

The Dynamics of Wave-Particle Duality

Adriano Orefice*, Raffaele Giovanelli and Domenico Ditto

Department of Agricultural and Environmental Sciences (DISAA) - University of Milan
Via G. Celoria 2, Milano, Italy 20133.

*Corresponding Author: adriano.orefice@unimi.it

Abstract. Both *classical* and *wave-mechanical* treatments of *monochromatic* wave-like features may be faced in terms of *exact* trajectories, mutually coupled by a “Wave Potential” function (*encoded in the structure itself of all Helmholtz-like equations*) which is the cause of any diffraction and/or interference process. In the case of Wave Mechanics, an *energy-dependent* Wave Potential establishes a bridge between de Broglie's matter waves and the relevant particles by piloting them along their path by means of an *energy-preserving* “gentle drive”, allowing an insight into the dynamical mechanism of wave-particle duality. Thanks to a numerical treatment which is not substantially less manageable than its classical counterpart each particle is seen to “dance”, under the action of the Wave Potential, “a wave-mechanical dance around the classical motion”.

Keywords: Helmholtz equation; Wave Potential; Hamilton-Jacobi equation; Wave Mechanics; de Broglie's duality; matter waves; Schrödinger equations; Klein-Gordon equation.

1. – Introduction

As is well expressed in Ref. [1], “*the knowledge of several routes and their connections is always helpful when traveling through the quantum territory*”. The authors of Ref.[1] had in mind, however, a “territory” crossed by routes declaredly leading to Bohm's Mechanics [2-7], i.e. to *probability-based guidance-laws and to their hydrodynamic flow-lines*.

In order to describe, on the contrary, a (*non-Bohmian*) *dynamical set of exact point-particle trajectories* running as close as possible to the classical ones, without resorting to wave-packet particle representations, we follow here a route beginning from the very foundations of Wave Mechanics [8, 9], i.e.

1. de Broglie's seminal conjecture $\vec{p} = \hbar \vec{k}$, to which we owe the concept of matter waves and wave-particle duality [10, 11];
2. Schrödinger's *time-independent* equation [12-15], bypassing (with its eigen-fuctions and eigen-values) the heuristic prescriptions of the “old” Quantum Mechanics: an equation whose *energy-dependence* suggests a crucial analogy with Classical Mechanics, to be exploited in the present paper;
3. Schrödinger's *time-dependent* equation, whose *energy-independence* suggests a probabilistic distribution over the entire eigen-energy spectrum; and finally
4. Davisson-Germer's experimental test [16] of the real existence of de Broglie's matter waves *in physical space*.

We demonstrate, to begin with, that *any Helmholtz-like equation* is associated with a characteristic Hamiltonian set of “rays” in physical space, allowing an *exact, trajectory-based* approach to any kind of *classical* wave-like process, beyond any geometrical optics approximation. Passing then to the *wave-mechanical* case, de Broglie's relation $\vec{p} = \hbar \vec{k}$ clearly tells us that the “Helmholtz rays” (with wave vector \vec{k}) of de Broglie's matter waves coincide with the *dynamical trajectories* of the associated point-particles with momentum \vec{p} , thus laying *the foundations of Wave Mechanics*. Each particle is guided along its path by the energy-preserving “gentle drive” of a matter wave having a well-established [16] physical existence. Our approach reaches the double goal of a clear insight into the mechanism of wave-particle duality and of a reasonably simple computability: a necessary condition for any theory of practical interest.

An *experimentally tested* Hamiltonian description of wave-like features holding beyond the eikonal approximation [17] was first contrived in Refs.[18-20] for the propagation of electromagnetic waves in thermonuclear plasmas. This method was applied until recently [21]

to the microwave diagnostics installed on the TORE SUPRA fusion facility, and extended in Ref. [22] to the treatment of the Schrödinger equation. The general demonstration of the equivalence between classical and wave-mechanical Helmholtz-like equations and exact, trajectory-based Hamiltonian systems was given in Refs.[23-27], and led to extensive application both in relativistic electrodynamics and in the analysis of experimental arrangements and devices employed for light transmission and guiding [28-30]. Let us finally remind that the equivalence between the time-dependent Schrödinger equation and Hamiltonian mechanics was demonstrated in Ref.[31] in terms of symplectic geometry, without giving rise, so far, to numerical applications.

We derive here the *exact* Hamiltonian systems concerning, respectively, *classical* electromagnetic waves (Sect.2), *non-relativistic* matter waves (Sect.3) and *relativistic* matter waves (Sect.4). Examples of numerical computation are given in Sect.5, and a comparison is made in Sects.6 and 7 with the hydrodynamic approach of the Bohmian theory.

2. - The case of classical waves

We shall refer in the present Section, in order to fix ideas, to a *stationary*, isotropic and (generally) inhomogeneous dielectric medium, sustaining a classical *monochromatic* electromagnetic wave whose electric and/or magnetic field components are represented in the form

$$\psi(\vec{r}, \omega, t) = u(\vec{r}, \omega) e^{-i\omega t}, \quad (1)$$

where $u(\vec{r}, \omega)$ is assumed to satisfy the Helmholtz equation

$$\nabla^2 u + (n k_0)^2 u = 0, \quad (2)$$

with

$$k_0 \equiv \frac{2\pi}{\lambda_0} = \frac{\omega}{c} \quad (3)$$

and with a (time-independent) refractive index $n \equiv n(\vec{r}, \omega)$. The time-independence of eq.(2) does NOT mean, of course, that no physical change is expected. Just as for the usual laws of Dynamics, once suitable boundary and initial conditions are assigned the ensuing motion, occurring in stationary *external* conditions, is exactly described. If we perform in fact, into eq.(2), the quite general replacement

$$u(\vec{r}, \omega) = R(\vec{r}, \omega) e^{i\varphi(\vec{r}, \omega)}, \quad (4)$$

with real amplitude $R(\vec{r}, \omega)$ and real phase $\varphi(\vec{r}, \omega)$, and separate real from imaginary parts, eq.(2) is seen to provide [23] a *kinematical* “ray-trajectory” system of the Hamiltonian form

$$\left\{ \begin{array}{l} \frac{d\vec{r}}{dt} = \frac{\partial D}{\partial \vec{k}} = \frac{c\vec{k}}{k_0} \\ \frac{d\vec{k}}{dt} = -\frac{\partial D}{\partial \vec{r}} = \vec{\nabla} \left[\frac{ck_0}{2} n^2(\vec{r}, \omega) - W(\vec{r}, \omega) \right] \\ \vec{\nabla} \cdot (R^2 \vec{k}) = 0 \\ k(t=0) \equiv k_0 = \omega / c \end{array} \right. \quad (5)$$

where

$$\vec{k} = \vec{\nabla} \varphi, \quad (6)$$

$$D(\vec{r}, \vec{k}, \omega) \equiv \frac{c}{2k_0} [k^2 - (n k_0)^2 - \frac{\nabla^2 R}{R}], \quad (7)$$

