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Height Pairings of 1-Motives

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Abstract

The purpose of this work is to generalize, in the context of 1-motives, the height pairings constructed by B. Mazur and J. Tate on abelian varieties (see [MT83]). Following their approach, we consider $\rho-$splittings of the Poincaré biextension of a 1-motive and require that they be compatible with the canonical linearization associated to the biextension. We establish results concerning the existence of such $\rho-$splittings. When ρ is unramified this is guaranteed if the monodromy pairing of the 1-motive considered is non-degenerate. For ramified ρ, the $\rho-$splitting is constructed from a pair of splittings of the Hodge filtrations of the de Rham realizations of the 1-motive and its dual. This generalizes previous results by R. Coleman [Col91] and Y. Zarhin [Zar90] for abelian varieties. These $\rho-$splittings are then used to define a global pairing between rational points of a 1-motive and its dual. We also provide local pairings between zero cycles and divisors on a variety, which is done by applying the previous results to its Picard and Albanese 1-motives.

Abstract

Lo scopo di questo lavoro è la generalizzazione, nel contesto degli 1-motivi, degli accoppiamenti di altezza costruiti da B. Mazur e J. Tate [MT83] sulle varietà abeliane. Seguendo il loro approccio, consideriamo $\rho-$splittings della biestensione di Poincaré de un 1-motivo e richiediamo che siano compatibili con la linearizzazione canonica associata alla biestensione. Stabiliamo quindi risultati riguardanti l'esistenza di tali $\rho-$splittings. Quando ρ è non ramificato, tale risultato segue se l'accoppiamento di monodromia dell'1-motivo preso in considerazione è non degenere. Per ρ ramificato, il $\rho-$splitting si costruisce a partire da una coppia di scissioni delle filtrazioni di Hodge delle realizzazioni di de Rham dell'1-motivo e del suo duale. In questo modo generalizziamo precedenti risultati di R. Coleman [Col91] and Y. Zarhin [Zar90] sulle varietà abeliane. Questi $\rho-$splittings vengono poi usati per definire un accoppiamento globale sui punti razionali di un 1-motivo e del suo duale. Infine forniamo accoppiamenti locali tra i zero-cicli e i divisori di una varietà, applicando i risultati precedenti ai suoi 1-motivi di Picard e d'Albanese.

Résumé

L'objectif de ce travail est la généralisation, dans le contexte des 1-motifs, des accouplements de hauteurs construits par B. Mazur et J. Tate [MT83] sur les variétés abéliennes. Suite à leur approche, nous considérons de $\rho-$splittings de la biextension de Poincaré d'un 1-motif et nous demandons qu'ils soient compatibles avec la linéarisation canonique associée à la biextension. Nous établissons donc des résultats concernant l'existence de tels $\rho-$splittings. Quand ρ est non-ramifié, celle-ci est garanti si l'accouplement de monodromie du 1-motif pris en considération est non-dégénéré. Pour ρ ramifié, le $\rho-$splitting se construit à partir d'une paire de scindages des filtrations de Hodge des réalisations de de Rham du 1-motif et de son dual. Ceci généralise des résultats précédents di R. Coleman [Col91] and Y. Zarhin [Zar90] pour les variétés abéliennes. Ces $\rho-$splittings sont ensuite utilisés pour définir un accouplement global entre les points rationnels d'un 1-motif et de son dual. Également, nous fournissons des accouplements locaux entre les zéro-cycles et les diviseurs sur une variété, qui est fait en appliquant les résultats précédents à ses 1-motifs de Picard et d'Albanese.

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Contents

Introduction

Parings via biextensions

In [MT83], Mazur and Tate provided constructions of global pairings on the rational points of a pair of abelian varieties over a global field, and also local pairings on the set of divisors and zero cycles with disjoint support on an abelian variety over a local field. This construction generalizes Néron's height pairing (see [MT83, Prop. 2.3.1]). Their approach involves the use of $\rho-$splittings of the set of sections of a biextension over a field K, where $\rho : K^* \to Y$ is a homomorphism from the non-zero elements of K to an abelian group Y. These $\rho-$splittings are basically bi-homomorphic maps which are compatible with the natural actions of K^* (see Definition 2.1.1 for a precise definition).

A key result concerning the existence of $\rho-$splittings is the following, which is Theorem 2.1.6 below (see also [MT83, §1.5]). Fix abelian varieties A and B over a field K which is complete with respect to a place v, either archimedean or discrete, and a biextension P of (A, B) by $\mathbb{G}_m$. Given an abelian group Y and a homomorphism $\rho : K^* \to Y$ it is possible to construct canonical $\rho-$splittings

$$\psi_\rho : P_K(K) \to Y$$

in the following three cases:

1) v is archimedean and $\rho(c) = 0$ for all c such that $|c|_v = 1$;

2) v is discrete, ρ is unramified and Y is uniquely divisible by N; and

3) v is discrete, the residue field of K is finite, A has semistable ordinary reduction and Y is uniquely divisible by M,

where N is an integer depending on A and M is an integer depending on A and B. If both 2) and 3) hold, they yield the same ψ_ρ.

Now consider the case of a global field F and let A and B be abelian varieties over F. For each place v, let F_v denote the completion of F with respect to v, and for v discrete let R_v be its ring of integers. Denote $\mathbb{A}_F$ the ring of adeles of F and consider a homomorphism

$$\rho = (\rho_v) : \mathbb{A}_F^* \to Y \tag{1}$$

which annihilates the image of R_v^*, for almost all discrete places v, and is such that the "sum formula" $\sum_v \rho_v(c) = 0$ holds, for all $c \in K^*$. Suppose, moreover, that for each place v the canonical ρ_v–splitting ψ_v of $P(F_v)$ exists. In this situation we have the following fact, which is stated in Lemma 2.3.1 (see also [MT83, Lemma 3.1]). There is a canonical pairing

$$\langle \cdot , \cdot \rangle : A(F) \times B(F) \to Y \tag{2}$$

such that if $x \in P(F)$ lies above $(a, b) \in A(F) \times B(F)$ then

$$\langle a, b\rangle = \sum_v \psi_v(x_v),$$

where $x_v \in P(F_v)$ is the image of x under the inclusion $F \subset F_v$.

Pairings via splittings of the Hodge filtration

Let K be a field which is the completion of a number field with respect to a discrete place v over a prime p and consider a continuous, ramified homomorphism $\rho : K^* \to \mathbb{Q}_p$. In this case, there is another way to construct ρ–splittings of the Poincaré biextension P_A associated to an abelian variety A over K. This is done by considering a splitting of the Hodge filtration of the first de Rham cohomology of A as follows (see [IW03] and the explanation in Chapter 3).

Recall that for the first de Rham cohomology K–vector space of A we have a canonical extension

$$0 \to \mathrm{H}^0(A, \Omega^1_{A/K}) \to \mathrm{H}^1_{\mathrm{dR}}(A) \to \mathrm{H}^1(A, \mathcal{O}_A) \to 0 \tag{3}$$

provided by the Hodge filtration of $\mathrm{H}^1_{\mathrm{dR}}(A)$. The universal vectorial extension (or UVE for short) of the dual abelian variety $A^\vee$, which we here denote $A^{\vee\natural}$, sits in the exact sequence

$$0 \to V(A^\vee) \to A^{\vee\natural} \to A^\vee \to 0, \tag{4}$$

where $V(A^\vee)$ is the vector group associated to the sheaf of invariant differentials on A. We know that the resulting exact sequence of Lie algebras induced by (4) corresponds to the Hodge filtration of A displayed by (3) (see [MM74, §4]). This allows us to obtain a (uniquely determined) splitting $\eta : A^\vee(K) \to A^{\vee\natural}(K)$ at the level of groups from any splitting $r : \mathrm{H}^1(A, \mathcal{O}_A) \to \mathrm{H}^1_{\mathrm{dR}}(A)$ of (3). Since $A^{\vee\natural}$ represents the functor $\underline{\mathrm{Extrig}}(A, \mathbb{G}_m)$, then a splitting of (4) gives us a multiplicative way of associating a rigidification to every extension of A by $\mathbb{G}_m$. Indeed, take a point $a^\vee \in A^\vee(K)$ and let $P_{A,a^\vee}$ be the extension corresponding to it. Then $\eta(a^\vee)$ is the extension $P_{A,a^\vee}$ together with a rigidification, which is equivalent to a splitting

$$t_{a^\vee} : \mathrm{Lie}\, P_{A,a^\vee}(K) \to \mathrm{Lie}\, K^*.$$

Again, extending these homomorphisms of Lie algebras to homomorphisms of the corresponding groups, we are able to construct a λ–splitting

$$\gamma : P_A(K) \to K,$$

where λ is a branch of the $p-$adic logarithm. Using the fact that every ramified homomorphism factors through a branch of the logarithm, we are able to construct the desired $\rho-$splitting.

In [Col91], Coleman proved that, when A has good ordinary reduction, the canonical $\lambda-$splitting of $P_A(K)$ constructed by Mazur and Tate comes from the splitting of (3) induced by the unit root subspace, which is the subspace of $\mathrm{H}^1_{\mathrm{dR}}(A)$ on which the Frobenius acts with slope 0 (see Section 3.1). In [IW03], Iovita and Werner generalized this result to abelian varieties A with semistable ordinary reduction (see Section 3.3). This is done using the Raynaud extension of A, which can be seen as a 1-motive whose abelian part has good ordinary reduction (see also [Wer98]).

Pairings of 1-motives

The purpose of this work is to generalize the previous constructions in the context of 1-motives (and this is the content of Chapter 4). Recall the following definition (see [Del74, §10.1] and also Chapter 1). A *1-motive* M over a field K consists of:

i) a *lattice* L over K, *i.e.* a group scheme which, locally for the étale topology on K, is isomorphic to a finitely generated free abelian constant group;

ii) a *semi-abelian scheme* G over K which is an extension of an abelian scheme A by a torus T; and

iii) a morphism of group schemes $u : L \to G$.

Associated to M, we have a *dual 1-motive* $M^\vee = [L^\vee \xrightarrow{u^\vee} G^\vee]$, where $G^\vee$ is an extension of the dual abelian variety $A^\vee$ by the Cartier dual $T^\vee$ of L, and $L^\vee$ is the Cartier dual of T. The Poincaré biextension P_A of $(A, A^\vee)$ by $\mathbb{G}_m$ induces by pullback a biextension P of $(G, G^\vee)$ by $\mathbb{G}_m$ which is endowed with trivializations over $L \times G^\vee$ and $G \times L^\vee$ that coincide on $L \times L^\vee$. Under the definition of biextension of complexes given by Deligne (see [Del74, §10.2] and also Definition 1.2.4), this says that P is a biextension of $(M, M^\vee)$ by $\mathbb{G}_m$.

In order to construct analogs of the previous constructions, we will need to define the group of *$K-$points of a 1-motive* M over K (see Section 4.1). We do this by setting

$$M(K) := \mathrm{Ext}^1(M^\vee, \mathbb{G}_m).$$

Notice that this generalizes Weil-Barsotti formula for abelian varieties. We have that the trivializations of P over $L \times G^\vee$ and $G \times L^\vee$ induce a canonical $L(K) \times L^\vee(K)-$linearization on the K^*-torsor $P(K)$ (see Section 4.2). Under certain conditions, this linearization in turn induces a quotient biextension $Q_M(K)$ of $(M(K), M^\vee(K))$. We then have that for any morphism $\rho : K^* \to Y$ of abelian groups, giving a $\rho-$splitting of $P(K)$ which is compatible with the $L(K) \times L^\vee(K)-$linearization is equivalent to giving a $\rho-$splitting of the quotient biextension $Q_M(K)$.

We first consider a field K which is the completion of a number field with respect to a discrete valuation v and analyze the conditions under which the canonical $v-$splitting of $P_A(K)$ constructed by Mazur and Tate induces a $v-$splitting of $Q_M(K)$ (see Section 4.3). Namely, if A has good reduction and the monodromy pairing associated to M (see [Ray94] and also Section 1.5) is non-degenerate, we can obtain a $v-$splitting of $Q_M(K)$ from the canonical $v-$splitting of $P_A(K)$; this is the content of Theorem 4.3.2. This goes along the same lines as the construction of $v-$splittings of the Poincaré biextension of an abelian variety with semistable ordinary reduction from its Raynaud extension given in [Wer98].

We are also interested in giving a construction of $\lambda-$splittings of $Q_M(K)$, for λ a branch of the $p-$adic logarithm, analogous to the one of abelian varieties which uses the Hodge filtration of the first de Rham cohomology group (see Section 4.4). For this, we have to consider the de Rham realization of 1-motives which also uses the concept of universal vectorial extension.

The *universal vectorial extension* (UVE) of a 1-motive M over K is a two term complex of group schemes

$$M^\natural = [L \xrightarrow{u^\natural} G^\natural]$$

which is an extension of M by a vector group $V(M)$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & L & = & L & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle u^\natural} & & \downarrow{\scriptstyle u} & & \\ 0 & \longrightarrow & V(M) & \longrightarrow & G^\natural & \longrightarrow & G & \longrightarrow & 0\ . \end{array} \qquad (5)$$

It is well known that the universal vectorial extension of a 1-motive always exists. Deligne then defines the *de Rham realization* of M as

$$\mathrm{T}_{\mathrm{dR}}(M) = \underline{\mathrm{Lie}} G^\natural$$

(see [Del74, §10.1.7] and also Definition 1.3.2). The *Hodge filtration* of $\mathrm{T}_{\mathrm{dR}}(M)$ is defined as

$$F^i\,\mathrm{T}_{\mathrm{dR}}(M) = \begin{cases} \mathrm{T}_{\mathrm{dR}}(M) & \text{if } i = -1, \\ V(M) & \text{if } i = 0, \\ 0 & \text{if } i \neq -1, 0. \end{cases}$$

We prove in Theorems 4.4.8 and 4.4.9 that we can obtain a $\lambda-$splitting of $Q_M(K)$ from a pair of splittings $\eta : G(K) \to G^\natural(K)$, $\eta : G^\vee(K) \to G^{\vee\natural}(K)$ of (5) which are dual with respect to Deligne's pairing

$$(\,\cdot\,,\,\cdot\,)_M^{Del} : \mathrm{T}_{\mathrm{dR}}(M) \otimes \mathrm{T}_{\mathrm{dR}}(M^\vee) \to \mathbb{G}_a.$$

This can be generalized to an arbitrary ramified continuous homomorphism $\rho : K^* \to \mathbb{Q}_p$ (see Corollary 4.4.10).

As an application, in Section 4.5 we use these $\rho-$splittings, for ramified or unramified ρ, to provide local pairings between zero cycles and divisors on a variety over a local field in the following three cases: proper varieties, smooth varieties and curves.

Finally, in Section 4.6 we consider a global field F endowed with a set of places and a homomorphism ρ as in (1). In this case we can construct a global pairing on the rational points of a 1-motive $M(F)$ and its dual $M^\vee(F)$ under the condition that the ρ_v-splittings ψ_v, for ramified ρ_v, are compatible with the $L(F_v) \times L^\vee(F_v)-$linearization. This is Corollary 4.6.2. The pairing is defined analogously to the case of abelian varieties, and therefore generalizes (2).

Chapter 1

1-motives

Let S be a scheme and consider the fppf site S_{fppf} on S. Remember that, by Yoneda's lemma, we have a fully faithful embedding

$$\begin{aligned} \text{CommGrpSch}/S &\hookrightarrow \text{AbSh}(S_{\text{fppf}}) \\ G &\mapsto \underline{G} := \text{Hom}_S(\,\cdot\,, G)\ , \end{aligned}$$

from the category of commutative group schemes over S to the category of abelian sheaves on the fppf site of S. On the other hand, for any additive category $\mathcal{A}$, we can consider the category $\mathrm{C^b}(\mathcal{A})$ of bounded chain complexes of $\mathcal{A}$. We have a fully faithful embedding

$$\begin{aligned} &\mathcal{A} \hookrightarrow \mathrm{C^b}(\mathcal{A}) \\ &A \mapsto A[0] = \ldots 0 \to A \to 0 \ldots\ , \end{aligned}$$

which associates to an object A in $\mathcal{A}$ the complex concentrated in degree 0 whose component in this degree is A. These embeddings clearly make the following diagram commute

$$\begin{array}{ccc} \text{CommGrpSch}/S & \hookrightarrow & \text{AbSh}(S_{\text{fppf}}) \\ \big\downarrow & & \big\downarrow \\ \mathrm{C^b}(\text{CommGrpSch}/S) & \hookrightarrow & \mathrm{C^b}(\text{AbSh}(S_{\text{fppf}})) =: \mathrm{C^b}(S_{\text{fppf}})\ , \end{array}$$

so we can, and will, identify an object in one of these categories with its image under any of the previous functors. Notice that composing the functor

$$\text{CommGrpSch}/S \hookrightarrow \mathrm{C^b}(S_{\text{fppf}})$$

resulting from the diagram with the canonical functor

$$\mathrm{C^b}(S_{\text{fppf}}) \to \mathrm{D^b}(\text{AbSh}(S_{\text{fppf}})) =: \mathrm{D^b}(S_{\text{fppf}})$$

we get an embedding as well. So we will also identify a commutative group scheme G over S with the element in the derived category having $\underline{G}[0]$ as representative.

1.1 Deligne's 1-motives

The definiton of 1-motive is due to Deligne. In [Del74, §10.1] he defines a 1-motive over an algebraically closed field K and constructs several realization functors, which associate to a 1-motive a vector space over K endowed with *Hodge* and *weight* filtrations. Namely, he constructs the Hodge realization for $K = \mathbb{C}$, and the $l-$adic and de Rham realizations for K of characteristic 0, and gives comparison isomorphisms relating them which are compatible with the filtrations. We will take here as definition of 1-motive what Deligne calls *1-motif lisse* (see Definition 1.1.1). This generalizes the previous definition of 1-motives to arbitrary base schemes. In this case, we also have an $l-$adic realization functor, for l a prime invertible in S, and a de Rham realization. We will concentrate on the de Rham realization; this will be defined in Section 1.3.

Definition 1.1.1 ([Del74, §10.1.10])**.** Let S be a scheme. A *1-motive* M over S consists of:

i) a *lattice* L over S, *i.e.* an $S-$group scheme which, locally for the étale topology on S, is isomorphic to a finitely generated free abelian constant group;

ii) a *semi-abelian scheme* G over S which is an extension of an abelian scheme A by a torus T over S ; and

iii) a morphism of $S-$group schemes $u : L \to G$.

A 1-motive is represented in a diagram as follows

$$\begin{array}{ccccccccc} & & & & L & & & & \\ & & & & \downarrow{\scriptstyle u} & \searrow{\scriptstyle v} & & & \\ 0 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0 \end{array} .$$

A 1-motive can be considered as a complex of $S-$group schemes or of the associated representable fppf sheaves on S. Using Raynaud's convention ([Ray94, §2]), we will consider L in degree -1 and G in degree 0. If $M = [L \xrightarrow{u} G]$ and $M' = [L' \xrightarrow{u'} G']$ are two 1-motives we define a *morphism* $\varphi : M \to M'$ as a morphism of the associated complexes of $S-$group schemes, *i.e.* φ is given by a pair of $S-$homomorphisms $f : L \to L'$ and $g : G \to G'$ such that

$$\begin{array}{ccc} L & \xrightarrow{u} & G \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ L' & \xrightarrow{u'} & G' \end{array}$$

commutes. This says that the category $\mathcal{M}_1(S)$ of 1-motives over S is a full subcategory of the category of complexes of fppf sheaves on S. Notice that $\mathcal{M}_1(S)$ has kernels and cokernels but, in general, images and coimages do not coincide, so $\mathcal{M}_1(S)$ is not an abelian category.

Associated to M, we have an increasing *weight filtration* by sub-1-motives defined as follows:

$$W_i(M) = \begin{cases} 0 & i \leq -3, \\ [0 \to T] & i = -2, \\ [0 \to G] & i = -1, \\ M & i \geq 0. \end{cases}$$

The graded pieces defining pure 1-motives are:

$$gr_i^W(M) = \begin{cases} 0 & i \leq -3 \text{ or } i \geq 1, \\ [0 \to T] & i = -2, \\ [0 \to A] & i = -1, \\ [L \to 0] & i = 0. \end{cases}$$

We will also consider, for a 1-motive M, the induced object $\iota(M)$ in the derived category $\mathrm{D^b}(S_{\mathrm{fppf}})$, providing a functor

$$\iota : \mathcal{M}_1(S) \to \mathrm{D^b}(S_{\mathrm{fppf}}).$$

The following proposition states that this functor is fully faithful.

Proposition 1.1.2 ([Ray94, Prop. 2.3.1])**.** *Let M and M' be two 1-motives over S. Then*

$$\mathrm{Hom}_{\mathcal{M}_1(S)}(M, M') \cong \mathrm{Hom}_{D^b(S_{\mathrm{fppf}})}(\iota(M), \iota(M')).$$

Corollary 1.1.3 ([Ray94, Cor. 2.3.3])**.** *Two $S-$1-motives whose images in $D^b(S_{\mathrm{fppf}})$ are isomorphic, are isomorphic as 1-motives.*

1.2 Cartier duality

Given a 1-motive M over S we can construct another 1-motive $M^\vee$ called the *Cartier dual* of M. This association in fact defines a contravariant functor on the category of 1-motives

$$(\,\cdot\,)^\vee : \mathcal{M}_1(S) \to \mathcal{M}_1(S)$$

with the property that there exist canonical isomorphisms $M^{\vee\vee} \cong M$ which are natural in M. This construction generalizes duality of abelian schemes, given by the functor $\underline{\mathrm{Ext}}^1_S(\,\cdot\,, \mathbb{G}_m)$, and Cartier duality of commutative groups schemes taking tori to lattices and viceversa, which is given by $\underline{\mathrm{Hom}}_S(\,\cdot\,, \mathbb{G}_m)$ (see (1.2)). As in the case of abelian schemes we have a canonical biextension of $(M, M^\vee)$ by $\mathbb{G}_m$, named *Poincaré biextension* by analogy with the case of abelian varieties, which expresses the duality relation. In Section 1.2.1, we will remember the theory of biextensions and give a generalization of this notion due to Deligne to be applied on 1-motives; then in Section 1.2.2 we will give the construction of the dual of a 1-motive.

1.2.1 Biextensions

We recall some general facts about torsors which can be found in [Gir71, §III]. We fix a site $\mathcal{S}$ and work in the topos $\mathrm{Sh}(\mathcal{S})$ of sheaves on $\mathcal{S}$. Let $S, H \in \mathrm{Sh}(\mathcal{S})$ with H a sheaf of abelian groups over S. By an $H-$*torsor* over S we will mean a sheaf P over S endowed with an $H-$action $m : H \times_S P \to P$ such that:

i) the morphism

$$\begin{aligned} H \times_S P &\to P \times_S P \\ (h, e) &\mapsto (m(h, e), e) \end{aligned}$$

is an isomorphism, and

ii) the structural morphism $P \to S$ is an epimorphism.

A *morphism of torsors* is a morphism of the corresponding sheaves which is compatible with the actions. The *trivial* $H-$*torsor* is just H with the action given by multiplication.

Condition i) is an algebraic condition and is equivalent to the following:

i') for any $T \in \mathrm{Sh}(\mathcal{S})$, the action of $H(T) = \mathrm{Hom}_S(T, H)$ on $P(T) = \mathrm{Hom}_S(T, P)$ is simply transitive.

Condition ii) is more of a topological condition and can be rephrased as:

ii') there exists an epimorphic family $\{S_i \to S\}$ such that

$$P(S_i) = \mathrm{Hom}_S(S_i, P) \neq \emptyset;$$

or equivalently,

ii") there exists an epimorphic family $\{S_i \to S\}$ such that $P \times_S S_i$ is isomorphic to the trivial $H \times_S S_i-$torsor over S_i.

Definition 1.2.1 ([SGA7-I, Def. 2.1])**.** Let H, A, B be sheaves of abelian groups on a site. A *biextension* of (A, B) by H is an $H_{A\times B}-$torsor P over $A \times B$ which is endowed with a structure of extension of B_A by H_A and a structure of extension of A_B by H_B, such that both structures are compatible.

Formally, this means the following. For $T \in \mathrm{Sh}(\mathcal{S})$ and morphisms $a \in A(T)$, $b \in B(T)$, denote by $P_{a,b}$ the fiber of P over $(a, b) \in (A \times B)(T)$, that is, the pullback of P along the morphism $(a, b) : T \to A \times B$. Then the structure of extension of B_A by H_A is given by a system of isomorphisms

$$\varphi_{a;b,b'} : P_{a,b} \wedge P_{a,b'} \to P_{a,bb'} ,$$

where $P_{a,b} \wedge P_{a,b'}$ denotes the contracted product, for every $T \in \mathrm{Sh}(\mathcal{S})$ and morphisms $a \in A(T)$ and $b, b' \in B(T)$. These are functorial in T and satisfy associativity and commutativity conditions expressed by the following commutative diagrams

$$\begin{CD} P_{a,b} \wedge P_{a,b'} \wedge P_{a,b''} @>{\varphi_{a;b,b'} \wedge Id}>> P_{a,bb'} \wedge P_{a,b''} \\ @V{Id \wedge \varphi_{a;b',b''}}VV @VV{\varphi_{a;bb',b''}}V \\ P_{a,b} \wedge P_{a,b'b''} @>{\varphi_{a;b,b'b''}}>> P_{a,bb'b''} \end{CD} \qquad \begin{CD} P_{a,b} \wedge P_{a,b'} @>{\varphi_{a;b,b'}}>> P_{a,bb'} \\ @V{\cong}VV @| \\ P_{a,b'} \wedge P_{a,b} @>{\varphi_{a;b',b}}>> P_{a,b'b} \end{CD} \ ,$$

for $a \in A(T)$ and $b, b', b'' \in B(T)$.

Symmetrically, the structure of extension of A_B by H_B is given by a system of isomorphisms

$$\psi_{a,a';b} : P_{a,b} \wedge P_{a',b} \to P_{aa',b} \,,$$

for every $T \in \mathrm{Sh}(\mathcal{S})$ and morphisms $a, a' \in A(T)$ and $b \in B(T)$. These are functorial in T and satisfy associativity and commutativity conditions expressed by the following commutative diagrams

$$\begin{CD} P_{a,b} \wedge P_{a',b} \wedge P_{a'',b} @>{\psi_{a,a';b} \wedge Id}>> P_{aa',b} \wedge P_{a'',b} \\ @V{Id \wedge \psi_{a',a'';b}}VV @VV{\psi_{aa',a'';b}}V \\ P_{a,b} \wedge P_{a'a'',b} @>{\psi_{a,a'a'';b}}>> P_{aa'a'',b} \end{CD} \qquad \begin{CD} P_{a,b} \wedge P_{a',b} @>{\psi_{a,a';b}}>> P_{aa',b} \\ @V{\cong}VV @| \\ P_{a',b} \wedge P_{a,b} @>{\psi_{a',a;b}}>> P_{a'a,b} \end{CD} \ ,$$

for $a, a', a'' \in A(T)$ and $b \in B(T)$.

These structures are said to be compatible if for every $T \in \mathrm{Sh}(\mathcal{S})$ and $a, a' \in A(T)$, $b, b' \in B(T)$ the following diagram commutes

$$\begin{CD} P_{a,b} \wedge P_{a,b'} \wedge P_{a',b} \wedge P_{a',b'} @>{\varphi_{a;b,b'} \wedge \varphi_{a';b,b'}}>> P_{a,bb'} \wedge P_{a',bb'} @>{\psi_{a,a';bb'}}>> P_{aa',bb'} \\ @V{\cong}VV @. @| \\ P_{a,b} \wedge P_{a',b} \wedge P_{a,b'} \wedge P_{a',b'} @>{\psi_{a,a';b} \wedge \psi_{a,a';b'}}>> P_{aa',b} \wedge P_{aa',b'} @>{\varphi_{aa';b,b'}}>> P_{aa',bb'} \end{CD} \ . \tag{1.1}$$

Example 1.2.2. A biextension of abelian groups is a biextension on the punctual topos, *i.e.* the category of sheaves on the site with one object $*$, one morphism $Id_* : * \to *$ and one covering $\{Id_*\}$. So, for abelian groups H, A, B, a biextension of (A, B) by H is given by a set P endowed with a simply transitive $H-$action and a surjective function $P \to A \times B$ such that the fibers P_a, for every $a \in A$, have the structure of extension of B by H, and the fibers P_b, for every $b \in B$ have the structure of extension of A by H .

Example 1.2.3. Fix a scheme S and consider the fppf site of S. Let A_S be an abelian scheme over S. The dual abelian variety of A_S is characterized by a pair $(A_S^\vee, P_{A_S})$, where $A_S^\vee$ is an abelian scheme over S and P_{A_S} is a biextension of $(A_S, A_S^\vee)$ by $\mathbb{G}_{m,S}$, called the

Poincaré biextension. We have a canonical isomorphism $A_S^\vee \cong \underline{\mathrm{Pic}}^0_{A/S}$ of fppf sheaves (see [FC90, Thm. 1.9]). When composing with the isomorphism $\underline{\mathrm{Pic}}^0_{A/S} \cong \underline{\mathrm{Ext}}^1_S(A_S, \mathbb{G}_{m,S})$ taking an invertible sheaf to the group of its nonzero sections we get the *Weil-Barsotti formula* $A_S^\vee \cong \underline{\mathrm{Ext}}^1_S(A_S, \mathbb{G}_{m,S})$, which maps a section $a^\vee \in A_S^\vee$ to the fiber $P_{A_S,a^\vee}$ of P_{A_S} over $a^\vee$.
It will be useful to note that since $\mathbb{G}_{m,S}$ is affine over S then P_{A_S} is representable. Moreover, since $\mathbb{G}_{m,S}$ is also smooth over S then P_{A_S} is already locally trivial with respect to the étale topology on S, *i.e.* P_{A_S} is a $\mathbb{G}_{m,S}-$torsor over $A_S \times A_S^\vee$ for the étale topology on S.
If $S = \mathrm{Spec}(K)$, where K is a discrete valuation field with ring of integers R, then P_{A_K} extends canonically to a biextension P_A of $(A^0, A^\vee)$ by $\mathbb{G}_{m,R}$, where A^0 denotes the identity component of the Néron model of A_K and $A^\vee$ denotes the Néron model of $A_K^\vee$. In particular, its generic fiber is P_{A_K} (see [SGA7-I, Exposé VIII, Thm. 7.1 (b)]).

In [Del74, §10.2], Deligne intoduced a generalization of the notion of biextension in which he considered complexes of sheaves. The theory developed can then be applied to 1-motives to obtain results analogous to those regarding biextensions of abelian varieties.

Definition 1.2.4. [Del74, §10.2.1] Let $C_1 = [A_1 \to B_1]$ and $C_2 = [A_2 \to B_2]$ be two complexes of sheaves of abelian groups concentrated in degrees 0 and -1. A *biextension* of (C_1, C_2) by a sheaf of abelian groups H consists of:

i) a biextension P of (B_1, B_2) by H,

ii) a trivialization (biadditive section) of the biextension of (B_1, A_2) by H, obtained as the pullback of P over $B_1 \times A_2$, and

iii) a trivialization of the biextension of (A_1, B_2) by H, obtained as the pullback of P over $A_1 \times B_2$.

We require the trivializations in ii) and iii) to coincide on $A_1 \times A_2$.

1.2.2 Construction of the dual 1-motive

First, we set the notation used in this section regarding $\underline{\mathrm{Hom}}$ and $\underline{\mathrm{Ext}}$ sheaves. Let $F^\bullet$ and $G^\bullet$ be objects in $\mathrm{D^b}(S_{\mathrm{fppf}})$. We have an internal Hom in $\mathrm{D^b}(S_{\mathrm{fppf}})$ denoted $R\underline{\mathrm{Hom}}_S(F^\bullet, G^\bullet)$. This complex induces abelian sheaves on the site S_{fppf}

$$\underline{\mathrm{Ext}}^i_S(F^\bullet, G^\bullet) = H^i(R\underline{\mathrm{Hom}}_S(F^\bullet, G^\bullet)). \tag{1.2}$$

This $\underline{\mathrm{Ext}}^i_S-$sheaf is just the sheafification of the presheaf

$$\begin{aligned} (S_{\mathrm{fppf}})^{opp} &\to \mathrm{Ab} \\ (T \xrightarrow{f} S) &\mapsto \mathrm{Ext}^i_T(F^\bullet|_T, G^\bullet|_T) \end{aligned}$$

with respect to the fppf topology on S, where

$$\mathrm{Ext}^i_S(F^\bullet, G^\bullet) := \mathrm{Hom}_{\mathrm{D^b}(S_{\mathrm{fppf}})}(F^\bullet, G^\bullet[i]) = \mathrm{Hom}_{\mathrm{D^b}(S_{\mathrm{fppf}})}(F^\bullet[-i], G^\bullet).$$

Sometimes we will consider the restriction of the fppf sheaf $\underline{\mathrm{Ext}}^i_S$ to the small Zariski site S_{Zar} on S. The Zariski sheaf on S_{Zar} thus obtained will be denoted $\underline{\mathrm{Ext}}^i_{S,\mathrm{Zar}}$.

Now we proceed to the construction of the Cartier dual. Let $M = [L \xrightarrow{u} G]$ be a 1-motive over a locally noetherian scheme S where G is an extension of an abelian variety A by a torus T, and denote $M_A = [L \xrightarrow{v} A]$. The dual of M will be a 1-motive $M^\vee = [L^\vee \xrightarrow{u^\vee} G^\vee]$, with $G^\vee$ an extension of $A^\vee$ by $T^\vee$, defined as follows.

i) The lattice $L^\vee$ is the Cartier dual of T, *i.e.* the group scheme which represents the sheaf $\underline{\mathrm{Hom}}_S(T, \mathbb{G}_{m,S})$.

ii) The torus $T^\vee$ is the Cartier dual of L, *i.e.* the group scheme which represents the sheaf $\underline{\mathrm{Hom}}_S(L, \mathbb{G}_{m,S})$.

iii) The abelian variety $A^\vee$ is the dual abelian variety of A. Remember that, by the Weil-Barsotti formula, $A^\vee$ represents the sheaf $\underline{\mathrm{Ext}}^1_S(A, \mathbb{G}_{m,S})$. Denote P_A the Poincaré biextension of $(A, A^\vee)$ by $\mathbb{G}_{m,S}$.

iv) To construct $G^\vee$, consider the exact sequence of complexes

$$0 \to A \to M_A \to L[1] \to 0\,. \tag{1.3}$$

Applying the $\delta-$functor $\underline{\mathrm{Ext}}^\bullet_S(\,\cdot\,, \mathbb{G}_{m,S})$ to (1.3) we get a long exact sequence

$$\begin{aligned} \ldots \underline{\mathrm{Hom}}_S(A, \mathbb{G}_{m,S}) \to \underline{\mathrm{Hom}}_S(L, \mathbb{G}_{m,S}) \to \underline{\mathrm{Ext}}^1_S(M_A, \mathbb{G}_{m,S}) \\ \to \underline{\mathrm{Ext}}^1_S(A, \mathbb{G}_{m,S}) \to \underline{\mathrm{Ext}}^1_S(L, \mathbb{G}_{m,S}) \ldots . \end{aligned}$$

The sheaf $\underline{\mathrm{Ext}}^1_S(M_A, \mathbb{G}_{m,S})$ happens to be representable and we denote $G^\vee$ the group scheme that represents it. Since $\underline{\mathrm{Hom}}_S(A, \mathbb{G}_{m,S}) = 0$, because A is proper and geometrically connected and $\mathbb{G}_{m,S}$ is affine, and $\underline{\mathrm{Ext}}^1_S(L, \mathbb{G}_{m,S}) = 0$ then $G^\vee$ is a semi-abelian scheme which is an extension of $A^\vee$ by $T^\vee$. This construction gives in fact an isomorphism of sheaves on S_{fppf}

$$\underline{\mathrm{Hom}}_S(L, A) \to \underline{\mathrm{Ext}}^1_S(A^\vee, T^\vee) \tag{1.4}$$

which is functorial in both components and in S, and whose evaluation at S sends $v : L \to A$ to $G^\vee$.
We have a Poincaré biextension P' of $(M_A, G^\vee)$ by $\mathbb{G}_m$, which is obtained from P_A by pullback.

v) To obtain the morphism $u^\vee : L^\vee \to G^\vee$ we proceed as follows. Consider the exact sequence of complexes

$$0 \to T \to M \to M_A \to 0\,. \tag{1.5}$$

Then the sequence (1.3) and the connecting morphism of (1.5) gives the following diagram in the derived category $\mathrm{D}^{\mathrm{b}}(S_{\mathrm{fppf}})$

$$\begin{array}{ccccccccc} & & & & T[1] & & & & \\ & & & & \uparrow & & & & \\ 0 & \longleftarrow & L[1] & \longleftarrow & M_A & \longleftarrow & A & \longleftarrow & 0\ , \end{array}$$

which after applying $\underline{\mathrm{Ext}}^1_S(\,\cdot\,,\mathbb{G}_{m,S})$ gives

$$\begin{array}{ccccccccc} & & & & L^\vee & & & & \\ & & & & {\scriptstyle u^\vee}\downarrow & \searrow^{v^\vee} & & & \\ 0 & \longrightarrow & T^\vee & \longrightarrow & G^\vee & \longrightarrow & A^\vee & \longrightarrow & 0\ . \end{array}$$

vi) Finally, we need to construct the Poincaré biextension P of $(M, M^\vee)$ by $\mathbb{G}_{m,S}$. Again, by pullback, we have that P' induces a biextension P'' of $(M, G^\vee)$ by $\mathbb{G}_{m,S}$. To prove that this induces a biextension of $(M, M^\vee)$ by $\mathbb{G}_{m,S}$ it remains to prove that P'' has a trivialization over $G \times L^\vee$ and that, on $L \times L^\vee$, it coincides with the trivialization over $L \times G^\vee$.

