

GENUS THREE CURVES AND 56 NODAL SEXTIC SURFACES

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ABSTRACT. Catanese and Tonoli showed that the maximal cardinality for an even set of nodes on a sextic surface is 56 and they constructed such nodal surfaces. In this paper we give an alternative, rather simple, construction for these surfaces starting from a non-hyperelliptic genus three curve. We illustrate our method by giving explicitly the equation of such a sextic surface starting from the Klein curve.

INTRODUCTION

A nodal surface is a projective surface with only ordinary double points as singularities. A set of nodes of a surface F is said to be even if there is a double cover $S \rightarrow F$ branched exactly in the nodes from that set. In [CT07], Catanese and Tonoli showed that an even set of nodes on a sextic surface has cardinality in $\{24, 32, 40, 56\}$. They also provided a construction of such 56 nodal surfaces, constructions for the other cases were already known. Their method is based on the paper [CC97], where it is shown that even sets of nodes correspond to certain symmetric maps between vector bundles. A careful study of the sheaves involved leads to certain matrices whose entries are homogeneous polynomials on \mathbb{P}^3 . The points in \mathbb{P}^3 where such a matrix has rank less than 6 is a sextic surface with an even set of 56 nodes. In this way, one can find explicit examples of such surfaces, but the equations tend to be rather complicated and it is not easy to understand the geometry of these surfaces.

Let F be a 56 nodal surface as constructed in [CT07] and let $f : S \rightarrow F$ be the double cover which is branched exactly over the nodes of F . The first Betti number of the smooth surface S is equal to 6, hence S has a 3-dimensional Albanese variety. With some trial and error, we then found the following construction for 56 nodal surfaces. A principally polarized abelian threefold A has a theta divisor Θ , defining the polarization, which can be taken to be symmetric, so $[-1]\Theta = \Theta$. The fixed points of $[-1]$ are the two-torsion points. There are exactly 28 such points on Θ precisely in the case that (A, Θ) is the Jacobian of a non-hyperelliptic genus three curve. Assume that we are in this case. Then $\bar{\Theta} = \Theta/[-1]$ has 28 nodes. This singular surface has been studied before, cf. [DO88, Chapter IX.6, Theorem 4 and Remark 6]. In particular it has an embedding into \mathbb{P}^6 where it is a Cartier divisor in a cone over a Veronese surface. This cone is the quotient of \mathbb{P}^3 by an involution which changes the sign of one of the homogeneous coordinates. The inverse image of $\bar{\Theta}$ in \mathbb{P}^3 is then a sextic surface F with an even set of 56 nodes, as we show in Section 1.

By construction, F has an involution with quotient $\bar{\Theta}$. This involution lifts to S and together with the covering involution of the map $S \rightarrow F$ generates a subgroup $(\mathbb{Z}/2\mathbb{Z})^2$ of $\text{Aut}(S)$. In Section 2 we study the cohomology of the quotients of S . We also show there that our construction and the one from Catanese and Tonoli produce the same surfaces. In the last section we give an explicit example, with a simple equation, of such a surface.

1. CONSTRUCTION OF A FAMILY OF 56 NODAL SEXTIC SURFACES

Let C be a smooth non-hyperelliptic curve of genus 3 and consider its Jacobian $A = \text{Jac}(C)$. The abelian variety A admits a principal polarization defined by the theta divisor Θ and we will identify $\Theta = S^2C$. We can choose Θ to be a symmetric divisor on A , i.e. $[-1]^*\Theta = \Theta$. The involution $[-1]$ on Θ corresponds to the involution $D \mapsto K_C - D$ on S^2C , where K_C is the canonical divisor on C .

The linear system $|2\Theta|$ is totally symmetric, and defines a morphism

$$\varphi_{2\Theta} : A \longrightarrow \mathbb{P}^7$$

which is the quotient map by the involution $[-1]$. Let $\bar{A} \cong A/[-1]$, the Kummer variety of A , be the image of $\varphi_{2\Theta}$. The singular locus of \bar{A} consists of 64 nodes, these are the images of the two-torsion points of A .