$$W(\vec{r}, \omega) \equiv - \frac{c}{2k_0} \frac{\nabla^2 R(\vec{r}, \omega)}{R(\vec{r}, \omega)}, \quad (8)$$

and a "ray" velocity $\vec{v}_{ray} = c \vec{k} / k_0$ is implicitly defined. It is easily seen that, as long as $k \equiv |\vec{k}| = k_0$, we'll have $v_{ray} \equiv |\vec{v}_{ray}| = c$. The function $W(\vec{r}, \omega)$ (which we call "Wave Potential"), represents a newly discovered intrinsic property encoded in any *Helmholtz-like equation*. It is seen to couple together the whole set of ray-trajectories, causing (thanks to its frequency dependence) any diffraction and/or interference process. Its gradient $\vec{\nabla} W$ turns out to be orthogonal to $\vec{k} \equiv \vec{\nabla} \varphi$, thus modifying the direction, but not the amplitude, of the wave vector \vec{k} itself. The time-integration of the Hamiltonian system (5) requires the full knowledge of the amplitude $R(\vec{r}, \omega)$ on an assigned starting wave-front, together with the ray positions $\vec{r}(t=0)$ and of the corresponding wave vectors $\vec{k}(t=0)$, orthogonal to the wave-front. Thanks to the third of eqs.(5) (expressing the constancy of the flux of $R^2 \vec{\nabla} \varphi$ along any tube formed by the field lines of the wave vector field $\vec{k} = \vec{\nabla} \varphi$, i.e. by the "ray-trajectories" themselves) one obtains, then, the amplitude $R(\vec{r}, \omega)$ over the next wave-front. The iteration of this procedure allows to build up, step by step, *both* the "Helmholtz trajectories" along which the "rays" are channeled *and* the time-table of the "ray motion" along them. We shall see at the end of Sect.4 what the "rays" are conveying.

When, in particular, the space variation length L of the amplitude $R(\vec{r}, \omega)$ satisfies the condition $k_0 L \gg 1$, the Wave Potential term $W(\vec{r}, \omega)$ may be dropped, thus removing any ray-coupling. The "rays" will therefore propagate independently of each other, without any diffraction and/or interference, according to the *eikonal equation* [17]

$$k^2 \equiv (n k_0)^2. \quad (9)$$

3. -The case of non relativistic de Broglie's matter waves

Let us pass now to the case of non-interacting particles of mass m and total energy E , launched with an initial momentum \vec{p}_0 (with $p_0 = \sqrt{2 m E}$) into an external force field deriving from a *time-independent* potential energy $V(\vec{r})$. The *classical* dynamical behavior of each particle is described, as is well known [17], by the time-independent Hamilton-Jacobi (HJ) equation

$$(\vec{\nabla} S)^2 = 2m[E - V(\vec{r})], \quad (10)$$

where the basic property of the HJ function $S(\vec{r}, E)$ is that the particle momentum is given by

$$\vec{p} = \vec{\nabla} S(\vec{r}, E). \quad (11)$$

The *classical* HJ surfaces $S(\vec{r}, E) = const$, perpendicular to the momentum of the moving particles, *pilot* them along stationary trajectories, according to the laws of Classical Mechanics. The origin of Wave Mechanics was due to considerations based on the analogy between the Maupertuis and Fermat variational principles in physical space [17], inducing Louis de Broglie [10, 11] to associate (in a single physical entity) each *moving* particle with a "matter wave" of the form, suggested by eqs.(1) and (4),

$$\psi = u(\vec{r}, \omega) e^{-i \omega t} \equiv R(\vec{r}, \omega) e^{i[\varphi(\vec{r}, \omega) - \omega t]} \quad (12)$$

under Planck's condition

$$\omega = E / \hbar, \quad (13)$$

according to the *basic conjecture*

$$\vec{p} = \hbar \vec{k}, \quad (14)$$

laying the *very foundations of Wave Mechanics* thanks to its merging of Maupertuis's and Fermat's principles. We have therefore, thanks to eqs.(6) and (11)-(14), the relations $\varphi = S(\vec{r}, E)/\hbar$ and

$$u(\vec{r}, E) \equiv R(\vec{r}, E) e^{\frac{i}{\hbar} S(\vec{r}, E)}, \quad (15)$$

showing that the HJ surfaces $S(\vec{r}, E) = \text{const}$ represent the phase-fronts of matter waves, while maintaining the *piloting* role played in the classical eq.(11). Eq.(14) provides both the structure (15) and the "guidance equation" of de Broglie's matter waves, addressing the particles with momentum \vec{p} along the wave-vector \vec{k} . Point-particles move, in other words, along stationary

trajectories orthogonal to the phase-fronts of matter waves with $\lambda \equiv \frac{2\pi}{k} = 2\pi\hbar/p$. The

successive step was performed by Schrödinger [12-15]), with the suggestion of *viewing the laws of Classical Mechanics* - represented here by eq.(10) - *as the eikonal approximation of a suitable Helmholtz-like equation*. Reminding that eq.(9) provides the eikonal limit of the Helmholtz equation (2), this suggestion simply amounts to performing, into eq.(2), the replacement, making use of eqs.(9)-(14),

$$(n k_0)^2 \rightarrow k^2 \equiv \frac{p^2}{\hbar^2} = \frac{2m}{\hbar^2} (E - V) \quad (16)$$

leading to the Helmholtz-like equation

$$\nabla^2 u(\vec{r}, E) + \frac{2m}{\hbar^2} [E - V(\vec{r})] u(\vec{r}, E) = 0. \quad (17)$$

which is the well-known *time-independent* (and *energy-dependent*) *Schrödinger equation*, holding for stationary matter waves. It's an eigen-value equation admitting in general both continuous and discrete energy and eigen-function spectra, which replace the heuristic prescriptions of the "old" quantum theory [8].

The *real existence of de Broglie's matter waves in physical space* was established in 1927 by the Davisson-Germer experiments [16] on electron diffraction by a nickel crystal.

Having in mind the dynamics of particles with an *assigned total energy, just as it usually occurs in classical physics*, let us now apply to the Helmholtz-like *energy-dependent* equation (17) the same procedure leading from the Helmholtz eq.(2) to the Hamiltonian ray-tracing system (5), by simply replacing eq.(15) into eq.(17) and separating real and imaginary parts. After having defined the energy-dependent "Wave Potential function"

$$W(\vec{r}, E) = -\frac{\hbar^2}{2m} \frac{\nabla^2 R(\vec{r}, E)}{R(\vec{r}, E)}. \quad (18)$$

and the energy function

$$H(\vec{r}, \vec{p}, E) \equiv \frac{p^2}{2m} + W(\vec{r}, E) + V(\vec{r}) \quad (19)$$

we get now the *dynamical* Hamiltonian system

$$\left\{ \begin{array}{l} \frac{d\vec{r}}{dt} = \frac{\partial H}{\partial \vec{p}} \equiv \frac{\vec{p}}{m} \\ \frac{d\vec{p}}{dt} = -\frac{\partial H}{\partial \vec{r}} \equiv -\vec{\nabla} [V(\vec{r}) + W(\vec{r}, E)] \\ \vec{\nabla} \cdot (R^2 \vec{p}) = 0 \\ p(t=0) \equiv p_0 = \sqrt{2mE} \end{array} \right. \quad (20)$$

providing, in strict analogy with the Helmholtz ray-tracing system (5), a stationary set of point-particle trajectories, together with the time-table of the particle motion along them. These trajectories are mutually coupled, once more, by the "Wave Potential" function $W(\vec{r}, E)$ of eq.(18), acting orthogonally to the particle motion and exerting an energy-conserving "gentle drive" which is the cause, thanks to its "monochromaticity", of any diffraction and/or interference process. Notice that this *energy-dependence* makes $W(\vec{r}, E)$ basically different (in spite of its formal analogy) from Bohm's *energy-independent* "Quantum Potential"

