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Rich Families of Projections and Retractions

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Title: Rich Families of Projections and Retractions

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Abstract: We deal with problems on non-separable Banach spaces and non-metrizable compact spaces. In particular these problems concern Banach spaces with a projectional skeleton and compact spaces with a retractional skeleton. A projectional (resp. retractional) skeleton is a family of continuous projections (resp. retractions) on a Banach (resp. compact) space, which satisfies certain compatibility properties. Banach spaces with projectional skeleton and compact spaces with retractional skeleton can be viewed as non-commutative version of Plichko Banach spaces and Valdivia compact spaces respectively.
The thesis is split into three chapters. Each chapter consists of a submitted/published paper concerning different problems in this area.
In the first chapter, *On the class of continuous images of non-commutative Valdivia compacta*, we investigate the stability of some topological properties in the class of weakly non-commutative Valdivia compacta (i.e. the class of spaces that are image of a non-commutative Valdivia compact space). We deal, among others, with arbitrary products, $[0, \eta)$-sums, Aleksandrov duplication.
In the second chapter, *New examples of non-commutative Valdivia compact spaces*, we characterize compact trees with a retractional skeleton. This characterization answers in the negative the following question:

Let X be a non-commutative Valdivia compact space that does not contain any copy of the ordinal space $[0, \omega_2]$. Is X necessarily Valdivia?

In the third chapter, *On compact trees with the coarse wedge topology*, we investigate in more detail the class of compact trees. We study the properties of Radon measures on compact trees, proving that each tree has the property (M). We characterize compact trees to be Valdivia and finally we prove that $C(T)$, the space of continuous functions on a compact tree T, is Plichko whenever T has height less than $\omega_1 \cdot \omega_0$.

Keywords: retractional skeleton, projectional skeleton, Valdivia compacta, Plichko spaces, tree.

Název práce: Bohaté systémy projekcí a retrakcí

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Abstrakt: V této disertaci se zabýváme neseparabilními Banachovými prostory a nemetrizovatelnými kompaktními prostory. Zkoumané problémy se týkají zejména Banachových prostorů s projekčním skeletonem a kompaktních prostorů s retrakčním skeletonem. Projekční (resp. retrakční) skeleton je systém spojitých projekcí (resp. retrakcí) na Banachově (resp. kompaktním) prostoru, který splňuje jisté podmínky kompatibility. Na tyto třídy lze nahlížet jako na nekomutativní verze Pličkových Banachových prostorů a Valdiviových kompaktních prostorů.
Disertace je rozdělena do tří kapitol. Každá z kapitol obsahuje jeden článek (publikovaný či zaslaný k publikaci) týkající se různých problémů z této oblasti.
V prvním článku *On the class of continuous images of non-commutative Valdivia compacta* zkoumáme stabilitu některých topologických vlastností v rámci třídy slabých nekomutativních Valdiviových kompaktů (tj. třídy prostorů, které jsou spojitým obrazem nekomutativního Valdiviova kompaktu). Zabýváme se, mimo jiné, libovolnými součiny, $[0,\eta)$-sumami či Aleksandrovovými zdvojeními.
V druhém článku *New examples of non-commutative Valdivia compact spaces* charakterizujeme kompaktní stromy s retrakčním skeletonem. Tato charakterizace dává negativní odpověď na otázku:

Nechť X je nekomutativní Valdiviův kompakt, který neobsahuje kopii prostoru $[0,\omega_2]$. Je X Valdiviův?

Ve třetím článku *On compact trees with the coarse wedge topology* důkladněji zkoumáme třídu kompaktních stromů. Studujeme vlastnosti Radonových měr na kompaktních stromech, ukazujeme, že každý strom má vlastnost (M). Dále charakterizujeme Valdiviovy kompaktní stromy a dokazujeme, že $C(T)$, prostor spojitých funkcí na kompaktním stromě T, je Pličkův, pokud T má výšku menší než $\omega_1 \cdot \omega_0$.

Klíčová slova: retrakční skeleton, projekční skeleton, Valdiviuv kompakt, Pličkuv prostor, strom.

Titolo: Rich Families of Projections and Retractions

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Abstract: Nel presente elaborato tratteremo di problemi su spazi di Banach non-separabili e spazi compatti non metrizzabili. In particolare questi problemi riguardano spazi di Banach con una projectional skeleton e spazi compatti con una retractional skeleton. Una projectional (risp. retractional) skeleton è una famiglia di proiezioni (risp. retrazioni) continue su uno spazio di Banach (risp. compatto), che soddisfa alcune proprietà di compatibilità. Gli spazi di Banach con projectional skeleton e i gli spazi compatti con retractional skeleton possono essere visti come versioni non commutative, rispettivamente degli spazi di Banach di Plichko e degli spazi compatti di Valdivia.
La tesi si sviluppa in tre capitoli, ciascuno dei quali presenta un articolo di ricerca riguardante problemi nell'ambito di cui sopra.
Nel primo capitolo, *On the class of continuous images of non-commutative Valdivia compacta*, si è studiata la stabilità di alcune proprietà topologiche nella classe di compatti debolmente Valdivia non commutativi (i.e. la classe degli spazi che sono immagine continua di uno spazio compatto di Valdivia non commutativo). In particolare si è trattato di prodotti arbitrari, $[0,\eta)$-somme e duplicazione di Aleksandrov.
Nel secondo capitolo, *New examples of non-commutative Valdivia compact spaces*, viene presentata una caratterizzazione per alberi compatti con una retractional skeleton. Questa caratterizzazione ha portato ad una risposta negativa alla domanda:

Sia X un compatto di Valdivia non commutativo che non contenga nessuna copia dello spazio $[0,\omega_2]$. X è necessariamente di Valdivia?

Nel terzo capitolo, *On compact tree with the coarse wedge topology*, si è studiato in maggior dettaglio la classe degli alberi compatti. Si è dimostrato che ogni albero compatto ha la proprietà (M), è stata data una caratterizzazione degli alberi compatti di Valdivia e, infine, si è provato che $C(T)$, lo spazio delle funzioni continue su un albero compatto T, è di Plichko se T ha altezza minore di $\omega_1 \cdot \omega_0$.

Parole chiave: retractional skeleton, projectional skeleton, Valdivia compacta, Plichko spaces, tree.

Contents

Chapter 1

Introduction

In the theory of Banach spaces, non-separable spaces turn out to be strictly related to non-metrizable compact spaces. Given a Banach space, its dual unit ball (endowed with the weak star topology) is compact by the Banach-Alaoglu Theorem. Moreover, such ball is metrizable if and only if the Banach space is separable. These are just two reasons why to combine the study of a class of non-separable Banach spaces with the related class of compact spaces. Separable Banach spaces (as well as the class of metrizable compact spaces) share many interesting topological and structural properties, e.g. having a locally uniformly rotund (LUR) renorming (Kadec, see [12]), having a Markuševič basis (M-basis) [26], in fact, as proved Terenzi years later [32], having a strong M-basis. On the other hand a non-separable Banach space usually does not share the same properties. This pushed academics to define different classes of non-separable Banach spaces and to investigate the inner relations with the related class of compacta.

The first two classes that were studied systematically are the class of Weakly Compactly Generated spaces (WCG) and the corresponding class of compact spaces, the Eberlein compacta [2], [24], [25], [5]. A compact space is called Eberlein if it is homeomorphic to a weakly compact subset of a Banach space, while a Banach space is WCG if it is equal to the closure of the linear span of some its weakly compact subsets. It is proved in [2] that given a compact space K, the space of continuous functions $C(K)$, equipped with the supremum norm, is WCG if and only if K is an Eberlein compactum. Furthermore it was proved that, a compact space is Eberlein if and only if it is homeomorphic to a weakly compact subset of $c_0(\Gamma)$, for some set Γ. All these results were proved using a new tool, introduced in [24] and [25]: Projectional Resolution of the Identity (PRI). If a Banach space, with density character equals to κ, admits a PRI, roughly speaking, it is possible to decompose the space in an increasing sequence of κ-many subspaces with smaller density character.

On the other hand, from the topological side, Σ-products of unit intervals and the class of Corson compacta contribute to the study of non-metrizable spaces, to cite just few works [7], [4] [13], [14]. We here briefly recall that a Corson compact space is a compact space homeomorphic to a subset of some Σ-product. Clearly, any metrizable compact space is Corson. Moreover, by [2], each Eberlein compact space is also Corson. It is easy to find an example of Eberlein compact space that is not metrizable (e.g. the unit ball of a nonseparable Hilbert space, equipped with the weak topology or the Alexandroff compactification of an uncountable set, endowed with the discrete topology). However, it is not that easy to find an example of Corson compact space that is not Eberlein. An example can be found in [1] by using adequate family introduced by Talagrand [31]. Another example, more relevant to the purpose of present work, is due to S. Todorčević: in [33] he used the path topology on trees in order to give an example of first countable Corson compact space with no dense metrizable subspace. It turns out that such example is not Eberlein [15].

At the begining of the nineties, M. Valdivia [34], [35] started the study of those compact spaces that admit an embedding into a product $\mathbb{R}^\Gamma$, for some set Γ, such that the intersection with the Σ-product is dense. This class of compact spaces has been called Valdivia compacta by R. Deville and G. Godefroy [11]. From that moment on, Valdivia compacta and the related class of Banach spaces, the so called Plichko spaces, have been widely studied see for example [20], [16], [17] and more recently [23], [18], [21], [6]. The class of Plichko spaces properly contains the class of weakly Lindelöf determined spaces (WLD), which was indroduced by S. Argyros and S. Mercourakis in [3]. In the same paper the authors proved that the class of WLD spaces is strictly related to the class of Corson compacta. In fact, they proved that a Banach space X is WLD if and only if its dual unit ball, endowed with the weak star topology, is Corson. Every Plichko space has a renorming such that admits a PRI and moreover the class of 1-Plichko spaces form a $\mathcal{P}$-Class. Therefore each Plichko space has an equivalent LUR renorming and admits a Markuševič basis.

W. Kubiś in 2009 [22], borrowing from model theory the concept of elementry submodels, defined the projectional skeletons that, loosely speaking, translate the convenience of the PRI and the Σ-spaces to a more general setting. In fact, a projectional skeleton on a Banach space is a collection of projections indexed on an up-directed σ-complete partially ordered set that satisfies certain compatibility conditions. Not every Banach has a projectional skeleton (for example ℓ_∞). If a Banach space has a projectional skeleton, then it has an equivalent renorming that admits a PRI. In addition, the induced subspace in the dual space has the same topological properties as the Σ-subspaces. Therefore these projectional skeletons do indeed generalize the concepts of PRI and Σ-subspace. The class of compact spaces related to the class of spaces admitting a projectional skeleton is called non-commutative Valdivia compacta and comprises exactly those compact spaces with a retractional skeleton. By a result in [23], it turns out that a compact space is Valdivia if and only if it admits a commutative retractional skeleton. If a compact space K has a retractional skeleton then $C(K)$ has a 1-projectional skeleton (i.e. each projection of the family has norm 1). It is proven in [23] that the space $[0, \omega_2]$ has a retractional skeleton, hence $C([0, \omega_2])$ has a 1-projectional skeleton. However $[0, \omega_2]$ is not Valdivia [20] and $C([0, \omega_2])$ is not a Plichko space [17]. The class of non-commutative Valdivia compacta does not coincide with the class of Valdivia compacta. The same can be said for Plichko spaces and Banach spaces with projectional skeleton. Relations between a compact space K, the space $C(K)$ and the space of probability on K in this context were studied in [8] and [9]. In [27] a new concept to decompose spaces was introduced. A countably compact space X is said monotonically retractable if for any countable set $A \subset X$ we can assign a retraction r_A and a countable family

$\mathcal{N}(A)$ that satisfy some compatibility conditions. It turns out that for countably compact spaces, being monotonically retractable is equivalent to admitting a retractional skeleton [10]. This provides an alternative approach that avoids the theory of elementary submodels.
The present work is devoted to the study and the development of these last classes of Banach and compact spaces. The thesis is a collection of my accepted and submitted papers related to this field. In particular the thesis contains:

- *On the class of continuous images of non-commutative Valdivia compacta*, published in Topology and its Applications, [29]. This paper forms the chapter 2;
- *New examples of non-commutative Valdivia compact spaces*, published in Fundamenta Mathematicae, [30]. This paper forms the chapter 3;
- *On compact trees with the coarse wedge topology*, submitted for a possible publication [28]. This paper forms the chapter 4.

Chapter 2 is devoted to continuous images of non-commutative Valdivia compacta, called there weakly non-commutative Valdivia compacta. This topic is interesting as the class of non-commutative Valdivia compacta is unstable under continuous images ([8, Example 3.8 (i)] can be obtained as continuous images of $[0, \omega_1]$). In order to study this class it is convenient to define the class of non-commutative Corson countably compact spaces, which contains all the induced subspaces of each non-commutative Valdivia compact space. The class of non-commutative Corson countably compact spaces and the class of weakly non-commutative Corson countably compact spaces and weakly non-commutative Valdivia compacta are stable under several topological properties. We highlight here some of such properties: taking countably closed subspaces, countable products and one-point modification of topological sums in the first two classes (Lemma 2.2.8 and Lemma 2.2.9), and one-point compactification and arbitrary products in weakly non-commutative Valdivia case, Proposition 2.3.6. More subtle and technical are the stability results under $[0, \eta)$-sums and Aleksandrov duplication. In section 4 we deal with the $[0, \eta)$-sums, this topological operation was introduced in [19]. It is a disjoint union of compact spaces indexed on an ordinal and endowed with a topology that mimics the ordinal one. We prove that the three classes of compact spaces mentioned above are stable under $[0, \eta)$-sums (Lemma 2.4.1, Lemma 2.4.2 and Proposition 2.4.5). This tool provides an overview on the relations between ordinal segments and the classes already mentioned (Theorem 2.4.6). In Section 2.5 it is shown that the classes of non-commutative Corson countably compact space and weakly non-commutative Corson countably compact space are closed under Aleksandrov duplication and that the duplication of any compact ordinal has a retractional skeleton, Theorem 2.5.1. Finally the last section is devoted to the study of the classes of Banach spaces related to the classes of topological spaces aforementioned.
In chapter 3 we investigate the class of trees endowed with the coarse wedge topology in the setting of non-commutative Valdivia compacta. This line of reaseach was motivated by the following question:

Let X be a non-commutative Valdivia compact space that does not contain any copy of the ordinal space $[0, \omega_2]$. Is X necessarily Valdivia?

In order to answer this question we give, in Theorem 3.3.1, a characterization of those trees that admit a retractional skeleton. The characterization can be roughly summarized in: a tree is a non-commutative Valdivia compact space if and only if each point of uncountable cofinality has finitely many successors. In the last section, after providing a necessary and a sufficient condition on trees to be Valdivia, we prove that a slight modification of the dyadic tree of height $\omega_1 + 2$, Example 3.4.3, is an example that answer in the negative the question above. This result is proved by combining Theorem 3.3.1 with a Baire-type result for trees Proposition 3.4.2.
In chapter 4 we investigate in more detail the class of compact trees endowed with the coarse wedge topology and the spaces of continuous functions on them. We show that any such tree has property (M), see Theorem 4.3.2. A key tool for the proof of this theorem is Proposition 4.3.1, which deals with the support of continuous measures. We further provide a characterization of Valdivia trees within trees of height strictly less than ω_2, contained in Theorem 4.4.2, which is a substantial improvement of the Theorem 3.4.1 from chapter 3. Finally we prove that the space $C(T)$ of continuous functions on a tree T is Plichko whenever the height of T is strictly less of $\omega_1 \cdot \omega_0$.
Each bibliography is referred to the corresponding chapter.

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Chapter 2

On the class of continuous images non-commutative Valdivia compacta

2.1 Introduction

In order to investigate structural properties of certain topological and Banach spaces, it is often convenient to define special families of retractions on them. For example Amir and Lindenstrauss used projectional resolution of identity (PRI) to characterize Eberlein compact spaces [1]. This line of research continued for a long time exploring relations between some classes of compact spaces and non-separable Banach spaces, for example Corson and Valdivia compact spaces, Weakly Lindelöf determined spaces (WLD) and Plichko spaces. This kind of spaces have been widely studied, we refer to [8] for a survey in these topics.
A compact space K is called Corson if it is homeomorphic to a subset of

$$\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma : \ \mathrm{supp}(x) \text{ is countable}\}$$

for a set Γ. A compact space K is called Valdivia if it is homeomorphic to some $K^{'} \subset \mathbb{R}^\Gamma$ with $K^{'} \cap \Sigma(\Gamma)$ dense in $K^{'}$.
In this work we will use retractional skeletons that yield a generalization of Valdivia and Corson compact spaces. In [11] the authors introduced the definition of retractional skeleton and they proved that a compact space is Valdivia if and only if it has a commutative retractional skeleton. In [2] it is proved that a compact space is Corson if and only if it has a full retractional skeleton. There is a dual formulation of retractional skeleton in Banach space, called projectional skeleton [10]. This definition is strictly related with Plichko spaces and weakly Lindelöf determined spaces (WLD), mentioned above. The paper is organized as follows.
In the remaining part of the introductory section notations and basic notions concerning topology and Banach space theory addressed in this paper are given.
In Section 2 the classes of non-commutative Corson countably compact spaces and weakly non-commutative Corson countably compact spaces are introduced. These notions are the non-commutative counterparts of similar notions introduced in [7]. Moreover several stability properties are studied.
In Section 3 the class of weakly non-commutative Valdivia compact spaces is introduced. Also in this case several stability properties are studied.

In Section 4 the definition of $[0,\eta)$-sum is recalled. Relations between $[0,\eta)$-sum and countably compact spaces are investigated. Results about ordinal spaces are given.
In Section 5 the definition of Aleksandrov duplicates is recalled. Some relations between Aleksandrov duplicates and retractional skeletons are given.
In Section 6 consequences of previous sections are studied in Banach space theory setting.
We denote with ω_0 the set of natural numbers (including 0) with the usual order. Given a set X we denote by $[X]^{\leq\omega_0}$ the family of all countable subsets of X and by $|X|$ the cardinality of the set X. As usual we denote with $\aleph_0$ the smallest infinite cardinal.
All the topological spaces are assumed to be Hausdorff and completely regular. Given a topological space T we denote by $\overline{A}$ the closure of $A \subset T$. We say that $A \subset T$ is countably closed if $\overline{C} \subset A$ for every $C \in [A]^{\leq\omega_0}$. A topological space T is a Fréchet-Urysohn space if for every $A \subset T$ and $x \in \overline{A}$ there is a sequence $\{x_n\}_{n\in\omega_0} \subset A$ such that $x_n \to x$. βT denotes the Čech-Stone compactification of T. We use S^1 to indicate the complex numbers with absolute value equal to one.
As in [2], we will use non-commutative Valdivia compacta to indicate the class of compact spaces with retractional skeleton.
Given a topological compact space K we use $C(K)$ to indicate the space of all real-valued continuous function on K or the space of all complex-valued continuous function on K with the usual norm. Additionally we will use $C(K,\mathbb{R})$ and $C(K,\mathbb{C})$ where we want to differentiate. By the Riesz representation theorem the elements of $C(K)^*$ are considered as measures. $P(K)$ stands for the space of probability measures with the weak*-topology. If $\mu \in C(K)^*$, we denote by $|\mu|$ its total variation. If μ is a non-negative measure, we denote by $\text{supp}\mu$ the support of the measure μ, i.e. the set of those points $x \in K$ such that each neighborhood of x has positive μ-measure. The support of a measure $\mu \in C(K)^*$ coincides with the support of its total variation $|\mu|$.
We shall consider Banach spaces over the field of real or complex numbers (most proofs work simultaneously for both cases, when necessary we will point out explicitly the differences). Given a Banach space X and a subset $A \subset X$ we denote by $\text{span}(A)$ and $\text{conv}(A)$ the linear hull and the convex hull respectively. B_X is the norm-closed unit ball of X (i.e. the set $\{x \in X : \|x\| \leq 1\}$). As usual X^* stands for the (topological) dual space of X. Given $A \subset X$ we denote by $A^\perp = \{x^* \in X^* : x^*(x) = 0, \forall x \in A\}$. A set $D \subset X^*$ is said *r-norming* if

$$\|x\| \leq r \sup\{|x^*(x)| : x^* \in D \cap B_{X^*}\}$$

for every $x \in X$. We say that a set $D \subset X^*$ is norming if it is *r-norming* for some $r \geq 1$.

2.2 Non-commutative Corson countably compact spaces

In this paper we will use retractional skeletons also in countably compact setting. We recall the following definition.

Definition 2.2.1. A retractional skeleton in a countably compact space X is a family of continuous retractions $\{r_s\}_{s\in\Gamma}$, indexed by an up-directed partially ordered set Γ, such that

(*i*) $r_s[X]$ is a metrizable compact space for each $s \in \Gamma$,

(*ii*) if $s, t \in \Gamma$, $s \leq t$ then $r_s = r_t \circ r_s = r_s \circ r_t$,

(*iii*) given $s_0 \leq s_1 \leq \ldots$ in Γ, $t = \sup_{n\in\omega_0} s_n$ exists and $r_t(x) = \lim_{n\to\infty} r_{s_n}(x)$ for every $x \in X$,

(*iv*) for every $x \in X$, $x = \lim_{s\in\Gamma} r_s(x)$.

We say that $D = \bigcup_{s\in\Gamma} r_s[X]$ is the set induced by the retractional skeleton $\{r_s\}_{s\in\Gamma}$ in X.
If $D = X$ we will say that $\{r_s\}_{s\in\Gamma}$ is a *full retractional skeleton* and X is a *non-commmutative Corson countably compact space.*

We recall some useful and well-known results about retractional skeletons.

Theorem 2.2.2. *[10, Theorem 32] Assume D is induced by a retractional skeleton in a compact space K. Then:*

(*i*) *D is dense in K and for every countable set $A \subset D$, $\overline{A}$ is metrizable and contained in D.*

(*ii*) *D is a Fréchet-Urysohn space.*

(*iii*) *D is a normal space and $K = \beta D$.*

In particular we observe that given a retractional skeleton, in a compact space X, its induced space D is countably compact.

Proposition 2.2.3. *[3, Proposition 4.5] Let X be a countably compact space. Then X has a full retractional skeleton if and only if it is induced by a retractional skeleton in βX.*
Moreover, if $\{r_s\}_{s\in\Gamma}$ is a full retractional skeleton in X, then there is a retractional skeleton $\{R_s\}_{s\in\Gamma}$ in βX inducing X such that $R_s \restriction_X = r_s$ for every $s \in \Gamma$.

We observe that non-commutative Corson countably compact spaces are a generalization of Corson countably compact spaces given in [7]. Moreover, let X be a countably compact space, it is a Corson countably compact space if and only if X has a commutative full retractional skeleton. In fact:
($\Rightarrow$) Let $h : X \to [0,1]^\Gamma$ be a continuous injection of X into $\Sigma(\Gamma)$, for some set Γ, then $\overline{h(X)}$ is a Valdivia compact space, hence by [11, Theorem 6.1] it has a commutative retractional skeleton such that its induced subspace is $h(X)$. Hence X has a commutative full retractional skeleton.
($\Leftarrow$) Suppose that X has a commutative full retractional skeleton. Then by Proposition 2.2.3, X is induced by a commutative retractional skeleton on βX. Hence by [11, Theorem 6.1], X is a dense Σ-subspace of βX, hence it is a Corson countably compact space.

Lemma 2.2.4. *[2, Lemma 3.5] Let K be a compact space, $F \subset K$ closed subset and let $D \subset K$ be such that D is induced by a retractional skeleton in K. If $D \cap F$ is dense in F, then $D \cap F$ is induced by a retractional skeleton in F.*

Proposition 2.2.5. *[10, Proposition 31] The class of non-commutative Valdivia compacta is closed under arbitrary products. Moreover if $\{K_n\}_{n\in\omega_0}$ is a countable family of non-commutative Valdivia compact spaces and $D_n \subset K_n$ is an induced subspace for every $n \in \omega_0$, then $D = \prod\limits_{n\in\omega_0} D_n$ is an induced space of $K = \prod\limits_{n\in\omega_0} K_n$.*

Now we give the definition of weakly non-commutative Corson countably compact space, which is a generalization of weakly Corson countably compact space introduced in [7].

Definition 2.2.6. Let X be a countably compact space, we say that it is a *weakly non-commutative Corson countably compact space* if there exists a continuous onto mapping $f : Y \to X$ such that Y is a non-commutative Corson countably compact space.

We now give the definition of the countably compact version of the one-point compactification.

Definition 2.2.7. Let $\{X_\alpha\}_{\alpha\in A}$ be a family of countably compact spaces, we say that $X = (\bigoplus\limits_{\alpha\in A} X_\alpha) \cup \{\infty\}$ is an *one-point countably compact modification of topological sum* if X is countably compact and each X_α is a clopen subset of X.

We observe that the previous definition is different from the definition of one-point modification given in [7]. Using that definition, Lemma 2.1 of [7] is not correct. In fact, let $A = \{1, 2\}$, $X_1 = X_2 = [0, \omega_1)$ with usual topology and X be the one-point compatification of $X_1 \oplus X_2$, moreover we observe that by [8, Example 1.10] the space X is not Valdivia. Let $\Gamma_i = [0, \omega_1)$ and

$$\begin{aligned} f_i :&X_i \to \Sigma(\Gamma_i) \\ &\alpha \mapsto \chi_{[0,\alpha)}, \end{aligned}$$

for $i = 1, 2$. As in [7, Lemma 2.1] we define $\Gamma = \{(i, \gamma) : i \in A, \gamma \in \Gamma_\alpha\} \cup \{(i, A) : i \in A\}$ and $f : X \to \mathbb{R}^\Gamma$

$$f(x)(i,\gamma) = \begin{cases} f_i(x)(\gamma) & x \in X_i, \gamma \in \Gamma_i \\ 1, & x \in X_i, \gamma = A \\ 0, & \text{otherwise.} \end{cases}$$

Claim: f is not continuous. In fact, let $\{\alpha\}_{\alpha<\omega_1} \subset X_1$, it is a converging net to ∞ in X. Using the definition of f, $\{f(\alpha)\}_{\alpha<\omega_1}$ does not converge to $f(\infty)$ in $\mathbb{R}^\Gamma$. Hence it cannot be continuous.
Finally, we observe that using Definition 2.2.7, Lemma 2.1 of [7] is correct.

Lemma 2.2.8. *The class of non-commutative Corson countably compact spaces is closed under*

(1) countably closed subspaces,

(2) countable products,

(3) finite topological sums,

(4) one-point countably compact modifications of topological sums,

(5) quotient images.

Proof. (1) Let Y be a countably closed subspace of a non-commutative Corson countably compact space X, then it is a countably compact space. We notice that Y is a closed subspace of X. In fact let $x \in \overline{Y}$: by Theorem 2.2.2 X is a Fréchet-Urysohn space, then there exists a sequence $\{x_n\}_{n\in\omega_0} \subset Y$ that converges to $x \in X$ and, since Y is countably closed, it follows that $x \in Y$. By Proposition 2.2.3 βX has a retractional skeleton with X as induced space. Since Y is closed in X we have $\overline{Y}^{\beta X} \cap X = Y$. Hence by Lemma 2.2.4 it follows that Y is a non-commutative Corson countably compact space.

(2) Let $\{X_n\}_{n\in\omega_0}$ be a countable family of non-commutative Corson countably compact spaces. For every $n \in \omega_0$, by Proposition 2.2.3, X_n is an induced subspace of the non-commutative Valdivia compact space βX_n. By Proposition 2.2.5 $\prod_{n\in\omega_0} \beta X_n$ is non-commutative Valdivia and $\prod_{n\in\omega_0} X_n$ is an induced subspace. Hence $\prod_{n\in\omega_0} X_n$ is a non-commutative Corson countably compact space.