Consider the hyperplane $H_{2\Theta}$ of \mathbb{P}^7 corresponding to the divisor 2Θ . The intersection of $H_{2\Theta}$ with \overline{A} is the image $\overline{\Theta} \cong \Theta/[-1]$ of Θ , with multiplicity two. As Θ contains 28 of the two-torsion points of A , the surface $\overline{\Theta}$ has 28 nodes. Equivalently, these are the images of the 28 odd theta characteristics in S^2C .

To describe this map $\varphi_{2\Theta}|_{\Theta} : \Theta \rightarrow \mathbb{P}^6$, notice that the adjunction formula on A shows that the canonical class of Θ is $K_{\Theta} = \mathcal{O}_{|\Theta}$. Thus $\mathcal{O}_{\Theta}(2\Theta) \cong \omega_{\Theta}^{\otimes 2}$. Moreover, the cohomology of the restriction sequence

$$0 \rightarrow \mathcal{O}_A(\Theta) \rightarrow \mathcal{O}_A(2\Theta) \rightarrow \mathcal{O}_{\Theta}(2\Theta) \rightarrow 0$$

combined with $H^i(A, \mathcal{O}_A(\Theta)) = 0$ for $i > 0$ (Kodaira vanishing or Riemann-Roch on A), shows that $h^0(\omega_{\Theta}^{\otimes 2}) = h^0(\mathcal{O}_{\Theta}(2\Theta)) = 7$. Hence, when restricted to Θ , the morphism $\varphi_{2\Theta}|_{\Theta} = \varphi_{2K_{\Theta}}$ is given by the complete linear system $|2K_{\Theta}|$.

To understand this morphism better, we first consider the map $\varphi_{K_{\Theta}}$. From the restriction sequence above, twisted by $\mathcal{O}_A(-\Theta)$, one deduces that $H^0(\Theta, \omega_{\Theta}) \cong H^1(A, \mathcal{O}_A)$ is three dimensional. The map $\varphi_{K_{\Theta}} : \Theta \rightarrow \mathbb{P}^2$ is the Gauss map, which is a morphism of degree $(\mathcal{O}_{|\Theta})^2 = \Theta^3 = 6$ which factors over $\overline{\Theta}$. As $\varphi_{K_{\Theta}}$ is surjective, the natural map $S^2H^0(\Theta, \omega_{\Theta}) \rightarrow H^0(\Theta, \omega_{\Theta}^{\otimes 2})$ is injective, thus the image has codimension one.

Let $t \in H^0(\Theta, \omega_{\Theta}^{\otimes 2})$ be a general section in the complement of the image of $S^2H^0(\Theta, \omega_{\Theta})$. Since $|2\Theta|$ is basepoint free, we may assume that the divisor B in Θ defined by $t = 0$ is smooth and does not pass through any two-torsion points. Since $|2\Theta|$ is totally symmetric, any divisor in this linear system is symmetric, that is, $[-1]B = B$. Let s_0, \dots, s_2 be a basis of $H^0(\Theta, \omega_{\Theta})$. Then we have:

$$\varphi_{2K_{\Theta}} : \Theta \rightarrow \mathbb{P}H^0(\Theta, \omega_{\Theta}^{\otimes 2}) \cong H_{2\Theta} \cong \mathbb{P}^6, \quad x \mapsto (\dots : s_i(x)s_j(x) : \dots : t(x))_{0 \leq i < j \leq 2}.$$

The image $\overline{\Theta}$ of Θ thus lies in a cone over the Veronese surface of \mathbb{P}^2 . This cone is the image Y of the weighted projective 3-space $\mathbb{P}(1, 1, 1, 2)$, which is embedded into \mathbb{P}^6 by the (very) ample generator $\mathcal{O}_Y(1)$ of its Picard group:

$$\mathbb{P}(1, 1, 1, 2) \rightarrow Y \subset \mathbb{P}^6, \quad (y_0 : y_1 : y_2 : y_3) \mapsto (\dots : y_i y_j : \dots : y_3)_{0 \leq i < j \leq 2}.$$