$Q(\vec{r}, t) = -\frac{\hbar^2}{2m} \frac{\nabla^2 \psi(\vec{r}, t)}{\psi(\vec{r}, t)}$ [1-7]. The third of eqs. (20) expresses the constancy of the flux of

$R^2 \vec{\nabla} S$ along any tube formed by the field lines of $\vec{p} = \vec{\nabla} S(\vec{r}, E)$, i.e. by the trajectories themselves. In complete analogy with Sect.2 the launching values $\vec{r}(t=0)$ and $\vec{p}(t=0)$ of each particle must be supplemented, in the time-integration of the system (20), with the assignment of the amplitude $R(\vec{r}, E)$ (and its derivatives) over a starting wave-front, orthogonal to $\vec{p}(t=0)$ at each point $\vec{r}(t=0)$. The third of eqs.(20) provides then, step by step, the function $R(\vec{r}, E)$ (with its derivatives) over the next wave-front, and so on. Notice that the Wave Potential (18) depends on the profile of the transverse particle distribution $R(\vec{r}, E)$, but not on its intensity, which may be quite arbitrarily chosen.

4.-The case of relativistic de Broglie's matter waves

Let us finally pass (maintaining the same notations of the previous Sections) to the relativistic dynamics of particles with rest mass m_0 and assigned energy E , launched, as in the previous case, into a force field deriving from a stationary potential energy $V(\vec{r})$, and moving according to the (relativistic) *time-independent* Hamilton-Jacobi equation [8, 9]

$$[\vec{\nabla} S(\vec{r}, E)]^2 = \left[\frac{E - V(\vec{r})}{c} \right]^2 - (m_0 c)^2 \quad . \quad (21)$$

After having repeated de Broglie's logical steps (11)-(14), we shall assume, with Schrödinger, that the relevant matter waves satisfy a *Helmholtz-like* equation of the form (2), reducing to *standard Mechanics* - represented now by eq.(21) - in its *eikonal approximation*, and perform therefore, into eq.(2), the replacement

$$(n k_0)^2 \rightarrow k^2 \equiv \frac{p^2}{\hbar^2} = \left[\frac{E - V(\vec{r})}{\hbar c} \right]^2 - \left(\frac{m_0 c}{\hbar} \right)^2 \quad (22)$$

thus obtaining the *time-independent* Klein-Gordon equation [8, 9]

$$\nabla^2 u + \left[\left(\frac{E - V(\vec{r})}{\hbar c} \right)^2 - \left(\frac{m_0 c}{\hbar} \right)^2 \right] u = 0, \quad (23)$$

holding for de Broglie's relativistic matter waves associated with particles of total energy E . By replacing eqs.(15) into eq.(23) and separating once more real and imaginary parts, eq.(23) leads [26] to the dynamical Hamiltonian system

$$\left\{ \begin{array}{l} \frac{d\vec{r}}{dt} = \frac{\partial H}{\partial \vec{p}} \equiv \frac{c^2 \vec{p}}{E-V(\vec{r})} \\ \frac{d\vec{p}}{dt} = -\frac{\partial H}{\partial \vec{r}} \equiv -\vec{\nabla} V(\vec{r}) - \frac{E}{E-V(\vec{r})} \vec{\nabla} W(\vec{r}, E) \\ \vec{\nabla} \cdot (R^2 \vec{p}) = 0 \\ p(t=0) \equiv p_0 = \sqrt{E/c^2 - (m_0 c)^2} \end{array} \right. \quad (24)$$

where

$$H(\vec{r}, \vec{p}, E) \equiv V(\vec{r}) + \sqrt{(pc)^2 + (m_0 c^2)^2 - \hbar^2 c^2 \frac{\nabla^2 R(\vec{r}, E)}{R(\vec{r}, E)}} \quad (25)$$

and a Wave Potential
$$W(\vec{r}, E) = -\frac{\hbar^2 c^2}{2E} \frac{\nabla^2 R(\vec{r}, E)}{R(\vec{r}, E)} \quad (26)$$

couples and guides, once more, without any wave-particle energy exchange, the particle trajectories, acting *orthogonally* to the particle motion. It is interesting to observe that the first of eqs.(24) turns out to coincide with the "guidance formula" presented by de Broglie in his relativistic "*double solution theory*" [32], and *doesn't coincide* with \vec{p}/m , while maintaining itself parallel to the momentum \vec{p} . Let us notice that, in the particular case of *massless* particles (i.e. for $m_0 = 0$), the Klein-Gordon equation (23), thanks to eq.(13), reduces to the form

$$\nabla^2 u + \left[\frac{\hbar\omega - V(\vec{r})}{\hbar c} \right]^2 u = 0, \quad (27)$$

structurally analogous to eq.(2), which may be viewed therefore as a suitable Klein-Gordon equation holding for *massless* point-particles. Einstein's first historical duality (conceived for the electromagnetic radiation field [33]) is therefore included in de Broglie's wave-particle mechanism.

5. - Numerical examples

We compute in the present Section, for a number of different experimental set-ups, the stationary sets of particle trajectories provided by the integration of the Hamiltonian system (20), which is a direct consequence, as we know, of Schrödinger's energy-dependent eq.(17). The very plausibility of the numerical results plays in favor of the underlying philosophy. No kind of measurement perturbation is taken into account, and the geometry is assumed, for simplicity sake, to allow a computation limited to the (x, z) -plane. Because of the orthogonality between the wave front and the particle momentum we shall make use, over the (x, z) plane, of the identities

$$\left(p_x \frac{\partial}{\partial x} + p_z \frac{\partial}{\partial z} \right) R = 0 \quad (28)$$

$$\nabla^2 R = (p_x / p_z)^2 \partial^2 R / \partial x^2 . \quad (29)$$

We show in **Fig.1** the diffraction of a Gaussian beam of the initial form $R(x; z=0) \propto \exp(-x^2/w_0^2)$, launched from the left hand side along the z -axis, in the absence of external fields (i.e. for $V(x, z) = 0$), with $p_x(t=0) = 0$ and $p_z(t=0) = \sqrt{2mE}$.