Take $x^\vee \in L^\vee$. If we interpret $x^\vee$ as a morphism $\chi : T \to \mathbb{G}_{m,S}$ then the image of $x^\vee$ under $u^\vee$ corresponds to the extension of M_A by $\mathbb{G}_{m,S}$ obtained as the pushout of (1.5) along $-\chi$ (see [AB05, §1.2])

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T & \longrightarrow & M & \longrightarrow & M_A & \longrightarrow & 0 \\ & & {\scriptstyle -\chi}\downarrow & \ulcorner & \downarrow{\scriptstyle \xi} & & \| & & \\ 0 & \longrightarrow & \mathbb{G}_{m,S} & \longrightarrow & P'_{u^\vee(x^\vee)} & \longrightarrow & M_A & \longrightarrow & 0\ . \end{array}$$

The morphism ξ gives a trivialization of the pullback $P''_{u^\vee(x^\vee)}$ of $P'_{u^\vee(x^\vee)}$ to M

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{G}_{m,S} & \longrightarrow & P''_{u^\vee(x^\vee)} & \longrightarrow & M & \longrightarrow & 0 \\ & & \| & & \downarrow & \lrcorner\ \swarrow_{\xi} & \downarrow & & \\ 0 & \longrightarrow & \mathbb{G}_{m,S} & \longrightarrow & P'_{u^\vee(x^\vee)} & \longrightarrow & M_A & \longrightarrow & 0\ . \end{array}$$

Since the pullback of P'' to $G \times \{x^\vee\}$ is isomorphic to $P''_{u^\vee(x^\vee)}$, we see that P'' has indeed a canonical trivialization over $G \times L^\vee$. Therefore, the biextension P is just P'' endowed with the trivialization on $G \times L^\vee$.

1.2.3 Symmetric avatar

By the previous discussion, we see that the Poincaré biextension of $(M, M^\vee)$ by $\mathbb{G}_{m,S}$ is the pullback of the Poincaré biextension P_A of $(A, A^\vee)$ by $\mathbb{G}_m$ to $G \times G^\vee$, which is endowed with trivializations on $L \times G^\vee$ and $G \times L^\vee$ coinciding on $L \times L^\vee$. In particular, the biextension P_A has a trivialization $\tau : L \times L^\vee \to P_A$ on $L \times L^\vee$, and in fact this is necessary and sufficient information to construct the liftings of $v : L \to A$ and $v^\vee : L^\vee \to A^\vee$ to G and $G^\vee$, respectively ([Bar07, §2.8]). So, the data of a 1-motive M and its dual $M^\vee$ is determined by the data

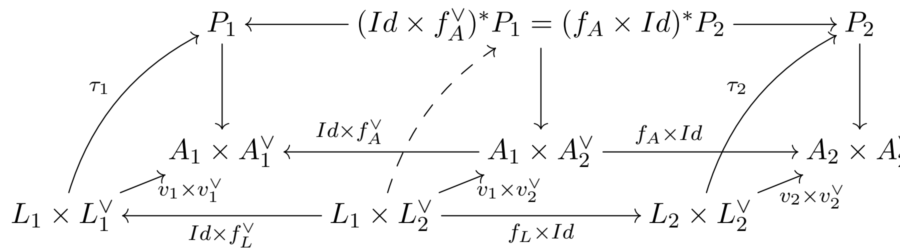
$$(L \xrightarrow{v} A, L^\vee \xrightarrow{v^\vee} A^\vee, L \times L^\vee \xrightarrow{\tau} P_A).$$

We can construct a category, denoted $\mathcal{M}_1^{\mathrm{sym}}$, whose objects are such tuples and whose morphisms $(v_1, v_1^\vee, \tau_1) \to (v_2, v_2^\vee, \tau_2)$ are given by maps $f_L : L_1 \to L_2$ and $f_A : A_1 \to A_2$, together with their duals $f_L^\vee : L_2^\vee \to L_1^\vee$ and $f_A^\vee : A_2^\vee \to A_1^\vee$, making the following diagrams commute

$$\begin{array}{ccc} L_1 & \xrightarrow{v_1} & A_1 \\ {\scriptstyle f_L}\downarrow & & \downarrow{\scriptstyle f_A} \\ L_2 & \xrightarrow{v_2} & A_2 \end{array} \qquad\qquad \begin{array}{ccc} L_1^\vee & \xrightarrow{v_1^\vee} & A_1^\vee \\ {\scriptstyle f_L^\vee}\uparrow & & \uparrow{\scriptstyle f_A^\vee} \\ L_2^\vee & \xrightarrow{v_2^\vee} & A_2^\vee \end{array}$$

and such that the trivializations τ_1 and τ_2 are compatible, *i.e.* they satisfy that

$$(Id \times f_A^\vee)^* \tau_1 = (f_A \times Id)^* \tau_2.$$



1.3 de Rham realization via the universal vectorial extension (UVE)

We define a *vector group scheme* over S as an $S-$group scheme that is locally isomorphic (for the Zariski topology) to a finite product of $\mathbb{G}_a$'s endowed with an action of the group ring $\mathbb{A}^1$ which is compatible with said local isomorphisms. If V is a vector group over S then the sheaf $\underline{\mathrm{Hom}}_S(\cdot, V)$ is a locally free $\mathcal{O}_S-$module of finite rank. Conversely, every locally free $\mathcal{O}_S-$module $\mathcal{E}$ of finite rank induces a vector group W whose sections over an $S-$scheme T are $W(T) = \Gamma(T, \mathcal{O}_T \otimes_{\mathcal{O}_S} \mathcal{E})$. We will often identify a vector group with the $\mathcal{O}_S-$module it represents.

Given a group scheme G over S such that $\underline{\mathrm{Hom}}_S(G,V)=0$ for all vector groups V, we can consider the problem of finding a group scheme $E(G)$ that is an extension of G by a vector group $V(G)$

$$0\to V(G)\to E(G)\to G\to 0 \tag{1.6}$$

with the property that the map from morphisms of $\mathcal{O}_S-$modules to extensions of commutative group schemes

$$\mathrm{Hom}_{\mathcal{O}_S}(V(G),V)\to \mathrm{Ext}^1_S(G,V) \tag{1.7}$$

induced by pushout is an isomorphism for all vector groups V. In case such $E(G)$ exists, this group is called the *universal vectorial extension* (UVE) of G and we can see that $E(G)$, as well as $V(G)$, are determined up to canonical isomorphism. In [MM74, §1.7] it is proved that if the following conditions are satisfied:

i) $\underline{\mathrm{Hom}}_{S,\mathrm{Zar}}(G,\mathbb{G}_{a,S})=0$, and

ii) $\underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(G,\mathbb{G}_{a,S})$ is a locally free $\mathcal{O}_S-$module of finite rank,

then

$$V(G):=\underline{\mathrm{Hom}}_{\mathcal{O}_S}(\underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(G,\mathbb{G}_{a,S}),\mathcal{O}_S)$$

is a vector group that satisfies

$$\begin{aligned}\underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(G,V)&=\underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(G,\mathbb{G}_{a,S})\otimes_{\mathcal{O}_S}V\\&=\underline{\mathrm{Hom}}_{\mathcal{O}_S}(V(G),\mathcal{O}_S)\otimes_{\mathcal{O}_S}V\\&=\underline{\mathrm{Hom}}_{\mathcal{O}_S}(V(G),V);\end{aligned}$$

and thus $E(G)$ is the extension corresponding to the identity morphism on $V(G)$. In particular, a semi-abelian scheme has a universal vectorial extension.

This definition can be generalized to 1-motives as follows (see [Del74, §10.1.7]).

Definition 1.3.1. Let S be a scheme and M a 1-motive over S. The *universal vectorial extension* (UVE) of M is a two term complex of $S-$group schemes

$$M^\natural=[L\xrightarrow{u^\natural}G^\natural]$$

which is an extension of M by a vector group $V(M)$ over S as complexes of $S-$group schemes

$$\begin{array}{ccccccccc}0&\longrightarrow&0&\longrightarrow&L&=\!=&L&\longrightarrow&0\\&&\downarrow&&\downarrow{\scriptstyle u^\natural}&&\downarrow{\scriptstyle u}&&\\0&\longrightarrow&V(M)&\longrightarrow&G^\natural&\longrightarrow&G&\longrightarrow&0\end{array}\tag{1.8}$$

such that (1.8) is universal in the sense that the map

$$\mathrm{Hom}_{\mathcal{O}_S}(V(M), V) \to \mathrm{Ext}^1_S(M, V) \tag{1.9}$$

induced by pushout is an isomorphism for all vector groups V.

If the following conditions are satisfied:

i) $\underline{\mathrm{Hom}}_{S,\mathrm{Zar}}(M, \mathbb{G}_{a,S}) = 0$, that is, the extensions of M by $\mathbb{G}_{a,S}$ do not have automorphisms, and

ii) $\underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(M, \mathbb{G}_{a,S})$ is a locally free $\mathcal{O}_S-$module of finite rank,

then

$$V(M) := \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(M, \mathbb{G}_{a,S}), \mathcal{O}_S)$$

is a vector group and

$$\begin{aligned} \underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(M, V) &= \underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(M, \mathbb{G}_{a,S}) \otimes_{\mathcal{O}_S} V \\ &= \underline{\mathrm{Hom}}_{\mathcal{O}_S}(V(M), \mathcal{O}_S) \otimes_{\mathcal{O}_S} V \\ &= \underline{\mathrm{Hom}}_{\mathcal{O}_S}(V(M), V). \end{aligned}$$

Therefore, (1.9) is an isomorphism and M has a universal vectorial extension.

To prove that condition i) is satisfied, observe that $\underline{\mathrm{Hom}}_{S,\mathrm{Zar}}(G, \mathbb{G}_{a,S}) = 0$ implies $\underline{\mathrm{Hom}}_{S,\mathrm{Zar}}(M, \mathbb{G}_{a,S}) = 0$. Now, to prove ii) consider the exact sequence

$$0 \to \underline{\mathrm{Hom}}_{S,\mathrm{Zar}}(L, \mathbb{G}_{a,S}) \to \underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(M, \mathbb{G}_{a,S}) \to \underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(G, \mathbb{G}_{a,S}) \to 0\,,$$

which is obtained from the exact sequence of complexes

$$0 \to G \to M \to L[1] \to 0 \tag{1.10}$$

once we notice that $\underline{\mathrm{Hom}}_{S,\mathrm{Zar}}(G, \mathbb{G}_{a,S}) = \underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(L, \mathbb{G}_{a,S}) = 0$. Then we have that $\underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(M, \mathbb{G}_{a,S})$ will be a locally free sheaf of $\mathcal{O}_S-$modules of finite rank if both $\underline{\mathrm{Hom}}_{S,\mathrm{Zar}}(L, \mathbb{G}_{a,S})$ and $\underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(G, \mathbb{G}_{a,S})$ have finite rank. Clearly, $\underline{\mathrm{Hom}}_{S,\mathrm{Zar}}(L, \mathbb{G}_{a,S})$ is as desired. For $\underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(G, \mathbb{G}_{a,S})$, notice that the exact sequence

$$0 \to T \to G \to A \to 0\,, \tag{1.11}$$

induces an isomorphism $\underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(A, \mathbb{G}_{a,S}) \cong \underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(G, \mathbb{G}_{a,S})$, since we have that $\underline{\mathrm{Hom}}_{S,\mathrm{Zar}}(T, \mathbb{G}_{a,S}) = \underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(T, \mathbb{G}_{a,S}) = 0$. Thus, $\underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(G, \mathbb{G}_{a,S})$ also has finite rank.

Definition 1.3.2. The *de Rham realization* of M is defined as the sheaf

$$\mathrm{T}_{\mathrm{dR}}(M) = \underline{\mathrm{Lie}} G^{\natural}\,.$$

The *Hodge filtration* of $\mathrm{T}_{\mathrm{dR}}(M)$ is defined as

$$F^i\,\mathrm{T}_{\mathrm{dR}}(M) = \begin{cases} \mathrm{T}_{\mathrm{dR}}(M) & \text{if } i = -1, \\ V(M) & \text{if } i = 0, \\ 0 & \text{if } i \neq -1, 0. \end{cases}$$

1.3.1 Relation between UVE of subquotients of M

The exact sequence (1.11) induces the following commutative diagram of group schemes with exact rows and columns, where we are denoting $G^{\#}$ the universal vectorial extension of the semi-abelian scheme G

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & T^{\#} & = & T & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V(G) & \longrightarrow & G^{\#} & \longrightarrow & G & \longrightarrow & 0 \\
 & & \| & & \downarrow & \lrcorner & \downarrow & & \\
0 & \longrightarrow & V(A) & \longrightarrow & A^{\#} & \longrightarrow & A & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & .
\end{array} \tag{1.12}$$

Remember that we have $\underline{\mathrm{Hom}}_{S,\mathrm{Zar}}(T, \mathbb{G}_{a,S}) = \underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(T, \mathbb{G}_{a,S}) = 0$; this implies that $V(T) = 0$, and $V(G) = V(A)$.

Similarly, the exact sequence (1.10) induces the following commutative diagram of complexes of group schemes with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V(G) & \longrightarrow & G^{\#} & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V(M) & \longrightarrow & M^{\natural} & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V(L[1]) & \longrightarrow & L[1]^{\natural} & \longrightarrow & L[1] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

whose degree 0 level is given by the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & V(G) & \longrightarrow & G^{\#} & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \Vert & & \\
0 & \longrightarrow & V(M) & \longrightarrow & \ulcorner G^{\natural} & \longrightarrow & G & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & V(L[1]) & = & V(L[1]) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & & .
\end{array} \tag{1.13}$$

Notice that $V(L[1])$ is given by

$$\begin{aligned}
V(L[1]) &= \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(L[1], \mathbb{G}_{a,S}), \mathcal{O}_S) \\
&= \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\underline{\mathrm{Hom}}_{S,\mathrm{Zar}}(L, \mathbb{G}_{a,S}), \mathcal{O}_S) \\
&= L \otimes \mathbb{G}_{a,S}.
\end{aligned}$$

Moreover, for $V = V(L[1])$ and $M = L[1]$, the isomorphism (1.9) sends the identity to the evaluation map in $\mathrm{Ext}^1_S(L[1], L \otimes \mathbb{G}_{a,S}) = \mathrm{Hom}_S(L, L \otimes \mathbb{G}_{a,S})$:

$$\begin{aligned}
ev : L &\to \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\underline{\mathrm{Hom}}_{S,\mathrm{Zar}}(L, \mathbb{G}_{a,S}), \mathcal{O}_S) = L \otimes \mathbb{G}_a \\
x &\mapsto \begin{pmatrix} \underline{\mathrm{Hom}}_{S,\mathrm{Zar}}(L, \mathbb{G}_{a,S}) \xrightarrow{ev(x)} \mathcal{O}_S \\ \hfill f \mapsto f(x) \end{pmatrix} .
\end{aligned}$$

This implies that $L[1]^{\natural} = [L \xrightarrow{ev} L \otimes \mathbb{G}_{a,S}]$.

In order to give an equivalent description of $V(M)$, we will recall the following

Definition 1.3.3. Let B be a commutative group scheme over S. We say that $\omega \in \Omega^i_{B/S}$ is *invariant* if $\tau_b^*(\omega) = \omega$ for all $b \in B$, where τ_b denotes translation by b. For $i = 1$, we denote by $\underline{\omega}_B$ the sheaf of invariant differentials on B.

Remember that, for B as above, $\underline{\omega}_B = e^*\Omega^1_{B/S}$, where $e : S \to B$ is the zero section (see [BLR90, §4.2]). By [Ber09, Prop. 2.3], we know that

$$\underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(M, \mathbb{G}_{a,S}) = \underline{\mathrm{Ext}}^1_{S,\mathrm{Zar}}(M_A, \mathbb{G}_{a,S}) = \underline{\mathrm{Lie}} G^{\vee}$$

and therefore $V(M) = \underline{\mathrm{Hom}}_{\mathcal{O}_S}(\underline{\mathrm{Lie}}(G^{\vee}), \mathcal{O}_S) = \underline{\omega}_{G^{\vee}}$. So, we have that

$$\begin{array}{ccccccccc}
0 & \longrightarrow & V(G) & \longrightarrow & V(M) & \longrightarrow & V(L[1]) & \longrightarrow & 0 \\
 & & \Vert & & \Vert & & \Vert & & \\
0 & \longrightarrow & \underline{\omega}_{A^{\vee}} & \longrightarrow & \underline{\omega}_{G^{\vee}} & \longrightarrow & \underline{\omega}_{T^{\vee}} & \longrightarrow & 0
\end{array}$$

is the exact sequence of invariant differentials induced by duality from the exact sequence of Lie algebras obtained from

$$0 \to T^\vee \to G^\vee \to A^\vee \to 0.$$

1.3.2 Interpretation of UVE via $\natural$−extensions

Let X be an S−scheme and consider the diagonal map $\Delta : X \to X \times_S X$. The subscheme $\Delta(X) \subset X \times_S X$ is the closed subscheme corresponding to the ideal sheaf $\mathcal{J} \subset \mathcal{O}_{X\times X}$ generated by elements of the form $1\otimes x - x\otimes 1$. We denote by $\Delta^1(X)$ the first infinitesimal neighborhood of $\Delta(X)$, that is, the closed subscheme corresponding to $\mathcal{J}^2$. For $i = 1, 2$, denote by

$$p_i : \Delta^1(X) \hookrightarrow X \times_S X \xrightarrow{pr_i} X$$

the morphisms induced by the usual projections pr_i.

In what follows we will be considering torsors in the étale site.

Definition 1.3.4. Let H be a smooth commutative group scheme over S and P an H_X−torsor over X. A *connection* on P is an isomorphism $\nabla : p_1^*P \to p_2^*P$ of $H_{\Delta^1(X)}$−torsors which restricts to the identity on X.

Torsors endowed with connections (P, ∇) are the objects of a category whose morphisms $\mathrm{Hom}((P, \nabla), (Q, \nabla'))$ are given by morphisms $\phi : P \to Q$ of H_X−torsors that make the following diagram commute

$$\begin{array}{ccc} p_1^*P & \xrightarrow{p_1^*\phi} & p_1^*Q \\ \nabla \downarrow & & \downarrow \nabla' \\ p_2^*P & \xrightarrow{p_2^*\phi} & p_2^*Q \end{array} .$$

A morphism $(P, \nabla) \to (Q, \nabla')$ in this category is called *horizontal*.

A *connection* on an $\mathcal{O}_X$−module $\mathcal{E}$ is an $\mathcal{O}_{\Delta^1(X)}$−isomorphism $\nabla : p_1^*\mathcal{E} \to p_2^*\mathcal{E}$ restricting to the identity on X. From this, we can obtain an $\mathcal{O}_S$−linear homomorphism

$$\bar{\nabla} : \mathcal{E} \to \Omega^1_{X/S} \otimes_{\mathcal{O}_S} \mathcal{E}$$

as follows. For $i = 1, 2$, denote by $j_i : \mathcal{O}_X \to p_{i*}\mathcal{O}_{\Delta^1(X)}$ the homomorphism of sheaves of rings obtained from $p_i : \Delta^1(X) \to X$ and by $j_i(\mathcal{E}) : \mathcal{E} \to p_{i*}p_i^*\mathcal{E}$ the corresponding morphism of $\mathcal{O}_X$−modules. Define

$$\bar{\nabla} := \nabla^{-1} \circ j_2(\mathcal{E}) - j_1(\mathcal{E}).$$

This morphism $\bar{\nabla}$ satisfies the “Leibniz rule”

$$\bar{\nabla}(fs) = f\bar{\nabla}(s) + df \otimes s,$$

where f and s are sections of $\mathcal{O}_S$ and $\mathcal{E}$, respectively, and $d : \mathcal{O}_S \to \Omega^1_{X/S}$ is the exterior derivative. A *horizontal* morphism $(\mathcal{E}, \nabla) \to (\mathcal{F}, \nabla')$ is a morphism $\phi : \mathcal{E} \to \mathcal{F}$ of $\mathcal{O}_S-$modules inducing a commutative diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{\quad\phi\quad} & \mathcal{F} \\ \bar{\nabla}\big\downarrow & & \big\downarrow\bar{\nabla}' \\ \Omega^1_{X/S} \otimes_{\mathcal{O}_S} \mathcal{E} & \xrightarrow{Id\otimes\phi} & \Omega^1_{X/S} \otimes_{\mathcal{O}_S} \mathcal{F} \end{array} .$$

When $H = \mathbb{G}_{m,S}$, the previously defined map $\nabla \mapsto \bar{\nabla}$ gives a one to one correspondence between connections on a $\mathbb{G}_{m,X}-$torsor P and connections on the invertible sheaf associated to P.

Let ∇ be a connection on the H_X-torsor P. We define the curvature of ∇, which will be an element of $\Gamma(X, \Omega^2_{X/S} \otimes \underline{\mathrm{Lie}}(H))$, as follows. First we consider the case $P = H_X$ the trivial torsor. In this case, ∇ is just an automorphism of $H_{\Delta^1(X)}$ over $\Delta^1(X)$ which restritcts to the identity over X

$$\begin{array}{ccccc} H_X & \longrightarrow & H_{\Delta^1(X)} & & \\ & \searrow & & \searrow & \\ Id\big\downarrow & X & \longrightarrow \nabla\big\downarrow\cong \longrightarrow & & \Delta^1(X) \\ & \nearrow & & \nearrow & \\ H_X & \longrightarrow & H_{\Delta^1(X)} & & \end{array} .$$

Therefore, ∇ is determined by the image of the zero section of $H_{\Delta^1(X)}$ over $\Delta^1(X)$, which in turn is determined by an element $\zeta \in \Gamma(\Delta^1(X), H)$. By the commutativity of the diagram, we see that in fact ζ is zero when precomposed with $X \to \Delta^1(X)$. Now observe that

$$\begin{aligned} \mathrm{Ker}(\Gamma(\Delta^1(X), H) \to \Gamma(X, H)) &= \mathrm{Hom}_{\mathcal{O}_X}(\underline{\omega}_H \otimes_{\mathcal{O}_S} \mathcal{O}_X, \Omega^1_{X/S}) \\ &= \Gamma(X, \Omega^1_{X/S} \otimes_{\mathcal{O}_S} \underline{\mathrm{Lie}}(H)), \end{aligned}$$

where the second equality comes from the duality between $\underline{\omega}_H$ and $\underline{\mathrm{Lie}}(H)$. We define the *curvature* of ∇ as the image of ζ in $\Gamma(X, \Omega^2_{X/S} \otimes_{\mathcal{O}_S} \underline{\mathrm{Lie}}(H))$ under $d \otimes Id : \Omega^1_{X/S} \otimes_{\mathcal{O}_S} \underline{\mathrm{Lie}}(H) \to \Omega^2_{X/S} \otimes_{\mathcal{O}_S} \underline{\mathrm{Lie}}(H)$.

To define the curvature of a connection ∇ on an arbitrary H_X-torsor P we consider its pullback to some base X', along an epimorphic étale map $f : X' \to X$, over which P becomes trivial. Choosing an isomorphism $f^*P \cong H_{X'}$ we can construct the curvature of f^*P by the procedure just described and obtain an element in $\Gamma(X', \Omega^2_{X'/S} \otimes_{\mathcal{O}_S} \underline{\mathrm{Lie}}(H)) \cong \Gamma(X', f^*\Omega^2_{X/S} \otimes_{\mathcal{O}_S} \underline{\mathrm{Lie}}(H))$ (the isomorphism comes from the fact that f is étale). Now, by applying descent we get an element in $\Gamma(X, \Omega^2_{X/S} \otimes_{\mathcal{O}_S} \underline{\mathrm{Lie}}(H))$, which we define to be the curvature of P (see [MM74, §3.1.4]).

When considering a connection $\bar{\nabla}$ on a $\mathcal{O}_S-$module $\mathcal{E}$, the *curvature* of $\bar{\nabla}$ is defined as the $\mathcal{O}_S-$linear map

$$K := \bar{\nabla}_1 \circ \bar{\nabla} : \mathcal{E} \to \Omega^2_{X/S} \otimes_{\mathcal{O}_S} \mathcal{E},$$

where $\bar{\nabla}_1 : \Omega^1_{X/S} \otimes_{\mathcal{O}_S} \mathcal{E} \to \Omega^2_{X/S} \otimes_{\mathcal{O}_S} \mathcal{E}$ is defined as

$$\bar{\nabla}_1(w \otimes s) = dw \otimes s - w \wedge \bar{\nabla}(s),$$

on sections w and s of $\Omega^1_{X/S}$ and $\mathcal{E}$, respectively.

Definition 1.3.5. A connection ∇ is called *integrable* if it has zero curvature. A $\natural - H_X-$*torsor* on X is an H_X-torsor endowed with an integrable connection.

The *trivial* $\natural - H_X-$*torsor* is the torsor H_X endowed with the connection $\nabla^0 = Id : H_{\Delta^1(X)} \to H_{\Delta^1(X)}$. A *trivialization* of a $\natural - H_X-$torsor (P, ∇) is an isomorphism $(P, \nabla) \to (H_X, \nabla^0)$ in the category of H_X-torsors endowed with connections, *i.e.* an isomorphism $P \to H_X$ of H_X-torsors which is horizontal with respect to the connections ∇ and ∇^0.

Definition 1.3.6. Let B be a group scheme over S and denote $\mu : B \times_S B \to B$ the group law. A $\natural-$*extension* is a $\natural - H_B-$torsor (P, ∇) endowed with a horizontal isomorphism

$$\beta : p_1^* P \wedge p_2^* P \to \mu^* P.$$

We will also say that P is endowed with a $\natural-$structure when there is no ambiguity.

Notice that, in the previous definition, β defines a group structure on P and makes it an extension of B by H. The following proposition gives a description of $\natural-$extensions of B by $\mathbb{G}_{m,S}$.

Proposition 1.3.7. *Let B be a commutative group scheme over S and E an extension of B by $\mathbb{G}_{m,S}$. Then there is a bijection between connections ∇ on E such that (E, ∇) is a $\natural-$extension of B by $\mathbb{G}_{m,S}$ and invariant differentials on E whose pullback to $\mathbb{G}_{m,S}$ is dz/z.*

Proof. See [Ber09, Prop. 3.4] and [Col91, Prop. 0.2.1]. □

Definition 1.3.8. Let P be a biextension of (B_1, B_2) by H and consider the morphisms defining the partial group structures

$$\beta_1 : (p_1 \times Id)^* P \wedge (p_2 \times Id)^* P \to (\mu_1 \times Id)^* P \text{ on } B_1 \times_S B_1 \times_S B_2,$$
$$\beta_2 : (Id \times p_1)^* P \wedge (Id \times p_2)^* P \to (Id \times \mu_2)^* P \text{ on } B_1 \times_S B_2 \times_S B_2,$$

where $\mu_i : B_i \times_S B_i \to B_i$, for $i = 1, 2$, are the group laws.

i) We say that P is a $\natural-$*biextension* of (B_1, B_2) by H if it is a $\natural - H_{B_1\times B_2}-$torsor such that β_1 and β_2 are horizontal. We will also say that P is endowed with a $\natural-$structure when there is no ambiguity.

ii) A $\natural-1-$*structure* on P is a connection ∇_1 on P which endows P with the structure of $\natural-$extension of B_{1,B_2} by H_{B_2} such that β_2 is horizontal. Analogously, a $\natural - 2-$*structure* on P is a connection ∇_2 on P which endows P with the structure of $\natural-$extension of B_{2,B_1} by H_{B_1} such that β_1 is horizontal.

Notice that P is a $\natural-$biextension if and only if it is endowed with a $\natural - 1-$structure and a $\natural-2-$structure. If ∇_1 is a $\natural-1-$structure on P then β_1 is automatically horizontal with respect to ∇_1. Similarly, if ∇_2 is a $\natural - 2-$structure on P then β_2 is automatically horizontal with respect to ∇_2.

Definition 1.3.9. i) A $\natural-$*extension of a complex* $[A \to B]$ by H is a $\natural-$extension (P, ∇) of B by H endowed with a trivialization of the pullback of (P, ∇) to A.

ii) A $\natural-$*biextension of complexes* $([A_1 \to B_1], [A_2 \to B_2])$ by H is a $\natural-$biextension (P, ∇) of (B_1, B_2) by H endowed with trivializations of the pullback of (P, ∇) to $A_1 \times_S B_2$ and $B_1 \times_S A_2$ which coincide on $A_1 \times_S A_2$.

We have the following description of the group scheme $G^\natural$ appearing in the universal vectorial extension of a 1-motive in terms of $\natural-$extensions of complexes.

Proposition 1.3.10. *Let $M = [L \xrightarrow{u} G]$ be a 1-motive over S with dual $M^\vee = [L^\vee \xrightarrow{u^\vee} G^\vee]$ and P the Poincaré biextension of $(M, M^\vee)$. Then the group scheme $G^\natural$ represents the presheaf*

$$\begin{aligned} S_{\mathrm{fppf}} &\to \mathrm{Ab} \\ S' &\mapsto \left\{ (g, \nabla) \;\middle|\; \begin{array}{l} g \in G(S') \textit{ and } \nabla \textit{ is a } \natural-\textit{structure on} \\ \textit{the extension } [L \to P_g] \textit{ of } M^\vee \textit{ by } \mathbb{G}_{m,S'} \end{array} \right\} \end{aligned}$$

Proof. See [Ber09, Prop. 3.8]. □

1.3.3 Deligne's pairing

Let $P^\natural$ be the biextension of $(M^\natural, M^{\vee\natural})$ by $\mathbb{G}_m$ obtained as the pullback of P. We use the following notation

$$0 \to \underline{\omega}_{G^\vee} \xrightarrow{\iota} G^\natural \xrightarrow{\rho} G \to 0,$$

$$0 \to \underline{\omega}_{G} \xrightarrow{\iota^\vee} G^{\vee\natural} \xrightarrow{\rho^\vee} G^\vee \to 0.$$

Denote P_ρ the pullback of P to $G^\natural \times_S G^\vee$; this is a biextension of $(M^\natural, M^\vee)$ by $\mathbb{G}_{m,S}$. By Proposition 1.3.10, the identity map in $G^\natural(G^\natural)$ induces a connection ∇_2 on the torsor P_ρ endowing it with the structure of $\natural-$extension of $M^\vee_{G^\natural}$ by $\mathbb{G}_{m,G^\natural}$. Notice that the pullback of P_ρ along $\rho^\vee$ equals $P^\natural$; we will see that $\rho^{\vee *}\nabla_2$ endows $P^\natural$ with a

$\natural - 2-$structure. First, we have that $(P^\natural, \rho^{\vee *}\nabla_2)$ is a $\natural-$extension of $G^{\vee\natural}_{G^\natural}$ by $\mathbb{G}_{m,G^\natural}$. It remains to prove that

$$\beta_1 : (p_1 \times Id)^* P^\natural \wedge (p_2 \times Id)^* P^\natural \to (\mu_{G^\natural} \times Id)^* P^\natural$$

is horizontal on $G^\natural \times_S G^\natural \times_S G^{\vee\natural}$. Notice that $p_1 + p_2 = \mu_{G^\natural}$ in $G^\natural(G^\natural \times_S G^\natural)$. The $\natural-$extension of $M^\vee_{G^\natural \times G^\natural}$ by $\mathbb{G}_{m,G^\natural \times G^\natural}$ induced by p_i, for $i = 1, 2$, is

$$(p_i \times Id)^*(P_\rho, \nabla_2) = (P_{\rho \circ p_i}, (p_i \times Id)^* \nabla_2),$$

and the one induced by $\mu_{G^\natural}$ is

$$((\mu_{G^\natural} \times Id)^*(P_\rho, \nabla_2) = (P_{\rho \circ \mu_{G^\natural}}, (\mu_{G^\natural} \times Id)^* \nabla_2).$$

Pulling back along $Id \times Id \times \rho^\vee$ we obtain the horizontality of β_1.

In a similar way, we obtain a connection ∇_1 on $P_{\rho^\vee}$ which endows $P^\natural$ with a $\natural - 1-$structure. The connections ∇_1 and ∇_2 then makes $P^\natural$ into a $\natural-$biextension of $(M^\natural, M^{\vee\natural})$ by $\mathbb{G}_m$. The following proposition generalizes this fact and gives the uniqueness of this $\natural-$structure when $\mathrm{Hom}(G_1^\#, \mathbb{G}_a) = \mathrm{Hom}(G_2^\#, \mathbb{G}_a) = 0$.

Proposition 1.3.11. *Let $M_1 = [L_1 \xrightarrow{u_1} G_1]$ and $M_2 = [L_2 \xrightarrow{u_2} G_2]$ be a pair of 1-motives over S. Let P be a biextension of (M_1, M_2) by $\mathbb{G}_{m,S}$ and denote by $P^\natural$ its pullback to $(M_1^\natural, M_2^\natural)$. Then $P^\natural$ is a $\natural-$biextension of $(M_1^\natural, M_2^\natural)$ by $\mathbb{G}_{m,S}$ in a canonical way. This is the unique $\natural-$structure on $P^\natural$ if* $\mathrm{Hom}(G_1^\#, \mathbb{G}_{a,S}) = \mathrm{Hom}(G_2^\#, \mathbb{G}_{a,S}) = 0$.

Proof. Clearly, $P^\natural$ is a biextension of $(M_1^\natural, M_2^\natural)$ by $\mathbb{G}_{m,S}$. Notice that by [SGA7-I, Exposé VIII, Cor. 3.5] we have an isomorphism

$$\mathrm{Biext}(G_1, [L_2 \to A_2]; \mathbb{G}_{m,S}) \cong \mathrm{Biext}(G_1, M_2; \mathbb{G}_{m,S})$$

induced by pullback along the natural morphism $M_2 \to [L_2 \to A_2]$. This means that P is the pullback of a biextension $\tilde{P}$ of $(G_1, [L_2 \to A_2])$ by $\mathbb{G}_{m,S}$. By [SGA7-I, Exposé VIII, §1.4], $\tilde{P}$ induces a morphism

$$\psi : G_1 \to \underline{\mathrm{Ext}}^1([L_2 \to A_2], \mathbb{G}_{m,S}) = G_2^\vee$$

satisfying that $\tilde{P}$ is the pullback along $\psi \times Id$ of the Poincaré biextension of $(G_2^\vee, [L_2 \to A_2])$. We define the group scheme C as the pullback

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\omega}_{G_2} & \longrightarrow & C & \longrightarrow & G_1 & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \psi} & & \\ 0 & \longrightarrow & \underline{\omega}_{G_2} & \longrightarrow & G_2^{\vee\natural} & \longrightarrow & G_2^\vee & \longrightarrow & 0 \end{array},$$

from which we can see that C is an extension of G_1 by $\underline{\omega}_{G_2}$. From Proposition 1.3.10, we see that

$$C(S) = \left\{ (g, \nabla) \;\middle|\; \begin{array}{c} g \in G_1(S) \text{ and } \nabla \text{ is a } \natural\text{−structure on} \\ \text{the extension } [L_2 \to P_g] \text{ of } M_2 \text{ by } \mathbb{G}_{m,S} \end{array} \right\}.$$

Define $u_C : L_1 \to C$ as $u_C(x) = (u_1(x), \nabla_0)$, where ∇_0 is the trivial connection on $[L_2 \to P_{u_1(x)}]$. Then we have that $[u_C : L_1 \to C]$ is an extension of M_1 by $\underline{\omega}_{G_2}$. By the universal property of the universal vectorial extension of M_1, we see that $[u_C : L_1 \to C]$ is the pushout of $M_1^\natural$ along a morphism $\underline{\omega}_{G_1^\vee} \to \underline{\omega}_{G_2}$. The morphism $G_1^\natural \to C$ thus obtained is an element in $C(G_1^\natural)$ and therefore induces a connection on the extension of $M_{2,G_1^\natural}$ by $\mathbb{G}_{m,G_1^\natural}$ induced by P. This connection in turn induces a $\natural - 2-$structure on $P^\natural$. In a similar way, we obtain a $\natural-1-$structure and hence the desired $\natural-$structure on $P^\natural$.

