else
        if W>N/2  % The ith penguin is a member of a big enough raft,
               so it tends to go home, forgetful of the other penguins
          if -3.5<P(i,k) & P(i,k)<0  % The environment can still affect
                                     the movement of the raft
            V(i)=vc-WAVE(k);
          else
            V(i)=vc;  % If the environment does not affect the movement,
                      the penguin moves at cruise velocity
          end
        else  % The raft is not big enough, so the strategic velocity
        of the ith penguin is influenced by the other penguins in sight
          for j=1 : N
            if -s<P(i,k)-P(j,k) & P(i,k)-P(j,k)<0
              M(i)=M(i)+delta/N;  % Each penguin in sight ahead adds a
                                delta/N to the strategic velocity of the
                                ith penguin
            else
              if 0<P(i,k)-P(j,k) & P(i,k)-P(j,k)<s
                M(i)=M(i)-delta/N;  % Each penguin in sight behind
                                subtracts a delta/N from the strategic
                                velocity of the ith penguin
              end
            end
          end
          if -3.5<P(i,k) & P(i,k)<0
            V(i)=eps+M(i)-WAVE(k);
          else
            V(i)=eps+M(i);
          end
        end
        P(i,k+1)=P(i,k)+V(i)*t;
      end
    else
      P(i,k+1)=H;
    end
    PAN=-1;
    W=0;
  end
  M=zeros(1,N);
  for i=2 : N
    for j=1 : i-1
      if -0.011<P(j,k+1)-P(i,k+1) & P(j,k+1)-P(i,k+1)<0.011
        P(j,k+1)=P(i,k+1);  % For simplicity, we assume that penguins
                  close enough occupy the same position, forming a raft
                  and moving together
      end
    end
  end
end
Q=P(1:N,1:length(TG));
plot(TG,Q)
```

7.9. Figures

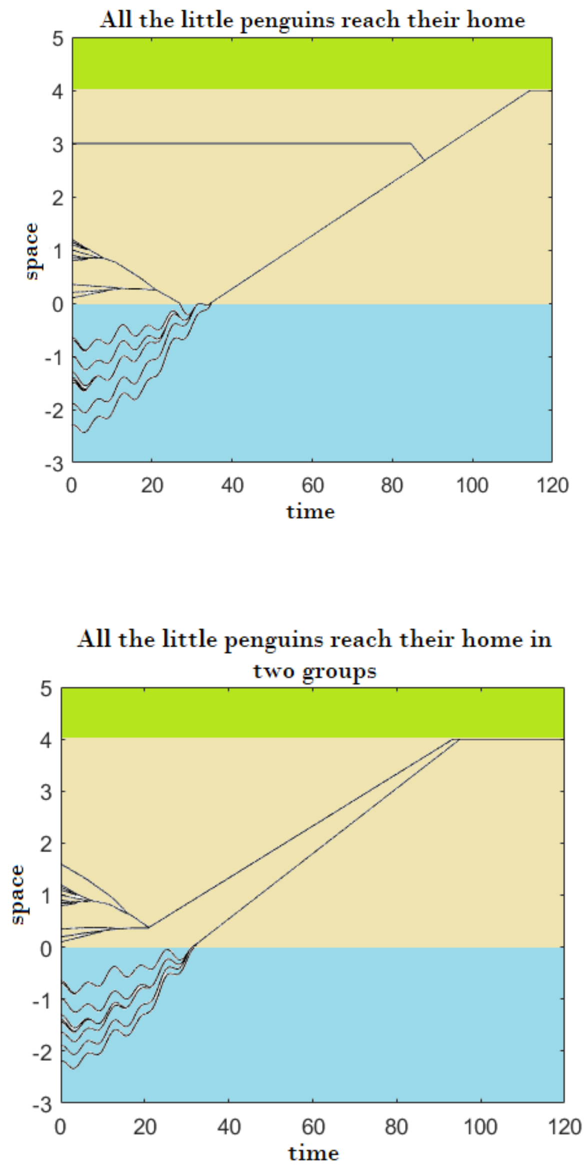


FIGURE 1. All the little penguins safely return home.

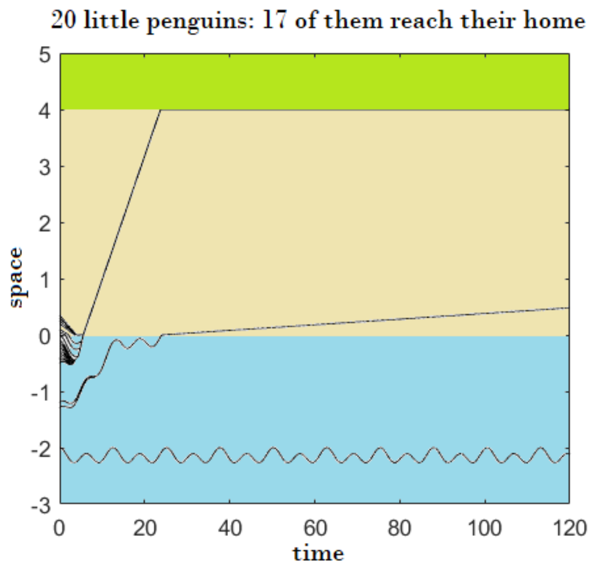


FIGURE 2. One penguin remains in the water.

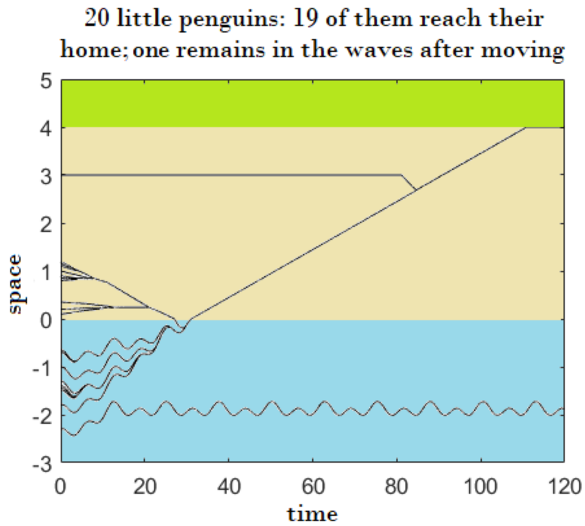
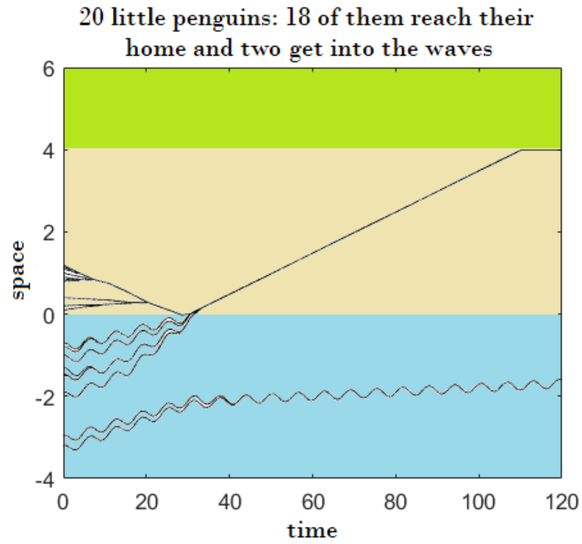


FIGURE 3. One penguin moves towards the others but remains in the water.



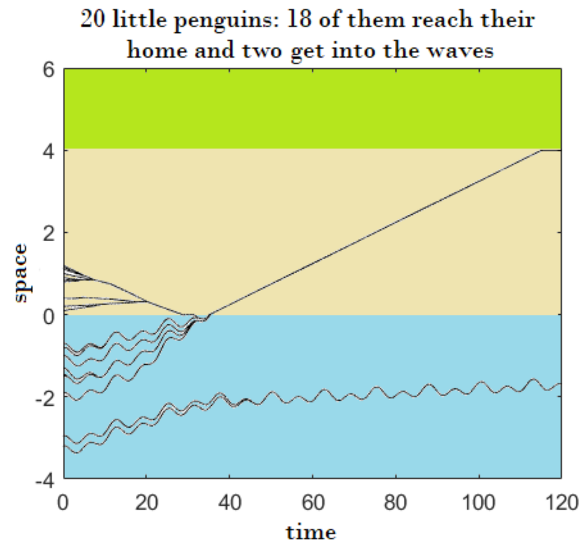


FIGURE 4. Two penguins are still in the water after a long time.

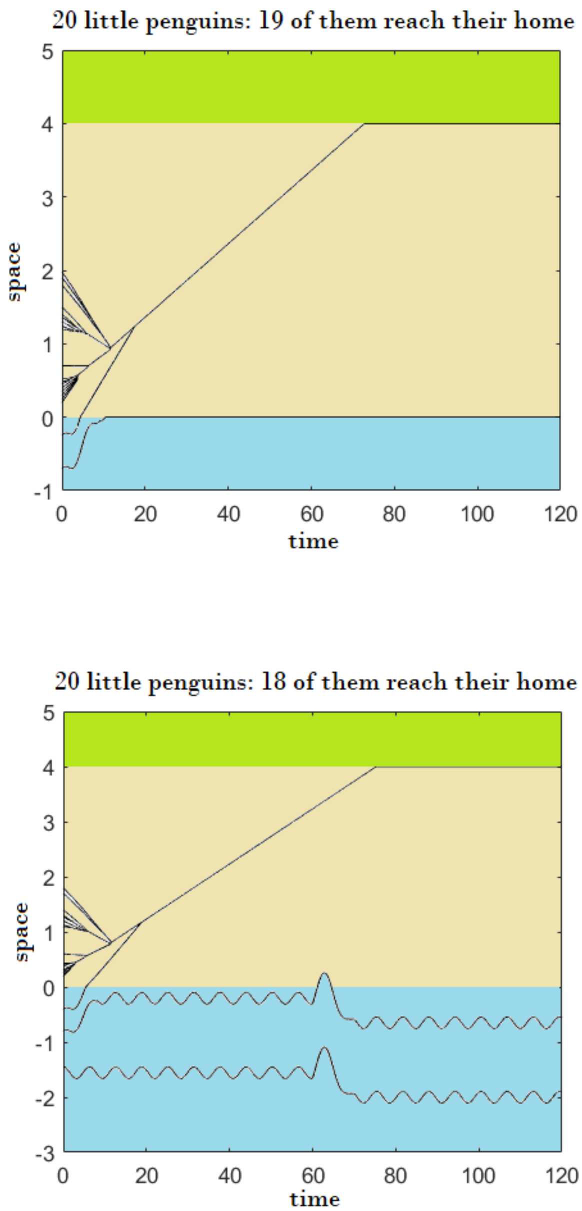


FIGURE 5. Effect of the waves on the movement of the penguins in the sea.

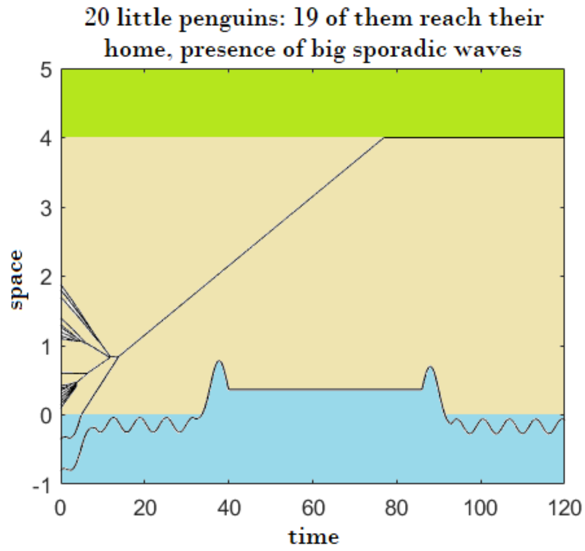


FIGURE 6. Effect of the waves on the movement of the penguins in the sea.

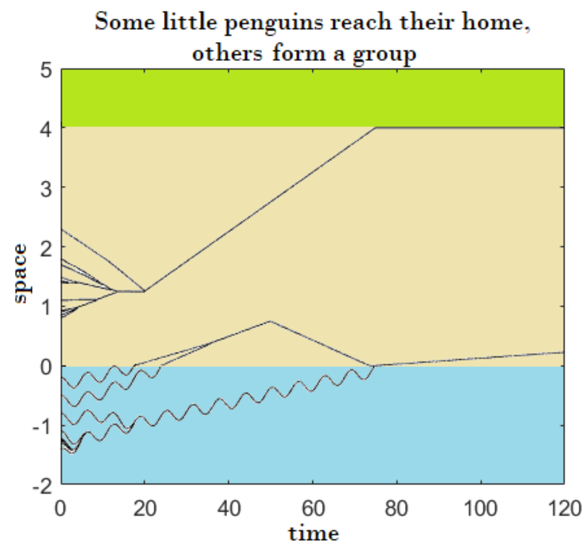
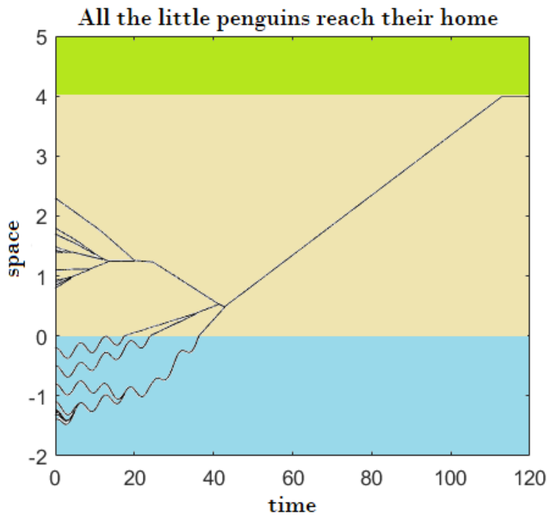


FIGURE 7. The penguins form smaller groups and move towards their home.



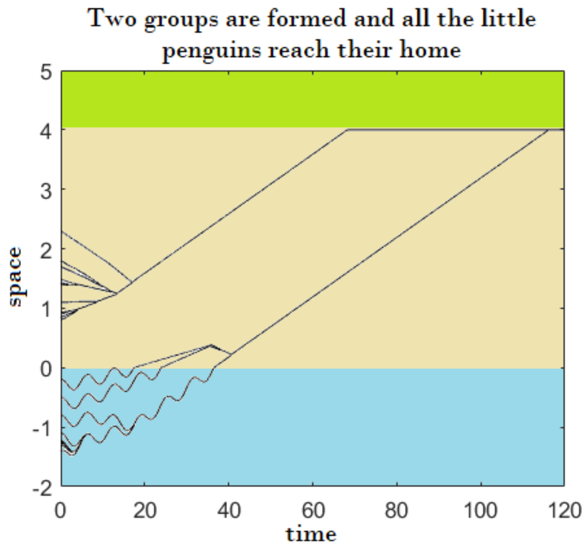


FIGURE 8. The penguins form groups of different sizes and reach their home.