As $\varphi_{K_{\Theta}}$ has no base points, the surface $\overline{\Theta} \subset Y$ does not contain the singular point $v = (0 : \dots : 0 : 1)$ of Y , the vertex of the cone over the Veronese surface. Hence, $\overline{\Theta}$ is a Cartier divisor on Y . The projection of $\overline{\Theta}$ from v onto the Veronese surface is the Gauss map $\varphi_{K_{\Theta}}$, which has degree $6/2 = 3$ on $\overline{\Theta}$. This implies that $\overline{\Theta}$ lies in the linear system on Y defined by three times the ample generator. Since the map $S^3H^0(Y, \mathcal{O}_Y(1)) \rightarrow H^0(Y, \mathcal{O}_Y(3))$ is surjective, we conclude that $\overline{\Theta}$ is defined by a weighted homogeneous polynomial p of degree six in $Y = \mathbb{P}(1, 1, 1, 2)$:

$$\overline{\Theta} = \{(y_0 : y_1 : y_2 : y_3) \in \mathbb{P}(1, 1, 1, 2) : p(y_0, \dots, y_3) = \sum_{i=0}^3 p_{2i}(y_0, y_1, y_2) y_3^{3-i} = 0\},$$

where each p_{2i} is homogeneous of degree $2i$ in y_0, y_1, y_2 . Since $v \notin \overline{\Theta}$, we may and will assume that $p_0 = 1$.

The weighted projective space $\mathbb{P}(1, 1, 1, 2)$ is also the quotient of \mathbb{P}^3 by the involution $i_3 : (x_0 : \dots : x_3) \mapsto (x_0 : x_1 : x_2 : -x_3)$, the quotient map is explicitly given by:

$$\overline{p} : \mathbb{P}^3 \rightarrow \mathbb{P}(1, 1, 1, 2), \quad (x_0 : x_1 : x_2 : x_3) \mapsto (\dots : x_i x_j : \dots : x_3^2)_{0 \leq i < j \leq 2}.$$

Now we define a surface F in \mathbb{P}^3 as $F := \overline{p}^{-1}(\overline{\Theta})$, thus F is defined by the sextic equation $P = 0$ where

$$P := p_6(x_0, x_1, x_2) + p_4(x_0, x_1, x_2)x_3^2 + p_2(x_0, x_1, x_2)x_3^4 + x_3^6.$$

The double cover $\overline{p} : F \rightarrow \overline{\Theta}$ is branched over the points where $x_3 = 0$, so the branch locus is the divisor $\overline{B} \subset \overline{\Theta}$ defined by $t = 0$. Here $\overline{B} = B/[-1]$, which is a smooth curve since by assumption B is smooth and does not pass through the 28 fixed points of $[-1]$ in Θ . Hence the singular locus of F consists of 56 nodes. The 28 nodes of $\overline{\Theta}$ form an even set since the double cover $\Theta \rightarrow \overline{\Theta}$ is branched only over the nodes. Hence the preimage $\Delta \subset F$ of these nodes is also an even set, cf. diagram 1, in fact F has a double cover S branched only over the nodes by pulling back the double cover $\Theta \rightarrow \overline{\Theta}$ along $\overline{p} : F \rightarrow \overline{\Theta}$.

We summarize the construction as follows:

Theorem 1.1. *There exists a family of 56 nodal sextic surfaces with the nodes forming an even set, which is parametrized by pairs (C, B) where C is a non-hyperelliptic curve of genus 3 and $B \in |2K_{S^2C}|$ is a general divisor. In particular, we have a $6 + 6 = 12$ dimensional family of such surfaces. Moreover, each surface in the family has an automorphism of order two.*