(3) Let $X_1, ..., X_n$ be a finite collection of non-commutative Corson countably compact spaces and define the topological sum $X = \bigoplus_{k=1}^{n} X_k$. Using the countably compactness of every X_k it is easy to prove that X is countably compact.
It remains to prove that X has full retractional skeleton. For every $k = 1, ..., n$ X_k has a full retractional skeleton, then let $(\Gamma_k, \leq_k)$ be an up-directed partially ordered set and $\{r^k_s\}_{s\in\Gamma_k}$ be a full retractional skeleton on X_k. Now we define a family of retractions on X, let

$$\Gamma = \{\gamma = (\gamma_1, ..., \gamma_n) \in \Gamma_1 \times ... \times \Gamma_n\}$$

equipped with the following order: given $\gamma, \delta \in \Gamma$ we will say that $\gamma \leq \delta$ if and only if $\gamma_k \leq_k \delta_k$ for every $k = 1, ..., n$.
For every $\gamma \in \Gamma$ we define $r_\gamma : X \to X$ as follows: given $x \in X$ we have $x \in X_k$ for some k, then we put

$$r_\gamma(x) = r^k_{\gamma_k}(x).$$

Since r_γ is continuous on every X_k, it is continuous on X. Moreover, since $\{r^k_s\}_{s\in\Gamma_k}$ is a full retractional skeleton on X_k for every $k = 1, ..., n$, it is easy to check that $\{r_\gamma\}_{\gamma\in\Gamma}$ is a full retractional skeleton on X.

(4) Let $\{X_\alpha\}_{\alpha\in A}$ be a family of non-commutative Corson countably compact spaces and $X = (\bigoplus_{\alpha\in A} X_\alpha) \cup \{\infty\}$ be a one-point countably compact modification of topological sum of them.
For every $\alpha \in A$ there exist an up-directed partially ordered set $(\Gamma_\alpha, \preceq_\alpha)$ and a

full retractional skeleton $\{r^\alpha_s\}_{s\in\Gamma_\alpha}$ on X_α.
We define $\Gamma^{'}_\alpha = \Gamma_\alpha \cup \{0\}$ and a relation $\leq_\alpha$ such that if we restrict $\leq_\alpha$ to $\Gamma_\alpha \times \Gamma_\alpha$ we have the same order of $(\Gamma_\alpha, \preceq_\alpha)$ and $0 \leq_\alpha s$, for every $s \in \Gamma_\alpha$. This way $(\Gamma^{'}_\alpha, \leq_\alpha)$ is an up-directed partially ordered set, for every $\alpha \in A$.
Let

$$\Gamma = \{\gamma \in \prod_{\alpha\in A} \Gamma^{'}_\alpha : |S(\gamma)| \leq \aleph_0\}$$

where $S(\gamma) = \{\alpha \in A : \gamma(\alpha) \neq 0\}$. Given $\gamma_1, \gamma_2 \in \Gamma$ we will say that $\gamma_1 \leq \gamma_2$ if and only if $\gamma_1(\alpha) \leq_\alpha \gamma_2(\alpha)$ for every $\alpha \in A$. For every $\gamma \in \Gamma$ we define the retraction $r_\gamma : X \to X$ as follows:

- if $x = \infty$, $r_\gamma(x) = \infty$,
- if $x \neq \infty$ there exists an $\alpha \in A$ such that $x \in X_\alpha$ then

$$r_\gamma(x) = \begin{cases} r^\alpha_{\gamma(\alpha)}(x) & \text{if } \gamma(\alpha) \neq 0, \\ \infty & \text{if } \gamma(\alpha) = 0, \end{cases}$$

<u>Claim</u>: for every $\gamma \in \Gamma$ the retraction r_γ is a continuous mapping.
Let $\gamma \in \Gamma$ we study the continuity of r_γ at each point:

- if $x = \infty$, let U be an open neighborhood of x then $X \setminus U$ is a countably compact space. Moreover, since each X_α is open and $X \setminus U$ is a countably compact space, we have that the cardinality of $F = \{\alpha \in A : (X\setminus U) \cap X_\alpha \neq \emptyset\}$ is finite. Let $V = X \setminus \bigcup_{\alpha\in F} X_\alpha$, it is a neighborhood of x and $V \subset U$. By definition of r_γ we have $r_\gamma(V) \subset V$; hence we have the continuity in x.
- if $x \neq \infty$, since each X_α is clopen and the restriction of r_γ on each X_α is continuous, we conclude that r_γ is continuous in x.

It proves the claim.
It remains to prove that $\{r_\gamma\}_{\gamma\in\Gamma}$ is a full retractional skeleton on X.

(*i*) Since $r_\gamma[X] = (\bigoplus_{\alpha\in S(\gamma)} r^\alpha_{\gamma(\alpha)}[X_\alpha]) \cup \{\infty\}$ is countably compact, regular and has a countable network we have that it is metrizable and compact.

(*ii*) Let $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1 \leq \gamma_2$. If $x = \infty$ it is trivial that $r_{\gamma_1}(x) = r_{\gamma_1} \circ r_{\gamma_2}(x) = r_{\gamma_2} \circ r_{\gamma_1}(x)$. Then, suppose $x \in X_\alpha$ for some $\alpha \in A$, three cases are possible:

  - $\gamma_1(\alpha) = 0$ and $\gamma_2(\alpha) = 0$ we have

$$r_{\gamma_1}(x) = r_{\gamma_1} \circ r_{\gamma_2}(x) = r_{\gamma_2} \circ r_{\gamma_1}(x) = \infty.$$

  - $\gamma_1(\alpha) = 0$ and $\gamma_2(\alpha) \neq 0$ we have

$$r_{\gamma_1}(r_{\gamma_2}(x)) = r_{\gamma_1}(r^\alpha_{\gamma_2(\alpha)}(x)) = \infty = r_{\gamma_1}(x)$$

  and

$$r_{\gamma_2}(r_{\gamma_1}(x)) = r_{\gamma_2}(\infty) = \infty = r_{\gamma_1}(x)$$

- $\gamma_1(\alpha) \neq 0$ and $\gamma_2(\alpha) \neq 0$ we have:

$$r_{\gamma_1}(r_{\gamma_2}(x)) = r^\alpha_{\gamma_1(\alpha)}(r^\alpha_{\gamma_2(\alpha)}(x)) = r^\alpha_{\gamma_1(\alpha)}(x) = r_{\gamma_1}(x)$$

  and

$$r_{\gamma_2}(r_{\gamma_1}(x)) = r^\alpha_{\gamma_2(\alpha)}(r^\alpha_{\gamma_1(\alpha)}(x)) = r^\alpha_{\gamma_1(\alpha)}(x) = r_{\gamma_1}(x).$$

(*iii*) Let $\gamma_1 \leq \gamma_2 \leq ...$ and $\gamma = \sup_{n\in\omega_0} \gamma_n$, then for every $\alpha \in A$ we have $\gamma(\alpha) = \sup_{n\in\omega_0} \gamma_n(\alpha)$. Let $x \in X$, two cases are possible

- $x = \infty$: we have $r_{\gamma_n}(x) = \infty$ for every $n \in \omega_0$ then $\lim\limits_{n\in\omega_0} r_{\gamma_n}(x) = x$.
- $x \neq \infty$: there exists $\alpha \in A$ such that $x \in X_\alpha$. If $\alpha \notin S(\gamma)$ then $\alpha \notin S(\gamma_n)$ for each $n \in \omega_0$; hence $r_\gamma(x) = r_{\gamma_n}(x) = \infty$. If $\alpha \in S(\gamma)$, then there exists $n_0 \in \omega_0$ such that for every $n \geq n_0$ we have $\alpha \in S(\gamma_n)$; hence using the fact that the family $\{r^\alpha_s\}_{s\in\Gamma_\alpha}$ is a full retractional skeleton on X_α we have that $r_\gamma(x) = r^\alpha_{\gamma(\alpha)}(x) = \lim\limits_{n\in\omega_0} r^\alpha_{\gamma_n(\alpha)}(x) = \lim\limits_{n\in\omega_0} r_{\gamma_n}(x)$.

(*iv*) For every $\alpha \in A$ and $x \in X_\alpha$ there exists $s \in \Gamma_\alpha$ such that $r^\alpha_s(x) = x$ then for every $\gamma \in \Gamma$ such that $s \leq_\alpha \gamma(\alpha)$ we have $r_\gamma(x) = x$. Therefore $\lim_{\gamma\in\Gamma} r_\gamma(x) = x$. The case $x = \infty$ is trivial.

Finally it is trivial that $X = \bigcup_{\gamma\in\Gamma} r_\gamma[X]$.

(5) It follows immediately combining [3, Theorem 1.1] and [13, Theorem 3.6].
This completes the proof. □

Using the same argument as in [7, Lemma 2.2] it is possible to prove the following result about stability properties of weakly non-commutative Corson countably compact spaces.

Lemma 2.2.9. *The class of weakly non-commutative Corson countably compact is closed under*

(1) countably closed subspaces,

(2) countable products,

(3) continuous images,

(4) finite unions,

(5) finite topological sums,

(6) one-point countably compact modifications of topological sums.

2.3 Weakly non-commutative Valdivia compact spaces

Now we give the definition of weakly non-commutative Valdivia compact space which is a generalization of the commutative one introduced in [7].

Definition 2.3.1. A compact space K is said *weakly non-commutative Valdivia compact* if it has a dense countably compact subspace which is weakly non-commutative Corson.

Next two results are the non-commutative version of [7, Proposition 3.1] and [8, Lemma 1.17].

Proposition 2.3.2. *A compact space K is weakly non-commutative Valdivia if and only if it is a continuous image of a non-commutative Valdivia compact.*

Proof. We start by proving the "if part". Let L be a non-commutative Valdivia compact space and $f : L \to K$ be a continuous onto mapping.
Let D be an induced subspace of L, hence by Theorem 2.2.2 it is a dense non-commutative Corson countably compact space. Since f is a continuous mapping we have that $f(D) \subset K$ is a dense weakly non-commutative Corson countably compact space. Hence K is a weakly non-commutative Valdivia compact space.
Conversely let D be a dense weakly non-commutative Corson countably compact subspace of K. Then there exist a non-commutative Corson countably compact space A and a continuous surjection $f : A \to D$. By Proposition 2.2.3 we have that βA is a non-commutative Valdivia compact space. Let $\beta f : \beta A \to K$ be the continuous extension of f.
Since D is dense in K, $D \subset \beta f(\beta A)$ and $\beta f(\beta A)$ is closed we have that βf is a surjection. Thus K is a continuous onto image of a non-commutative Valdivia compact space. ☐

We prefer to omit the proof of the following result because it is completely analogous to [8, Lemma 1.17].

Proposition 2.3.3. *Let K be a compact space with a countable dense set of G_δ points. If K is a continuous image of a non-commutative Valdivia compact space then K is metrizable.*

Corollary 2.3.4. *Let K be a compact space with a countable dense set of G_δ points. If K is a continuous image of a non-commutative Corson countably compact space then K is metrizable.*

Using Proposition 2.3.3, Corollary 2.3.4 and [8, Example 1.18] we have some examples of compact spaces which are neither weakly non-commutative Corson nor weakly non-commutative Valdivia.

Proposition 2.3.5. *Let $\{K_\alpha\}_{\alpha\in A}$ be a family of non-commutative Valdivia compact spaces. Then the one-point compactification of $K = \bigoplus_{\alpha\in A} K_\alpha$ is a non-commutative Valdivia compact space.*

We will not provide the full proof because we use the same idea of Lemma 2.2.8, point (4).

Proof. We will use the same notations of point (4) Lemma 2.2.8. We define the up-directed partially ordered set Γ and the family of retractions $\{r_\gamma\}_{\gamma\in\Gamma}$ as well. Moreover, we observe that each K_α is clopen in the one-point compactification of

K, hence the continuity of every r_γ follows in the same way of point (4) Lemma 2.2.8.
It remains to prove that $\{r_\gamma\}_{\gamma\in\Gamma}$ is a retractional skeleton. Points $(i), (ii)$ and (iii) follow as in point (4) Lemma 2.2.8.
(iv) If $x = \infty$, $r_\gamma(x) = \infty$ for every $\gamma \in \Gamma$, hence it is clear that $\lim\limits_{\gamma\in\Gamma} r_\gamma(x) = x$. Suppose otherwise $x \in K_\alpha$ for some $\alpha \in A$, by definition of $(\Gamma, \leq)$ there exists $\gamma_0 \in \Gamma$ such that $\gamma(\alpha) \neq 0$ for every $\gamma \geq \gamma_0$, hence we have $r_\gamma(x) = r^\alpha_{\gamma(\alpha)}(x)$ for such γ. Since $\{r^\alpha_s\}_{s\in\Gamma_\alpha}$ is a retractional skeleton on K_α, we deduce that $\lim\limits_{\gamma\in\Gamma} r_\gamma(x) = \lim\limits_{s\in\Gamma_\alpha} r^\alpha_s(x) = x$. □

Proposition 2.3.6. *The class of weakly non-commutative Valdivia compact spaces is closed under*

(1) arbitrary products,

(2) one point compactifications of arbitrary topological sums,

(3) continuous images,

(4) finite unions.

Moreover if K is weakly non-commutative Valdivia compact and L is a subset of K which can be written as the closure of the union of an arbitrary family of G_δ subsets of K, then L is weakly non-commutative Valdivia as well.

Proof. (1) Let $\{K_\alpha\}_{\alpha\in A}$ be a family of weakly non-commutative Valdivia compact spaces. Then there exists a continuous onto mapping $f_\alpha : L_\alpha \to K_\alpha$, where L_α is a non-commutative Valdivia compact space for every $\alpha \in A$.
We define $K = \prod\limits_{\alpha\in A} K_\alpha$ and $L = \prod\limits_{\alpha\in A} L_\alpha$. By Proposition 2.2.5 L is a non-commutative Valdivia compact space. Finally let $f : L \to K$ defined by $f(y)(\alpha) = f_\alpha(y(\alpha))$, it is clearly onto and moreover, since it is coordinatewise continuous, it is continuous.

(2) It follows, using Proposition 2.3.5, by the same argument of the previous point.

Points (3)-(4) are trivial. Finally let K be a weakly non-commutative Valdivia compact and $L \subseteq K$ such that $L = \overline{\bigcup_{\beta\in B} Z_\beta}$, where B is a set and Z_β is a G_δ subset of K, for every $\beta \in B$. Let $D \subset K$ be a dense weakly non-commutative Corson countably compact subspace. Now we want to prove that $L \cap D$ is dense in L. In fact

$$L \cap D = \overline{\bigcup_{\beta\in B} Z_\beta} \cap D \supset (\bigcup_{\beta\in B} Z_\beta) \cap D = \bigcup_{\beta\in B} (Z_\beta \cap D).$$

Taking the closure

$$\overline{L \cap D} \supset \overline{\bigcup_{\beta\in B} (Z_\beta \cap D)} \supset \bigcup_{\beta\in B} \overline{(Z_\beta \cap D)} \supset \bigcup_{\beta\in B} Z_\beta.$$

In the last part we have used [2, Lemma 3.3]. Therefore we have $L \supset \overline{L \cap D} \supset L$. Moreover $L \cap D$ is a closed subspace of D, hence it is weakly non-commutative Corson, therefore L is weakly non-commutative Valdivia. □

Now we want to use the previous results to prove the equivalent of [7, Theorem 3.6] in the non-commutative case.

Theorem 2.3.7. *Let K be a compact space. Consider the following assertions.*

(1) K is weakly non-commutative Valdivia.

(2) $(B_{C(K,\mathbb{C})^*}, w^*)$ is weakly non-commutative Valdivia.

(3) $(B_{C(K,\mathbb{R})^*}, w^*)$ is weakly non-commutative Valdivia.

(4) $P(K)$ is weakly non-commutative Valdivia.

Then $1 \Rightarrow 2 \Leftrightarrow 3 \Leftrightarrow 4$. If K has a dense set of G_δ points, then all three assertions are equivalent.

Proof. (1) $\Rightarrow$ (2) Let K be a weakly non-commutative Valdivia compact space, then it is a continuous image of a non-commutative Valdivia compact L. Using [10, Proposition 28] $C(L,\mathbb{C})$ has 1-projectional skeleton then it is clear that $(B_{C(L,\mathbb{C})^*}, w^*)$ is non-commutative Valdivia. Hence, since $(B_{C(K,\mathbb{C})^*}, w^*)$ is a continuous image of $(B_{C(L,\mathbb{C})^*}, w^*)$, it is a weakly non-commutative Valdivia compact space.
(2) $\Rightarrow$ (3) Suppose that $(B_{C(K,\mathbb{C})^*}, w^*)$ is a weakly non-commutative Valdivia compact space.
We want to show that $(B_{C(K,\mathbb{R})^*}, w^*)$ is a continuous images of $(B_{C(K,\mathbb{C})^*}, w^*)$. To do that, consider the following map:

$$\varphi : (B_{C(K,\mathbb{C})^*}, w^*) \to (B_{C(K,\mathbb{R})^*}, w^*)$$
$$\mu \mapsto \mathrm{Re}(\mu).$$

It is clear that it is a surjection. To prove that φ is a continuous mapping, it is sufficient to observe that for every $f \in C(K,\mathbb{R})$ we have $\mathrm{Re}(\mu)(f) = \mathrm{Re}\mu(f)$.
(3) $\Rightarrow$ (4) Since $P(K) = \{\mu \in C(K)^* : \|\mu\| \leq 1 \,\&\, \mu(1_K) = 1\}$, $P(K)$ is a weak* closed weak* G_δ subset of $(B_{C(K)^*}, w^*)$. Hence, Proposition 2.4.3 gives the assertion.
(4) $\Rightarrow$ (2) Suppose $P(K)$ is weakly non-commutative Valdivia. By Proposition 2.3.6 the space $P(K) \times S^1$ is weakly non-commutative Valdivia, finally by (1) $\Rightarrow$ (4) $P(P(K) \times S^1)$ is weakly non-commutative Valdivia. By [12, Proposition 2.38] the barycenter mapping

$$r : P(B_{C(K,\mathbb{C})^*}) \to (B_{C(K,\mathbb{C})^*}, w^*)$$

is surjective and continuous. Moreover, since $P(K) \times S^1$ contains all the extreme points of $(B_{C(K,\mathbb{C})^*}, w^*)$, using [12, Theorem 2.31] we obtain that the restriction of r to $P(P(K) \times S^1)$ is surjective as well. Hence, since $P(P(K) \times S^1)$ is weakly non-commutative Valdivia, we have that $(B_{C(K,\mathbb{C})^*}, w^*)$ is weakly non-commutative Valdivia, too.
(4) $\Rightarrow$ (1) If K has a dense set of G_δ points, it follows in the same way as [7, Theorem 3.6]. □

Now we give the non-commutative version of [7, Theorem 3.7], we recall the definition of property (M).

Definition 2.3.8. A compact space K is said to have the *property* (M) if every Radon probability measure on K has separable support.

Theorem 2.3.9. *Let K be a weakly non-commutative Corson compact space with property (M), then $(B_{C(K)^*}, w^*)$ is weakly non-commutative Corson as well.*

Proof. The proof of the real case follows as in the commutative case [7, Theorem 3.7], using Proposition 2.3.10 below instead of [8, Proposition 5.1].
Suppose that K is a weakly non-commutative Corson compact space with property (M), then using the real case and Lemma 2.2.9, $(B_{C(K,\mathbb{R})^*}, w^*) \times (B_{C(K,\mathbb{R})^*}, w^*)$ is weakly non-commutative Corson as well. Now consider

$$\psi : (B_{C(K,\mathbb{R})^*}, w^*) \times (B_{C(K,\mathbb{R})^*}, w^*) \to (C(K,\mathbb{C})^*, w^*),$$

defined by $\psi(\mu,\nu) = \mu + i\nu$. ψ is clearly continuous, hence $\psi((B_{C(K,\mathbb{R})^*}, w^*) \times (B_{C(K,\mathbb{R})^*}, w^*))$ is weakly non-commutative Corson. Finally, since $(B_{C(K,\mathbb{C})^*}, w^*)$ is a weak* compact space and $(B_{C(K,\mathbb{C})^*}, w^*) \subset \psi((B_{C(K,\mathbb{R})^*}, w^*) \times (B_{C(K,\mathbb{R})^*}, w^*))$, it is a weak* closed subspace of $\psi((B_{C(K,\mathbb{R})^*}, w^*) \times (B_{C(K,\mathbb{R})^*}, w^*))$. Therefore, by Lemma 2.2.9, $(B_{C(K,\mathbb{C})^*}, w^*)$ is a weakly non-commutative Corson compact space. $\square$

Now we give the non-commutative version of [8, Proposition 5.1]. For sake of completeness we will give the full proof although the last part is the same of the commutative one.

Proposition 2.3.10. *Let K be a non-commutative Valdivia compact and $D \subset K$ be an induced subspace. Then the set*

$$S = \{\mu \in C(K)^* : \ \mathit{supp}(\mu) \text{ is a separable subset of } D\}$$

is 1-norming and induced by a 1-projectional skeleton in $C(K)$.

Proof. The real case follows by combining the first part of the complex case below and the second part of [8, Proposition 5.1].
Let $\{r_s\}_{s\in\Gamma}$ be a retractional skeleton on K such that $D = \bigcup_{s\in\Gamma} r_s[K]$. It is standard to define a 1-projectional skeleton $\{P_s\}_{s\in\Gamma}$ in $C(K,\mathbb{C})$ as follow

$$P_s(f) = f \circ r_s.$$

Let $S = \bigcup_{s\in\Gamma} P_s^*(C(K,\mathbb{C})^*)$ be the induced subspace. It is well known that it is 1-norming (hence weak*-dense in $C(K,\mathbb{C})^*$) linear and weak*-countably closed. Moreover, $(B_{C(K,\mathbb{C})^*}, w^*)$ has retractional skeleton and $S \cap (B_{C(K,\mathbb{C})^*}, w^*)$ is an induced subspace.
Now we want to prove that

$$S = \{\mu \in C(K,\mathbb{C})^* : \ \text{supp}\mu \text{ is a separable subset of } D\}.$$

We will prove the double inclusion.

- "$\supseteq$" Let μ be a real measure in the set on the right-hand side, then using the same argument of [8, Proposition 5.1] we obtain $\mu \in S$.
  Now, let μ be a complex measure in the set on the right-hand side, then, by the previous sentence, its total variation $|\mu|$ belongs to S . Hence there exists $s_0 \in \Gamma$ such that $P_s^*|\mu| = |\mu|$ for every $s \geq s_0$, in particular, by Riesz representation theorem, for every $f \in C(K, \mathbb{C})$ we have
  $$\int_K f d|\mu| = \int_K f \circ r_s d|\mu|. \tag{2.1}$$
  Moreover, by the Radon-Nikodým theorem there exists a measurable function h such that $d\mu = h\, d|\mu|$ and $|h(x)| = 1$ for every $x \in K$.
  Claim: There exists a $t \in \Gamma$ such that for every continuous function f the equality
  $$\int_K f \cdot h\, d|\mu| = \int_K (f \circ r_t) \cdot h\, d|\mu|.$$
  holds.
  Indeed, let $t \in \Gamma$ such that $t \geq s_0$ and $\operatorname{supp}\mu \subset r_t[K]$: such t exists by the σ-completeness of Γ and the separability of $\operatorname{supp}\mu$. Finally, let $f \in C(K, \mathbb{C})$ and $\varepsilon > 0$ then by the density of continuous function in $L^1(|\mu|)$ there exists $g \in C(K, \mathbb{C})$ such that $\int_K |f \cdot h - g|\, d|\mu| < \varepsilon$; then using (2.1) and the fact that $\operatorname{supp}\mu \subset r_t[K]$, we obtain
  $$\begin{aligned}\left|\int_K f \cdot h - (f \circ r_t) \cdot h\, d|\mu|\right| &\leq \int_K |f \cdot h - g|\, d|\mu| + \int_K |(f \circ r_t) \cdot h - g \circ r_t|\, d|\mu| \\ &< \varepsilon + \int_{r_t[K]} |(f \circ r_t) \cdot h - g \circ r_t|\, d|\mu| \\ &= \varepsilon + \int_{r_t[K]} |f \cdot h - g|\, d|\mu| < 2\varepsilon.\end{aligned}$$
  It proves the claim. Therefore $P_t^*\mu = \mu$, hence $\mu \in S$.

- "$\subseteq$" Let $S' = \operatorname{span}\{\delta_x : x \in D\}$, since S is linear and $\delta_x \in S$ for every $x \in D$ we have $S' \subset S$. Moreover since D is dense in K we have that S' is 1-norming. Then $S' \cap B_{C(K,\mathbb{C})^*}$ is weak* dense in $B_{C(K,\mathbb{C})^*}$. In particular, every $\mu \in S \cap B_{C(K,\mathbb{C})^*}$ belongs to the weak* closure of $S' \cap B_{C(K,\mathbb{C})^*}$. Hence, since $S \cap B_{C(K,\mathbb{C})^*}$ is a weak* Fréchet-Urysohn space, there exists a sequence $\{\mu_n\}_{n \in \omega_0} \subset S' \cap B_{C(K,\mathbb{C})^*}$ such that $\mu_n \overset{w^*}{\to} \mu$.
  Let
  $$C = \{x \in K : \exists\, n \in \omega_0\ \mu_n(\{x\}) \neq 0\},$$
  clearly $|C| \leq \aleph_0$. Moreover μ is supported by $\overline{C}$. Therefore, since $\overline{C} \subset D$ is separable, it is metrizable. Since $\operatorname{supp}\mu \subset \overline{C}$ and $\overline{C}$ is metrizable and separable, we have that $\operatorname{supp}\mu$ is separable as well.

□

2.4 $[0, \eta)$-sums

We recall the definition of $[0, \eta)$-sum, introduced in [7]. Given an ordinal η we will denote with $I(\eta)$ the subset of all isolated ordinals less than η. Let $\{X_\alpha\}_{\alpha \in I(\eta)}$ be a family of topological spaces, the $[0, \eta)$-sum is the set

$$X = \{(\alpha, x) : x \in X_\alpha, \alpha < \eta \text{ isolated}\} \cup \{(\alpha, \alpha) : \alpha < \eta \text{ limit}\}$$

equipped with the following topology. Whenever α is isolated, the set $\{\alpha\} \times X_\alpha$ is canonically homeomorphic to X_α and clopen in X. A neighborhoods basis for (α, α) if α is limit is formed by sets

$$B_\gamma((\alpha, \alpha)) = \{(\beta, x) \in X : \gamma < \beta < \alpha\}.$$

Since in our setting X_α is Hausdorff and completely regular, X is Hausdorff and completely regular too.
Let η be an uncountable cofinality ordinal and X_α be a countably compact space, for every $\alpha \in I(\eta)$. Let Y be the $[0, \eta)$-sum of $\{X_\alpha\}_{\alpha \in I(\eta)}$ and X be the topological subspace of Y defined by

$$\begin{aligned} X = &\{(\alpha, x) : x \in X_\alpha,\ \alpha < \eta \text{ isolated}\} \cup \\ &\{(\alpha, \alpha) : \alpha < \eta \text{ limit with countable cofinality}\}. \end{aligned}$$

We will say that X is the *countably* $[0, \eta)$*-sum* of $\{X_\alpha\}_{\alpha \in I(\eta)}$. We observe that the countably $[0, \eta)$-sum is a countably closed subset of the $[0, \eta)$-sum. Then, since, by [7, Lemma 2.3], the $[0, \eta)$-sum is countably compact, we conclude that the countably $[0, \eta)$-sum is countably compact as well.