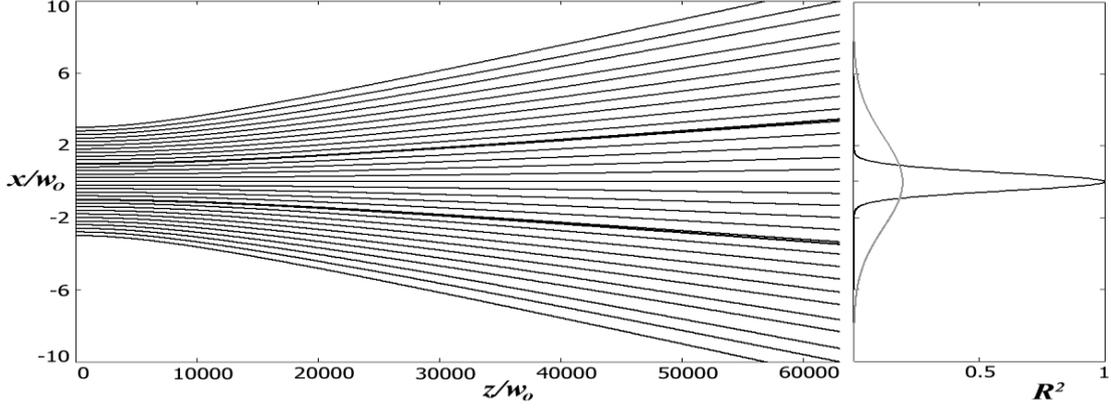


FIG.1. Diffraction of a Gaussian beam.

The numerical computations are performed by assuming a ratio $\lambda_0/w_0 \cong 10^{-4}$ between the beam wavelength λ_0 and "waist" w_0 , and by plotting the figures in terms of the dimensionless coordinates x/w_0 and z/w_0 .

The (axially symmetrical) particle beam travels under the coupling role of the Wave Potential, which induces a "diffractive widening" due to a progressive conversion of p_z into p_x , while respecting the overall conservation of the beam energy and momentum.

We plotted on the right-hand side of **Fig.1** the initial and final transverse intensity distributions R^2 of the beam, and on the left-hand side the relevant trajectory pattern. The two heavy lines represent the trajectories starting (at $z/w_0 = 0$) from the so-called "waist" positions $x/w_0 = \pm 1$, whose numerical values turn out to be in excellent agreement with their well-known "paraxial" [34] analytical expression

$$\frac{x}{w_0} = \pm \sqrt{1 + \left(\frac{\lambda_0 z}{\pi w_0^2} \right)^2}. \quad (30)$$

Fig.2 refers, in its turn, to the diffraction/interference of two neighbouring Gaussians beams, symmetrical with respect to the axis $x/w_0 = 0$. We plotted on the right-hand side the initial and final transverse intensity profiles of the beams, and, on the left-hand side, the relevant trajectory pattern.

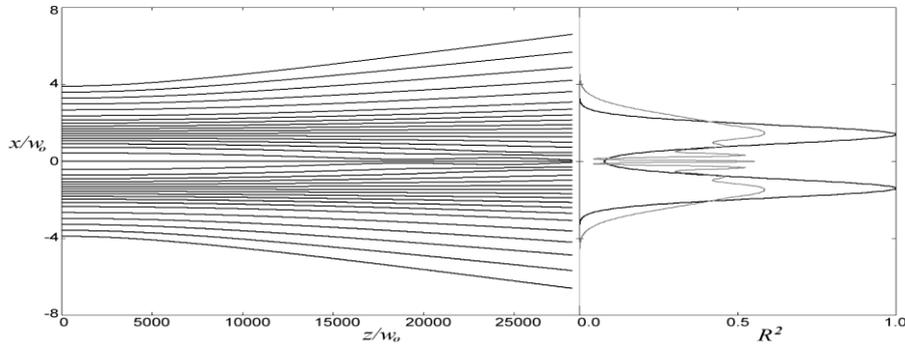


FIG.2. The case of two neighbouring Gaussians beams symmetric with respect to the axis $x = 0$.

Fig.3 presents, in its turn, the launch, stopping and "backward fall" of a single Gaussian beam of particles with total energy E , traveling under a constant force field $-F_{oz}$ acting in the negative z -direction.

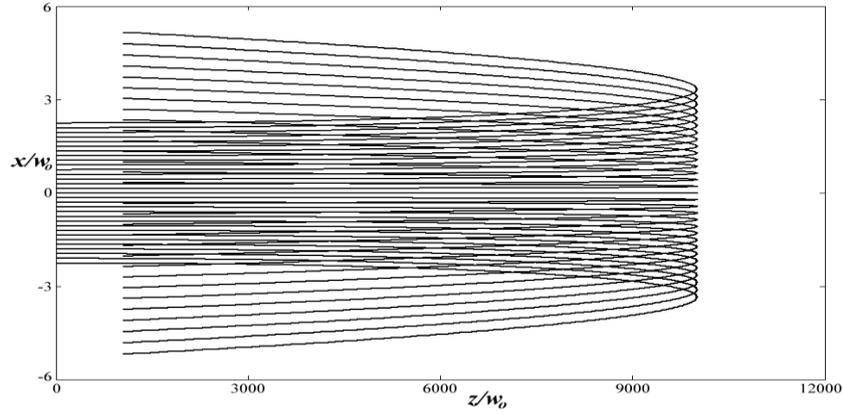


FIG.3. Beam launched into a constant field $-F_{oz}$.

Starting from $z/w_0 = 0$ the beam travels, for a while, in the positive z direction, and is stopped and pushed back, for $z \cong E/F_{oz}$, by the force field, while continuing its diffractive widening due to the Wave Potential. We shall consider now

1) the case of a *potential barrier* of the form

$$V(z) = V_0 \exp[-2(z - z_B)^2 / d^2] \quad (31)$$

(where the parameters z_B and d determine the position of the *peak* and of the distance between the flexes, respectively), and

2) the case of a "step-like" potential of the form

$$V(z) = V_0 \left\{ 1 + \exp\left[-\alpha \frac{z - z_S}{w_0}\right] \right\}^{-1} \quad (32)$$

where α and z_S determine, respectively, the slope and the *flex* position of the continuous line connecting the two asymptotic levels where $V(z \rightarrow -\infty) = 0$ and $V(z \rightarrow \infty) = V_0$.

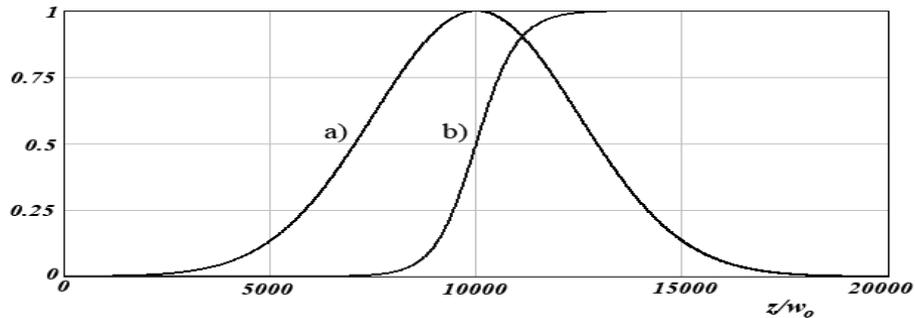


FIG.4. Profiles $V(z)/V_0$ of (a) the **barrier** and (b) the **step-like** potentials.