APPENDIX A

Measure theoretic boundary

Since

$$|E\Delta F| = 0 \implies \operatorname{Per}(E,\Omega) = \operatorname{Per}(F,\Omega) \quad \text{and} \quad \operatorname{Per}_s(E,\Omega) = \operatorname{Per}_s(F,\Omega), \tag{A.1}$$

we can modify a set making its topological boundary as big as we want, without changing its (fractional) perimeter.
For example, let $E \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then, if we set

$$F := (E \setminus \mathbb{Q}^n) \cup (\mathbb{Q}^n \setminus E),$$

we have $|E\Delta F| = 0$ and hence we get (A.1). However $\partial F = \mathbb{R}^n$.

For this reason one considers measure theoretic notions of interior, exterior and boundary, which solely depend on the class of χ_E in $L^1_{loc}(\mathbb{R}^n)$.
In some sense, by considering the measure theoretic boundary $\partial^- E$ defined below we can also minimize the size of the topological boundary (see (A.6)). Moreover, this measure theoretic boundary is actually the topological boundary of a set which is equivalent to E. Thus we obtain a "good" representative for the class of E.

We refer to [99, Section 3.2] (see also step two in the proof of [79, Proposition 12.19] and [68, Proposition 3.1]). For some details about the good representative of an s-minimal set, see the Appendix of [43].

Definition A.0.1. *Let $E \subseteq \mathbb{R}^n$. For every $t \in [0,1]$ we define the set*

$$E^{(t)} := \left\{ x \in \mathbb{R}^n \,\Big|\, \exists \lim_{r\to 0} \frac{|E \cap B_r(x)|}{\omega_n r^n} = t \right\}, \tag{A.2}$$

of points density t of E. We also define the essential boundary of E as

$$\partial_e E := \mathbb{R}^n \setminus \big(E^{(0)} \cup E^{(1)}\big).$$

Using the Lebesgue's points Theorem for the characteristic function χ_E, we see that the limit in (A.2) exists for a.e. $x \in \mathbb{R}^n$ and

$$\lim_{r\to 0} \frac{|E \cap B_r(x)|}{\omega_n r^n} = \begin{cases} 1 & \text{for a.e. } x \in E, \\ 0 & \text{for a.e. } x \in \mathcal{C}E. \end{cases}$$

So

$$|E\Delta E^{(1)}| = 0, \qquad |\mathcal{C}E\Delta E^{(0)}| = 0 \qquad \text{and } |\partial_e E| = 0.$$

In particular a set E is equivalent to the set $E^{(1)}$ of its points of density 1.
Roughly speaking, the sets $E^{(0)}$ and $E^{(1)}$ can be thought of as a measure theoretic version of, respectively, the exterior and the interior of the set E. However, notice that both $E^{(1)}$ and $E^{(0)}$ in general are not open.

We have another natural way to define measure theoretic versions of interior, exterior and boundary.

Definition A.0.2. *Given a set $E \subseteq \mathbb{R}^n$, we define the measure theoretic interior and exterior of E by*

$$E_{int} := \{x \in \mathbb{R}^n \,|\, \exists\, r > 0,\ |E \cap B_r(x)| = \omega_n r^n\}$$

and

$$E_{ext} := \{x \in \mathbb{R}^n \,|\, \exists\, r > 0,\ |E \cap B_r(x)| = 0\},$$

respectively. Then we define the measure theoretic boundary of E as

$$\begin{aligned}\partial^- E &:= \mathbb{R}^n \setminus (E_{ext} \cup E_{int}) \\ &= \{x \in \mathbb{R}^n \,|\, 0 < |E \cap B_r(x)| < \omega_n r^n \text{ for every } r > 0\}.\end{aligned}$$

Notice that E_{ext} and E_{int} are open sets and hence $\partial^- E$ is closed. Moreover, since

$$E_{ext} \subseteq E^{(0)} \qquad \text{and} \qquad E_{int} \subseteq E^{(1)}, \tag{A.3}$$

we get

$$\partial_e E \subseteq \partial^- E.$$

We observe that

$$F \subseteq \mathbb{R}^n \text{ s.t. } |E\Delta F| = 0 \quad \Longrightarrow \quad \partial^- E \subseteq \partial F. \tag{A.4}$$

Indeed, if $|E\Delta F| = 0$, then $|F \cap B_r(x)| = |E \cap B_r(x)|$ for every $r > 0$. Thus for any $x \in \partial^- E$ we have

$$0 < |F \cap B_r(x)| < \omega_n r^n,$$

which implies

$$F \cap B_r(x) \neq \varnothing \quad \text{and} \quad \mathcal{C}F \cap B_r(x) \neq \varnothing \quad \text{for every } r > 0,$$

and hence $x \in \partial F$.

In particular, $\partial^- E \subseteq \partial E$.

Moreover

$$\partial^- E = \partial E^{(1)}. \tag{A.5}$$

Indeed, since $|E\Delta E^{(1)}| = 0$, we already know that $\partial^- E \subseteq \partial E^{(1)}$. The converse inclusion follows from (A.3) and the fact that both E_{ext} and E_{int} are open.
From (A.4) and (A.5) we obtain

$$\partial^- E = \bigcap_{F \sim E} \partial F, \tag{A.6}$$

where the intersection is taken over all sets $F \subseteq \mathbb{R}^n$ such that $|E\Delta F| = 0$, so we can think of $\partial^- E$ as a way to minimize the size of the topological boundary of E. In particular

$$F \subseteq \mathbb{R}^n \text{ s.t. } |E\Delta F| = 0 \quad \Longrightarrow \quad \partial^- F = \partial^- E.$$

From (A.3) and (A.5) we see that we can take $E^{(1)}$ as "good" representative for E, obtaining Remark MTA.

Recall that the support of a Radon measure μ on $\mathbb{R}^n$ is defined as the set

$$\text{supp}\ \mu := \{x \in \mathbb{R}^n \,|\, \mu(B_r(x)) > 0 \text{ for every } r > 0\}.$$

Notice that, being the complementary of the union of all open sets of measure zero, it is a closed set. In particular, if E is a Caccioppoli set, we have

$$\text{supp}\ |D\chi_E| = \{x \in \mathbb{R}^n \,|\, \text{Per}(E, B_r(x)) > 0 \text{ for every } r > 0\}, \tag{A.7}$$

and it is easy to verify that

$$\partial^- E = \text{supp}\ |D\chi_E| = \overline{\partial^* E},$$

where $\partial^* E$ denotes the reduced boundary (see, e.g., [79, Chapter 15]). Moreover, $\partial^* E \subseteq \partial_e E$ and by Federer's Theorem (see, e.g., [79, Theorem 16.2]) we have

$$\mathcal{H}^{n-1}(\partial_e E \setminus \partial^* E) = 0.$$

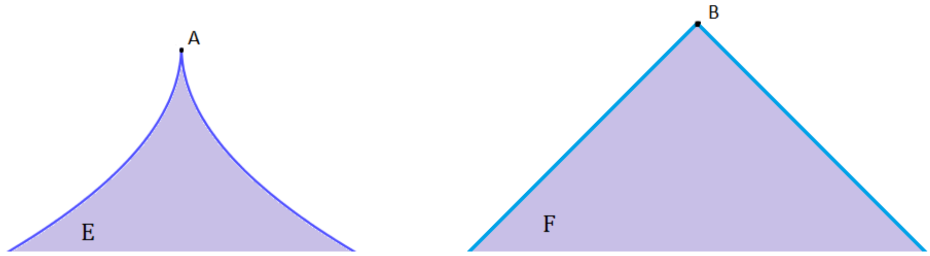


FIGURE 1. *The point A belongs to $\partial^- E$ but $A \notin \partial_e E$. The point B belongs to $\partial_e F$ but $B \notin \partial^* F$.*

We remark that in general the inclusions

$$\partial^* E \subseteq \partial_e E \subseteq \partial^- E \subseteq \partial E$$

are all strict. Indeed, we have already observed in the previous discussion that in general $\partial^- E$ is much smaller than the topological boundary ∂E. In order to have an example of a point $p \in \partial^- E \setminus \partial_e E$ it is enough to consider sublinear cusps. For example, if $E := \{(x, y) \in \mathbb{R}^2 \,|\, y < -|x|^{\frac{1}{2}}\}$ and $p := (0, 0)$, then it is easy to verify that $p \in E^{(0)}$ and hence $p \notin \partial_e E$. On the other hand, $p \in \partial^- E$. Finally, the vertex of an angle is an example of a point $p \in \partial_e E \setminus \partial^* E$ (see, e.g., [79, Example 15.4]).

APPENDIX B

Some geometric observations

We collect here some useful results and observations of a geometric nature, concerning in particular the signed distance function.

B.1. Signed distance function

Given $\varnothing \neq E \subseteq \mathbb{R}^n$, the distance function from E is defined as

$$d_E(x) = d(x, E) := \inf_{y \in E} |x - y|, \qquad \text{for } x \in \mathbb{R}^n.$$

The signed distance function from ∂E, negative inside E, is then defined as

$$\bar{d}_E(x) = \bar{d}(x, E) := d(x, E) - d(x, \mathcal{C}E).$$

For the details about the main properties of the signed distance function we refer, e.g., to [4, 66] and [10].

We also define the sets

$$E_r := \{x \in \mathbb{R}^n \,|\, \bar{d}_E(x) < r\},$$

for every $r \in \mathbb{R}$, and

$$N_\varrho(\partial E) := \{|\bar{d}_E| < \varrho\} = \{x \in \mathbb{R}^n \,|\, d(x, \partial E) < \varrho\},$$

for every $\varrho > 0$, which is usually called the tubular ϱ-neighborhood of ∂E.

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. By definition we can locally describe Ω near its boundary as the subgraph of appropriate Lipschitz functions. To be more precise, we can find a finite open covering $\{C_{\varrho_i}\}_{i=1}^m$ of $\partial\Omega$ made of cylinders, and Lipschitz functions $\varphi_i : B'_{\varrho_i} \longrightarrow \mathbb{R}$ such that $\Omega \cap C_{\varrho_i}$ is the subgraph of φ_i. That is, up to rotations and translations,

$$C_{\varrho_i} = \{(x', x_n) \in \mathbb{R}^n \,|\, |x'| < \varrho_i,\ |x_n| < \varrho_i\},$$

and

$$\Omega \cap C_{\varrho_i} = \{(x', x_n) \in \mathbb{R}^n \,|\, x' \in B'_{\varrho_i},\ -\varrho_i < x_n < \varphi_i(x')\},$$
$$\partial\Omega \cap C_{\varrho_i} = \{(x', \varphi_i(x')) \in \mathbb{R}^n \,|\, x' \in B'_{\varrho_i}\}.$$

Let L be the sup of the Lipschitz constants of the functions φ_i.

We observe that [48, Theorem 4.1] guarantees that also the bounded open sets Ω_r have Lipschitz boundary, when r is small enough, say $|r| < r_0$.
Moreover these sets Ω_r can locally be described, in the same cylinders C_{ϱ_i} used for Ω, as subgraphs of Lipschitz functions φ_i^r which approximate φ_i (see [48] for the precise statement) and whose Lipschitz constants are less than or equal to L.
Notice that

$$\partial\Omega_r = \{\bar{d}_\Omega = r\}.$$

Now, since in C_{ϱ_i} the set Ω_r coincides with the subgraph of φ_i^r, we have

$$\mathcal{H}^{n-1}(\partial\Omega_r \cap C_{\varrho_i}) = \int_{B'_{\varrho_i}} \sqrt{1 + |\nabla \varphi_i^r|^2}\, dx' \leq M_i,$$

with M_i depending on ϱ_i and L but not on r.
Therefore

$$\mathcal{H}^{n-1}(\{\bar{d}_\Omega = r\}) \le \sum_{i=1}^{m} \mathcal{H}^{n-1}(\partial\Omega_r \cap C_{\varrho_i}) \le \sum_{i=1}^{m} M_i$$

independently on r, proving the following

PROPOSITION B.1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then there exists $r_0 = r_0(\Omega) > 0$ such that Ω_r is a bounded open set with Lipschitz boundary for every $r \in (-r_0, r_0)$ and*

$$\sup_{|r|<r_0} \mathcal{H}^{n-1}(\{\bar{d}_\Omega = r\}) < \infty.$$

B.1.1. Smooth domains. In this section we collect some properties of the signed distance function from the boundary of a regular open set.

We begin by recalling the notion of (uniform) interior ball condition.

DEFINITION B.1.2. *We say that an open set $\mathcal{O}$ satisfies an interior ball condition at $x \in \partial\mathcal{O}$ if there exists a ball $B_r(y)$ s.t.*

$$B_r(y) \subseteq \mathcal{O} \qquad \text{and} \qquad x \in \partial B_r(y).$$

We say that the condition is "strict" if x is the only tangency point, i.e.

$$\partial B_r(y) \cap \partial\mathcal{O} = \{x\}.$$

The open set $\mathcal{O}$ satisfies a uniform (strict) interior ball condition of radius r if it satisfies the (strict) interior ball condition at every point of $\partial\mathcal{O}$, with an interior tangent ball of radius at least r.
In a similar way one defines exterior ball conditions.

We remark that if $\mathcal{O}$ satisfies an interior ball condition of radius r at $x \in \partial\mathcal{O}$, then the condition is strict for every radius $r' < r$.

REMARK B.1.3. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^2 boundary. It is well known that Ω satisfies a uniform interior and exterior ball condition. We fix $r_0 = r_0(\Omega) > 0$ such that Ω satisfies a strict interior and a strict exterior ball contition of radius $2r_0$ at every point $x \in \partial\Omega$. Then

$$\bar{d}_\Omega \in C^2(N_{2r_0}(\partial\Omega)), \tag{B.1}$$

(see, e.g., [66, Lemma 14.16]).

We remark that the distance function $d(\,\cdot\,, E)$ is differentiable at $x \in \mathbb{R}^n \setminus \overline{E}$ if and only if there is a unique point $y \in \partial E$ of minimum distance, i.e.