2. COVERINGS OF $\overline{\Theta}$

First of all, we provide another construction of the double cover $f : S \rightarrow F$ branched over the set Δ of 56 nodes of F . Let $\pi_F : \tilde{F} \rightarrow F$ be a minimal resolution of singularities and let $N_i \subset \tilde{F}$ be the inverse image of the node $p_i \in F$. Since the nodes form an even set, the divisor $\tilde{\Delta} = \sum_{i=1}^{56} N_i$ is even, that is, it is 2-divisible in $\text{Pic}(\tilde{F})$. Thus \tilde{F} admits a smooth double cover $\tilde{f} : \tilde{S} \rightarrow \tilde{F}$ branched along $\tilde{\Delta}$. Let $E_i = \tilde{f}^{-1}(N_i)$, so $\tilde{f}^* N_i = 2E_i$. Since p_i are nodes, the exceptional curves N_i are (-2) -curves, so

$$E_i \cdot E_i = \frac{1}{4} \tilde{f}^* N_i \cdot \tilde{f}^* N_i = \frac{1}{2} \tilde{f}^*(N_i \cdot N_i) = -1$$

and E_i are (-1) -curves. The surface S can be obtained by blowing down this set of (-1) -curves on the smooth surface \tilde{S} , so it is also smooth and it is a double cover of F , giving a commutative diagram

$$\begin{array}{ccc} \tilde{S} & \xrightarrow{\pi_S} & S \\ \tilde{f} \downarrow & & \downarrow f \\ \tilde{F} & \xrightarrow{\pi_F} & F \end{array}$$

From the definition of S as the base change along $\overline{p} : F \rightarrow \overline{\Theta}$ of the double cover $\Theta \rightarrow \overline{\Theta}$, it follows that the covering $S \rightarrow \overline{\Theta}$ is a $(\mathbb{Z}/2\mathbb{Z})^2$ -covering. Let ι_1 and ι_2 be involutions on S with quotient surface F and Θ respectively. Let $\iota_3 = \iota_1 \iota_2$, then ι_3 is an involution and we define $T := S/\iota_3$.

This gives a commutative diagram

$$(1) \quad \begin{array}{ccc} S & \xrightarrow{p} & \Theta \\ \downarrow f & \searrow & \downarrow \phi \\ F & \xrightarrow{\overline{p}} & \overline{\Theta} \end{array} \quad \begin{array}{c} \\ \\ \\ \\ \end{array} \quad \begin{array}{c} \\ \\ T \\ \\ \end{array}$$

Proposition 2.1. *The double cover $S \rightarrow T$ is unramified. In particular, T is smooth and $T \rightarrow \overline{\Theta}$ is branched along \overline{B} and the 28 nodes.*

Proof. The ramification locus of $S \rightarrow T$ is the fixed locus R_3 of ι_3 , which is precisely the points $s \in S$ such that $\iota_3 \in \text{Stab}_{(\mathbb{Z}/2\mathbb{Z})^2}(s)$. The fixed loci of ι_1 and ι_2 are $f^{-1}\Delta$ and $p^{-1}B$ respectively. Since the branch curve B does not contain any of the 28 two-torsion points on Θ , the intersection of the fixed loci $f^{-1}\Delta \cap p^{-1}B = \emptyset$. Hence, there are no points $s \in S$ such that $\text{Stab}_{(\mathbb{Z}/2\mathbb{Z})^2}(s) = (\mathbb{Z}/2\mathbb{Z})^2$. In particular, $R_3 = \{s \in S \mid \text{Stab}_{(\mathbb{Z}/2\mathbb{Z})^2}(s) = \langle \iota_3 \rangle\}$ is disjoint from $f^{-1}\Delta \cup p^{-1}B$. Since the ramification locus of $S \rightarrow \overline{\Theta}$ is precisely the union of that of f and p , we conclude that $R_3 = \emptyset$ and $S \rightarrow T$ is unramified. \square

We now consider the Hodge numbers of the surfaces in diagram 1.