We plot in **Fig.4**, for $z_B/w_0 = z_S/w_0 = 10^4$, the profiles of the respective ratios $V(z)/V_0$, and consider the launch from the left hand side of the Gaussian beam of **Fig.1**, with energy E and waist w_0 , into these external fields.

In the case of the *potential barrier* (31) the beam gradually widens under the diffractive action of the Wave Potential, and is stopped and thrown back when $E = V(z) < V_0$, with a behavior quite similar to the one of **Fig.3**. When $E/V_0 \gg 1$, on the other hand, the beam overcomes the top of the barrier and undergoes a strong acceleration beyond it. We omit in both cases, for brevity sake, the relevant plots, limiting ourselves to the remarkable case $E/V_0 \cong 1$ (**Fig.5**).

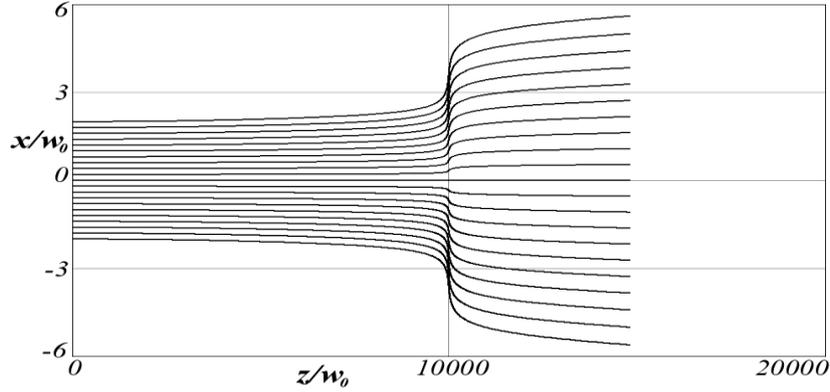


FIG.5. Potential barrier: case $E/V_0 \cong 1$

Here, in a narrow region around the peak position $z = z_B \cong 10^4 w_0$, both the external force $F_z(z)$ and p_z are very close to zero. The beam motion basically occurs, under the action of its Wave Potential, along the x -axis, and is *evanescent* in the z -direction. After this brief interlude, the beam is strongly accelerated along z for $z > z_B$.

In the case of the *step-like potential* (32), the beam gradually widens under the diffractive action of the Wave Potential, and is stopped and thrown back, once more, for $E = V(z) < V_0$, with a behavior similar to the one of **Fig.3**. When $E/V_0 > 1$, on the other hand, the beam travels forward, under the diffractive widening action of the Wave Potential, with an energy reduced by the overcoming of the potential step. We omit, in both cases, the relevant plots, limiting once more ourselves to the remarkable case $E/V_0 \cong 1$ (**Fig.6**). When $V(z)$ is close to V_0 both the external force $F_z(z)$ and p_z are close to zero. The beam basically moves, under the action of the Wave Potential alone, along the x -axis, while its forward motion becomes, *and remains*, evanescent.

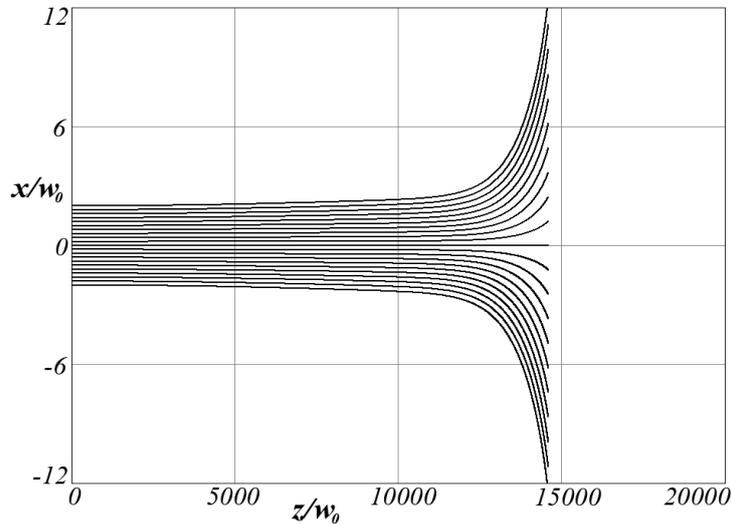


FIG.6. Step-like potential: case $E/V_0 \cong 1$

As a further applicative example, let us come to a collimated matter wave beam launched into a potential field $V(x, z)$ representing a *lens-like* focalizing structure, of which we omit here, for simplicity sake, the analytical expression (see, for instance, Refs.[18] and [22]). **Fig.7** and **Fig.8** present, in this case, the results obtained by neglecting, respectively, and by taking into account, in eqs.(25), the Wave Potential term $W(\vec{r}, E)$. Its diffractive effect is seen to replace the *point-like* focus of geometrical optics by a *finite focal waist*, strictly reminding the case of the Luneburg lenses considered in Refs.[29, 30].

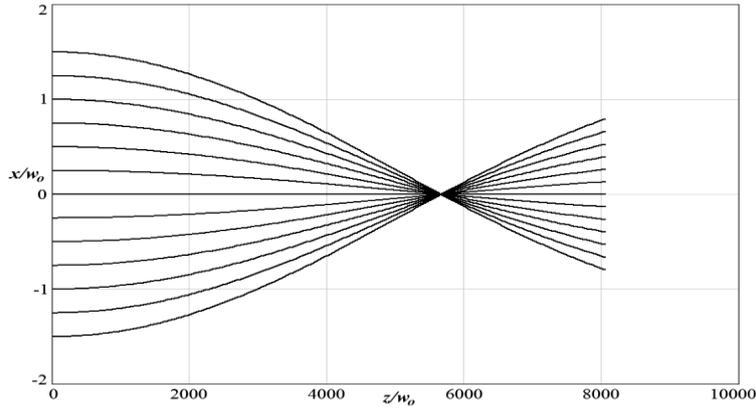


FIG.7. Lens-like potential: point-like focusing of a collimated beam in the absence of Wave Potential.