$$d(x, E) = |x - y|.$$

In this case, the two points x and y are related by the formula

$$y = x - d(x, E)\nabla d(x, E).$$

This generalizes to the signed distance function. In particular, if Ω is bounded and has C^2 boundary, then we can define a C^1 projection function from the tubular $2r_0$-neighborhood $N_{2r_0}(\partial\Omega)$ onto $\partial\Omega$ by assigning to a point x its unique nearest point $\pi(x)$, that is

$$\pi : N_{2r_0}(\partial\Omega) \longrightarrow \partial\Omega, \qquad \pi(x) := x - \bar{d}_\Omega(x)\nabla\bar{d}_\Omega(x).$$

We also remark that on $\partial\Omega$ we have that $\nabla\bar{d}_\Omega = \nu_\Omega$ and that

$$\nabla\bar{d}_\Omega(x) = \nabla\bar{d}_\Omega(\pi(x)) = \nu_\Omega(\pi(x)), \qquad \forall x \in N_{2r_0}(\partial\Omega).$$

Thus $\nabla \bar{d}_\Omega$ is a vector field which extends the outer unit normal to a tubular neighborhood of $\partial\Omega$, in a C^1 way.

Notice that given a point $y \in \partial\Omega$, for every $|\delta| < 2r_0$ the point $x := y + \delta\nu_\Omega(y)$ is such that $\bar{d}_\Omega(x) = \delta$ (and y is its unique nearest point). Indeed, we consider for example $\delta \in (0, 2r_0)$. Then we can find an exterior tangent ball

$$B_{2r_0}(z) \subseteq \mathcal{C}\Omega, \qquad \partial B_{2r_0}(z) \cap \partial\Omega = \{y\}.$$

Notice that the center of the ball must be

$$z = y + 2r_0\nu_\Omega(y).$$

Then, for every $\delta \in (0, 2r_0)$ we have

$$B_\delta(y + \delta\nu_\Omega(y)) \subseteq B_{2r_0}(y + 2r_0\nu_\Omega(y)) \subseteq \mathcal{C}\Omega, \qquad \partial B_\delta(y + \delta\nu_\Omega(y)) \cap \partial\Omega = \{y\}.$$

This proves that

$$|\bar{d}_\Omega(y + \delta\nu_\Omega(y))| = d(x, \partial\Omega) = \delta.$$

Finally, since the point x lies outside Ω, its signed distance function is positive.

Remark B.1.4. Since $|\nabla \bar{d}_\Omega| = 1$, the bounded open sets

$$\Omega_\delta := \{\bar{d}_\Omega < \delta\}$$

have C^2 boundary

$$\partial\Omega_\delta = \{\bar{d}_\Omega = \delta\},$$

for every $\delta \in (-2r_0, 2r_0)$.

As a consequence, we know that for every $|\delta| < 2r_0$ the set Ω_δ satisfies a uniform interior and exterior ball condition of radius $r(\delta) > 0$. Moreover, we have that $r(\delta) \geq r_0$ for every $|\delta| \leq r_0$ (see also [**90**, Appendix A] for related results).

Lemma B.1.5. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^2 boundary. Then for every $\delta \in [-r_0, r_0]$ the set Ω_δ satisfies a uniform interior and exterior ball condition of radius at least r_0, i.e.*

$$r(\delta) \geq r_0 \qquad \textit{for every } |\delta| \leq r_0.$$

Proof. Take for example $\delta \in [-r_0, 0)$ and let $x \in \partial\Omega_\delta = \{\bar{d}_\Omega = \delta\}$. We show that Ω_δ has an interior tangent ball of radius r_0 at x. The other cases are proven in a similar way.

Consider the projection $\pi(x) \in \partial\Omega$ and the point

$$x_0 := x - r_0\nabla \bar{d}_\Omega(x) = \pi(x) - (r_0 + |\delta|)\nu_\Omega(\pi(x)).$$

Then

$$B_{r_0}(x_0) \subseteq \Omega_\delta \quad \text{ and } \quad x \in \partial B_{r_0}(x_0) \cap \partial\Omega_\delta.$$

Indeed, notice that, as remarked above,

$$d(x_0, \partial\Omega) = |x_0 - \pi(x)| = r_0 + |\delta|.$$

Thus, by the triangle inequality we have that

$$d(z, \partial\Omega) \geq d(x_0, \partial\Omega) - |z - x_0| > |\delta|, \qquad \text{for every } z \in B_{r_0}(x_0),$$

so $B_{r_0} \subseteq \Omega_\delta$. Moreover, by definition of x_0 we have

$$x \in \partial B_{r_0}(x_0) \cap \partial\Omega_\delta$$

and the desired result follows. □

To conclude, we remark that the sets $\overline{\Omega_{-\delta}}$ are retracts of Ω, for every $\delta \in (0, r_0]$. Indeed, roughly speaking, each set $\overline{\Omega_{-\delta}}$ is obtained by deforming Ω in normal direction, towards the interior. An important consequence is that if Ω is connected then $\overline{\Omega_{-\delta}}$ is path connected.

To be more precise, we have the following:

PROPOSITION B.1.6. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with C^2 boundary. Let $\delta \in (0, r_0]$ and define*

$$\mathcal{D} : \Omega \longrightarrow \overline{\Omega_{-\delta}}, \qquad \mathcal{D}(x) := \begin{cases} x, & x \in \Omega_{-\delta}, \\ x - \big(\delta + \bar{d}_\Omega(x)\big)\nabla \bar{d}_\Omega(x), & x \in \Omega \setminus \Omega_{-\delta}. \end{cases}$$

Then $\mathcal{D}$ is a retraction of Ω onto $\overline{\Omega_{-\delta}}$, i.e. it is continuous and $\mathcal{D}(x) = x$ for every $x \in \overline{\Omega_{-\delta}}$. In particular, if Ω is connected, then $\overline{\Omega_{-\delta}}$ is path connected.

PROOF. Notice that the function

$$\Phi(x) := x - \big(\delta + \bar{d}_\Omega(x)\big)\nabla \bar{d}_\Omega(x)$$

is continuous in $\Omega \setminus \Omega_{-\delta}$ and $\Phi(x) = x$ for every $x \in \partial\Omega_{-\delta}$. Therefore the function $\mathcal{D}$ is continuous.

We are left to show that

$$\mathcal{D}(\Omega \setminus \Omega_{-\delta}) \subseteq \partial\Omega_{-\delta}.$$

For this, it is enough to notice that

$$\mathcal{D}(x) = \pi(x) - \delta\nu_\Omega(\pi(x)) \qquad \text{for every } x \in \Omega \setminus \Omega_{-\delta}.$$

To conclude, suppose that Ω is connected and recall that if an open set $\Omega \subseteq \mathbb{R}^n$ is connected, then it is also path connected. Thus $\overline{\Omega_{-\delta}}$, being the continuous image of a path connected space, is itself path connected. □

B.2. Sliding the balls

We now point out the following useful geometric result, which has been exploited in Chapter 3.

LEMMA B.2.1. *Let $F \subseteq \mathbb{R}^n$ be such that*[1]

$$B_\delta(p) \subseteq F_{ext} \quad \textit{for some } \delta > 0 \qquad \textit{and} \qquad q \in \overline{F},$$

and let $c : [0, 1] \longrightarrow \mathbb{R}^n$ be a continuous curve connecting p to q, that is

$$c(0) = p \qquad \textit{and} \qquad c(1) = q.$$

Then there exists $t_0 \in [0, 1)$ such that $B_\delta\big(c(t_0)\big)$ is an exterior tangent ball to F, that is

$$B_\delta\big(c(t_0)\big) \subseteq F_{ext} \qquad \textit{and} \qquad \partial B_\delta\big(c(t_0)\big) \cap \partial F \neq \varnothing. \tag{B.2}$$

PROOF. Define

$$t_0 := \sup\Big\{\tau \in [0, 1] \,\Big|\, \bigcup_{t\in[0,\tau]} B_\delta\big(c(t)\big) \subseteq F_{ext}\Big\}. \tag{B.3}$$

We begin by proving that

$$B_\delta\big(c(t_0)\big) \subseteq F_{ext}. \tag{B.4}$$

[1]Concerning the statement of Lemma B.2.1, we recall that the notation $\overline{F}$ denotes the closure of the set F, when F is modified, up to sets of measure zero, in such a way that F is assumed to contain its measure theoretic interior F_{int} and to have empty intersection with the exterior F_{ext}, according to the setting described in Remark MTA. For instance, if F is a segment in $\mathbb{R}^2$, this convention implies that $F_{int} = \varnothing$, $F_{ext} = \mathbb{R}^2$ and so F and $\overline{F}$ in this case also reduce to the empty set.

If $t_0 = 0$, this is trivially true by hypothesis. Thus, suppose that $t_0 > 0$ and assume by contradiction that

$$B_\delta\big(c(t_0)\big) \cap \overline{F} \neq \varnothing.$$

Then there exists a point

$$y \in \overline{F} = F_{int} \cup \partial F \quad \text{s.t.} \quad d := |y - c(t_0)| < \delta.$$

By exploiting the continuity of c, we can find $t \in [0, t_0)$ such that

$$|y - c(t)| \leq |y - c(t_0)| + |c(t_0) - c(t)| \leq d + \frac{\delta - d}{2} < \delta,$$

and hence $y \in B_\delta\big(c(t)\big)$. However, this is in contradiction with the fact that, by definition of t_0, we have $B_\delta\big(c(t)\big) \subseteq F_{ext}$. This concludes the proof of (B.4).

We point out that, since $q \in \overline{F}$, by (B.4) we have that $t_0 < 1$.

Now we prove that t_0 as defined in (B.3) satisfies (B.2).

Notice that by (B.4) we have

$$\overline{B_\delta\big(c(t_0)\big)} \subseteq \overline{F_{ext}} = F_{ext} \cup \partial F. \tag{B.5}$$

Suppose that

$$\partial B_\delta\big(c(t_0)\big) \cap \partial F = \varnothing.$$

Then (B.5) implies that

$$\overline{B_\delta\big(c(t_0)\big)} \subseteq F_{ext},$$

and, since F_{ext} is an open set, we can find $\tilde{\delta} > \delta$ such that

$$B_{\tilde{\delta}}\big(c(t_0)\big) \subseteq F_{ext}.$$

By continuity of c we can find $\varepsilon \in (0, 1 - t_0)$ small enough such that

$$|c(t) - c(t_0)| < \tilde{\delta} - \delta, \qquad \forall\, t \in [t_0, t_0 + \varepsilon].$$

Therefore

$$B_\delta\big(c(t)\big) \subseteq B_{\tilde{\delta}}\big(c(t_0)\big) \subseteq F_{ext}, \qquad \forall\, t \in [t_0, t_0 + \varepsilon],$$

and hence

$$\bigcup_{t \in [0, t_0 + \varepsilon]} B_\delta\big(c(t)\big) \subseteq F_{ext},$$

which is in contradiction with the definition of t_0. Thus

$$\partial B_\delta\big(c(t_0)\big) \cap \partial F \neq \varnothing,$$

which concludes the proof. $\square$

APPENDIX C

Collection of useful results on nonlocal minimal surfaces

Here, we collect some auxiliary results on nonlocal minimal surfaces. In particular, we recall the representation of the fractional mean curvature when the set is a graph and a useful and general version of the maximum principle.

C.1. Explicit formulas for the fractional mean curvature of a graph

We denote

$$Q_{r,h}(x) := B'_r(x') \times (x_n - h, x_n + h),$$

for $x \in \mathbb{R}^n$, $r, h > 0$. If $x = 0$, we write $Q_{r,h} := Q_{r,h}(0)$. Let also

$$g_s(t) := \frac{1}{(1+t^2)^{\frac{n+s}{2}}} \qquad \text{and} \qquad G_s(t) := \int_0^t g_s(\tau)\,d\tau.$$

Notice that

$$0 < g_s(t) \le 1, \quad \forall\, t \in \mathbb{R} \qquad \text{and} \qquad \int_{-\infty}^{+\infty} g_s(t)\,dt < \infty,$$

for every $s \in (0,1)$.

In this notation, we can write the fractional mean curvature of a supergraph as follows:

Proposition C.1.1. *Let $F \subseteq \mathbb{R}^n$ and $p \in \partial F$ such that*

$$F \cap Q_{r,h}(p) = \{(x', x_n) \in \mathbb{R}^n \,|\, x' \in B'_r(p'),\, v(x') < x_n < p_n + h\},$$

for some $v \in C^{1,\alpha}(B'_r(p'))$. Then for every $s \in (0, \alpha)$

$$\begin{aligned} H_s[F](p) = 2 \int_{B'_r(p')} \Big\{ G_s\Big(\frac{v(y') - v(p')}{|y'-p'|}\Big) - G_s\Big(\nabla v(p') \cdot \frac{y'-p'}{|y'-p'|}\Big) \Big\} \frac{dy'}{|y'-p'|^{n-1+s}} \\ + \int_{\mathbb{R}^n \setminus Q_{r,h}(p)} \frac{\chi_{\mathcal{C}F}(y) - \chi_F(y)}{|y-p|^{n+s}}\,dy. \end{aligned} \tag{C.1}$$

This explicit formula was introduced in [25] (see also [2]) when $\nabla v(p) = 0$. In [9], the reader can find the formula for the case of non-zero gradient.

Remark C.1.2. In the right hand side of (C.1) there is no need to consider the principal value, since the integrals are summable. Indeed,

$$\begin{aligned} \Big| G_s\Big(\frac{v(y') - v(p')}{|y'-p'|}\Big) - G_s\Big(\nabla v(p') \cdot \frac{y'-p'}{|y'-p'|}\Big) \Big| &= \Big| \int_{\nabla v(p') \cdot \frac{y'-p'}{|y'-p'|}}^{\frac{v(y')-v(p')}{|y'-p'|}} g_s(t)\,dt \Big| \\ &\le \Big| \frac{v(y') - v(p') - \nabla v(p') \cdot (y'-p')}{|y'-p'|} \Big| \le \|v\|_{C^{1,\alpha}(B'_r(p'))} |y'-p'|^\alpha, \end{aligned}$$

for every $y' \in B'_r(p')$. As for the last inequality, notice that by the Mean value Theorem we have

$$v(y') - v(p') = \nabla v(\xi) \cdot (y'-p'),$$

for some $\xi \in B'_r(p')$ on the segment with end points y' and p'. Thus

$$\begin{aligned}|v(y') - v(p') - \nabla v(p') \cdot (y' - p')| &= |(\nabla v(\xi) - \nabla v(p')) \cdot (y' - p')| \\ &\le |\nabla v(\xi) - \nabla v(p')||y' - p'| \le \|\nabla v\|_{C^{0,\alpha}(B'_r(p'))}|\xi - p'|^\alpha |y' - p'| \\ &\le \|v\|_{C^{1,\alpha}(B'_r(p'))}|y' - p'|^{1+\alpha}.\end{aligned}$$

We denote for simplicity

$$\mathcal{G}(s, v, y', p') := G_s\Big(\frac{v(y') - v(p')}{|y' - p'|}\Big) - G_s\Big(\nabla v(p') \cdot \frac{y' - p'}{|y' - p'|}\Big). \tag{C.2}$$

With this notation, we have

$$|\mathcal{G}(s, v, y', p')| \le \|v\|_{C^{1,\alpha}(B'_r(p'))}|y' - p'|^\alpha. \tag{C.3}$$

C.2. Interior regularity theory and its influence on the Euler-Lagrange equation inside the domain

In this Appendix we give a short review of the the Euler-Lagrange equation in the interior of the domain. In particular, by exploiting results which give an improvement of the regularity of ∂E, we show that an s-minimal set is a classical solution of the Euler-Lagrange equation almost everywhere.