Lemma 2.4.1. *Let η be an uncountable cofinality ordinal and $\{X_\alpha\}_{\alpha \in I(\eta)}$ be a family of non-commutative Corson countably compact spaces. Let X be the countably $[0, \eta)$-sum of $\{X_\alpha\}_{\alpha \in I(\eta)}$. Then X is a non-commutative Corson countably compact space.*

Proof. For every isolated $\alpha < \eta$ there exist an up-directed partially ordered set $(\Gamma_\alpha, \preceq_\alpha)$ and a full retractional skeleton $\{r_s^\alpha\}_{s \in \Gamma_\alpha}$ on X_α.
We define $\Gamma_\alpha^{'} = \Gamma_\alpha \cup \{0\}$ and a relation $\leq_\alpha$ such that if we restrict $\leq_\alpha$ to $\Gamma_\alpha \times \Gamma_\alpha$ we have the same order of $(\Gamma_\alpha, \preceq_\alpha)$ and $0 \leq_\alpha s$, for every $s \in \Gamma_\alpha$. This way $(\Gamma_\alpha^{'}, \leq_\alpha)$ is an up-directed partially ordered set.
For every $\gamma \in \prod_{\alpha \in I(\eta)} \Gamma_\alpha^{'}$ let us define $S(\gamma) = \{\alpha \in I(\eta) : \gamma(\alpha) \neq 0\}$. Now we are going to define the partially ordered set and the family of retractions on X. In order to do this, let

$$\begin{aligned} \Gamma = &\{\gamma \in \prod_{\alpha \in I(\eta)} \Gamma_\alpha^{'} : |S(\gamma)| \leq \aleph_0,\ 0 \in S(\gamma),\ \text{and if } \alpha \in S(\gamma) \\ &\text{then } \beta \in S(\gamma) \text{ whenever } \alpha < \beta < \alpha + \omega_0\}. \end{aligned}$$

Given $\gamma_1, \gamma_2 \in \Gamma$ we will say that $\gamma_1 \leq \gamma_2$ if and only if $\gamma_1(\alpha) \leq_\alpha \gamma_2(\alpha)$ for every $\alpha \in I(\eta)$. We observe that, if $\gamma_1 \leq \gamma_2$ then $S(\gamma_1) \subset S(\gamma_2)$. By the definition of $(\Gamma_\alpha^{'}, \leq_\alpha)$ it is clear that $(\Gamma, \leq)$ is an up-directed partially ordered set.

For every $\gamma \in \Gamma$ we define $r_\gamma : X \to X$ as follows: let $(\alpha, x) \in X$ and $\beta_\alpha = \sup([0,\alpha] \cap S(\gamma))$

$$r_\gamma(\alpha, x) = \begin{cases} (\alpha, r^\alpha_{\gamma(\alpha)}(x)) & \text{if } \alpha \text{ is isolated and } \gamma(\alpha) \neq 0; \\ (\beta_\alpha, \beta_\alpha) & \text{if } \alpha \text{ is isolated and } \gamma(\alpha) = 0 \text{ or } \alpha \text{ is limit.} \end{cases}$$

Claim: for every $\gamma \in \Gamma$ the retraction r_γ is a continuous mapping.
Let $\gamma \in \Gamma$ and let $\{(\alpha_\lambda, x_\lambda)\}_{\lambda \in \Lambda}$ be a net converging to (α, x). If α is isolated then $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ is eventually constant and equal to α;

- if $\alpha \in S(\gamma)$, by continuity of $r^\alpha_{\gamma(\alpha)}$, we have $\lim_{\lambda\in\Lambda} r_\gamma(\alpha_\lambda, x_\lambda) = r_\gamma(\alpha, x)$;
- if $\alpha \notin S(\gamma)$ we have that $r_\gamma(\alpha_\lambda, x_\lambda)$ is eventually constant, hence it is clear that $\lim_{\lambda\in\Lambda} r_\gamma(\alpha_\lambda, x_\lambda) = r_\gamma(\alpha, x)$.

In the case of α limit we can suppose without loss of generality that $\alpha_\lambda \leq \alpha$ for every $\lambda \in \Lambda$. Two cases are possible:

- if $\sup([0,\alpha] \cap S(\gamma)) = \alpha$, since α has countably cofinality, by definition of supremum there exists a sequence $\{\xi_n\}_{n\in\omega_0} \in [0,\alpha] \cap S(\gamma)$ such that ξ_n is increasing and convergent to α. For every $n \in \omega_0$, by definition of convergence of $\{\alpha_\lambda\}_{\lambda\in\Lambda}$ there exists λ_n such that $\alpha_\lambda \in [\xi_n, \alpha]$ for every $\lambda \geq \lambda_n$. Therefore $\lim_{\lambda\in\Lambda} r_\gamma(\alpha_\lambda, x_\lambda) = (\alpha, \alpha) = r_\gamma(\alpha, x)$.
- if $\beta = \sup([0,\alpha] \cap S(\gamma)) < \alpha$ then for a sufficiently large λ we have $\beta < \alpha_\lambda \leq \alpha$; hence by definition of r_γ we have $\lim\limits_{\lambda\in\Lambda} r_\gamma(\alpha_\lambda, x_\lambda) = r_\gamma(\alpha, x) = (\beta, \beta)$.

Therefore r_γ is continuous for every $\gamma \in \Gamma$.

It remains to prove that $\{r_\gamma\}_{\gamma\in\Gamma}$ is a full retractional skeleton.

(*i*) for every $\gamma \in \Gamma$ the subspace $r_\gamma[X]$ is a closed subset of X, hence it is countably compact. Moreover, since it has a countable network, it is compact and metrizable.

(*ii*) Let $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1 \leq \gamma_2$, then $S(\gamma_1) \subset S(\gamma_2)$. Let $(\alpha, x) \in X$

- if α is limit let $\beta_\alpha = \sup([0,\alpha] \cap S(\gamma_1)) = \sup([0,\alpha] \cap S(\gamma_1) \cap S(\gamma_2))$ and $\beta'_\alpha = \sup([0,\alpha] \cap S(\gamma_2))$. We observe that $\beta_\alpha \leq \beta'_\alpha \leq \alpha$ and that the intersection of the interval $(\beta_\alpha, \beta'_\alpha)$ and $S(\gamma_1)$ is empty. Hence:

$$r_{\gamma_1} \circ r_{\gamma_2}(\alpha, \alpha) = r_{\gamma_1}(\beta'_\alpha, \beta'_\alpha) = (\beta_\alpha, \beta_\alpha) = r_{\gamma_1}(\alpha, \alpha),$$
$$r_{\gamma_2} \circ r_{\gamma_1}(\alpha, \alpha) = r_{\gamma_2}(\beta_\alpha, \beta_\alpha) = (\beta_\alpha, \beta_\alpha) = r_{\gamma_1}(\alpha, \alpha).$$

- if α is isolated three cases are possible:
  (1) if $\gamma_1(\alpha) = 0$ and $\gamma_2(\alpha) = 0$, let $\beta_\alpha = \sup([0,\alpha] \cap S(\gamma_1))$ and $\beta'_\alpha = \sup([0,\alpha] \cap S(\gamma_2))$, thus using the same observation of the previous point we have:

$$r_{\gamma_2}(r_{\gamma_1}(\alpha, x)) = r_{\gamma_2}(\beta_\alpha, \beta_\alpha) = (\beta_\alpha, \beta_\alpha) = r_\gamma(\alpha, x),$$
$$r_{\gamma_1}(r_{\gamma_2}(\alpha, x)) = r_{\gamma_2}(\beta'_\alpha, \beta'_\alpha) = (\beta_\alpha, \beta_\alpha) = r_\gamma(\alpha, x);$$

(2) if $\gamma_1(\alpha) = 0$ and $\gamma_2(\alpha) \neq 0$, let $\beta_\alpha = \sup([0,\alpha] \cap S(\gamma_1))$. Thus we have

$$r_{\gamma_1}(\alpha, x) = (\beta_\alpha, \beta_\alpha),$$

since $S(\gamma_1) \subset S(\gamma_2)$

$$r_{\gamma_2}(r_{\gamma_1}(\alpha, x)) = r_{\gamma_2}(\beta_\alpha, \beta_\alpha) = (\beta_\alpha, \beta_\alpha)$$

and at the end we have

$$r_{\gamma_1}(r_{\gamma_2}(\alpha, x)) = r_{\gamma_1}(\alpha, r^\alpha_{\gamma_2(\alpha)}(x)) = (\beta_\alpha, \beta_\alpha);$$

(3) if $\gamma_1(\alpha) \neq 0$ and $\gamma_2(\alpha) \neq 0$, since for every $\alpha \in I(\eta)$ the family $\{r^\alpha_s\}_{s\in\Gamma_\alpha}$ is a retractional skeleton on X_α, we have

$$\begin{aligned} r_{\gamma_1} \circ r_{\gamma_2}(\alpha, x) &= (\alpha, r^\alpha_{\gamma_1} \circ r^\alpha_{\gamma_2}(x)) = (\alpha, r^\alpha_{\gamma_1}(x)) = r_{\gamma_1}(\alpha, x),\\ r_{\gamma_2} \circ r_{\gamma_1}(\alpha, x) &= (\alpha, r^\alpha_{\gamma_2} \circ r^\alpha_{\gamma_1}(x)) = (\alpha, r^\alpha_{\gamma_1}(x)) = r_{\gamma_1}(\alpha, x). \end{aligned}$$

(iii) Let $\gamma_1 \leq \gamma_2 \leq ... \leq \gamma_n \leq ...$ and define $\gamma(\alpha) = \sup_{n<\omega_0} \gamma_n(\alpha)$, for every $\alpha \in I(\eta)$. Since $S(\gamma_n) \subset S(\gamma_{n+1})$ for every $n < \omega_0$ we have that $\bigcup_{n\in\omega_0} S(\gamma_n) = S(\gamma)$ is countable, $0 \in S(\gamma)$ and if $\alpha \in S(\gamma)$ there exists $n \in \omega_0$ such that $\alpha \in S(\gamma_n)$, hence $[\alpha, \alpha + \omega_0) \subset S(\gamma_n) \subset S(\gamma)$, thus $\gamma \in \Gamma$. Moreover, let $(\alpha, x) \in X$ then

- if $\alpha \in S(\gamma_{n_0})$ for some $n_0 < \omega_0$ then $\alpha \in S(\gamma_n)$ for every $n > n_0$. Thus we have

$$\begin{aligned} \lim_{n\in\omega_0} r_{\gamma_n}(\alpha, x) = \lim_{n>n_0} r_{\gamma_n}(\alpha, x) &= \lim_{n>n_0} (\alpha, r^\alpha_{\gamma_n(\alpha)}(x))\\ &= (\alpha, r^\alpha_{\gamma(\alpha)}(x)) = r_\gamma(\alpha, x). \end{aligned}$$

- if $\alpha \notin S(\gamma)$, then

$$\begin{aligned} \lim_{n\in\omega_0} r_{\gamma_n}(\alpha, x) &= \lim_{n\in\omega_0} (\sup([0,\alpha] \cap S(\gamma_n)), \sup([0,\alpha] \cap S(\gamma_n)))\\ &= \sup_{n<\omega_0} (\sup([0,\alpha] \cap S(\gamma_n)), \sup([0,\alpha] \cap S(\gamma_n)))\\ &= (\sup([0,\alpha] \cap (\bigcup_{n\in\omega_0} S(\gamma_n))), \sup([0,\alpha] \cap (\bigcup_{n\in\omega_0} S(\gamma_n))))\\ &= r_\gamma(\alpha, x). \end{aligned}$$

(iv) For every $(\alpha, x) \in X$ there exists a $\gamma \in \Gamma$ such that $r_\gamma(\alpha, x) = (\alpha, x)$. In fact

- if α is isolated, since $\{r^\alpha_s\}_{s\in\Gamma_\alpha}$ is a full retractional skeleton on X_α there exists $s \in \Gamma_\alpha$ such that $r^\alpha_s(x) = x$. Hence there exists $\gamma \in \Gamma$ with $\gamma(\alpha) \geq s$ such that $r_\gamma(\alpha, x) = (\alpha, x)$;
- if α is limit, there exists a sequence $\{\alpha_n\}_{n\in\omega_0}$ of isolated points such that $\alpha_n \nearrow \alpha$, moreover there exists a $\gamma \in \Gamma$ such that $\{\alpha_n\}_{n\in\omega_0} \subset S(\gamma)$. Hence, we have $r_\gamma(\alpha, \alpha) = (\alpha, \alpha)$.

Hence $\lim\limits_{\gamma\in\Gamma} r_\gamma(\alpha,x)=(\alpha,x)$. Therefore (iv) is proved.

Finally it is clear that $X=\bigcup\limits_{\gamma\in\Gamma} r_\gamma[X]$. $\square$

Lemma 2.4.2. *Let η be an uncountable cofinality ordinal and X_α be a weakly non-commutative Corson countably compact space for every isolated $\alpha<\eta$. Let X be the $[0,\eta+1)$-sum of $\{X_\alpha\}_{\alpha\in I(\eta)}$. Then X is a weakly non-commutative Corson countably compact space.*

Proof. For every $\alpha\in I(\eta)$ there are a non-commutative Corson countably compact space Y_α and a continuous surjection $f_\alpha: Y_\alpha\to X_\alpha$.
Now, we are going to define a suitable family of countably compact spaces indexed on $I(\eta)$. Let $\alpha\in I(\eta)$, put $Z_\alpha=Y_\alpha$ if $\alpha<\omega_0$, $Z_\alpha=\{\alpha\}$ if $\alpha=\beta+1$ and β is limit finally put $Z_\alpha=Y_\beta$ if $\alpha=\beta+1>\omega_0$ and β is not limit.
Let X be the $[0,\eta+1)$-sum of $\{X_\alpha\}_{\alpha\in I(\eta)}$ and $Y=Z\oplus\{(\eta+1,\eta+1)\}$ where Z is the countably $[0,\eta)$-sum of $\{Z_\alpha\}_{\alpha\in I(\eta)}$. By Lemma 2.4.1 and Lemma 2.2.8 Y is a non-commutative Corson countably compact space.
Finally, we define a mappings $f: Y\to X$ as follows:

$$f(\alpha,x)=\begin{cases}(\alpha,\alpha) & \text{if } \alpha \text{ is limit;}\\ (\beta,\beta) & \text{if } \alpha=\beta+1 \text{ and } \beta \text{ is limit;}\\ (\beta,f_\beta(x)) & \text{if } \alpha=\beta+1,\ \beta>\omega_0 \text{ and } \beta \text{ is not limit;}\\ (\alpha,f_\alpha(x)) & \text{if } \alpha<\omega_0.\end{cases}$$

Since every limit ordinal is covered by its successor and every f_α is surjective, we have that f is surjective.
Claim: f is continuous.
Let $\{(\alpha_\lambda,x_\lambda)\}_{\lambda\in\Lambda}$ be a converging net to $(\alpha,x)\in Y$. Two cases are possible:

(1) α is not limit, then, since Z_α is clopen there exists a $\lambda_0\in\Lambda$ such that $x_\lambda\in Z_\alpha$ for every $\lambda\geq\lambda_0$. Let $\alpha=\beta+1$, we split in other three cases:

- β is limit, by definition of the function f, we have $f(\alpha_\lambda,x_\lambda)=(\beta,\beta)$ for every $\lambda\geq\lambda_0$. Hence we have $\lim_{\lambda\in\Lambda} f(\alpha_\lambda,x_\lambda)=(\beta,\beta)=f(\alpha,x)$;
- β is not limit and $\beta>\omega_0$, by definition $f(\alpha_\lambda,x_\lambda)=(\beta,f_\beta(x_\lambda))$ for every $\lambda\geq\lambda_0$. Hence, by continuity of f_β we have $\lim_{\lambda\in\Lambda} f(\alpha_\lambda,x_\lambda)=(\beta,f_\beta(x))=f(\alpha,x)$;
- $\alpha<\omega_0$ by definition $f(\alpha_\lambda,x_\lambda)=(\alpha,f_\alpha(x_\lambda))$ for every $\lambda\geq\lambda_0$. Hence, by continuity of f_α we have $\lim_{\lambda\in\Lambda} f(\alpha_\lambda,x_\lambda)=(\alpha,f_\alpha(x))=f(\alpha,x)$.

(2) If α is limit, the using definition of the function f and the topology on X we have $\lim_{\lambda\in\Lambda} f(\alpha_\lambda,x_\lambda)=(\alpha,\alpha)=f(\alpha,\alpha)$.

Therefore X is a weakly non-commutative Corson countably compact space. $\square$

The following Lemma is an analogue of [7, Lemma 2.4]. It gives a characterization of non-commutative Corson countably compact spaces in the class of weakly non-commutative countably compact.

Lemma 2.4.3. *Let X be a weakly non-commutative Corson countably compact space. The following assertions are equivalent:*

1. *X is non-commutative Corson.*
2. *X is Fréchet-Urysohn.*
3. *X has countable tightness.*

Proof. ($1 \Rightarrow 2$) By Proposition 2.2.3 X is induced by a retractional skeleton in βX, hence by Theorem 2.2.2, it is a Fréchet-Urysohn space.
($2 \Rightarrow 3$) is trivial.
($3 \Rightarrow 1$) Since X is a weakly non-commutative Corson countably compact space, there are a non-commutative Corson countably compact space and a continuous onto mapping $f : Y \to X$. Let $F \subset Y$ be a closed subspace. We claim that $f(F)$ is closed in X. Let $x \in \overline{f(F)}$, by countably tightness of X there exists a countable subset $C \subset f(F)$ such that $x \in \overline{C}$. Since f is a onto mapping, there exists a countable set $D \subset F$ such that $f(D) = C$. Then $\overline{D} \subset F$ and, by Theorem 2.2.2 $\overline{D}$ is a metrizable countably compact space, hence it is compact. Thus $f(\overline{D})$ is compact and contains x. Since $x \in f(\overline{D}) \subset f(F)$, we deduce that $f(F)$ is closed. Therefore, since f is a continuous closed onto mapping, it is a quotient mapping. Hence, the assertion follows by Lemma 2.2.8. □

Theorem 2.4.4. *(1) The ordinal space $[0, \eta]$ is non-commutative Corson if and only if it is Corson if and only if $\eta < \omega_1$.*
The ordinal space $[0, \eta]$ is a weakly non-commutative Corson for every ordinal η.

(2) Let X be a countably compact set of ordinals. Then X is a weakly non-commutative Corson countably compact space.

(3) Let X be a countably compact set of ordinals. Then X is a non-commutative Corson countably compact if and only if every ordinal of uncountable cofinality is isolated in X.

Proof. (1) The first part is trivial. If η has uncountable cofinality, the assertion follows by Lemma 2.4.2. Isolated and countable cofinality cases follow by the uncountable cofinality case and Lemma 2.2.9.

(2) Let X be a countably compact set of ordinals. Let $\theta = \sup(X)$, and let Y be the closure of X in $[0, \theta]$.
Since Y is a well-ordered compact space it is homeomorphic to $[0, \eta]$ for some ordinal η. By previous point Y is weakly non-commutative Corson. By Lemma 2.2.9 X is a countably closed subspace of Y, hence X is a weakly non-commutative Corson countably compact space.

(3) It follows by previous point and Lemma 2.4.3.

□

Using the previous result and Lemma 2.2.9 we observe that [2, Example 3.8] is an example of weakly non-commutative Corson countably compact space that is not a non-commutative Valdivia compact. We observe that, for every set Γ, the space $[0,1]^\Gamma$ is a Valdivia compact space, therefore it is a non-commutative Valdivia compact space. Moreover, there exists Γ such that $[0,1]^\Gamma$ contains a subspace homeomorphic to a compact space that is not weakly non-commutative Corson. Finally, recalling that the class of weakly non-commutative Corson countably compact spaces is closed under countably closed subspaces (Lemma 2.2.9), we have an example of non-commutative Valdivia compact space that is not weakly non-commutative Corson compact space. Thus, the two classes are independent.

Now, we give the (weakly) non-commutative Valdivia version of Lemma 2.4.1. The proof follows straightforwardly by defining the same family of retractions of Lemma 2.4.1.

Proposition 2.4.5. *Let η be an uncountable cofinality ordinal. Any $[0,\eta+1)$-sum of (weakly) non-commutative Valdivia compact is again a (weakly) non-commutative Valdivia compact.*

Next result gives a characterization of compact ordinal segments in the non-commutative setting.

Theorem 2.4.6. *Let η be an ordinal. Then the following hold*

(*i*) $[0,\eta]$ *is non-commutative Valdivia.*

(*ii*) $[0,\eta]$ *is weakly non-commutative Valdivia.*

(*iii*) $[0,\eta]$ *is weakly non-commutative Corson.*

Proof. It trivially follows by Theorem 2.4.4 and Proposition 2.4.5. □

2.5 Aleksandrov duplicates

We recall the definition of Aleksandrov duplicate $AD(X)$ of the space X. It is the space $X \times \{0,1\}$ with the topology in which all points of $X \times \{1\}$ are isolated, and neighborhoods of $(x,0)$ are $(U \times \{0,1\}) \setminus \{(x,1)\}$, for every U neighborhood of $x \in X$. We denote by π the projection from $AD(X)$ onto X.

It is well known and easy to check that if X is compact (countably compact) then $AD(X)$ is compact (countably compact).

Now we give the main result of this section about the relations between retractional skeletons and Alexandrov duplicates of topological spaces.

Theorem 2.5.1. *(1) Let X be a (weakly) non-commutative Corson countably compact space, then $AD(X)$ is a (weakly) non-commutative Corson countably compact space as well.*

(2) Let K be a non-comutative Valdivia compact space and D be an induced subspace such that $|K \setminus D| < \aleph_0$, then $AD(K)$ is a non-commutative Valdivia compact space.

(3) Let K be a compact space, if $AD(K)$ is a non-commutative Valdivia compact space then so is K.

(4) Let η be an ordinal, then $AD([0,\eta])$ is a non-commutative Valdivia compact space.

Proof. (1) Let X be a non-commutative Corson countably compact space, then the assertion follows easily by [4, Theorem 3.1].

Let X be a weakly non-commutative Corson countably compact space, then there exists a continuous onto mapping $f : Y \to X$, where Y is a non-commutative Corson countably compact space. Let $AD(X)$ and $AD(Y)$ be the Aleksandrov duplicates of X and Y respectively. Hence, by the previous point, $AD(Y)$ is non-commutative Corson countably compact. Let $g : AD(Y) \to AD(X)$ be a map defined by $g(x,i) = (f(x), i)$, it is clearly onto. The continuity follows by the continuity of f and the definition of the topology of Aleksandrov duplicate. Therefore $AD(X)$ is a weakly non-commutative Corson countably compact space.

(2) Let K be a non-commutative Valdivia compact space. By hypothesis there exists a retractional skeleton $\{r_s\}_{s\in\Gamma}$ such that $|K \setminus D| < \aleph_0$ and $D = \bigcup_{s\in\Gamma} r_s[K]$. We observe that the restriction $\{r_s \restriction_D\}_{s\in\Gamma}$ is a full retractional skeleton on the countably compact space D. Hence using [4, Proposition 2.7] there exists a full retractional skeleton on D, $\{r_A : A \in [D]^{\leq\omega_0}\}$ such that for every $A \in [D]^{\leq\omega_0}$ we have $r_A(x) = x$ for every $x \in A$. Now, using Theorem 2.2.2 and Proposition 2.2.3, we extend every $r_A \in \{r_A : A \in [D]^{\leq\omega_0}\}$ to K as $\{R_A : A \in [D]^{\leq\omega_0}\}$ such that $R_A \restriction_D = r_A$. For every $A \in [D \times \{0,1\}]^{\leq\omega_0}$ we define $\hat{R}_A : AD(K) \to AD(K)$ as follows

$$\hat{R}_A(x,i) = \begin{cases} (R_{\pi(A)}(x), i) & \text{if } x \in \pi(A); \\ (x,1) & \text{if } x \in K \setminus D \text{ and } i = 1; \\ (R_{\pi(A)}(x), 0) & \text{otherwise.} \end{cases}$$

Using the same argument of [4, Theorem 3.1] it is possible to prove that the family of retractions $\{\hat{R}_A\}_{A\in[D\times\{0,1\}]^{\leq\omega_0}}$ is a retractional skeleton on $AD(K)$.

(3) We use the same idea as in [6, Theorem 2.13]. Let D be an induced subspace of $AD(K)$, we will prove that $D \cap K \times \{0\}$ is dense in $K \times \{0\}$; then using Lemma 2.2.4 we have that K is a non-commutative Valdivia compact space.

Let U be a non-empty open subset of K. Since K is completely regular there exists a non-empty open subspace $V \subset K$, such that $\overline{V} \subset U$. Two cases are possible:

(a) V is finite. Every point of V is a G_δ point of $AD(K)$, hence $V \times \{0\} \subset D$. Hence $U \cap D \neq \emptyset$.

(b) V is infinite. $V \times \{1\} \subset D$ is infinite and has a cluster point $(x,i) \in D$, by Theorem 2.2.2. By definition of Aleksandrov duplicate topology, i must be equal to 0 and $(x,0) \in \overline{V} \times \{0\}$. Hence $U \cap D \neq \emptyset$.

By above observation K is a non-commutative Valdivia compact space.

(4) Let η be an ordinal. Let $\mathcal{A}$ be the family of all closed countable subsets A of $[0, \eta]$ such that $0 \in A$ and any isolated point of A is isolated in $[0, \eta]$. Let $\mathcal{A}$ be ordered by inclusion. The family of mappings $\{r_A\}_{A \in \mathcal{A}}$, where $r_A : [0, \eta] \to [0, \eta]$ and $r_A(\alpha) = \max([0, \alpha] \cap A)$, is a retractional skeleton on $[0, \eta]$, see [9] or [11]. Let $AD(K)$ be the Aleksandrov duplicate of $K = [0, \eta]$. Let us define

$$\Gamma = \{(A, B) \in \mathcal{A} \times [K]^{\leq \omega_0} : \forall \beta (\neq \eta) \in B, \, \beta + 1 \in A\}.$$

Define the following order on Γ: $(A_1, B_1) \leq (A_2, B_2)$ if and only if $A_1 \subset A_2$ and $B_1 \subset B_2$. This way $(\Gamma, \leq)$ is an up-directed partially ordered set.
For every $\gamma = (A, B) \in \Gamma$ we define $r_\gamma : AD(K) \to AD(K)$ as follows

$$r_\gamma(x, i) = \begin{cases} (r_A(x), 0) & \text{if } (x, i) \in K \times \{0\} \cup (K \times \{1\} \setminus B \times \{1\}) \\ (x, i) & \text{if } (x, i) \in B \times \{1\} \end{cases}$$

<u>Claim:</u> For every $\gamma \in \Gamma$, r_γ is continuous. Let $\gamma = (A, B) \in \Gamma$ and let $\{(x_\lambda, i_\lambda)\}_{\lambda \in \Lambda}$ be a net converging to $(x, i) \in AD(K)$. We can suppose without loss of generality that $x_\lambda \leq x$ for every $\lambda \in \Lambda$. Now we study all possible cases

- if $i = 1$ then $\{(x_\lambda, i_\lambda)\}_{\lambda \in \Lambda}$ is eventually constant. Hence it follows that $\lim_{\lambda \in \Lambda} r_\gamma(x_\lambda, i) = r_\gamma(x, i)$.
- if $i = 0$ then $x_\lambda \to x$ in K, two cases are possible:
  (a) suppose that $(x_\lambda, i_\lambda) \notin B \times \{1\}$ for sufficiently large λ. Then we conclude by continuity of r_A;
  (b) suppose that $(x_\lambda, i_\lambda) \in B \times \{1\}$ for a cofinal set $\Lambda_1 \subset \Lambda$. Since $x_\lambda \nearrow x$ for $\lambda \in \Lambda_1$, we have $x_\lambda + 1 \nearrow x$ as well. Hence we have $x \in A$, then by definition $r_\gamma(x, 0) = (x, 0) = (r_A(x), 0)$. Finally, since $r_\gamma(x_\lambda, i_\lambda)$ is equal to $(x_\lambda, 1)$ for $\lambda \in \Lambda_1$ and equal to $(r_A(x_\lambda), 0)$ otherwise, we have that $\lim_{\lambda \in \Lambda} r_\gamma(x_\lambda, i_\lambda) = (x, 0) = (r_A(x), 0)$, then we are done.