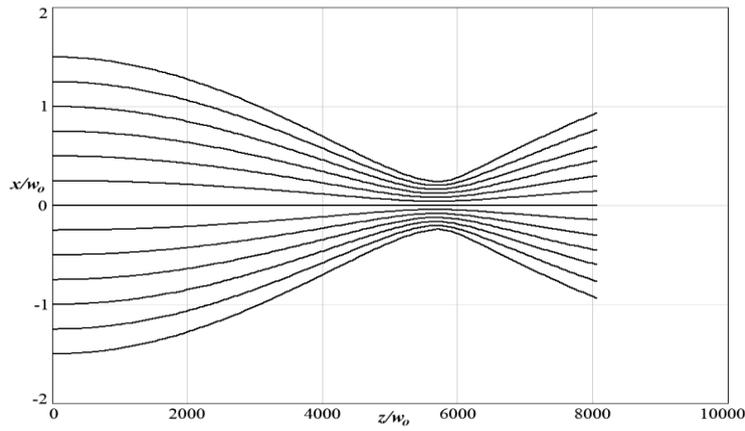


FIG.8. Lens-like potential: finite-focusing of a collimated beam in the presence of Wave Potential

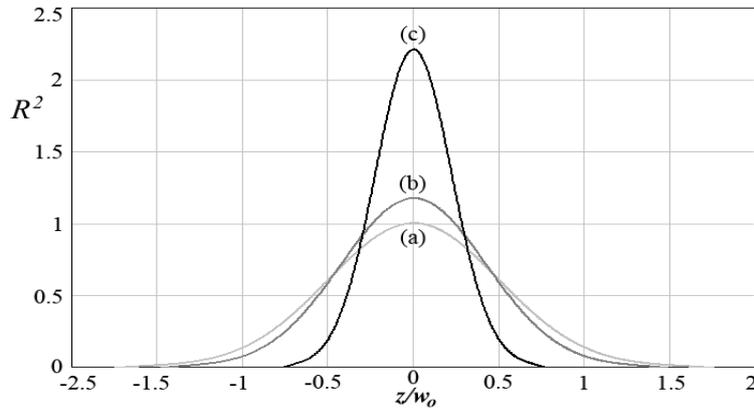


FIG.9. Lens-like potential: intensity-sharpening of a finite-focused beam in the presence of Wave Potential.

Fig.9 shows, in its turn, the progressive sharpening of the beam intensity while reaching its finite waist.

A final application of eqs.(20), clearly showing the diffractive role of the Wave Potential, is obtained by applying to a collimated Gaussian beam, launched along the z -direction, a potential of the form

$$V(z) = m \omega^2 z^2 / 2 \quad , \quad (33)$$

suggested by the classical case of particles harmonically oscillating around $z=0$, with a period $T=2\pi/\omega$, under an elastic force $-m\omega^2 z$. The particles moving along the beam axis are standard linear oscillators [8, 9] with well-known quantized energies

$$E_n = (n+1/2)\hbar\omega, \quad (34)$$

which we assume to be shared by the whole beam. We show in **Figs.10** and **11**, respectively, the launch of the beam from $z=0$ and its successive oscillations around such a position. The beam is seen to oscillate between the positions

$$z_{\pm} \cong \pm \sqrt{2E_n/m\omega^2}, \quad (35)$$

while progressively widening under the diffractive action of the Wave Potential.

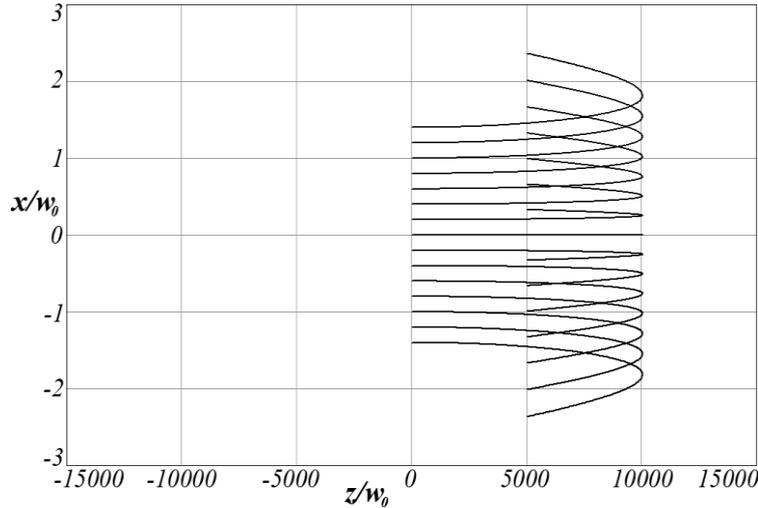


Fig.10. Harmonic potential: launching of the beam.

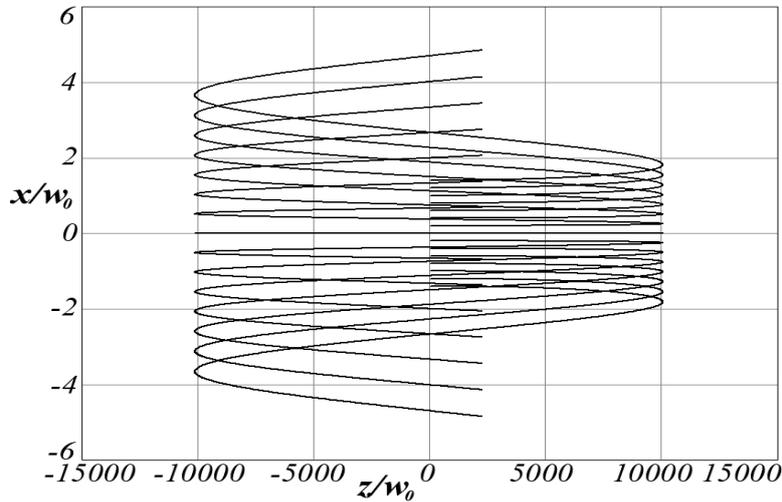


Fig.11. Harmonic potential: oscillations and diffractive widening of the beam.

6. – Born's Wave Function and Bohm's approach

Even though, as we have seen, the wave-mechanical dynamics *in stationary external fields* is already adequately described by the *energy-dependent* Schrödinger equation (17), one more interesting equation may be obtained from eq.(17) itself, making use of (12) and (13), in the form

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(\vec{r}) \psi \quad (36)$$

which is the standard *energy-independent* Schrödinger equation. Referring now, in order to fix ideas, to a discrete energy spectrum of eq.(17), and defining both the eigen-frequencies $\omega_n \equiv E_n / \hbar$ and the eigen-functions

$$\psi_n(\vec{r}, t) = u_n(\vec{r}) e^{-i\omega_n t} \equiv u_n(\vec{r}) e^{-iE_n t / \hbar}, \quad (37)$$

any linear superposition (with arbitrary constant coefficients c_n) of the form

$$\psi(\vec{r}, t) = \sum_n c_n \psi_n(\vec{r}, t), \quad (38)$$

turns out to be, as is well known [8, 9], a (deterministically evolving) solution of eq.(36). Such a function is a weighted average over the whole set of eigen-functions $\psi_n(\vec{r}, t)$, where the coefficients c_n (in duly normalized form) may represent either a set of experimental results (in view of a statistical treatment) or an *ad hoc* mathematical assembling, in view of the construction of a particular “packet” of wave-trains. Each eigen-function is acted on by its own, energy preserving, Wave Potential $W(\vec{r}, E_n)$, and has therefore its own trajectories, provided by eqs. (20). In **Born's** words [35], indeed, “*it's very attractive to interpret a particle of matter as a wave-packet due to the superposition of a number of wave trains. But this tentative interpretation comes up against insurmountable difficulties, since a wave packet of this kind is in general very soon dissipated*”.