To show uniqueness it is enough to consider the trivial $\mathbb{G}_{m,S}-$torsor on $G_1^\natural \times_S G_2^\natural$. Let ∇ be a connection inducing a structure of $\natural-$biextension of $(M_1^\natural, M_2^\natural)$ by $\mathbb{G}_{m,S}$ on $\mathbb{G}_{m,S} \times_S G_1^\natural \times_S G_2^\natural$. The connection ∇ is determined by a global differential $\omega \in \Omega^1_{G_1^\natural \times G_2^\natural / S}$ which is the sum of a global invariant differential $\omega_1 \in \Omega^1_{G_1^\natural \times G_2^\natural / G_2^\natural}$, corresponding to the $\natural - 1-$structure, and a global invariant differential $\omega_2 \in \Omega^1_{G_1^\natural \times G_2^\natural / G_1^\natural}$, corresponding to the $\natural - 2-$structure. Working (Zariski) locally on S we may assume that the sheaf of differentials of $G_1^\natural$ is free over S. Since $\Omega^1_{G_1^\natural \times G_2^\natural / G_2^\natural} \cong p_1^* \Omega^1_{G_1^\natural / S}$ we may express $\omega_1 = \sum a_j \nu_j$, where $\{\nu_j\}$ is the pullback of a basis of invariant differentials on $\Omega^1_{G_1^\natural / S}$ and a_j is the pullback of a global section of $G_2^\natural$. From the horizontality of β_2 we get that a_j corresponds to a morphism of $S-$group schemes $G_2^\natural \to \mathbb{G}_{a,S}$. Since we have an extension

$$0 \to G_2^{\#} \to G_2^\natural \to L_2 \otimes \mathbb{G}_{a,S} \to 0$$

and every morphism of group schemes $G_2^{\#} \to \mathbb{G}_{a,S}$ is trivial then a_j induces a homomorphism $\bar{a}_j : L_2 \otimes \mathbb{G} \to \mathbb{G}_{a,S}$. Now the condition that the pullback of ω to $G_1^\natural \times L_2$ is trivial is equivalent to the condition that the pullback of ω_1 to $G_1^\natural \times L_2$ is trivial, since ω_2 is automatically trivial because L_2 is étale over S and therefore $\Omega^1_{G_1^\natural \times L_2 / G_1^\natural} = 0$. Notice that the pullback of ω_1 to $G_1^\natural \times L_2$ has the expression $\sum \bar{a}_j \nu_j$; since this is trivial, then $a_j = \bar{a}_j = 0$ and therefore $\omega_1 = 0$. Similarly, we see that $\omega_2 = 0$, which proves that $\omega = 0$. □

In the situation of Proposition 1.3.11, let ∇ be the connection on $P^\natural$ inducing the canonical $\natural-$structure. The curvature of ∇ is an invariant 2-form on $G_1^\natural \times G_2^\natural$ and so induces an alternating pairing R on $\underline{\text{Lie}} G_1^\natural \times \underline{\text{Lie}} G_2^\natural$ with values in $\underline{\text{Lie}} \mathbb{G}_{m,S}$. Since the restriction of R to $\underline{\text{Lie}} G_1^\natural$ and $\underline{\text{Lie}} G_2^\natural$ is zero, it induces a pairing

$$\Phi : \underline{\text{Lie}} G_1^\natural \times \underline{\text{Lie}} G_2^\natural \to \underline{\text{Lie}} \mathbb{G}_{m,S}.$$

Deligne's pairing on the de Rham realizations of M_1 and M_2 is then defined as (see [Del74, p. 66])

$$(\,\cdot\,,\,\cdot\,)_M^{Del} := -\Phi : \mathrm{T}_{\mathrm{dR}}(M_1) \times \mathrm{T}_{\mathrm{dR}}(M_2) \to \underline{\mathrm{Lie}}\mathbb{G}_{m,S}.$$

1.4 Albanese and Picard 1-motives

We define Albanese and Picard 1-motives following [BS01]. Let X be an equidimensional variety over an algebraically closed field K of characteristic 0. Let $S \subset X$ be the singular locus and $f : \tilde{X} \to X$ a resolution of singularities of X. Denote $\tilde{S} := f^{-1}(\mathrm{S})$ the reduced inverse image. We consider a smooth compactification $\bar{X}$ of $\tilde{X}$ with boundary $Y = \bar{X} - \tilde{X}$, which we assume to be a divisor on $\bar{X}$. Denote $\bar{S}$ the Zariski closure of $\tilde{S}$ in $\bar{X}$. We can choose the resolution $\tilde{X}$ and compactification $\bar{X}$ of X so that $\bar{X}$ is projective and $\bar{S} + Y$ is a reduced normal crossing divisor in $\bar{X}$; we call such a compactification a *good normal crossing compactification* of the resolution $\tilde{X}$.

For a $K-$scheme V and a closed subscheme $i : Z \hookrightarrow V$ consider the group

$$\mathrm{Pic}(V, Z) = \mathbb{H}^1(V, \mathbb{G}_{m,V} \to i_*\mathbb{G}_{m,Z})$$

which consists of isomorphism classes of pairs $(\mathcal{L}, \varphi)$ where $\mathcal{L}$ is an invertible sheaf on V and $\varphi : \mathcal{L}|_Z \cong \mathcal{O}_Z$ is a trivialization on Z. We have that the fpqc-sheaf

$$T \mapsto \mathrm{Pic}(\bar{X} \times_K T, Y \times_K T)$$

is representable by a $K-$group scheme which is locally of finite type over K and whose group of $K-$points is $\mathrm{Pic}(\bar{X}, Y)$ (see [BS01, Lemma 2.1]). We have the following proposition describing the structure of its identity component, which we will denote $\mathrm{Pic}^0(\bar{X}, Y)$.

Proposition 1.4.1 ([BS01, Prop. 2.2])**.** *Let $Y = \cup Y_i$, where Y_i are the (smooth) irreducible components of Y. Then $\mathrm{Pic}^0(\bar{X}, Y)$ fits in the exact sequence*

$$0 \to T(\bar{X}, Y) \to \mathrm{Pic}^0(\bar{X}, Y) \to A(\bar{X}, Y) \to 0,$$

where:

i) $T(\bar{X}, Y)$ is the torus

$$T(\bar{X}, Y) := \mathrm{Coker}\left((\pi_{\bar{X}})_*\mathbb{G}_{m,\bar{X}} \to (\pi_Y)_*\mathbb{G}_{m,Y}\right),$$

where $\pi_{\bar{X}} : \bar{X} \to \mathrm{Spec}\, K$ and $\pi_Y : Y \to \mathrm{Spec}\, K$ are the structure morphisms; and

ii) $A(\bar{X}, Y)$ is the abelian variety

$$A(\bar{X}, Y) := \mathrm{Ker}^0\left(\mathrm{Pic}^0(\bar{X}) \to \oplus_i \mathrm{Pic}^0(Y_i)\right),$$

which is the identity component of the kernel.

Consider the group of Weil (or equivalently Cartier) divisors $\mathrm{Div}(\bar{X})$ on $\bar{X}$. Denote $\mathrm{Div}(\bar{X}, Y)$ the subgroup consisting of divisors on $\bar{X}$ such that $\mathrm{supp}(D) \cap Y = \emptyset$. If $D \in \mathrm{Div}(\bar{X}, Y)$ then $\mathcal{O}(D)$ has a section trivializing it on $\bar{X} - D$ (and hence also on Y), therefore $(\mathcal{O}_{\bar{X}}(D), 1)$ determines an element $[D] \in \mathrm{Pic}^0(\bar{X}, Y)$. Denote $\mathrm{Div}^0(\bar{X}, Y) \subset \mathrm{Div}(\bar{X}, Y)$ the subgroup of divisors D such that $[D] \in \mathrm{Pic}^0(\bar{X}, Y)$. We will say that an element in $\mathrm{Div}^0(\bar{X}, Y)$ is *algebraically equivalent to 0 relative to* Y.

Denote $\mathrm{Div}_{\bar{S}}(\bar{X}, Y) \subset \mathrm{Div}(\bar{X}, Y)$ the subgroup consisting of divisors D on $\bar{X}$ such that $\mathrm{supp}(D) \subset \bar{S}$ and $\mathrm{supp}(D) \cap Y = \emptyset$. Consider the push-forward of Weil divisors $f_* : \mathrm{Div}_{\tilde{S}}(\tilde{X}) \to \mathrm{Div}_S(X)$ and denote $\mathrm{Div}_{\tilde{S}/S}(\tilde{X}, Y)$ its kernel. We denote by $\mathrm{Div}^0_{\bar{S}/S}(\bar{X}, Y)$ the intersection of $\mathrm{Div}_{\tilde{S}/S}(\bar{X}, Y)$ and $\mathrm{Div}^0_{\bar{S}}(\bar{X}, Y)$, *i.e.* the group of divisors on $\bar{X}$ which are linear combinations of compact divisorial components in $\tilde{S}$ which have trivial push-forward under f and which are algebraically equivalent to zero relative to Y.

Definition 1.4.2. [BS01, Def. 2.3] The *homological Picard 1-motive* of X is defined as

$$\mathrm{Pic}^-(X) = [u : \mathrm{Div}^0_{\bar{S}/S}(\bar{X}, Y) \to \mathrm{Pic}^0(\bar{X}, Y)],$$

where $u(D) = [D]$. The *cohomological Albanese 1-motive* $\mathrm{Alb}^+(X)$ of X is defined as the Cartiel dual of $\mathrm{Pic}^-(X)$.

The Albanese and Picard 1-motives defined above are generalizations of a construction of Deligne regarding the "motivic cohomology" of a curve; we remember his construction here (see [Del74, §10.3]). Let C_0 be a curve over K, *i.e.* a purely 1-dimensional variety. We have the following commutative diagram

$$\begin{array}{ccc} C' & \xhookrightarrow{j'} & \bar{C}' \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\bar{\pi}} \\ C & \xhookrightarrow{j} & \bar{C} \\ {\scriptstyle\pi_0}\big\downarrow & & \\ C_0 & & \end{array}$$

where C is the semi-normalization of C_0, C' is the normalization of C (and hence of C_0), $\bar{C}'$ is a smooth compactification of C' and $\bar{C}$ is a compactification of C. Denote S the set of singular points of C, $S' := \pi^{-1}(S)$ and $F := \bar{C}' - C' = \bar{C} - C$. In this case, we have

$$\mathrm{Pic}^-(C_0) = [u : \mathrm{Div}^0_{S'/S}(\bar{C}', F) \to \mathrm{Pic}^0(\bar{C}', F)]$$

and

$$\mathrm{Alb}^+(C_0) = [u^\vee : \mathrm{Div}^0_F(\bar{C}') \to \mathrm{Pic}^0(\bar{C})].$$

Notice that $\mathrm{Pic}^-(C_0) = \mathrm{Pic}^-(C)$ and $\mathrm{Alb}^+(C_0) = \mathrm{Alb}^+(C)$.

We have a canonical identification with Deligne's definition of the motivic H^1 of C_0 (see [BS01, Prop. 3.2])

$$\mathrm{H}^1_m(C_0)(1) \cong \mathrm{Alb}^+(C_0).$$

Remark 1.4.3. This constructions can also be done over an arbitrary field of characteristic 0 (*cf.* [BS01, §7]).

1.5 Good and semistable reduction of 1-motives

Let K be a field which is complete with respect to a discrete valuation v. We denote $\bar{K}$ the algebraic closure, R the ring of $v-$integers, π a uniformizer and $k = R/\pi R$ the residue field of characteristic $p > 0$.

First, we remember some concepts concerning reduction of abelian varieties ([ST68]). Let A_K be an abelian variety over K and denote A its Néron model over R. Then A_K has *good reduction over* R if A is an abelian scheme and *semi-stable reduction* if the connected component of the special fiber of A is a semi-abelian variety over k. We also say that A_K has *potentially good reduction over* R if it acquires good reduction over a finite extension of K.

Now we give some definitions regarding reduction of particular types of commutative groups schemes due to Raynaud [Ray94].

Definition 1.5.1. We say that:

1) A lattice L_K over K has *good reduction over* R if it is an unramified finite representation of $Gal(\bar{K}/K)$ on $\mathbb{Z}^r$, where r is the rank of L_K, *i.e.* the inertia subgroup of $Gal(\bar{K}/K)$ acts trivially on $\mathbb{Z}^r$. In this case, L_K extends to a lattice L over R.

2) A torus T_K over K has *good reduction over* R if its group of characters has good reduction over R. In this case, T_K extends to a torus over R.

3) A semi-abelian variety G_K over K has:

   i) *good reduction over* R if it extends to a semi-abelian scheme G over R;

   ii) *potentially good reduction* if it acquires good reduction over a finite extension of K; and

   iii) *semi-stable reduction* if it extends to a smooth scheme over R whose connected component of the special fiber is semi-abelian.

For locally constant group schemes and tori, the concept of potentially good reduction becomes trivial, since such group schemes always acquire good reduction over a finite extension of K. Observe that if G_K is a semi-abelian variety over K, extension of A_K by T_K, such that A_K and T_K have good reductions A and T over R then G_K also extends to a semi-abelian scheme G over R, which is an extension of A by T. This is because the good reduction of T_K and A_K imply good reduction of $L_K^\vee$ and $A_K^\vee$ and of the morphism $v_K^\vee : L_K^\vee \to A_K^\vee$, which extends to a morphism $v^\vee : L^\vee \to A^\vee$ over R. Then the isomorphism (1.4) induces an abelian scheme G over R. By functoriality, G is indeed an extension of G_K to R, as we can see by the following commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_R(L^\vee, A^\vee) & \xrightarrow{\cong} & \mathrm{Ext}^1_R(A,T) \\ \cong\big\downarrow & & \big\downarrow \\ \mathrm{Hom}_K(L_K^\vee, A_K^\vee) & \xrightarrow{\cong} & \mathrm{Ext}^1_K(A_K,T_K) \end{array} \qquad \begin{array}{ccc} v^\vee & \longmapsto & G \\ \big\downarrow & & \big\downarrow \\ v_K^\vee & \longmapsto & G_K \end{array} .$$

Notice that the left vertical arrow is an isomorphism because of the universal property of Néron models. Therefore, we see that G_K has good reduction if and only if T_K and A_K have good reduction. Observe also that G_K having semi-stable reduction is equivalent to T_K having good reduction and A_K having semi-stable reduction.

In [Ray94, §4], Raynaud gives the following definitions generalizing the concept of good reduction to 1-motives.

Definition 1.5.2. Let $M_K = [u_K : L_K \to G_K]$ be a 1-motive over K. We say that:

1) M_K has *good reduction over* R if it extends to a 1-motive $M = [u : L \to G]$ over R.

2) M_K has *potentially good reduction over* R if it acquires good reduction after a finite extension of K.

3) M_K has *semi-stable reduction over* R if L_K has good reduction and G_K has semi-stable reduction.

4) M_K is *strict* if G_K has potentially good reduction.

Notice that M_K having good reduction over R is equivalent to L_K, T_K and A_K having good reduction and $u_K : L_K \to G_K$ extending to a morphism $u : L \to G$ over R. In this case, if M is the extension of M_K over R then this is unique up to unique isomorphism. Also notice that M_K has potentially good reduction if and only if A_K has potentially good reduction and, over a finite extension K' of K, $u_{K'} : L_{K'} \to G_{K'}$ extends to a morphism over R', where R' is the ring of integers of K'.

We have a GAGA type functor that associates to any $K-$scheme X locally of finite type a rigid analytic space X_{rig} over K. This functor is flat, it maps exact sequences of $K-$group schemes (for the fppf, étale topology, resp.) to exact sequences of rigid $K-$groups (for the fppf, étale topology, resp.). Thus, for every 1-motive $M_K = [u_K : L_K \to G_K]$ we have a rigid 1-motive $M_{K,\mathrm{rig}} = [u_{K,\mathrm{rig}} : L_{K,\mathrm{rig}} \to G_{K,\mathrm{rig}}]$ over K. This functor is compatible with the weight filtration.

Theorem 1.5.3 ([Ray94, Thm. 4.2.2]). *There is a canonical way to associate to a 1-motive $M_K = [u_K : L_K \to G_K]$ a strict 1-motive $M'_K = [u'_K : L'_K \to G'_K]$. This association is functorial. Furthermore, we have a canonical morphism of rigid 1-motives*

$$can : M'_{K,\mathrm{rig}} \to M_{K,\mathrm{rig}},$$

which is an isomorphism in the derived category of bounded complexes of fppf sheaves on the small rigid site of Spec(K).

1.5.1 Geometric monodromy of 1-motives

Consider a strict 1-motive $M_K = [u_K : L_K \to G_K]$ with dual $M_K^\vee = [u_K^\vee : L_K^\vee \to G_K^\vee]$. Let P_K be the Poincaré biextension of $(M_K, M_K^\vee)$ by $\mathbb{G}_{m,K}$. Remember that the 1-motive M_K is described by its symmetric avatar $(v_K : L_K \to A_K, v_K^\vee : L_K^\vee \to A_K^\vee, \tau_K : L_K \times L_K^\vee \to P_K)$.

If A_K has good reduction then P_{A_K} extends to a biextension P_A of $(A, A^\vee)$ by $\mathbb{G}_{m,R}$. Suppose that L_K and G_K have good reduction. Then taking the valuation of τ_K we get a canonical bilinear map

$$\mu_0 : L_K \times L_K^\vee \to \mathbb{Z},$$

which is compatible with the Galois actions on L_K and $L_K^\vee$. Notice that it can be considered as a morphism on the tensor product $\mu_0 : L_K \otimes L_K^\vee \to \mathbb{Z}$. In the general case where L_K and G_K only have potentially good reduction, we canonically extend the valuation from K to $\bar{K}$ taking values in $\mathbb{Q}$. Again, taking the valuation of τ_K we get a canonical bilinear map

$$\mu : L_K \times L_K^\vee \to \mathbb{Q}$$

which is compatible with the Galois actions on L_K and $L_K^\vee$. We can also consider μ as a morphism on the tensor product $\mu : L_K \otimes L_K^\vee \to \mathbb{Q}$, since μ is bilinear. When L_K and G_K have good reduction, μ factors through $\mathbb{Z}$, recovering μ_0. We call μ the *geometric monodromy* of M_K.

If we replace K by a finite extension K' with ramification index e, then the geometric monodromy of $M_{K'}$ becomes $e\mu$. In particular, if L_K and G_K acquire good reduction after a finite extension of K with ramification index e then $e\mu$ takes values in $\mathbb{Z}$.

Proposition 1.5.4 ([Ray94, Prop. 4.3.1]). *Let $M_K = [u_K : L_K \to G_K]$ be a strict 1-motive.*

i) M_K has potentially good reduction if and only if the geometric monodromy μ of M_K is zero.

ii) Suppose L_K and G_K have good reduction. Then M_K has good reduction if and only if μ_0 is zero.

Proof. It is enough to prove 2. Let L and G be the extensions of L_K and G_K over R. Then M_K has good reduction if and only if $u_K : L_K \to G_K$ extends to a morphism $u : L \to G$ over R. This happens if and only if the trivialization $\tau_K : L_K \times L_K^\vee \to P_{A_K}$ extends to a morphism $\tau : L \times L^\vee \to P_A$ over R, and this is equivalent to μ_0 being zero. □

Let M_K be a strict 1-motive whose monodromy factors through $\mathbb{Z}$. Defining

$$\tilde{L}_K := \mathrm{Ker}\begin{pmatrix} L_K \xrightarrow{\mu_L} \underline{\mathrm{Hom}}_K(L_K^\vee, \mathbb{Z}) \\ x \mapsto (x^\vee \mapsto \mu(x, x^\vee)) \end{pmatrix},$$

$$\tilde{L}_K^\vee := \mathrm{Ker}\begin{pmatrix} L_K^\vee \xrightarrow{\mu_{L^\vee}} \underline{\mathrm{Hom}}_K(L_K, \mathbb{Z}) \\ x^\vee \mapsto (x \mapsto \mu(x, x^\vee)) \end{pmatrix}$$

and $\bar{L}_K := \mathrm{Coker}(\tilde{L}_K \hookrightarrow L_K)$, $\bar{L}_K^\vee := \mathrm{Coker}(\tilde{L}_K^\vee \hookrightarrow L_K^\vee)$ we get exact sequences

$$0 \to \tilde{L}_K \to L_K \to \bar{L}_K \to 0 \quad \text{and} \quad 0 \to \tilde{L}_K^\vee \to L_K^\vee \to \bar{L}_K^\vee \to 0. \tag{1.14}$$

Note that $\tilde{L}_K$ and $\tilde{L}_K^\vee$ are lattices, since L_K and $L_K^\vee$ are. Also, $\bar{L}_K$ is a lattice because, being the image of μ_L, it injects into $\underline{\mathrm{Hom}}_K(L_K^\vee, \mathbb{Z})$, which is a lattice. Similarly, $\bar{L}_K^\vee$ is a lattice. We have that $\bar{L}_K$ and $\bar{L}_K^\vee$ have the same rank. Finally, notice that the morphism $\tilde{\mu} : \tilde{L}_K \times \tilde{L}_K^\vee \to \mathbb{Z}$, which is obtained by restricting μ, is zero. The monodromy thus induces a map $\bar{\mu} : \bar{L}_K \times \bar{L}_K^\vee \to \mathbb{Z}$.

If we define $\bar{T}_K := \underline{\mathrm{Hom}}_K(\bar{L}_K^\vee, \mathbb{G}_{m,K})$ and $\tilde{T}_K := \underline{\mathrm{Hom}}_K(\tilde{L}_K^\vee, \mathbb{G}_{m,K})$, and analogously for $\bar{T}_K^\vee$ and $\tilde{T}_K^\vee$, we also get exact sequences of tori

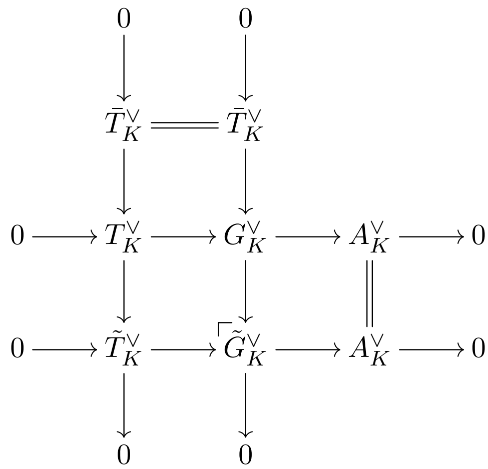
$$0 \to \bar{T}_K \to T_K \to \tilde{T}_K \to 0 \quad \text{and} \quad 0 \to \bar{T}_K^\vee \to T_K^\vee \to \tilde{T}_K^\vee \to 0. \tag{1.15}$$

We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{L}_K & \longrightarrow & L_K & \longrightarrow & \bar{L}_K & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longleftarrow & \tilde{T}_K & \longleftarrow & T_K & \longleftarrow & \bar{T}_K & \longleftarrow & 0, \end{array}$$

where the maps are induced by $\pi^\mu : L_K \times L_K^\vee \to \mathbb{G}_{m,K}$, $\pi^{\tilde{\mu}} : \tilde{L}_K \times \tilde{L}_K^\vee \to \mathbb{G}_{m,K}$ and $\pi^{\bar{\mu}} : \bar{L}_K \times \bar{L}_K^\vee \to \mathbb{G}_{m,K}$. Observe that since $\tilde{\mu} = 0$ then $\tilde{L}_K \to \tilde{T}_K$ is the constant map with value 1. Therefore, this is saying that the middle map $L_K \to T_K$ is trivial on $\tilde{L}_K$ and factors through $\bar{L}_K \to \bar{T}_K$.

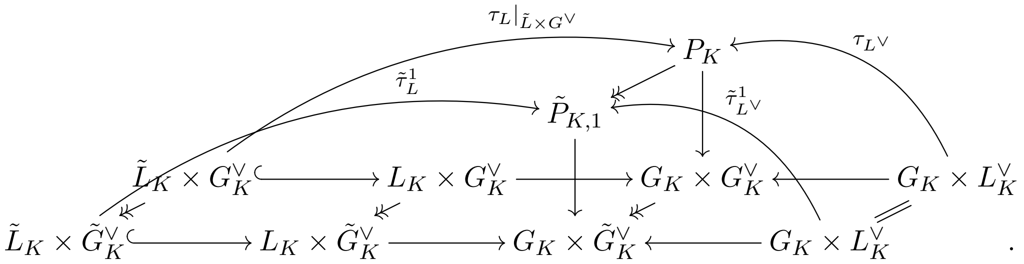
Notice that we have a 1-motive $\tilde{M}_{K,1} = [\tilde{u}_1 : \tilde{L}_K \to G_K]$ with $\tilde{u}_1 := u|_{\tilde{L}}$. This is the "biggest" sub-1-motive of M whose monodromy is zero and therefore has good reduction. Its dual is of the form $\tilde{M}^\vee_{K,1} = [\tilde{u}^\vee_1 : L^\vee_K \to \tilde{G}^\vee_K]$, where $\tilde{G}^\vee_K$ is the pushout



and $\tilde{u}^\vee_1$ is the composition of $u^\vee$ with the projection $G^\vee_K \to \tilde{G}^\vee_K$. We have the following short exact sequences of 1-motives

$$0 \to \tilde{M}_{K,1} \to M_K \to \bar{L}_K[1] \to 0 \quad \text{and} \quad 0 \to \bar{T}^\vee_K \to M^\vee_K \to \tilde{M}^\vee_{K,1} \to 0.$$

The pullback of P_{A_K} along $G_K \times \tilde{G}^\vee_K \to A_K \times A^\vee_K$ gives the biextension of $(G_K, \tilde{G}^\vee_K)$ by $\mathbb{G}_{m,K}$ underlying the Poincaré biextension $\tilde{P}_{K,1}$ of $(\tilde{M}_{K,1}, \tilde{M}^\vee_{K,1})$. The trivializations $\tilde{\tau}^1_L$ and $\tilde{\tau}^1_{L^\vee}$ of $\tilde{P}_{K,1}$ make the following diagram commute

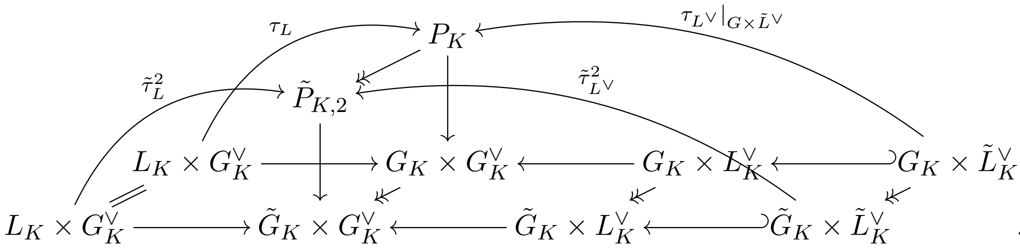


We also have a 1-motive $\tilde{M}_{K,2} = [\tilde{u}_2 : L_K \to \tilde{G}_K]$, where $\tilde{G}_K$ is the pushout of G_K along $T_K \to \tilde{T}_K$ and $\tilde{u}_2$ is the composition of u with the projection $G_K \to \tilde{G}_K$. Its dual is $\tilde{M}^\vee_{K,2} = [\tilde{u}^\vee_2 : \tilde{L}^\vee \to G^\vee]$ with $\tilde{u}^\vee_2 := u^\vee|_{\tilde{L}^\vee}$. As before, we have the following short exact sequences of 1-motives

$$0 \to \bar{T}_K \to M_K \to \tilde{M}_{K,2} \to 0 \quad \text{and} \quad 0 \to \tilde{M}^\vee_{K,2} \to M^\vee_K \to \bar{L}^\vee_K[1] \to 0.$$

The biextension of $(\tilde{G}_K, G^\vee_K)$ by $\mathbb{G}_{m,K}$ underlying the Poincaré biextension $\tilde{P}_{K,2}$ of $(\tilde{M}_{K,2}, \tilde{M}^\vee_{K,2})$ is given by the pullback of P_{A_K} along the projection $\tilde{G}_K \times G^\vee_K \to A_K \times A^\vee_K$.

The trivializations $\tilde{\tau}_L^2$ and $\tilde{\tau}_{L^\vee}^2$ of $\tilde{P}_{K,2}$ make the following diagram commute



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Theorem 1.5.5 ([Ray94, §4.5.1]). *Let $M_K = [u_K : L_K \to G_K]$ be a strict 1-motive such that the monodromy factors through $\mathbb{Z}$ and fix a uniformizer π of R. Then we have a canonical decomposition*

$$u_K = u_K^1 + u_K^2,$$

where u_K^2 factors through the torus T_K and $M_K^1 = [u_K^1 : L_K \to G_K]$ has potentially good reduction.

Proof. Consider the 1-motive $M_K = [u_K : L_K \to G_K]$ and its dual $M_K^\vee = [u_K^\vee : L_K^\vee \to G_K^\vee]$. Let P_K be the Poincaré biextension of $(M_K, M_K^\vee)$ by $\mathbb{G}_{m,K}$ with trivialization $\tau_K : L_K \times L_K^\vee \to P_K$.

If $\mu : L_K \times L_K^\vee \to \mathbb{Z}$ is the monodromy of M_K then we modify the trivialization τ_K to obtain a new morphism

$$\begin{aligned} \tau_K^1 : L_K \times L_K^\vee &\to P_K \\ (x, x^\vee) &\mapsto \pi^{-\mu(x,x^\vee)}\tau_K(x, x^\vee). \end{aligned}$$

This trivialization corresponds to a 1-motive $M_K^1 = [u_K^1 : L_K \to G_K]$ whose monodromy is zero and thus has potentially good reduction.

Define $u_K^2 = u_K - u_K^1 : L_K \to G_K$. This morphism factors through T_K and is given by the following formula

$$\begin{aligned} u_K^2 : L_K &\to T_K = \underline{\mathrm{Hom}}_K(L^\vee, \mathbb{G}_{m,K}) \\ x &\mapsto (x^\vee \mapsto \pi^{\mu(x,x^\vee)}). \end{aligned}$$

Moreover, with the previously used notation, we have that u_K^2 factors through

$$u_K^2 : L_K \to \bar{L}_K \to \bar{T}_K \to T_K,$$

where the middle morphism $\bar{L}_K \to \bar{T}_K$ is given by $\bar{x} \mapsto (\bar{x}^\vee \mapsto \pi^{\bar{\mu}(\bar{x},\bar{x}^\vee)})$. □

Chapter 2

Pairings via biextensions

In [MT83], Mazur and Tate introduced the notion of $\rho-$splittings associated to homomorphisms $\rho : K^* \to Y$ from the non-zero elements of a field to an abelian group. We review this definition, as well as important results concerning $\rho-$splittings in Section 2.1. These $\rho-$splittings are then used to define local pairings on the set of divisors and zero cycles with disjoint support, whose construction is given in Section 2.2, and also global pairings on the $K-$rational points of a pair of abelian varieties over K, which is the content of Section 2.3.

2.1 $\rho-$splittings of biextensions

We fix abelian groups U, V, W and a biextension X of (U, V) by W.

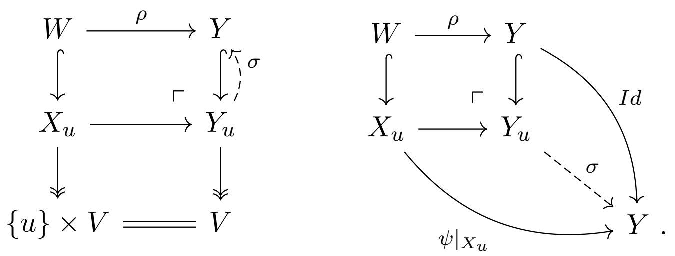
Definition 2.1.1. Let $\rho : W \to Y$ be a homomorphism. A $\rho-$*splitting* of X is a map $\psi : X \to Y$ such that

i) $\psi(w + x) = \rho(w) + \psi(x)$, for $w \in W$, $x \in X$, and

ii) for each $u \in U$ (resp. $v \in V$) the restriction of ψ to X_u (resp. X_v) is a group homomorphism,

where X_u (resp. X_v) denotes the part of X above $\{u\} \times V$ (resp. $U \times \{v\}$).

Remark 2.1.2. Observe that the restriction of ψ to X_u provides a trivialization of the pushout Y_u of X_u along ρ, seen as an extension of V by Y, and similarly for X_v



The following lemmas tells us about some situations in which we can extend $\rho-$splittings uniquely. These will be used in the proof of Theorem 2.1.6. As before, consider a biextension X of (U, V) by W. For an integer m we have a map

$$(m, 1) : X \to X$$

which is defined on the fibers X_v, for $v \in V$, as multiplication by m. Similarly, we have a map

$$(1, n) : X \to X$$

defined on fibers X_u, for $u \in U$, as multiplication by n. Denote

$$(m, n) := (m, 1) \circ (1, n) = (1, n) \circ (m, 1) : X \to X.$$

Notice that (m, n) takes the fiber $X_{u,v}$ to $X_{mu,nv}$. This map also satisfies

$$(m', n') \circ (m, n) = (m'm, n'n)$$

and, for $w \in W$,

$$(m, n)(w + x) = mnw + (m, n)x\,.$$

Moreover, if $\rho : W \to Y$ is a homomorphism and ψ a $\rho-$splitting then

$$\psi((m, n)x) = mn\psi(x)\,.$$

Lemma 2.1.3. *Let $U^0 \subset U$ and $V^0 \subset V$ be subgroups and denote X^0 the pullback of X over $U^0 \times V^0$. If n, m are positive integers such that $mU \subset U^0$ and $nV \subset V^0$ and $\rho : W \to Y$ is a homomorphism into a group Y uniquely divisible by mn then a $\rho-$splitting $\psi_0 : X^0 \to Y$ extends uniquely to a $\rho-$splitting $\psi : X \to Y$.*

Proof. The relation $(m, n)X \subset X^0$ forces ψ to take the value

$$\psi(x) = \frac{1}{mn}\psi_0((m, n)x)$$

on $x \in X$. One can easily check that this indeed defines a $\rho-$splitting. ☐

Lemma 2.1.4. *Let $W' \subset W$ be a subgroup and $X' \subset X$ a subset such that X' is a biextension of (U, V) by W'. Consider a homomorphism $\rho : W \to Y$ and denote $\rho' := \rho|_{W'}$. Then a $\rho'-$splitting ψ' of X' extends uniquely to a $\rho-$splitting ψ of X.*

Proof. Expressing W as a disjoint union $W = \bigcup(w_i + W')$ of cosets of W' gives us an expression of X as a disjoint union $X = \bigcup(w_i + X')$ of sets (the equality being true on each fiber over $U \times V$). Then every $x \in X$ is of the form $x = w_i + x'$, for some $x' \in X'$. This forces ψ to take the value

$$\psi(x) = \rho(w_i) + \psi'(x')$$

on an arbitrary element $x \in X$. One can check that this defines a $\rho-$splitting. ☐

In what follows, we will work with biextensions of abelian varieties over fields. Let K be a field which is complete with respect to a place v which is either archimedean or discrete. Consider an abelian variety A_K over K. If v is archimedean, we set $A := A_K$. If v is discrete, denote R the ring of $v-$integers in K, π a uniformizer in R and $k = R/\pi R$ the residue field. We denote by A the Néron model of A_K over R. We will also denote by A^0 the identity component of A, *i.e.* the open subgroup scheme of A whose closed fiber $A^0_k := A^0 \times_R \operatorname{Spec}(k)$ is connected. For a scheme U over R, we will denote $U_k := U \times_R \operatorname{Spec}(k)$.

Definition 2.1.5. An abelian variety A_K over a field K which is complete with respect to a discrete valuation is said to have *semistable ordinary reduction* if the residue field k has characteristic $p > 0$ and its closed fiber A_k satisfies the following equivalent conditions:

i) the formal completion A^f_k of A_k at the origin is isomorphic to a product of copies of $\mathbb{G}^f_m$ over the algebraic closure $\bar{k}$ of k,

ii) the connected component of the kernel of the homomorphism "multiplication by p"
$$p : A^0_k \to A^0_k$$
is the dual of an étale group scheme over k,

iii) A^0_k is an extension over k of an ordinary abelian variety by a torus T_A.

Observe that if A_K has semistable ordinary reduction then in particular it has semistable reduction, and it has good reduction if and only if the torus $T_A = 0$.