First of all, we recall the definition of supersolution.

Definition C.2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $s \in (0, 1)$. A set E is an s-supersolution in Ω if* $\mathrm{Per}_s(E, \Omega) < \infty$ *and*

$$\mathrm{Per}_s(E, \Omega) \le \mathrm{Per}_s(F, \Omega) \quad \textit{for every set } E \textit{ s.t. } E \subseteq F \textit{ and } F \setminus \Omega = E \setminus \Omega. \tag{C.4}$$

We remark that (C.4) is equivalent to

$$A \subseteq \mathcal{C}E \cap \Omega \qquad \Longrightarrow \qquad \mathcal{L}_s(A, E) - \mathcal{L}_s(A, \mathcal{C}(E \cup A)) \le 0.$$

In a similar way one defines s-subsolutions.

In [21] it is shown that a set E which is an s-supersolution in Ω is also a viscosity supersolution of the equation $H_s[E] = 0$ on $\partial E \cap \Omega$. To be more precise

Theorem C.2.2 (Theorem 5.1 of [21]). *Let E be an s-supersolution in the open set Ω. If $x_0 \in \partial E \cap \Omega$ and E has an interior tangent ball at x_0, contained in Ω, i.e.*

$$B_r(y) \subseteq E \cap \Omega \quad s.t. \quad x_0 \in \partial E \cap \partial B_r(y),$$

then

$$\liminf_{\varrho \to 0^+} H_s^\varrho[E](x_0) \ge 0. \tag{C.5}$$

In particular, E is a viscosity supersolution in the following sense.

Corollary C.2.3. *Let E be an s-supersolution in the open set Ω and let F be an open set such that $F \subseteq E$. If $x \in (\partial E \cap \partial F) \cap \Omega$ and ∂F is $C^{1,1}$ near x, then $H_s[F](x) \ge 0$.*

Proof. Since ∂F is $C^{1,1}$ near x, F has an interior tangent ball at x. In particular, notice that this ball is tangent also to E at x (from the inside). Thus by Theorem C.2.2

$$\liminf_{\varrho \to 0^+} H_s^\varrho[E](x) \ge 0.$$

Now notice that

$$F \subseteq E \qquad \Longrightarrow \qquad \chi_{\mathcal{C}F} - \chi_F \ge \chi_{\mathcal{C}E} - \chi_E,$$

so

$$H_s^\delta[F](x) \ge H_s^\delta[E](x) \qquad \forall\, \delta > 0.$$

Since $H_s[F](x)$ is well defined, it is then enough to pass to the limit $\delta \to 0$. ∎

Remark C.2.4. Similarly, for an s-subsolution E which has an exterior tangent ball at x_0 we obtain

$$\limsup_{\varrho\to 0^+} H_s^\varrho[E](x_0) \le 0. \tag{C.6}$$

Now we recall the following two regularity results. If E is s-minimal, having a tangent ball (either interior or exterior) at some point $x_0 \in \partial E\cap\Omega$ is enough (via an improvement of flatness result) to have $C^{1,\alpha}$ regularity in a neighborhood of x_0 (see [21, Corollary 6.2]). Moreover, bootstrapping arguments prove that $C^{0,1}$ regularity guarantees C^∞ regularity (according to [58, Theorem 1.1]).

It is also convenient to recall the notion of locally s-minimal set, which is useful when considering an unbounded domain Ω.
We say that a set $E \subseteq \mathbb{R}^n$ is locally s-minimal in an open set $\Omega \subseteq \mathbb{R}^n$ if E is s-minimal in every bounded open set $\Omega' \Subset \mathbb{R}^n$.

Exploiting the regularity results that we recalled above, we obtain the following:

Theorem C.2.5. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let E be locally s-minimal in Ω. If $x_0 \in \partial E\cap\Omega$ and E has either an interior or exterior tangent ball at x_0, then there exists $r>0$ such that $\partial E\cap B_r(x_0)$ is C^∞ and*

$$H_s[E](x) = 0 \qquad \textit{for every} \quad x \in \partial E\cap B_r(x_0). \tag{C.7}$$

Proof. Since $x_0 \in \partial E\cap\Omega$ and Ω is open, we can find $r>0$ such that $B_r(x_0)\Subset\Omega$. The set E is then s-minimal in $B_r(x_0)$. Moreover, by hypothesis we have a tangent ball (either interior or exterior) to E at x_0. Also notice that we can suppose that the tangent ball is contained in $B_r(x_0)$.
Thus, by [21, Corollary 6.2] and [58, Theorem 1.1], we know that ∂E is C^∞ in $B_r(x_0)$ (up to taking another $r>0$ small enough).

In particular, $H_s[E](x)$ is well defined for every $x\in\partial E\cap B_r(x_0)$ and E has both an interior and an exterior tangent ball at every $x\in\partial E\cap B_r(x_0)$ (both contained in $B_r(x_0)$).
Therefore, since an s-minimal set is both an s-supersolution and an s-subsolution, by (C.5) and (C.6), we obtain

$$0 \le \liminf_{\varrho\to 0^+} H_s^\varrho[E](x) = H_s[E](x) = \limsup_{\varrho\to 0^+} H_s^\varrho[E](x) \le 0,$$

for every $x\in\partial E\cap B_r(x_0)$, proving (C.7). □

Furthermore, we recall that if $E \subseteq \mathbb{R}^n$ is s-minimal in Ω, then the singular set $\Sigma(E;\Omega)\subseteq\partial E\cap\Omega$ has Hausdorff dimension at most $n-3$ (by the dimension reduction argument developed in [21, Section 10] and [92, Corollary 2]).

Now suppose that E is locally s-minimal in an open set Ω. We observe that we can find a sequence of bounded open sets with Lipschitz boundaries $\Omega_k \Subset \Omega$ such that $\bigcup \Omega_k = \Omega$ (see, e.g., Corollary 2.2.6). Since E is s-minimal in each Ω_k and $\Sigma(E;\Omega) = \bigcup \Sigma(E;\Omega_k)$, we get in particular

$$\mathcal{H}^{n-2}(\Sigma(E;\Omega)) \le \sum_{k=1}^\infty \mathcal{H}^{n-2}(\Sigma(E;\Omega_k)) = 0 \tag{C.8}$$

(and indeed $\Sigma(E;\Omega)$ has Hausdorff dimension at most $n-3$, since we have inequality (C.8) with $n-d$ in place of $n-2$, for every $d\in[0,3)$).

As a consequence, a (locally) s-minimal set is a classical solution of the Euler-Lagrange equation, in the following sense

THEOREM C.2.6. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let E be locally s-minimal in Ω. Then*

$$H_s[E](x) = 0 \qquad \textit{for every } x \in (\partial E \cap \Omega) \setminus \Sigma(E;\Omega),$$

and hence in particular for $\mathcal{H}^{n-1}$-a.e. $x \in \partial E \cap \Omega$.

C.3. Boundary Euler-Lagrange inequalities for the fractional perimeter

We recall that a set E is locally s-minimal in an open set Ω if it is s-minimal in every bounded open set compactly contained in Ω. In this section we show that the Euler-Lagrange equation of a locally s-minimal set E holds (at least as an inequality) also at a point $p \in \partial E \cap \partial\Omega$, provided that the boundary ∂E and the boundary $\partial\Omega$ do not intersect "transversally" in p.

To be more precise, we prove the following

THEOREM C.3.1. *Let $s \in (0,1)$. Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $E \subseteq \mathbb{R}^n$ be locally s-minimal in Ω. Suppose that $p \in \partial E \cap \partial\Omega$ is such that $\partial\Omega$ is $C^{1,1}$ in $B_{R_0}(p)$, for some $R_0 > 0$. Assume also that*

$$B_{R_0}(p) \setminus \Omega \subseteq \mathcal{C}E. \tag{C.9}$$

Then

$$H_s[E](p) \le 0.$$

Moreover, if there exists $R \in (0, R_0)$ such that

$$\partial E \cap \big(\Omega \cap B_r(p)\big) \neq \varnothing \qquad \textit{for every } r \in (0,R), \tag{C.10}$$

then

$$H_s[E](p) = 0.$$

We remark that by hypothesis the open set $B_{R_0}(p) \setminus \overline{\Omega}$ is tangent to E at p, from the outside. Therefore, either (C.10) holds true, meaning roughly speaking that the boundary of E detaches from the boundary of Ω at p (towards the interior of Ω), or ∂E coincides with $\partial\Omega$ near p.

FIGURE 1. *Examples of a set which satisfies* (C.10) *(on the left) and of a set whose boundary sticks to that of Ω near p (on the right)*

Roughly speaking, the idea of the proof of Theorem C.3.1 is the following. The set $\mathcal{O} := B_{R_0}(p) \setminus \overline{\Omega}$ plays the role of an obstacle in the minimization of the s-perimeter in $B_{R_0}(p)$. The (local) minimality of E in Ω, together with hypothesis (C.9), implies that E solves this geometric obstacle type problem, which has been investigated in [**20**]. As a consequence, the set E is a viscosity subsolution in $B_{R_0}(p)$ and we obtain that $H_s[E](p) \le 0$. Furthermore, the regularity result proved in [**20**] guarantees that ∂E is $C^{1,\sigma}$, with $\sigma > s$, near p. Thus, if ∂E satisfies (C.10), then we can exploit the Euler-Lagrange equation inside Ω and the continuity of $H_s[E]$ to prove that $H_s[E](p) = 0$.

We now proceed to give a rigorous proof of Theorem C.3.1.

PROOF OF THEOREM C.3.1. We begin by observing that we can find a bounded and connected open set $\Omega' \subseteq \Omega$ such that

$$\partial\Omega' \text{ is } C^{1,1} \qquad \text{and} \qquad \Omega' \cap B_{\frac{R_0}{2}}(p) = \Omega \cap B_{\frac{R_0}{2}}(p).$$

Then, since E is locally s-minimal in Ω, we know that it is locally s-minimal also in Ω'. Hence, since Ω' is bounded and has regular boundary, by Theorem 2.1.7 we find that E is actually s-minimal in Ω'. Moreover $p \in \partial E \cap \partial\Omega'$ and

$$B_{\frac{R_0}{2}}(p) \setminus \Omega' = B_{\frac{R_0}{2}}(p) \setminus \Omega \subseteq B_{R_0}(p) \setminus \Omega \subseteq \mathcal{C}E.$$

Therefore, we can suppose without loss of generality that Ω is a bounded and connected open set with $C^{1,1}$ boundary $\partial\Omega$ and that E is s-minimal in Ω.

As observed in the proof of [43, Theorem 5.1], the minimality of E and hypothesis (C.9) imply that the set $\mathcal{C}E$ is a solution, in $B_{\frac{R_0}{4}}(p)$, of the geometric obstacle type problem considered in [20].

More precisely, we remark that we can find a bounded and connected open set $\mathcal{O}$ with $C^{1,1}$ boundary, such that

$$\mathcal{O} \cap B_{\frac{R_0}{4}}(p) = B_{\frac{R_0}{4}}(p) \setminus \overline{\Omega}.$$

Then hypothesis (C.9) guarantees that

$$\mathcal{O} \cap B_{\frac{R_0}{4}}(p) \subseteq \mathcal{C}E.$$

Now, by arguing as in the proof of [43, Theorem 5.1], we find that the minimality of E (hence also of $\mathcal{C}E$) in Ω implies that

$$\operatorname{Per}_s\Big(\mathcal{C}E, B_{\frac{R_0}{4}}(p)\Big) \leq \operatorname{Per}_s\Big(F, B_{\frac{R_0}{4}}(p)\Big),$$

for every $F \subseteq \mathbb{R}^n$ such that

$$F \setminus B_{\frac{R_0}{4}}(p) = \mathcal{C}E \setminus B_{\frac{R_0}{4}}(p) \quad \text{and} \quad \mathcal{O} \cap B_{\frac{R_0}{4}}(p) \subseteq F.$$

In particular, as observed in [20] (see the comment (2.2) there), the set $\mathcal{C}E$ is a viscosity supersolution in $B_{\frac{R_0}{4}}(p)$, meaning that the set E is a viscosity subsolution in $B_{\frac{R_0}{4}}(p)$. Now, since the set Ω has $C^{1,1}$ boundary, we can find an exterior tangent ball at $p \in \partial\Omega$. By hypothesis (C.9), this means that we can find an exterior tangent ball at $p \in \partial E$ and hence we have

$$\limsup_{\varrho \to 0^+} H_s^\varrho[E](p) \leq 0. \tag{C.11}$$

Furthermore, [20, Theorem 1.1] guarantees that ∂E is $C^{1,\sigma}$ in $B_{R_0'}(p)$ for some $R_0' \in (0, R_0)$, and $\sigma := \frac{1+s}{2}$ (see also [43, Theorem 5.1]). In particular, since $\sigma > s$, we know that the s-fractional mean curvature of E is well defined at p. Therefore (C.11) actually implies that $H_s[E](p) \leq 0$, as claimed.

Now we suppose in addition that (C.10) holds true, i.e. that

$$\partial E \cap \big(\Omega \cap B_r(p)\big) \neq \varnothing \qquad \text{for every } r \in (0, R),$$

with $R < R_0'$. By [58, Theorem 1.1] we know that $\partial E \cap \big(B_R(p) \cap \Omega\big)$ is C^∞. In particular, as observed in Theorem C.2.5, we know that every point $x \in \partial E \cap \big(B_R(p) \cap \Omega\big)$ satisfies the Euler-Lagrange equation in the classical sense, i.e.