This proves the claim. It remains to show that $\{r_\gamma\}_{\gamma \in \Gamma}$ is a retractional skeleton.

(*i*) Since $r_\gamma[AD(K)]$ is compact and countable, it is metrizable.

(*ii*) Let $\gamma_1 = (A_1, B_1)$, $\gamma_2 = (A_2, B_2) \in \Gamma$ such that $\gamma_1 \leq \gamma_2$. Let $(x, i) \in AD(K)$. Then, if $i = 0$ we have $r_{\gamma_1}(x, 0) = (r_{A_1}(x), 0) = (r_{A_1} \circ r_{A_2}(x), 0) = r_{\gamma_1} \circ r_{\gamma_2}(x, 0)$ and $r_{\gamma_2} \circ r_{\gamma_1}(x, 0) = (r_{A_2} \circ r_{A_1}(x), 0) = (r_{A_1}(x), 0) = r_{\gamma_1}(x, 0)$. If $i = 1$ three cases are possible

  (a) $x \in B_1$ then $x \in B_2$ as well and the equality is trivial.
  (b) $x \notin B_1$ and $x \in B_2$ then we have $r_{\gamma_2} \circ r_{\gamma_1}(x, 1) = (r_{A_2} \circ r_{A_1}(x), 0) = (r_{A_1}(x), 0) = r_{\gamma_1}(x, 1)$ and $r_{\gamma_1} \circ r_{\gamma_2}(x, 1) = r_{\gamma_1}(x, 1)$.
  (c) $x \notin B_2$, then $r_{\gamma_1}(x, 1) = (r_{A_1}(x), 0) = (r_{A_1} \circ r_{A_2}(x), 0) = r_{\gamma_1} \circ r_{\gamma_2}(x, 1)$ and $r_{\gamma_2} \circ r_{\gamma_1}(x, 1) = (r_{A_2} \circ r_{A_1}(x), 0) = (r_{A_1}(x), 0) = r_{\gamma_1}(x, 1)$.

(iii) Let $\gamma_1 \leq \gamma_2 \leq \ldots$ with $\gamma_n = (A_n, B_n)$ for every $n < \omega_0$ and $\gamma = \sup_{n<\omega_0}(A_n, B_n) = (\overline{\bigcup_{n<\omega_0} A_n}, \bigcup_{n<\omega_0} B_n)$.
We observe that

- $|\overline{\bigcup_{n<\omega_0} A_n}| \leq \aleph_0$,
- $0 \in \overline{\bigcup_{n<\omega_0} A_n}$
- every isolated point of $\overline{\bigcup_{n<\omega_0} A_n}$ belongs to some A_k, hence it is isolated also in K.

Moreover $|\bigcup_{n<\omega_0} B_n| \leq \aleph_0$ and for every $\beta \in B$ there exists $k \in \omega_0$ such that $\beta \in B_k$, hence $\beta + 1 \in A_k \subset \overline{\bigcup_{n<\omega_0} A_n}$. Therefore $\gamma \in \Gamma$.
Let $(x, i) \in AD(K)$, if $i = 0$ or $i = 1$ and $x \notin B_n$ for every $n \in \omega_0$ we have

$$\lim_{n\in\omega_0} r_{\gamma_n}(x, i) = \lim_{n\in\omega_0} (r_{A_n} x, 0) = r_\gamma(x, i).$$

If $i = 1$ and $x \in B_{n_0}$ for some $n_0 < \omega_0$ then

$$\lim_{n\in\omega_0} r_{\gamma_n}(x, 1) = (x, 1) = r_\gamma(x, 1).$$

(iv) Let $(x, i) \in AD(K)$, if $i = 0$ then

$$\lim_{\gamma\in\Gamma} r_\gamma(x, 0) = \lim_{A\in\mathcal{A}} (r_A(x), 0) = (x, 0).$$

If $i = 1$ set $\gamma_0 = (\{0, x+1\}, \{x\})$ if $x < \eta$ and $\gamma_0 = (0, \{\eta\})$ if $x = \eta$. Then for $\gamma \geq \gamma_0$ we have $r_\gamma(x, 1) = (x, 1)$. Therefore $\lim_{\gamma\in\Gamma} r_\gamma(x, 1) = (x, 1)$ and *(iv)* is proved.

Hence $\{r_\gamma\}_{\gamma\in\Gamma}$ is a retractional skeleton on $AD(K)$.
This completes the proof. □

2.6 Associated Banach spaces

In this section we generalize Section 4 of [7] to the non-commutative case. Though the ideas, of following proofs, are the same of [7] we prefer to give every result because they are interesting in the non-separable Banach space theory setting.

Proposition 2.6.1. *Let X be a Banach space, then the following assertions are equivalent.*

(i) *X is linearly isometric to a subspace of a Banach space Y, that has a 1-projectional skeleton.*

(ii) *(B_{X^*}, w^*) contains a dense convex symmetric weakly non-commutative Corson countably compact space.*

(iii) *(B_{X^*}, w^*) is a weakly non-commutative Valdivia space.*

Proof. $(i) \Rightarrow (ii)$ Let i be the isometric injection of X into Y and i^* be the adjoint surjection of Y^* onto X^*.
By [2, Proposition 3.14] there exists a convex symmetric set R induced by a retractional skeleton in (B_{Y^*}, w^*). Hence by Theorem 2.2.2 R is a dense non-commutative Corson countably compact space.
Since i^* is a linear w^*-w^*-continuous mapping and $i^*(B_{Y^*}) = B_{X^*}$, we have that $i^*(R)$ is a convex symmetric dense weakly non-commutative Corson countably compact space.
$(ii) \Rightarrow (iii)$ Easily follows by definition of weakly non-commutative Valdivia compact space.
$(iii) \Rightarrow (i)$ Suppose (B_{X^*}, w^*) is weakly non-commutative Valdivia. Then there exist a non-commutative Valdivia compact space K and a continuous onto mapping $T : K \to (B_{X^*}, w^*)$. By [2, Proposition 3.15] $C(K)$ has a 1-projectional skeleton. Moreover, using the adjoint map of T we have that $C(B_{X^*}, w^*)$ is an isometric subspace of $C(K)$. Observing that, by Hahn-Banach extension theorem, we have $X \hookrightarrow C(B_{X^*}, w^*)$, $X \hookrightarrow C(K)$ isometrically. □

Definition 2.6.2. Let X be a Banach space, we will call it *non-commutative weakly WLD* if and only if (B_{X^*}, w^*) is a weakly non-commutative Corson compact.

Proposition 2.6.3. *The class of non-commutative weakly WLD Banach spaces is closed under isomorphisms, subspaces and quotients.*

Proof. Let X be a non-commutative weakly WLD Banach space and the map $T : Y \to X$ be an isomorphism. Using Lemma 2.2.9 and observing that $(T^*)^{-1}(B_{Y^*})$ is a weak*-closed subspace of λB_{X^*}, for some $\lambda \in \mathbb{R}^+$, we have that B_{Y^*} is a weakly non-commutative Corson compact space. Therefore Y is a non-commutative weakly WLD Banach space.
Let X be a non-commutative weakly WLD Banach space and Y be a closed subspace of X. Let $i : Y \to X$ be the canonical embedding then $i^*(B_{X^*}) = B_{Y^*}$. Using Lemma 2.2.9 we have that B_{Y^*} is a weakly non-commutative Corson compact. Therefore Y is a non-commutative weakly WLD Banach space.
Let X be a non-commutative weakly WLD Banach space and Y be a closed subspace. Since $(X/Y)^*$ is canonically isometric and weak* homeomorphic to $Y^\perp$ and $Y^\perp$ is a weak*-closed subspace of X^*, we have by Lemma 2.2.9 that $B_{(X/Y)^*}$ is weakly non-commutative Corson. Therefore X/Y is a non-commutative weakly WLD Banach space. □

Proposition 2.6.4. *Let X be a Banach space such that X^* contains a convex weak* -compact subset that is not weakly non-commutative Valdivia. Then there is an equivalent norm on $X \times \mathbb{R}$ such that it cannot be linearly isometric to any subspace of Y, where Y is any Banach space with 1-projectional skeleton.*

Proof. Let $K \subset X^*$ be a convex weak* compact that is not weakly non-commutative Valdivia. Put

$$B = \operatorname{conv}\left(\left(\frac{1}{2}B_{X^*} \times [-\frac{1}{2}, \frac{1}{2}]\right) \cup (K \times \{1\}) \cup (-K \times \{1\})\right).$$

Since $(\frac{1}{2}B_{X^*} \times [-\frac{1}{2}, \frac{1}{2}]) \subset B \subset (B_{X^*} \times [-1, 1])$, we have that B is a dual unit ball of $X \times \mathbb{R}$. Furthermore, since $K \times \{1\} = \bigcap_{n \in \omega_0} K \times (1 - \frac{1}{n}, 1]$, it is a weak* closed weak* G$_\delta$ subset of B. Suppose by contradiction that B is weakly non-commutative Valdivia. By Proposition 2.3.6 K must be a weakly non-commutative Valdivia compact space, which is a contradiction. Then the statement follows by Proposition 2.6.1. ☐

We observe that it is possible to prove the previous result in the complex setting, we define

$$B = \operatorname{conv}(\frac{1}{2}B_{X^*} \times \overline{D(0, 1/2)} \cup \overline{\operatorname{conv} \bigcup_{|\alpha|=1} \{\alpha K \times \{\alpha\}\}}^{w^*}),$$

where $D(0, 1/2)$ is the complex disk with centre in the origin and radius equal to $1/2$. Then the proof follows using the same construction as in [5, Theorem 4].

Corollary 2.6.5. *Let K be a compact space such that there exists a closed subset $L \subset K$ with a dense set of (relatively) G_δ points which is not weakly non-commutative Valdivia. Then there exists an equivalent norm on $C(K)$ such that it cannot be linearly isometric to any subspace of Y, where Y is any Banach space with 1-projectional skeleton.*

Proof. By Theorem 2.3.7 $P(L)$ is not weakly non-commutative Valdivia. Since $P(L)$ can be identified with a convex weak* compact subset of $P(K)$, the statement follows by Proposition 2.6.4. ☐

Proposition 2.6.6. *Let X, Y be Banach spaces. Suppose that Y has a 1-projectional skeleton, the norm on X is Gâteaux smooth and X is isometric to some subspace of Y. Then X is non-commutative weakly WLD.*

Proof. Let D be a dense weakly non-commutative Corson countably compact subspace of B_{X^*}. Let $x^* \in S_{X^*}$ be a functional that attains its norm at some point $x_0 \in S_X$. Since the hyperplane $\{y^* \in X^* : y^*(x_0) = 1\}$ is weak* G_δ and the norm on X is Gâteaux differentiable at x_0, then

$$\{x^*\} = \{y^* \in X^* : y^*(x_0) = 1\} \cap B_{X^*}.$$

Therefore x^* is a weak* G_δ point of B_{X^*}. Hence it belongs to D. By the Bishop-Phelps theorem the subset of norm-attaining functionals is norm dense in S_{X^*}. Since D is closed with respect to limits of sequences we have that $S_{X^*} \subset D$. Using the corollary of Josefson-Nissenzweig theorem we have that $B_{X^*} \subset D$. Hence we are done. ☐

Proposition 2.6.7. *Let K be a weakly non-commutative Corson compact space with property (M). If L is a compact space such that $C(L)$ is isomorphic to $C(K)$, then L is weakly non-commutative Corson.*

Proof. Since K is a weakly non-commutative Corson compact space with property (M), using Theorem 2.3.9 we have that $C(K, \mathbb{C})$ is non-commutative weakly WLD.

Since the class of non-commutative weakly WLD spaces is closed under isomorphisms we have that $C(L, \mathbb{C})$ is non-commutative weakly WLD. Therefore by definition of non-commutative weakly WLD we have that $(B_{C(L,\mathbb{C})^*}, w^*)$ is a weakly non-commutative Corson compact space. Finally, since L is a closed subset of $(B_{C(L,\mathbb{C})^*}, w^*)$, using Lemma 2.2.9, we have that L is weakly non-commutative Corson compact as well. The proof holds for the real case as well.

□

Observing that for every ordinal number η the topological space $[0, \eta]$ has the property (M) we have the following result.

Corollary 2.6.8. *Let L be a compact space. Suppose that $C(L)$ is isomorphic to $C([0, \eta])$, for some ordinal number η. Then L is a weakly non-commutative Corson compact space.*

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Chapter 3

New examples of non-commutative Valdivia compact spaces

3.1 Introduction

The class of Valdivia compact spaces plays an important role in the study of non-separable Banach spaces. It has been investigated for example in [4] or more recently in [6]. For a detailed survey of this subject we refer to [7]. A related class of compact spaces is the class of the non-commutative Valdivia compacta. As in [2], for non-commutative Valdivia compacta we mean the class of compact spaces with retractional skeleton. These two classes share several topological properties (see for example [2] and [12]). The Banach spaces associated to Valdivia compacta ⟨respectively non-commutative Valdivia compacta⟩ are called Plichko spaces ⟨respectively Banach spaces with projectional skeletons⟩. These two classes share several structural and geometrical properties (see [8] and [3]). The definition of retractional skeleton was introduced by Kubiś and Michalewski in [9]. In the same paper the authors proved that a compact space is Valdivia if and only if it has a commutative retractional skeleton. The two classes do not agree: the ordinal space $[0,\omega_2]$, endowed with the interval topology, is an easy well-known example of a non-commutative Valdivia compact space that is not Valdivia (see [9, Example 6.4]). Nevertheless, since $[0,\omega_2]$ is involved in any such example, the following question seems to be of interest:

Let X be a non-commutative Valdivia compact space that does not contain any copy of the ordinal space $[0,\omega_2]$. Is X necessarily Valdivia?

The present paper answers in the negative the question above. That can be done via a characterization of trees, endowed with coarse wedge topology, with retractional skeleton. That characterization can have an independent interest.

The coarse wedge topology (also known as path topology) has been studied in detail in [13] for providing a minimal size counterexample to preservation of Lindelöf property in products. In [14] is contained an example of tree endowed with the same topology which is a first countable Corson compact space with no dense metrizable subspace.

The paper is organized as follows. In the remaining part of the introductory sec-

tion, notations and basic notions addressed in this paper are given. In section 2 we provide basic definitions on trees and we recall the basic results on the coarse wedge topology, finally we recall some useful results about Valdivia and non-commutative Valdivia compacta classes. Section 3 is devoted to proving the characterization of trees with retractional skeleton. In section 4 we investigate the relations between trees and Valdivia compact spaces, in particular we present an example that gives a negative answer to the question above.
We denote with ω the set of natural numbers (including 0) with the usual order. Given a set X we denote by $|X|$ the cardinality of the set X, by $[X]^{\leq\omega}$ ($[X]^{<\omega}$) the family of all countable (finite) subsets of X.
All the topological spaces are assumed to be Hausdorff and completely regular. Given a topological space X we denote by $\overline{A}$ the closure of $A \subset X$. We say that $A \subset X$ is countably closed if $\overline{C} \subset A$ for every $C \in [A]^{\leq\omega}$. A topological space X is a Fréchet-Urysohn space if for every $A \subset X$ and $x \in \overline{A}$ there is a sequence $\{x_n\}_{n\in\omega} \subset A$ such that $x_n \to x$. βX denotes the Čech-Stone compactification of X.

3.2 Basic notions

In the first part of this section we recall some basic definitions about trees and the topology on them. The second part is devoted to giving the basic definitions and properties of classes of compact spaces which we are interested in.

Definition 3.2.1. A *tree* is a partially ordered set $(T, \leq)$, such that for each $x \in T$ the set $\{y \in T : y < x\}$ of all predecessors of x is well-ordered by $<$.

If a tree has only one minimal member, it is said to be *rooted* and the minimal member is called the *root* of the tree, in this case we will denote the root by 0. Maximal chains are called *branches*. Given a tree T we will use the following notations:

- the height of x in T, denoted by $\mathrm{ht}(x, T)$, is the order type of $\{y \in T : y < x\}$;
- for each ordinal α, the α-th level of T, is the set defined by $\mathrm{Lev}_\alpha(T) = \{x \in T : \mathrm{ht}(x, T) = \alpha\}$;
- by $\mathrm{cf}(x)$ we indicate the cofinality of $\mathrm{ht}(x, T)$;
- the height of T, denoted by $\mathrm{ht}(T)$, is the least α such that $\mathrm{Lev}_\alpha(T) = \emptyset$;
- we denote by $\mathrm{ims}(x) = \{y \in T : x < y \,\&\, \mathrm{ht}(y, T) = \mathrm{ht}(x, T) + 1\}$ the set of the immediate successors of x.

For $t \in T$ we put $V_t = \{s \in T : s \geq t\}$ and $\hat{t} = \{s \in T : s \leq t\}$. In this work we consider T endowed with the *coarse wedge topology*. Now we recall the definition of such a topology and its properties, we refer to [13] and [11] for the details. The coarse wedge topology on a tree T is the one whose subbase is the set of all V_t and their complements, where t is either minimal or on a successor level. Let us describe a local base at each point $t \in T$. If $\mathrm{ht}(t, T)$ is a successor or t is the minimal element, a local base at t is formed by sets of the form

$$W_t^F = V_t \setminus \bigcup\{V_s : s \in F\},$$

where F is a finite set of immediate successors of t. In case $\mathrm{ht}(t,T)$ is limit, a local base at t is formed by sets of the form

$$W_s^F = V_s \setminus \bigcup\{V_r : r \in F\},$$

where $s < t$, $\mathrm{ht}(s,T)$ is a successor and F is a finite set of immediate successor of t. From now on every tree is considered endowed with the coarse wedge topology. We are interested in examples of compact spaces, therefore we recall the characterization of the trees which are compact in the coarse wedge topology.

Definition 3.2.2. A tree is called *chain complete* if every chain has a supremum.

Theorem 3.2.3. *[11, Corollary 3.5], A tree T is compact Hausdorff in the coarse wedge topology if and only if T is chain complete and has finitely many minimal elements.*

From now on we will consider only chain complete trees with a unique minimal element. We observe that if T is as in the previous theorem, it is a zero dimensional compact space. Let us now give the definition and a characterization of Valdivia compact spaces.

Definition 3.2.4. For any set Γ we put $\Sigma(\Gamma) = \{x \in [0,1]^\Gamma : |\{\gamma \in \Gamma : x(\gamma) \neq 0\}| \leq \omega_0\}$. Let K be a compact space, we say that $A \subset K$ is a Σ-subset of K if there is a homeomorphic injection h of K into some $[0,1]^\Gamma$ such that $h(A) = h(K) \cap \Sigma(\Gamma)$. K is called *Valdivia compact space* if it has a dense Σ-subset.

A family of sets $\mathcal{U}$ is T_0-separating in X if for every two distinct elements $x, y \in X$ there is $U \in \mathcal{U}$ satisfying $|\{x,y\} \cap U| = 1$. A family $\mathcal{U}$ is point countable on $D \subset X$ if

$$|\{U \in \mathcal{U} : x \in U\}| \leq \omega$$

for every $x \in D$. Given $x \in X$ we indicate by $\mathcal{U}(x) = \{U \in \mathcal{U} : x \in U\}$. We recall a useful characterization of Valdivia compact spaces.

Theorem 3.2.5. *[7, Proposition 1.9], Let K be a compact space and D be a dense subset of K. Then D is a Σ-subset of K if and only if there is a family $\mathcal{U}$ of open F_σ subsets of K which is T_0-separating and D is the set of point countability, i.e., $D = \{x \in K : \mathcal{U}(x)$ is countable$\}$.*

It is easy to see that the family $\mathcal{U}$ in the previous theorem may consist of sets from a given base. Therefore, if K is zero-dimensional, $\mathcal{U}$ may consists of clopen sets, see [5, Theorem 19.11] for the details. Now let us recall the notion of retractional skeleton which is used to define a non-commutative counterpart of Valdivia compact spaces.

Definition 3.2.6. A *retractional skeleton* in a compact space K is a family of continuous retractions $\{r_s\}_{s\in\Gamma}$, indexed by an up-directed partially ordered set Γ, such that:

(*i*) $r_s[K]$ is a metrizable compact space for each $s \in \Gamma$,

(*ii*) $s, t \in \Gamma$, $s \leq t$ then $r_s = r_t \circ r_s = r_s \circ r_t$,

(*iii*) given $s_0 \leq s_1 \leq ...$ in Γ, $t = \sup_{n \in \omega} s_n$ exists and $r_t(x) = \lim_{n \to \infty} r_{s_n}(x)$ for every $x \in K$,

(*iv*) for every $x \in K$, $x = \lim_{s \in \Gamma} r_s(x)$.

We say that $D = \bigcup_{s \in \Gamma} r_s[K]$ is the set induced by the retractional skeleton $\{r_s\}_{s \in \Gamma}$ in K.

By [9] K is Valdivia if and only if it has a commutative retractional skeleton. More precisely, a dense set $D \subset K$ is a Σ-subset if and only if it is induced by a commutative retractional skeleton. Compact spaces having a retractional skeleton (not necessarily commutative) are called, following [2], non-commutative Valdivia compact spaces. We recall some useful and well-known results about retractional skeletons.

Theorem 3.2.7. *[8, Theorem 32] Assume D is induced by a retractional skeleton in a compact space K. Then:*

(*i*) *D is dense in K and for every countable set $A \subset D$, $\overline{A}$ is metrizable and contained in D.*

(*ii*) *D is a Fréchet-Urysohn space.*

(*iii*) *D is a normal space and $K = \beta D$.*

In particular we observe that, given a retractional skeleton in a compact space X, its induced subset D is countably compact.

3.3 Trees and Retractional Skeletons

Since we are interested in compact Hausdorff spaces, by Theorem 3.2.3 we assume that every tree is rooted, chain complete and endowed with the coarse wedge topology.
We start by stating the main result of this section, which provides a characterization of trees with retractional skeleton.

Theorem 3.3.1. *Let T be a tree. T has a retractional skeleton if and only if it satisfies the following condition:*

$$t \text{ has at most finitely many immediate successors whenever } cf(t) \geq \omega_1. \qquad (*)$$

We observe that the "only if" part follows from the following proposition:

Proposition 3.3.2. *Let T be a tree. If T has a retractional skeleton then it satisfies $(*)$ and, moreover, the induced subset is $D = \{t \in T : cf(t) \leq \omega\}$.*

Proof. Let D be the induced subset of a retractional skeleton on T. Let $t \in T$ in a successor level, we want to prove that $t \in D$. Two cases are possible:

- suppose that $\operatorname{ims}(t)$ is finite, then t is isolated and therefore, since D is dense, we have $t \in D$;
- suppose there exists an infinite subset $\{t_n\}_{n\in\omega} \subset \operatorname{ims}(t)$. For every $n \in \omega$, by the density of D, there exists $s_n \in V_{t_n} \cap D$. We observe that the sequence $\{s_n\}_{n\in\omega}$ converges to t, hence, since D is countably closed, we have $t \in D$.

Since D is countably closed $t \in D$ whenever $\operatorname{cf}(t) = \omega$. Since D is a Fréchet-Urysohn space, if $\operatorname{cf}(t) \geq \omega_1$ we have that $t \notin D$. This implies that $\operatorname{ims}(t)$ is finite whenever $\operatorname{cf}(t) \geq \omega_1$. □

The remaining part of this section is devoted to prove the "if part" of Theorem 3.3.1. The strategy is to find a suitable class of subsets of the tree that play the role of the set of indices of the retractional skeleton.

Definition 3.3.3. Let T be a tree that satisfies $(*)$. Define the following mappings:

- $\wedge : T \times T \to T$ such that $t \wedge s = \max(\hat{s} \cap \hat{t})$, for every $s, t \in T$.
- $\varphi : T \to [T]^{\leq\omega}$ such that $\varphi(t) = \{0\}$ if $\operatorname{cf}(t) \neq \omega$ and $\varphi(t) = \{t_n\}_{n\in\omega}$ if $\operatorname{cf}(t) = \omega$, where $\{t_n\}_{n\in\omega} \subset T$ is a sequence that converges to t, $t_n < t$ for every $n \in \omega$ and t_n belongs to a successor level for every $n \in \omega$.
- $\psi : T \to [T]^{<\omega}$ such that $\psi(t) = \{0\}$ if $\operatorname{cf}(t) < \omega_1$, $\psi(t) = \operatorname{ims}(t)$ if $\operatorname{cf}(t) \geq \omega_1$.

We denote by $\mathcal{A}(T)$ the class of subsets A of a tree T that satisfy the following properties:

(a) A is countable,

(b) $0 \in A$,

(c) if $s, t \in A$, then $s \wedge t \in A$,

(d) if $t \in A$, then $\varphi(t) \subset A$,

(e) if $t \in A$, then $\psi(t) \subset A$.

Proposition 3.3.4. *Let T be a tree that satisfies $(*)$ and S be a countable subset of T, then there exists an $A \in \mathcal{A}(T)$ such that $S \subset A$.*

Proof. We will use an induction argument, consider $S_0 = S \cup \{0\}$ and

$$S_{n+1} = S_n \cup \wedge(S_n \times S_n) \cup \psi(S_n) \cup \varphi(S_n).$$

Define $A = \bigcup_{n\in\omega} S_n$, it remains to prove that $A \in \mathcal{A}(T)$:

(a) given a countable set N, we have that $\wedge(N \times N), \psi(N), \varphi(N)$ are countable. This implies that A is countable.

(b) $0 \in S_0 \subset A$;

(c) let $s, t \in A$, then there exists $n \in \omega$ such that $s, t \in S_n$, therefore $s \wedge t \in S_{n+1} \subset A$;

(d) let $t \in A$, then there exists $n \in \omega$ such that $t \in S_n$, therefore $\varphi(t) \subset S_{n+1} \subset A$;

(e) if $t \in A$, then there exists $n \in \omega$ such that $t \in S_n$, therefore $\psi(t) \subset S_{n+1} \subset A$.

This completes the proof. □

It follows from Proposition 3.3.4 that $\mathcal{A}(T)$, ordered by inclusion, is an up-directed partially ordered set. Moreover, it is obvious that $\mathcal{A}(T)$ is σ-complete.
Now we are going to list some properties of $\overline{A}$, with $A \in \mathcal{A}(T)$ and the closure is taken in T with respect to the coarse wedge topology.

Proposition 3.3.5. *Let T be a tree that satisfies $(*)$. Let $A \in \mathcal{A}(T)$ and $D = \{t \in T : cf(t) \leq \omega\}$, then the following properties hold:*

1. *$\overline{A}$ is separable,*
2. *$0 \in \overline{A}$,*
3. *if $t \in \overline{A}$ and $cf(t) = \omega$, then there exists a sequence $\{t_n\}_{n\in\omega} \subset A$ that converges to t, $t_n < t$ for every $n \in \omega$ and each element of $\{t_n\}_{n\in\omega}$ belongs to a successor level,*
4. *if $t \in \overline{A} \setminus A$, then $cf(t) = \omega$,*
5. *if $t, s \in \overline{A}$, then $s \wedge t \in \overline{A}$,*
6. *if $t \in \overline{A}$, then $\psi(t) \subset \overline{A}$,*
7. *$\overline{A} \cap D = \overline{A \cap D}$.*

Proof. (1) and (2) are clear.