Proposition 2.2. *The smooth surfaces Θ , S , T and \tilde{F} have Hodge numbers:*

| | $h^{1,0}$ | $h^{2,0}$ | $h^{1,1}$ |
|-------------|-----------|-----------|-----------|
| Θ | 3 | 3 | 10 |
| S | 3 | 10 | 38 |
| \tilde{S} | 3 | 10 | 94 |
| \tilde{F} | 0 | 10 | 86 |
| T | 0 | 3 | 16 |

Proof. The cohomologies of \mathcal{O}_Θ are computed using the short exact sequence

$$0 \rightarrow \mathcal{O}_A(-\Theta) \rightarrow \mathcal{O}_A \rightarrow \mathcal{O}_\Theta \rightarrow 0.$$

Since Θ is ample on A and $K_A = 0$, $h^i(\mathcal{O}_A(-\Theta)) = 0$ for $i < 3$ by Kodaira vanishing and, using Serre duality, $h^3(\mathcal{O}_A(-\Theta)) = h^0(\mathcal{O}_A(\Theta)) = 1$ since Θ is a principal polarization. Moreover, $h^i(\mathcal{O}_A) = \binom{3}{i}$, hence $h^{1,0}(\Theta) = h^1(\mathcal{O}_\Theta) = 3$ and $h^{2,0}(\Theta) = 3$. As

$$\chi_{\text{top}}(\Theta) = 2 - 4h^{1,0}(\Theta) + 2h^{2,0}(\Theta) + h^{1,1}(\Theta) = 2 - 12 + 6 + h^{1,1}(\Theta) = h^{1,1}(\Theta) - 4,$$

we can compute $h^{1,1}(\Theta)$ from Noether's formula

$$\chi(\mathcal{O}_\Theta) = \frac{\chi_{\text{top}}(\Theta) + K_\Theta^2}{12} \Rightarrow h^{1,1}(\Theta) = 12\chi(\mathcal{O}_\Theta) - K_\Theta^2 + 4 = 12 - 6 + 4 = 10 .$$

For the double cover $p : S \rightarrow \Theta$, branched over the divisor B , there is an isomorphism

$$p_*\mathcal{O}_S = \mathcal{O}_\Theta \oplus \mathcal{L}^{-1}, \quad \mathcal{L} \cong \omega_\Theta ,$$

so $\mathcal{L}^{\otimes 2} = \mathcal{O}_\Theta(B)$. Thus $h^{i,0}(S) = h^i(\mathcal{O}_\Theta) + h^i(\mathcal{L}^{-1})$. As $\mathcal{L} = \omega_\Theta$ is ample, by Kodaira vanishing we get $h^i(\mathcal{L}^{-1}) = 0$ for $i < 2$. Hence, by Riemann-Roch

$$h^2(\mathcal{L}^{-1}) = \chi(\omega_\Theta^{-1}) = \chi(\mathcal{O}_\Theta) + \frac{K_\Theta \cdot (K_\Theta + K_\Theta)}{2} = 7 .$$

On the canonical bundles, we have an isomorphism $\omega_S = p^*(\omega_\Theta \otimes \mathcal{L})$. Thus, $K_S^2 = p^*(2K_\Theta)^2 = 8K_\Theta^2 = 48$ and we obtain $h^{1,1}(S) = 38$ by Noether's formula.

The blowup $\pi_S : \tilde{S} \rightarrow S$ at 56 points does not change $h^{1,0}$ and $h^{2,0}$, and $h^{1,1}(\tilde{S}) = h^{1,1}(S) + 56$.

Since $\pi_F : \tilde{F} \rightarrow F$ is a blowup at isolated rational singularities, we have $h^i(\mathcal{O}_{\tilde{F}}) = h^i(\mathcal{O}_F)$. The latter can be computed using the short exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-6) \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_F \longrightarrow 0 .$$

Since the singularity is canonical, we have $\omega_{\tilde{F}} = \pi_F^*\omega_F = \pi_F^*(\omega_{\mathbb{P}^3} \otimes \mathcal{O}_{\mathbb{P}^3}(F))|_F = \pi_F^*\mathcal{O}_{\mathbb{P}^3}(2)|_F$ by the adjunction formula. Thus, $K_{\tilde{F}}^2 = (\mathcal{O}(2)|_F \cdot \mathcal{O}(2)|_F) = 2 \cdot 2 \cdot 6 = 24$. By Noether's formula, we obtain $h^{1,1}(\tilde{F}) = 86$.