$$H_s[E](x) = 0 \qquad \text{for every } x \in \partial E \cap \big(B_R(p) \cap \Omega\big). \tag{C.12}$$

Since $\partial E \cap B_R(p)$ is $C^{1,\sigma}$, with $\sigma > s$, we also know that $H_s[E] \in C(\partial E \cap B_R(p))$ (by, e.g., Proposition 3.1.11 or [43, Lemma 3.4]). Finally, we observe that by (C.10) we can find a sequence of points $x_k \in \partial E \cap \big(B_R(p) \cap \Omega\big)$ such that $x_k \longrightarrow p$. Then, by the continuity of $H_s[E]$ and (C.12) we get

$$H_s[E](p) = \lim_{k\to\infty} H_s[E](x_k) = 0,$$

concluding the proof. □

C.4. A maximum principle

By exploiting the Euler-Lagrange equation, we can compare an s-minimal set with half spaces. We show that if E is s-minimal in Ω and the exterior data $E_0 := E \setminus \Omega$ lies above a half-space, then also $E \cap \Omega$ must lie above that same half-space. This is indeed a very general principle, that we now discuss in full detail. To this aim, it is convenient to point out that if $E \subseteq F$ and the boundaries of the two sets touch at a common point x_0 where the s-fractional mean curvatures coincide, then the two sets must be equal. The precise result goes as follows:

LEMMA C.4.1. *Let $E, F \subseteq \mathbb{R}^n$ be such that $E \subseteq F$ and $x_0 \in \partial E \cap \partial F$. Then*

$$H_s^\varrho[E](x_0) \geq H_s^\varrho[F](x_0) \qquad \text{for every } \varrho > 0. \tag{C.13}$$

Furthermore, if

$$\liminf_{\varrho\to 0^+} H_s^\varrho[F](x_0) \geq a \quad \text{and} \quad \limsup_{\varrho\to 0^+} H_s^\varrho[E](x_0) \leq a, \tag{C.14}$$

then $E = F$, the fractional mean curvature is well defined in x_0 and $H_s[E](x_0) = a$.

PROOF. To get (C.13) it is enough to notice that

$$E \subseteq F \quad \Longrightarrow \quad \big(\chi_{\mathcal{C}E}(y) - \chi_E(y)\big) \geq \big(\chi_{\mathcal{C}F}(y) - \chi_F(y)\big) \qquad \forall\, y \in \mathbb{R}^n.$$

Now suppose that (C.14) holds true. Then by (C.13) we find that

$$\exists \lim_{\varrho\to 0^+} H_s[E](x_0) = \lim_{\varrho\to 0^+} H_s[F](x_0) = a.$$

To conclude, notice that if the two curvatures are well defined (in the principal value sense) in x_0 and are equal, then

$$\begin{aligned} 0 \leq &\int_{\mathcal{C}B_\varrho(x_0)} \frac{\big(\chi_{\mathcal{C}E}(y) - \chi_E(y)\big) - \big(\chi_{\mathcal{C}F}(y) - \chi_F(y)\big)}{|x_0 - y|^{n+s}}\,dy \\ &= H_s^\varrho[E](x_0) - H_s^\varrho[F](x_0) \xrightarrow{\varrho\to 0^+} 0, \end{aligned}$$

which implies that $\chi_E(y) = \chi_F(y)$ for a.e. $y \in \mathbb{R}^n$, i.e. $E = F$. □

PROPOSITION C.4.2. *[Maximum Principle] Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with $C^{1,1}$ boundary. Let $s \in (0,1)$ and let E be s-minimal in Ω. If*

$$\{x \cdot \nu \leq a\} \setminus \Omega \subseteq \mathcal{C}E, \tag{C.15}$$

for some $\nu \in \mathbb{S}^{n-1}$ and $a \in \mathbb{R}$, then

$$\{x \cdot \nu \leq a\} \subseteq \mathcal{C}E.$$

PROOF. First of all, we remark that up to a rotation and translation, we can suppose that $\nu = e_n$ and $a = 0$. Furthermore we can assume that

$$\inf_{x\in\overline{\Omega}} x_n < 0,$$

otherwise there is nothing to prove.

If $E \cap \Omega = \varnothing$, i.e. $\Omega \subseteq \mathcal{C}E$, we are done. Thus we can suppose that $E \cap \Omega \neq \varnothing$. Since $\overline{E} \cap \overline{\Omega}$ is compact, we have

$$b := \min_{x \in \overline{E} \cap \overline{\Omega}} x_n \in \mathbb{R}.$$

Now we consider the set of points which realize the minimum above, namely we set

$$\mathcal{P} := \{p \in \overline{E} \cap \overline{\Omega} \,|\, p_n = b\}.$$

Notice that

$$\big\{x_n \le \min\{b, 0\}\big\} \subseteq \mathcal{C}E, \tag{C.16}$$

so we are reduced to prove that $b \ge 0$.

We argue by contradiction and suppose that $b < 0$. We will prove that $\mathcal{P} = \varnothing$. We remark that $\mathcal{P} \subseteq \partial E \cap \overline{\Omega}$.

Indeed, if $p \in \mathcal{P}$, then by (C.16) we have that $B_\delta(p) \cap \{x_n \le b\} \subseteq \mathcal{C}E$ for every $\delta > 0$, so $|B_\delta(p) \cap \mathcal{C}E| \ge \frac{\omega_n}{2}\delta^n$ and $p \notin E_{int}$. Therefore, since $\overline{E} = E_{int} \cup \partial E$, we find that $p \in \partial E$.

Roughly speaking, we are sliding upwards the half-space $\{x_n \le t\}$ until we first touch the set $\overline{E}$. Then the contact points must belong to the boundary of E.

Notice that the points of $\mathcal{P}$ can be either inside Ω or on $\partial\Omega$. In both cases we can use the Euler-Lagrange equation to get a contradiction. The precise argument goes as follows.

First, if $p = (p', b) \in \partial E \cap \Omega$, then since $H := \{x_n \le b\} \subseteq \mathcal{C}E$, we can find an exterior tangent ball to E at p (contained in Ω), so $H_s[E](p) = 0$.

On the other hand, if $p \in \partial E \cap \partial\Omega$, then $B_{|b|}(p) \setminus \Omega \subseteq \mathcal{C}E$ and hence (by [43, Theorem 5.1]) $\partial E \cap B_r(p)$ is $C^{1,\frac{s+1}{2}}$ for some $r \in (0, |b|)$, and $H_s[E](p) \le 0$ by Theorem (C.3.1) .

In both cases, we have that

$$p \in \partial H \cap \partial E, \quad H \subseteq \mathcal{C}E \quad \text{and} \quad H_s[\mathcal{C}E](p) = -H_s[E](p) \ge 0 = H_s[H](p),$$

and hence Lemma C.4.1 implies $\mathcal{C}E = H$. However, since $b < 0$, this contradicts (C.15).

This proves that $b \ge 0$, thus concluding the proof. □

From this, we obtain a strong comparison principle with planes, as follows:

Corollary C.4.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with $C^{1,1}$ boundary. Let $E \subseteq \mathbb{R}^n$ be s-minimal in Ω, with $\{x_n \le 0\} \setminus \Omega \subseteq \mathcal{C}E$. Then*

(i) if $|(\mathcal{C}E \setminus \Omega) \cap \{x_n > 0\})| = 0$, then $E = \{x_n > 0\}$;

(ii) if $|(\mathcal{C}E \setminus \Omega) \cap \{x_n > 0\}| > 0$, then for every $x = (x', 0) \in \Omega \cap \{x_n = 0\}$ there exists $\delta_x \in (0, d(x, \partial\Omega))$ s.t. $B_{\delta_x}(x) \subseteq \mathcal{C}E$. Thus

$$\{x_n \le 0\} \cup \bigcup_{(x',0) \in \Omega} B_{\delta_x}(x) \subseteq \mathcal{C}E. \tag{C.17}$$

Proof. First of all, Proposition C.4.2 guarantees that

$$\{x_n \le 0\} \subseteq \mathcal{C}E.$$

(i) Notice that since E is s-minimal in Ω, also $\mathcal{C}E$ is s-minimal in Ω. Thus, since $\{x_n > 0\} \setminus \Omega \subseteq E = \mathcal{C}(\mathcal{C}E)$, we can use again Proposition C.4.2 (notice that $\{x_n = 0\}$ is a set of measure zero) to get $\{x_n > 0\} \subseteq E$, proving the claim.

(ii) Let $x \in \{x_n = 0\} \cap \Omega$.

We argue by contradiction. Suppose that $|B_\delta(x) \cap E| > 0$ for every $\delta > 0$. Notice that, since $B_\delta(x) \cap \{x_n \le 0\} \subseteq \mathcal{C}E$ for every $\delta > 0$, this implies that $x \in \partial E \cap \Omega$. Moreover, we can find an exterior tangent ball to E in x, namely

$$B_\varepsilon(x - \varepsilon\, e_n) \subseteq \{x_n \le 0\} \cap \Omega \subseteq \mathcal{C}E \cap \Omega.$$

Thus the Euler-Lagrange equation gives $H_s[E](x) = 0$.

Let $H := \{x_n \leq 0\}$. Since $x \in \partial H$, $H \subseteq \mathcal{C}E$ and also $H_s[H](x) = 0$, Lemma C.4.1 implies $\mathcal{C}E = H$. However this contradicts the hypothesis

$$|(\mathcal{C}E \setminus \Omega) \cap \{x_n > 0\}| > 0,$$

which completes the proof. □

APPENDIX D

Some auxiliary results

D.1. Useful integral inequalities

We collect here some useful inequalities which we have exploited at various places within the thesis.

We begin with the following simple integral inequality.

LEMMA D.1.1. *Let $n \geq 1$, $s \in (0,1)$ and $A, B \subseteq \mathbb{R}^n$ be bounded sets. Then*

$$\int_A \int_B \frac{dx\,dy}{|x-y|^{n-1+s}} \leq \frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{1-s} \min\left\{|A|, |B|\right\} \operatorname{diam}(A \cup B)^{1-s}.$$

PROOF. Suppose without loss of generality that $|A| \leq |B|$ and set $D := \operatorname{diam}(A \cup B)$. Then, by changing variables conveniently we estimate

$$\int_A \int_B \frac{dx\,dy}{|x-y|^{n-1+s}} \leq \int_A \left(\int_{B_D} \frac{dz}{|z|^{n-1+s}} \right) dx = \mathcal{H}^{n-1}(\mathbb{S}^{n-1})|A| \int_0^D \frac{d\varrho}{\varrho^s},$$

which directly leads to the conclusion. □

Now we prove that a measurable function with finite $W^{s,p}$-seminorm is actually L^p-summable and hence belongs to the fractional Sobolev space $W^{s,p}$. The proof follows by arguing as in the proof [38, Theorem 8.2] (see in particular the formula (8.3) there).

LEMMA D.1.2. *Let $p \in [1, \infty)$, $s \in (0,1)$ and let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Let $u : \Omega \to \mathbb{R}$ be a measurable function such that*

$$[u]^p_{W^{s,p}(\Omega)} = \int_\Omega \int_\Omega \frac{|u(x) - u(\xi)|^p}{|x-\xi|^{n+sp}}\, dx\, d\xi < +\infty.$$

Then $u \in W^{s,p}(\Omega)$. More precisely, if $E \subseteq \Omega$ is any measurable set such that

$$|E| > 0 \quad \text{and} \quad \int_E |u(\xi)|\, d\xi < +\infty, \tag{D.1}$$

then, if we denote

$$M_E := \int_E u(\xi)\, d\xi,$$

we have

$$\|u\|^p_{L^p(\Omega)} \leq \frac{2^{p-1}}{|E|} \left\{ (\operatorname{diam} \Omega)^{n+sp} [u]^p_{W^{s,p}(\Omega)} + |\Omega| \frac{|M_E|^p}{|E|^{p-1}} \right\}. \tag{D.2}$$

PROOF. First of all, we remark that since u is measurable there exists at least one set E satisfying (D.1). Indeed, for every $k \in \mathbb{N}$ we can consider the set

$$E_k := \{x \in \Omega \,|\, |u(x)| \leq k\},$$

which is measurable. Since u is finite almost everywhere in Ω, there exists $h \in \mathbb{N}$ such that $|E_h| > 0$. Then, notice that

$$\int_{E_h} |u(\xi)|\, d\xi \leq |E_h| h \leq |\Omega| h < +\infty,$$

so that E_h satisfies (D.1).

Now let E be any set satisfying (D.1) and define the constant

$$c := \frac{1}{|E|}\int_E u(\xi)\,d\xi = \frac{M_E}{|E|},$$

which is finite by hypothesis.

By exploiting Holder's inequality we find

$$|u(x)-c|^p = \frac{1}{|E|^p}\Big|\int_E \big(u(x)-u(\xi)\big)d\xi\Big|^p \le \frac{1}{|E|}\int_E |u(x)-u(\xi)|^p\,d\xi,$$

for every $x \in \Omega$. Integrating in x over Ω we obtain

$$\int_\Omega |u(x)-c|^p\,dx \le \frac{1}{|E|}\int_\Omega\int_E |u(x)-u(\xi)|^p\,dx\,d\xi.$$

Since $|x-\xi| \le \operatorname{diam}\Omega$ for every $x \in \Omega$ and $\xi \in E \subseteq \Omega$, we conclude that

$$\int_\Omega |u(x)-c|^p\,dx \le \frac{1}{|E|}\int_\Omega\int_E |u(x)-u(\xi)|^p\,dx\,d\xi \le \frac{(\operatorname{diam}\,\Omega)^{n+sp}}{|E|}[u]^p_{W^{s,p}(\Omega)}.$$

Finally, we observe that

$$\int_\Omega |c|^p\,dx = |\Omega|\left(\frac{|M_E|}{|E|}\right)^p.$$

Therefore

$$\begin{aligned}\|u\|^p_{L^p(\Omega)} &\le 2^{p-1}\int_\Omega |u(x)-c|^p\,dx + 2^{p-1}\int_\Omega |c|^p\,dx\\ &\le 2^{p-1}\frac{(\operatorname{diam}\,\Omega)^{n+sp}}{|E|}[u]^p_{W^{s,p}(\Omega)} + 2^{p-1}|\Omega|\,\frac{|M_E|^p}{|E|^p},\end{aligned}$$

proving (D.2) and concluding the proof of the Lemma. □

Now we prove a "global version" of Lemma D.1.2 in which we use the nonlocal functional

$$\mathcal{N}_s(u,\Omega) := \iint_{\mathbb{R}^{2n}\setminus(\mathcal{C}\Omega)^2} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy,$$

with $s \in (0,1)$, in place of the Gagliardo seminorm. We recall the following definition,

$$L^2_s(\mathbb{R}^n) := \left\{u:\mathbb{R}^n \to \mathbb{R}\,\Big|\,\|u\|^2_{L^2_s(\mathbb{R}^n)} := \int_{\mathbb{R}^n}\frac{|u(\xi)|^2}{1+|\xi|^{n+2s}}d\xi < \infty\right\}.$$

Lemma D.1.3. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and let $s \in (0,1)$. If $u : \mathbb{R}^n \to \mathbb{R}$ is a measurable function such that $\mathcal{N}_s(u,\Omega) < \infty$, then $u \in L^2_s(\mathbb{R}^n)$. More precisely, if $E \subseteq \Omega$ is any measurable set such that*

$$|E| > 0 \quad \text{and} \quad \int_E |u(\xi)|\,d\xi < \infty, \tag{D.3}$$

then, if we denote

$$M_E := \int_E u(\xi)\,d\xi,$$

we have

$$\|u\|^2_{L^2_s(\mathbb{R}^n)} \le \frac{C}{|E|}\left\{\mathcal{N}_s(u,\Omega) + \frac{M_E^2}{|E|}\right\},$$

for some $C = C(n,s,\Omega) > 0$.