Born himself proposed however, for the function (38), an interpretation [36] going much beyond a simple superposition, assuming it to represent the *most complete possible description* of the physical state of a particle whose energy is not determined, in the form of a simultaneous permanence (before observation) in its full set of eigenstates, according to the (duly normalized) probabilities $|c_n|^2$. Since eq.(36) is not, in itself, an ordinary-looking wave equation, its solution (38) is not, in itself, a wave. Such a mathematical object was called “**Born Wave-Function**”, and became the pivot of Standard Quantum Mechanics (**SQM**), an intrinsically probabilistic theory which became the pillar of any further development of microphysics.

We find it useful to remind here a well known comment written by Jaynes [37] in 1989: “*Our present quantum mechanical formalism is an omelette that nobody has seen how to unscramble, and whose unscrambling is a prerequisite for any future advance in basic physical theory*”. The pre-eminent role and the probabilistic nature assigned to the *energy-independent* Schrödinger equation (36) -scrambling all eigen-energies together - suggested, indeed, to extend the same nature to the *energy-dependent* eq.(17), thus compromising the specific *guiding role* of the *energy-dependent* matter waves (14, 15) conceived by de Broglie and experimentally identified by Davisson and Germer.

An attempt to rescue this guiding role, while maintaining the properties of **Born's** Wave-Function, was performed in **Bohm's** approach [2, 3], where $\psi(\vec{r}, t)$ was written in the form

$$\psi(\vec{r}, t) \equiv \sum_n c_n \psi_n(\vec{r}, t) = R(\vec{r}, t) e^{iG(\vec{r}, t) / \hbar} \quad (39)$$

with real $R(\vec{r}, t)$ and $G(\vec{r}, t)$. The function $R^2 \equiv |\psi|^2 \equiv \psi \psi^*$, in agreement with the SQM interpretation, was assumed to represent, in Bohm's words [2], “*the probability density for particles belonging to a statistical ensemble*”. We made use here of the term $G(\vec{r}, t)$, instead of Bohm's original term $S(\vec{r}, t)$, in order to avoid its instinctive *a priori* identification with the Hamilton-Jacobi function of eqs.(14) and (15). **Bohm's** expression (39) was “shaped”, in fact, on eq.(15) (i.e. on de Broglie's definite-energy matter waves, whose physical *reality* was established by the Davisson-Germer experiments) with the aim of depicting **Born's** “Wave-Function” as a *single pilot wave*, hopefully sharing and generalizing the nature and experimental evidence of de Broglie's guiding waves.

Since however, as we said before, the function $\psi(\vec{r},t)$ is not, in itself, the solution of a usual wave equation, let us look for its effective nature and relevance. By applying simple analysis to eq.(39) we get

$$\vec{\nabla} G(\vec{r},t)/m \equiv \frac{\hbar}{mi} \text{Im} \left(\frac{\vec{\nabla} \psi}{\psi} \right) \equiv \frac{\hbar}{2mi} \frac{\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*}{\psi \psi^*} . \quad (40)$$

Reminding therefore that the quantity $\vec{J} \equiv \frac{\hbar}{2mi} (\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*)$ represents, in SQM, a *probability current density* [8, 9], the term

$$\vec{\nabla} G(\vec{r},t)/m \equiv J/R^2 \quad (41)$$

is seen to coincide with the velocity $\vec{v}_{prob}(\vec{r},t)$ at which “*the probability density is transported*” [5], in agreement *both* with Bohm's probabilistic assumption on the function R^2 and with the fluid-like equations (which we omit here for simplicity) obtained by Bohm [2] by replacing eq.(39) into eq.(36) and separating real from imaginary parts.

7.- Discussion

The basic equations of the *hydrodynamic* Bohmian theory, together with the ones of our *dynamical* system (20), may be summarized, respectively, in the following **Tables I** and **II**. No time dependence is taken into account in the external potential $V(\vec{r})$, since it wouldn't be strictly justified [38]. We remind however that, according to de Broglie [32], “*it is plausible to admit that, when V depends on time, the form of the equations be preserved as the general form of wave propagation in the Wave Mechanics of a single particle*”.

TAB. I. Bohmian hydrodynamic description	TAB. II. Dynamic point-particle description
$\frac{d\vec{r}}{dt} = \frac{\hbar}{2mi} \frac{\psi^* \vec{\nabla} \psi - \psi \vec{\nabla} \psi^*}{ \psi ^2}$ $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V(\vec{r}) \psi$	$\frac{d\vec{r}}{dt} = \frac{\vec{p}}{m}$ $\frac{d\vec{p}}{dt} = -\vec{\nabla} [V(\vec{r}) + W(\vec{r}, E)]$ $\vec{\nabla} \cdot (R^2 \vec{p}) = 0$

The first equation of TAB.I (to be time-integrated in parallel with the second one) is based on the *bold equalization* of the velocity of a (hopefully and permanently) narrow wave-packet centered at \vec{r} with the *fluid-like* velocity $\vec{v}_{prob}(\vec{r},t)$, providing a (basically) hydrodynamic description which allows, to a certain extent, a pictorial visualization of SQM. This equalization represents however a handmade, external intervention mimicking, in agreement with our comment on eq.(39), the role of a “*guidance velocity*” *à la de Broglie*.

The equations of TAB.I are quite easily written, but not as much easily solved, and may give rise, in general, to analytical and computational troubles [4, 5]. We remind, in this connection, that the original formulation of the Bohmian theory involved an *energy-independent* “Quantum

Potential" function $Q(\vec{r}, t) = -\frac{\hbar^2}{2m} \frac{\nabla^2 \psi(\vec{r}, t)}{\psi(\vec{r}, t)}$, formally similar to (but utterly different from) our

"Wave Potential" function $W(\vec{r}, E)$, whose energy-dependence makes it apt to describe diffraction and/or interference processes. The exploitation of Bohm's Quantum Potential was faced, for instance, by means of iterative solutions leading to complex quantum trajectories [39], but didn't lead in general to easily practicable numerical procedures. The equation set of TAB.I, bypassing the use of $Q(\vec{r}, t)$, represents therefore a more tractable approach.

The Hamiltonian system of TAB.II, on the other hand, requiring no simultaneous solution of a Schrödinger equation and no bold conception, concerns the exact *dynamical* trajectories along which point-particles with an assigned total energy E are driven by the dual mechanism of de Broglie's matter waves. Its numerical treatment - including $W(\vec{r}, E)$ - is *not substantially less manageable than its classical counterpart*, as is shown by the examples of Sect.5. Generalizing Kerner's comment [40] on the quantum solutions holding in the particular case of forced and/or damped oscillators, we may say that the particles, under the action of the Wave Potential, "*dance a wave-mechanical dance around their classical motion*".

In the light of the emergence of "weak" experimental measurements, made possible by an increasingly powerful technology [41-43], the coarse grain *hydrodynamic* approach of Tab.I and the fine grain *dynamic* approach of Tab.II may work together to give an overview of Bohm's [11, 12] "hidden" variables.