Finally, we use the eigenspace decomposition of the cohomologies on S to compute the Hodge numbers of T . Let $G = (\mathbb{Z}/2\mathbb{Z})^2$. The G -action on S induces a decomposition

$$H^i(S, \mathcal{O}_S) = \bigoplus_{\chi \in G^*} H^i(S, \mathcal{O}_S)_\chi$$

where $G^* = \{1, \chi_1, \chi_2, \chi_3\}$ is the character group and χ_i is chosen such that, if H_i is the stabilizer of ι_i , then $G|_{H_i}^* = \{1, \chi_i\}$. Hence,

$$h^i(\mathcal{O}_F) = h^i(\mathcal{O}_S)_1 + h^i(\mathcal{O}_S)_{\chi_1}, \quad h^i(\mathcal{O}_\Theta) = h^i(\mathcal{O}_S)_1 + h^i(\mathcal{O}_S)_{\chi_2}, \quad h^i(\mathcal{O}_T) = h^i(\mathcal{O}_S)_1 + h^i(\mathcal{O}_S)_{\chi_3} .$$

From the Hodge numbers $h^{1,0}$ and $h^{2,0}$ of S, T and \tilde{F} (notice $h^{i,0}(F) = h^{i,0}(\tilde{F})$), we obtain that

$$h^{1,0}(S)_\chi = \begin{cases} 3 & \chi = \chi_2, \\ 0 & \chi \neq \chi_2, \end{cases} \quad \text{and} \quad h^{2,0}(S)_\chi = \begin{cases} 3 & \chi = 1, \\ 7 & \chi = \chi_1, \\ 0 & \chi = \chi_2, \chi_3 . \end{cases}$$

Hence, $h^{1,0}(T) = 0$ and $h^{2,0}(T) = 3$. Since $S \rightarrow T$ is an unramified double cover, we have an equality $\chi_{\text{top}}(S) = 2\chi_{\text{top}}(T)$. This allows us to compute $h^{1,1}(T) = 16$. \square

A consequence of the fact that we have a morphism $p : S \rightarrow \Theta$ and $h^{1,0}(S) = h^{1,0}(\Theta)$, is that the Albanese map of S factors over the Albanese map for Θ , which is just the inclusion $\Theta \hookrightarrow A$, hence $A = \text{Alb}(S)$.

We now deduce that the 12 dimensional family of 56 nodal sextics we constructed coincides with the family constructed by Catanese and Tonoli in [CT07, Main Theorem B]. Notice that they obtain a 27 dimensional subvariety of the space of sextic surfaces parametrizing 56 nodal sextics, but modulo the action of $\text{Aut}(\mathbb{P}^3)$ one again finds a $27 - 15 = 12$ dimensional family. We were not able to relate their construction to ours. However, when using their Macaulay scripts (which can be found in the eprint arXiv:math/0510499) we noticed that it does produce sextics which are invariant under the involution $x_0 \mapsto -x_0$ in \mathbb{P}^3 .

Corollary 2.3. *The family of sextics with an even set of 56 nodes from [CT07, Main Theorem B] coincides with the family constructed in Theorem 1.1.*

Proof. For a double cover $f : S \rightarrow F$ of a 56 nodal sextic surface F , branched exactly over the nodes of F , the 'quadratic' sheaf \mathcal{F} on F defined by $f_*\mathcal{O}_S = \mathcal{O}_F \oplus \mathcal{F}$ must satisfy $(\tau, a) = (3, 3)$ or $(\tau, a) = (3, 4)$, where $2\tau = h^1(F, \mathcal{F}(1))$ and $a = h^1(F, \mathcal{F})$, cf. [CT07, Theorem 2.5]. The family constructed in [CT07] is the one with invariants $(\tau, a) = (3, 3)$. For our surfaces we have $h^{1,0}(S) = h^1(F, f_*\mathcal{O}_S) = h^1(F, \mathcal{O}_F) + h^1(F, \mathcal{F})$ so we get $h^1(F, \mathcal{F}) = 3$, which shows that they are in the same family. \square

3. AN EXPLICIT EXAMPLE

Let C be a non-hyperelliptic genus three curve, we will also denote the canonical model of C , a quartic curve in \mathbb{P}^2 , by C . Recall that $\Theta = S^2C$, the symmetric product of C .