If B_K is a second abelian variety over K then giving a biextension P_K of (A_K, B_K) by $\mathbb{G}_{m,K}$ is equivalent to giving a morphism $\varphi_K : B_K \to A^\vee_K$, and P_K corresponds to the pullback of the Poincaré biextension P_{A_K} along $(Id, \varphi_K) : A_K \times_K B_K \to A_K \times_K A^\vee_K$ (see [SGA7-I, Exposé VIII, §1.4]). Notice that if v is discrete then, by the Néron mapping property, φ_K extends to a morphism $\varphi : B \to A^\vee$ between the Néron models of B_K and A_K. In this case, we will denote P the pullback of the biextension P_A of $A^0 \times_R A^\vee$ to $A^0 \times_R B$, which is a biextension of (A^0, B) by $\mathbb{G}_{m,R}$ (see Example 1.2.3). Similarly as before, if v is archimedean we set $P := P_K$.

We fix abelian varieties A_K and B_K over K and a biextension P_K of (A_K, B_K) by $\mathbb{G}_{m,K}$. Given an abelian group Y and a homomorphism $\rho : K^* \to Y$ we can construct canonical $\rho-$splittings $\psi_\rho : P_K(K) \to Y$ in some particular cases. We introduce the necessary notation. Suppose that v is discrete. We denote m_A the exponent of $A_k(k)/A^0_k(k)$, *i.e.* the smallest positive integer such that $m_A(A_k(k)/A^0_k(k)) = 0$. Now suppose also that k is finite and let T_A denote the maximal torus in A_k. We denote n_A the exponent of $A^0_k(k)/T_A(k)$. We define m_B and n_B analogously.

Theorem 2.1.6 ([MT83, §1.5]). *There exists a canonical $\rho-$splitting*

$$\psi_\rho : P_K(K) \to Y$$

in the following three cases:

1) *v is archimedean and $\rho(c) = 0$ for c such that $|c|_v = 1$;*

2) *v is discrete, $\rho(R^*) = 0$, i.e. ρ is unramified, and Y is uniquely divisible by m_A;*

3) *v is discrete, k is finite, A_K has semistable ordinary reduction and Y is uniquely divisible by $m_A m_B n_A n_B$.*

If both 2) and 3) hold, they yield the same ψ_ρ.

Proof. Case 1): In this case, $K = \mathbb{R}$ or $\mathbb{C}$ and we have a factorization

$$\rho : K^* \xrightarrow{v} \mathbb{R} \xrightarrow{\rho_1} Y\,,$$

where $v : K^* \to \mathbb{R}$ is the homomorphism defined as $v(c) = -\log|c|$. We can then define the $\rho-$splitting as

$$\psi_\rho : P_K(K) \xrightarrow{\psi_v} \mathbb{R} \xrightarrow{\rho_1} Y\,,$$

where ψ_v is the unique $v-$splitting of $P_K(K)$ that is continuous.

Case 2): We have the following commutative diagram, where $P^0(K)$ is the pullback of $P_K(K)$ over $A^0(R) \times B_K(K)$:

$$\begin{array}{ccccc}
R^* & \hookrightarrow & K^* & \hookrightarrow & K^* \\
\downarrow & & \downarrow & & \downarrow \\
P(R) & \hookrightarrow & P^0(K) & \hookrightarrow & P_K(K) \\
\downarrow & & \downarrow & \lrcorner & \downarrow \\
A^0(R) \times B(R) & = & A^0(R) \times B_K(K) & \hookrightarrow & A_K(K) \times B_K(K)\,.
\end{array}$$

Notice that $\rho|_{R^*} = 0$. Then the constant function 0 is a $\rho|_{R^*}-$splitting of $P(R)$. By Lemma 2.1.3 and 2.1.4 we can uniquely extend this $\rho|_{R^*}-$splitting to a $\rho-$splitting ψ_ρ of $P_K(K)$.

Explicitly, ψ_ρ is defined as follows. Let $x \in P_K(K)$. Then $(m_A, 1)x \in P^0(K)$ and so there exists an integer r such that $\pi^{-r} + (m_A, 1)x \in P(R)$. Hence, ψ_ρ is given by the formula

$$\psi_\rho(x) = \frac{r}{m_A}\rho(\pi)\,.$$

Case 3): We will denote the ring R as $\widehat{R}$ when viewed as an adic-ring. We will also be considering the formal completion of $\mathbb{G}_{m,R}$ along its special fiber $\mathbb{G}_{m,k}$, which we will denote $\widehat{\mathbb{G}}_m$. Let T_A and T_B denote the maximal tori in the special fibers of A and B, respectively. Denote A^t (resp. B^t) the formal completion of A (resp. B) along T_A (resp. T_B). From P we can obtain a biextension P^t of (A^t, B^t) by $\widehat{\mathbb{G}}_m$ in the category of formal groups over $\widehat{R}$ as the formal completion of P along the inverse image of $T_A \times T_B$ in P. Since A_K has semistable ordinary reduction, P^t has a unique trivialization $\psi : P^t \to \widehat{\mathbb{G}}_m$ (see [MT83, §5.11.1]). Taking points in $\widehat{R}$ we get a biextension $P^t(\widehat{R})$ of $(A^t(\widehat{R}), B^t(\widehat{R}))$ by $\widehat{\mathbb{G}}_m(\widehat{R}) = R^*$ with trivialization $\psi_R : P^t(\widehat{R}) \to R^*$. So, we can define a $\rho-$splitting of $P^t(\widehat{R})$ as the composition

$$\rho|_{R^*} \circ \psi_R : P^t(\widehat{R}) \to R^* \to Y.$$

We can add a smaller biextension to the diagram of case 2)

$$\begin{array}{ccc}
R^* & = & R^* \\
\downarrow & & \downarrow \\
P^t(\widehat{R}) & \hookrightarrow & P(R) \\
\downarrow & \lrcorner & \downarrow \\
A^t(\widehat{R}) \times B^t(R) & \hookrightarrow & A^0(R) \times B(R)
\end{array}$$

in which the bottom square is a pullback. Since $n_A A^0(R) \subset A^t(\widehat{R})$ and $m_B n_B B(R) \subset B^t(R)$, then Lemma 2.1.3 and 2.1.4 allow us to uniquely extend the $\rho-$splitting of $P^t(\widehat{R})$ to a $\rho-$splitting ψ_ρ of $P_K(K)$.

As before, we can give an explicit formula for ψ_ρ. Let $x \in P_K(K)$. There exists an integer r such that $y := \pi^{-r} + (m_A, 1)x \in P(R)$ and we have that $(n_A, m_B n_B)y \in P^t(\widehat{R})$. Therefore,

$$\psi(x) = \frac{r}{m_A}\rho(\pi) + \frac{1}{m_A n_A m_B n_B}\rho|_{R*} \circ \psi_R((n_A, m_B n_B)y)\,.$$

If we are simultaneously in cases 2) and 3) then $\rho|_{R*} \circ \psi_R = 0$ and it is clear from the explicit formulas of ψ_ρ that both $\rho-$splittings coincide. □

The canonical $\rho-$splittings satisfy the following functorial properties. In each situation considered we suppose that the canonical splittings exist, *i.e.* that we are in one of the cases of Theorem 2.1.6.

1) *Change of value group*: Let $c : Y \to Y'$ be a homomorphism. Then

$$\psi_{c\rho} = c\psi_\rho\,.$$

2) *Linearity in* ρ: Let $\rho' : K^* \to Y$ and $c : Y \to Y$ be homomorphisms. Then

$$\psi_{c\rho+\rho'} = c\psi_\rho + \psi_{\rho'}\,.$$

3) *Change of field*: Let $\sigma : K \to L$ be a continuous homomorphism of local fields and, for this case, we consider a homomorphism $\rho : L^* \to Y$. Then

$$\psi_{\rho\circ\sigma} = \psi_\rho \circ \sigma\,,$$

i.e, the following diagram is commutative

$$\begin{array}{ccc} P_K(K) & \xrightarrow{\sigma} & P_K(L) \\ {\scriptstyle \psi_{\rho\circ\sigma}}\downarrow & & \downarrow{\scriptstyle \psi_\rho} \\ Y & = & Y \end{array}\,.$$

4) *Change of abelian variety*: Let A'_K, B'_K be a second pair of abelian varieties over K and $f : A'_K \to A_K$, $g : B'_K \to B_K$ homomorphisms over K. Denote P'_K the biextension of (A'_K, B'_K) by $\mathbb{G}_{m,K}$ obtained as the pullback of P_K along $f \times g$

$$\begin{array}{ccc} P'_K & \xrightarrow{\phi} & P_K \\ \downarrow & & \downarrow \\ A'_K \times B'_K & \xrightarrow{f\times g} & A_K \times B_K \end{array}\,.$$

If ψ'_ρ denotes the canonical $\rho-$splitting of $P'_K(K)$ then

$$\psi'_\rho = \psi_\rho \circ \phi\,.$$

5) *Symmetry*: The biextension P_K induces a biextension ${}^sP_K$ of (B_K, A_K) by $\mathbb{G}_{m,K}$. We have a canonical bijection ${}^sP_K(K) \cong P_K(K)$, switching the two extension structures in Definition 1.2.1. So, any $\rho-$splitting of $P_K(K)$ is a $\rho-$splitting of ${}^sP_K(K)$.

2.2 Local pairings on abelian varieties

In this section, we will omit the subscript indicating the base field. We will review the construction of local pairings on zero cycles and divisors from $\rho-$splittings given in [MT83, §2].

Let K be any field, A an abelian variety over K, $A^\vee$ its dual and P_A the Poincaré biextension. Denote $Z_0(A)_0$ the group of zero cycles of degree 0 of A over K and $\mathrm{Div}^0(A)$ the group of (Cartier) divisors of A. Remember that any element $\mathfrak{a} \in Z_0(A)_0$ is a formal finite sum $\mathfrak{a} = \sum n_i(a_i)$, where $\sum n_i = 0$ and $a_i \in A(K)$ for all i. We also have a morphism

$$\begin{aligned} S : Z_0(A)_0 &\to A(K) \\ \sum n_i(a_i) &\mapsto \sum n_i a_i \end{aligned}$$

induced by the group structure of $A(K)$, and a morphism

$$Cl : \mathrm{Div}^0(A) \to A^\vee(K)$$

induced by the isomorphism $A^\vee(K) \cong \mathrm{Pic}^0(A) \cong \mathrm{Div}^0(A)/\,\mathrm{Prin}(A)$, where $\mathrm{Prin}(A)$ denotes the group of principal divisors of A. Denote $(Z_0(A)_0 \times \mathrm{Div}^0(A))' \subset Z_0(A)_0 \times \mathrm{Div}^0(A)$ the subset consisting of pairs $(\mathfrak{a}, D)$ such that $\mathrm{supp}(\mathfrak{a}) \cap \mathrm{supp}(D) = \emptyset$. We have a function

$$\begin{aligned}[\,\cdot\,,\,\cdot\,,\,\cdot\,] : (Z_0(A)_0 \times \mathrm{Div}^0(A))' \times K^* &\to P_A(K) \\ (\mathfrak{a}, D, c) &\mapsto c + \sum n_i s_D(a_i)\,,\end{aligned}$$

where, if $d := Cl(D) \in A^\vee(K)$, then s_D is a rational section of the invertible sheaf on A corresponding to $P_{A,d}$ (s_D is determined up to multiplication by an element in K^* and its domain is $A - \mathrm{supp}(D)$). There is the following

Proposition 2.2.1 ([MT83, §2.1])**.** $[\,\cdot\,,\,\cdot\,,\,\cdot\,]$ *is well-defined and surjective.*

The previous result says that every element in $P_A(K)$ can be expressed as $[\mathfrak{a}, D, c]$, although not in a unique way.

The function $[\,\cdot\,,\,\cdot\,,\,\cdot\,]$ satisfies the following properties (see [MT83, §2.1]):

1) $p([\mathfrak{a}, D, c]) = (S(\mathfrak{a}), Cl(D))$, where $p : P_A(K) \to A(K) \times A^\vee(K)$ is the structural morphism.

2) $[\mathfrak{a}, D, c] = c + [\mathfrak{a}, D, 1]$.

3) $[\mathfrak{a}, D_1, 1] + [\mathfrak{a}, D_2, 1] = [\mathfrak{a}, D_1 + D_2, 1]$.

4) $[\mathfrak{a}_1, D, 1] + [\mathfrak{a}_2, D, 1] = [\mathfrak{a}_1 + \mathfrak{a}_2, D, 1]$.

5) If f is a rational function on A such that $\mathrm{supp}(f) \cap \mathrm{supp}(\mathfrak{a}) = \emptyset$ then

$$[\mathfrak{a}, (f), 1] = [\mathfrak{a}, 0, f(\mathfrak{a})]\,,$$

where $f(\mathfrak{a}) := \prod f(a_i)^{n_i}$ if $\mathfrak{a} = \sum n_i(a_i)$.

6) For each $D \in \mathrm{Div}^0(A)$ and each $a_0 \in (A - \mathrm{supp}(D))(K)$ there is a $K - morphism$

$$\begin{aligned}g_{a_0,D} : A - \mathrm{supp}(D) &\to P \\ a &\mapsto [(a) - (a_0), D, 1]\,.\end{aligned}$$

7) For $a \in A(K)$ we have

$$[\mathfrak{a}_a, D_a, c] = [\mathfrak{a}, D, c]\,,$$

where $\mathfrak{a}_a$ and D_a denote the images of $\mathfrak{a}$ and D, respectively, under translation by a.

Notice that properties 2, 3 and 4 imply that multiplication on $P_{A,a}(K)$, for $a := S(\mathfrak{a})$, is given by

$$[\mathfrak{a}, D_1, c_1] + [\mathfrak{a}, D_2, c_2] = [\mathfrak{a}, D_1 + D_2, c_1c_2]$$

with unit $[\mathfrak{a}, 0, 1]$, and that multiplication on $P_d(K)$, for $d := Cl(D)$, is given by

$$[\mathfrak{a}_1, D, c_1] + [\mathfrak{a}_2, D, c_2] = [\mathfrak{a}_1 + \mathfrak{a}_2, D, c_1c_2]$$

with unit $[0, D, 1]$.

Proposition 2.2.2. *Properties 1-6, for K and its algebraic extensions, characterize the function $[\,\cdot\,,\,\cdot\,,\,\cdot\,]$.*

Proof. Suppose $[\,\cdot\,,\,\cdot\,,\,\cdot\,]_1$ and $[\,\cdot\,,\,\cdot\,,\,\cdot\,]_2$ are two functions satisfying properties 1-6 above and define

$$\begin{aligned}\delta : (Z_0(A)_0 \times \mathrm{Div}^0(A))' \times K^* &\to K^* \\ (\mathfrak{a}, D, c) &\mapsto [\mathfrak{a}, D, c]_1 - [\mathfrak{a}, D, c]_2\,.\end{aligned}$$

Indeed, $\delta(\mathfrak{a}, D, c) \in K^*$ because $[\mathfrak{a}, D, c]_1, [\mathfrak{a}, D, c]_2 \in p^{-1}(\{S(\mathfrak{a})\} \times \{Cl(D)\}) \cong K^*$ and so there is a unique $c' \in K^*$ such that $[\mathfrak{a}, D, c]_1 = [\mathfrak{a}, D, c]_2 + c'$.

Notice that properties 3 and 4 make δ additive on $\mathfrak{a}$ and D. Property 2 implies that $\delta(\mathfrak{a}, D, c) = \delta(\mathfrak{a}, D, 1)$, and so $\delta(\mathfrak{a}, D, c)$ is independent of c. By property 5, we also have that $\delta(\mathfrak{a}, D, c)$ only depends on the class of D and not on D itself. To verify this, let $D \in \mathrm{Div}^0(A)$ be a divisor and f a rational function on A such that D, as well as (f), have support disjoint from $\mathfrak{a}$. Then we have

$$\begin{aligned}\delta(\mathfrak{a}, D + (f), 1) &= \delta(\mathfrak{a}, D, 1) + \delta(\mathfrak{a}, (f), 1) \\ &= \delta(\mathfrak{a}, D, 1) + \delta(\mathfrak{a}, 0, f(\mathfrak{a}))\,.\end{aligned}$$

But $\delta(\mathfrak{a}, 0, f(\mathfrak{a})) = \delta(\mathfrak{a}, 0, 1) = 0$, since both $[\mathfrak{a}, 0, 1]_1$ and $[\mathfrak{a}, 0, 1]_2$ are the unit of $P_{A,a}(K)$, where $a := S(\mathfrak{a})$.

Now choose divisors D_i such that $Cl(D_i) = Cl(D)$, 0 belongs to $(A - \mathrm{supp}(D_i))(K)$ and $\bigcap \mathrm{supp}(D_i) = \emptyset$. For each i we define

$$\begin{aligned}h_i := g^1_{0,D_i} - g^2_{0,D_i} : A - \mathrm{supp}(D_i) &\to \mathbb{G}_m \\ a &\mapsto \delta((a) - (0), D_i, 1)\,,\end{aligned}$$

where g^j_{0,D_i} is the morphism in property 6 associated to $[\ ,\ ,\ ]_j$. As we have noted, δ depends only on the class of D, so the h_i fit together to define a morphism $h : A \to \mathbb{G}_m$. Since A is proper and $\mathbb{G}_m$ is affine then h is constant. But we know that $h(0) = 0$, so h must be the zero morphism. Finally, since the cycles $(a) - (0)$ generate $Z_0(A)_0$ and δ is additive on the first term, we have that $\delta = 0$. □

Now consider a homomorphism $\rho : K^* \to Y$, with Y an arbitrary abelian group. From a $\rho-$spltting $\psi : P_A(K) \to Y$ of $P_A(K)$ we can define a function

$$\begin{aligned}[\,\cdot\,,\,\cdot\,]_\psi : (Z_0(A)_0 \times \mathrm{Div}^0(A))' &\to Y\\ (\mathfrak{a}, D) &\mapsto \psi([\mathfrak{a}, D, 1])\,.\end{aligned}$$

This map satisfies the following properties:

a) $[\,\cdot\,,\,\cdot\,]_\psi$ is biadditive.

b) $[\mathfrak{a}, (f)]_\psi = \rho(f(\mathfrak{a}))$.

c) $[\mathfrak{a}_a, D_a]_\psi = [\mathfrak{a}, D]_\psi$.

Conversely, from a function $[\,\cdot\,,\,\cdot\,] : (Z_0(A)_0 \times \mathrm{Div}^0(A))' \to Y$ which satisfies a)-c) we can obtain a $\rho-$splitting

$$\begin{aligned}\psi : P_A(K) &\to Y\\ [\mathfrak{a}, D, c] &\mapsto \rho(c) + [\mathfrak{a}, D]\,.\end{aligned}$$

These two constructions are inverses of each other.

If K is a field which is complete with respect to an archimedean or discrete valuation and we are in one of the cases of Theorem 2.1.6 in which there exists a canonical $\rho-$splitting ψ_ρ then we define the *canonical $\rho-$pairing* as

$$\begin{aligned}[\,\cdot\,,\,\cdot\,]_\rho : (Z_0(A)_0 \times \mathrm{Div}^0(A))' &\to Y\\ (\mathfrak{a}, D) &\mapsto \psi_\rho([\mathfrak{a}, D, 1])\,.\end{aligned}$$

Remark 2.2.3. If $v(x) = -\log|x|$ then the canonical $v-$pairing $[\mathfrak{a}, D]_v$ coincides with Néron's symbol $(D, \mathfrak{a})_v$ (see [MT83, Prop. 2.3.1]).

2.3 Global pairing on abelian varieties

Let F be a global field endowed with a set of places which are either archimedean or discrete satisfying that $|c|_v = 1$, for every $c \in F^*$ and almost all places v. For each place v let F_v denote the completion of F with respect to v; for v discrete denote R_v the ring of integers of F_v. Consider a homomorphism $\rho = (\rho_v) : \mathbb{A}_F^* \to Y$ from the invertible elements of the ring of adeles $\mathbb{A}_F$ of F to an abelian group Y which annihilates the image of R_v^*, for almost all discrete places v, as well as the image of F^* under the canonical homomorphisms, and satisfies the "sum formula" $\sum_v \rho_v(c) = 0$ for all $c \in F^*$.

Let A_F and B_F be abelian varieties over F and P_F a biextension of (A_F, B_F) by $\mathbb{G}_{m,F}$. Suppose we are given, for each place v, a ρ_v-splitting ψ_v of $P_{F_v}(F_v)$ such that $\psi_v(P(R_v)) = 0$, for almost all v. In this situation we have the following result.

Lemma 2.3.1 ([MT83, Lemma 3.1]). *With the above notation, there is a pairing*

$$\langle \cdot\,,\,\cdot \rangle : A_F(F) \times B_F(F) \to Y$$

such that if $x \in P_F(F)$ lies above $(a,b) \in A_F(F) \times B_F(F)$ then

$$\langle a, b \rangle = \sum_v \psi_v(x_v)\,,$$

where $x_v \in P_{F_v}(F_v)$ is the image of x under the inclusion $F \subset F_v$.

Proof. First, we check that the sum defining $\langle \cdot\,,\,\cdot \rangle$ is finite. We have that for every $x \in P_F(F)$ we can find a finitely generated subring $S \subset F$ satisfying the following: there exist abelian schemes A_S, B_S over S and a biextension P_S of (A_S, B_S) by $\mathbb{G}_{m,S}$ whose pullbacks along $\operatorname{Spec} F \to \operatorname{Spec} S$ are A_F, B_F and P_F, respectively, and such that $x \in P_S(S)$. Fix $x \in P_F(F)$ and let $c_1, \ldots, c_n \in F^*$ be the generators of the subring S satisfying the previous property. Then $S \subset R_v$ for almost all v, since $|c_i|_v = 1$ for all i and almost all v. This means that we can define a map $\psi : P_F(F) \to Y$ by

$$\psi(x) = \sum_v \psi_v(x_v)\,.$$

Notice that ψ is constant on each fiber of $P_F(F)$ over $A_F(F) \times B_F(F)$ because of the sum formula: for all $c \in F^*$ we have

$$\begin{aligned}
\psi(x+c) &= \sum_v \psi_v(x_v + c)\\
&= \sum_v \psi_v(x_v) + \sum_v \rho_v(c)\\
&= \sum_v \psi_v(x_v)\\
&= \psi(x).
\end{aligned}$$

Then ψ induces a map from the image of $P_F(F) \to A_F(F) \times B_F(F)$ to Y. Since P_F is a line bundle on $A_F \times B_F$ minus its zero section, we have local sections, which implies that $P_F(F) \to A_F(F) \times B_F(F)$ is surjective. Therefore, ψ defines a map $A_F(F) \times B_F(F) \to Y$ which must be additive because each ψ_v is a ρ_v-splitting. □

If, in the situation of Lemma 2.3.1, the ρ_v-splitting ψ_v is the canonical ρ_v-splitting, for all v, then the pairing $\langle \cdot\,,\,\cdot \rangle$ is called the *canonical $\rho-$pairing.*

Chapter 3

Pairings via splittings of the Hodge filtration

Let K be a field which is the completion of a number field with respect to a non-archimedean valuation v over a prime p. Consider an abelian variety A_K over K with dual $A_K^\vee$ and let P_{A_K} be the Poincaré biextension of $(A_K, A_K^\vee)$ by $\mathbb{G}_{m,K}$. Recall that the Hodge filtration of $\mathrm{H}^1_{\mathrm{dR}}(A_K)$ is given by the following extension

$$0 \to \mathrm{H}^0(A_K, \Omega^1_{A_K}) \to \mathrm{H}^1_{\mathrm{dR}}(A_K) \to \mathrm{H}^1(A_K, \mathcal{O}_{A_K}) \to 0,$$

which can be identified with the exact sequence of Lie algebras associated to the universal vectorial extension of $A_K^\vee$ (see [MM74, §4])

$$0 \to V(A_K^\vee) \to A_K^{\vee\natural} \to A_K^\vee \to 0.$$

By Proposition 1.3.10, we know that $A_K^{\vee\natural}$ represents a sheaf whose global sections are extensions of A_K by $\mathbb{G}_{m,K}$ endowed with an integrable connection. By [MM74, §3] these in turn correspond to rigidified extensions, *i.e.* extensions E of A_K by $\mathbb{G}_{m,K}$ together with an homomorphism $\mathrm{Inf}^1(A_K/K) \to E$ of $K-$pointed $K-$schemes making the following diagram commute

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{G}_{m,K} & \longrightarrow & E & \longrightarrow & A_K & \longrightarrow & 0 \\
 & & & & \uparrow & & \uparrow & & \\
 & & & & \mathrm{Inf}^1(A_K/K) & = & \mathrm{Inf}^1(A_K/K) & & ,
\end{array}$$

where $\mathrm{Inf}^1(A_K/K)$ denotes the first infinitesimal neighborhood of the zero section of A_K over K. We have that a rigidification of E is the same as a splitting of the induced sequence of Lie algebras

$$0 \longrightarrow \mathrm{Lie}\,\mathbb{G}_{m,K} \longrightarrow \mathrm{Lie}\,E \longrightarrow \mathrm{Lie}\,A_K \longrightarrow 0 \ . \qquad (t : \mathrm{Lie}\,E \dashrightarrow \mathrm{Lie}\,\mathbb{G}_{m,K})$$

From now on, we will also call *rigidification* such a morphism of Lie algebras.

Let $\rho : K^* \to \mathbb{Q}_p$ be a morphism of Lie groups. Then from a rigidification $t : \mathrm{Lie}\, E \to \mathrm{Lie}\, \mathbb{G}_{m,K}$ as before we can obtain a morphism $\gamma : E(K) \to \mathbb{Q}_p$ of Lie groups extending ρ in such a way that $\mathrm{Lie}\, \gamma = \mathrm{Lie}\, \rho \circ t$; moreover, this is the unique morphism with these properties (see [Zar90, Thm. 3.1.7]). We have two cases:

i) ρ is an unramified morphism, *i.e.* $\rho(R^*) = 0$, where R is the ring of $v-$integers of K: Then $\rho = \rho(\pi)v$, where π is a uniformizer and $v : K^* \to \mathbb{Q}$ is the valuation map (see [Zar90, p. 318]). Since $\mathbb{Q}$ is a discrete Lie group then $\mathrm{Lie}\, v = 0$ and also $\mathrm{Lie}\, \gamma = \mathrm{Lie}\, \rho \circ t = 0$. Therefore, $\gamma : E(K) \to \mathbb{Q}_p$ is the unique morphism of Lie groups extending ρ. In particular, γ does not depend on the rigidification t.

ii) ρ is a ramified morphism: Then $\rho = \delta \circ \lambda$, where $\lambda : K^* \to K$ is a branch of the $p-$adic logarithm and $\delta : K \to \mathbb{Q}_p$ is a $\mathbb{Q}_p-$linear map (see [Zar90, p. 319]). By uniqueness we have that $\gamma = \delta \circ \gamma_\lambda$, where $\gamma_\lambda : E(K) \to K$ is the unique morphism of Lie groups extending λ such that $\mathrm{Lie}\, \gamma_\lambda = \mathrm{Lie}\, \lambda \circ t$. Notice that since $\mathrm{Lie}\, \lambda$ is an isomorphism, we can recover the rigidification t from γ_λ as $t = (\mathrm{Lie}\, \lambda)^{-1} \circ \mathrm{Lie}\, \gamma_\lambda$. This gives us a bijection between rigidifications $t : \mathrm{Lie}\, E \to \mathrm{Lie}\, \mathbb{G}_{m,K}$ of E and morphisms $\gamma : E(K) \to K$ of Lie groups extending λ such that $\mathrm{Lie}\, \gamma_\lambda = \mathrm{Lie}\, \lambda \circ t$.

Now, consider a splitting $r : \mathrm{H}^1(A_K, \mathcal{O}_{A_K}) \to \mathrm{H}^1_{\mathrm{dR}}(A_K)$ of the Hodge filtration of $\mathrm{H}^1_{\mathrm{dR}}(A_K)$. As we said before, this is the same as a splitting of

$$0 \to \mathrm{Lie}(V(A_K^\vee)) \to \mathrm{Lie}(A_K^{\vee\natural}) \to \mathrm{Lie}(A_K^\vee) \to 0$$

which, again by [Zar90, Thm. 3.1.7], comes from a splitting $\eta : A_K^\vee(K) \to A_K^{\vee\natural}(K)$ at the level of groups. Therefore, for each $a^\vee \in A_K^\vee(K)$ we have a splitting $t_{a^\vee}$ of the exact sequence of Lie algebras

$$0 \longrightarrow \mathrm{Lie}\, \mathbb{G}_{m,K} \longrightarrow \mathrm{Lie}\, P_{A_K, a^\vee} \longrightarrow \mathrm{Lie}\, A_K \longrightarrow 0 \qquad \left(t_{a^\vee} : \mathrm{Lie}\, P_{A_K,a^\vee} \dashrightarrow \mathrm{Lie}\, \mathbb{G}_{m,K}\right)$$

which can be extended to a morphism $\gamma_{a^\vee} : P_{A_K, a^\vee}(K) \to \mathbb{Q}_p$. This allows us to define a $\rho-$splitting of $P_{A_K}(K)$ as

$$\begin{aligned} \psi : P_{A_K}(K) &\to \mathbb{Q}_p \\ x &\mapsto \gamma_{a^\vee}(x), \end{aligned} \tag{3.1}$$

where $a^\vee$ is such that $x \in P_{A_K, a^\vee}(K)$ (see [IW03, p. 7]). Observe that in the case that ρ is ramified we have that $\psi = \delta \circ \psi_\lambda$, where $\psi_\lambda : P_{A_K}(K) \to K$ is the $\lambda-$splitting of $P_{A_K}(K)$ obtained from $\eta : A_K^\vee(K) \to A_K^{\vee\natural}(K)$.

Notice that case i) above implies the uniqueness of $\rho-$splittings in the unramified case, which is Mazur and Tate's result (see Theorem 2.1.6, case 2). On the other hand, case ii) says that, in general, when ρ is ramified different splittings of the Hodge filtration give rise to different $\rho-$splittings. A natural thing would be to wonder what splitting of the Hodge filtration of $\mathrm{H}^1_{\mathrm{dR}}(A_K)$ induces Mazur and Tate's canonical $\rho-$splitting in the ramified case. Section 3.1 is devoted to give an answer to this question; we follow Coleman [Col91] in proving that the unit root splitting of the Hodge filtration induces Mazur and Tate's canonical $\lambda-$splitting when A_K has good ordinary reduction. Case ii) above could also suggest that there is even a bijection between splittings $\eta : A_K^\vee(K) \to A_K^{\vee\natural}(K)$ of the Hodge filtration of $\mathrm{H}^1_{\mathrm{dR}}(A_K)$, endowing each fiber $P_{A_K,a^\vee}(K)$ of $P_{A_K}(K)$ with a rigidification, and $\lambda-$splittings $\psi : P_{A_K}(K) \to K$. However, this is not true in general and only happens under certain conditions. To see what these conditions are, consider a $\lambda-$splitting $\psi : P_{A_K}(K) \to K$ and define

$$\begin{aligned} \eta : A_K^\vee(K) &\to A_K^{\vee\natural}(K) \\ a^\vee &\mapsto (P_{A_K,a^\vee}, t_{a^\vee}), \end{aligned} \tag{3.2}$$

where $t_{a^\vee}$ is the rigidification

$$t_{a^\vee} := (\mathrm{Lie}\,\lambda)^{-1} \circ \mathrm{Lie}(\psi|_{P_{A_K,a^\vee}(K)}) : \mathrm{Lie}\,P_{A,a^\vee}(K) \to \mathrm{Lie}\,K \xrightarrow{\cong} \mathrm{Lie}\,K^*.$$

We can see that η is indeed a section of the projection $A_K^{\vee\natural}(K) \to A_K^\vee(K)$. If η were analytic then it would induce a morphism of Lie algebras

$$\mathrm{Lie}\,\eta : \mathrm{Lie}\,A_K^\vee \to \mathrm{Lie}\,A_K^{\vee\natural}$$

and hence a splitting of the Hodge filtration of $\mathrm{H}^1_{\mathrm{dR}}(A_K)$. In this case, this construction would be the converse of the previous one $r \mapsto \psi$. In Section 3.3 we follow the proof in [IW03] that the unit root splitting of the Hodge filtration induces Mazur and Tate's canonical $\lambda-$splitting also when A_K has semistable ordinary reduction. For this, it is necessary to prove first that the map $\eta : A_K^\vee(K) \to A_K^{\vee\natural}(K)$ obtained from Mazur and Tate's canonical $\lambda-$splitting is analytic, and then to show that $\mathrm{Lie}\,\eta$ is the unit root splitting.

3.1 Comparison with Mazur and Tate's construction for the case of good ordinary reduction

When the abelian variety has good ordinary reduction, Mazur and Tate's canonical pairing corresponds to the splitting of the Hodge filtration given by the unit root subspace. This is proved in [Col91]. We explain the proof in this section.

We will fix a base scheme S for the entirety of this section. Let A be an abelian scheme over S and $A^\vee$ its dual. Let $A^\natural$ and $A^{\vee\natural}$ be the universal vectorial extensions of A and $A^\vee$, respectively. They fit into exact sequences of group schemes

$$0 \to \underline{\omega}_{A^\vee} \to A^\natural \xrightarrow{\theta} A \to 0, \tag{3.3}$$

$$0 \to \underline{\omega}_A \to A^{\vee\natural} \xrightarrow{\theta^\vee} A^\vee \to 0, \tag{3.4}$$

where $\underline{\omega}_A$ and $\underline{\omega}_{A^\vee}$ are the sheaves of invariant differentials on A and $A^\vee$, respectively. Notice that we have a canonical isomorphism $\underline{\omega}_A = e^*\Omega^1_{A/S}$, where $e : S \to A$ is the zero section, and similarly for $A^\vee$ (see [BLR90, §4.2]).

We will be working with special types of differentials, which we define next.

Definition 3.1.1. Let X be a scheme over S and H, P schemes over X such that P is an $H-$torsor for the fppf topology on X. Let $m : H \times_X P \to P$ be the $H-$action on P and denote by $pr_1 : H \times_X P \to H$ and $pr_2 : H \times_X P \to P$ the projections.

i) We say that a differential $\eta \in \Omega^1_{P/S}$ is *$H-$invariant* if $m^*(\eta) - pr_2^*(\eta)$ is a global section of $pr_1^*(\Omega^1_{H/S})$.

ii) When $H = G_{m,X}$ and η is a $\mathbb{G}_{m,X}-$invariant differential it follows that $m^*(\eta) - pr_2^*(\eta) = f dz/z$, for some $f \in \mathcal{O}_X(X)$, where z is the standard parameter on $\mathbb{G}_{m,X}$. In this case, f is called the *residue* of η and if $f = 1$ then η is called *normal invariant differential.*

Let A, B and H be commutative group schemes over S and P a scheme over S which has the structure of biextension of (A, B) by H in the category of algebraic groups over S, formal groups over S or Lie groups over Spec$(\mathbb{C}_p)$; in particular, P is an $H_{A\times B}-$torsor over $A \times B$. Let $e_A : A \to P$ and $e_B : B \to P$ denote the zero sections.

iii) We say that $\eta \in \Omega^1_{P/S}$ is *bi-invariant* if it is $H_{A\times B}-$invariant, its images in $\Omega^1_{P/A}$ and $\Omega^1_{P/B}$ are invariant and $d\eta \in \Omega^2_{P/S}$ is the pullback of an invariant 2-form on $A \times B$ (see Definition 1.3.3).

iv) When H is the multiplicative group in the category considered, then $\eta \in \Omega^1_{P/S}$ is called *normal bi-invariant differential* if it is bi-invariant, it has residue one, *i.e.* it is a normal invariant differential, and both $e_A^*(\eta) \in \Omega^1_{A/S}$ and $e_B^*(\eta) \in \Omega^1_{B/S}$ are zero.

We consider the Poincaré biextension P_A of $(A, A^\vee)$. Denote $p : P_A \to A \times A^\vee$ the associated morphism and $p^\natural : P_A^\natural \to A^\natural \times A^{\vee\natural}$ its pullback to $A^\natural \times A^{\vee\natural}$. There exists a canonical connection on $P_A^\natural$ whose curvature is the pullback of the invariant 2-form γ on $A^{\vee\natural} \times A^\natural$ which gives Deligne's pairing $(\,\cdot\,,\,\cdot\,)_A^{Del}$ on the one-dimensional de Rham cohomology. Associated to this connection, there is a canonical normal bi-invariant 1-form $\eta \in \Omega^1_{P_A^\natural/S}$ such that $d\eta = p^{\natural *}\gamma$. Notice that Deligne's pairing in this case is perfect; this motivates the following definition.

As in Section 2.1, we will denote X^f the formal completion at the identity of a smooth commutative group scheme X over S. We have that P_A^f is a biextension of $(A^f, A^{\vee f})$ by $\mathbb{G}_m^f$.