References

1. Benseny, A., Albareda, G., Sanz, A. S., Mompert, J., and Oriols, X., *Applied Bohmian Mechanics*, Eur. Phys. Journal D, **68**:286 (2014).
2. Bohm, D. J., *A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables I*, Phys. Rev. **85**, 166 (1952)
3. Bohm, D. J., *A Suggested Interpretation of the Quantum Theory in Terms of "Hidden" Variables II*, Phys. Rev. **85**, 180 (1952)
4. Holland, P. R., *The Quantum Theory of Motion*, Cambridge University Press (1993)
5. Dürr, D., and Teufel S., *Bohmian Mechanics*, Springer-Verlag (2009).
6. Wyatt, R. E., *Quantum Dynamics with Trajectories. Introduction to Quantum Hydrodynamics*, Springer (2010).
7. A.A. V.V., *Applied Bohmian Mechanics: from Nanoscale Systems to Cosmology*, ed. by X. Oriols and J. Mompert, Pan Stanford Publishing (2012)
8. Persico, E., *Fundamentals of Quantum Mechanics*, Prentice-Hall, Inc. (1950).
9. Messiah, A., *Mécanique Quantique*, Dunod (1959).
10. de Broglie, L., *A Tentative Theory of Light Quanta*, Phil. Mag. **47**, 446 (1924).
11. de Broglie, L., *Recherches sur la théorie des quanta*, Thèse de doctorat (1924); same text on Annales de Physique **3**, 22 (1925).
12. Schrödinger, E., *Quantisierung als Eigenwertproblem I*, Annalen der Physik **79**, 361 (1926).
13. Schrödinger, E., *Quantisierung als Eigenwertproblem II*, Annalen der Physik **79**, 489 (1926).
14. Schrödinger, E., *Quantisierung als Eigenwertproblem III*, Annalen der Physik **80**, 437 (1926).
15. Schrödinger, E., *Quantisierung als Eigenwertproblem IV*, Annalen der Physik **81**, 109 (1926).
16. Davisson, C. J., and Germer, L. H., *The Scattering of Electrons by a Single Crystal of Nickel*, Nature **119**, 558 (1927).
17. Goldstein, H., *Classical Mechanics*, Addison-Welsey (1965)
18. Nowak, S., and Orefice, A., *Quasioptical treatment of electromagnetic Gaussian Beams in inhomogeneous and anisotropic plasmas*, Phys. Fluids **B 5** (7), 1945 (1993).
19. Nowak, S., and Orefice, A., *Three-dimensional propagation and absorption of high-frequency Gaussian beams in magnetoactive plasmas*, Phys. Plasmas **1** (5), 1242 (1994).
20. Nowak, S., and Orefice, A., *Quasi-optical description of reflection, focusing and plasma cut-offs of electron cyclotron Gaussian beams*, J. Plasma Phys. **57**, 349 (1997).
21. Honoré, C., Hennequin, P., Truc, A., and Quéméneur, A., *Quasi-optical Gaussian beam-tracing to evaluate Doppler backscattering conditions*, Nuclear Fusion **46**, S-809 (2006).
22. Orefice, A., and Nowak, S., *Hamiltonian Approach to Electron Diffraction*, Phys. Essays **10**, 364 (1997).

23. Orefice, A., Giovanelli, R., and Ditto, D., *Complete Hamiltonian Description of Wave-Like Features in Classical and Quantum Physics*, Found. Phys. **39**, 256 (2009).
24. Orefice, A., Giovanelli, R., and Ditto, D., *Beyond the Eikonal Approximation in Classical Optics and Quantum Physics*, Chapt.7, pg.425-453 of Ref.7 (2012).
25. Orefice, A., Giovanelli, R., and Ditto, D., *Annales de la Fondation L. de Broglie* **38**, 7 (2013).
26. Orefice, A., Giovanelli, R., and Ditto, D., *From Classical to Wave-Mechanical Dynamics*, *Annales de la Fondation L. de Broglie* **40**, 95 (2015).
27. Orefice, A., Giovanelli, R., and Ditto, D., *Is wave mechanics consistent with classical logic?* Phys. Essays **28**, 515 (2015).
28. Yamaleev, R. M., and Rodriguez-Dominguez, A.R., *As Regards the Speed in a Medium of the Electromagnetic Radiation Field*, J. of Modern Phys., **7**, 1320 (2016).
29. Mattheakis, M. M., Tsironis, G. P. and Kovanis, V.I., *Lunenburg lens waveguide networks*, J. Opt. **14**, 114006 (2012).
30. Mattheakis, M. M. and Tsironis, G. P., *Extreme waves and branching flows in optical media*, Chapt. 18, pg.425 of the Volume "*Quodons in Mica*", ed. by Archilla, J.F.R., Jiménez, N., Sánchez-Morcillo, V.J., and Garcia-Raffi, L.M., Springer Series in Material Science, Vol.221 (2015).
31. de Gosson, M. A., and Hiley B. J., *Imprints of the Quantum World in Classical Mechanics*, Found. Phys.**41**, 1415 (2011).
32. de Broglie, L., *Une tentative d'interprétation causale et non-linéaire de la Mécanique Ondulatoire*, Gauthier-Villars (1956).
33. Einstein, A., *Über einen die Erzeugung und Verwandlung des Lichtes betreffenden heuristischen Gesichtspunkt*, Ann. der Physik **17**, 132 (1905).
34. Kogelnik, H., and Li,T., *Laser Beams and Resonators*, Applied Optics **5**, 1550 (1966).
35. Born, M., *Atomic Physics* (Blackie & Son Ltd, London (1935).
36. Born, M., *Quantenmechanik der Stoßvorgänge*, Zeitschrift für Physik, **38**, 803 (1926)
37. Jaynes, E.T., *Probability in Quantum Theory*, Proc. Workshop on *Complexity, Entropy and the Physics of Information*, Santa Fe, New Mexico (1989)
38. Schiff, L. I., *Quantum Mechanics*, Mc Graw-Hill (1952).
39. Goldfarb, Y., Degani, I., and Tannor, D. J., *Bohmian Mechanics with Complex Action: a New Trajectory-Based Formulation of Quantum Mechanics*, J. Chem. Phys. **125**, 231103 (2006).
40. Kerner, E. H., *Note On The Forced And Damped Oscillator In Quantum Mechanics*, Can. J. Phys. **36**, 371 (1958).
41. Hosten, O., and Kwiat, P., *Observation of the Spin Hall Effect of Light via Weak Measurements*, Science **319**, 787 (2008).
42. Steinberg, A. M., *Quantum measurement: a light touch*, Nature **463**, 890 (2010).
43. Kocsis, S., Braverman, B., Ravets, S., Stevens, M. J., Mirin, R. P., Shalm, L. K., and Steinberg, A. M., *Observing the Average Trajectories of Single Photons in a Two-Slit Interferometer*, Science **332**, 1170 (2011).