PROOF. The proof is similar to that of Lemma D.1.2. Again, since u is measurable we know that there exists at least one set $E \subseteq \Omega$ satisfying (D.3).

Now we take a set $E \subseteq \Omega$ which satisfies (D.3), we define the constant

$$c := \frac{1}{|E|} \int_E u(\xi)\, d\xi = \frac{M_E}{|E|},$$

and we remark that

$$|u(x) - c|^2 \le \frac{1}{|E|} \int_E |u(x) - u(\xi)|^2\, d\xi,$$

for every $x \in \mathbb{R}^n$. Integrating in x over $\mathbb{R}^n$, against the weight $1/(1 + |x|^{n+2s})$, we find

$$\int_{\mathbb{R}^n} \frac{|u(x) - c|^2}{1 + |x|^{n+2s}}\, dx \le \frac{1}{|E|} \int_E \int_{\mathbb{R}^n} \frac{|u(x) - u(\xi)|^2}{1 + |x|^{n+2s}}\, d\xi\, dx. \tag{D.4}$$

Now notice that, since Ω is bounded, there exists a constant $C = C(n, s, \Omega) > 0$ such that for every $\xi \in \Omega$ and every $x \in \mathbb{R}^n$, it holds

$$\frac{1}{1 + |x|^{n+2s}} \le C \frac{1}{|x - \xi|^{n+2s}}.$$

Thus, from (D.4) we obtain

$$\int_{\mathbb{R}^n} \frac{|u(x) - c|^2}{1 + |x|^{n+2s}}\, dx \le \frac{C}{|E|} \int_E \int_{\mathbb{R}^n} \frac{|u(x) - u(\xi)|^2}{|x - \xi|^{n+2s}}\, d\xi\, dx \le \frac{C}{|E|} \mathcal{N}_s(u, \Omega).$$

Finally, notice that

$$\int_{\mathbb{R}^n} \frac{|u(x)|^2}{1 + |x|^{n+2s}}\, dx \le 2 \int_{\mathbb{R}^n} \frac{|u(x) - c|^2}{1 + |x|^{n+2s}}\, dx + 2 \int_{\mathbb{R}^n} \frac{|c|^2}{1 + |x|^{n+2s}}\, dx,$$

and

$$\int_{\mathbb{R}^n} \frac{|c|^2}{1 + |x|^{n+2s}}\, dx = \frac{M_E^2}{|E|^2} \int_{\mathbb{R}^n} \frac{1}{1 + |x|^{n+2s}}\, dx.$$

This concludes the proof of the Lemma. □

D.1.1. Fractional Hardy-type inequality. We point out the following fractional Hardy-type inequality, which is stated, e.g., in [50]—see formula (17) there. Since the proof for the case $p = 1$ is hard to find in the literature, we provide a simple argument based on the fractional Hardy inequality on half-spaces ensured by [61, Theorem 1.1].

We recall that $\bar{d}_\Omega$ denotes the signed distance function from $\partial\Omega$, negative inside Ω—see Appendix B.1. Let us also observe that

$$|\bar{d}_\Omega(x)| = \operatorname{dist}(x, \partial\Omega).$$

THEOREM D.1.4. *Let $n \ge 1$, $p \ge 1$ and let $s \in (0, 1)$ be such that $sp < 1$. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Then, there exists a constant $C = C(n, s, p, \Omega) \ge 1$ such that*

$$\int_\Omega \frac{|u(x)|^p}{|\bar{d}_\Omega(x)|^{sp}}\, dx \le C \|u\|^p_{W^{s,p}(\Omega)} \tag{D.5}$$

for every $u \in W^{s,p}(\Omega)$.

PROOF. We first prove (D.5) for a function $u \in C_c^\infty(\Omega)$, then we extend it to the whole space $W^{s,p}(\Omega)$ by density.

Let $\{B^{(j)}\}_{j=1}^N$ be a sequence of balls of the form $B^{(j)} = B_r(x^{(j)})$, with $N \in \mathbb{N}$, $r > 0$, and $x^{(j)} \in \partial\Omega$, for which there exist Lipschitz isomorphisms

$$T_j : B'_2 \times (-2, 2) \longrightarrow 2B^{(j)} := B_{2r}(x^{(j)})$$

satisfying

$$\begin{aligned} T_j(U_2) &= 2B^{(j)}, && \text{with } U_2 := B_2' \times (-2,2),\\ T_j(U_2^+) &= \Omega \cap 2B^{(j)}, && \text{with } U_2^+ := B_2' \times (0,2),\\ T_j(U_2^0) &= \partial\Omega \cap 2B^{(j)}, && \text{with } U_2^0 := B_2' \times \{0\}, \end{aligned}$$

and such that $\partial\Omega \subseteq \cup_{j=1}^N B^{(j)}$.

Let $\varepsilon > 0$ be such that $\Omega \setminus \cup_{j=1}^N B^{(j)} \Subset \Omega_{-\varepsilon}$ and set $B^{(0)} := \Omega_{-\varepsilon}$. Clearly,

$$\int_{B^{(0)}} \frac{|u(x)|^p}{|\bar{d}_\Omega(x)|^{sp}}\,dx \le \varepsilon^{-sp} \int_{B^{(0)}} |u(x)|^p\,dx \le C\|u\|^p_{L^p(\Omega)}, \tag{D.6}$$

where, from now on, C denotes any constant larger than 1, whose value depend at most on n, s, p, and Ω.

Notice that $\{B^{(j)}\}_{j=0}^N$ is an oper cover of Ω and let $\{\eta_j\}_{j=0}^N$ be a smooth partition of unity on Ω subordinate to $\{B^{(j)}\}_{j=0}^N$.

For $j = 1, \ldots, N$, we define $v_j := \eta_j u \in C_c^\infty(\Omega \cap B^{(j)})$. Changing variables through T_j, we have

$$\int_{\Omega\cap B^{(j)}} \frac{|v_j(x)|^p}{|\bar{d}_\Omega(x)|^{sp}}\,dx = \int_{T_j^{-1}(\Omega\cap B^{(j)})} \frac{|v_j(T_j(\bar{x}))|^p}{|\bar{d}_\Omega(T_j(\bar{x}))|^{sp}} |\det DT_j(\bar{x})|\,d\bar{x}.$$

Notice that for every $x \in \Omega \cap B^{(j)}$ there exists $D_j(x) \in \partial\Omega \cap 2B^{(j)}$ such that $|\bar{d}_\Omega(x)| = |x - D_j(x)|$. Since T_j is bi-Lipschitz and $T_j^{-1}(D_j(x)) \in B_2' \times \{0\}$, we have

$$\begin{aligned} |\bar{d}_\Omega(T_j(\bar{x}))| &= |T_j(\bar{x}) - D_j(T_j(\bar{x}))| = |T_j(\bar{x}) - T_j(T_j^{-1}(D_j(T_j(\bar{x}))))|\\ &\ge C^{-1}|\bar{x} - T_j^{-1}(D_j(T_j(\bar{x})))| \ge C^{-1}\bar{x}_n \end{aligned}$$

for every $\bar{x} \in T_j^{-1}(\Omega \cap B^{(j)})$. Accordingly, writing $w_j := v_j \circ T_j$ we get

$$\int_{\Omega\cap B^{(j)}} \frac{|v_j(x)|^p}{|\bar{d}_\Omega(x)|^{sp}}\,dx \le C \int_{U_2^+} \frac{|w_j(\bar{x}))|^p}{|\bar{x}_n|^{sp}}\,d\bar{x}.$$

Let us observe that w_j is supported inside $T_j^{-1}(\Omega\cap B^{(j)})$. We now employ the fractional Hardy inequality on half-spaces—e.g., [61, Theorem 1.1]—and deduce that

$$\int_{\Omega\cap B^{(j)}} \frac{|v_j(x)|^p}{|\bar{d}_\Omega(x)|^{sp}}\,dx \le C \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \frac{|w_j(\bar{x}) - w_j(\bar{y})|^p}{|\bar{x} - \bar{y}|^{n+sp}}\,d\bar{x}\,d\bar{y}, \tag{D.7}$$

where $\mathbb{R}^n_+ = \{z \in \mathbb{R}^n \,|\, z_n > 0\}$ and it is understood that w_j is extended by 0 in $\mathbb{R}^n_+ \setminus U_2^+$. We point out that—since $T_j^{-1}(B^{(j)}) \Subset U_2$ and $T_j^{-1}(\Omega \cap B^{(j)}) \subseteq U_2^+$—we have

$$\operatorname{dist}\big(T_j^{-1}(\Omega \cap B^{(j)}), \mathbb{R}^n_+ \setminus U_2^+\big) > 0.$$

Thus, using that w_j is supported inside $T_j^{-1}(\Omega \cap B^{(j)})$, we estimate

$$\begin{aligned} \int_{\mathbb{R}^n_+}\int_{\mathbb{R}^n_+} \frac{|w_j(\bar{x}) - w_j(\bar{y})|^p}{|\bar{x} - \bar{y}|^{n+sp}}\,d\bar{x}\,d\bar{y} &\le \int_{U_2^+}\int_{U_2^+} \frac{|w_j(\bar{x}) - w_j(\bar{y})|^p}{|\bar{x} - \bar{y}|^{n+sp}}\,d\bar{x}\,d\bar{y}\\ &\quad + 2\int_{T_j^{-1}(\Omega\cap B^{(j)})} \left(\int_{\mathbb{R}^n_+\setminus U_2^+} \frac{|w_j(\bar{x})|^p}{|\bar{x} - \bar{y}|^{n+sp}}\,d\bar{y}\right) d\bar{x}\\ &\le \int_{U_2^+}\int_{U_2^+} \frac{|w_j(\bar{x}) - w_j(\bar{y})|^p}{|\bar{x} - \bar{y}|^{n+sp}}\,d\bar{x}\,d\bar{y} + C\|w_j\|^p_{L^p(U_2^+)}. \end{aligned} \tag{D.8}$$

By combining (D.7) with (D.8) and switching back to the variables in Ω, we easily find that

$$\int_{\Omega\cap B^{(j)}} \frac{|v_j(x)|^p}{|\bar{d}_\Omega(x)|^{sp}}\, dx \leq C\left(\int_{\Omega\cap 2B^{(j)}}\int_{\Omega\cap 2B^{(j)}} \frac{|v_j(x)-v_j(y)|^p}{|x-y|^{n+sp}}\, dx\, dy + \|v_j\|^p_{L^p(\Omega\cap 2B^{(j)})}\right).$$

Recalling that $v_j = \eta_j u$ and $\operatorname{supp}(\eta_j) \Subset B^{(j)}$, a simple computation then leads us to

$$\int_{\Omega\cap B^{(j)}} \frac{|v_j(x)|^p}{|\bar{d}_\Omega(x)|^{sp}}\, dx \leq C\|u\|^p_{W^{s,p}(\Omega)} \quad \text{for all } j = 1,\ldots,N.$$

Then, estimate (D.5) for $u \in C^\infty_c(\Omega)$ follows by putting together this with (D.6) and using that $\{\eta_j\}$ is a partition of unity, whereas the general case of $u \in W^{s,p}(\Omega)$ is obtained by density. More precisely, let $u \in W^{s,p}(\Omega)$ and notice that by the density of $C^\infty_c(\Omega)$ in $W^{s,p}(\Omega)$—see, e.g., Theorem D.2.1—we can find $\{u_k\} \subseteq C^\infty_c(\Omega)$ such that

$$\lim_{k\to\infty} \|u-u_k\|_{W^{s,p}(\Omega)} = 0.$$

Up to passing to a subsequence, we can further suppose that $u_k \to u$ a.e. in Ω. Then, by Fatou's Lemma we find

$$\int_\Omega \frac{|u(x)|^p}{|\bar{d}_\Omega(x)|^{sp}} dx \leq \liminf_{k\to\infty} \int_\Omega \frac{|u_k(x)|^p}{|\bar{d}_\Omega(x)|^{sp}} dx \leq \lim_{k\to\infty} C\|u_k\|^p_{W^{s,p}(\Omega)} = C\|u\|^p_{W^{s,p}(\Omega)},$$

concluding the proof of the Theorem. □

Corollary D.1.5. *Let $n \geq 1$ and let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary. Let $p \geq 1$ and $s \in (0,1)$ be such that $sp < 1$. Then*

$$\int_\Omega\left(\int_{\mathcal{C}\Omega} \frac{|u(x)|^p}{|x-y|^{n+sp}} dy\right) dx \leq C(n,s,p,\Omega)\|u\|^p_{W^{s,p}(\Omega)}, \tag{D.9}$$

for every $u \in W^{s,p}(\Omega)$.

Proof. It is enough to notice that

$$\begin{aligned}\int_\Omega\left(\int_{\mathcal{C}\Omega} \frac{|u(x)|^p}{|x-y|^{n+sp}} dy\right) dx &\leq \int_\Omega\left(\int_{\mathcal{C}B_{|\bar{d}_\Omega(x)|}(x)} \frac{dy}{|x-y|^{n+sp}}\right)|u(x)|^p\, dx \\ &= \frac{\mathcal{H}^{n-1}(\mathbb{S}^{n-1})}{sp}\int_\Omega \frac{|u(x)|^p}{|\bar{d}_\Omega(x)|^{sp}} dx.\end{aligned}$$

Then the conclusion follows from Theorem D.1.4. □

D.1.2. Fractional Poincaré-type inequality. For the convenience of the reader, we provide a proof of the following well known fractional Poincaré-type inequality.