(3) If $\mathrm{cf}(t) = \omega$ and $t \in A$, the assertion follows by taking $\varphi(t)$. Suppose that $t \in \overline{A} \setminus A$ and $\mathrm{cf}(t) = \omega$.
The first step is to prove that for every $s < t$, with s on a successor level, there exists $w \in A$ such that $s \leq w < t$. In order to do this, we take an open neighborhood W_s^F such that if $x \in W_s^F \cap A$, then either $x < t$ or x and t are incomparable. Since $t \in \overline{A} \setminus A$, such a neighborhood exists. Let $x \in W_s^F \cap A$, if $x < t$, then we take $w = x$. Suppose now that $x \in W_s^F \cap A$ and x and t are incomparable. Let $w = x \wedge t$ and consider W_r^F where $w \leq r < t$ and r is on a successor level, hence there exists $w_1 \in W_r^F \cap A$ such that $w = x \wedge w_1$. Therefore $w \in A$ and $s \leq w < t$.
So, we can find a sequence $\{s_n\}_{n\in\omega} \subset T$ that converges to t, each s_n belongs to a successor level and $s_n < t$ for every $n \in \omega$. By the previous consideration there exists $w_n \in A$ such that $s_n \leq w_n < t$ for every $n \in \omega$. The assertion follows by considering three different cases:

- if w_n is on a successor level, then $t_n = w_n$;
- if $\mathrm{cf}(w_n) = \omega$, then there exists $t_n \in \varphi(w_n)$ such that $s_n < t_n < t$;
- if $\mathrm{cf}(w_n) \geq \omega_1$, then there exists $t_n \in \psi(w_n)$ such that $s_n < t_n < t$.

(4) Suppose that $t \in \overline{A}$ and t on a successor level. If t is isolated, then obviously $t \in A$. So, suppose that t is an accumulation point. Let $x \in V_t \cap A$ and $y \in W_t^F \cap A$, where $F \subset \mathrm{ims}(t)$ and $x \notin W_t^F$. Since $t = x \wedge y$ and $A \in \mathcal{A}(T)$, we have $t \in A$. On the other hand, let $t \in \overline{A}$ and $\mathrm{cf}(t) \geq \omega_1$. Fix some $s < t$ on a successor level and set $F =\mathrm{ims}(t)$. Since A is countable, we have that $A \cap W_s^F \setminus \{t\} = \{s_i\}_{i \in \omega}$. Let $\overline{s}_i = t \wedge s_i$, and let $\overline{s} = \sup\{\overline{s}_i\}$. Since $\mathrm{cf}(t) \geq \omega_1$, we have $\overline{s} < t$. Hence $W_{\overline{s}+1}^F \cap A = \{t\}$, where $\overline{s}+1 \in \mathrm{ims}(\overline{s})$ and $\overline{s}+1 < t$. Therefore $t \in A$. We conclude that if $t \in \overline{A} \setminus A$, then $\mathrm{cf}(t) = \omega$.

(5) Let $s, t \in \overline{A}$, we may assume without loss of generality that s and t are incomparable. Let $s_0, t_0 \in \mathrm{ims}(s \wedge t)$ such that $s_0 \leq s$ and $t_0 \leq t$. Thus we have that $s \in V_{s_0}$, $t \in V_{t_0}$ and $V_{s_0} \cap V_{t_0} = \emptyset$. Therefore there are $s_1, t_1 \in A$ such that $s_1 \in V_{s_0}$ and $t_1 \in V_{t_0}$, hence $t \wedge s = t_1 \wedge s_1 \in A$.

(6) It follows from point (4).

(7) Since D is countably closed and $A \cap D$ is countable, we have $\overline{A \cap D} \subset D$, moreover, since $A \cap D \subset A$, we have $\overline{A \cap D} \subset \overline{A}$; therefore $\overline{A \cap D} \subset \overline{A} \cap D$. On the other hand, if $t \in \overline{A} \setminus A$, by (3) and (4), we have $\mathrm{cf}(t) = \omega$, hence, by definition of D, $t \in D$ which implies $\overline{A} \cap D \subset \overline{A \cap D}$. □

Proposition 3.3.6. *Let T be a tree that satisfies $(*)$, $A \in \mathcal{A}(T)$ and $r_A : T \to T$ defined by $r_A(t) = \max\{\hat{t} \cap \overline{A} \cap D\}$. Then r_A is a continuous retraction with range $\overline{A} \cap D$.*

Proof. We observe that by point (7) of Proposition 3.3.5, the mapping r_A is well-defined for every $A \in \mathcal{A}(T)$. Now we are going to prove the continuity of each r_A. Let $t \in T$:

- if $t \notin \overline{A \cap D}$, there exists an open neighborhood W_s^F of t that does not intersect $\overline{A \cap D}$. Since r_A is constant on W_s^F, r_A is continuous at t;
- if $t \in \overline{A \cap D}$, then we consider three different cases:
  1. suppose that t is on a successor level and $\mathrm{ims}(t)$ is finite, then t is an isolated point, therefore r_A is continuous at t;
  2. suppose that t is on a successor level and $\mathrm{ims}(t)$ is infinite. Let W_t^F be an open neighborhood of $r_A(t) = t$. Observing that $r_A(s) \geq r_A(t)$ for every $s \in W_t^F$ and if, $y \in F$, then $z \notin W_t^F$ for every $z \in V_y$, we obtain $r_A(W_t^F) \subset W_t^F$. Therefore r_A is continuous at t;
  3. suppose that $\mathrm{cf}(t) = \omega$. Let W_s^F be an open neighborhood of $r_A(t) = t$, hence, by the point (3) of Proposition 3.3.5, there exists a successor point $w \in A$ such that $s \leq w < t$. Hence $r_A(W_w^F) \subset W_s^F$. Therefore r_A is continuous at t.

This proves the continuity. □

Now we can provide the "if part" of Theorem 3.3.1:

Proof of Theorem 3.3.1. Suppose that T satisfies condition $(*)$, consider the family of retractions $\{r_A\}_{A\in\mathcal{A}(T)}$ as in Proposition 3.3.6 and $D = \{t \in T : \mathrm{cf}(t) \leq \omega\}$. We have to prove $(i) - (iv)$ of Definition 3.2.6.

(i) Let $A \in \mathcal{A}(T)$, we have $r_A[T] = \overline{A} \cap D$. Since $\overline{A} \cap D$ is closed and it has at most countable chains, by [10, Theorem 2.8], we have that $r_A[T]$ is a Corson compact space. Moreover, since $\overline{A}$ is separable, we have that $r_A[T]$ is metrizable.

(ii) Let $A, B \in \mathcal{A}(T)$ such that $A \leq B$, then for every $t \in T$ we have $r_A(t) \leq r_B(t) \leq t$. We observe that $r_A(t) \in \overline{A} \cap D \subset \overline{B} \cap D$, therefore $r_A(t) = r_B(r_A(t))$. If $r_B(t) \in \overline{A}$, then $r_A(t) = r_A(r_B(t))$. On the other hand if $r_B(t) > r_A(t)$, then there are no points $s \in \overline{A} \cap D$ such that $r_A(t) < s < r_B(t)$. Hence $r_A(t) = r_A(r_B(t))$.

(iii) Consider $A_1 \leq A_2 \leq ... \leq A_n \leq ...$ and $A = \sup_{n\in\omega} A_n$. Since $\overline{A_n} \cap D \subset \overline{A_m} \cap D \subset \overline{A} \cap D$ if $n \leq m$, we have, for every $t \in T$, $r_{A_n}(t) \leq r_{A_m}(t) \leq r_A(t)$. Hence $\lim_{n\in\omega} r_{A_n}(t) = \sup_{n\in\omega} r_{A_n}(t)$. Suppose, by contradiction, $\sup_{n\in\omega} r_{A_n}(t) < s \leq r_A(t)$ for some s in a successor level. Since V_s is an open neighborhood of $r_A(t)$ and $A = \bigcup_{n\in\omega} A_n$ is dense in $\overline{A}$, there exists $n \in \omega$ and $s_1 \in A_n \cap V_s$. Consider $W_s^F \subset V_s$ such that $s_1 \notin W_s^F$. By the same reasons of above there exists $m \geq n$ and $s_2 \in A_m$ such that $s_2 \in W_s^F$. Thus $s_0 = s_1 \wedge s_2$ belongs to A_m and $s \leq s_0 \leq t$. This implies $r_{A_m}(t) \geq s > \sup_{n\in\omega} r_{A_n}(t)$. A contradiction.

(iv) Let $t \in T$, we split in two cases:

- suppose $\mathrm{cf}(t) \geq \omega_1$, then for every open set $t \in W_s^F$ there exists a successor point $w < t$ and $w \in W_s^F$. Hence by Proposition 3.3.4 there exists $A \in \mathcal{A}(T)$ such that $w \in A$. Hence $r_B(t) \in W_s^F$ for every $B \supseteq A$;
- suppose $\mathrm{cf}(t) \leq \omega$, then by Proposition 3.3.4 there exists $A \in \mathcal{A}(T)$ such that $t \in A$. Hence, $r_B(t) = t$ for every $B \supseteq A$.

Therefore $\{r_A\}_{A\in\mathcal{A}(T)}$ is a retractional skeleton on T. □

3.4 Valdivia compact trees

In this section we investigate the relations between Valdivia compact spaces and trees. We start by proving that every tree of height at most $\omega_1 + 1$ is Valdivia. On the other hand if a tree is Valdivia then its height must be less than $\omega_2 + 1$.

Theorem 3.4.1. *Let T be a tree, then the following assertions hold:*

(1) *if $ht(T) \leq \omega_1 + 1$, then T is a Valdivia compact space;*

(2) *if T is Valdivia, then $ht(T) < \omega_2 + 1$.*

Proof. (1) We will use Theorem 3.2.5 to prove that T is Valdivia. Let $D = \{t \in T : \mathrm{cf}(t) \leq \omega\}$, D is dense and since $\mathrm{ht}(T) \leq \omega_1 + 1$, if $t \in D$, then $\mathrm{ht}(t, T) < \omega_1$. Let $\mathcal{U} = \{V_t : t \in D \;\&\; \mathrm{ht}(t, T) \text{ is a successor ordinal}\}$. We want to prove that the family $\mathcal{U}$ is point countable on D. Let $t \in D$ and $s \in T$ such that $V_s \in \mathcal{U}$. We consider three cases:

- s, t are incomparable, then $V_s \notin \mathcal{U}(t)$;
- $s < t$, then $V_s \in \mathcal{U}(t)$;
- $t < s$, then $V_s \notin \mathcal{U}(t)$.

Hence, since if $t \in D$ then $\mathrm{ht}(t, T) < \omega_1$, we have $|\mathcal{U}(t)| \leq \omega$ if $t \in D$. This proves that $\mathcal{U}$ is point countable on D. It remains to prove that $\mathcal{U}$ is T_0-separating. Let $s, t \in T$ be distinct elements, we consider two cases:

- $\mathrm{ht}(s, T)$ is a successor ordinal. In the case that s and t are incomparable or $t < s$, since $V_s \in \mathcal{U}$ and $t \notin V_s$, we have $|V_s \cap \{s, t\}| = 1$. Suppose that $s < t$, then there exists an element r such that $\mathrm{ht}(r, T)$ is a successor ordinal and $s < r \leq t$. Hence $V_r \in \mathcal{U}$, $t \in V_r$ and $s \notin V_r$;
- $\mathrm{ht}(s, T)$ and $\mathrm{ht}(t, T)$ are limit ordinals. If $t < s$ there exists $r \in T$ such that $\mathrm{ht}(r, T)$ is a successor ordinal and $t < r < s$. Therefore $V_r \in \mathcal{U}$ and $|V_r \cap \{s, t\}| = 1$. Otherwise, if s and t are incomparable there exists $r \in T$ such that $\mathrm{ht}(r, T)$ is a successor ordinal, $r < t$ and r and s are incomparable. Hence $|V_r \cap \{s, t\}| = 1$.

(2) Since T is Valdivia, by Proposition 3.3.2 we have that $D = \{t \in T : \mathrm{cf}(t) \leq \omega\}$ is the Σ-subset of T. Suppose, by contradiction, that $\mathrm{ht}(T) \geq \omega_2 + 1$. Then there exists $t \in T$ such that $\mathrm{ht}(t, T) = \omega_2$. This means that $\hat{t}$, with the subspace topology, is homeomorphic to $[0, \omega_2]$, with interval topology. Since $\hat{t} \cap D$ is dense in $\hat{t}$, we have that $\hat{t}$ would be a Valdivia compact space with the subspace topology. This is a contradiction because $[0, \omega_2]$ is not Valdivia.

This concludes the proof. □

In the rest of this section we provide an example of tree with retractional skeleton that gives a negative answer to the question posed in the introduction.
Let $X = \prod_{\alpha<\omega_1} X_\alpha$, with $|X_\alpha| \geq 2$ for every $\alpha < \omega_1$, endowed with the topology whose basis is the collection of all sets $V(g, \alpha) = \{f \in X : g \restriction \alpha = f \restriction \alpha\}$, where $g \in X$ and $\alpha < \omega_1$. This is a slight generalization of the generalized Baire space endowed with the *bounded topology.* We refer to [1] for a detailed reference in this field.
We observe that the sets $V(g, \alpha)$ are clopen and that already the sets $V(g, \alpha)$, α successor ordinal, form a basis of the topology.

Proposition 3.4.2. *Let* $\{A_\xi\}_{\xi<\omega_1}$ *be a family of open dense subsets of* X*, then* $\bigcap_{\xi<\omega_1} A_\xi$ *is dense in* X*.*

Proof. Let U be an open subset of X. By transfinite induction on $\xi < \omega_1$ we define a transfinite sequence of elements $\{f_\xi\}_{\xi<\omega_1} \subset X$ and a transfinite increasing sequence of ordinals $\{\alpha_\xi\}_{\xi<\omega_1}$ such that $f_{\xi_1} \restriction \alpha_{\xi_1} = f_{\xi_2} \restriction \alpha_{\xi_1}$ whenever $\xi_1 < \xi_2$. We define those sequences in the following way:

- since $U \cap A_0$ is open, there exist $f_0 \in X$ and $\alpha_0 < \omega_1$ such that $V(f_0, \alpha_0) \subset U \cap A_0$;

- $\xi = \gamma + 1$. Since $A_\xi \cap V(f_\gamma, \alpha_\gamma)$ is an open non-empty subspace of X, there exist $f_{\gamma+1}$ and $\alpha_{\gamma+1}$ such that $V(f_{\gamma+1}, \alpha_{\gamma+1}) \subset A_\xi \cap V(f_\gamma, \alpha_\gamma)$ and $f_{\gamma+1} \restriction \alpha_\gamma = f_\gamma \restriction \alpha_\gamma$;
- ξ is limit. Let $\alpha_\xi = \sup_{\gamma<\xi} \alpha_\gamma$ and $f_\xi \in X$ such that $f_\xi \restriction \alpha_\gamma = f_\gamma \restriction \alpha_\gamma$ for every $\gamma < \xi$. Therefore we have that $V(f_\xi, \alpha_\xi) \subset \bigcap_{\gamma<\xi} A_\gamma$.

Finally, let $f_{\omega_1} \in X$ defined by $f_{\omega_1} \restriction \alpha_\xi = f_\xi \restriction \alpha_\xi$, for every $\xi < \omega_1$. f_{ω_1} belongs to U and to A_ξ for every $\xi < \omega_1$. This gives us the assertion. $\square$

We recall that, given a cardinal κ and an ordinal α, the full κ-ary tree of height α is the tree of all transfinite sequences $f : \beta \to \kappa$, for some ordinal $\beta < \alpha$, with the following order: $f \leq g$ if and only if $\operatorname{dom}(f) \subset \operatorname{dom}(g)$ and $g \restriction \operatorname{dom}(f) = f$.

Example 3.4.3. Let T be a subtree of the binary tree of height $\omega_1 + 2$ such that for every $t \in \mathrm{Lev}_{\omega_1}(T)$ there exists a unique immediate successor, we denote the immediate successor of $t \in \mathrm{Lev}_{\omega_1}(T)$ by $t + 1$. Consider T endowed with the coarse wedge topology. We have the following:

1. T has retractional skeleton by using Theorem 3.3.1;
2. T does not contain any copy of $[0, \omega_2]$, since the character of T is equal to ω_1;
3. nevertheless T is not Valdivia.

Let us finally prove point 3. Suppose by contradiction that T is Valdivia. Let D be a dense Σ-subset of T. By Proposition 3.3.2 necessarily $D = \{t \in T : \mathrm{cf}(t) \leq \omega\}$. By Theorem 3.2.5, let $\mathcal{U}$ be a T_0-separating family of clopen sets witnessing that D is a Σ-subset. We may assume that the elements of $\mathcal{U}$ are of the form W_s^F where $s \in T$ is on a successor level and $F \subset T$ is a finite. We observe that if $W_s^F \in \mathcal{U}$ and $W_s^F \cap \mathrm{Lev}_{\omega_1}(T) \neq \emptyset$, then $t + 1 \in W_s^F$ for all but finitely many $t \in W_s^F \cap \mathrm{Lev}_{\omega_1}(T)$. Fix $t \in \mathrm{Lev}_{\omega_1}(T)$, since $t + 1 \in D$ and $\mathcal{U}$ is point countable on D, there exists $\theta(t) < \omega_1$ such that $t + 1 \notin U$ whenever $U \in \mathcal{U}(t)$ and $U \cap \mathrm{Lev}_{\theta(t)}(T) = \emptyset$. By previous observation the mapping $\theta : \mathrm{Lev}_{\omega_1}(T) \to \omega_1$ is well defined.
We observe that $\mathrm{Lev}_{\omega_1}(T)$ with the subspace topology is homeomorphic to 2^{ω_1} endowed with the bounded topology. Since $\mathrm{Lev}_{\omega_1}(T) = \bigcup_{\alpha<\omega_1} \theta^{-1}(\alpha)$, there exists, by Proposition 3.4.2, a $\alpha < \omega_1$ such that $\theta^{-1}(\alpha) \subset \mathrm{Lev}_{\omega_1}(T)$ is somewhere dense. Let A be an open subset of T such that $A \cap \theta^{-1}(\alpha)$ is dense in $A \cap \mathrm{Lev}_{\omega_1}(T)$. Since $\alpha < \omega_1$, there exists an open set $V_{t_0} \subset A$, where $\alpha < \mathrm{ht}(t_0, T) < \omega_1$ and $\mathrm{ht}(t_0, T)$ is a successor, such that $V_{t_0} \cap \theta^{-1}(\alpha)$ is dense in $V_{t_0} \cap \mathrm{Lev}_{\omega_1}(T)$.
Claim: there exists an element U_0 of the family $\mathcal{U}$ such that $U_0 \subset V_{t_0}$.
Therefore, since U_0 is a clopen subset of T contained in V_{t_0} and $\theta^{-1}(\alpha)$ is dense in V_{t_0}, the intersection $U_0 \cap \theta^{-1}(\alpha)$ is an infinite subset of $\mathrm{Lev}_{\omega_1}(T)$. Further we have that $\mathrm{ht}(t_0, T) > \alpha$, hence for every $s \in U_0 \cap \theta^{-1}(\alpha)$ we have $s + 1 \notin U_0$, a contradiction.
It remains to prove the claim. Since $\mathcal{U}$ is point countable on D, in particular at t_0, and $|V_{t_0} \cap \theta^{-1}(\alpha)| > \omega$, there exist a $w \in V_{t_0} \cap \theta^{-1}(\alpha)$ and a $W_{s_0}^F \in \mathcal{U}$ such that $w \in W_{s_0}^F$ and $t_0 \notin W_{s_0}^F$. It is enough to prove that $t_0 < s_0$. Since $t_0 \notin W_{s_0}^F$, we have that $t_0 \neq s_0$, therefore suppose that $s_0 < t_0$. Then there exists $r \in F$, on a successor level, such that $s_0 < r \leq t_0$. Since $t_0 < w$, we have that $w \notin W_s^F$, a contradiction. Thus $t_0 < s_0$ and therefore $W_{s_0}^F \subset V_{t_0}$.

Remark 3.4.4. We have assumed that every tree was rooted, the above results can be proved also if the tree T has finitely many minimal elements. In fact if T has finitely many minimal elements, then it can be viewed as the topological direct sum of rooted trees.

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Chapter 4

On compact trees with the coarse wedge topology

4.1 Introduction

Large families of continuous projections often appear in the theory of non-separable Banach spaces. In this spirit W. Kubiś introduced the concept of projectional skeletons in [11], where he adapted the definition of retractional skeleton (see [12], [2]) from the topological setting to Banach spaces. Roughly speaking, a projectional skeleton decomposes the Banach space into smaller separable subspaces. The existence of a projectional skeleton allowed us to transfer structural properties, such as having a countably norming M-basis as well as having locally uniformly rotund (LUR) renorming, from the subspace to the whole space. More recently, along the same line, the class of Banach spaces admitting a projectional skeleton has been investigated by different authors, we refer for example to [3], [4] and [9]. A large subclass is that of Plichko spaces, which are characterized by having a commutative projectional skeleton [11, Theorem 27]. This class was introduced by Plichko in [16] and was studied under equivalent definitions in [5], [20] and [21]; we refer to [10] for a detailed survey in this area. Plichko spaces and the related class of compact spaces, the so called Valdivia compacta, appear in many different areas, see[8]. It is known that there exist Banach spaces with projectional skeletons $\langle$compact spaces with retractional skeletons$\rangle$ that are not Plichko $\langle$Valdivia$\rangle$. A simple example of non Valdivia compact space with rectractional skeleton is the compact ordinal interval $[0, \omega_2]$, [12]. It is more difficult to prove that that the Banach space $C([0, \omega_2])$, which has a projectional skeleton, is not Plichko [7]. Recently new examples of non Valdivia compact spaces with retractional skeletons have been studied in [17]. This was done by studying the class of trees endowed with the coarse wedge topology ([14],[15]), also known as path topology ([19],[18],[6]), in the context of Valdivia compacta. In the same paper it was proved that every tree with height less or equal than $\omega_1 + 1$ is Valdivia and no Valdivia tree has height greater than ω_2. Moreover, an example of non Valdivia tree with heigth $\omega_1 + 2$ was provided. In the present paper we follow the same research line and we investigate in much more details the classes of Valdivia and Plichko spaces. In particular we investigate Radon measures on such trees. We prove that every compact tree has property (M), i.e. each Radon measure defined on it has separable support. This result is used to investigate the space of continuous functions on compact trees. We prove that $C(T)$ is Plichko whenever the height of T is less than $\omega_1 \cdot \omega_0$. Finally we extend the Theorem 4.1 of [17], characterizing Valdivia compact trees with height less than ω_2. It turns out that this characterization depends only on the behaviour of the tree on the levels with uncountable cofinality.

We now outline how the paper is organized. In the remaining part of the introductory section notation and basic notions addressed in this paper are given. Section 2 contains details of notation, basic definitions and some preliminary results on trees. In section 3 we investigate the class of Radon measures on trees, proving that each compact tree has the so called property (M). Section 4 is devoted to characterizing Valdivia compact trees with height less than ω_2. In section 5 we deal with the class of continuous functions on a compact tree. It is shown that, if $C(T)$ is 1-Plichko, then T is Valdivia. We also prove that, if T is an arbitrary tree with height less than $\omega_1 \cdot \omega_0$, then $C(T)$ is a Plichko space.

We denote with ω_0 the set of natural numbers (including 0) with the usual order. Given a set X we denote by $|X|$ the cardinality of the set X, by $[X]^{\leq\omega}$ the family of all countable subsets of X.

All the topological spaces are assumed to be Hausdorff and completely regular. Given a topological space X we denote by $\overline{A}$ the closure of $A \subset X$. We say that $A \subset X$ is countably closed if $\overline{C} \subset A$ for every $C \in [A]^{\leq\omega}$.

Given a topological compact space $\langle$loc. compact space$\rangle$ K we use $C(K)$ $\langle C_0(K)\rangle$ to indicate the space of all real-valued or complex-valued continuous functions on K $\langle$all real-valued or complex-valued continuous functions on K vanishing at infinity$\rangle$ with the usual norm. By the Riesz representation theorem the elements of $C(K)^*$ are considered as measures. If $\mu \in C(K)^*$, we denote by $\|\mu\|$ its norm. If μ is a non-negative measure, we denote by $\mathrm{supp}(\mu)$ the support of the measure μ, i.e. the set of those points $x \in K$ such that each neighborhood of x has positive μ-measure. The support of a measure $\mu \in C(K)^*$ coincides with the support of its total variation $|\mu|$.

Given a Banach space X and a subset $A \subset X$ we denote by $\mathrm{span}(A)$ the linear hull of A. B_X is the norm-closed unit ball of X (i.e. the set $\{x \in X : \|x\| \leq 1\}$). As usual X^* stands for the (topological) dual space of X. A set $D \subset X^*$ is said λ-norming if

$$\|x\| \leq \lambda \sup\{|x^*(x)| : x^* \in D \cap B_{X^*}\}$$

for every $x \in X$. We say that a set $D \subset X^*$ is norming if it is λ-norming for some $\lambda \geq 1$. A subspace $S \subset X^*$ is called a Σ-*subspace* of X^* if there is a set $M \subset X$ such that $\overline{\mathrm{span}}(M) = X$ and that

$$S = \{f \in X^* : \{m \in M : f(m) \neq 0\} \text{ is countable}\}.$$

A Banach space X is called *Plichko* $\langle$resp. λ-*Plichko*$\rangle$ *space* if X^* admits a norming $\langle\lambda$-norming, respectively$\rangle$ Σ-subspace. Let Γ be an arbitrary set, we put

$$\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma : |\{\gamma \in \Gamma : x(\gamma) \neq 0\}| \leq \omega_0\}.$$

Let K be a compact space, we say that $A \subset K$ is a Σ-*subset* of K if there exists a homeomorphic injection h of K into some $\mathbb{R}^\Gamma$ such that $h(A) = h(K) \cap \Sigma(\Gamma)$. K is called *Valdivia compact space* if K has a dense Σ-subset.

4.2 Basic notions on trees

A *tree* is a partially ordered set $(T,<)$ such that the set of predecessors $\{s \in T : s < t\}$ of any $t \in T$ is well-ordered by $<$. A tree T is said *rooted* if it has only one minimal element, called *root*. T is called *chain complete* if every chain has a supremum. For any element $t \in T$, $\mathrm{ht}(t,T)$ denotes the order type of $\{s \in T : s < t\}$. For any ordinal α, the set $\mathrm{Lev}_\alpha(T) = \{t \in T : \mathrm{ht}(t,T) = \alpha\}$ is called the *αth level* of T. The height of T is denoted by $\mathrm{ht}(T)$, and it is the least α such that $\mathrm{Lev}_\alpha(T) = \emptyset$. For an element $t \in T$, $\mathrm{cf}(t)$ denotes the cofinality of $\mathrm{ht}(t,T)$ and $\mathrm{ims}(t) = \{s \in T : t < s, \mathrm{ht}(s,T) = \mathrm{ht}(t,T)+1\}$ denotes the set of *immediate successors* of t. Given a subset S of a tree T, and an element $t \in S$, we denote by $\mathrm{ims}_S(t)$ the set of immediate successors of t in S with the inerithed order. Let T be a tree of height α, let $\beta < \alpha$, we denote by $T_\beta = \bigcup_{\gamma \leq \beta} \mathrm{Lev}_\gamma(T)$ and by $T_{<\beta} = \bigcup_{\gamma<\beta} \mathrm{Lev}_\gamma(T)$.

For $t \in T$ we put $V_t = \{s \in T : s \geq t\}$ and $\hat{t} = \{s \in T : s \leq t\}$. In the present work we consider T endowed with the *coarse wedge topology.* Now we recall the definition of such a topology and its properties, we refer to [18] and [15] for the details. The coarse wedge topology on a tree T is the one whose subbase is the set of all V_t and their complements, where t is either minimal or on a successor level. If $\mathrm{ht}(t,T)$ is a successor or t is the minimal element, a local base at t is formed by sets of the form

$$W_t^F = V_t \setminus \bigcup\{V_s : s \in F\},$$

where F is a finite set of immediate successors of t. In case $\mathrm{ht}(t,T)$ is limit, a local base at t is formed by sets of the form

$$W_s^F = V_s \setminus \bigcup\{V_r : r \in F\},$$

where $s < t$, $\mathrm{ht}(s,T)$ is a successor and F is a finite set of immediate successor of t. Since we are interested in compact spaces, we recall that, by [15, Corollary 3.5], a tree T is compact Hausdorff in the coarse wedge topology if and only if T is chain complete and has finitely many minimal elements. For this reason, from now on we will consider only chain complete trees with a unique minimal element. In these settings the operation $t \wedge s = \max(\hat{t} \cap \hat{s})$ is well-defined for every $s,t \in T$.