Definition 3.1.2. Let $r : A^f \to A^{\natural f}$, $r^\vee : A^{\vee f} \to A^{\vee \natural f}$ be a pair of formal splittings of (3.3) and (3.4), respectively. We will say that $(r, r^\vee)$ are *dual* if

$$(\,\cdot\,,\,\cdot\,)_A^{Del} \circ (\mathrm{Lie}\, r \times \mathrm{Lie}\, r^\vee) = 0.$$

Theorem 3.1.3. *There is a bijection between pairs of formal splittings of (3.3) and (3.4) and normal bi-invariant differentials on P_A^f. Moreover, dual formal splittings correspond to closed differentials.*

Proof. Let $r : A^f \to A^{\natural f}$ and $r^\vee : A^{\vee f} \to A^{\vee \natural f}$ be formal splittings of (3.3) and (3.4), respectively. From these we obtain a morphism of biextensions $t^f : P_A^f \to P_A^\natural$, which is a formal section of $P_A^\natural \to P_A$

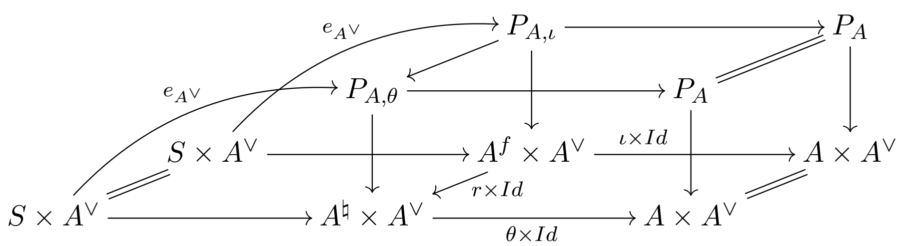
$$\begin{array}{ccccc} P_A^f & \xrightarrow{t^f} & P_A^\natural & \longrightarrow & P_A \\ \downarrow{\scriptstyle p^f} & & \downarrow{\scriptstyle p^\natural} & & \downarrow{\scriptstyle p} \\ A^f \times A^{\vee f} & \xrightarrow{r \times r^\vee} & A^\natural \times A^{\vee\natural} & \longrightarrow & A \times A^\vee \end{array}.$$

Then $t^{f*}\eta$ is a normal bi-invariant differential on P_A^f.

Now suppose that ω is a normal bi-invariant differential on P_A^f. Denote by $\iota : A^f \to A$ and $\iota^\vee : A^{\vee f} \to A^\vee$ the natural immersions. Let $P_{A,\iota}$ be the pullback of P_A to $A^f \times A^\vee$

$$\begin{array}{ccccc} P_A^f & \longrightarrow & P_{A,\iota} & \longrightarrow & P_A \\ \downarrow{\scriptstyle p^f} & & \downarrow & & \downarrow{\scriptstyle p} \\ A^f \times A^{\vee f} & \xrightarrow{Id \times \iota^\vee} & A^f \times A^\vee & \xrightarrow{\iota \times Id} & A \times A^\vee \end{array}.$$

The image of $\omega \in \Omega^1_{P_A^f/S}$ in $\Omega^1_{P_A^f/A^f}$ extends uniquely to an invariant differential $\omega_2 \in \Omega^1_{P_{A,\iota}/A^f}$. We have that ω_2 pulls back to $dz/z \in \Omega^1_{\mathbb{G}_m/S}$, since this is true for ω. Then ω_2 is a normal invariant differential and endows $P_{A,\iota}$ with a $\natural - 2-$structure. This gives an element of $A^\natural(A^f)$, *i.e.* a map $r : A^f \to A^\natural$. Since $r^* : A^\natural(A^\natural) \to A^\natural(A^f)$ maps $Id \mapsto r$ then the pullback of $\eta_2 \in \Omega^1_{P_{A,\theta}/A^\natural}$ along $r \times Id$ is ω_2.



.

Notice that the pullback of $P_{A,\iota}$ to $S \times A^\vee$ is trivialized by $e_{A^\vee} : A^\vee \to P_{A,\iota}$ and that the pullback of ω_2 to $\mathbb{G}_m \times (S \times A^\vee)$ is dz/z, because $e_{A^\vee}^*(\omega_2) = e_{A^\vee}^*(\eta_2) = 0$. Since the differential dz/z corresponds to the trivial $\natural-$structure then this means that the morphism $A^\natural(A^f) \to A^\natural(S)$ sends r to the zero section of $A^\natural$, that is, the composition $S \to A^f \to A^\natural$ equals the zero section and therefore r factors through $A^{\natural f}$.

We follow the same procedure to obtain a map $r^\vee : A^{\vee f} \to A^{\vee\natural f}$. Similarly, if $\omega_1 \in \Omega^1_{PA,\iota^\vee/A^{\vee f}}$ denotes the differential extending the image of ω in $\Omega^1_{P^f_A/A^{\vee f}}$ then the pullback of $\eta_1 \in \Omega^1_{P_{A,\theta^\vee}/A^{\vee\natural}}$ along $Id \times r^\vee : A \times A^{\vee f} \to A \times A^{\vee\natural}$ coincides with ω_1. It remains to prove that r and $r^\vee$ are homomorphisms. This follows from the fact that $r^*\underline{\omega}_{A^\natural/S} \subset \underline{\omega}_{A^f/S}$, $r^{\vee *}\underline{\omega}_{A^{\vee\natural}/S} \subset \underline{\omega}_{A^{\vee f}/S}$ and [Col91, Lemma 2.3].

To prove the last part, observe that from the equalities

$$d\omega_2 = d(r \times Id)^*\eta_2 = (r \times Id)^* d\eta_2,$$
$$d\omega_1 = d(Id \times r^\vee)^*\eta_1 = (Id \times r^\vee)^* d\eta_1$$

we obtain that

$$d\omega = dt^{f*}\eta = t^{f*}d\eta = t^{f*}p^{\natural *}\gamma = p^{f*}(r \times r^\vee)^*\gamma.$$

Since p^f is faithfully flat, ω is closed if and only if $(r \times r^\vee)^*\gamma = 0$, which is equivalent to r and $r^\vee$ being dual with respect to $(\cdot\,,\cdot)^{Del}_A$. □

In the case that S is a scheme over $\mathbb{Q}$, the homomorphisms of commutative formal groups over S are in one-to-one correspondence with homomorphisms of their Lie algebras, which yields the following proposition. Here, dual splittings of the Hodge filtration on the one-dimensional de Rham cohomology of A and $A^\vee$ are defined in a manner analogous to that of Definition 3.1.2.

Proposition 3.1.4. *If S is a $\mathbb{Q}-$scheme, there is a bijection between formal splittings of* (3.3) *and* (3.4) *and splittings of the Hodge filtration*

$$0 \to \underline{\omega}_{A^\vee} \to \underline{\mathrm{H}}^1_{\mathrm{dR}}(A) \to R^1\pi_*(\mathcal{O}_A) \to 0, \tag{3.5}$$
$$0 \to \underline{\omega}_A \to \underline{\mathrm{H}}^1_{\mathrm{dR}}(A^\vee) \to R^1\pi^\vee_*(\mathcal{O}_{A^\vee}) \to 0, \tag{3.6}$$

where $\pi : A \to S$ and $\pi^\vee : A^\vee \to S$ are the structural morphisms, and under this correspondance dual formal splittings correspond to dual splittings.

Proof. This follows from the fact that morphisms of formal group schemes are in bijection with morphisms of their Lie algebras; and from the fact that if X is a commutative group scheme which is an extension of an abelian variety by a vector group then $\mathrm{Lie}(X^f) \cong \mathrm{Lie}(X)$ (see [Iov00, Lemma 2.2]). □

Let $\mathrm{Log} : \mathbb{G}^f_m \to \mathbb{G}^f_a$ be the formal logarithm; this is the unique homomorphism over $\mathbb{Q}$ satisfying $d\,\mathrm{Log}(z) = dz/z$. We have that Log is an isomorphism with inverse $\exp : \mathbb{G}^f_a \to \mathbb{G}^f_m$. We have the following

Proposition 3.1.5. *If S is a $\mathbb{Q}-$scheme, there is a bijection between:*

i) closed normal bi-invariant differentials on P_A^f,

ii) $\operatorname{Log}-$*splittings of* P_A^f, *and*

iii) splittings of P_A^f.

Proof. i) $\implies$ *ii):* Let ω be a closed normal bi-invariant differential on P_A^f. We will prove that the formal solution $\tau : P_A^f \to \mathbb{G}_a^f$ of $d\tau = \omega$ such that $\tau(e) = 0$ is a $\operatorname{Log}-$splitting of P_A^f. Let $m : \mathbb{G}_m \times P_A \to P_A$ be the map giving the $\mathbb{G}_m-$action on P_A and $m_A : P_A \times_A P_A \to P_A$, $m_{A^\vee} : P_A \times_{A^\vee} P_A \to P_A$ the partial group structures. The fact that ω is a $\mathbb{G}_m-$invariant differential with residue one implies that there exists a global section $c \in \Gamma(S, \mathcal{O}_S)$ such that

$$\tau \circ m = \operatorname{Log} \circ p_1 + \tau \circ p_2 + c, \tag{3.7}$$

where $p_1 : \mathbb{G}_m \times P_A \to \mathbb{G}_m$ and $p_2 : \mathbb{G}_m \times P_A \to P_A$ are the projections. On the other hand, the fact that ω is bi-invariant implies that there exist global sections $c_A \in \Gamma(A^f, \mathcal{O}_{A^f})$ and $c_{A^\vee} \in \Gamma(A^{\vee f}, \mathcal{O}_{A^{\vee f}})$ such that

$$\tau \circ m_A = \tau \circ p_{A,1} + \tau \circ p_{A,2} + c_A, \tag{3.8}$$

$$\tau \circ m_{A^\vee} = \tau \circ p_{A^\vee,1} + \tau \circ p_{A^\vee,2} + c_{A^\vee}, \tag{3.9}$$

where $p_{A,i} : P_A \times_A P_A \to P_A$ and $p_{A^\vee,i} : P_A \times_{A^\vee} P_A \to P_A$ are the projections. Evaluating $\tau \circ m$ at $(1, x) \in \mathbb{G}_m^f \times P_A^f$ and from (3.7) we get that

$$\begin{aligned} \tau(x) &= \tau \circ m(1, x) \\ &= \operatorname{Log}(1) + \tau(x) + c \\ &= \tau(x) + c, \end{aligned}$$

and therefore $c = 0$. Now notice that $d(\tau \circ e_A) = e_A^* \omega = 0$ and $\tau(e) = 0$ implies $\tau \circ e_A = 0$. Evualuating (3.8) at (e_A, e_A) we get that

$$\begin{aligned} 0 &= \tau \circ m_A(e_A, e_A) \\ &= \tau \circ e_A + \tau \circ e_A + c_A \\ &= c_A, \end{aligned}$$

and so $c_A = 0$. Similarly, from (3.9) we get that $c_{A^\vee} = 0$. Therefore, τ equals Log when restricted to $\mathbb{G}_m^f$ and is compatible with the partial group structures, which proves our claim.

ii) $\implies$ *iii):* If τ is a $\operatorname{Log}-$splitting of P_A^f then $\exp(\tau)$ is the desired formal splitting of P_A.

iii) $\implies$ *i):* Let $\sigma : P_A^f \to \mathbb{G}_m^f$ be a formal splitting of P_A. Then $\sigma^* dz/z$ is a closed normal bi-invariant differential on P_A^f. □

We can summarize the previous results in the following diagram

$$
\begin{array}{ccc}
\left\{\begin{array}{c}\text{Dual splittings of the}\\ \text{Hodge filtration}\\ \text{(3.5) and (3.6)}\end{array}\right\} & \hookrightarrow & \left\{\begin{array}{c}\text{Splittings of the}\\ \text{Hodge filtration}\\ \text{(3.5) and (3.6)}\end{array}\right\} \\
{\scriptstyle \text{Prop. 3.1.4}}\updownarrow & & \updownarrow{\scriptstyle \text{Prop. 3.1.4}} \\
\left\{\begin{array}{c}\text{Dual formal splittings of}\\ \text{the universal vectorial}\\ \text{extension (3.3) and (3.4)}\end{array}\right\} & \hookrightarrow & \left\{\begin{array}{c}\text{Formal splittings of}\\ \text{the universal vectorial}\\ \text{extension (3.3) and (3.4)}\end{array}\right\} \\
{\scriptstyle \text{Thm. 3.1.3}}\updownarrow & & \updownarrow{\scriptstyle \text{Thm. 3.1.3}} \\
\left\{\begin{array}{c}\text{Closed normal bi-invariant}\\ \text{differentials on } P_A^f\end{array}\right\} & \hookrightarrow & \left\{\begin{array}{c}\text{Normal bi-invariant}\\ \text{differentials on } P_A^f\end{array}\right\} \\
{\scriptstyle \text{Prop. 3.1.5}}\updownarrow & & \\
\left\{\text{Log} - \text{splittings of } P_A^f\right\} & & \\
{\scriptstyle \text{Prop. 3.1.5}}\updownarrow & & \\
\left\{\text{Splittings of } P_A^f\right\} & &
\end{array}
\tag{3.10}
$$

Let $\mathbb{R}_p$ denote the ring of integers of $\mathbb{C}_p$. We will now focus on the case $S = \mathrm{Spec}(\mathbb{R}_p)$. We define the unit root subspace of $\mathrm{H}^1_{\mathrm{dR}}(A_{\mathbb{R}_p})$ as the subspace on which the Frobenius operator acts with slope 0.

Proposition 3.1.6. *If A is an abelian variety over $\mathbb{C}_p$ with good ordinary reduction over $\mathbb{R}_p$ then there exists a unique formal splitting r of* (3.3) *which induces the splitting of*

$$0 \to \underline{\omega}_{A^\vee}(\mathbb{R}_p) \to \mathrm{H}^1_{\mathrm{dR}}(A_{\mathbb{R}_p}) \to \mathrm{H}^1(A_{\mathbb{R}_p}, \mathcal{O}_A) \to 0 \tag{3.11}$$

determined by the unit root subspace of $\mathrm{H}^1_{\mathrm{dR}}(A_{\mathbb{R}_p})$. *Similarly, there exists a unique formal splitting $r^\vee$ of (3.4) which induces the splitting of*

$$0 \to \underline{\omega}_{A}(\mathbb{R}_p) \to \mathrm{H}^1_{\mathrm{dR}}(A^\vee_{\mathbb{R}_p}) \to \mathrm{H}^1(A^\vee_{\mathbb{R}_p}, \mathcal{O}_{A^\vee}) \to 0 \tag{3.12}$$

determined by the unit root subspace of $\mathrm{H}^1_{\mathrm{dR}}(A^\vee_{\mathbb{R}_p})$. *These splittings are dual.*

Proof. This is a consequence of the fact that A^f, and similarly $A^{\vee f}$, is of multiplicative type (since the residue field of $\mathbb{C}_p$ is algebraically closed then A^f and $A^{\vee f}$ are in fact split formal tori, *i.e.* isomorphic to a product of $\mathbb{G}_m^f$). So, the category of biextensions of $(A^f, A^{\vee f})$ by $\mathbb{G}_m^f$ is equivalent to the punctual category, that is, there is a unique biextension of $(A^f, A^{\vee f})$ by $\mathbb{G}_m^f$, which is the trivial one, and this has a unique trivialization. By Proposition 3.1.5 and Theorem 3.1.3, this corresponds to a pair of dual splittings of the universal vectorial extensions (3.3) and (3.4), and by Proposition 3.1.4 these correspond in turn to dual splittings of the Hodge filtration (3.11) and (3.12). □

This proposition implies that, in the case that A has ordinary good reduction, all the sets appearing on the left row in diagram (3.10) contain exactly one element. This uniqueness result is the formal analog to the one given by Mazur and Tate, which is stated in Theorem 2.1.6, case 3 (see [MT83, §1.5.3]). To see this correspondance, we will have to work in the category of rigid analytic group schemes over $\mathbb{C}_p$ and obtain similar constructions.

By [Col91, Lemma 3.1.1], we can uniquely extend the splittings of (3.11) and (3.12) given by the unit root subsapce to splittings

$$0 \longrightarrow \underline{\omega}_{A^\vee}(\mathbb{C}_p) \longrightarrow A^\natural(\mathbb{C}_p) \underset{\overset{s}{\dashleftarrow}}{\longrightarrow} A(\mathbb{C}_p) \longrightarrow 0$$

$$0 \longrightarrow \underline{\omega}_{A}(\mathbb{C}_p) \longrightarrow A^{\vee\natural}(\mathbb{C}_p) \underset{\overset{s^\vee}{\dashleftarrow}}{\longrightarrow} A^\vee(\mathbb{C}_p) \longrightarrow 0 \ .$$

Notice that we can also take the unique formal splittings r and $r^\vee$ of the universal vectorial extensions of A and $A^\vee$ given by Proposition 3.1.6 and extend them to splittings $A(\mathbb{C}_p) \to A^\natural(\mathbb{C}_p)$ and $A^\vee(\mathbb{C}_p) \to A^{\vee\natural}(\mathbb{C}_p)$, respectively. These extensions are also uniquely determined and by construction they coincide with the previous ones, s and $s^\vee$.

From s and $s^\vee$ we obtain a morphism t of biextensions in the category of Lie groups over $\mathbb{C}_p$ that is rigid analytic locally and extends t^f, which is the pullback of $r \times r^\vee$ along p^f

$$\begin{array}{ccc} P_A^f & \xrightarrow{t^f} & P_A^\natural \\ {\scriptstyle p^f}\downarrow & & \downarrow{\scriptstyle p^\natural} \\ A^f \times A^{\vee f} & \xrightarrow{r \times r^\vee} & A^\natural \times A^{\vee\natural} \end{array} \qquad \begin{array}{ccc} P_A(\mathbb{C}_p) & \xrightarrow{t} & P_A^\natural(\mathbb{C}_p) \\ \downarrow & & \downarrow \\ A(\mathbb{C}_p) \times A^\vee(\mathbb{C}_p) & \xrightarrow{s \times s^\vee} & A^\natural(\mathbb{C}_p) \times A^{\vee\natural}(\mathbb{C}_p) \end{array} \ .$$

By [Col91, Prop. 3.1.2], $t^*\eta$ is a closed normal bi-invariant differential on $P_A(\mathbb{C}_p)$ that extends the closed normal bi-invariant differential $t^{f*}\eta$ on P_A^f.

Let $\lambda : \mathbb{C}_p^* \to \mathbb{C}_p$ be a branch of the p-adic logarithm, *i.e.* a locally analytic homomorphism extending $\mathrm{Log} : \mathbb{G}_m^f(\mathbb{C}_p) \to \mathbb{C}_p$. Then the $\mathrm{Log}\,-$splitting $\tau^f : P_A^f(\mathbb{C}_p) \to \mathbb{G}_m^f(\mathbb{C}_p)$ corresponding to $t^{f*}\eta$ by Proposition 3.1.5 can be uniquely extended to a locally analytic $\lambda-$splitting $\tau : P_A(\mathbb{C}_p) \to \mathbb{C}_p$ such that $d\tau = t^*\eta$ (see [Col91, Prop. 3.2.1]).

Proposition 3.1.7. *Suppose A has good ordinary reduction. Then the $\lambda-$splitting ψ : $P_A(\mathbb{C}_p) \to \mathbb{C}_p$ constructed by Mazur and Tate in [MT83, §1.5] is equal to τ.*

Proof. Let $\sigma : P_A^f \to \mathbb{G}_m^f$ be the unique formal splitting of P_A. By construction, $\psi = \mathrm{Log} \circ \sigma$ on $P_A^f(\mathbb{C}_p)$. From the bijections defined in Proposition 3.1.5 we see that $\sigma^* dz/z = t^{f*}\eta$. Since $d\,\mathrm{Log}(z) = dz/z$ then, on $P_A^f(\mathbb{C}_p)$, we have that $d\psi = d(\mathrm{Log} \circ \sigma) = \sigma^* dz/z = t^{f*}\eta = d\tau$. By the uniqueness of the formal solution of $d\tau = t^{f*}\eta$ we obtain the equality

$\psi = \tau$ on $P_A^f(\mathbb{C}_p)$. Since $\psi(e) = \tau(e) = 0$ and they are both analytic then by uniqueness of extensions we conclude that $\psi = \tau$ (see [Col91, Prop. 3.2.1]). □

3.2 Local $p-$adic height pairings on Jacobians of curves

In this section we give the construction of the $p-$adic height pairings on pairs of zero cycles on curves described in [CG89] using splittings of the Hodge filtration. When the curve has good ordinary reduction and the splitting of the Hodge filtration is the unit root splitting this corresponds to the local pairing on zero cycles and divisors defined in Section 2.2, and thus provides a geometric interpretation of said pairing.

Let p be a prime and $\mathbb{Q}_p$ the field of p-adic numbers. As before, we consider a non-archimedean local field K of characteristic 0 with valuation ring R, uniformizer π and residue field $k = R/\pi R$ of order q. We fix a continuous homomorphism

$$\rho : K^* \to \mathbb{Q}_p.$$

If p does not divide q then ρ is trivial on R^*, *i.e.* ρ is unramified. In this case, ρ is determined by its value on π, since the subgroup $(\pi) \times R^*$ has finite index in K^*. Precisely, ρ is given by

$$\rho = \rho(\pi)v,$$

where v is the valuation on K (see [Zar90, p. 318]).

Let C be a complete non-singular, geometrically connected curve over K, and assume that C has a $K-$rational point. We have two cases.

If p does not divide q then there is a unique function

$$[\,\cdot\,,\,\cdot\,] : \big\{(a,b)\,\big|\,a,\,b \text{ are relatively prime divisors of degree 0 on } C\big\} \to \mathbb{Q}_p$$

which is continuous, symmetric, bi-additive (when all the terms are defined) and satisfies

$$[(f), b] = \rho(f(b))$$

for $f \in K(C)^*$ (see [CG89, Prop. 1.2]). The uniqueness of this function gives the equality with the Mazur and Tate's pairing in the unramified case.

The rest of this section will be devoted to studying the case where p divides q.

3.2.1 Splittings and normalized differentials

Definition 3.2.1. A differential on C over K is called:

i) *of the first kind* if it is regular everywhere;

ii) *of the third kind* if it is regular everywhere, except possibly for simple poles with integral residues; and

iii) *logarithmic* if it is of the form df/f, for $f \in K(C)^*$.

We have a short exact sequence corresponding to the Hodge filtration of $\mathrm{H}^1_{\mathrm{dR}}(C)$

$$0 \to \mathrm{H}^{1,0}(C) \to \mathrm{H}^1_{\mathrm{dR}}(C) \to \mathrm{H}^1(C, \mathcal{O}_C) \to 0, \tag{3.13}$$

where $\mathrm{H}^{1,0}(C) = \mathrm{H}^0(C, \Omega^1_{C/K})$ is the group of differentials of the first kind on C over K. Denote $T(K)$ the group of differentials of the third kind on C and $T_l(K)$ the subgroup of $T(K)$ consisting of logarithmic differentials. Then the residual divisor homomorphism gives rise to the exact sequence

$$0 \to \mathrm{H}^{1,0}(C) \to T(K) \xrightarrow{\mathrm{Res}} \mathrm{Div}^0(C) \to 0. \tag{3.14}$$

Since $T_l(K) \cap \mathrm{H}^{1,0}(C) = \{0\}$ and $\mathrm{Res}(df/f) = (f)$, we have an induced exact sequence

$$0 \to \mathrm{H}^{1,0}(C) \to T(K)/T_l(K) \to J(K) \to 0, \tag{3.15}$$

where J is the Jacobian of C. Sequence (3.15) may be identified with the $K-$rational points of the exact sequence corresponding to the universal vectorial extension of J

$$0 \to V(J) \to J^\natural \to J \to 0, \tag{3.16}$$

where $V(J)$ is the vector group associated to the vector space $\mathrm{H}^{1,0}(C) = \mathrm{H}^0(C, \Omega^1_{C/K})$. Since $\mathrm{Lie}(J^\natural) \cong \mathrm{H}^1_{\mathrm{dR}}(C)$ (see [MM74, Prop. 4.1.4]), $\mathrm{Lie}\, V(J) = V(J)(K) = \mathrm{H}^{1,0}(C)$ and $\mathrm{Lie}\, J \cong \mathrm{H}^1(C, \mathcal{O}_C)$, because $\mathrm{H}^1(C, \mathcal{O}_C)$ can be canonically identified with the tangent space at the origin of J, then the resulting exact sequence of Lie algebras associated to (3.16) is sequence (3.13).

For any commutative p-adic Lie group G we have a logarithmic homomorphism from an open subgroup of $G(K)$ to $\mathrm{Lie}\, G$. When $G = J^\natural$ or J this open subgroup has finite index, so it can be uniquely extended to the whole group $\log_G : G(K) \to \mathrm{Lie}\, G$. The following proposition relates the universal vectorial extension of J with the Hodge filtration of C via the logarithm.

Proposition 3.2.2. *There is a canonical homomorphism*

$$\Psi : T(K)/T_l(K) \to \mathrm{H}^1_{\mathrm{dR}}(C)$$

which is the identity on differentials of the first kind and makes the following diagram commute

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V(J)(K) & \longrightarrow & J^\natural(K) & \longrightarrow & J(K) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \Psi = \log_{J^\natural}} & & \downarrow{\scriptstyle \log_J} & & \\ 0 & \longrightarrow & \mathrm{H}^{1,0}(C) & \longrightarrow & \mathrm{H}^1_{\mathrm{dR}}(C) & \longrightarrow & \mathrm{H}^1(C, \mathcal{O}_C) & \longrightarrow & 0 \end{array} .$$

Notice that a splitting $\sigma : \mathrm{H}^1_{\mathrm{dR}}(C) \to \mathrm{H}^{1,0}(C)$ of (3.13) is equivalent to a direct sum decomposition $\mathrm{H}^1_{\mathrm{dR}}(C) = \mathrm{H}^{1,0}(C) \oplus W$, where $W = \ker(\sigma)$. We have the following result.

Proposition 3.2.3. *The choice of W gives a section*

$$\begin{gathered} \mathrm{Div}^0(C) \to T(K) \\ \mathfrak{a} \mapsto \omega_{\mathfrak{a}} \end{gathered}$$

of the residual divisor homomorphism $\mathrm{Res} : T(K) \to \mathrm{Div}^0(C)$. *Moreover, if (f) is a principal divisor then $\omega_{\mathfrak{a}} = df/f$, i.e. it defines a section of $T(k)/T_l(k) \to J(k)$.*

Proof. We define $\omega_{\mathfrak{a}} \in T(K)$ as the differential of the third kind such that $\mathrm{Res}(\omega_{\mathfrak{a}}) = \mathfrak{a}$ and $\Psi(\omega_{\mathfrak{a}}) \in W$. Indeed, $\omega_{\mathfrak{a}}$ exists and is uniquely determined by these two conditions because $\mathrm{Res}^{-1}(\mathfrak{a})$ is a principal homogeneous space for $\mathrm{H}^{1,0}(C)$. □

We will call *normalized differentials* the differentials $\omega_{\mathfrak{a}}$ of the third kind of Proposition 3.2.3 associated to divisors $\mathfrak{a}$ on C. Notice that the section $\mathrm{Div}^0(C) \to T(K)$ of Proposition 3.2.3 is the one obtained by lifting the splitting $\sigma' : \mathrm{H}^1(C, \mathcal{O}_C) \to \mathrm{H}^1_{\mathrm{dR}}(C)$ to the upper row of the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^{1,0}(C) & \longrightarrow & T(K) & \xrightarrow{\mathrm{Res}} & \mathrm{Div}^0(C) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & V(J)(K) & \longrightarrow & J^{\natural}(K) & \longrightarrow & J(K) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \Psi=\log_E} & & \downarrow{\scriptstyle \log_J} & & \\
0 & \longrightarrow & \mathrm{H}^{1,0}(C) & \underset{\sigma}{\longleftarrow\!\!\longrightarrow} & \mathrm{H}^1_{\mathrm{dR}}(C) & \underset{\sigma'}{\longleftarrow\!\!\longrightarrow} & \mathrm{H}^1(C, \mathcal{O}_C) & \longrightarrow & 0 \, .
\end{array}$$

3.2.2 The local pairing

Assume that C has good reduction modulo π, *i.e.* there exists a smooth and proper scheme over $\mathrm{Spec}\, R$ whose generic fiber is C. We fix a direct sum decomposition $\mathrm{H}^1_{\mathrm{dR}}(C) = \mathrm{H}^{1,0}(C) \oplus W$. Since $\rho : K^* \to \mathbb{Q}_p$ takes values in a torsion-free group, its restriction to R^* factors as

$$\begin{array}{ccc}
R^* & \xrightarrow{\rho|_{R^*}} & \mathbb{Q}_p \\
{\scriptstyle \log} \searrow & & \nearrow {\scriptstyle \delta} \\
 & K &
\end{array} \quad ,$$

where $\log : R^* \to K$ is the unique homomorphism extending the convergent series for $\log(1 + x)$ on $1 + \pi R$ (see [Zar90, p. 319]). The map δ is $\mathbb{Q}_p$−linear and uniquely determined by ρ. We fix an extension $\lambda : \mathbb{C}_p^* \to \mathbb{C}_p$ of log which satisfies $\lambda(K^*) \subset K$ and makes the following diagram commute

$$\begin{array}{ccc} K^* & \xrightarrow{\quad \rho \quad} & \mathbb{Q}_p \\ & {\scriptstyle \lambda|_{K^*}} \searrow \quad \nearrow {\scriptstyle \delta} & \\ & K & \end{array} .$$

To construct the pairing between relatively prime divisors of degree 0 on C, we will need to integrate the normalized differentials previously constructed. Consider a differential ω of the third kind on C, denote $\mathfrak{a} := \mathrm{Res}(\omega)$ its residue and let Y be an affinoid obtained from C by removing finitely many residue disks whose union contains $\mathrm{supp}(\mathfrak{a})$. In [CG89, Prop. 4.1] it is stated the existence of a locally analytic function $F : Y(\mathbb{C}_p) \to \mathbb{C}_p$ satisfying $dF = \omega$. Using this function, we can define the integral of ω as

$$\int_{\mathfrak{b}} \omega = \sum (ord_y \mathfrak{b}) F(y),$$

where $\mathfrak{b}$ is a divisor of degree 0 on Y and the sum is taken over $y \in Y(\mathbb{C}_p)$. We can generalize this definition to divisors $\mathfrak{b}$ that are relatively prime to $\mathfrak{a} = \mathrm{Res}(\omega)$ but that may not be supported on any Y; these integrals will depend on the branch of the $p-$adic logarithm chosen.

Let $\mathfrak{a}, \mathfrak{b}$ be relatively prime divisors of degree 0 on C and let $\omega_{\mathfrak{a}}$ be the normalized differential of the third kind determined by W. We define

$$[\mathfrak{a}, \mathfrak{b}] : \delta\bigg(\int_{\mathfrak{b}} \omega_{\mathfrak{a}} \bigg).$$

Proposition 3.2.4. *The function*

$$[\,\cdot\,,\,\cdot\,] : \big\{ (\mathfrak{a}, \mathfrak{b}) \,\big|\, a,\ b \text{ are relatively prime divisors of degree 0 on } C \big\} \to \mathbb{Q}_p$$

is continuous, biadditive and satisfies

$$[(f), \mathfrak{b}] = \rho(f(\mathfrak{b}))$$

for $f \in K(C)^*$. It is symmetric if W is isotropic with respect to the cup product pairing on the first de Rham cohomology group of C.

Proof. The continuity and bilinearity follow from construction. Symmetry follows from the reciprocity law for differentials of the third kind (see [CG89, Prop. 4.5]). □

By [Col91, Thm. 3.3.1], we have that if C has good ordinary reduction and W is the unit root subspace then this is the local pairing on zero cycles and divisors constructed as in Section 2.2 from Mazur and Tate's canonical $\rho-$splitting $P_J(K) \to \mathbb{Q}_p$ of the Poincaré biextension of the Jacobian J of C (see also Proposition 3.1.7). In this case, the local pairing is symmetric.

Remark 3.2.5. The hypothesis of good reduction is necessary for the existence of the locally analytic function $F : Y(\mathbb{C}_p) \to \mathbb{C}_p$ used to define the integral of a normalized differential. Ordinary reduction is used in the comparison to Mazur and Tate's pairing.

3.3 Comparison with Mazur and Tate's construction for the case of semistable ordinary reduction

When the abelian variety has semistable ordinary reduction, Mazur and Tate's canonical pairing corresponds to the splitting of the Hodge filtration given by the unit root subspace, generalizing what we know in the case of good ordinary reduction. In this section we follow the proof of this result as given in [IW03].

3.3.1 Splittings of the Hodge filtration

Let K be a field which is the completion of a number field with respect to a non-archimedean place v over a prime p. Denote R the ring of integers of K and k its residue field. Consider an abelian variety A_K over K with semistable ordinary reduction, let $A_K^\vee$ be its dual and P_{A_K} the Poincaré biextension. Denote $A, A^\vee$ the Néron models of $A_K, A_K^\vee$, respectively, and suppose that the maximal tori in A_k^0 and $A_k^{\vee 0}$ are split.

We recall some facts about the rigid analytic uniformization of A_K and $A_K^\vee$. There is an extension of algebraic groups over K

$$0 \to T_K \xrightarrow{j} G_K \xrightarrow{q} B_K \to 0,$$

such that T_K is a split torus of dimension d over K and B_K is an abelian variety over K with good reduction. We also have a short exact sequence in the category of rigid analytic groups

$$0 \to \Gamma_{\mathrm{rig}} \xrightarrow{i} G_{\mathrm{rig}} \xrightarrow{\pi} A_{\mathrm{rig}} \to 0,$$

where Γ_K is the constant group scheme corresponding to a free $\mathbb{Z}-$module of rank d. Dually, we have an exact sequence of algebraic groups

$$0 \to T_K^\vee \xrightarrow{j^\vee} G_K^\vee \xrightarrow{q^\vee} B_K^\vee \to 0,$$

where $T_K^\vee$ is a split torus and $B_K^\vee$ an abelian variety over K with good reduction; and a short exact sequence of rigid analytic groups

$$0 \to \Gamma_{\mathrm{rig}}^\vee \xrightarrow{i^\vee} G_{\mathrm{rig}}^\vee \xrightarrow{\pi^\vee} A_{\mathrm{rig}}^\vee \to 0.$$

These objects are related in the following way. $\Gamma_K^\vee$ is the character group of T_K, and Γ_K is the character group of $T_K^\vee$; thus $T_K^\vee$ also has dimension d. The extension G_K corresponds to the homomorphism $\Gamma_K^\vee \xrightarrow{i^\vee} G_K^\vee \xrightarrow{q^\vee} B_K^\vee$, and the extension $G_K^\vee$ corresponds to the homomorphism $\Gamma_K \xrightarrow{i} G_K \xrightarrow{q} B_K$. This means that, when considered as rigid 1-motives, $[i^\vee : \Gamma^\vee_{\mathrm{rig}} \to G^\vee_{\mathrm{rig}}]$ is the dual of $[i : \Gamma_{\mathrm{rig}} \to G_{\mathrm{rig}}]$.