Proposition D.1.6. *Let $\Omega \subseteq \mathcal{O} \subseteq \mathbb{R}^n$ be bounded open sets such that $|\mathcal{O}\setminus\Omega| > 0$ and let $p \in [1,\infty)$ and $s \in (0,1)$. Let $u : \mathcal{O} \longrightarrow \mathbb{R}$ be such that $u = 0$ almost everywhere in $\mathcal{O}\setminus\Omega$. Then*

$$\|u\|^p_{L^p(\Omega)} \leq \frac{(\operatorname{diam}\mathcal{O})^{n+sp}}{|\mathcal{O}\setminus\Omega|}\int_\Omega\int_{\mathcal{O}\setminus\Omega} \frac{|u(x)|^p}{|x-y|^{n+sp}}\, dx\, dy \leq \frac{(\operatorname{diam}\mathcal{O})^{n+sp}}{|\mathcal{O}\setminus\Omega|}[u]^p_{W^{s,p}(\mathcal{O})}. \tag{D.10}$$

Proof. Notice that

$$|u(x)| = |u(x)-u(y)| \qquad \text{for almost every } (x,y) \in \Omega\times(\mathcal{O}\setminus\Omega).$$

Hence

$$|u(x)|^p = \frac{1}{|\mathcal{O}\setminus\Omega|}\int_{\mathcal{O}\setminus\Omega} |u(x)-u(y)|^p\, dy = \frac{1}{|\mathcal{O}\setminus\Omega|}\int_{\mathcal{O}\setminus\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}|x-y|^{n+sp}\, dy.$$

Since

$$|x-y| \le \text{diam } \mathcal{O} \qquad \forall\,(x,y) \in \Omega \times (\mathcal{O} \setminus \Omega),$$

we obtain

$$|u(x)|^p \le \frac{(\text{diam } \mathcal{O})^{n+sp}}{|\mathcal{O} \setminus \Omega|} \int_{\mathcal{O}\setminus\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dy.$$

Integrating over Ω gives

$$\|u\|^p_{L^p(\Omega)} \le \frac{(\text{diam } \mathcal{O})^{n+sp}}{|\mathcal{O} \setminus \Omega|} \int_{\Omega}\int_{\mathcal{O}\setminus\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}\, dx\, dy,$$

hence the claim. □

D.2. Density of compactly supported smooth functions

As customary, we denote by $W^{s,p}_0(\Omega)$ the closure of $C^\infty_c(\Omega)$ in $W^{s,p}(\Omega)$ with respect to the usual $W^{s,p}$-norm.

The aim of this section consists in providing a proof of the well known fact that, when $sp < 1$, the space $C^\infty_c(\Omega)$ is dense in $W^{s,p}(\Omega)$. Roughly speaking, this means that, in this case, the space $W^{s,p}(\Omega)$ has no well defined trace on $\partial\Omega$.

THEOREM D.2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $p \in [1,\infty)$, $s \in (0,1)$. Then*

$$sp < 1 \quad \Longrightarrow \quad W^{s,p}_0(\Omega) = W^{s,p}(\Omega),$$

i.e. $C^\infty_c(\Omega)$ is dense in $W^{s,p}(\Omega)$.

The proof of this well known theorem is the consequence of the following results.

LEMMA D.2.2. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $p \in [1,\infty)$, $s \in (0,1)$. Then*

$$W^{s,p}(\Omega) \cap L^\infty(\Omega) \quad \textit{is dense in} \quad W^{s,p}(\Omega).$$

PROOF. Given $u \in W^{s,p}(\Omega)$, consider the functions

$$u_k := \begin{cases} u & \text{in } \{|u| \le k\}, \\ k & \text{in } \{u \ge k\}, \\ -k & \text{in } \{u \le -k\}. \end{cases}$$

Then

$$|u_k|^p \le |u|^p \quad \text{a.e. in } \Omega \quad \text{and} \quad u_k \longrightarrow u \quad \text{a.e. in } \Omega,$$

hence

$$\lim_{k\to\infty} \|u-u_k\|_{L^p(\Omega)} = 0,$$

by the dominated convergence Theorem. Similarly, since

$$\frac{|u_k(x)-u_k(y)|^p}{|x-y|^{n+sp}} \le \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} \quad \text{for a.e. } (x,y) \in \Omega\times\Omega,$$

by using again the dominated convergence Theorem, we find

$$\lim_{k\to\infty} [u-u_k]_{W^{s,p}(\Omega)} = 0,$$

concluding the proof. □

Now we consider a symmetric mollifier, that is $\eta \in C_c^\infty(\mathbb{R}^n)$ such that

$$\eta \geq 0, \quad \int_{\mathbb{R}^n} \eta \, dx = 1, \quad \eta(x) = \eta(-x) \quad \text{and supp}\, \eta \subseteq B_1. \tag{D.11}$$

We set

$$\eta_\varepsilon(x) := \frac{1}{\varepsilon^n} \eta\Big(\frac{x}{\varepsilon}\Big),$$

for every $\varepsilon \in (0,1)$.

We recall the following well known result:

Lemma D.2.3. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and let $p \in [1,\infty)$, $s \in (0,1)$. Then for every $u \in W^{s,p}(\Omega)$ it holds*

$$\lim_{\varepsilon \to 0^+} \|u - u * \eta_\varepsilon\|_{W^{s,p}(\Omega')} = 0 \qquad \forall\, \Omega' \Subset \Omega.$$

We only observe that the proof of Lemma D.2.3 can be obtained by arguing as in the proof of point (i) of Lemma 2.3.2.

Proposition D.2.4. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with Lipschitz boundary and let $p \in [1,\infty)$, $s \in (0,1)$. Then*

$$sp < 1 \quad \Longrightarrow \quad C_c^\infty(\Omega) \quad \textit{is dense in} \quad W^{s,p}(\Omega) \cap L^\infty(\Omega).$$

Proof. Let $\sigma := sp \in (0,1)$ and let $u \in W^{s,p}(\Omega) \cap L^\infty(\Omega)$. For $\delta > 0$ small enough, let

$$u_\delta := u \chi_{\Omega_{-\delta}}.$$

Then

$$\lim_{\delta \to 0^+} \|u - u_\delta\|_{W^{s,p}(\Omega)} = 0. \tag{D.12}$$

Indeed

$$\|u - u_\delta\|^p_{L^p(\Omega)} \leq \|u\|^p_{L^\infty(\Omega)} |\Omega \setminus \Omega_{-\delta}| \xrightarrow{\delta \to 0^+} 0.$$

We remark that, since Ω is bounded and has Lipschitz boundary, and since $\sigma \in (0,1)$, by Lemma 2.2.7 we have

$$\int_{\Omega_{-\delta}} \int_{\Omega \setminus \Omega_{-\delta}} \frac{dx\, dy}{|x-y|^{n+\sigma}} \leq C(n,\Omega,\sigma)\delta^{1-\sigma}. \tag{D.13}$$

Then

$$\begin{aligned}\int_\Omega \int_\Omega & \frac{|u(x)(1-\chi_{\Omega_{-\delta}}(x)) - u(y)(1-\chi_{\Omega_{-\delta}}(y))|^p}{|x-y|^{n+\sigma}} dx\, dy \\ &= 2 \int_{\Omega_{-\delta}} \int_{\Omega \setminus \Omega_{-\delta}} \frac{|u(y)|^p}{|x-y|^{n+\sigma}} dx\, dy + [u]^p_{W^{s,p}(\Omega \setminus \Omega_{-\delta})} \\ &\leq 2\|u\|^p_{L^\infty(\Omega)} C(n,\Omega,\sigma)\delta^{1-\sigma} + [u]^p_{W^{s,p}(\Omega \setminus \Omega_{-\delta})}.\end{aligned}$$

Notice that, since $|\Omega \setminus \Omega_{-\delta}| \xrightarrow{\delta \to 0^+} 0$, we get by the dominated convergence Theorem

$$\lim_{\delta \to 0^+} [u]^p_{W^{s,p}(\Omega \setminus \Omega_{-\delta})} = 0.$$

Therefore

$$\lim_{\delta \to 0^+} [u - u_\delta]_{W^{s,p}(\Omega)} = 0,$$

proving (D.12).

Now we consider the ε-regularization of the function u_δ.

Notice that for every $\varepsilon \in (0, \delta/4)$

$$\text{supp}(u_\delta * \eta_\varepsilon) \Subset \Omega_{-\frac{\delta}{2}},$$

since the ε-convolution enlarges the support at most to an ε-neighborhood of the original function. It is well known that—since u_δ is compactly supported inside Ω—we have

$$\lim_{\varepsilon\to 0^+} \|u_\delta - u_\delta * \eta_\varepsilon\|_{L^p(\Omega)} = 0. \tag{D.14}$$

Moreover

$$\|u_\delta * \eta_\varepsilon\|_{L^\infty(\Omega)} \le \|u_\delta\|_{L^\infty(\Omega)} \le \|u\|_{L^\infty(\Omega)}.$$

Thus, by (D.13)

$$[u_\delta - u_\delta * \eta_\varepsilon]^p_{W^{s,p}(\Omega)} \le [u_\delta - u_\delta * \eta_\varepsilon]^p_{W^{s,p}(\Omega_{-\delta/2})} + 2\|u\|^p_{L^\infty(\Omega)} C(n,\Omega,\sigma)\left(\frac{\delta}{2}\right)^{1-\sigma}.$$

By Lemma D.2.3 we have

$$\lim_{\varepsilon\to 0^+} [u_\delta - u_\delta * \eta_\varepsilon]_{W^{s,p}(\Omega_{-\delta/2})} = 0.$$

Hence, recalling (D.14), we can find $\varepsilon_\delta \in (0, \delta/4)$ small enough such that, if we set

$$\tilde{u}_\delta := u_\delta * \eta_{\varepsilon_\delta} \in C^\infty_c(\Omega),$$

then

$$\|u_\delta - \tilde{u}_\delta\|_{L^p(\Omega)} \le \delta \quad \text{and} \quad [u_\delta - \tilde{u}_\delta]^p_{W^{s,p}(\Omega)} \le \delta + C\delta^{1-\sigma}. \tag{D.15}$$

Then, by (D.15) and (D.12) we obtain

$$\lim_{\delta\to 0^+} \|u - \tilde{u}_\delta\|_{W^{s,p}(\Omega)} = 0,$$

concluding the proof. □

Bibliography

[1] Nicola Abatangelo. Large S-harmonic functions and boundary blow-up solutions for the fractional Laplacian. *Discrete Contin. Dyn. Syst.*, 35(12):5555–5607, 2015.

[2] Nicola Abatangelo and Enrico Valdinoci. A notion of nonlocal curvature. *Numer. Funct. Anal. Optim.*, 35(7-9):793–815, 2014.

[3] Luigi Ambrosio. *Corso introduttivo alla teoria geometrica della misura ed alle superfici minime*. Appunti dei Corsi Tenuti da Docenti della Scuola. [Notes of Courses Given by Teachers at the School]. Scuola Normale Superiore, Pisa, 1997.

[4] Luigi Ambrosio and Norman Dancer. *Calculus of variations and partial differential equations*. Springer-Verlag, Berlin, 2000. Topics on geometrical evolution problems and degree theory, Papers from the Summer School held in Pisa, September 1996, Edited by G. Buttazzo, A. Marino and M. K. V. Murthy.

[5] Luigi Ambrosio, Guido De Philippis, and Luca Martinazzi. Gamma-convergence of nonlocal perimeter functionals. *Manuscripta Math.*, 134(3-4):377–403, 2011.

[6] I. Athanasopoulos, L. A. Caffarelli, C. Kenig, and S. Salsa. An area-Dirichlet integral minimization problem. *Comm. Pure Appl. Math.*, 54(4):479–499, 2001.

[7] M. Ballerini, N. Cabibbo, R. Candelier, A. Cavagna, E. Cisbani, I. Giardina, V. Lecomte, A. Orlandi, G. Parisi, A. Procaccini, M. Viale, and V. Zdravkovic. Interaction ruling animal collective behavior depends on topological rather than metric distance: Evidence from a field study. *Proceedings of the National Academy of Sciences*, 105(4):1232–1237, 2008.

[8] Viorel Barbu. *Differential equations*. Springer Undergraduate Mathematics Series. Springer, Cham, 2016. Translated from the 1985 Romanian original by Liviu Nicolaescu.

[9] Begoña Barrios, Alessio Figalli, and Enrico Valdinoci. Bootstrap regularity for integro-differential operators and its application to nonlocal minimal surfaces. *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)*, 13(3):609–639, 2014.

[10] Giovanni Bellettini. *Lecture notes on mean curvature flow, barriers and singular perturbations*, volume 12 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, 2013.

[11] Andrew Berdahl, Colin J. Torney, Christos C. Ioannou, Jolyon J. Faria, and Iain D. Couzin. Emergent sensing of complex environments by mobile animal groups. *Science*, 339(6119):574–576, 2013.

[12] Andrea L. Bertozzi, Jesus Rosado, Martin B. Short, and Li Wang. Contagion shocks in one dimension. *J. Stat. Phys.*, 158(3):647–664, 2015.

[13] E. Bombieri, E. De Giorgi, and M. Miranda. Una maggiorazione a priori relativa alle ipersuperfici minimali non parametriche. *Arch. Rational Mech. Anal.*, 32:255–267, 1969.

[14] Jean Bourgain, Haïm Brezis, and Petru Mironescu. Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications. *J. Anal. Math.*, 87:77–101, 2002. Dedicated to the memory of Thomas H. Wolff.

[15] Claudia Bucur and Luca Lombardini. Asymptotics as $s \searrow 0$ of the nonlocal nonparametric Plateau problem with obstacles. *In preparation.*

[16] Claudia Bucur, Luca Lombardini, and Enrico Valdinoci. Complete stickiness of nonlocal minimal surfaces for small values of the fractional parameter. *arXiv preprint arXiv:1612.08295*, 2016. To appear in Annales de l'Institut Henri Poincaré Analyse Non Linéaire.

[17] Claudia Bucur and Enrico Valdinoci. *Nonlocal diffusion and applications*, volume 20 of *Lecture Notes of the Unione Matematica Italiana*. Springer, [Cham]; Unione Matematica Italiana, Bologna, 2016.

[18] Xavier Cabré, Eleonora Cinti, and Joaquim Serra. Stable s-minimal cones in $\mathbb{R}^3$ are flat for $s \sim 1$. *arXiv preprint arXiv:1710.08722*, 2017.

[19] Xavier Cabré and Matteo Cozzi. A gradient estimate for nonlocal minimal graphs. *arXiv preprint arXiv:1711.08232*, 2017.

[20] Luis Caffarelli, Daniela De Silva, and Ovidiu Savin. Obstacle-type problems for minimal surfaces. *Communications in Partial Differential Equations*, 41(8):1303–1323, 2016.

[21] Luis Caffarelli, Jean-Michel Roquejoffre, and Ovidiu Savin. Nonlocal minimal surfaces. *Comm. Pure Appl. Math.*, 63(9):1111–1144, 2010.

[22] Luis Caffarelli, Ovidiu Savin, and Enrico Valdinoci. Minimization of a fractional perimeter-Dirichlet integral functional. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 32(4):901–924, 2015.