We show how to find the global sections $H^0(\Theta, \omega_\Theta^{\otimes 2})$ in terms of the geometry of C , following [BV96]. Note that if we map $S^2C \rightarrow \text{Jac}(C)$ by $p + q \mapsto p + q - t$ where $t \in S^2C$ is an odd theta characteristic (so $2t \equiv K_C$), then the image of S^2C is a symmetric theta divisor. Let $d = z_1 + \dots + z_4$ be an effective canonical divisor on C , $D = \sum(z_i + C)$ be the corresponding divisor on S^2C and Δ be the diagonal in S^2C . Then, $2K_{S^2C} = 2D - \Delta$. By [BV96, Lemma 4.7], we have the restriction sequence

$$0 \longrightarrow \mathcal{O}_{S^2C}(2K_{S^2C}) \longrightarrow \mathcal{O}_{S^2C}(2D) \longrightarrow \mathcal{O}_\Delta(2D) \cong \mathcal{O}_C(4d) \longrightarrow 0.$$

and

$$H^0(S^2C, \omega_{S^2C}^{\otimes 2}) \cong \ker(S^2H^0(C, \omega_C^{\otimes 2}) \xrightarrow{\mu} H^0(C, \omega_C^{\otimes 4})).$$

where μ is the multiplication map. Note that $h^0(C, \omega_C^{\otimes 2}) = 6$, so $\dim S^2H^0(C, \omega_C^{\otimes 2}) = 21$, and $h^0(C, \omega_C^{\otimes 4}) = 14$. By the same lemma, μ is surjective so indeed $h^0(S^2C, \omega_{S^2C}^{\otimes 2}) = 7$.

Let $\sigma_0, \dots, \sigma_2$ be a basis of $H^0(C, \omega_C)$. It induces a basis $\sigma_i \otimes \sigma_j$ of $H^0(C^2, \omega_{C^2}) = H^0(C, \omega_C)^{\otimes 2}$. The sections of $H^0(\Theta, \omega_\Theta) \cong \wedge^2 H^0(C, \omega_C)$ define the Gauss map $S^2C \cong \Theta \rightarrow \mathbb{P}^2$. Explicitly, the Gauss map is induced by the map

$$C \times C \longrightarrow \mathbb{P}^2, \quad (x, y) \longmapsto (p_{12} : p_{13} : p_{23}), \quad p_{ij}(x, y) := \sigma_i(x)\sigma_j(y) - \sigma_j(x)\sigma_i(y).$$

The six products $p_{ij}p_{kl}$ span a six dimensional subspace of $\ker(\mu)$ which is the image of $S^2H^0(\Theta, \omega_\Theta)$ in $H^0(\Theta, \omega_\Theta^{\otimes 2})$.

Let $f(z)$ be a homogeneous quartic polynomial in $\mathbb{C}[z_0, z_1, z_2]$ such that $f(\sigma_0(x), \sigma_1(x), \sigma_2(x)) = 0$ for all $x \in C$, that is, f defines the curve $C \subset \mathbb{P}^2$. Choose any polynomial $g(u, v)$ of bidegree $(2, 2)$ in $\mathbb{C}[u_0, u_1, u_2, v_0, v_1, v_2]$ such that $g(z, z) = f(z)$ and let $g_s(u, v) := g(u, v) + g(v, u)$, then $\tilde{g}(x, y) := g_s(\sigma_0(x), \dots, \sigma_2(y)) \in S^2H^0(C, \omega_C^{\otimes 2})$ and lies in $\ker(\mu)$. Thus the choice of \tilde{g} provides the section t used to construct the map φ_{2K_Θ} , any other choice of \tilde{g} is of the form $\lambda\tilde{g} + \sum \lambda_{ij}p_{ij}$ for complex numbers λ, λ_{ij} with $\lambda \neq 0$.