Let X_K be a commutative group variety over K. As in the previous sections, we denote by $\underline{\omega}_{X_K} = e^*\Omega^1_{X_K/K}$ the space of invariant differentials of X_K, where $e : \mathrm{Spec}(K) \to X_K$ is the zero section. We have the following commutative diagrams with exact rows and columns, with the rows induced by the Hodge filtration (H is defined as the vector space making the horizontal sequence exact)

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \underline{\omega}_{B_K}(K) & \xrightarrow{i_B} & \mathrm{H}^1_{\mathrm{dR}}(B_K) & \xrightarrow{\delta} & \mathrm{H}^1(B_K, \mathcal{O}_B) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle q^*} & & \cong\downarrow{\scriptstyle \beta} & & \\
0 & \longrightarrow & \underline{\omega}_{G_K}(K) & \xrightarrow{i_G} & \mathrm{H}^1_{\mathrm{dR}}(G_K) & \longrightarrow & H & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle j^*} & & & & \\
 & & \underline{\omega}_{T_K}(K) & \xrightarrow{\cong} & \mathrm{H}^1_{\mathrm{dR}}(T_K) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathrm{Hom}_K(\Gamma, K) & = & \mathrm{Hom}_K(\Gamma, K) & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{\omega}_{A_K}(K) & \xrightarrow{i_A} & \mathrm{H}^1_{\mathrm{dR}}(A_K) & \longrightarrow & \mathrm{H}^1(A_K, \mathcal{O}_A) & \longrightarrow & 0 \\
 & & \cong\downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \gamma} & & \\
0 & \longrightarrow & \underline{\omega}_{G_K}(K) & \xrightarrow{i_G} & \mathrm{H}^1_{\mathrm{dR}}(G_K) & \longrightarrow & H & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & & .
\end{array}
$$

With these diagrams we can lift any splitting $r : \mathrm{H}^1(B_K, \mathcal{O}_B) \to \mathrm{H}^1_{\mathrm{dR}}(B_K)$ of the Hodge filtration of $\mathrm{H}^1_{\mathrm{dR}}(B_K)$ to a splitting $L(r) : \mathrm{H}^1(A_K, \mathcal{O}_A) \to \mathrm{H}^1_{\mathrm{dR}}(A_K)$ of the Hodge filtration of $\mathrm{H}^1_{\mathrm{dR}}(A_K)$. $L(r)$ is then the unique splitting of the Hodge filtration of $\mathrm{H}^1_{\mathrm{dR}}(A_K)$ that makes the following diagram commutative

$$\begin{array}{ccc} \mathrm{H}^1_{\mathrm{dR}}(A_K) & \xleftarrow{L(r)} & \mathrm{H}^1(A_K, \mathcal{O}_A) \\ \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle \gamma} \\ \mathrm{H}^1_{\mathrm{dR}}(G_K) & & H \\ \uparrow{\scriptstyle q^*} & & \downarrow{\scriptstyle \beta^{-1}} \\ \mathrm{H}^1_{\mathrm{dR}}(B_K) & \xleftarrow{r} & \mathrm{H}^1(B_K, \mathcal{O}_B) \end{array} \tag{3.17}$$

For $X_K = A_K$, G_K or B_K, the $K-$vector space $\mathrm{H}^1_{\mathrm{dR}}(X_K)$ can be endowed with a Frobenius operator $\varphi_X : \mathrm{H}^1_{\mathrm{dR}}(X_K) \to \mathrm{H}^1_{\mathrm{dR}}(X_K)$. Let $W_X \subset \mathrm{H}^1_{\mathrm{dR}}(X_K)$ be the unit root subspace, *i.e.* the subspace on which φ_X acts with slope 0. We put $H(X_K) := H$ if $X_K = G_K$ and $H(X_K) := \mathrm{H}^1(X_K, \mathcal{O}_X)$ if $X_K = A_K$ or B_K. Then we define $r_X : H(X_K) \to \mathrm{H}^1_{\mathrm{dR}}(X_K)$ to be the unique splitting such that $\mathrm{Im}(r_X) \subset W_X$. We denote by $s_X : \mathrm{H}^1_{\mathrm{dR}}(X_K) \to \underline{\omega}_{X_K}(K)$ the retraction induced by r_X. Either r_X or s_X will be called "the unit root splitting" of X_K. An important property of r_B is the following.

Theorem 3.3.1. *Let* $r_B : \mathrm{H}^1(B_K, \mathcal{O}_B) \to \mathrm{H}^1_{\mathrm{dR}}(B_K)$ *be the unit root splitting of* B. *Then* $L(r_B)$ *is equal to* r_A, *the unit root splitting of* $\mathrm{H}^1_{\mathrm{dR}}(A_K)$.

Proof. See [IW03, Thm. 2.2]. □

3.3.2 Mazur-Tate height pairing and the unit root splitting

We fix a continuous, ramified homomorphism $\rho : K^* \to \mathbb{Q}_p$. By [Zar90, p. 319], ρ factors as $\rho = \delta \circ \lambda$, where $\lambda : K^* \to K$ is a branch of the $p-$adic logarithm and δ is a $\mathbb{Q}_p-$linear map. Now, let τ_A and τ_B denote the canonical $\lambda-$splittings of $P_{A_K}(K)$ and $P_{B_K}(K)$, respectively, constructed in Theorem 2.1.6, which exist since B_K has good ordinary reduction and A_K has semistable ordinary reduction. Then $\psi_A = \delta \circ \tau_A$ and $\psi_B = \delta \circ \tau_B$ are the canonical $\rho-$splittings inducing the Mazur-Tate height pairings on A_K and B_K, respectively. Notice that since B_K has good ordinary reduction then, by Proposition 3.1.7, τ_B (resp. ψ_B) is the $\lambda-$splitting (resp. $\rho-$splitting) induced by the unit root splitting r_B as in (3.1).

The uniformization maps π and $\pi^\vee$ induce isomorphisms on rigid analytic open subgroups

$$G_{\mathrm{rig}} \supset \bar{G} \cong \bar{A} \subset A_{\mathrm{rig}},$$
$$G^\vee_{\mathrm{rig}} \supset \bar{G^\vee} \cong \bar{A^\vee} \subset A^\vee_{\mathrm{rig}},$$

where $\bar{X}$ denotes the rigid generic fiber of the formal completion of X along its special fiber X_k. By [Wer98, Prop. 3.1], we have a uniquely determined isomorphism of biextensions of $(G_{\mathrm{rig}}, G^\vee_{\mathrm{rig}})$ by $\mathbb{G}_{m,\mathrm{rig}}$

$$\theta : (\pi \times \pi^\vee)^* P_A^{\mathrm{rig}} \to (q \times q^\vee)^* P_B^{\mathrm{rig}}.$$

Let $a^\vee \in \bar{A}^\vee(K)$. Denote $\iota : P^{\text{rig}}_{\bar{A}\times\{a^\vee\}} \hookrightarrow (\pi\times\pi^\vee)^* P^{\text{rig}}_{\bar{A}\times A^\vee}$ the map induced by $\bar{A}\cong\bar{G}\hookrightarrow G$ and $\bar{A}^\vee\cong\bar{G}^\vee\hookrightarrow G^\vee$, and $pr : (q\times q^\vee)^* P^{\text{rig}}_B \to P^{\text{rig}}_B$ the projection. The map θ relates τ_A and τ_B by the following formula

$$\tau_B(pr\circ\theta\circ\iota(x)) = \tau_A(x), \tag{3.18}$$

where $x \in P_{\bar{A}\times\{a^\vee\}}(K)$.

We are interested in the section $\eta_A : A^\vee_K(K) \to A^{\vee\natural}_K(K)$ of $A^{\vee\natural}_K(K) \to A^\vee_K(K)$ which is induced by the Mazur-Tate $\lambda-$splitting τ_A as in (3.2). Equation (3.18) helps us give a description of η_A on rational points of $\bar{A}^\vee_K$ as follows. Let $\eta_B : B^\vee_K(K) \to B^{\vee\natural}_K(K)$ be the unique splitting on $K-$points of the universal vectorial extension of $B^\vee_K$ such that $\operatorname{Lie}\eta_B = r_B$. For $a^\vee \in \bar{A}^\vee_K(K)$, denote the corresponding point in $\bar{G}^\vee_K$ by $g^\vee$ and put $b^\vee := q^\vee(g^\vee)$. Then $\eta_A(a^\vee)$ is the extension $P_{A_K,a^\vee}$ together with the rigidification induced via θ by the one on $P_{B_K,b^\vee}$ given by $\eta_B(b^\vee)$. Notice that, since $A^{\vee\natural}_K$ is an extension of $A^\vee_K$ by a vector group, η_A corresponds to the unique extension to $A^\vee_K(K)$ of the map just described on $\bar{A}^\vee_K(K)$. As mentioned at the beginning of this chapter, we need to prove that η_A is an analytic map.

Lemma 3.3.2. *The $\lambda-$splitting $\tau_A : P_{A_K}(K) \to K$ is an analytic map.*

Proof. See [IW03, Lemma 3.1]. □

Proposition 3.3.3. *Let $\tau : P_A(K) \to K$ be an analytic $\lambda-$splitting à la Mazur-Tate and let $\eta : A^\vee_K(K) \to A^{\vee\natural}_K(K)$ be the section of $A^{\vee\natural}_K(K) \to A^\vee_K(K)$ induced by τ as in* (3.2)*. Then η is an analytic map.*

Proof. See [IW03, Prop. 3.2]. □

Corollary 3.3.4. *The section $\eta_A : A^\vee_K(K) \to A^{\vee\natural}_K(K)$ is analytic.*

Theorem 3.3.5. *The Mazur-Tate height pairing on A_K coincides with the height pairing defined by the unit root splitting r_A on $\mathrm{H}^1_{\mathrm{dR}}(A_K)$.*

Proof. Since by Corollary 3.3.4, η_A is an analytic map, it induces a map of Lie algebras $\operatorname{Lie}\eta_A$ which corresponds to a splitting of the Hodge filtration of A_K. Then, by Theorem 3.3.1, it suffices to show that $\operatorname{Lie}\eta_A = L(r_B)$, *i.e.* that $\operatorname{Lie}\eta_A$ is the unique splitting that makes a diagram like (3.17) commute. This is Theorem 3.6 of [IW03]. □

Chapter 4

Pairings of 1-motives

In this chapter we focus on giving generalizations of the previous results to 1-motives. In order to do this, first we define the set of rational points of a 1-motive, which is done in Section 4.1. Then in Section 4.2 we proceed to study linearizations of biextensions and their compatibility with $\rho-$splittings, since this will be relevant to subsequents sections. In Section 4.3 and 4.4 we analyze the conditions under which it is possible to obtain $\rho-$splittings of the Poincaré biextension of a 1-motive in the unramified and ramified case, respectively. Finally, in Section 4.5 and 4.6 we show how to construct local and global pairings, respectively, from $\rho-$splittings.

4.1 Rational points of 1-motives

Let $M_S = [u_S : L_S \to G_S]$ be a 1-motive over S and $M_S^\vee = [u_S^\vee : L_S^\vee \to G_S^\vee]$ its dual. The following definition is inspired by [Del79, §4.3] (see also [BB]).

Definition 4.1.1. We define the group of *$S-$points of M_S* as

$$M_S(S) := \mathrm{Ext}^1_S(M_S^\vee, \mathbb{G}_{m,S}).$$

By exactness of Cartier duality (see [BK16, Prop. 1.13.5 (c)]) we have canonical isomorphisms

$$\begin{aligned} M_S(S) &:= \mathrm{Ext}^1_S(M_S^\vee, \mathbb{G}_{m,S}) \\ &\cong \mathrm{Hom}_S(M_S^\vee, \mathbb{G}_{m,S}[-1]) \\ &\cong \mathrm{Hom}_S(\mathbb{Z}, M_S) \\ &\cong \mathrm{Ext}^1_S(\mathbb{Z}[1], M_S) \\ &\cong \mathbb{H}^0_{\mathrm{fppf}}(S, M_S), \end{aligned}$$

where the Hom and Ext^1 groups are considered in the derived category $\mathrm{D^b}(S_{\mathrm{fppf}})$ of abelian sheaves on the fppf site of S. Note that for $M_S = [0 \to A_S]$ an abelian scheme over S, the previous isomorphisms reduce to Weil-Barsotti formula

$$\begin{aligned} M_S(S) :=& \operatorname{Ext}^1_S(A^\vee_S, \mathbb{G}_{m,S}) \\ \cong& \operatorname{Hom}_S(\mathbb{Z}, A_S) \\ \cong& A_S(S). \end{aligned}$$

Notice that the short exact sequence of complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & L^\vee_S & = & L^\vee_S & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle u^\vee} & & \downarrow{\scriptstyle v^\vee} & & \\ 0 & \longrightarrow & T^\vee_S & \longrightarrow & G^\vee_S & \longrightarrow & A^\vee_S & \longrightarrow & 0 \end{array}$$

induces a long exact sequence

$$0 \to \operatorname{Hom}_S(M^\vee_S, \mathbb{G}_{m,S}) \to L_S(S) \to G_S(S) \to M_S(S) \to \operatorname{Ext}^1_S(T^\vee_S, \mathbb{G}_{m,S}) \to \dots .$$

Now consider the case $S = \operatorname{Spec}(K)$, for K a field. If $T^\vee_K$ is split, or equivalently if L_K is split, then by the previous exact sequence we have that

$$M_K(K) = G_K(K)/\operatorname{Im}(u_K(K)), \tag{4.1}$$

where $u_K(K)$ is the morphism on $K-$points induced by u_K. In the rest of this chapter we will assume that all tori and lattices are split.

4.2 Linearizations of biextensions

In this section we consider a field K and commutative group schemes over K.

Definition 4.2.1. Let $C = [u : A \to B], C' = [u' : A' \to B']$ be complexes of commutative group schemes over K. Let

$$\sigma : A \times B \to B \tag{4.2}$$
$$(a, b) \mapsto u(a) + b \tag{4.3}$$

be the $A-$action on B induced by u, and let $\sigma' : A' \times B' \to B'$ be defined analogously. An $A \times A'-$*linearization of a biextension* P of (B, B') by $\mathbb{G}_m$ is given by an $A \times A'-$action on P

$$\alpha : (A \times A') \times P \to P$$

satisfying the following conditions:

i) $\mathbb{G}_m-$*equivariance*: the following diagram commutes

$$\begin{array}{ccccc} (A \times A') \times \mathbb{G}_m \times P & \xrightarrow{\cong} & \mathbb{G}_m \times (A \times A') \times P & \xrightarrow{Id \times \alpha} & \mathbb{G}_m \times P \\ {\scriptstyle Id \times m}\downarrow & & & & \downarrow{\scriptstyle m} \\ (A \times A') \times P & & \xrightarrow{\quad\alpha\quad} & & P \end{array} .$$

Equivalently, for $a \in A$, $a' \in A'$, $c \in \mathbb{G}_m$ and $w \in P$,

$$\alpha(a, a', c + w) = c + \alpha(a, a', w).$$

ii) *Compatibility with σ and σ'*: the diagram

$$\begin{array}{ccccc} (A \times A') \times P & & \xrightarrow{\quad\quad\alpha\quad\quad} & & P \\ {\scriptstyle Id\times p}\big\downarrow & & & & \big\downarrow{\scriptstyle p} \\ (A \times A') \times B \times B' & \xrightarrow{\cong} & (A \times B) \times (A' \times B') & \xrightarrow{\sigma\times\sigma'} & B \times B' \end{array}$$

is a pullback, where p is the natural projection. In particular, for $a \in A$ and $a' \in A'$, if $w \in P$ lies above $(b, b') \in B \times B'$ then $\alpha(a, a', w)$ lies above $(\sigma(a, b), \sigma'(a', b'))$.

iii) *Compatibility with the partial group structures of P*: the following diagrams commute

$$\begin{array}{ccc} (A \times A' \times P) \times_{A\times B} (A \times A' \times P) & \xrightarrow{\alpha\times\alpha} & P \times_B P \\ \big\downarrow{\scriptstyle \cong} & & \\ A \times (A' \times A') \times (P \times_B P) & & \big\downarrow{\scriptstyle +_1} \\ {\scriptstyle Id\times +_{A'}\times +_1}\big\downarrow & & \\ A \times A' \times P & \xrightarrow{\quad\quad\alpha\quad\quad} & P \end{array}$$

$$\begin{array}{ccc} (A \times A' \times P) \times_{A'\times B'} (A \times A' \times P) & \xrightarrow{\alpha\times\alpha} & P \times_{B'} P \\ \big\downarrow{\scriptstyle \cong} & & \\ (A \times A) \times A' \times (P \times_{B'} P) & & \big\downarrow{\scriptstyle +_2} \\ {\scriptstyle +_A\times Id\times +_2}\big\downarrow & & \\ A \times A' \times P & \xrightarrow{\quad\quad\alpha\quad\quad} & P \end{array} \quad ,$$

where $+_1$ and $+_2$ denote the partial group structures on the fibers of P. Equivalently, for $a \in A$, $a'_1, a'_2 \in A'$ and $w_1, w_2 \in P$ lying above $b \in B$,

$$\alpha(a, a'_1 + a'_2, w_1 +_1 w_2) = \alpha(a, a'_1, w_1) +_1 \alpha(a, a'_2, w_2),$$

and for $a_1, a_2 \in A$, $a' \in A'$ and $w_1, w_2 \in P$ lying above $b' \in B'$,

$$\alpha(a_1 + a_2, a', w_1 +_2 w_2) = \alpha(a_1, a', w_1) +_2 \alpha(a_2, a', w_2).$$

Conditions i) and ii) are equivalent to α being an $A \times A'-$linearization of the line bundle P in the sense of Definition 1.6 in [MFK94, p. 30]; this can be interpreted as saying that α is a “bundle action” lifting the group actions σ and σ'. In the rest of this chapter, we will only use the term *linearization* in the sense of Definition 4.2.1. Notice that σ is a group homomorphism and, moreover, every $A-$action on B which is a homomorphism is induced by a homomorphism $u : A \to B$ as in (4.2); and similarly for σ'. Condition iii) may then be interpreted as a lifting to P of the compatibility of σ and σ' with the group structures of B and B'.

Proposition 4.2.2. *Let $C = [u : A \to B], C' = [u' : A' \to B']$ be complexes of commutative group schemes over K, P a biextension of (B, B') by $\mathbb{G}_m$ and σ, σ' be the actions induced by u, u' as in (4.2), respectively. Given a biextension structure of (C, C') by $\mathbb{G}_m$ on P with trivializations*

$$\tau_A : A \times B' \to P$$
$$\tau_{A'} : B \times A' \to P$$
$$\tau : A \times A' \to P$$

we can define an $A \times A'-$linearization of P as

$$\alpha_\tau : (A \times A') \times P \to P$$
$$(a, a', w) \mapsto [\tau_{A'}(b, a') +_2 \tau(a, a')] +_1 [w +_2 \tau_A(a, b')],$$

where $w \in P$ lies above $(b, b') \in B \times B'$.

Proof. Notice that by diagram (1.1), which expresses the compatibility between $+_1$ and $+_2$, we have the equality

$$\alpha_\tau(a, a', w) = [\tau_{A'}(b, a') +_1 w] +_2 [\tau(a, a') +_1 \tau_A(a, b')]. \tag{4.4}$$

First, we prove that α_τ is an action. Let $a_1, a_2 \in A$, $a'_1, a'_2 \in A'$ and $w \in P$ lying above $(b, b') \in B \times B'$. Then

$$\begin{aligned}
\alpha_\tau(a_1, a'_1, \alpha_\tau(a_2, a'_2, w)) &= \alpha_\tau(a_1, a'_1, [\tau_{A'}(b, a'_2) +_1 w] +_2 \tau_A(a_2, u'(a'_2) + b')) \\
&= \tau_{A'}(u(a_1) + u(a_2) + b, a'_1) +_1 \{[\tau_{A'}(b, a'_2) +_1 w] \\
&\quad +_2 \tau_A(a_2, u'(a'_2) + b') +_2 \tau_A(a_1, u'(a'_2) + b')\} \\
&= \{\tau_{A'}(b, a'_1) +_2 \tau(a_1 + a_2, a'_1)\} +_1 \{[\tau_{A'}(b, a'_2) +_1 w] \\
&\quad +_2 \tau_A(a_1 + a_2, u'(a'_2) + b')\} \\
&= \{\tau_{A'}(b, a'_1 + a'_2) +_1 w\} +_2 \tau_A(a_1 + a_2, u'(a'_1 + a'_2) + b') \\
&= \alpha_\tau(a_1 + a_2, a'_1 + a'_2, w),
\end{aligned}$$

$$\begin{aligned}
\alpha_\tau(0, 0, w) &= [\tau_{A'}(b, 0) +_2 \tau(0, 0)] +_1 [w +_2 \tau_A(0, b')] \\
&= w.
\end{aligned}$$

Now we prove that α_τ satisfies conditions i)-iii) of Definition 4.2.1.

i) For $c \in \mathbb{G}_m$, $a \in A$, $a' \in A'$ and $w \in P$ lying above $(b, b') \in B \times B'$ we have

$$\begin{aligned}
\alpha_\tau(a, a', c + w) &= [\tau_{A'}(b, a') +_2 \tau(a, a')] +_1 [c + w +_2 \tau_A(a, b')] \\
&= c + [\tau_{A'}(b, a') +_2 \tau(a, a')] +_1 [w +_2 \tau_A(a, b')] \\
&= c + \alpha_\tau(a, a', w).
\end{aligned}$$

ii) Notice that

$$\begin{aligned} pr_B \circ p \circ \alpha_\tau(a, a', w) &= pr_B \circ p(\tau_{A'}(b, a') +_2 \tau(a, a')) \\ &= u(a) + b \\ &= \sigma(a, b). \end{aligned}$$

This equality and the alternate definition of α_τ given by equation (4.4) allows us to conclude that

$$p \circ \alpha_\tau(a, a', w) = (\sigma(a, b), \sigma'(a', b')).$$

iii) Let $w_1, w_2 \in P$ lying over $(b, b'_1), (b, b'_2) \in B \times B'$, respectively. Then

$$\begin{aligned} \alpha_\tau(a, a'_1 + a'_2, w_1 +_1 w_2) &= \tau_{A'}(u(a) + b, a'_1 + a'_2) +_1 [\{w_1 +_1 w_2\} \\ &\quad +_2 \tau_A(a, b'_1 + b'_2)] \\ &= \tau_{A'}(u(a) + b, a'_1 + a'_2) +_1 [\{w_1 +_1 w_2\} \\ &\quad +_2 \{\tau_A(a, b'_1) +_1 \tau_A(a, b'_2)\}] \\ &= \tau_{A'}(u(a) + b, a'_1) +_1 \tau_{A'}(u(a) + b, a'_2) \\ &\quad +_1 \{w_1 +_2 \tau_A(a, b'_1)\} +_1 \{w_2 +_2 \tau_A(a, b'_2)\} \\ &= \alpha_\tau(a, a'_1, w_1) +_1 \alpha_\tau(a, a'_2, w_2). \end{aligned}$$

Using the alternate definition of α_τ given by equation (4.4), it becomes clear from the previous calculations that we also have

$$\alpha_\tau(a_1 + a_2, a', w_1 +_2 w_2) = \alpha_\tau(a_1, a', w_1) +_2 \alpha_\tau(a_2, a', w_2).$$

□

Remark 4.2.3. The previous definition of an $A \times A'-$linearization of P from its biextension structure of (C, C') by $\mathbb{G}_m$ follows [Wer98, p. 306].

Proposition 4.2.4. *Let $C = [u : A \to B], C' = [u' : A' \to B']$ be complexes of commutative group schemes over K, P a biextension of (B, B') by $\mathbb{G}_m$ and σ, σ' be the actions induced by u, u' as in* (4.2)*, respectively. Given an $A \times A'-$linearization*

$$\alpha : (A \times A') \times P \to P$$

of P we can define a biextension structure of (C, C') by $\mathbb{G}_m$ on P as the one determined by the trivializations

$$\begin{aligned} \tau_{\alpha,A} : A \times B' &\to P \\ (a, b') &\mapsto \alpha(a, 0, 0_{b'}) \end{aligned}$$

$$\begin{aligned} \tau_{\alpha,A'} : B \times A' &\to P \\ (b, a') &\mapsto \alpha(0, a', 0_b), \end{aligned}$$

where $0_b, 0_{b'}$ are the zero elements in the groups $(P_b, +_1), (P_{b'}, +_2)$, respectively.

Proof. Since P is already a biextension of (B, B') by $\mathbb{G}_m$ then $\tau_{\alpha,A}, \tau_{\alpha,A'}$ determine a biextension structure on P of $([A \xrightarrow{u} B], [A' \xrightarrow{u'} B'])$ by $\mathbb{G}_{m,K}$ if and only if they are biadditive sections of the corresponding pullbacks of P which coincide on $A \times A'$.

i) *$\tau_{\alpha,A}$ and $\tau_{\alpha,A'}$ are biadditive*: For $a, a_1, a_2 \in A$ and $b', b'_1, b'_2 \in B'$ we have

$$\begin{aligned}\tau_{\alpha,A}(a_1 + a_2, b') &= \alpha(a_1 + a_2, 0, 0_{b'})\\ &= \alpha(a_1, 0, 0_{b'}) +_2 \alpha(a_2, 0, 0_{b'})\\ &= \tau_{\alpha,A}(a_1, b') +_2 \tau_{\alpha,A}(a_2, b'),\end{aligned}$$

$$\begin{aligned}\tau_{\alpha,A}(a, b'_1 + b'_2) &= \alpha(a, 0, 0_{b'_1+b'_2})\\ &= \alpha(a, 0, 0_{b'_1} +_1 0_{b'_2})\\ &= \alpha(a, 0, 0_{b'_1}) +_1 \alpha(a, 0, 0_{b'_2})\\ &= \tau_{\alpha,A}(a, b'_1) +_2 \tau_{\alpha,A}(a, b'_2).\end{aligned}$$

We remark the use of the equality

$$0_{b'_1+b'_2} = 0_{b'_1} +_1 0_{b'_2}. \tag{4.5}$$

This comes from the fact that $+_1$ induces an isomorphism $P_{b'_1} \wedge P_{b'_2} \cong P_{b'_1+b'_2}$, since P is a biextension of (B, B') by $\mathbb{G}_m$. This means that every $w \in P_{b'_1+b'_2}$ is of the form $w = w_1 +_1 w_2$, for some $w_1 \in P_{b'_1}$ and $w_2 \in P_{b'_2}$, from which it is immediate that $0_{b'_1} +_1 0_{b'_2}$ satisfies the characterizing property of the identity $0_{b'_1+b'_2}$ of $(P_{b'_1+b'_2}, +_2)$.

In a similar way, we see that $\tau_{\alpha,A'}$ is biadditive.

ii) *$p \circ \tau_{\alpha,A} = u \times Id$ and $p \circ \tau_{\alpha,A'} = Id \times u'$*: Clearly, we have

$$\begin{aligned}p \circ \tau_{\alpha,A}(a, b') &= p(\alpha(a, 0, 0_{b'}))\\ &= (\sigma(a, 0), \sigma'(0, b'))\\ &= (u(a), b')\\ &= (u \times Id)(a, b').\end{aligned}$$

Similarly, we see that $p \circ \tau_{\alpha,A'} = Id \times u'$.

iii) *$\tau_{\alpha,A} \circ (Id \times u') = \tau_{\alpha,A'} \circ (u \times Id)$*: First, notice that the identity 0_1 of the group $(P_{\{0\}\times B'}, +_1)$ is equal to the identity 0_2 of $(P_{B\times\{0\}}, +_2)$. Indeed, we have

$$0_2 +_1 0_2 = 0_2 = 0_2 +_1 0_1,$$

where the first equality comes from (4.5), which clearly implies $0_1 = 0_2$. We will use the notation $0_0 := 0_1 = 0_2$. Notice also that property iii) of Definition 4.2.1 implies that

$$\alpha(a, 0, \cdot\,) : (P_{\{0\}\times B'}, +_1) \to (P_{\{u(a)\}\times B'}, +_1)$$

is a group homomorphism. In particular,

$$\alpha(a, 0, 0_0) = 0_{u(a)}.$$

By the same reasoning, we also get that

$$\alpha(0, a', 0_0) = 0_{u'(a')}.$$

From this, we are able to do the necessary calculations:

$$\begin{aligned}
\tau_{\alpha,A} \circ (Id \times u')(a, a') &= \tau_{\alpha,A}(a, u'(a')) \\
&= \alpha(a, 0, 0_{u'(a')}) \\
&= \alpha(a, 0, \alpha(0, a', 0_0)) \\
&= \alpha(a, a', 0_0) \\
&= \alpha(0, a', \alpha(a, 0, 0_0)) \\
&= \alpha(0, a', 0_{u(a)}) \\
&= \tau_{\alpha,A'}(u(a), a') \\
&= \tau_{\alpha,A'} \circ (u \times Id)(a, a').
\end{aligned}$$

□

Proposition 4.2.5. *Let $C = [u : A \to B], C' = [u' : A' \to B']$ be complexes of commutative group schemes over K and P a biextension of (B, B') by $\mathbb{G}_m$. Then the map*

$$(\tau_A, \tau_{A'}) \mapsto \alpha_\tau$$

from biextension structures on P of (C, C') by $\mathbb{G}_m$ to $A \times A'-$linearizations of P defined in Proposition 4.2.2 is a bijection. Moreover, its inverse is the map

$$\alpha \mapsto (\tau_{\alpha,A}, \tau_{\alpha,A'})$$

defined in Proposition 4.2.4.

Proof. Let

$$\begin{aligned}
\tau_A &: A \times B' \to P, \\
\tau_{A'} &: B \times A' \to P
\end{aligned}$$

be the trivializations associated to a biextension structure on P of (C, C') by $\mathbb{G}_m$. We will prove that

$$\begin{aligned}
\tau_{\alpha_\tau,A} &= \tau_A, \\
\tau_{\alpha_\tau,A'} &= \tau_{A'}.
\end{aligned}$$

Indeed, for all $a \in A$ and $b' \in B'$ we have

$$\begin{aligned}\tau_{\alpha_\tau,A}(a,b') &= \alpha_\tau(a,0,0_{b'})\\ &= [\tau_{A'}(0,0) +_2 \tau(a,0)] +_1 [0_{b'} +_2 \tau_A(a,b')]\\ &= \tau_A(a,b')\end{aligned}$$

and, similarly, $\tau_{\alpha_\tau,A'}(b,a') = \tau_{A'}(b,a')$, for all $b \in B$ and $a' \in A'$.

Now, let

$$\alpha : (A \times A') \times P \to P$$

be an $A \times A'-$linearization of P. We will prove that

$$\alpha_{\tau_\alpha} = \alpha.$$

First, notice that for all $a \in A$ and $w \in P$ lying over $(b,b') \in B \times B'$ we have

$$\begin{aligned}\alpha_{\tau_\alpha}(a,0,w) &= [\tau_{\alpha,A'}(b,0) +_2 \tau_\alpha(a,0)] +_1 [w +_2 \tau_{\alpha,A}(a,b')]\\ &= \alpha(0,0,w) +_2 \alpha(a,0,0_{b'})\\ &= \alpha(a,0,w)\end{aligned}$$

and, similarly, $\alpha_{\tau_\alpha}(0,a',w) = \alpha(0,a',w)$, for all $a' \in A'$ and $w \in P$. Then

$$\begin{aligned}\alpha_{\tau_\alpha}(a,a',w) &= \alpha_{\tau_\alpha}(a,0,\alpha_{\tau_\alpha}(0,a',w))\\ &= \alpha(a,0,\alpha(0,a',w))\\ &= \alpha(a,a',w).\end{aligned}$$

□

Proposition 4.2.6. *Let $C = [u : A \to B], C' = [u' : A' \to B']$ be complexes of commutative group schemes over K, with u and u' injective, P a biextension of (B,B') by $\mathbb{G}_m$ and σ, σ' the actions induced by u, u', respectively, as in* (4.2)*. Then an $A \times A'-$linearization of P canonically induces a biextension $Q(K)$ of $(B(K)/\operatorname{Im}(u(K)), B'(K)/\operatorname{Im}(u'(K)))$ by K^*.*

Proof. Let α be the $A \times A'-$linearization of P. We know that $P(K)$ is a biextension of abelian groups of $(B(K), B'(K))$ by K^*. Furthermore, it is endowed with the $A(K) \times A'(K)-$linearization

$$\alpha(K) : (A(K) \times A'(K)) \times P(K) \to P(K)$$

coming from α, lifting the actions

$$\begin{aligned}\sigma(K) : A(K) \times B(K) \to B(K),\\ \sigma'(K) : A'(K) \times B'(K) \to B'(K).\end{aligned}$$

Notice that these actions are induced by the morphisms $u(K) : A(K) \to B(K)$, $u'(K) : A'(K) \to B'(K)$. We define $Q(K)$ as the set consisting of the orbits

$$[w] := \{\alpha(a, a', w) | a \in A(K), a' \in A'(K)\}$$

of elements $w \in P(K)$ under $\alpha(K)$. Since the set of orbits $B(K)/A(K)$ of $B(K)$ under the $A(K)-$action $\sigma(K)$ is isomorphic to the quotient of groups $B(K)/\operatorname{Im}(u(K))$, and similarly $B'(K)/A'(K) \cong B'(K)/\operatorname{Im}(u'(K))$, we see that $Q(K)$ is naturally a K^*-torsor over $B(K)/\operatorname{Im}(u(K)) \times B'(K)/\operatorname{Im}(u'(K))$. To see that it is a biextension it is then enough to prove that $+_1$ and $+_2$ induce partial group structures on $Q(K)$. For this, take elements $w_1, w_2 \in P(K)$ lying above $(b_1, b_1'), (b_2, b_2') \in B(K) \times B'(K)$, respectively, and satisfying that the orbits of b_1 and b_2 under σ are equal. This is equivalent to having

$$b_1 = \sigma(a, b_2) = u(a) + b_2, \tag{4.6}$$

for an $a \in A(K)$ (notice that this a is unique, because of the injectivity of u). Then w_1 and $\alpha(a, 0, w_2)$ project to the same element in $B(K)$ and we define

$$[w_1] +_1 [w_2] := [w_1 +_1 \alpha(a, 0, w_2)].$$

Since

$$\begin{aligned} [w_2 +_1 \alpha(-a, 0, w_1)] &= [\alpha(a, 0, w_2 +_1 \alpha(-a, 0, w_1))] \\ &= [\alpha(a, 0, w_2) +_1 w_1] \end{aligned}$$

we see that this definition is commutative. To prove that it is well defined, let $a_1, a_2 \in A(K)$ and $a_1', a_2' \in A'(K)$. If $a \in A(K)$ is the element satisfying (4.6) then, since

$$\begin{aligned} \sigma(a_1, b_1) &= u(a_1) + b_1 \\ &= u(a_1) + (u(a) + b_2) \\ &= u(a + a_1 - a_2) + (u(a_2) + b_2) \\ &= \sigma(a + a_1 - a_2, \sigma(a_2, b_2)), \end{aligned}$$

we have

$$\begin{aligned} [\alpha(a_1, a_1', w_1)] +_1 [\alpha(a_2, a_2', w_2)] &= [\alpha(a_1, a_1', w_1) +_1 \alpha(a + a_1 - a_2, 0, \alpha(a_2, a_2', w_2)] \\ &= [\alpha(a_1, a_1', w_1) +_1 \alpha(a + a_1, a_2', w_2)] \\ &= [\alpha(a_1, a_1', w_1) +_1 \alpha(a_1, a_2', \alpha(a, 0, w_2))] \\ &= [\alpha(a_1, a_1' + a_2', w_1 +_1 \alpha(a, 0, w_2))] \\ &= [w_1 +_1 \alpha(a, 0, w_2)] \\ &= [w_1] +_1 [w_2]. \end{aligned}$$

Therefore, the definition of $[w_1] +_1 [w_2]$ does not depend on the representatives chosen.

Analogously, if $w_1, w_2 \in P(K)$ lie over $(b_1, b'_1), (b_2, b'_2) \in B(K) \times B'(K)$, respectively, and the orbits of b'_1 and b'_2 under σ' are equal then we define

$$[w_1] +_2 [w_2] := [w_1 +_2 \alpha(0, a', w_2)],$$

where $a' \in A'(K)$ is the unique element satisfying $b'_1 = \sigma'(a', b'_2)$. In the same way as before, we see that this defines an abelian partial group structure on $Q(K)$. □

Corollary 4.2.7. *Let $M = [u : L \to G]$ and $M' = [u' : L' \to G']$ be two 1-motives over K, with u and u' injective, and P a biextension of (M, M') by $\mathbb{G}_m$. Then there is a canonical biextension $Q(K)$ of $(M(K), M'(K))$ by K^*.*

Proof. Let α_τ be the $L \times L'$−linearization of P induced by its biextension structure of (M, M') by $\mathbb{G}_m$, as constructed in Proposition 4.2.2. Then α_τ induces by Proposition 4.2.6 a quotient biextension $Q(K)$ of $(M(K), M'(K))$ by K^* (remember that under our hypothesis, equality (4.1) holds). □

When considering a 1-motive $M = [u : L \to G]$ and its dual $M^\vee = [u^\vee : L^\vee \to G^\vee]$ we have a canonical biextension of $(M, M^\vee)$ by $\mathbb{G}_m$, which is the Poincaré biextension P. Thus, by the previous corollary, we have an induced biextension of $(M(K), M^\vee(K))$ by K^*, in the case that u and $u^\vee$ are injective.

Definition 4.2.8. If u and $u^\vee$ are injective, we denote by $Q_M(K)$ the canonical biextension of $(M(K), M^\vee(K))$ by K^* induced by the Poincaré biextension P of $(M, M^\vee)$.

Definition 4.2.9. Let $C = [u : A \to B], C' = [u' : A' \to B']$ be complexes of commutative group schemes over K, P a biextension of (B, B') by $\mathbb{G}_m$. Let Y be an abelian group and $\rho : K^* \to Y$ a homomorphism. We will say that a ρ−splitting $\psi : P(K) \to Y$ of $P(K)$ is *compatible* with an $A \times A'$−linearization α of P if

$$\psi(\alpha_{(a,a')}(w)) = \psi(w),$$

for all $a \in A(K)$, $a' \in A'(K)$ and $w \in P(K)$.

Notice that, assuming u and u' injective, ψ is compatible with an $A \times A'$−linearization if and only if it induces a ρ−splitting on the quotient biextension $Q(K)$.