[23] Luis Caffarelli and Luis Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.

[24] Luis Caffarelli and Enrico Valdinoci. Uniform estimates and limiting arguments for nonlocal minimal surfaces. *Calc. Var. Partial Differential Equations*, 41(1-2):203–240, 2011.

[25] Luis Caffarelli and Enrico Valdinoci. Regularity properties of nonlocal minimal surfaces via limiting arguments. *Adv. Math.*, 248:843–871, 2013.

[26] Shiing-shen Chern. On the curvatures of a piece of hypersurface in euclidean space. *Abh. Math. Sem. Univ. Hamburg*, 29:77–91, 1965.

[27] Andre Chiaradia, John McBride, Tanya Murray, and Peter Dann. Effect of fog on the arrival time of little penguins Eudyptula minor: a clue for visual orientation? *J. of Ornithol.*, 148(2):229–233, 2007.

[28] Eleonora Cinti, Joaquim Serra, and Enrico Valdinoci. Quantitative flatness results and BV-estimates for stable nonlocal minimal surfaces. *arXiv preprint arXiv:1602.00540*, 2016. To appear in Journal of Differential Geometry.

[29] Matteo Cozzi. On the variation of the fractional mean curvature under the effect of $C^{1,\alpha}$ perturbations. *Discrete Contin. Dyn. Syst.*, 35(12):5769–5786, 2015.

[30] Matteo Cozzi, Serena Dipierro, and Enrico Valdinoci. Planelike interfaces in long-range Ising models and connections with nonlocal minimal surfaces. *J. Stat. Phys.*, 167(6):1401–1451, 2017.

[31] Matteo Cozzi, Alberto Farina, and Luca Lombardini. Bernstein-Moser-type results for nonlocaal minimal graphs. *arXiv preprint arXiv:1807.05774*, 2018. To appear in Communications in Analysis and Geometry.

[32] Emiliano Cristiani, Benedetto Piccoli, and Andrea Tosin. Modeling self-organization in pedestrians and animal groups from macroscopic and microscopic viewpoints. In *Mathematical modeling of collective behavior in socio-economic and life sciences*, Model. Simul. Sci. Eng. Technol., pages 337–364. Birkhäuser Boston, Inc., Boston, MA, 2010.

[33] T. Daniel, A. Chiaradia, M. Logan, G. Quinn, and R. Reina. Synchronized group association in little penguins, Eudyptula Minor. *Anim. behav.*, 74(5):1241–1248, 2007.

[34] Guy David. $C^{1+\alpha}$-regularity for two-dimensional almost-minimal sets in $\mathbb{R}^n$. *J. Geom. Anal.*, 20(4):837–954, 2010.

[35] Juan Dávila. On an open question about functions of bounded variation. *Calc. Var. Partial Differential Equations*, 15(4):519–527, 2002.

[36] Juan Dávila, Manuel del Pino, and Juncheng Wei. Nonlocal s-minimal surfaces and Lawson cones. *J. Differential Geom.*, 109(1):111–175, 2018.

[37] Françoise Demengel. Fonctions à hessien borné. *Ann. Inst. Fourier (Grenoble)*, 34(2):155–190, 1984.

[38] Eleonora Di Nezza, Giampiero Palatucci, and Enrico Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[39] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Invent. Math.*, 98(3):511–547, 1989.

[40] Serena Dipierro, Alessio Figalli, Giampiero Palatucci, and Enrico Valdinoci. Asymptotics of the s-perimeter as $s \searrow 0$. *Discrete Contin. Dyn. Syst.*, 33(7):2777–2790, 2013.

[41] Serena Dipierro, Luca Lombardini, Pietro Miraglio, and Enrico Valdinoci. The Phillip Island penguin parade (a mathematical treatment). *ANZIAM J.*, 60(1):27–54, 2018.

[42] Serena Dipierro, Ovidiu Savin, and Enrico Valdinoci. A nonlocal free boundary problem. *SIAM J. Math. Anal.*, 47(6):4559–4605, 2015.

[43] Serena Dipierro, Ovidiu Savin, and Enrico Valdinoci. Graph properties for nonlocal minimal surfaces. *Calc. Var. Partial Differential Equations*, 55(4):Art. 86, 25, 2016.

[44] Serena Dipierro, Ovidiu Savin, and Enrico Valdinoci. All functions are locally s-harmonic up to a small error. *J. Eur. Math. Soc. (JEMS)*, 19(4):957–966, 2017.

[45] Serena Dipierro, Ovidiu Savin, and Enrico Valdinoci. Boundary behavior of nonlocal minimal surfaces. *J. Funct. Anal.*, 272(5):1791–1851, 2017.

[46] Serena Dipierro and Enrico Valdinoci. Continuity and density results for a one-phase nonlocal free boundary problem. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 34(6):1387–1428, 2017.

[47] Serena Dipierro and Enrico Valdinoci. Nonlocal minimal surfaces: interior regularity, quantitative estimates and boundary stickiness. In Giampiero Palatucci and Tuomo Kuusi, editors, *Recent Developments in the Nonlocal Theory*, chapter 4, pages 165–209. De Gruyter Open, 2018.

[48] Pavel Doktor. Approximation of domains with Lipschitzian boundary. *Čas. Pěstování Mat.*, 101:237–255, 1976.

[49] Ning Du, Jintu Fan, Huijun Wu, Shuo Chen, and Yang Liu. An improved model of heat transfer through penguin feathers and down. *J. Theoret. Biol.*, 248(4):727–735, 2007.

[50] Bartł omiej Dyda. A fractional order Hardy inequality. *Illinois J. Math.*, 48(2):575–588, 2004.

[51] Kenneth Falconer. *Fractal geometry*. John Wiley & Sons, Ltd., Chichester, third edition, 2014. Mathematical foundations and applications.

[52] Daniel Faraco and Keith M. Rogers. The Sobolev norm of characteristic functions with applications to the Calderón inverse problem. *Q. J. Math.*, 64(1):133–147, 2013.

[53] Alberto Farina. A Bernstein-type result for the minimal surface equation. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 14(4):1231–1237, 2015.

[54] Alberto Farina. A sharp bernstein-type theorem for entire minimal graphs. *arXiv preprint arXiv:1707.0411*, 2017. To appear in Calc. Var. Partial Differential Equations.

[55] Alberto Farina and Enrico Valdinoci. Flatness results for nonlocal minimal cones and subgraphs. *arXiv preprint arXiv:1706.05701*, 2017.

[56] D. R. Farine, P. O. Montiglio, and O. Spiegel. From individuals to groups and back: the evolutionary implications of group phenotypic composition. *Trends in Ecol. and Evolut.*, 30(10):609–621, 2015.

[57] Patricio Felmer and Alexander Quaas. Boundary blow up solutions for fractional elliptic equations. *Asymptot. Anal.*, 78(3):123–144, 2012.

[58] Alessio Figalli and Enrico Valdinoci. Regularity and Bernstein-type results for nonlocal minimal surfaces. *J. Reine Angew. Math.*, 729:263–273, 2017.

[59] A. F. Filippov. *Differential equations with discontinuous righthand sides*, volume 18 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1988. Translated from the Russian.

[60] Alessio Fiscella, Raffaella Servadei, and Enrico Valdinoci. Density properties for fractional Sobolev spaces. *Ann. Acad. Sci. Fenn. Math.*, 40(1):235–253, 2015.

[61] Rupert L. Frank and Robert Seiringer. Sharp fractional Hardy inequalities in half-spaces. In *Around the research of Vladimir Maz'ya. I*, volume 11 of *Int. Math. Ser. (N. Y.)*, pages 161–167. Springer, New York, 2010.

[62] R. C. Gerum, B. Fabry, C. Metzner, M. Beaulieu, A. Ancel, and D. P. Zitterbart. The origin of traveling waves in an emperor penguin huddle. *New J. of Phys.*, 15(12):125022, 2013.

[63] Y. Gheraibia and A. Moussaoui. *Penguins Search Optimization Algorithm (PeSOA)*. in Recent Trends in Applied Artificial Intelligence: 26th International Conference on Industrial, Engineering and Other Applications of Applied Intelligent Systems, IEA/AIE 2013, Amsterdam, The Netherlands, June 17-21, 2013. Springer Berlin Heidelberg, 2013.

[64] Mariano Giaquinta and Enrico Giusti. On the regularity of the minima of variational integrals. *Acta Math.*, 148:31–46, 1982.

[65] Mariano Giaquinta and Luca Martinazzi. *An introduction to the regularity theory for elliptic systems, harmonic maps and minimal graphs*, volume 11 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie) [Lecture Notes. Scuola Normale Superiore di Pisa (New Series)]*. Edizioni della Normale, Pisa, second edition, 2012.

[66] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition.

[67] D. Giling, R. D. Reina, and Z. Hogg. Anthropogenic influence on an urban colony of the little penguin *eudyptula minor*. *Marine Freshwater Res.*, 59:647–651, 2008.

[68] Enrico Giusti. *Minimal surfaces and functions of bounded variation*, volume 80 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1984.

[69] Enrico Giusti. *Direct methods in the calculus of variations*. World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[70] E. Gonzalez, U. Massari, and M. Miranda. On minimal cones. *Appl. Anal.*, 65(1-2):135–143, 1997.

[71] R. Hegselmann and U. Krause. Opinion dynamics and bounded confidence: models, analysis and simulation. *J. Artif. Soc. and Soc. Simul.*, 5(3):1–33, 2002.

[72] Qingkai Kong. *A short course in ordinary differential equations*. Universitext. Springer, Cham, 2014.

[73] S. M. Laaksonen, A. Chiaradia, and R. D. Reina. Behavioural plasticity of a multihabitat animal, the little penguin, eudyptula minor, in response to tidal oscillations on its interhabitat transitions. *Preprint*, 2016.

[74] Erik Lindgren. Hölder estimates for viscosity solutions of equations of fractional p-Laplace type. *NoDEA Nonlinear Differential Equations Appl.*, 23(5):Art. 55, 18, 2016.

[75] Luca Lombardini. An introduction to the extension operator of the fractional Laplacian. *in preparation.*

[76] Luca Lombardini. Approximation of sets of finite fractional perimeter by smooth sets and comparison of local and global s-minimal surfaces. *Interfaces Free Bound.*, 20(2):261–296, 2018.

[77] Luca Lombardini. Fractional perimeters from a fractal perspective. *arXiv preprint arXiv:1508.06241*, 2018. To appear in Advanced Nonlinear Studies.

[78] A. J. J. Macintosh, L. Pelletier, A. Chiaradia, A. Kato, and Y. Ropert-Coudert. Temporal fractals in seabird foraging behaviour: diving through the scales of time. *Sci. Rep.*, 3(1884):1–10, 2013.

[79] Francesco Maggi. *Sets of finite perimeter and geometric variational problems*, volume 135 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2012. An introduction to geometric measure theory.

[80] Pertti Mattila. *Geometry of sets and measures in Euclidean spaces*, volume 44 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1995. Fractals and rectifiability.

[81] V. Maz'ya and T. Shaposhnikova. On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces. *J. Funct. Anal.*, 195(2):230–238, 2002.

[82] N. Miller, S. Garnier, and I. D. Couzin. Both information and social cohesion determine collective decisions in animal groups. *Proc. Natl. Acad. Sci. USA*, 110(13):5263–5268.

[83] Mario Miranda. Distribuzioni aventi derivate misure insiemi di perimetro localmente finito. *Ann. Scuola Norm. Sup. Pisa (3)*, 18:27–56, 1964.

[84] Jürgen Moser. On Harnack's theorem for elliptic differential equations. *Comm. Pure Appl. Math.*, 14:577–591, 1961.

[85] Augusto C. Ponce. A new approach to Sobolev spaces and connections to Γ-convergence. *Calc. Var. Partial Differential Equations*, 19(3):229–255, 2004.

[86] Romain Ragonnet, Romain Jumentier, and Bernhard Beckermann. La marche de l'empereur. *Matapli*, (102):71–82, 2013.

[87] A. M. Reynolds, Y. Ropert-Coudert, A. Kato, A. Chiaradia, and A. J. J. MacIntosh. A priority-based queuing process explanation for scale-free foraging behaviours. *Anim. Behav.*, (108):67–71, 2015.

[88] A. Rodríguez, A. Chiaradia, P. Wasiak, L. Renwick, and P. Dann. Waddling on the dark side: Ambient light affects attendance behavior of little penguins. *J. Biol. Rhythms*, 31(2):194–204, 2016.

[89] Xavier Ros-Oton and Joaquim Serra. The Dirichlet problem for the fractional Laplacian: regularity up to the boundary. *J. Math. Pures Appl. (9)*, 101(3):275–302, 2014.

[90] Xavier Ros-Oton and Enrico Valdinoci. The Dirichlet problem for nonlocal operators with singular kernels: convex and nonconvex domains. *Adv. Math.*, 288:732–790, 2016.

[91] Ovidiu Savin and Enrico Valdinoci. Γ-convergence for nonlocal phase transitions. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 29(4):479–500, 2012.

[92] Ovidiu Savin and Enrico Valdinoci. Regularity of nonlocal minimal cones in dimension 2. *Calc. Var. Partial Differential Equations*, 48(1-2):33–39, 2013.

[93] Ovidiu Savin and Enrico Valdinoci. Density estimates for a variational model driven by the Gagliardo norm. *J. Math. Pures Appl. (9)*, 101(1):1–26, 2014.

[94] Winfried Sickel. Pointwise multipliers of Lizorkin-Triebel spaces. In *The Maz'ya anniversary collection, Vol. 2 (Rostock, 1998)*, volume 110 of *Oper. Theory Adv. Appl.*, pages 295–321. Birkhäuser, Basel, 1999.

[95] L. A. Sidhu, E. A. Catchpole, and P. Dann. Modelling banding effect and tag loss for Little Penguins *eudyptula minor*. *ANZIAM J. Electron. Suppl.*, 52((C)):C206–C221, 2010.

[96] Elizabeth A. Skewgar. *Behavior of Magellanic penguins at sea*. ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–University of Washington.

[97] Italo Tamanini. Regularity results for almost minimal oriented hypersurfaces in $\mathbb{R}^n$. *Quaderni del Dipartimento di Matematica, Universitá di Lecce*, 1984.

[98] Enrico Valdinoci. A fractional framework for perimeters and phase transitions. *Milan J. Math.*, 81(1):1–23, 2013.

[99] Augusto Visintin. Generalized coarea formula and fractal sets. *Japan J. Indust. Appl. Math.*, 8(2):175–201, 1991.

[100] Georg Sebastian Weiss. Partial regularity for a minimum problem with free boundary. *J. Geom. Anal.*, 9(2):317–326, 1999.