The map $\varphi_{2K_\Theta} : \Theta \rightarrow \mathbb{P}^6$ is therefore induced by the map

$$C \times C \longrightarrow \mathbb{P}^6, \quad (x, y) \longmapsto (\dots : p_{ij}(x, y)p_{kl}(x, y) : \dots : \tilde{g}(x, y)).$$

A homogeneous polynomial P in seven variables is an equation for the image of this map if $P(\dots, p_{ij}(u, v)p_{kl}(u, v), \dots, \tilde{g}(u, v))$ lies in the ideal of $\mathbb{C}[u_0, \dots, v_2]$ generated by $f(u)$ and $f(v)$.

An explicit example, worked out using the computer program Magma [BCP97], is provided by the choice $f = z_0z_1^3 + z_1z_2^3 + z_2z_0^3$, which defines the Klein curve in \mathbb{P}^2 . We will take $g = u_0u_1v_1^2 + u_1u_2v_2^2 + u_2u_0v_0^2$ and the map φ_{2K_Θ} is given by:

$$(y_{00} : y_{01} : \dots : y_{22} : y_g) = (p_{01}^2 : p_{01}p_{02} : p_{01}p_{12} : p_{02}^2 : p_{02}p_{12} : p_{12}^2 : \tilde{g}).$$

One of the equations for the image is

$$y_{00}^2y_{02} - y_{12}y_{22}^2 - y_{01}y_{11}^2 - 5y_{01}^2y_{22} + (-y_{00}y_{01} + y_{02}y_{22} - y_{11}y_{12})y_g - y_g^3$$

(this equation thus defines the image in $\mathbb{P}(1, 1, 1, 2) \subset \mathbb{P}^6$). Next we pull this equation back to \mathbb{P}^3 along the map \bar{p} by substituting $y_{ij} = x_i x_j$ and $y_g = x_3^2$, moreover we change the sign of x_1 in order to simplify the equation and we obtain

$$Q := x_0^5x_2 + x_0x_1^5 + x_1x_2^5 - 5x_0^2x_1^2x_2^2 + (x_0^3x_1 + x_0x_2^3 + x_1^3x_2)x_3^2 - x_3^6.$$

The singular locus of the surface F defined by $Q = 0$ consists of 56 nodes and these are thus an even set of nodes. To find all the nodes, we observe that $\text{Aut}(F)$ contains a subgroup G_{336} of order 336 with generators

$$g_7 := \text{diag}(\omega, \omega^4, \omega^2, 1), \quad g_2 := \frac{1}{\sqrt{-7}} \begin{pmatrix} a & c & b & 0 \\ c & b & a & 0 \\ b & a & c & 0 \\ 0 & 0 & 0 & \sqrt{-7} \end{pmatrix}, \quad \begin{cases} a = \omega^2 - \omega^5, \\ b = \omega - \omega^6, \\ c = \omega^4 - \omega^3, \end{cases}$$

where ω is a primitive seventh root of unity. One of the nodes is $(1 : 1 : 1 : 1)$ and G_{336} acts transitively on the 56 nodes, the stabilizer of a node is isomorphic to the symmetric group S_3 . The covering involution $\text{diag}(1, 1, 1, -1)$ generates the center of G_{336} and $G_{336} \cong \{\pm 1\} \times G_{168}$ where $G_{168} \cong SL(3, \mathbb{F}_2)$ is the automorphism group of the Klein curve. The equation of F can be written as $p_6 + p_4x_3^2 - x_3^6$, the discriminant of the cubic polynomial $p_6 + p_4T - T^3$ has degree 12 in $\mathbb{C}[x_0, x_1, x_2]$ and the curve it defines is the dual of the Klein curve (as expected from the presence of the Gauss map).

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