Proposition 4.2.10. *Let $C = [u : A \to B], C' = [u' : A' \to B']$ be complexes of commutative group schemes over K and P a biextension of (B, B') by $\mathbb{G}_m$. Let Y be an abelian group and $\rho : K^* \to Y$ a homomorphism. Then a ρ−splitting ψ of $P(K)$ is compatible with an $A \times A'$−linearization α of P if and only if*

$$\psi \circ \tau_{\alpha,A}(K) = \psi \circ \tau_{\alpha,A'}(K) = 0.$$

Proof. Suppose ψ is compatible with the $A \times A'$−linearization α. For all $a \in A(K)$ and $b' \in B'(K)$ we have

$$\begin{aligned}\psi(\tau_{\alpha,A}(a, b')) &= \psi(\alpha(a, 0, 0_{b'})) \\ &= \psi(0_{b'}) \\ &= 0,\end{aligned}$$

and similarly for $\tau_{\alpha,A'}$.

Conversely, suppose $\psi \circ \tau_{\alpha,A}(K) = \psi \circ \tau_{\alpha,A'}(K) = 0$. Then for all $a \in A(K)$, $a' \in A'(K)$ and $w \in P(K)$ we have

$$\begin{aligned}\psi(\alpha_{(a,a')}(w)) &= \psi(\tau_{\alpha,A'}(u(a)+b,a') +_1 [w +_2 \tau_{\alpha,A}(a,b')]) \\ &= \psi(\tau_{\alpha,A'}(u(a)+b,a')) + \psi(w +_2 \tau_{\alpha,A}(a,b')) \\ &= \psi(\tau_{\alpha,A'}(u(a)+b,a')) + \psi(w) + \psi(\tau_{\alpha,A}(a,b')) \\ &= \psi(w).\end{aligned}$$

□

4.3 $\rho-$splittings: unramified case

Let K be a field which is complete with respect to a discrete place, with ring of integers R, and let π be a uniformizer. We consider a 1-motive $M_K = [u_K : L_K \to G_K]$ over K with dual $M_K^\vee = [u_K^\vee : L_K^\vee \to G_K^\vee]$

$$M_K = \left[\begin{array}{c} L_K \\ u_K\downarrow \searrow \\ 0 \longrightarrow T_K \to G_K \to A_K \longrightarrow 0 \end{array}\right] \quad M_K^\vee = \left[\begin{array}{c} L_K^\vee \\ u_K^\vee\downarrow \searrow \\ 0 \longrightarrow T_K^\vee \to G_K^\vee \to A_K^\vee \longrightarrow 0 \end{array}\right].$$

In what follows we will assume that M_K is strict. Denote the valuation map as $v : K^* \to \mathbb{Q}$. Since the canonical $v-$splittings are well behaved under finite field extensions, we will even assume that G_K has good reduction in the sense of Definition 1.5.2. This implies that $M_K^\vee$ is also strict and, moreover, that $G_K^\vee$ has good reduction.

We denote by ψ_A the canonical Mazur-Tate $v-$splitting of $P_{A_K}(K)$, which exists by Theorem 2.1.6, case 2. We continue to denote μ the monodromy of M_K, which we have defined as the composition

$$\mu : L_K(K) \times L_K^\vee(K) \xrightarrow{\tau} P_{A_K}(K) \xrightarrow{\psi_A} \mathbb{Q},$$

where τ is the trivialization appearing in the symmetric avatar of M_K (see Section 1.5). Notice that under our assumptions μ factors through $\mathbb{Z}$, since L_K and G_K have good reduction. Denote by ψ the composition

$$\psi : P_K(K) \to P_{A_K}(K) \xrightarrow{\psi_A} \mathbb{Q}.$$

We readily see that this is a $v-$splitting of $P_K(K)$, since it is the composition of a morphism of biextensions and a $v-$splitting.

Lemma 4.3.1. *Let M_K be a 1-motive with good reduction over R. Then ψ is compatible with the $L_K(K) \times L_K^\vee(K)-$linearization of $P_K(K)$ if and only if $L_K = L_K^\vee = 0$.*

Proof. Notice that the Poincaré biextensions P_K and P_{A_K}, as well as the trivializations τ_L and $\tau_{L^\vee}$ of P_K, extend to R, since M_K has good reduction. This yields the following commutative diagram

$$\begin{array}{ccccccc} & & R^* & = & R^* & \xrightarrow{v|_{R^*}=0} & \\ & & \downarrow & & \downarrow & & \\ & \xrightarrow{\tau_L(R)} & P(R) & \longrightarrow & P_A(R) & \xrightarrow[0]{} & \mathbb{Q} \\ & & \downarrow & & \downarrow & & \\ L(R) \times G^\vee(R) & \xrightarrow{u \times Id} & G(R) \times G^\vee(R) & \longrightarrow & A(R) \times A^\vee(R) & & \end{array}.$$

The isomorphism $A_K(K) \cong A(R)$ induces an isomorphism between the quotients

$$\frac{T_K^\vee(K)}{T^\vee(R)} \cong \frac{G_K^\vee(K)}{G^\vee(R)}.$$

Since the restriction of $\psi \circ \tau_L$ to $L_K(K) \times T_K^\vee(K)$ is the map given by Cartier duality

$$L_K(K) \times T_K^\vee(K) \cong \mathrm{Hom}_K(T^\vee, \mathbb{G}_m) \times T_K^\vee(K) \xrightarrow{ev} K^* \xrightarrow{v} \mathbb{Q},$$

where ev denotes the evaluation map, then for every nonzero $x \in L_K(K)$ we can always find $g^\vee \in G_K^\vee(K)$ such that $\psi \circ \tau_L(x, g^\vee) \neq 0$. Therefore, $L_K \neq 0$ implies that ψ is not compatible with the $L_K(K) \times L_K^\vee(K)-$linearization, by Proposition 4.2.10. Since we have the similar statement for $L_K^\vee$, we get that compatibility of ψ with the $L_K(K) \times L_K^\vee(K)-$linearization implies that $L_K = L_K^\vee = 0$. The converse is clear. □

Recall the short exact sequences of lattices (1.14) as defined in Section 1.5.1

$$0 \to \tilde{L}_K \to L_K \to \bar{L}_K \to 0 \quad \text{and} \quad 0 \to \tilde{L}_K^\vee \to L_K^\vee \to \bar{L}_K^\vee \to 0.$$

Let $\{\bar{x}_1, \ldots, \bar{x}_n\}$ and $\{\bar{x}_1^\vee, \ldots, \bar{x}_n^\vee\}$ be basis of $\bar{L}_K(K)$ and $\bar{L}_K^\vee(K)$, respectively. Choose a preimage $x_i \in L_K(K)$ of $\bar{x}_i \in \bar{L}_K(K)$ and $x_i^\vee \in L^\vee(K)$ of $\bar{x}_i^\vee \in \bar{L}^\vee(K)$, for every i. Notice that the set $\{x_1, \ldots, x_n\} \bigcup \tilde{L}_K(K)$ generates $L(K)$ as a $\mathbb{Z}-$module and, similarly, $\{x_1^\vee, \ldots, x_n^\vee\} \bigcup \tilde{L}_K^\vee(K)$ generates $L^\vee(K)$. Define the matrix

$$\Sigma = (\mu(x_i, x_j^\vee))_{i,j}.$$

Notice that it depends only on the $\bar{x}_i, \bar{x}_j^\vee$ and not on their preimages. Observe also that it is invertible in $M_n(\mathbb{Q})$ because it is the matrix associated to the non-degenerate pairing induced by μ on $\bar{L}_K \times \bar{L}_K^\vee$. Thus, we can define a map

$$\psi_M : P_K(K) \to \mathbb{Q}$$
$$w \mapsto \psi(w) - (\psi \circ \tau_L^\vee(g, x_j^\vee))_j \; \Sigma^{-1} \; (\psi \circ \tau_L(x_i, g^\vee))_i^T \;,$$

where $w \in P_K(K)$ lies above $(g, g^\vee) \in G_K(K) \times G_K^\vee(K)$. This definiton follows [Wer98, Prop. 42].

Theorem 4.3.2. *The map $\psi_M : P_K(K) \to \mathbb{Q}$ is a v–splitting of $P_K(K)$ which is compatible with the $L_K(K) \times L_K^\vee(K)$–linearization of $P_K(K)$ if and only if the monodromy pairing μ is non-degenerate. In particular, if u_K and $u_K^\vee$ are injective then it induces a v–splitting of $Q_M(K)$.*

Proof. First we prove that ψ_M is a v–splitting. If $c \in K^*$ and $w \in P_K(K)$ lies above $(g, g^\vee) \in G_K(K) \times G_K^\vee(K)$ then

$$\begin{aligned}
\psi_M(c+w) &= \psi(c+w) - (\psi \circ \tau_L^\vee(g, x_j^\vee))_j \ \Sigma^{-1} \ (\psi \circ \tau_L(x_i, g^\vee))_i^T \\
&= \rho(c) + (\psi(w) - (\psi \circ \tau_L^\vee(g, x_j^\vee))_j \ \Sigma^{-1} \ (\psi \circ \tau_L(x_i, g^\vee))_i^T) \\
&= \rho(c) + \psi_M(w).
\end{aligned}$$

If $w, w' \in P_K(K)$ map to $(g, g^\vee), (g', g^\vee) \in G_K(K) \times G_K^\vee(K)$, resp., then

$$\begin{aligned}
\psi_M(w+w') &= \psi(w+w') - (\psi \circ \tau_L^\vee(g+g', x_j^\vee))_j \ \Sigma^{-1} \ (\psi \circ \tau_L(x_i, g^\vee))_i^T \\
&= \psi(w) + \psi(w') - (\psi \circ \tau_L^\vee(g, x_j^\vee) \\
&\quad + \psi \circ \tau_L^\vee(g', x_j^\vee))_j \ \Sigma^{-1} \ (\psi \circ \tau_L(x_i, g^\vee))_i^T \\
&= (\psi(w) - (\psi \circ \tau_L^\vee(g, x_j^\vee))_j \ \Sigma^{-1} \ (\psi \circ \tau_L(x_i, g^\vee))_i^T \\
&\quad + (\psi(w') - (\psi \circ \tau_L^\vee(g', x_j^\vee))_j \ \Sigma^{-1} \ (\psi \circ \tau_L(x_i, g^\vee))_i^T) \\
&= \psi_M(w) + \psi_M(w').
\end{aligned}$$

The case where w, w' belong to the same fiber over $G_K(K)$ is proved in the same way. Therefore, ψ_M is indeed a v–splitting.

We only prove that $\psi_M \circ \tau_L = 0$, the case $\psi_M \circ \tau_{L^\vee} = 0$ being analogous. Consider the following commutative diagram

$$\begin{array}{ccccc}
\tilde{L}_K \times \bar{T}_K^\vee & \hookrightarrow & \tilde{L}_K \times G_K^\vee & \twoheadrightarrow & \tilde{L}_K \times \tilde{G}_K^\vee \\
\downarrow & & \downarrow & & \downarrow \\
L_K \times \bar{T}_K^\vee & \hookrightarrow & L_K \times G_K^\vee & \twoheadrightarrow & L_K \times \tilde{G}_K^\vee \\
\downarrow & & \downarrow & & \downarrow \\
\bar{L}_K \times \bar{T}_K^\vee & \hookrightarrow & \bar{L}_K \times G_K^\vee & \twoheadrightarrow & \bar{L}_K \times \tilde{G}_K^\vee .
\end{array}$$

We first prove that $\psi_M \circ \tau_L = 0$ on elements of the form $(x_l, g^\vee) \in L(K) \times G^\vee(K)$. Indeed, the equality

$$\psi \circ \tau_L^\vee(u(x_l), x_j^\vee) = \psi_A \circ \tau(x_l, x_j^\vee) = \mu(x_l, x_j^\vee)$$

yields

$$\begin{aligned}
\psi_M \circ \tau_L(x_l, g^\vee) &= \psi \circ \tau_L(x_l, g^\vee) - (\psi \circ \tau_L^\vee(u(x_l), x_j^\vee))_j \ \Sigma^{-1} \ (\psi \circ \tau_L(x_i, g^\vee))_i^T \\
&= \psi \circ \tau_L(x_l, g^\vee) - (\mu(x_l, x_j^\vee))_j \ \Sigma^{-1} \ (\psi \circ \tau_L(x_i, g^\vee))_i^T \\
&= \psi \circ \tau_L(x_l, g^\vee) - \psi \circ \tau_L(x_l, g^\vee) \\
&= 0,
\end{aligned}$$

since $(\mu(x_l, x_j^\vee))_j\, \Sigma^{-1}$ is the $l-$th unit vector. Remember that the images of x_l generate $\bar{L}(K)$, so this implies that $\psi_M \circ \tau_L = 0$ on $L_K(K) \times G_K^\vee(K)$ if and only if $\psi_M \circ \tau_L = 0$ on $\tilde{L}_K(K) \times G_K^\vee(K)$.

Now, we have that $\psi_M \circ \tau_L = \psi \circ \tau_L$ on $\tilde{L}_K(K) \times G_K^\vee(K)$. This follows from the equality

$$(\psi \circ \tau_L^\vee(u(x), x_j^\vee))_j\, \Sigma^{-1}\, (\psi \circ \tau_L(x_i, g^\vee))_i^T = (\mu(x, x_j^\vee))_j\, \Sigma^{-1}\, (\psi \circ \tau_L(x_i, g^\vee))_i^T$$

and the fact that $\mu(x, x_j^\vee) = 0$ for $x \in \tilde{L}(K)$. Observe that this implies that $\psi_M \circ \tau_L = 0$ on $\tilde{L}_K(K) \times G_K^\vee(K)$ if and only if $\psi \circ \tau_L = 0$ on $\tilde{L}_K(K) \times G_K^\vee(K)$.

Let $\bar{t} \in \bar{T}_K^\vee(K)$. Recall that ψ restricted to $L_K \times T_K^\vee$ is Cartier duality. Then since every $x \in \tilde{L}_K(K)$ maps to zero on $\bar{L}_K(K)$, we have

$$\psi \circ \tau_L(x, \bar{t}^\vee) = 0,$$

for all $x \in \tilde{L}_K(K)$. Therefore, the restriction of $\psi \circ \tau_L$ to $\tilde{L}_K(K) \times G_K^\vee(K)$ induces a bilinear map φ on $\tilde{L}_K(K) \times \tilde{G}_K^\vee(K)$

$$\begin{tikzcd}
\tilde{L}_K(K) \times \bar{T}_K^\vee(K) \arrow[r, hook] \arrow[dr, "0"'] & \tilde{L}_K(K) \times G_K^\vee(K) \arrow[r, two heads] \arrow[d, "\psi\circ\tau_L"] & \tilde{L}_K(K) \times \tilde{G}_K^\vee(K) \arrow[dl, dashed, "\varphi"] \\
 & \mathbb{Q} &
\end{tikzcd}.$$

This implies that $\psi \circ \tau_L = 0$ on $\tilde{L}_K(K) \times G_K^\vee(K)$ if and only if $\varphi = 0$.

Now, consider the 1-motive $\tilde{M}_K = [\tilde{L}_K \to G_K]$, which has good reduction over K with dual $\tilde{M}_K^\vee = [L_K^\vee \to \tilde{G}_K^\vee]$ (see Section 1.5.1). By the commutativity of the diagram

$$\begin{tikzcd}
\tilde{L}_K \times G_K^\vee \arrow[r, hook] \arrow[d, two heads] & L_K \times G_K^\vee \arrow[r] \arrow[d, two heads] & G_K \times G_K^\vee \arrow[r, two heads] \arrow[d, two heads] & A_K \times A_K^\vee \arrow[d, equal] \\
\tilde{L}_K \times \tilde{G}_K^\vee \arrow[r, hook] & L_K \times \tilde{G}_K^\vee \arrow[r] & G_K \times \tilde{G}_K^\vee \arrow[r, two heads] & A_K \times A_K^\vee
\end{tikzcd}$$

we see that the pullback along $G_K \times G_K^\vee \to G_K \times \tilde{G}_K^\vee$ of the Poincaré biextension $\tilde{P}_K$ of $(\tilde{M}_K, \tilde{M}_K^\vee)$ is the biextension of $(\tilde{M}_K, M_K^\vee)$ by $\mathbb{G}_{m,K}$ obtained from the Poincaré biextension P_K of $(M_K, M_K^\vee)$ by restricting the trivializations. So, if we denote $\tilde{\psi}$ the $v-$splitting of $\tilde{P}_K(K)$ defined as the composition

$$\tilde{\psi} : \tilde{P}_K(K) \to P_{A_K}(K) \xrightarrow{\psi_A} \mathbb{Q},$$

we have that $\tilde{\psi} \circ \tilde{\tau}_L = \varphi$, where $\tilde{\tau}_L : \tilde{L}_K \times \tilde{G}_K^\vee \to \tilde{P}_K$ denotes the trivialization. Therefore, $\psi \circ \tau_L = 0$ on $\tilde{L}_K(K) \times G_K^\vee(K)$ if and only if $\tilde{\psi} \circ \tilde{\tau}_L = 0$. By Lemma 4.3.1, this happens if and only if $\tilde{L}_K = 0$.

Putting everything together, we see that $\psi_M \circ \tau_L = 0$ if and only if $\tilde{L}_K = 0$. The desired result is then obtained by applying Proposition 4.2.10. $\square$

Corollary 4.3.3. *If the monodromy pairing μ is non-degenerate then for any unramified morphism $\rho : K^* \to \mathbb{Q}$ we can construct a $\rho-$splitting of $P_K(K)$ which is compatible with the $L_K(K) \times L_K^\vee(K)-$linearization. In particular, if u_K and $u_K^\vee$ are injective then there is a canonical $\rho-$splitting of $Q_M(K)$.*

Proof. Since $\rho(R^*) = 0$ then ρ factors through the valuation v

$$\begin{array}{ccc} K^* & \xrightarrow{\quad\rho\quad} & \mathbb{Q} \\ & {}_{v}\searrow \qquad \nearrow_{\delta} & \\ & \mathbb{Q} & \end{array} .$$

Let ψ_M be the $v-$splitting of $P_K(K)$ constructed in Theorem 4.3.2, which is compatible with the $L_K(K) \times L_K^\vee(K)-$linearization. Then clearly $\delta \circ \psi_M$ is a $\rho = \delta \circ v-$splitting which is compatible with the linearization and thus induces a $\rho-$splitting of $Q_M(K)$. $\square$

4.4 $\rho-$splittings: ramified case

Let K be a finite extension of $\mathbb{Q}_p$ and consider a branch $\lambda : K^* \to K$ of the $p-$adic logarithm. For a commutative algebraic group H over K we will denote by $\lambda_H : H(K) \to \operatorname{Lie} H$ the uniquely determined $K-$analytic homomorphism extending λ as constructed in [Zar96]. Let $M = [L \xrightarrow{u} G]$ be a 1-motive over K and $M^\vee = [L^\vee \xrightarrow{u^\vee} G^\vee]$ its dual. Let $M^\natural = [L \xrightarrow{u^\natural} G^\natural]$ and $M^{\vee\natural} = [L \xrightarrow{u^{\vee\natural}} G^{\vee\natural}]$ be their corresponding universal vectorial extensions. Remember that we have exact sequences

$$0 \to \underline{\omega}_{G^\vee} \to G^\natural \to G \to 0, \tag{4.7}$$
$$0 \to \underline{\omega}_G \to G^{\vee\natural} \to G^\vee \to 0. \tag{4.8}$$

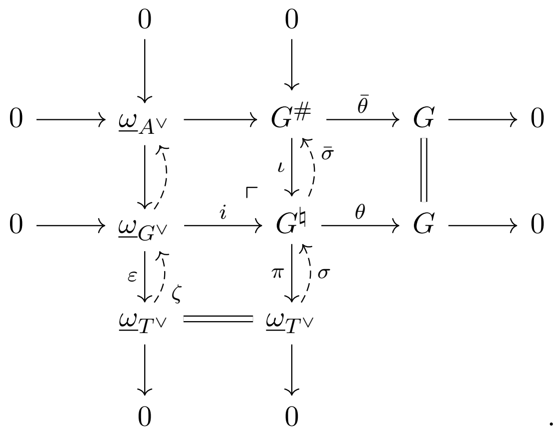
For the rest of this section, we fix splittings of the following exact sequences of vector group schemes over K

$$0 \to \underline{\omega}_A \to \underline{\omega}_G \to \underline{\omega}_T \to 0, \tag{4.9}$$
$$0 \to \underline{\omega}_{A^\vee} \to \underline{\omega}_{G^\vee} \to \underline{\omega}_{T^\vee} \to 0. \tag{4.10}$$

These induce the following isomorphisms:

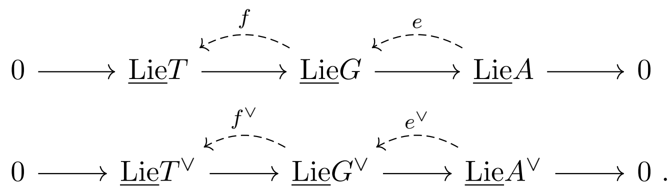
i) $\underline{\omega}_G \cong \underline{\omega}_A \times \underline{\omega}_T$ and $\underline{\omega}_{G^\vee} \cong \underline{\omega}_{A^\vee} \times \underline{\omega}_{T^\vee}$ of vector group schemes.

ii) $G^\natural \cong \underline{\omega}_{T^\vee} \times G^{\#}$ and $G^{\vee\natural} \cong \underline{\omega}_T \times G^{\vee\#}$ of commutative group schemes, obtained by defining $\sigma := i \circ \zeta$ and $\bar{\sigma}$ as the induced retraction in diagram (4.11), and similarly for $G^{\vee\natural}$:

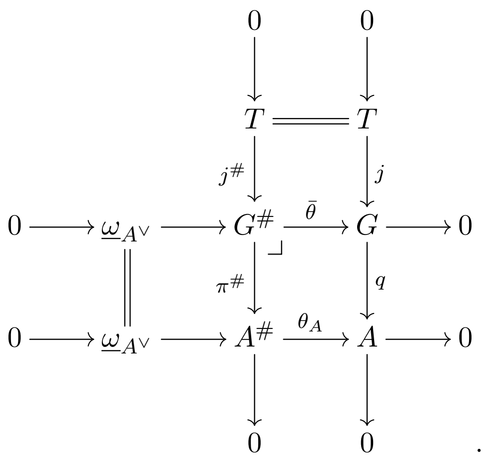


(4.11)

iii) $\underline{\mathrm{Lie}}G \cong \underline{\mathrm{Lie}}A \times \underline{\mathrm{Lie}}T$ and $\underline{\mathrm{Lie}}G^\vee \cong \underline{\mathrm{Lie}}A^\vee \times \underline{\mathrm{Lie}}T^\vee$ as Lie algebras, obtained from i) by duality:



To fix notations, we recall diagram (1.12):



The morphisms in the diagram for $G^{\vee\#}$ are denoted analogously.

We will continue to denote Deligne's pairing associated to M and its dual as

$$(\,\cdot\,,\,\cdot\,)_M^{Del} : \mathrm{T}_{\mathrm{dR}}(M) \times \mathrm{T}_{\mathrm{dR}}(M^\vee) = \underline{\mathrm{Lie}}G^\natural \times \underline{\mathrm{Lie}}G^{\vee\natural} \to \mathbb{G}_a.$$

Deligne's pairing associated to A and its dual will be denoted as

$$(\,\cdot\,,\,\cdot\,)_A^{Del} : \mathrm{T}_{\mathrm{dR}}(A) \times \mathrm{T}_{\mathrm{dR}}(A^\vee) = \underline{\mathrm{Lie}}A^\# \times \underline{\mathrm{Lie}}A^{\vee\#} \to \mathbb{G}_a.$$

We will also be considering a pairing on the toric parts, defined as follows.

Definition 4.4.1. Define $T^\natural := \underline{\omega}_{T^\vee} \times T$ and $T^{\vee\natural} := \underline{\omega}_T \times T^\vee$. Let $\alpha_{T^\vee}$ be the invariant differential of $T^\vee$ over $\underline{\omega}_{T^\vee}$ which corresponds to the identity map on $\underline{\omega}_{T^\vee}$, and define α_T analogously. Denote by Φ_T the pairing on $\underline{\mathrm{Lie}}T^\natural \times \underline{\mathrm{Lie}}T^{\vee\natural}$ determined by the curvature of the invariant differential $\alpha_{T^\vee} + \alpha_T$. We define

$$(\,\cdot\,,\,\cdot\,)_T := -\Phi_T : \underline{\mathrm{Lie}}T^\natural \times \underline{\mathrm{Lie}}T^{\vee\natural} \to \mathbb{G}_a.$$

The following lemma gives an explicit description of $(\,\cdot\,,\,\cdot\,)_T$.

Lemma 4.4.2. *Let $L \cong \mathbb{Z}^r$ and $T \cong \mathbb{G}_m^d$, so that $L^\vee \cong \mathbb{Z}^d$ and $T^\vee \cong \mathbb{G}_m^r$. Then the pairing*

$$(\,\cdot\,,\,\cdot\,)_T : \underline{\mathrm{Lie}}T^\natural \times \underline{\mathrm{Lie}}T^{\vee\natural} \cong (\mathbb{G}_a^r \times \mathbb{G}_a^d) \times (\mathbb{G}_a^d \times \mathbb{G}_a^r) \to \mathbb{G}_a$$

is given by the matrix

$$\Sigma = \begin{array}{c} r\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \\ d\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \end{array} \left(\begin{array}{ccc|ccc} \multicolumn{3}{c}{\overbrace{\hphantom{0\quad\ddots\quad 0}}^{d}} & \multicolumn{3}{c}{\overbrace{\hphantom{-1\quad\ddots\quad 0}}^{r}} \\ 0 & & 0 & -1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & 0 & 0 & & -1 \\ 1 & & 0 & 0 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & 1 & 0 & & 0 \end{array}\right).$$

Proof. In this case we have that the global differential $\alpha_{T^\vee} + \alpha_T$ on $T^\natural \times T^{\vee\natural} = (\mathbb{G}_a^r \times \mathbb{G}_m^d) \times (\mathbb{G}_a^d \times \mathbb{G}_m^r)$ equals

$$\alpha_{T^\vee} + \alpha_T = \sum_{i=1}^r x_i \frac{dt_i}{t_i} + \sum_{j=1}^d y_j \frac{dz_j}{z_j},$$

where x_i (resp. y_j) are the parameters of $\mathbb{G}_a^r$ (resp. $\mathbb{G}_a^d$) and t_i (resp. z_j) are the parameters of $\mathbb{G}_m^r$ (resp. $\mathbb{G}_m^d$) (see [Ber09, Ex. 4.4]), and its curvature is

$$d(\alpha_{T^\vee} + \alpha_T) = \sum_{i=1}^r dx_i \frac{dt_i}{t_i} + \sum_{j=1}^d dy_j \frac{dz_j}{z_j}\,.$$

This 2-form corresponds to the following alternating bilinear map on $(\underline{\mathrm{Lie}}T^\natural \times \underline{\mathrm{Lie}}T^{\vee\natural}) \times (\underline{\mathrm{Lie}}T^\natural \times \underline{\mathrm{Lie}}T^{\vee\natural})$

$$R_T : (\mathbb{G}_a^r \times \mathbb{G}_a^d \times \mathbb{G}_a^d \times \mathbb{G}_a^r) \times (\mathbb{G}_a^r \times \mathbb{G}_a^d \times \mathbb{G}_a^d \times \mathbb{G}_a^r) \to \mathbb{G}_a$$

$$\begin{pmatrix} \{a_i dx_i\}_i, \{b_j \frac{dz_j}{z_j}\}_j, \{c_j dy_j\}_j, \{e_i \frac{dt_i}{t_i}\}_i \\ \{a'_i dx_i\}_i, \{b'_j \frac{dz_j}{z_j}\}_j, \{c'_j dy_j\}_j, \{e'_i \frac{dt_i}{t_i}\}_i \end{pmatrix}^T \mapsto \begin{array}{c} \sum_i a_i e'_i - \sum_i e_i a'_i \\ + \sum_j c_j b'_j - \sum_j b_j c'_j\,. \end{array}$$

Therefore, the bilinear map on $\underline{\mathrm{Lie}}T^\natural \times \underline{\mathrm{Lie}}T^{\vee\natural}$ induced by $d(\alpha_{T^\vee} + \alpha_T)$ is

$$\begin{aligned}\Phi_T(\{a_i dx_i\}_i, \{b_j \frac{dz_j}{z_j}\}_j, \{c'_j dy_j\}_j, \{e'_i \frac{dt_i}{t_i}\}_i) &= R_T \begin{pmatrix} \{a_i dx_i\}_i, \{b_j \frac{dz_j}{z_j}\}_j, 0, 0 \\ 0, 0, \{c'_j dy_j\}_j, \{e'_i \frac{dt_i}{t_i}\}_i \end{pmatrix}^T \\ &= \sum_i a_i e'_i - \sum_j b_j c'_j \end{aligned}$$

and we see that $(\,\cdot\,,\,\cdot\,)_T = -\Phi_T$ is indeed given by the matrix Σ. $\square$

Define

$$\begin{aligned} &j^\natural := Id \times j^\# : T^\natural = \underline{\omega}_{T^\vee} \times T \to \underline{\omega}_{T^\vee} \times G^\# = G^\natural, \\ &q^\natural := \pi^\# \circ \bar{\sigma} : G^\natural \to A^\#, \end{aligned}$$

and similarly for $j^{\vee\natural}, q^{\vee\natural}$. Notice that the exactness of the upper and lower rows in the diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & T & \xrightarrow{j^\#} & G^\# & \xrightarrow{\pi^\#} & A^\# & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T^\natural & \xrightarrow{j^\natural} & G^\natural & \xrightarrow{q^\natural} & A^\# & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & & & \\ & & \underline{\omega}_{T^\vee} & = & \underline{\omega}_{T^\vee} & & & & \\ & & \downarrow & & \downarrow & & & & \\ & & 0 & & 0 & & & & \end{array}$$

induces the exactness of the middle row. We also define

$$f^\natural := Id \times (f \circ \mathrm{Lie}\,\bar{\theta}) : \underline{\mathrm{Lie}}G^\natural = \underline{\omega}_{T^\vee} \times \underline{\mathrm{Lie}}G^\# \to \underline{\mathrm{Lie}}T^\natural = \underline{\omega}_{T^\vee} \times \underline{\mathrm{Lie}}T,$$

and similarly for $f^{\vee\natural}$. We have that $f^\natural$ splits the exact sequence of Lie algebras

$$0 \longrightarrow \mathrm{Lie}\,T^\natural \underset{\mathrm{Lie}\,j^\natural}{\overset{f^\natural}{\rightleftarrows}} \mathrm{Lie}\,G^\natural \xrightarrow[\mathrm{Lie}\,q^\natural]{} \mathrm{Lie}\,A^\# \longrightarrow 0\,,$$

since $(f \circ \mathrm{Lie}\,\bar{\theta}) \circ \mathrm{Lie}\,j^\# = f \circ \mathrm{Lie}\,j = Id$, and we have a similar statement for $f^{\vee\natural}$. Consider the morphisms

$$\underline{\mathrm{Lie}}T^\natural \times \underline{\mathrm{Lie}}T^{\vee\natural} \xleftarrow{f^\natural \times f^{\vee\natural}} \underline{\mathrm{Lie}}G^\natural \times \underline{\mathrm{Lie}}G^{\vee\natural} \xrightarrow{\mathrm{Lie}\,q^\natural \times \mathrm{Lie}\,q^{\vee\natural}} \underline{\mathrm{Lie}}A^\# \times \underline{\mathrm{Lie}}A^{\vee\#}.$$

We have the following

Lemma 4.4.3. *For all* $(h, h^\vee) \in \underline{\mathrm{Lie}}G^\natural \times \underline{\mathrm{Lie}}G^{\vee\natural}$ *we have*

$$(h, h^\vee)^{Del}_M = (f^\natural(h), f^{\vee\natural}(h^\vee))_T + (\mathrm{Lie}\, q^\natural(h), \mathrm{Lie}\, q^{\vee\natural}(h^\vee))^{Del}_A.$$

Proof. The split exact sequence

$$0 \longrightarrow G^{\#} \overset{\bar\sigma}{\underset{\iota}{\longrightarrow}} G^\natural \overset{\sigma}{\underset{\pi}{\longrightarrow}} \underline{\omega}_{T^\vee} \longrightarrow 0$$

induces an isomorphism

$$\begin{aligned} G^\natural(G^\natural) &\cong \underline{\omega}_{T^\vee}(G^\natural) \oplus G^{\#}(G^\natural) \\ Id &\mapsto \pi \oplus \bar\sigma. \end{aligned}$$

Notice that $\bar\sigma \in G^{\#}(G^\natural)$ and $Id \in A^{\#}(A^{\#})$ map to the same element in $A^{\#}(G^\natural)$, namely $q^\natural$:

$$\begin{array}{ccccc} G^{\#}(G^\natural) & \xrightarrow{\pi^{\#}\circ\_} & A^{\#}(G^\natural) & \xleftarrow{\_\circ q^\natural} & A^{\#}(A^{\#}) \\ \bar\sigma & \longmapsto & \pi^{\#} \circ \bar\sigma = q^\natural & \longleftarrow\!\shortmid & Id \\ \| & & \| & & \| \\ ([L^\vee_{G^\natural} \to q^{\natural *} P_{A^{\#}\times A^\vee}], q^{\natural *}\nabla_{A,2}) & & (q^{\natural *} P_{A^{\#}\times A^\vee}, q^{\natural *}\nabla_{A,2}) & & (P_{A^{\#}\times A^\vee}, \nabla_{A,2})\ . \end{array}$$

Let $\beta_{T^\vee}$ be the invariant differential of $G^\vee$ over $G^\natural$ corresponding to $\zeta \circ \pi \in \underline{\omega}_{G^\vee}(G^\natural)$, and define β_T analogously. Then the identity in $G^\natural(G^\natural)$ corresponds to the extension $[L^\vee_{G^\natural} \to P_{G^\natural \times G^\vee}]$ of $[L^\vee_{G^\natural} \to G^\vee_{G^\natural}]$ by $G_{m,G^\natural}$ endowed with a connection ∇_2 which can be expressed as a sum

$$\nabla_2 = \beta_{T^\vee} + q^{\natural *}\nabla_{A,2}.$$

Since we have the dual statement, then the canonical connection ∇ of $P^\natural$ can be expressed as a sum

$$\nabla = (\beta_{T^\vee} + \beta_T) + (q^\natural \times q^{\vee\natural})^* \nabla_A.$$

This together with the split exact sequence

$$0 \longrightarrow \underline{\mathrm{Lie}}T \overset{f}{\underset{\mathrm{Lie}\, j}{\longrightarrow}} \underline{\mathrm{Lie}}G \overset{e}{\underset{\mathrm{Lie}\, q}{\longrightarrow}} \underline{\mathrm{Lie}}A \longrightarrow 0$$

gives the desired result. □

Definition 4.4.4. Let $\eta : G(K) \to G^\natural(K)$ and $\eta^\vee : G^\vee(K) \to G^{\vee\natural}(K)$ be a pair of splittings of the exact sequences

$$0 \to \underline{\omega}_{G^\vee}(K) \to G^\natural(K) \to G(K) \to 0, \tag{4.12}$$

$$0 \to \underline{\omega}_G(K) \to G^{\vee\natural}(K) \to G^\vee(K) \to 0. \tag{4.13}$$

We say that $(\eta, \eta^\vee)$, or also that $(\mathrm{Lie}\, \eta, \mathrm{Lie}\, \eta^\vee)$, are *dual* with respect to Deligne's pairing $(\,\cdot\,,\,\cdot\,)^{Del}_M$ if

$$(\,\cdot\,,\,\cdot\,)^{Del}_M \circ (\mathrm{Lie}\, \eta, \mathrm{Lie}\, \eta^\vee) = 0.$$

We define *dual* splittings with respect to $(\,\cdot\,,\,\cdot\,)^{Del}_A$ and $(\,\cdot\,,\,\cdot\,)_T$ in a similar way.

In order to prove Proposition 4.4.6 below, we will need the following

Lemma 4.4.5. *Let V, G and H be commutative group schemes over K, with V a vector group, such that we have an exact sequence*

$$0 \to V \to G \to H \to 0.$$

Let $r : \operatorname{Lie} H \to \operatorname{Lie} G$ be a splitting of

$$0 \to \operatorname{Lie} V \to \operatorname{Lie} G \to \operatorname{Lie} H \to 0.$$

Then there exists a canonical homomorphism $\eta : H(K) \to G(K)$ which is a splitting of

$$0 \to V(K) \to G(K) \to H(K) \to 0$$

and satisfies that $\operatorname{Lie} \eta = r$.