Given a subset S of a tree T, there are two natural topologies on S: the subspace topology and the coarse wedge topology generated by the inerithed order. We are going to prove that these topologies sometimes coincide.

Lemma 4.2.1. *Let S be closed subset of a tree T. Suppose that S is closed under $\wedge$ (i.e. if $s,t \in S$, then $s \wedge t \in S$). Then the subspace topology is equivalent to the coarse wedge topology on S.*

Proof. We firstly observe that if S is a branch of T, then the two topologies are equivalent to the interval topology. Now we are going to prove that if S is endowed with the coarse wedge topology, then it is a compact space. We observe that, since T is chain complete, any chain in S has a supremum in T. By the closedness of S, the supremum belongs to S. Moreover, since S is closed under the $\wedge$ operation and T is rooted, we deduce that S is rooted too. Therefore, by [15, Corollary 3.5]

S, endowed with the coarse wedge topology, is a compact Hausdorff space. Now we are going to prove that the coarse wedge topology on S is coarser than the subspace topology.
Let $x \in S$ and suppose that x is on a successor level in T. Since S is closed, x is also on a successor level in S. Let $W_x^F \subset S$ be an open basic neighborhood of x, where $F = \{t_i\}_{i=1}^n \subset \mathrm{ims}_S(x)$. For each $t_i \in T$ there exists a unique $u(t_i) \in \mathrm{ims}_T(x)$ such that $t_i \geq u(t_i)$. Let $F_1 = \{u(t_i)\}_{i=1}^n$, hence we have $W_x^F \supset W_x^{F_1} \cap S$.
Let $x \in S$ and suppose that x belongs to a limit level in T. Now two cases are possible:

- x is on a limit level in S. Let $W_s^F \subset S$ be an open basic neighborhood of x, where $s < x$ is on a successor level in S and $F = \{t_i\}_{i=1}^n \subset \mathrm{ims}_S(x)$. As previous case let $F_1 = \{u(t_i)\}_{i=1}^n$ and $\{s+1\} = \hat{x} \cap \mathrm{Lev}_{\mathrm{ht}(s,T)+1}(T)$. Hence $W_{s+1}^{F_1}$ is an open basic neighborhood of x in T and $W_{s+1}^{F_1} \cap S \subset W_s^F$.
- x is on a successor level in S. Let $W_x^F \subset S$ be an open basic neighborhood of x, where $F = \{t_i\}_{i=1}^n \subset \mathrm{ims}_S(x)$. Since x is on a successor level in S, x has an unique immediate predecessor, say $x-1$. Since x is on a limit level in T and S is closed in T, there exists $s \in T$ on a successor level such that $x-1 < s < x$ and $W_s^x \cap S = \emptyset$. In fact, let $x-1 < s < x$ and suppose that $y \in W_s^x \cap S$, then we get $x-1 < s \leq x \wedge y < x$. Since $x \wedge y \in S$ we find a contradiction because of the maximality of $x-1$. Hence $W_s^{F_1} \cap S \subset W_x^F$, where $F_1 = \{u(t_i)\}_{i=1}^n$.

Since S is a compact Hausdorff space in both topologies, we obtain the assertion. □

As a consequence we obtain the following result.

Corollary 4.2.2. *Let C be a countable subset of a tree T. Then $\overline{C}$ is a metrizable subspace of T.*

Proof. Let C be a countable subset of T. Let $C_\wedge$ be the smallest subset of T containing C and closed under the $\wedge$ opertion. It is clear that $C_\wedge$ is a countable subset. Now we are going to prove that $\overline{C_\wedge}$ is closed under $\wedge$. Let $s, t \in \overline{C_\wedge}$ and consider $s \wedge t$. Suppose that s, t are incomparable elements, otherwise the assertion would follows immediately. Let $\{u(t)\} = \mathrm{ims}(s \wedge t) \cap \hat{t}$ and $\{u(s)\} = \mathrm{ims}(s \wedge t) \cap \hat{s}$. Since $s, t \in \overline{C_\wedge}$, there are $s_1, t_1 \in C_\wedge$ such that $s_1 \in V_{u(s)}$ and $t_1 \in V_{u(t)}$. Therefore we have $s \wedge t = s_1 \wedge t_1 \in C_\wedge$.
Obeserving that any chain of $\overline{C_\wedge}$ is at most countable, combining Lemma 4.2.1 and [14, Theorem 2.8] we obtain that $\overline{C_\wedge}$ is a separable Corson compact space, hence metrizable. Since $\overline{C} \subset \overline{C_\wedge}$, we obtain the assetion. □

We conclude this section providing a well-known characterization of Valdivia compact spaces. Let X be a topological space, a family $\mathcal{U}$ of subsets of X is T_0-separating in X if for every two distinct elements $x, y \in X$ there is $U \in \mathcal{U}$ satisfying $|\{x, y\} \cap U| = 1$. A family $\mathcal{U}$ is point countable on $D \subset X$ if

$$|\{U \in \mathcal{U} : x \in U\}| \leq \omega_0$$

for every $x \in D$. Since we are interested in compact trees, we are going to state [10, Proposition 1.9] in these terms.

Theorem 4.2.3. *, Let T be a tree and $D = \{t \in T : \mathrm{cf}(t) \leq \omega_0\}$. Then the following are equivalent:*

- (i) *T is Valdivia.*
- (ii) *D is a Σ-subset of T.*
- (iii) *there is a T_0-separating family of basic clopen sets point-countable exactly on D.*

The equivalence between (i) and (ii) follows from [17, Proposition 3.2], while as a particular case of [10, Proposition 1.9] we obtain the equivalence between (ii) and (iii).

4.3 Radon measures on trees

In this section we are going to describe the Radon measures on trees, this is particularly useful to investigate the space of continuous functions on a tree. In particular it follows form Theorem 4.3.2 that every tree has the property (M) (i.e. each Radon measure has separable support). We recall that given a compact space K and a Borel measure μ on it, it is possible to define the restriction of μ to a Borel subset of K in the following way: let B be a Borel subset of K, $\mu \restriction_B (A) = \mu(A \cap B)$, for each Borel set $A \subset K$. Given a measurable subset $B \subset K$, $\mu \restriction_B$ is a Borel measure on K and clearly $\mathrm{supp}(\mu \restriction_B) \subset \overline{B}$.

Let K be a compact space and $\{G_\alpha\}_{\alpha \in A}$ be a family of disjoint open subsets of K. Let μ be a non-negative Radon measure on K, then $\mu(\bigcup_{\alpha \in A} G_\alpha) = \sum_{\alpha \in A} \mu(G_\alpha)$. In fact, for every $\varepsilon > 0$ there exists a compact space $K_\varepsilon \subset \bigcup_{\alpha \in A} G_\alpha$ such that $\mu(\bigcup_{\alpha \in A} G_\alpha \setminus K_\varepsilon) < \varepsilon$. Therefore, since K_ε is compact, it intersects at most finitely many open sets. Hence there is a finite set $F \subset A$ such that $\mu(\bigcup_{\alpha \in A \setminus F} G_\alpha) < \varepsilon$. This means that $\mu(G_\alpha) > 0$ for countably many $\alpha \in A$. Since F is finite and μ is σ-additive, we obtain

$$\mu(\bigcup_{\alpha \in A} G_\alpha) = \mu(\bigcup_{\alpha \in A \setminus F} G_\alpha) + \mu(\bigcup_{\alpha \in F} G_\alpha) < \varepsilon + \sum_{\alpha \in F} \mu(G_\alpha).$$

Hence we get $\mu(\bigcup_{\alpha \in A} G_\alpha) \leq \sum_{\alpha \in A} \mu(G_\alpha)$.The opposite inequality trivially follows from the monotonicity of the measure.

Next result gives a representation of Radon measures on trees which have the height with uncountable cofinality.

Proposition 4.3.1. *Let T be a tree with height equal to $\eta + 1$, where* $\mathrm{cf}(\eta) \geq \omega_1$, *and μ be a continuous Radon measure on T. Then there exists $\beta < \eta$ such that* $\mathrm{supp}(\mu) \subset T_\beta$.

Proof. Since any signed measure is a difference of two non-negative measures and any complex measure is a linear combination of four non-negative measures it is enough to prove the result for non-negative measures. For these reasons we assume that $\mu \geq 0$.

We observe that, for every $\alpha < \eta$, the subset T_α is closed. Hence the mapping $h_\mu : [0,\eta) \to \mathbb{R}^+$ defined by $h_\mu(\alpha) = \mu(T_\alpha)$ is well-defined and non-decreasing. Therefore, since the cofinality of η is uncountable, h_μ is eventually constant. This means that there exists an $\alpha_0 < \eta$, such that $h_\mu(\alpha) = h_\mu(\alpha_0)$ for every $\alpha \geq \alpha_0$.
We claim that $\mu = \mu \restriction_{T_{\alpha_0}}$, i.e. $\mu - \mu \restriction_{T_{\alpha_0}} = 0$. Now we replace μ by $\mu - \mu \restriction_{T_{\alpha_0}}$. Then we have that $\mu \restriction_{T_\alpha} = 0$ for each $\alpha < \eta$. We suppose, by contradiction, $\mu \neq 0$.
Define the mapping $g_\mu(\alpha) = \sup_{t \in \mathrm{Lev}_{\alpha+1}(T)} \mu(V_t)$. Since, by assumption, $\mu \geq 0$, it is clear that g is non-increasing, hence it is eventually constant. Therefore there exist $\beta_0 < \eta$ and $c \geq 0$ such that $g(\alpha) = c$ for every $\alpha \in [\beta_0, \eta)$.
Moreover, since for any $\alpha < \eta$ we have

$$0 < \mu(T) = \mu(T_\alpha) + \sum_{t \in \mathrm{Lev}_{\alpha+1}(T)} \mu(V_t) = \sum_{t \in \mathrm{Lev}_{\alpha+1}(T)} \mu(V_t),$$

the function g is strictly positive on $[0,\eta)$, hence c is strictly greater than 0. Further, since the series $\sum_{t \in \mathrm{Lev}_{\alpha+1}(T)} \mu(V_t)$ converges, the supremum in the definition of $g(\alpha)$ is attained and moreover it is attained for finitely many $t \in \mathrm{Lev}_{\alpha+1}(T)$.
Set $A = \{t \in T \setminus T_{\beta_0} : \mu(V_t) = c\}$. Then, the following three conditions hold:

- (i) for each $\alpha \in (\beta_0, \eta)$, $A \cap \mathrm{Lev}_\alpha(T)$ is a non-empty finite set;
- (ii) if $t \in A$, $s \in T \setminus T_{\beta_0}$, $s \leq t$, then $s \in A$;
- (iii) $s \in A \cap \mathrm{Lev}_\alpha(T)$, $\beta > \alpha$, then there exists at most one $t \in A \cap \mathrm{Lev}_\beta(T)$ such that $s \leq t$.

It follows that $\alpha \mapsto |A \cap \mathrm{Lev}_\alpha(T)|$, with $\alpha > \beta_0$, is a non-decreasing function, thus eventully constant. Hence there exist $\beta_1 > \beta_0$ and $k \in \omega_0$ such that $|A \cap \mathrm{Lev}_\alpha(T)| = k$ for $\alpha \in [\beta_1, \eta)$.
Let $t \in A \cap \mathrm{Lev}_{\beta_1}(T)$, it follows that $V_t \cap A$ is a linearly ordered subset of T with supremum $s \in \mathrm{Lev}_\eta(T)$. Then by the regularity of μ we have $\mu(\{s\}) = c > 0$. A contradiction. □

Theorem 4.3.2. *Let T be a tree and μ be a Radon measure on T. Then the support of μ is metrizable.*

Proof. By Corollary 4.2.2 it is enough to show that $\mathrm{supp}(\mu)$ is contained in a separable subset of T. Since the discrete part of μ is countably supported, each signed measure is a linear combination of two non-negative measures and each complex measure is a linear combination of four non-negative measures, we may assume without loss of generality that μ is a non-atomic non-negative measure. We use a tranfinite induction argument on the height of the tree.

Let T be a tree with height equal to $\alpha = \omega_1 + 1$ and μ be a Radon measure on it. Hence by Proposition 4.3.1, there exists $\beta < \omega_1$ such that $\mathrm{supp}(\mu) \subset T_\beta$. We define $P \subset T$ as follows: $t \in P$ if and only if t belongs to a successor level, and $\mu(V_t) > 0$. We observe that if $t \in P$, then $\mathrm{ht}(t, T) \leq \beta$, for this reason P is countable. In fact, suppose that P is uncountable, hence there exists $\beta_0 < \beta$ such that $|\mathrm{Lev}_{\beta_0}(T) \cap P| > \omega_0$. But this is not possible because of the finiteness of μ. We observe that $\mathrm{supp}(\mu) \subset \overline{P}$. In fact let $x \in \mathrm{supp}(\mu)$:

- if x is on a successor level, then $\mu(V_x) > 0$ hence $x \in P$;
- if $\mathrm{cf}(x) = \omega_0$, then for every $s < x$ on a successor level, we have $\mu(V_s) > 0$, hence $s \in P$ and $x \in \overline{P}$.

Since P is countable, $\overline{P}$ is separable and compact. Hence $\mathrm{supp}(\mu)$ is metrizable.

Let α be an ordinal and assume that for every $\beta < \alpha$ and for every tree T such that $\mathrm{ht}(T) \leq \beta + 1$ we have that each Radon measure on T has a separable support. Let T be a tree with height equal to $\alpha + 1$ and μ be a Radon measure on it. We study the three different cases.

- Suppose that $\alpha = \beta + 1$ for some β. Then, since μ is a continuous measure and each point on $\mathrm{Lev}_\alpha(T)$ is isolated, we have that $\mathrm{supp}(\mu) = \mathrm{supp}(\mu \restriction_{T_\beta})$. Hence, since $\mu \restriction_{T_\beta}$ is a Radon measure on the tree T_β, we have that $\mathrm{supp}(\mu)$ is separable for the induction hypothesis.

- Suppose that α is a limit ordinal. If $\mathrm{cf}(\alpha) \geq \omega_1$, by Theorem 4.3.1, there exists an ordinal $\gamma < \alpha$ such that $\mathrm{supp}(\mu) = \mathrm{supp}(\mu \restriction_{T_\gamma})$. Hence by the induction hypothesis we have that $\mathrm{supp}(\mu)$ is separable.

- Suppose that $\mathrm{cf}(\alpha) = \omega_0$. Let $\{\alpha_n\}_{n\in\omega_0}$ be an increasing sequence of ordinals such that $\sup_{n\in\omega_0} \alpha_n = \alpha$. For each $n \in \omega_0$ let $S_n = \{t \in \mathrm{Lev}_{\alpha_n+1}(T) : \mu(V_t) > 0\}$. Since μ is a non-negative Radon measure, we have $|S_n| \leq \omega_0$ for each $n \in \omega_0$. Moreover we observe that if $t \in S_{n+1}$ there exists a unique $s \in S_n$ such that $s < t$.
  Claim: $\mathrm{supp}(\mu) \subset \overline{(\bigcup_{n\in\omega_0} \mathrm{supp}(\mu \restriction_{T_{\alpha_n}})) \cup (\bigcap_{n\in\omega_0} \bigcup_{t\in S_n} V_t)}$.
  Let $x \in \mathrm{supp}(\mu)$ and suppose that $\mathrm{ht}(x, T) = \alpha$, hence for each $n \in \omega_0$ there exists a unique $t_x \in S_n$ such that $x \in V_{t_x}$. Therefore $x \in \bigcap_{n\in\omega_0} \bigcup_{t\in S_n} V_t$. Now suppose that $\mathrm{ht}(x, T) < \alpha$. Hence there exists $k \in \omega_0$ such that $\alpha_k \leq \mathrm{ht}(x, T) < \alpha_{k+1}$ (without loss of generality we may assume that $\alpha_0 = 0$). For every basic open neighborhood W_t^F of x let $A(t, F) = S_{k+1} \cap W_t^F$. If for some W_t^F we have $A(t, F) = \emptyset$, then easily we get $x \in \mathrm{supp}(\mu \restriction_{T_{\alpha_{k+1}}})$. Let us suppose that $A(t, F) \neq \emptyset$ for each W_t^F. Let W_t^F be an arbitrary basic open neighborhood of x. By the assumption $A(t, F) \neq \emptyset$, so we can fix $s \in A(t, F)$. Since $\mu(V_s) > 0$, we get $\mathrm{supp}(\mu) \cap V_s \neq \emptyset$. If $\mathrm{supp}(\mu) \cap V_s \cap \mathrm{Lev}_\alpha(T) \neq \emptyset$, then $W_t^F \cap \left(\bigcap_{n\in\omega_0} \bigcup_{t\in S_n} V_t\right) \neq \emptyset$. If $\mathrm{supp}(\mu) \cap V_s \cap \mathrm{Lev}_\alpha(T)$ is empty, then there exists a minimal $i > k + 1$ such that $(\mathrm{supp}(\mu) \cap V_s) \subset T_{<\alpha_i}$. Hence there exist an element $p \in (\mathrm{supp}(\mu) \cap V_s)$ such that $\alpha_{i-1} \leq \mathrm{ht}(p, T) < \alpha_i$ and an open neighborhood of p, $W_{t_p}^{F_p}$ such that the corresponding $A(t_p, F_p)$ is empty, hence $p \in W_t^F \cap \mathrm{supp}(\mu \restriction_{T_{\alpha_i}})$. Therefore each open neighborhood of x intersects $(\bigcup_{n\in\omega_0} \mathrm{supp}(\mu \restriction_{T_{\alpha_n}})) \cup (\bigcap_{n\in\omega_0} \bigcup_{t\in S_n} V_t)$. Hence we obtain:

$$x \in \overline{(\bigcup_{n\in\omega_0} \mathrm{supp}(\mu \restriction_{T_{\alpha_n}})) \cup (\bigcap_{n\in\omega_0} \bigcup_{t\in S_n} V_t)}.$$

  It remains to prove that $(\bigcup_{n\in\omega_0} \mathrm{supp}(\mu \restriction_{T_{\alpha_n}})) \cup (\bigcap_{n\in\omega_0} \bigcup_{t\in S_n} V_t)$ is separable. By the induction hypotesis $\mathrm{supp}(\mu \restriction_{T_{\alpha_n}})$ is separable for every $n \in \omega_0$. In order to prove that $\bigcap_{n\in\omega_0}\bigcup_{t\in S_n} V_t$ is separable we define a continuous, one-to-one mapping into the product space $\prod_{n\in\omega_0} S_n$, each S_n is considered with the discrete topology, so the product space is separable and metrizable. It is clear that if $x\in\bigcap_{n\in\omega_0}\bigcup_{t\in S_n} V_t$, then $x\in\mathrm{Lev}_\alpha(T)$ and for every $n\in\omega_0$ there exists a unique $t_{x,n}\in S_n$ such that $t_{x,n}<x$. Now we define $\phi:\bigcap_{n\in\omega_0}\bigcup_{t\in S_n} V_t\to\prod_{n\in\omega_0} S_n$ as follows $\phi(x)=\{t_{x,n}\}_{n\in\omega_0}$. The mapping ϕ is clearly one-to-one. Now we are going to prove that ϕ is a homeomorphism onto its image. Let us prove the contiuity of ϕ. Let $x\in\bigcap_{n\in\omega_0}\bigcup_{t\in S_n} V_t$ and

$$O(\phi(x),j)=\{t\in\prod_{n\in\omega_0} S_n : t(k)=\phi(x)(k) \text{ for } k<j\}$$

be a basic open neghborhood of $\phi(x)$. Let $U_{t_{x,j}}=V_{t_{x,j}}\cap(\bigcap_{n\in\omega_0}\bigcup_{t\in S_n} V_t)$.Then $\phi(U_{t_{x,j}})\subset O(f(x),j)$. Finally let us prove that ϕ is an open mapping onto its image. It is enough to show that the image of a basic open set is open. A basis of the topology of $\bigcap_{n\in\omega_0}\bigcup_{t\in S_n} V_t$ is formed by the sets $V_s\cap(\bigcap_{n\in\omega_0}\bigcup_{t\in S_n} V_t)$, where s is on a successor level. Note that

$$V_s\cap(\bigcap_{n\in\omega_0}\bigcup_{t\in S_n} V_t)=\bigcap_{n\in\omega_0}\bigcup_{t\in S_n}(V_s\cap V_t)=\bigcap_{n,\alpha_n>\mathrm{ht}(s,T)}\ \bigcup_{t\in S_n,t\geq s} V_t.$$

Let $x\in\phi(V_s\cap(\bigcap_{n\in\omega_0}\bigcup_{t\in S_n} V_t))$, then there exists $y\in V_s\cap(\bigcap_{n\in\omega_0}\bigcup_{t\in S_n} V_t)$ such that $\phi(y)=x$. Then $O(\phi(y),k)\cap\phi(V_s\cap(\bigcap_{n\in\omega_0}\bigcup_{t\in S_n} V_t))\subset\phi(V_s\cap(\bigcap_{n\in\omega_0}\bigcup_{t\in S_n} V_t))$, where $k\in\omega_0$ and satisfies $\alpha_k>\mathrm{ht}(s,T)$. This means that $\phi(V_s\cap(\bigcap_{n\in\omega_0}\bigcup_{t\in S_n} V_t))$ is open in the subspace topology, therefore ϕ is a homeomorphism onto its image. Since $\prod_{n\in\omega_0} S_n$ is separable, we obtain the assertion.

□

4.4 Characterization of Valdivia compact trees

The purpose of this section is to decribe the relations between trees and the class of Valdivia compacta. We will characterize trees of height less than ω_2. We recall the definition of ω_1-relatively discrete subset.

Definition 4.4.1. Let X be a topological space. We say that a subset $A\subset X$ is ω_1-*relatively discrete* if it can be written as union of ω_1-many relatively discrete subsets of X.

The main results of this section are contained in the following theorem.

Theorem 4.4.2. *Let T be a tree. Let $R=\{t\in T:\mathrm{cf}(t)=\omega_1\,\&\,\mathrm{ims}(t)\neq\emptyset\}$. Consider the following conditions:*

(i) $|\mathrm{ims}(t)|<\omega_0$ *for every* $t\in R$*;*

(ii) $R\cap\mathrm{Lev}_\alpha(T)$ *is* ω_1*-relatively discrete for each* $\alpha<\omega_1$ *with* $\mathrm{cf}(\alpha)=\omega_1$*.*

Then the following two statements hold.

(1) *If* T *is Valdivia, then* $\mathrm{ht}(T) \leq \omega_2$ *and* $(i), (ii)$ *hold.*

(2) *If* $\mathrm{ht}(T) < \omega_2$ *and* $(i), (ii)$ *hold, then* T *is Valdivia.*

The proof of this theorem is split in two parts. Since the first statement does not require any extra result we prove it here, while we postpone the second part at the end of the section because two lemmata are needed.

Proof of Theorem 4.4.2, (1). Let T be a Valdivia compact tree. Since T is Valdivia, by [17, Theorem 4.1], we have $\mathrm{ht}(T) \leq \omega_2$. Moreover, T has a retractional skeleton, hence by [17, Theorem 3.1], we have $|\operatorname{ims}(t)| < \omega_0$ for every $t \in \mathrm{Lev}_\alpha(T)$ with $\mathrm{cf}(\alpha) = \omega_1$. Hence, in particular $|\operatorname{ims}(t)| < \omega_0$ for every $t \in R$, then (i) is fulfilled. We are going to prove that (ii) holds as well. In order to do that let $\alpha < \omega_2$ and $\mathrm{cf}(\alpha) = \omega_1$. Since T is a Valdivia compact space, by Theorem 4.2.3 the initial part $T_{\alpha+1}$ is Valdivia as well. Hence, there exists a family $\mathcal{U}_\alpha$ of clopen subsets of $T_{\alpha+1}$ that is T_0-separating and point countable on $D_\alpha = \{t \in T_{\alpha+1} : \mathrm{cf}(t) \leq \omega_0\}$. We may assume, without loss of generality, that each element $U \in \mathcal{U}$ is of the form W_s^F for some $s \in T_{\alpha+1}$ and a finite subset $F \subset T_{\alpha+1}$, whose elements are bigger than s and on a successor level.
For every $t \in R \cap \mathrm{Lev}_\alpha(T)$ there is $\eta(t) < \alpha$, such that if $U \in \mathcal{U}_\alpha$, $t \in U$, $\mathrm{ht}(\min U, T_{\alpha+1}) > \eta(t)$, then $U \cap \operatorname{ims}(t) = \emptyset$. Indeed, since for every $s \in \operatorname{ims}(t)$, $s \in D_\alpha$, we have that s is contained in countably many elements of $\mathcal{U}_\alpha$. For this reason there are only countably many elements of U containing both t and s. It is enough to take

$$\eta(t) = \sup\{\mathrm{ht}(p, T_{\alpha+1}) : p < t,\ W_p^F \in \mathcal{U}, t \in W_p^F, \operatorname{ims}(t) \cap W_p^F \neq \emptyset\}.$$

Let $R_\eta = \{t \in R \cap \mathrm{Lev}_\alpha(T) : \eta(t) = \eta\}$.
Let $t \in R_\eta$, since $t \notin D_\alpha$, there exists an unbounded subset S_t of $\hat{t}$ such that for each $s \in S_t$ there exists $W_s^F \in \mathcal{U}_\alpha$ with $t \in W_s^F$. In particular, since S_t is unbounded, there exists $s_0 \in S_t$, and an open basic subset $W_{s_0}^F \in \mathcal{U}_\alpha$, with $\mathrm{ht}(s_0, T_{\alpha+1}) > \eta$. Since F is finite and $\operatorname{ims}(p) \cap W_{s_0}^F = \emptyset$ if $p \in R_\eta$, we have that $|W_{s_0}^F \cap R_\eta| < \omega_0$. Therefore there exists $r \in T_{\alpha+1}$ on a successor level such that $s_0 \leq r < t$ and $V_t \cap R_\eta = \{t\}$. Hence R_η is relatively discrete for each $\eta < \omega_1$, this gives us the assertion. ☐

We observe that the second statement of Theorem 4.4.2 cannot be reversed. In fact there are several examples of Valdivia trees with height equal to ω_2. Here we provide an easy example of such a space. Let X be the topological sum of the ordinal intervals $X_\alpha = [0, \alpha]$ where $\alpha < \omega_2$. Let $X_0 = X \cup \{\infty\}$ be the one-point compactification of X. By [10, Theorem 3.35], X_0 is a Valdivia compact space. Consider the following relation on X_0:

- ∞ is the least element,
- $x < y$ in X if and only if there exists $\alpha < \omega_2$ such that $x, y \in X_\alpha$ and $x < y$ in X_α.

It is clear that $(X_0, <)$ is a tree and, if it is endowed with the coarse wedge topology, is homeomorphic to X_0 with the topology given by the compactification. Therefore

we obtained the desired tree.
Much more interesting is the following problem that as far as we know seems to be open.

Problem 4.4.3. *Can the first statement of Theorem 4.4.2 be reversed?*

In order to prove the second statement of Theorem 4.4.2, we need to describe a natural way to extend relatively open subsets to the whole tree. Let T be a tree of height equal to α and let $\beta < \alpha$ on a successor level. Let $U \subset T_\beta$ be a relatively open set in T_β. We extend U to the whole tree as follows:

$$\widetilde{U} = U \cup (\bigcup_{x \in \mathrm{Lev}_\beta(T) \cap U} V_x).$$

It is clear that $\widetilde{U}$ is open in T. Given a family $\mathcal{U}_\beta$ of open subsets of T_β we denote by $\widetilde{\mathcal{U}}_\beta$ the family of the extended elements of $\mathcal{U}_\beta$.
Given a family $\mathcal{U}$ of clopen subsets of T we put $\mathcal{U}(t) = \{U \in \mathcal{U} : t \in U\}$, for every $t \in T$. If $A, B \subset T$ and $A \cap B = \{t\}$, by an abuse of the notation, $\mathcal{U}(A \cap B)$ means $\mathcal{U}(t)$. We need two technical lemmata.

Lemma 4.4.4. *Let T be a tree with height greater than η, where $\mathrm{cf}(\eta) \geq \omega_1$. Let N be a countable subset of $\mathrm{Lev}_\eta(T)$, then there exists $\delta < \eta$ such that if $t_1, t_2 \in N$, then $\mathrm{ht}(t_1 \wedge t_2, T) < \delta$.*

Proof. Let $N = \{t_n\}_{n \in \omega_0}$ and suppose that $\{t_n\}_{n \in \omega_0}$ is a one-to-one sequence. Define $\delta_n^m = \mathrm{ht}(t_n \wedge t_m, T)$. The assertion follows taking $\delta = (\sup_{n,m \in \omega_0, n \neq m} \delta_n^m) + 1$. If N is finite, we use the same argument as in the infinte case. ☐

Lemma 4.4.5. *Let T be a tree of height $\eta + 2$ where $\eta < \omega_2$ and $\mathrm{cf}(\eta) = \omega_1$. Suppose that:*

1. *$R = \{t \in \mathrm{Lev}_\eta(T) : \mathrm{ims}(t) \neq \emptyset\}$ has cardinality at most ω_1;*
2. *$|\mathrm{ims}(t)| < \omega_0$ for every $t \in R$;*
3. *$T_{\gamma+1}$ is a Valdivia compactum for every $\gamma < \eta$.*

Then T is a Valdivia compact space.

Proof. Let $R = \{t_\alpha\}_{\alpha < \omega_1}$, let us observe that this transfinite sequence might be not one-to-one, and $\{\eta_\gamma\}_{\gamma < \omega_1}$ be a continuous increasing transfinite sequence converging to η. We may suppose that $\eta_0 = 0$. First we define an auxiliary function $\theta : [0, \omega_1) \to [0, \eta)$. Let us define the mapping θ by using a transfinite recursion argument. Let $\theta(0) = 0$ and for each $\zeta < \omega_1$ we set

$$\theta(\zeta) = \max(\eta_\zeta, \sup\{\mathrm{ht}(t_\beta \wedge t_\gamma, T) + 1 : \beta, \gamma < \zeta, t_\beta \neq t_\gamma\}, \sup\{\theta(\xi) + 1 : \xi < \zeta\}).$$

We observe that θ satisfies the following conditions:

- for every $\alpha < \omega_1$ and $\beta, \gamma < \alpha$ ($t_\beta \neq t_\gamma$), $\mathrm{ht}(t_\beta \wedge t_\gamma, T) < \theta(\alpha)$,

- θ is increasing, continuous and $\sup_{\zeta<\omega_1}\theta(\zeta)=\eta$.

Let us prove that θ is continuous, the other properties of θ are clear. Let $\zeta<\omega_1$ be a limit ordinal, we need to show that $\sup_{\xi<\zeta}\theta(\xi)=\theta(\zeta)$. We observe that $\eta_\zeta=\sup_{\xi<\zeta}\eta_\xi\le\sup_{\xi<\zeta}(\theta(\xi)+1)$, furthermore we have $\sup\{\mathrm{ht}(t_\beta\wedge t_\gamma,T)+1:\beta,\gamma<\zeta,t_\beta\ne t_\gamma\}=\sup_{\xi<\zeta}\sup\{\mathrm{ht}(t_\beta\wedge t_\gamma,T)+1:\beta,\gamma<\xi,t_\beta\ne t_\gamma\}\le\sup_{\xi<\zeta}\theta(\xi)\le\sup_{\xi<\zeta}(\theta(\xi)+1)$. Hence, by definition of $\theta(\zeta)$, we obtain $\sup_{\xi<\zeta}(\theta(\xi)+1)=\theta(\zeta)$. This proves the continuity.
Since θ is continuous, for every $t\in T$ such that $\mathrm{ht}(t,T)<\eta$ and t on a successor level, there exists a unique $\alpha<\omega_1$ such that $\mathrm{ht}(t,T)\in[\theta(\alpha),\theta(\alpha+1))$. Moreover, under the same hypothesis, there exists at most one $\beta<\alpha$ such that $t<t_\beta$.
Since $T_{\eta_\gamma+1}$ is Valdivia, there exists a family $\mathcal{U}_\gamma$ of clopen subsets of $T_{\eta_\gamma+1}$ which is T_0-separating and point-countable on $D_\gamma=\{t\in T_{\eta_\gamma+1}:\mathrm{cf}(t)\le\omega_0\}$ for every $\gamma<\omega_1$. We may suppose that the elements of $\mathcal{U}_\gamma$ are of the form W_s^F for every $\gamma<\omega_1$, and moreover we may suppose that each element of $T_{\eta_\gamma+1}$ is contained in some element of the family $\mathcal{U}_\gamma$ (for example adding to the family $\mathcal{U}_\gamma$ the element $T_{\eta_\gamma+1}$).
In order to define a family $\mathcal{U}$ of clopen subsets of T which is T_0-separating and point-countable on $D=\{t\in T:\mathrm{cf}(t)\le\omega_0\}$, we are going to select and opportunely modify a suitable subfamily of $\bigcup_{\gamma<\omega_1}\mathcal{U}_\gamma$.
Let $\gamma<\omega_1$ be a successor ordinal and $U\in\mathcal{U}_\gamma$. Let $\alpha<\omega_1$ be the minimal ordinal such that $\eta_{(\gamma-1)}+1<\theta(\alpha+1)$. Then for every $t\in\mathrm{Lev}_{\eta_{(\gamma-1)}+1}(T)$ let $U_t=U\cap V_t$. Define $I(\omega_1)$ as the set of successor ordinals less than ω_1.

$$\begin{aligned}\mathcal{U}=&\{\{t\}:t\in\mathrm{Lev}_{\eta+1}(T)\}\\ &\cup\bigcup_{\gamma\in I(\omega_1)}\bigcup_{U\in\mathcal{U}_\gamma}\{\widetilde{U}_t:t\in\mathrm{Lev}_{\eta_{(\gamma-1)}+1}(T),\ \widetilde{U}_t\cap\{t_\beta\}_{\beta<\alpha}=\emptyset\}\\ &\cup\bigcup_{\gamma\in I(\omega_1)}\bigcup_{U\in\mathcal{U}_\gamma}\{\widetilde{U}_t\setminus\mathrm{ims}(t_\xi):t\in\mathrm{Lev}_{\eta_{(\gamma-1)}+1}(T),\ \{t_\xi\}=\widetilde{U}_t\cap\{t_\beta\}_{\beta<\alpha}\}.\end{aligned}$$

Now we are going to prove that T is Valdivia. First we show that $\mathcal{U}$ is T_0-separating. Let $s,t\in T$ be distinct elements and suppose that $\mathrm{ht}(s,T)\le\mathrm{ht}(t,T)$:

- if $t\in T_{\eta_\gamma+1}\setminus T_{\eta_{\gamma-1}}$ for some $\gamma\in I(\omega_1)$, the assertion follows from the fact that the family $\mathcal{U}_\gamma$ is T_0-separating on $T_{\eta_\gamma+1}$;

- if $t\in\mathrm{Lev}_{\eta_\gamma}(T)$ with γ limit, then we observe that $\mathrm{ht}(s\wedge t,T)<\mathrm{ht}(t,T)$. Since η_γ is limit too, there is $\xi<\gamma$ successor ordinal such that $\mathrm{ht}(s\wedge t,T)<\eta_{\xi-1}$. Let $\{u\}=\hat{t}\cap\mathrm{Lev}_{\eta_\xi+1}$ and $\{v\}=\hat{t}\cap\mathrm{Lev}_{\eta_{(\xi-1)}+1}$, fix $U\in\mathcal{U}_\xi$ such that $u\in U$. Then $\widetilde{U}_v$ $\langle$resp. $\widetilde{U}_v\setminus\mathrm{ims}(p)$ for some $p\in R\rangle$ contains t and not s;

- if $t\in\mathrm{Lev}_\eta(T)$ we use the same argument as in the previous item;

- if $t\in\mathrm{Lev}_{\eta+1}(T)$, then trivially $\{t\}\in\mathcal{U}$.

It remains to prove that $\mathcal{U}$ is point-countable on D, let $t\in D$, then we consider the following two cases:

- suppose that $\mathrm{ht}(t, T) < \eta$. We observe that $|\{\gamma < \omega_1 : \mathrm{ht}(t, T) > \eta_\gamma\}| \leq \omega_0$, let us define $\gamma_0 = \sup\{\gamma < \omega_1 : \mathrm{ht}(t, T) > \eta_\gamma\}$. Hence if $t \in U$ and $U \in \mathcal{U}$ we have that U is extended from an element of a family $\mathcal{U}_\xi$ where $\xi \leq \gamma_0$. Since $\mathcal{U}_\xi$ is point-countable on $D_\xi \subset T_{\eta_\xi+1}$ we have:

$$|\mathcal{U}(t)| \leq |\bigcup_{\xi\leq\gamma_0} \mathcal{U}_\xi((\hat{t} \cap \mathrm{Lev}_{\eta_\xi+1}(T)))| \leq \omega_0.$$

- $t \in \mathrm{ims}(t_\beta)$ for some $\beta < \omega_1$. Let $X \in \mathcal{U}$ such that $t \in X$. Then there are the following possibilities:

  1. $X = \{t\}$, exactly one element of $\mathcal{U}$ has this form;
  2. there exist $\xi < \omega_1$ and $s \in \mathrm{Lev}_{\eta_{(\xi-1)}+1}(T)$ such that $X = \widetilde{U}_s$, for some $U \in \mathcal{U}_\xi$. Therefore we have $\xi < \beta$, moreover we observe that: $(\hat{t} \cap \mathrm{Lev}_{\eta_{(\xi+1)}+1}(T)) \subset U_s$ and $|\mathcal{U}_\xi(\hat{t} \cap \mathrm{Lev}_{\eta_{(\xi+1)}+1}(T))| \leq \omega_0$. Hence there are at most countably many elements of this form;
  3. there are $\xi < \omega_1$, $s \in \mathrm{Lev}_{\eta_{(\xi-1)}+1}(T)$ and $p \in R$ such that $X = \widetilde{U}_s \setminus \mathrm{ims}(p)$. Then we have $\xi < \beta$ and since $(\hat{t} \cap \mathrm{Lev}_{\eta_{(\xi+1)}+1}(T)) \subset U_s$ and $|\mathcal{U}_\xi(\hat{t} \cap \mathrm{Lev}_{\eta_{(\xi+1)}+1}(T))| \leq \omega_0$, there are at most countably many sets of this form.

Therefore $\mathcal{U}$ is point-countable on D, hence T is Valdivia. □

Now we are ready to prove the statement (2) in Theorem 4.4.2.

Proof of Theorem 4.4.2, (2). In order to prove the second part of the theorem we are going to use a transfinite induction argument on the height of the tree. Let T be a tree as in the hypothesis, by [17, Theorem 4.1], if $\mathrm{ht}(T) \leq \omega_1+1$, then T is Valdivia.

Suppose that the assertion is true for each tree T that satisfies $\mathrm{ht}(T) \leq \alpha + 2$. Then we will prove the assetion for each tree T that satisfies $\mathrm{ht}(T) \leq \alpha + 3$.
Let T be a tree that satisfies $\mathrm{ht}(T) = \alpha + 3$, then, by induction hypothesis, $T_{\alpha+1}$ is a Valdivia compact space. Hence, by Theorem 4.2.3, there exists a family $\mathcal{U}_\alpha$ of clopen subsets of T_α which is T_0-separating and point-countable on $D_\alpha = \{t \in T_{\alpha+1} : \mathrm{cf}(t) \leq \omega_0\}$. The family $\mathcal{U} = \widetilde{\mathcal{U}}_\alpha \cup \{\{t\} : t \in \mathrm{Lev}_{\alpha+2}(T)\}$ is a family of clopen subset of T. It is easy to prove that $\mathcal{U}$ is T_0-separating and point-countable on $D = \{t \in T : \mathrm{cf}(t) \leq \omega_0\}$. Therefore T is Valdivia.

Suppose that the assertion is true for each tree T that satisfies $\mathrm{ht}(T) < \alpha$ for some limit ordinal α, then we will prove the assetion for each tree T that satisfies $\mathrm{ht}(T) \leq \alpha + 2$. Therefore, suppose T is a tree of height equal to $\alpha + 2$. Let us consider the two different cases.
Suppose that α is a limit ordinal with countable cofinality. Then there exists an increasing sequence of ordinals $\{\alpha_n\}_{n\in\omega_0}$ converging to α, we may suppose that $\alpha_0 = 0$. By induction hypotesis the subtrees T_{α_n+1} are Valdivia compact spaces. Hence, by Theorem 4.2.3, there exists a T_0-separating family of clopen subsets $\mathcal{U}_{\alpha_n}$ that is point countable on $D_n = \{t \in T_{\alpha_n+1} : \mathrm{cf}(t) \leq \omega_0\}$, for each $n \in \omega_0$. Now we are going

to prove that the family $\mathcal{U} = \bigcup_{n\in\omega_0} \widetilde{\mathcal{U}}_{\alpha_n} \cup \{\{t\} : t \in \mathrm{Lev}_{\alpha+1}(T)\}$, is a T_0-separating family of clopen subset of T that is point countable on $D = \{t \in T : \mathrm{cf}(t) \leq \omega_0\}$. Let $t \in T$, then let us consider the three possibilities:

- if $t \in \mathrm{Lev}_\alpha(T)$, then $\mathcal{U}(t) = \bigcup_{n\in\omega_0}\{\widetilde{U} : U \in \mathcal{U}_n(\hat{t} \cap \mathrm{Lev}_{\alpha_n+1}(T))\}$;
- if $t \in \mathrm{Lev}_{\alpha+1}(T)$, then $\mathcal{U}(t) = \{\{t\}\} \cup \bigcup_{n\in\omega_0}\{\widetilde{U} : U \in \mathcal{U}_n(\hat{t} \cap \mathrm{Lev}_{\alpha_n+1}(T))\}$;
- if $t \in D \cap T_{\alpha_k+1}$, then $t \in D \cap T_{\alpha_n+1}$ for every $n \in \omega_0$, hence $\mathcal{U}(t) = \bigcup_{n\in\omega_0}\{\widetilde{U} : U \in \mathcal{U}_n(t)\}$;

in all cases $\mathcal{U}(t)$ is countable, hence $\mathcal{U}$ is point-countable on D. It is possible to prove that $\mathcal{U}$ is T_0-separating on T in the same way as in Lemma 4.4.5.
Suppose that α is a limit ordinal with uncountable cofinality. Then there exists an increasing continuous transfinite sequence of ordinals $\{\alpha_\gamma\}_{\gamma<\omega_1}$ converging to α. We may suppose that $\alpha_0 = 0$. Since T satisfies (ii) we have that $\mathrm{Lev}_\alpha(T) \cap R = \bigcup_{\xi<\omega_1} A_\xi$, where A_ξ is relatively discrete in T for each $\xi < \omega_1$, we may suppose that the family $\{A_\xi\}_{\xi<\omega_1}$ is disjoint. We observe that if $A \subset \mathrm{Lev}_\alpha(T)$ is relatively discrete, then for each $t \in A$ there exists $s_t < t$, on a successor level, such that $V_{s_t} \cap A = \{t\}$. Let $A_\beta = \{t \in A : \alpha_\beta < \mathrm{ht}(s_t, T) \leq \alpha_{\beta+1}\}$. Then $A = \bigcup_{\beta<\omega_1} A_\beta$ and for each $s \in T$ with $\mathrm{ht}(s,T) > \alpha_{\beta+1}$ we have that $V_s \cap A_\beta$ contains at most one point. Therefore, each A_ξ in the definition of $R \cap \mathrm{Lev}_\alpha(T)$ can be decomposed in ω_1-many pieces as above. Hence we may suppose, without loss of generality, that for each $\xi < \omega_1$ there is $\beta(\xi) < \alpha$ such that for any $s \in T$ with $\mathrm{ht}(s,T) > \beta(\xi)$ we have $|V_s \cap A_\xi| \leq 1$. Moreover we may suppose that the function β is non-decreasing (replace $\beta(\xi)$ by $\sup\{\beta(\gamma) : \gamma \leq \xi\}$).
Firstly let us suppose that the function β is bounded by an ordinal $\beta_0 < \alpha$.
Let $p \in \mathrm{Lev}_{\beta_0+1}(T)$. Since the height of t is greater than β_0, we have $|V_p \cap A_\xi| \leq 1$ for every $\xi < \omega_1$. Whence we get $|V_p \cap \bigcup_{\xi<\omega_1} A_\xi| \leq \omega_1$. By induction hypothesis T_{β_0+1} is Valdivia, hence there exists a family $\mathcal{U}_0$ of clopen subsets of T_{β_0+1} which is T_0-separating and point-countable on $D_0 = \{t \in T_{\beta_0+1} : \mathrm{cf}(t) \leq \omega_0\}$. Further, for any $p \in \mathrm{Lev}_{\beta_0+1}(T)$, the subset $V_p \subset T$ is isomorphic to a tree satisfying the assumptions of Lemma 4.4.5. Hence V_p is a Valdivia compact space. Therefore there is a family of clopen sets $\mathcal{U}_p$ that is T_0-separating and point countable on $D_p = \{t \in V_p : \mathrm{cf}(t) \leq \omega_0\}$. We may assume without loss of generality that $V_p \in \mathcal{U}_p$, for every $p \in \mathrm{Lev}_{\beta_0+1}(T)$. Defining $\mathcal{U} = \widetilde{\mathcal{U}}_0 \cup (\bigcup_{p\in\mathrm{Lev}_{\beta_0+1}(T)} \mathcal{U}_p)$ we obtain a family of clopen subsets of T. Let $t \in D = \{t \in T : \mathrm{cf}(t) \leq \omega_0\}$. We consider the two cases:

- if $\mathrm{ht}(t,T) \geq \beta_0 + 1$. Let $\{p_t\} = \hat{t} \cap \mathrm{Lev}_{\beta_0+1}(T)$, then we have
$$|\mathcal{U}(t)| \leq |\mathcal{U}_0(p_t)| + |\mathcal{U}_{p_t}(t)| \leq \omega_0;$$
- if $\mathrm{ht}(t,T) < \beta_0 + 1$. Then
$$|\mathcal{U}(t)| \leq |\mathcal{U}_0(t)| \leq \omega_0.$$

Hence $\mathcal{U}$ is point-countable on D. We continue by proving that $\mathcal{U}$ is T_0-separating, let $s, t \in T$ and suppose that $\mathrm{ht}(s,T) \leq \mathrm{ht}(t,T)$.

- if $s, t \in T_{\beta_0+1}$ or $s, t \in V_p$ for some $p \in \mathrm{Lev}_{\beta_0+1}(T)$. Then use the fact that $\mathcal{U}_0$ $\langle$respectively $\mathcal{U}_p\rangle$ is T_0-separating on T_{β_0+1} $\langle V_p$, resp.$\rangle$;
- otherwise, we have $t \in V_p$, for some $p \in \mathrm{Lev}_{\beta_0+1}(T)$, then $V_p \in \mathcal{U}$ and $s \notin V_p$.

Hence the family $\mathcal{U}$ is T_0-separating. Therefore T is Valdivia.
Let us suppose that the mapping β is unbounded. We recall that β has the following property: if $t \in T$ and $\mathrm{ht}(t, T) > \beta(\xi)$, for some $\xi < \omega_1$, we have $|V_t \cap (\cup_{\eta \leq \xi} A_\eta)| \leq \omega_0$. We are going to define a family $\{S_\xi\}_{\xi < \omega_1}$ of subsets of T that satisfies the following properties:

(a) $S_\xi \subset T_{<\alpha}$;

(b) $S_\xi \cap S_\eta = \emptyset$ for $\xi \neq \eta$;

(c) if $t \in \bigcup_{\gamma<\xi} S_\gamma$ for some $\xi < \omega_1$, then $\{s \in T : s < t\} \subset \bigcup_{\gamma<\xi} S_\gamma$;

(d) if $t \in T_{<\alpha} \setminus (\bigcup_{\gamma \leq \xi} S_\gamma)$, then $\mathrm{ht}(t, T) > \beta(\xi + 1)$;

(e) if $t \in S_\xi$ for some $\xi < \omega_1$, then $V_t \cap (\bigcup_{\gamma<\xi} A_\gamma)$ is at most singleton.

We use a transfinite induction argument. Firstly we define

$$S_0 = \{t \in T : \mathrm{ht}(t, T) \leq \beta(1)\}.$$

We observe that $S_0 = T_{\beta(1)}$ and $\beta(1) < \alpha$, hence S_0 satisfies $(a)-(e)$. Let us suppose that for every $\gamma < \eta$, S_γ has been already defined such that $(a) - (e)$ are fulfilled. Now we are going to define S_η, let us consider the two cases:

- $\eta = \gamma + 1$. Let $M_\eta = \{t \in T : t \text{ is minimal in } T_{<\alpha} \setminus (\bigcup_{\zeta \leq \gamma} S_\zeta)\}$. For every $t \in M_\eta$ we have, by induction hypothesis, $\mathrm{ht}(t, T) > \beta(\eta)$. Hence $|V_t \cap (\cup_{\zeta \leq \eta} A_\zeta)| \leq \omega_0$, therefore by Lemma 4.4.4 there exists $\delta(t) < \alpha$ such that $\delta(t) > \beta(\eta)+1$ and if $t_0, t_1 \in V_t \cap (\cup_{\zeta \leq \eta} A_\zeta)$ we have $\mathrm{ht}(t_0 \wedge t_1, T) < \delta(t)$. Let $z(t) = \max\{\delta(t), \beta(\eta + 1)\}$ and
$$S_\eta = \bigcup_{t \in M_\eta} (V_t \cap T_{z(t)}).$$
Now we are going to prove that S_η satisfies $(a) - (e)$. For every $t \in M_\eta$ we have $\beta(\eta + 1) \leq z(t) < \alpha$, hence (a) and (d) are satisfied. By construction $S_\eta \subset T_{<\alpha} \setminus (\bigcup_{\zeta \leq \gamma} S_\zeta)$, therefore (b) is satisfied.
By definition of S_η and the induction hypothesis it follows that if $t \in \bigcup_{\gamma<\eta+1} S_\gamma$, then $\{s \in T : s < t\} \subset \bigcup_{\gamma<\eta+1} S_\gamma$, hence S_η satisfies (c).
Finally we are going to prove that S_η satisfies (e). Let $t \in S_\eta$, since $\mathrm{ht}(t, T) > \beta(\eta)$ we have $|V_t \cap A_\zeta| \leq 1$ for every $\zeta < \eta$. Suppose that there are $s_1 \in A_{\zeta_1}$ and $s_2 \in A_{\zeta_2}$, with $\zeta_1 < \zeta_2 < \eta$ such that $s_1, s_2 \in V_t \cap (\cup_{\zeta<\eta} A_\zeta)$. It follows that $\mathrm{ht}(s_1 \wedge s_2, T) \geq \mathrm{ht}(t, T)$. Further let $p \in \hat{t} \cap M_\gamma$, by construction we have $z(p) < \mathrm{ht}(t, T)$. Since $\zeta_1 < \zeta_2 \leq \gamma$ we have $\mathrm{ht}(s_1 \wedge s_2, T) < z(p) < \mathrm{ht}(t, T)$. That is clearly a contradiction. Hence $|V_t \cap (\cup_{\zeta<\eta} A_\zeta)| \leq 1$;

- η is limit. Let $M^0_\eta = \{t \in T : t \text{ is minimal in } T_{<\alpha} \setminus (\bigcup_{\gamma<\eta} S_\gamma)\}$. Let $t \in M^0_\eta$, by induction hypothesis, we have $\mathrm{ht}(t, T) > \beta(\gamma)$, for every $\gamma < \eta$. Hence $|V_t \cap (\cup_{\gamma<\eta} A_\gamma)| \leq \omega_0$, therefore by Lemma 4.4.4 there exists $\delta(t) < \alpha$ such that $\delta(t) > \sup_{\gamma<\eta}(\beta(\gamma))+1$ and if $t_0, t_1 \in V_t \cap (\cup_{\zeta<\eta} A_\zeta)$ we have $\mathrm{ht}(t_0 \wedge t_1, T) < \delta(t)$. Let $z(t) = \max\{\delta(t), \beta(\eta)\}$ and
$$S^0_\eta = \bigcup_{t \in M^0_\eta} (V_t \cap T_{z(t)}).$$
Let $M_\eta = \{t \in T : t \text{ is minimal in } T_{<\alpha} \setminus (S^0_\eta \cup (\bigcup_{\gamma<\eta} S_\gamma))\}$. Let $t \in M_\eta$, we have $\mathrm{ht}(t, T) > \beta(\eta)$. Hence $|V_t \cap (\cup_{\zeta\leq\eta} A_\zeta)| \leq \omega_0$, therefore by Lemma 4.4.4 there exists $\delta(t) < \alpha$ such that $\delta(t) > \beta(\eta) + 1$ and if $t_0, t_1 \in V_t \cap (\cup_{\zeta\leq\eta} A_\zeta)$ we have $\mathrm{ht}(t_0 \wedge t_1, T) < \delta(t)$. Let $z(t) = \max\{\delta(t), \beta(\eta + 1)\}$ and
$$S_\eta = S^0_\eta \cup \bigcup_{t \in M_\eta} (V_t \cap T_{z(t)}).$$
Since the definition of S_η is similar to the one given in previous case, it is possible to prove that conditions $(a) - (d)$ are satisfied by S_η.
We now show that S_η satisfies (e). Let $t \in S_\eta$, for every $\gamma < \eta$ we have $\mathrm{ht}(t, T) > \beta(\gamma)$ and in particular we have $|V_t \cap A_\gamma| \leq 1$. As in the successor case suppose there are $s_1 \in A_{\gamma_1}$ and $s_2 \in A_{\gamma_2}$, with $\gamma_1 < \gamma_2 < \eta$ such that $s_1, s_2 \in V_t \cap (\cup_{\gamma<\eta} A_\gamma)$. It follows that $\mathrm{ht}(s_1 \wedge s_2, T) \geq \mathrm{ht}(t, T)$. Let $p \in \hat{t} \cap M_{\gamma_2+1}$, by construction we have $z(p) < \mathrm{ht}(t, T)$. Since $\gamma_1 < \gamma_2 \leq \gamma_2 + 1$ we have $\mathrm{ht}(s_1 \wedge s_2, T) < z(p) < \mathrm{ht}(t, T)$. That is clearly a contradiction. Hence $|V_t \cap (\cup_{\gamma<\eta} A_\gamma)| \leq 1$.

By construction $\{S_\xi\}_{\xi<\omega_1}$ is a pairwise disjoint family of subsets of $T_{<\alpha}$. We observe that $T_{<\alpha} = \bigcup_{\xi<\omega_1} S_\xi$, in fact, since the function β is unbounded, for every $t \in T_{<\alpha}$ there exists $\xi < \omega_1$ such that $\mathrm{ht}(t, T) < \beta(\xi)$ and therefore $t \in \cup_{\gamma<\xi} S_\gamma$. From these two observations it follows that for every $t \in T_\alpha$ on a successor level there exists a unique $\xi < \omega_1$ such that $t \in S_\xi$.
By the transfinite induction hypothesis, there is a family $\mathcal{U}_\gamma$ of clopen subsets of $T_{\alpha_\gamma+1}$ which is T_0-separating and point-countable on $D_\gamma = \{t \in T_{\alpha_\gamma+1} : \mathrm{cf}(t) \leq \omega_0\}$, for every $\gamma < \omega_1$. As above we are assuming that the elements of each $\mathcal{U}_\gamma$ are of the form W^F_s.
Let $\phi : T_{<\alpha} \to [0, \omega_1)$ be the function that satisfies $t \in S_{\phi(t)}$, for every $t \in T_{<\alpha}$. Let $I(\omega_1)$ the set of successors ordinals less than ω_1. Let $\gamma \in I(\omega_1)$, we define $U_p = V_p \cap U$ for any $U \in \mathcal{U}_\gamma$ and $p \in \mathrm{Lev}_{\alpha_{(\gamma-1)}+1}(T)$. Now we are going to define a family $\mathcal{U}$ of clopen subsets of T:
$$\begin{aligned}\mathcal{U} =& \{\{t\} : t \in \mathrm{Lev}_{\alpha+1}(T)\} \\ &\cup \bigcup_{\gamma\in I(\omega_1)} \bigcup_{U\in\mathcal{U}_\gamma} \{\widetilde{U}_p : p \in \mathrm{Lev}_{\eta_{(\gamma-1)}+1}(T),\ \widetilde{U}_p \cap \bigcup_{\eta<\phi(\min(U_p))} A_\eta = \emptyset\} \\ &\cup \bigcup_{\gamma\in I(\omega_1)} \bigcup_{U\in\mathcal{U}_\gamma} \{\widetilde{U}_p \setminus \mathrm{ims}(s) : p \in \mathrm{Lev}_{\eta_{(\gamma-1)}+1}(T),\ \{s\} = \widetilde{U}_p \cap \bigcup_{\eta<\phi(\min(U_p))} A_\eta\}.\end{aligned}$$
It remains to prove that $\mathcal{U}$ is a family of clopen subsets which is T_0-separating and point-countable on $D = \{t \in T : \mathrm{cf}(t) \leq \omega_0\}$. It is possible to prove that $\mathcal{U}$ is

T_0-separating in the same way as in Lemma 4.4.5. Now we are going to prove that $\mathcal{U}$ is point-countable on D. Let $t \in D$ and consider the two cases.

- Suppose that $\mathrm{ht}(t, T) < \alpha$. We observe that $|\{\gamma < \omega_1 : \mathrm{ht}(t, T) > \alpha_\gamma\}| \leq \omega_0$, let us define $\gamma_0 = \sup\{\gamma < \omega_1 : \mathrm{ht}(t, T) > \alpha_\gamma\}$. Hence we have

  $$\mathcal{U}(t) = \bigcup_{\xi \leq \gamma_0} \{\widetilde{U}_p : p \in \mathrm{Lev}_{\alpha_{\xi-1}+1}(T),\, U \in \mathcal{U}_\xi,\, \hat{t} \cap \mathrm{Lev}_{\alpha_\xi+1}(T) \subset U\}.$$

  Since $\mathcal{U}_\xi$ is point-countable on $D_\xi \subset T_{\alpha_\xi+1}$, for each $\xi < \gamma_0$ we have that $\mathcal{U}(t)$ is countable as well.

- Let $t \in \mathrm{Lev}_{\alpha+1}(T)$, then there exists $s \in R \cap \mathrm{Lev}_\alpha(T)$ such that $t \in \mathrm{ims}(s)$. There exists $\gamma < \omega_1$ such that $s \in A_\gamma$. Take any element X of the family $\mathcal{U}$ containing t, then there are the following possibilities:
  - $X = \{t\}$, exactly one element of $\mathcal{U}$ has this form;
  - there exist $\eta < \omega_1$ and $p \in \mathrm{Lev}_{\alpha_{(\eta-1)}+1}(T)$ such that $X = \widetilde{U}_p$ for some $U \in \mathcal{U}_\eta$. Hence $\eta < \gamma$, and moreover we have $(\hat{t} \cap \mathrm{Lev}_{\alpha_{(\eta+1)}+1}(T)) \subset U_p$ and $|\mathcal{U}_\eta(\hat{t} \cap \mathrm{Lev}_{\alpha_{(\eta+1)}+1})| \leq \omega_0$. It follows that there are at most countably many elements of $\mathcal{U}$ containing t of this form;
  - there are $\eta < \omega_1$, $p \in \mathrm{Lev}_{\alpha_{(\eta-1)}+1}(T)$ and $u \in R$ such that $X = \widetilde{U}_p \backslash \mathrm{ims}(u)$, for some $U \in \mathcal{U}_\xi$. We have $\eta < \gamma$ and since $(\hat{t} \cap \mathrm{Lev}_{\alpha_{(\eta+1)}+1}(T)) \subset U_p$ and $|\mathcal{U}_\eta(\hat{t} \cap \mathrm{Lev}_{\alpha_{(\eta+1)}+1})| \leq \omega_0$, it follows that there are at most countably many elements of $\mathcal{U}$ containing t of this form.

Therefore T is Valdivia. This concludes the proof. $\square$

4.5 Banach spaces of continuous functions on trees

In this section we deal with the space of continuous functions on a tree T. We will prove that Valdivia compact trees can be characterized by their space of continuous functions, in fact we will prove that T is Valdivia if and only if $C(T)$ is 1-Plichko (T has a retractional skeleton if and only if $C(T)$ has a 1-projectional skeleton). Notice that, in general, this is not true: there are examples of a non Valdivia compacta K such that $C(K)$ is 1-Plichko see [1], [8], [13].
In the final part of this section, we will prove that each $C(T)$ space, where T is a tree with height less than $\omega_1 \cdot \omega_0$, is a Plichko space. Using this result, we observe that, the tree T defined as in [17, Example 4.3], is an example of compact space with retractional skeletons none of which is commutative, however $C(T)$ is a Plichko space, therefore it has a commutative projectional skeleton (see [11, Theorem 27]).

Theorem 4.5.1. *Let T be a tree. Then T is a Valdivia compact space if and only if $C(T)$ is a 1-Plichko space.*

Proof. The "only if" part is a particular case of [10, Theorem 5.2]. Suppose that $C(T)$ is a 1-Plichko space and let $S \subset C(T)^*$ be a 1-norming Σ-subspace. The

compact space T embeds canonically into $B_{C(T)^*}$ by identifying each $t \in T$ with the Dirac measure concentrated on $\{t\}$. This embedding will be denoted by δ. We are going to prove that $\delta(t) = \delta_t \in S$ whenever $t \in T$ is on a successor level.
Let $t \in T$ on a successor level:

- suppose that $\operatorname{ims}(t)$ is finite, it follows that t is isolated. Hence, since S is 1-norming, we obtain $\delta_t \in S$;

- on the other hand suppose that $\operatorname{ims}(t)$ is infinite. Let $\{t_n\}_{n\in\omega_0}$ be an infinite subset of $\operatorname{ims}(t)$. Since V_{t_n} is a clopen subset of T we have that the function $f_n = 1_{V_{t_n}}$ is continuous for every $n \in \omega_0$. Hence, since S is a 1-norming Σ-subspace, there exists $\mu_n \in S \cap B_{C(T)^*}$ such that $\mu_n(f_n) = \mu_n(V_{t_n}) = 1$. Observing that $\|\mu_n\| = |\mu_n|(T) \leq 1$ and $1 = |\mu_n(V_{t_n})| \leq |\mu_n|(V_{t_n})$, we easily obtain that $\operatorname{supp}(\mu_n) \subset V_{t_n}$.
  Now let $f \in C(T)$ and $\varepsilon > 0$, define
  $$Z_\varepsilon(f,t) = \{s \in \operatorname{ims}(t) : \sup_{p\in V_s} |f(p) - f(t)| > \varepsilon\}.$$
  By the continuity of f, the set $Z_\varepsilon(f,t)$ is finite. Hence there exists $n_0 \in \omega_0$ such that
  $$\sup_{p\in V_{t_n}} |f(p) - f(t)| < \varepsilon$$
  for every $n \geq n_0$. Let $n \geq n_0$, we obtain:
  $$\begin{aligned} |\mu_n(f) - f(t)| &= \left| \int_{V_{t_n}} f(x) d\mu_n(x) - f(t) \right| = \left| \int_{V_{t_n}} f(x) - f(t) d\mu_n(x) \right| \\ &\leq \int_{V_{t_n}} |f(x) - f(t)| d\mu_n(x) < \varepsilon. \end{aligned}$$
  Hence the sequence $\mu_n(f)$ converges to $\delta_t(f)$ for every $f \in C(T)$. By the weak* countably closedness of S it follows that $\delta_t \in S$.

Therefore, since by [10, Theorem 5.2] $B_{C(T)^*}$ is a Valdivia compact space with $B_{C(T)^*} \cap S$ as Σ-subset and $S \cap \delta(T)$ is dense in $\delta(T)$, we obtain that $\delta(T)$ is a Valdivia compact space. □

We observe that the same result can be done in the non-commutative setting. Using [2, Proposition 3.15] instead of [10, Theorem 5.2] we obtain the following result.

Theorem 4.5.2. *Let T be a tree. Then T has a retractional skeleton if and only if $C(T)$ has a 1-projectional skeleton.*

Now we are going to investigate the space of continuous function on trees with height less than $\omega_1 \cdot \omega_0$. It turns out that all such spaces are Plichko.

Theorem 4.5.3. *Let T be a tree such that* $\operatorname{ht}(T) < \omega_1 \cdot \omega_0$*, then $C(T)$ is a Plichko space.*

The previous theorem follow immediatly from the next technical proposition, where, for every tree T of height less than $\omega_1 \cdot \omega_0$, a norming Σ-subspace of $C(T)^*$ is explicitly described.

Proposition 4.5.4. *Let T be a tree and suppose that $\mathrm{ht}(T) \leq (\omega_1 \cdot n) + 1$, for some $n \geq 1$. Then*

$$\Lambda = \{\mu \in C(T)^* : \forall j \in \{1, ..., n-1\}, \forall t \in \mathrm{Lev}_{\omega_1 \cdot j}(T) :\ \mu(V_t) = 0\}$$

is a $(2n-1)$-norming Σ-subspace of $C(T)^*$. If $\mathrm{ht}(T) > (\omega_1 \cdot (n-1)) + 1$, then the norming constant is exactly $2n-1$.

Lemma 4.5.5. *Let T be a tree such that $\mathrm{ht}(T) \leq \omega_1 + 1$ and $D = \{t \in T : \mathrm{cf}(t) \leq \omega_0\}$. Then the set*

$$S = \{\mu \in C(T)^* : \mathrm{supp}(\mu) \subset D\}$$

is a 1-norming Σ-subspace of $C(T)^*$.

Proof. Since $\mathrm{ht}(T) \leq \omega_1 + 1$, by [17, Theorem 4.1], T is a Valdivia compact space and D is a dense Σ-subspace. Hence, by [10, Proposition 5.1] we have that

$$S = \{\mu \in C(T)^* : \mathrm{supp}(\mu) \text{ is a separable subset of } D\}$$

is a 1-norming Σ-subspace of $C(T)^*$. Finally the assertion follows by Theorem 4.3.2. $\square$

Proof of Proposition 4.5.4. If $n = 1$ the assertion follows from Lemma 4.5.5, hence we assume that $n \geq 2$ and that T is a tree with $\omega_1 \cdot (n-1) + 1 < \mathrm{ht}(T) \leq (\omega_1 \cdot n) + 1$. Let $S_0 = \emptyset$, $S_i = T_{\omega_1 \cdot i}$ for each $i \leq n-1$ and $S_n = T$. Then we obtain the following:

- S_i is a closed subset of T for every $i \in \{1, ..., n\}$;
- S_1 is isomorphic to a tree of height $\omega_1 + 1$, hence, by Lemma 4.5.5, $C(S_1)$ is a 1-Plichko space with $\Sigma_1 = \{\mu \in C(S_1)^* : \mathrm{supp}(\mu) \subset D \cap S_1\}$ as Σ-subspace;
- for every $i \in \{1, ..., n-1\}$, the subset $S_{i+1} \setminus S_i$ is a locally compact space and $C_0(S_{i+1} \setminus S_i)$ is a 1-Plichko space. In fact, let $t \in \mathrm{Lev}_{(\omega_1 \cdot i)+1}(T)$ and $U_t = V_t \cap S_{i+1}$. It is clear that U_t is a closed subset of T and further it is isomorphic to a tree of height less or equal than $\omega_1 + 1$. Hence, by Lemma 4.5.5, we obtain that U_t is a Valdivia compact space and $C(U_t)$ is a 1-Plichko space with $\Sigma_{i,t} = \{\mu \in C(U_t)^* : \mathrm{supp}(\mu) \subset D \cap U_t\}$ as Σ-subspace. Moreover we observe that $S_{i+1} \setminus S_i$ is the topological sum of all U_t, therefore $C_0(S_{i+1} \setminus S_i)$ is the c_0-sum of $C(U_t)$ and its dual is the ℓ_1-sum of $C(U_t)^*$. Hence, by [10, Theorem 4.31] we obtain that $C_0(S_{i+1} \setminus S_i)$ is a 1-Plichko space and

  $$\Sigma_i = \{(\mu_t)_{t \in \mathrm{Lev}_{\omega_1 \cdot i}(T)} \in C_0(S_{i+1} \setminus S_i)^* : (\forall t \in \mathrm{Lev}_{\omega_1 \cdot i}(T))(\mu_t \in \Sigma_{i,t})\}$$

  is its Σ-subspace.

Let $i \leq n-1$ and $r_i : T \to T$ be the continuous retraction defined by:

$$r_i(t) = \begin{cases} t & \text{if } t \in S_i, \\ s & \text{if } s \leq t \text{ and } s \in \mathrm{Lev}_{\omega_1 \cdot i}(T). \end{cases}$$

For simplicity we define $r_n : T \to T$ as the identity map. These continuous retractions induce continuous linear projections on $C(T)$ and such projections are defined by $P_i(f) = f \circ r_i$. Then for every $f \in C(T)$ and every $i \leq n-1$ the following conditions hold:

- $f \restriction_{S_i} = P_i f \restriction_{S_i}$;
- $(f - P_i f) \restriction_{S_{i+1}} = (P_{i+1} f - P_i f) \restriction_{S_{i+1}}$;
- $(f - P_i f) \restriction_{S_{i+1} \setminus S_i} \in C_0(S_{i+1} \setminus S_i)$.

In order to get an isomorphism between $C(T)$ and a 1-Plichko space we are going to define the following map:

$$\begin{aligned} G : C(T) &\to C(S_1) \oplus_\infty C_0(S_2 \setminus S_1) \oplus_\infty ... \oplus_\infty C_0(S_n \setminus S_{n-1}) \\ f &\mapsto (f \restriction_{S_1}, (f - P_1 f) \restriction_{S_2 \setminus S_1}, ..., (f - P_{n-1} f) \restriction_{S_n \setminus S_{n-1}}). \end{aligned}$$

This easily implies that the norm of G is at most 2. Now we are going to define the inverse of the mapping G. For simplicity we denote by

$$W = C(S_1) \oplus_\infty C_0(S_2 \setminus S_1) \oplus_\infty ... \oplus_\infty C_0(S_n \setminus S_{n-1}).$$

Let $(f_1, f_2, ..., f_n) \in W$, we define its preimage $f \in C(T)$ as follows:

$$f(t) = \begin{cases} f_1(t) & \text{if } t \in S_1, \\ f_{i+1}(t) + \sum_{j=1}^{i} f_j(r_j(t)) & \text{if } t \in S_{i+1} \setminus S_i. \end{cases}$$

It follows that the norm of the inverse of G is at most n. Therefore G is an isomorphism. Since each component of W is a 1-Plichko space, we have that W is 1-Plichko, therefore $C(T)$ is a Plichko space. Moreover, the subspace $\Sigma = \{(\mu_i)_{i=1}^n \in W^* : \mu_i \in \Sigma_i\} \subset W^*$ is a 1-norming Σ-subspace of W. In order to compute the exact value of the norming constant of the Σ-subspace $G^*(\Sigma)$, we are going to describe the adjoint map of G:

$$\begin{aligned} G^*(\mu_1, \dots, \mu_n)(f) = (\mu_1, \dots, \mu_n)(Gf) &= \mu_1(f \restriction_{S_1}) + \sum_{j=1}^{n-1} \mu_{j+1}((f - P_j f) \restriction_{S_{j+1} \setminus S_j}) \\ &= \int_{S_1} f d\mu_1 + \sum_{j=1}^{n-1} \int_{S_{j+1} \setminus S_j} (f - P_j f) d\mu_{j+1} \\ &= \int_T f d(\sum_i^n \mu_i) - \sum_{j=1}^{n-1} \int_T f dr_j(\mu_{j+1}), \end{aligned}$$

Hence

$$G^*(\mu_1, \dots, \mu_n) = \sum_{i=1}^{n} \mu_i - \sum_{j=1}^{n-1} r_j(\mu_{j+1}).$$

Where $r_i(\mu_j)(A) = \mu_j(r_i^{-1}(A))$ for every measurable subset A of T. Now we are going to give a representation of the inverse of G^*. Let $\mu = G^*(\mu_1, ..., \mu_n) = \sum_{i=1}^{n} \mu_i - \sum_{j=1}^{n-1} r_j(\mu_{j+1})$ and $k \leq n-1$, then

$$r_k(\mu) = \sum_{i=1}^{n} r_k(\mu_i) - \sum_{j=1}^{n-1} r_k(r_j(\mu_{j+1})).$$

Further we observe that $r_k(\mu_i) = \mu_i$ for $i \leq k$ and

$$r_k(r_j(\mu_j + 1)) = \begin{cases} r_j(\mu_{j+1}), & \text{if } j < k, \\ r_k(\mu_{j+1}), & \text{if } j \geq k. \end{cases}$$

Hence we obtain

$$\begin{aligned} r_k(\mu) &= \sum_{i=1}^{k} \mu_i + \sum_{i=k+1}^{n} r_k(\mu_i) - \sum_{j=1}^{k-1} r_j(\mu_{j+1}) - \sum_{j=k}^{n-1} r_k(\mu_{j+1}) \\ &= \sum_{i=1}^{k} \mu_i - \sum_{j=1}^{k-1} r_j(\mu_{j+1}). \end{aligned}$$

Now we take the restriction of μ to $S_i \setminus S_{i-1}$:

$$\begin{aligned} &\mu \restriction_{S_1} = \mu_1 - r_1(\mu_2), \\ &\mu \restriction_{S_i \setminus S_{i-1}} = \mu_i - r_i(\mu_{i+1}), \text{ for } i \in \{2, ..., n-1\}, \\ &\mu \restriction_{T \setminus S_{n-1}} = \mu_n. \end{aligned}$$

Hence, combining these two formulae we obtain

$$\begin{aligned} &\mu_1 = r_1(\mu), \\ &\mu_i = r_i(\mu) - \mu \restriction_{S_{i-1}}, \text{ for } i \in \{2, ..., n-1\}, \\ &\mu_n = \mu \restriction_{T \setminus S_{n-1}} . \end{aligned}$$

Therefore the inverse of G^* can be represented as

$$\mu \mapsto (r_1(\mu), r_2(\mu) - \mu \restriction_{S_1}, \dots, r_{n-1}(\mu) - \mu \restriction_{S_{n-2}}, \mu \restriction_{T \setminus S_{n-1}}).$$

Hence we have the following:

$$\begin{aligned} G^*(\Sigma) &= \{\mu \in C(T)^* : (r_1(\mu), r_2(\mu) - \mu \restriction_{S_1}, \dots, r_{n-1}(\mu) - \mu \restriction_{S_{n-2}}, \mu \restriction_{T \setminus S_{n-1}}) \in \Sigma\} \\ &= \{\mu \in C(T)^* : \forall j \in \{1, \dots, n\}, \forall B \subset \mathrm{Lev}_{\omega_1 \cdot j}(T) : \mu(\bigcup_{t \in B} V_t) = 0\} \\ &= \{\mu \in C(T)^* : \forall j \in \{1, \dots, n\}, \forall t \in \mathrm{Lev}_{\omega_1 \cdot j}(T) : \mu(V_t) = 0\}. \end{aligned}$$

Indeed, the first equality is obvious. Let us prove the second one:

- $\subset$: Let $\mu \in G^*(\Sigma)$ and $B \subset \mathrm{Lev}_{\omega_1 \cdot j}(T)$, for some $j \in \{1, ..., n-1\}$. Then, since $(r_j(\mu) - \mu \restriction_{S_{j-1}}) \in \Sigma_j$ we have $(r_j(\mu) - \mu \restriction_{S_{j-1}})(B) = 0$. Hence $0 = (r_j(\mu) - \mu \restriction_{S_{j-1}})(B) = \mu(\bigcup_{t \in B} V_t) - \mu(B \cap S_{j-1}) = \mu(\bigcup_{t \in B} V_t)$. If $j = n$ we have $\mu \restriction_{T \setminus S_{n-1}} \in \Sigma_n$, hence $0 = \mu \restriction_{T \setminus S_{n-1}} (B) = \mu(B)$.

- $\supset$: Let $\mu \in C(T)^*$ such that for each $j \in \{1, .., n\}$ and $B \subset \mathrm{Lev}_{\omega_1 \cdot j}(T)$, $\mu(\bigcup_{t \in B} V_t) = 0$ holds. We are going to show that for every $j \in \{1, ..., n-1\}$ and every $t \in \mathrm{Lev}_{\omega_1 \cdot j}(T)$ there exists $s < t$ such that $(r_j(\mu) - \mu \restriction_{S_{j-1}})(V_s) = 0$. In fact, let $s < t$ such that $\mathrm{ht}(s, T) > \omega_1 \cdot (j-1)$, then

  $$\begin{aligned}(r_j(\mu) - \mu \restriction_{S_{j-1}})(V_s) &= r_j(\mu)(V_s) = \mu(V_s) = \mu(V_s \cap T_{<\omega_1 \cdot j}) + \mu((V_s \cap (T \setminus T_{<\omega_1 \cdot j})) \\ &= \mu(V_s \cap T_{<\omega_1 \cdot j}).\end{aligned}$$

  Hence, by Proposition 4.3.1, there exists $s < t$ such that $(r_j(\mu) - \mu \restriction_{S_{j-1}})(V_s) = 0$. Therefore $(r_j(\mu) - \mu \restriction_{S_{j-1}}) \in \Sigma_j$. For the same reason we have $\mu \restriction_{T \setminus S_{n-1}} \in \Sigma_n$.

Let us prove the last one:

- $\subset$: is trivial.

- $\supset$: Let $\mu \in C(T)^*$ such that for any $j \in \{1, ..., n\}$ and $t \in \mathrm{Lev}_{\omega_1 \cdot j}(T)$, $\mu(V_t) = 0$ holds. Fix $j \in \{1, ..., n\}$. Let us define the continuous part of μ by μ_c, by Proposition 4.3.1, $\mathrm{supp}(\mu_c \restriction S_j)$ is contained in T_α, where $\alpha < \omega_1 \cdot j$. Hence we may suppose that $\mu(V_t \cap S_j) = 0$ whenever $\mathrm{ht}(t, T) > \alpha$. Hence for each relatively open subset $A \in \mathrm{Lev}_{\omega_1 \cdot}(T)$ we have $\mu(A) = 0$, therefore, by the completeness of μ, we have the same conclusion for each subset of $\mathrm{Lev}_{\omega_1 \cdot j}(T)$.

Hence $\Lambda = G^*(\Sigma)$ is a Σ-subspace of $C(T)^*$. Now are going to show that the norming constant of Λ is equal to $2n-1$. Let $f \in C(T)$, without loss of generality, we may suppose $\|f\| = 1$. Let $t \in T$ such that $|f(t)| = 1$, then there exists $i \in \{0, ..., n-1\}$ such that $\omega_1 \cdot i \leq \mathrm{ht}(t, T) < \omega_1 \cdot (i+1)$. For simplicity we set $t_i = t$. Let us consider $\mu \in \Lambda$ defined by:

$$\mu = (\sum_{k=1}^{i} \delta_{t_k} - \delta_{s_k}) + \delta_{t_0}$$

Where $\{s_k\} = \hat{t}_i \cap \mathrm{Lev}_{\omega_1 \cdot k}(T)$ and t_k is such that $s_k < t_k < s_{k+1}$ $\langle$resp. $t_0 < s_1\rangle$ and $f(s_{k+1}) = f(t_k)$ for $k \geq 1$ $\langle$for $k = 0$, respectively$\rangle$. Therefore we obtain $2n-1 \geq 2i+1 = \|\mu\|$ and easily we get $|\mu(f)| = 1$. Hence the norming constant of Λ is at most $2n-1$. On the other hand, suppose that $\mathrm{ht}(T) > (\omega_1 \cdot (n-1)) + 1$ and let $t_{n-1} \in T$ such that $\mathrm{ht}(t_{n-1}, T) > (\omega_1 \cdot (n-1)) + 1$ and

$$\begin{aligned}\{s_i\} &= \hat{t}_{n-1} \cap \mathrm{Lev}_{\omega_1 \cdot i}(T) \text{ for } i = 1, ..., n-1, \\ \{t_i\} &= \hat{t}_{n-1} \cap \mathrm{Lev}_{(\omega_1 \cdot i)+1}(T) \text{ for } i = 1, ..., n-2.\end{aligned}$$

Let us consider the following continuous map:

$$f(t) = 1_{V_{t_{n-1}}}(t) + \delta_1 1_{T \setminus V_{t_1}}(t) + \sum_{i=1}^{n-2} \delta_{i+1} 1_{V_{t_i} \setminus V_{t_{i+1}}}(t),$$

where $\delta_1 = 1/(2n-1)$ and $\delta_i = (2i-1)\delta_1$ for $i = 1, .., n-1$. Let $\mu \in \Lambda$ such that $\mu(f) = 1$. We put

$$\begin{aligned}
&\mu(T \setminus V_{s_1}) = a_0, \\
&\mu(V_{s_i} \setminus V_{t_i}) = b_i \text{ for } i = 1, ..., n-1, \\
&\mu(V_{t_i} \setminus V_{s_{i+1}}) = a_i \text{ for } i = 1, ..., n-2, \\
&\mu(V_{t_{n-1}}) = a_{n-1}
\end{aligned}$$

Since $\mu \in \Lambda$ we have $b_i = -a_i$ for every $i = 1, ..., n-1$. Therefore we obtain:

$$\begin{aligned}
1 = \mu(f) &= a_{n-1} + \sum_{i=1}^{n} \delta_i (a_{i-1} - a_i) \\
&= \delta_1 a_0 + (1 - \delta_{n-1}) a_{n-1} + \sum_{i=1}^{n-1} a_i (\delta_{i+1} - \delta_i) \\
&\leq (|a_0| + \sum_{i=1}^{n-1} 2|a_i|) \cdot \max\{\delta_1, \frac{1-\delta_{n-1}}{2}\} \\
&= (|a_0| + \sum_{i=1}^{n-1} 2|a_i|) \cdot \frac{1}{2n-1}.
\end{aligned}$$

Hence $\|\mu\| \geq 2n-1$. This concludes the proof. □

Combining Theorem 4.4.2 and Theorem 4.5.1 we have several examples of trees T, also with height bigger than $\omega_1 \cdot \omega_0$, such that $C(T)$ is a 1-Plichko space. However, in the final part of the proof of Theorem 4.5.3, the norming constant of that particular Σ-subspace grows as $2n-1$. This means that, in general, this is not the optimal choice for the Σ-subspace. This fact naturally rises the following question.

Problem 4.5.6. *Let T be a tree with height equal to $\omega_1 \cdot \omega_0$. Is $C(T)$ necessarily Plichko?*

Remark 4.5.7. We have assumed that every tree was rooted, the above results can be proved also if the tree T has finitely many minimal elements. In fact if T has finitely many minimal elements, then it can be viewed as the topological direct sum of rooted trees.

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