Proof. By functoriality of the logarithm and the fact that $\lambda_V = Id$, because V is a vector group (see [Zar96]), we have the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & \overset{\bar{r}\circ\lambda_G}{\dashleftarrow} & & \overset{\eta}{\dashleftarrow} & & & \\
0 & \longrightarrow & V(K) & \xrightarrow{\;i\;} & G(K) & \xrightarrow{\;p\;} & H(K) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \lambda_G} & & \downarrow{\scriptstyle \lambda_H} & & \\
0 & \longrightarrow & \operatorname{Lie} V & \xrightarrow{\operatorname{Lie} i} & \operatorname{Lie} G & \xrightarrow{\operatorname{Lie} p} & \operatorname{Lie} H & \longrightarrow & 0\;, \\
 & & & \underset{\bar{r}}{\dashleftarrow} & & \underset{r}{\dashleftarrow} & & &
\end{array}$$

where $\bar{r} : \operatorname{Lie} G \to \operatorname{Lie} V$ is the retraction induced by r and $\eta : H(K) \to G(K)$ is the section induced by $\bar{r} \circ \lambda_G$. Since

$$\begin{aligned}
\operatorname{Lie}(\bar{r} \circ \lambda_G) &= \operatorname{Lie} \bar{r} \circ \operatorname{Lie} \lambda_G \\
&= \bar{r} \circ Id \\
&= \bar{r}
\end{aligned}$$

we have that $\operatorname{Lie} \eta = r$. Finally, to prove that it is a splitting notice that

$$\begin{aligned}
(\bar{r} \circ \lambda_G) \circ i &= \bar{r} \circ \operatorname{Lie} i \\
&= Id.
\end{aligned}$$

Therefore, we also have $p \circ \eta = Id$. □

Proposition 4.4.6. *Let $\eta : G(K) \to G^\natural(K)$ and $\eta^\vee : G^\vee(K) \to G^{\vee\natural}(K)$ be a pair of splittings of the exact sequences* (4.12) *and* (4.13)*, respectively. Then we can define new splittings $\tilde{\eta} : G(K) \to G^\natural(K)$ and $\tilde{\eta}^\vee : G^\vee(K) \to G^{\vee\natural}(K)$ of* (4.12) *and* (4.13)*, splittings $\eta_T : T(K) \to T^\natural(K)$ and $\eta_T^\vee : T^\vee(K) \to T^{\vee\natural}(K)$ of the exact sequences*

$$\begin{aligned}
0 \to \underline{\omega}_{T^\vee}(K) \to T^\natural(K) \to T(K) \to 0, \\
0 \to \underline{\omega}_{T}(K) \to T^{\vee\natural}(K) \to T^\vee(K) \to 0
\end{aligned}$$

and splittings $\eta_A : A(K) \to A^{\#}(K)$ and $\eta_A^\vee : A^\vee(K) \to A^{\vee\#}(K)$ of the exact sequences

$$0 \to \underline{\omega}_{A^\vee}(K) \to A^{\#}(K) \to A(K) \to 0,$$
$$0 \to \underline{\omega}_{A}(K) \to A^{\vee\#}(K) \to A^\vee(K) \to 0$$

such that the following diagram commutes

$$\begin{array}{ccccc}
\mathrm{Lie}\, T \times \mathrm{Lie}\, T^\vee & \xleftarrow{f\times f^\vee} & \mathrm{Lie}\, G \times \mathrm{Lie}\, G^\vee & \xrightarrow{\mathrm{Lie}\, q \times \mathrm{Lie}\, q^\vee} & \mathrm{Lie}\, A \times \mathrm{Lie}\, A^\vee \\
\downarrow{\scriptstyle \mathrm{Lie}\,\eta_T \times \mathrm{Lie}\,\eta_T^\vee} & & \downarrow{\scriptstyle \mathrm{Lie}\,\tilde\eta \times \mathrm{Lie}\,\tilde\eta^\vee} & & \downarrow{\scriptstyle \mathrm{Lie}\,\eta_A \times \mathrm{Lie}\,\eta_A^\vee} \\
\mathrm{Lie}\, T^\natural \times \mathrm{Lie}\, T^{\vee\natural} & \xleftarrow{f^\natural\times f^{\vee\natural}} & \mathrm{Lie}\, G^\natural \times \mathrm{Lie}\, G^{\vee\natural} & \xrightarrow{\mathrm{Lie}\, q^\natural \times \mathrm{Lie}\, q^{\vee\natural}} & \mathrm{Lie}\, A^{\#} \times \mathrm{Lie}\, A^{\vee\#}
\end{array} \tag{4.14}$$

Moreover, if $(\mathrm{Lie}\,\eta, \mathrm{Lie}\,\eta^\vee)$ *are dual with respect to* $(\cdot\,,\cdot)_M^{Del}$ *then* $(\mathrm{Lie}\,\eta_T, \mathrm{Lie}\,\eta_T^\vee)$ *are dual with respect to* $(\cdot\,,\cdot)_T$*,* $(\mathrm{Lie}\,\eta_A, \mathrm{Lie}\,\eta_A^\vee)$ *are dual with respect to* $(\cdot\,,\cdot)_A^{Del}$ *and therefore* $(\mathrm{Lie}\,\tilde\eta, \mathrm{Lie}\,\tilde\eta^\vee)$ *are also dual with respect to* $(\cdot\,,\cdot)_M^{Del}$*.*

Proof. Define $r_A : \mathrm{Lie}\, A \to \mathrm{Lie}\, A^{\#}$ and $r_T : \mathrm{Lie}\, T \to \mathrm{Lie}\, T^\natural$ such that they make the following diagram commute

$$\begin{array}{ccccc}
\mathrm{Lie}\, T & \xrightarrow{\mathrm{Lie}\, j} & \mathrm{Lie}\, G & \xleftarrow{e} & \mathrm{Lie}\, A \\
{\scriptstyle r_T}\downarrow & & \downarrow{\scriptstyle \mathrm{Lie}\,\eta} & & \downarrow{\scriptstyle r_A} \\
\mathrm{Lie}\, T^\natural & \xleftarrow{f^\natural} & \mathrm{Lie}\, G^\natural & \xrightarrow{\mathrm{Lie}\, q^\natural} & \mathrm{Lie}\, A^{\#}\ .
\end{array}$$

Notice that $r_T : \mathrm{Lie}\, T \to \mathrm{Lie}\, T^\natural$ is given by $r_T(z) = (\mathrm{Lie}(\pi \circ \eta \circ j)(z), z)$. From the following equalities

$$\begin{aligned}
pr_2 \circ r_T &= pr_2 \circ f^\natural \circ \mathrm{Lie}\,\eta \circ \mathrm{Lie}\, j \\
&= f \circ \mathrm{Lie}\,\bar\theta \circ \mathrm{Lie}\,\bar\sigma \circ \mathrm{Lie}\,\eta \circ \mathrm{Lie}\, j \\
&= f \circ \mathrm{Lie}\,\theta \circ \mathrm{Lie}\,\eta \circ \mathrm{Lie}\, j \\
&= f \circ \mathrm{Lie}\, j \\
&= Id
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Lie}\,\theta_A \circ r_A &= \mathrm{Lie}\,\theta_A \circ \mathrm{Lie}\, q^\natural \circ \mathrm{Lie}\,\eta \circ e \\
&= \mathrm{Lie}\,\theta_A \circ \mathrm{Lie}\,\pi^{\#} \circ \mathrm{Lie}\,\bar\sigma \circ \mathrm{Lie}\,\eta \circ e \\
&= \mathrm{Lie}\, q \circ \mathrm{Lie}\,\bar\theta \circ \mathrm{Lie}\,\bar\sigma \circ \mathrm{Lie}\,\eta \circ e \\
&= \mathrm{Lie}\, q \circ \mathrm{Lie}\,\theta \circ \mathrm{Lie}\,\eta \circ e \\
&= \mathrm{Lie}\, q \circ e \\
&= Id
\end{aligned}$$

we have that r_T and r_A are splittings of

$$0 \to \underline{\omega}_{T^\vee}(K) \to \operatorname{Lie} T^\natural \xrightarrow{pr_2} \operatorname{Lie} T \to 0,$$
$$0 \to \underline{\omega}_{A^\vee}(K) \to \operatorname{Lie} A^\# \xrightarrow{\theta_A} \operatorname{Lie} A \to 0,$$

respectively. By Lemma 4.4.5 there exist canonical homomorphisms $\eta_T : T(K) \to T^\natural(K)$ and $\eta_A : A(K) \to A^\#(K)$ which are splittings of

$$0 \to \underline{\omega}_{T^\vee}(K) \to T^\natural(K) \to T(K) \to 0,$$
$$0 \to \underline{\omega}_{A^\vee}(K) \to A^\#(K) \to A(K) \to 0,$$

respectively, satisfying $\operatorname{Lie} \eta_T = r_T$, $\operatorname{Lie} \eta_A = r_A$. Notice that $\eta_T : T(K) \to T^\natural(K)$ is given by $\eta_T(t) = (\pi \circ \eta \circ j(t), t)$. We define $\eta_T^\vee : T^\vee(K) \to T^{\vee\natural}(K)$ and $\eta_A^\vee : A^\vee(K) \to A^{\vee\#}(K)$ analogously.

Now suppose that $(\operatorname{Lie} \eta, \operatorname{Lie} \eta^\vee)$ are dual with respect to $(\cdot\,,\,\cdot)_M^{Del}$. We will first prove that $(\operatorname{Lie} \eta_T, \operatorname{Lie} \eta_T^\vee)$ are dual splittings. For any $z \in \operatorname{Lie} T$ and $z^\vee \in \operatorname{Lie} T^\vee$ we have

$$\begin{aligned}
0 &= (\operatorname{Lie}(\eta \circ j)(z), \operatorname{Lie}(\eta^\vee \circ j^\vee)(z^\vee))_M^{Del} \\
&= (f^\natural \circ \operatorname{Lie}(\eta \circ j)(z), f^{\vee\natural} \circ \operatorname{Lie}(\eta^\vee \circ j^\vee)(z^\vee))_T \\
&\quad + (\operatorname{Lie}(q^\natural \circ \eta \circ j)(z), \operatorname{Lie}(q^{\vee\natural} \circ \eta^\vee \circ j^\vee)(z^\vee))_A^{Del} \\
&= (\operatorname{Lie} \eta_T(z), \operatorname{Lie} \eta_T^\vee(z^\vee))_T \\
&\quad + (\operatorname{Lie}(\pi^\# \circ \bar{\sigma} \circ \eta \circ j)(z), \operatorname{Lie}(\pi^{\vee\#} \circ \bar{\sigma}^\vee \circ \eta^\vee \circ j^\vee)(z^\vee))_A^{Del}.
\end{aligned}$$

Notice that $\operatorname{Lie}(\pi^\# \circ \bar{\sigma} \circ \eta \circ j)(z) \in \operatorname{Lie} A^\#$ is such that

$$\begin{aligned}
\operatorname{Lie} \theta_A \circ \operatorname{Lie}(\pi^\# \circ \bar{\sigma} \circ \eta \circ j)(z) &= \operatorname{Lie}(q \circ \bar{\theta} \circ \bar{\sigma} \circ \eta \circ j)(z) \\
&= \operatorname{Lie}(q \circ \theta \circ \eta \circ j)(z) \\
&= \operatorname{Lie}(q \circ j)(z) \\
&= 0,
\end{aligned}$$

and similarly $\operatorname{Lie} \theta_A^\vee \circ \operatorname{Lie}(\pi^{\vee\#} \circ \bar{\sigma}^\vee \circ \eta^\vee \circ j^\vee)(z^\vee) = 0$. Here we use the fact that $\theta = \bar{\theta} \circ \bar{\sigma}$, which holds because of the way that we have defined $\bar{\sigma}$. This means that

$$\operatorname{Lie}(\pi^\# \circ \bar{\sigma} \circ \eta \circ j)(z) = (\omega, 0) \in \operatorname{Lie} A^\#$$

is the trivial extension of $A^\vee$ by $\mathbb{G}_a$ endowed with a $\natural-$structure coming from an invariant differential $\omega \in \underline{\omega}_{A^\vee}$; and in the same way

$$\operatorname{Lie}(\pi^{\vee\#} \circ \bar{\sigma}^\vee \circ \eta^\vee \circ j^\vee)(z^\vee) = (\omega^\vee, 0) \in \operatorname{Lie} A^{\vee\#},$$

for an invariant differential $\omega^\vee \in \underline{\omega}_A$. Therefore,

$$\begin{aligned}
0 &= (\operatorname{Lie} \eta_T(z), \operatorname{Lie} \eta_T^\vee(z^\vee))_T + ((\omega, 0), (\omega^\vee, 0))_A^{Del} \\
&= (\operatorname{Lie} \eta_T(z), \operatorname{Lie} \eta_T^\vee(z^\vee))_T,
\end{aligned}$$

i.e. $(\operatorname{Lie}\eta_T, \operatorname{Lie}\eta_T^\vee)$ are dual splittings with respect to $(\,\cdot\,,\,\cdot\,)_T$.

To prove that $(\operatorname{Lie}\eta_A, \operatorname{Lie}\eta_A^\vee)$ are dual splittings with respect to $(\,\cdot\,,\,\cdot\,)_A^{Del}$ consider $b \in \operatorname{Lie} A$ and $b^\vee \in \operatorname{Lie} A^\vee$. Then we have

$$\begin{aligned}
0 &= (\operatorname{Lie}\eta \circ e(b), \operatorname{Lie}\eta^\vee \circ e^\vee(b^\vee))_M^{Del} \\
&= (f^\natural \circ \operatorname{Lie}\eta \circ e(b), f^{\vee\natural} \circ \operatorname{Lie}\eta^\vee \circ e^\vee(b^\vee))_T \\
&\quad + (\operatorname{Lie}(q^\natural \circ \eta) \circ e(b), \operatorname{Lie}(q^{\vee\natural} \circ \eta^\vee) \circ e^\vee(b^\vee))_A^{Del} \\
&= (f^\natural \circ \operatorname{Lie}\eta \circ e(b), f^{\vee\natural} \circ \operatorname{Lie}\eta^\vee \circ e^\vee(b^\vee))_T + (\operatorname{Lie}\eta_A(b), \operatorname{Lie}\eta_A^\vee(b^\vee))_A^{Del}.
\end{aligned}$$

Observe that $f^\natural \circ \operatorname{Lie}\eta \circ e(b) \in \operatorname{Lie} T^\natural$ is such that

$$\begin{aligned}
pr_2 \circ (f^\natural \circ \operatorname{Lie}\eta \circ e)(b) &= f \circ \operatorname{Lie}(\bar\theta \circ \bar\sigma \circ \eta) \circ e(b) \\
&= f \circ \operatorname{Lie}(\theta \circ \eta) \circ e(b) \\
&= f \circ e(b) \\
&= 0,
\end{aligned}$$

and similarly $pr_2 \circ f^{\vee\natural} \circ \operatorname{Lie}\eta^\vee \circ e^\vee(b^\vee) = 0$. This means that

$$\begin{aligned}
f^\natural \circ \operatorname{Lie}\eta \circ e(b) &= (\omega, 0) \in \operatorname{Lie} T^\natural, \\
f^{\vee\natural} \circ \operatorname{Lie}\eta^\vee \circ e^\vee(b^\vee) &= (\omega^\vee, 0) \in \operatorname{Lie} T^{\vee\natural},
\end{aligned}$$

for invariant differentials $\omega \in \underline{\omega}_{T^\vee}$ and $\underline{\omega}^\vee \in \underline{\omega}_T$. Therefore,

$$\begin{aligned}
0 &= (f^\natural \circ \operatorname{Lie}\eta \circ e(b), f^{\vee\natural} \circ \operatorname{Lie}\eta^\vee \circ e^\vee(b^\vee))_T + (\operatorname{Lie}\eta_A(b), \operatorname{Lie}\eta_A^\vee(b^\vee))_A^{Del} \\
&= ((\omega, 0), (\omega^\vee, 0))_T + (\operatorname{Lie}\eta_A(b), \operatorname{Lie}\eta_A^\vee(b^\vee))_A^{Del} \\
&= (\operatorname{Lie}\eta_A(b), \operatorname{Lie}\eta_A^\vee(b^\vee))_A^{Del},
\end{aligned}$$

i.e. $(\operatorname{Lie}\eta_A, \operatorname{Lie}\eta_A^\vee)$ are dual splittings with respect to $(\,\cdot\,,\,\cdot\,)_A^{Del}$.

If we define

$$\begin{aligned}
\tilde r : \operatorname{Lie} G \cong \operatorname{Lie} T \times \operatorname{Lie} A &\to \operatorname{Lie} G^\natural \cong \operatorname{Lie} T^\natural \times \operatorname{Lie} A^{\#} \\
h = (z, b) &\mapsto (r_T(z), r_A(b)),
\end{aligned}$$

and similarly for $\tilde r^\vee$, then these are sections of $\operatorname{Lie}\theta$ and $\operatorname{Lie}\theta^\vee$, respectively, they make diagram 4.14 commute and are dual with respect to $(\,\cdot\,,\,\cdot\,)_M^{Del}$. We can now define $\tilde\eta : G(K) \to G^\natural(K)$ as the uniquely determined section of θ such that $\operatorname{Lie}\tilde\eta = \tilde r$, and similarly for $\tilde\eta^\vee$. □

We will use the following result in the proof of Theorem 4.4.8.

Proposition 4.4.7. *Let $\eta_A : A(K) \to A^{\#}(K)$ and $\eta_A^{\vee} : A^{\vee}(K) \to A^{\vee\#}(K)$ be a pair of splittings of the exact sequences*

$$0 \to \underline{\omega}_{A^{\vee}}(K) \to A^{\#}(K) \to A(K) \to 0,$$
$$0 \to \underline{\omega}_{A}(K) \to A^{\vee\#}(K) \to A^{\vee}(K) \to 0,$$

respectively, such that they are dual with respect to $(\cdot\,,\cdot)_A^{Del}$. Then the following $\lambda-$splittings of $P_A(K)$ are equal:

i) Define ψ_A as the $\lambda-$splitting extending the $\mathrm{Log}-$*splitting of P_A^f induced by* $(\mathrm{Lie}\,\eta_A, \mathrm{Lie}\,\eta_A^{\vee})$ *by applying Theorem 3.1.3 and Proposition 3.1.5 (see diagram* (3.10)*).*

ii) Let $y \in P_A(K)$ lie above $(a, a^{\vee}) \in A(K) \times A^{\vee}(K)$ and denote $s_{a^{\vee}}$ the rigidification of $P_{A,a^{\vee}}$ corresponding to $\eta_A^{\vee}(a^{\vee})$. Then we define

$$\psi_A^1(y) = s_{a^{\vee}} \circ \lambda_{P_{A,a^{\vee}}}(y).$$

$$\begin{array}{ccc}
K^* & \xrightarrow{\quad\lambda\quad} & K \\
\big\downarrow & & \big\downarrow \,\uparrow s_{a^{\vee}} \\
P_{A,a^{\vee}}(K) & \xrightarrow{\lambda_{P_{A,a^{\vee}}}} & \mathrm{Lie}\,P_{A,a^{\vee}} \\
\big\downarrow & & \big\downarrow \\
A(K) \times \{a^{\vee}\} & \xrightarrow{\lambda_A} & \mathrm{Lie}\,A
\end{array}$$

iii) Let $y \in P_A(K)$ lie above $(a, a^{\vee}) \in A(K) \times A^{\vee}(K)$ and denote s_a the rigidification of $P_{a,A^{\vee}}$ corresponding to $\eta_A(a)$. Then we define

$$\psi_A^2(y) = s_a \circ \lambda_{P_{a,A^{\vee}}}(y).$$

$$\begin{array}{ccc}
K^* & \xrightarrow{\quad\lambda\quad} & K \\
\big\downarrow & & \big\downarrow \,\uparrow s_{a} \\
P_{a,A^{\vee}}(K) & \xrightarrow{\lambda_{P_{a,A^{\vee}}}} & \mathrm{Lie}\,P_{a,A^{\vee}} \\
\big\downarrow & & \big\downarrow \\
\{a\} \times A^{\vee}(K) & \xrightarrow{\lambda_{A^{\vee}}} & \mathrm{Lie}\,A^{\vee}
\end{array}$$

Proof. We will prove that $\psi_A = \psi_A^2$. As always, denote $(P_A^{\natural}, \nabla_A)$ the canonical $\natural-$biextension of $(A^{\#}, A^{\vee\#})$ by $\mathbb{G}_m$. This is equivalent to endowing $P^{\natural}$ with canonical $\natural - i-$structures, for $i = 1, 2$, given by connections $\nabla_{A,i}$, respectively. Then $\nabla_{A,1}$ gives $P_A^{\natural}$ the structure of $\natural-$extension of $A^{\#}_{A^{\#\vee}}$ by $\mathbb{G}_{m,A^{\#\vee}}$ and $\nabla_{A,2}$ the structure of $\natural-$extension of $A^{\#\vee}_{A^{\#}}$ by $\mathbb{G}_{m,A^{\#}}$. Now, let $\omega \in \Omega^1_{P_A^{\natural}/K}$ be the closed normal bi-invariant differential corresponding to ∇_A (see Section 3.1) and $\omega_1 \in \Omega^1_{P_A^\natural/A^{\vee\#}}$, resp. $\omega_2 \in \Omega^1_{P_A^\natural/A^{\#}}$, the closed normal invariant differential corresponding to $\nabla_{A,1}$, resp. $\nabla_{A,2}$. Remember that ψ_A is the uniquely determined locally analytic $\lambda-$splitting satisfying $d\psi_A = p^*\omega$.

$$\begin{array}{ccc} P_A(K) & \xrightarrow{\quad p \quad} & P_A^\natural(K) \\ \downarrow & \lrcorner & \downarrow \\ A(K)\times A^\vee(K) & \xrightarrow{\eta_A\times\eta_A^\vee} & A^\#(K)\times A^{\vee\#}(K) \end{array}$$

Let $a \in A(K)$ and consider the fiber $P_{a,A^\vee}$ of P_A over $\{a\}\times A^\vee$. Denote by $\psi_{A,a}$ the restriction of ψ_A to $P_{a,A^\vee}(K)$. Since the restriction of $p^*\omega$ to $P_{a,A^\vee}$ equals the restriction of $p^*\omega_2$ to $P_{a,A^\vee}$, which we will denote $a^*p^*\omega_2$, then $\psi_{A,a}$ is the unique solution to

$$d\psi_{A,a} = a^*p^*\omega_2. \tag{4.15}$$

Finally observe that $\eta_A(a) = (P_{a,A^\vee}, \eta_A(a)^*\nabla_{A,2})$, where $\eta_A(a)^*\nabla_{A,2}$ is the restriction of $\nabla_{A,2}$ to $P_{\eta(a),A^\vee}$. This connection corresponds to the closed normal invariant differential $\eta_A(a)^*\omega_2$, which equals $a^*p^*\omega_2$. Thus we see that equation (4.15) is equivalent to

$$\operatorname{Lie}\psi_{A,a} = s_a.$$

$$\begin{array}{ccc} K^* & \xrightarrow{\quad\lambda\quad} & K \\ \big\downarrow & & \| \\ P_A(K) & \xrightarrow{\quad\psi_A\quad} & K \\ \big\downarrow & & \\ A(K)\times A^\vee(K) & & \end{array} \qquad \begin{array}{ccc} K^* & \xrightarrow{\quad\lambda\quad} & K \\ \big\downarrow & \nearrow_{\psi_{A,a}} & \big\downarrow \uparrow \operatorname{Lie}\psi_{A,a}=s_a \\ P_{a,A^\vee}(K) & \xrightarrow[\lambda_{P_{a,A^\vee}}]{} & \operatorname{Lie} P_{a,A^\vee} \\ \big\downarrow & & \big\downarrow \\ \{a\}\times A^\vee(K) & \xrightarrow{\lambda_{A^\vee}} & \operatorname{Lie} A^\vee \end{array}$$

Therefore,

$$\begin{aligned} \psi_A^2(y) &= s_a \circ \lambda_{P_{a,A^\vee}}(y) \\ &= \operatorname{Lie}\psi_{A,a} \circ \lambda_{P_{a,A^\vee}}(y) \\ &= \psi_{A,a}(y). \end{aligned}$$

By an analogous argument we see that $\psi_A^1 = \psi_A$. □

Theorem 4.4.8. *Let* $r : \operatorname{Lie} G \to \operatorname{Lie} G^\natural$ *and* $r^\vee : \operatorname{Lie} G^\vee \to \operatorname{Lie} G^{\vee\natural}$ *be a pair of splittings of the exact sequences*

$$\begin{aligned} 0 \to \underline{\omega}_{G^\vee}(K) \to \operatorname{Lie} G^\natural \to \operatorname{Lie} G \to 0, \\ 0 \to \underline{\omega}_{G}(K) \to \operatorname{Lie} G^{\vee\natural} \to \operatorname{Lie} G^\vee \to 0, \end{aligned}$$

respectively, which are dual with respect to $(\cdot\,,\cdot)^{Del}_M$. *Then we have an induced* $\lambda-$*splitting*

$$\psi : P(K) \to K.$$

Proof. From the splittings of the exact sequences of vector groups (4.9) and (4.10) fixed at the beginning of the section, we also get the following isomorphisms of Lie algebras for every $g \in G(K)$ and $g^\vee \in G^\vee(K)$

$$\begin{aligned} \operatorname{Lie} P_g &\cong \operatorname{Lie} T^\vee \times \operatorname{Lie} P_{a,A^\vee}, \\ \operatorname{Lie} P_{g^\vee} &\cong \operatorname{Lie} T \times \operatorname{Lie} P_{A,a^\vee}, \end{aligned}$$

where we are denoting by $a \in A(K)$ the image of g, $a^\vee \in A^\vee(K)$ the image of $g^\vee$ and $P_{a,A^\vee}$ the pullback of P_A to $\{a\} \times A^\vee$. These are obtained from iii) by pullback:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & K & = & K & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \operatorname{Lie} T^\vee & \longrightarrow & \operatorname{Lie} P_g & \longrightarrow & \operatorname{Lie} P_{a,A^\vee} & \longrightarrow & 0 \\
 & & \cong\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \{g\} \times \operatorname{Lie} T^\vee & \xrightarrow[\operatorname{Lie} j^\vee]{} & \{g\} \times \operatorname{Lie} G^\vee & \xrightarrow[\operatorname{Lie} q^\vee]{} & \{a\} \times \operatorname{Lie} A^\vee & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and similarly for $\operatorname{Lie} P_{g^\vee}$.

By 4.4.5, there exist homomorphisms $\eta : G(K) \to G^\natural(K)$ and $\eta^\vee : G^\vee(K) \to G^{\vee\natural}(K)$ such that $\operatorname{Lie} \eta = r$ and $\operatorname{Lie} \eta^\vee = r^\vee$. Let η_T, $\eta_T^\vee$, η_A, $\eta_A^\vee$, $\tilde\eta$ and $\tilde\eta^\vee$ be as in Proposition 4.4.6. Consider the following diagram with the previously introduced notation

$$\begin{array}{ccccccc}
 & & G(K) & & & & \\
 & & \downarrow{\scriptstyle\tilde\eta} & & & & \\
\underline{\omega}_{T^\vee}(K) & \xleftarrow{\pi} & G^\natural(K) & \xrightarrow{\bar\sigma} & G^\#(K) & \xrightarrow{\pi^\#} & A^\#(K)\,.
\end{array}
\qquad (q^\natural = \pi^\# \circ \bar\sigma : G^\natural(K) \to A^\#(K))$$

Take $g \in G(K)$ mapping to $a \in A(K)$. Then $\pi \circ \tilde\eta(g) \in \underline{\omega}_{T^\vee}(K)$ is an invariant differential and we denote by $s_g^1 : \operatorname{Lie} T^\vee \to K$ the corresponding morphism of Lie algebras. On the other hand, by 1.3.10, we have $q^\natural \circ \tilde\eta(g) = (a, \nabla_g) \in A^\#(K)$, with ∇_g an integrable connection on $P_{a,A^\vee}$. By 1.3.7, ∇_g corresponds to a normal invariant differential on $P_{a,A^\vee}$ which in turn corresponds to a homomorphism $s_g^2 : \operatorname{Lie} P_{a,A^\vee} \to K$. We define

$$\begin{aligned} s_g : \operatorname{Lie} P_g \cong \operatorname{Lie} T^\vee \times \operatorname{Lie} P_{a,A^\vee} &\to K \\ z = (z^1, z^2) &\mapsto s_g^1(z^1) + s_g^2(z^2). \end{aligned}$$

This is a rigidification of P_g, considered as an extension of $G^\vee$ by $\mathbb{G}_m$. We define the rigidification $s_{g^\vee} : \operatorname{Lie} P_{g^\vee} \to K$ of $P_{g^\vee}$ analogously as

$$\begin{aligned} s_{g^\vee} : \operatorname{Lie} P_{g^\vee} \cong \operatorname{Lie} T \times \operatorname{Lie} P_{A,a^\vee} &\to K \\ z = (z^1, z^2) &\mapsto s^1_{g^\vee}(z^1) + s^2_{g^\vee}(z^2), \end{aligned}$$

where $s^1_{g^\vee} : \operatorname{Lie} T \to K$ is the morphism corresponding to the invariant differential $\pi^\vee \circ \tilde{\eta}^\vee(g^\vee) \in \underline{\omega}_T(K)$ and $s^2_{g^\vee} : \operatorname{Lie} P_{A,a^\vee} \to K$ is the morphism corresponding to the integrable connection on $P_{A,a^\vee}$ given by $q^{\vee\natural} \circ \tilde{\eta}^\vee(g^\vee) \in A^{\vee\#}(K)$.

Let $y \in P(K)$ lie above $(g, g^\vee) \in G(K) \times G^\vee(K)$. Then we define

$$\psi(y) = s_g \circ \lambda_{P_g}(y). \tag{4.16}$$

$$\begin{array}{ccc} K^* & \xrightarrow{\quad\lambda\quad} & K \\ \big\downarrow & & \big\downarrow \uparrow s_g \\ P_g(K) & \xrightarrow{\quad\lambda_{P_g}\quad} & \operatorname{Lie} P_g \\ \big\downarrow & & \big\downarrow \\ \{g\} \times G^\vee(K) & \xrightarrow{\lambda_{G^\vee}} & \operatorname{Lie} G^\vee \end{array} \tag{4.17}$$

Notice that we could also have defined ψ as

$$\psi(y) = s_{g^\vee} \circ \lambda_{P_{g^\vee}}(y). \tag{4.18}$$

$$\begin{array}{ccc} K^* & \xrightarrow{\quad\lambda\quad} & K \\ \big\downarrow & & \big\downarrow \uparrow s_{g^\vee} \\ P_{g^\vee}(K) & \xrightarrow{\quad\lambda_{P_{g^\vee}}\quad} & \operatorname{Lie} P_{g^\vee} \\ \big\downarrow & & \big\downarrow \\ G(K) \times \{g^\vee\} & \xrightarrow{\lambda_{G}} & \operatorname{Lie} G \end{array} \tag{4.19}$$

Claim. *Both definitions* (4.16) *and* (4.18) *of* $\psi : P(K) \to K$ *are equal.*

Proof. Denote $(z^1_g, z^2_g) := \lambda_{P_g}(y) \in \operatorname{Lie} P_g \cong \operatorname{Lie} T^\vee \times \operatorname{Lie} P_{a,A^\vee}$ and $(z^1_{g^\vee}, z^2_{g^\vee}) := \lambda_{P_{g^\vee}}(y) \in \operatorname{Lie} P_{g^\vee} \cong \operatorname{Lie} T \times \operatorname{Lie} P_{A,a^\vee}$. To prove the claim it suffices to show that $s^1_g(z^1_g) = s^1_{g^\vee}(z^1_{g^\vee})$ and $s^2_g(z^2_g) = s^2_{g^\vee}(z^2_{g^\vee})$.

a) $s^1_g(z^1_g) = s^1_{g^\vee}(z^1_{g^\vee})$: Consider $(f \circ \lambda_G(g), f^\vee \circ \lambda_{G^\vee}(g^\vee)) \in \operatorname{Lie} T \times \operatorname{Lie} T^\vee$. Then

$$\begin{aligned} \operatorname{Lie} \eta_T(f \circ \lambda_G(g)) &= f^\natural \circ \operatorname{Lie} \tilde{\eta}(\lambda_G(g)) \\ &= (\pi \circ \tilde{\eta}(g), f \circ \lambda_G(g)) \in \underline{\omega}_{T^\vee}(K) \times \operatorname{Lie} T = \operatorname{Lie} T^\natural, \end{aligned}$$

and similarly $\operatorname{Lie}\eta_T^\vee(f^\vee \circ \lambda_{G^\vee}(g^\vee)) = (\pi^\vee \circ \tilde{\eta}^\vee(g^\vee), f^\vee \circ \lambda_{G^\vee}(g^\vee))$. Then by Lemma 4.4.2 we have

$$(\operatorname{Lie}\eta_T(f \circ \lambda_G(g)), \operatorname{Lie}\eta_T^\vee(f^\vee \circ \lambda_{G^\vee}(g^\vee)))_T = s_g^1(f^\vee \circ \lambda_{G^\vee}(g^\vee)) - s_{g^\vee}^1(f \circ \lambda_G(g)).$$

Noticing that $z_{g^\vee}^1 = f \circ \lambda_G(g)$ and $z_g^1 = f^\vee \circ \lambda_{G^\vee}(g^\vee)$ we get the desired equality.

b) $s_g^2(z_g^2) = s_{g^\vee}^2(z_{g^\vee}^2)$: Let $y_A \in P_A(K)$ be the image of y. Then $z_g^2 = \lambda_{P_{a,A^\vee}}(y_A)$ and $z_{g^\vee}^2 = \lambda_{P_{A,a^\vee}}(y_A)$. Notice that we have

$$\operatorname{Lie} q^\natural \circ \operatorname{Lie}\tilde{\eta}(\lambda_G(g)) = \operatorname{Lie}\eta_A \circ \operatorname{Lie} q(\lambda_G(g)) = \operatorname{Lie}\eta_A(\lambda_A(a)),$$

and similarly $\operatorname{Lie} q^{\vee\natural} \circ \operatorname{Lie}\tilde{\eta}^\vee(\lambda(g)) = \operatorname{Lie}\eta_A^\vee(\lambda(a^\vee))$. Therefore, if we denote by s_a the rigidification corresponding to $\eta_A(a)$ and $s_{a^\vee}$ the rigidification corresponding to $\eta_A^\vee(a^\vee)$ then $s_a = s_g^2$ and $s_{a^\vee} = s_{g^\vee}^2$. Finally, by Proposition 4.4.7, we get that

$$\begin{aligned} s_g^2(z_g^2) &= s_a \circ \lambda_{P_{a,A^\vee}}(y_A) \\ &= s_{a^\vee} \circ \lambda_{P_{A,a^\vee}}(y_A) \\ &= s_{g^\vee}^2(z_{g^\vee}^2). \end{aligned}$$

□

It only remains to check that ψ is in fact a $\lambda-$splitting. Using, for example, definition (4.16) we get that for all $c \in K^*$ and $y \in P(K)$ lying above $(g, g^\vee) \in G(K) \times G^\vee(K)$

$$\begin{aligned} \psi(c+y) &= s_g \circ \lambda_{P_g}(c+y) \\ &= s_g \circ \lambda_{P_g}(c) + s_g \circ \lambda_{P_g}(y) \\ &= \lambda(c) + \psi(y), \end{aligned}$$

where the last equality holds because of the commutativity of diagram (4.17). Also, for $y, y' \in P_g(K)$,

$$\begin{aligned} \psi(y+y') &= s_g \circ \lambda_{P_g}(y +_1 y') \\ &= s_g \circ \lambda_{P_g}(y) + s_g \circ \lambda_{P_g}(y') \\ &= \psi(y) + \psi(y'). \end{aligned}$$

Finally, from definition (4.18) it follows that ψ is also compatible with the group structure of $P(K)$ relative to $G^\vee(K)$. □

Theorem 4.4.9. *In the situation of Theorem 4.4.8, assume that η and $\eta^\vee$ make the following diagrams commute*

$$\begin{array}{ccc} L(K) & = & L(K) \\ {\scriptstyle u(K)}\downarrow & & \downarrow{\scriptstyle u^\natural(K)} \\ G(K) & \xrightarrow{\eta} & G^\natural(K) \end{array} \qquad \begin{array}{ccc} L^\vee(K) & = & L^\vee(K) \\ {\scriptstyle u^\vee(K)}\downarrow & & \downarrow{\scriptstyle u^{\vee\natural}(K)} \\ G^\vee(K) & \xrightarrow{\eta^\vee} & G^{\vee\natural}(K) \end{array}$$

and, moreover, that $\eta = \tilde{\eta}$, $\eta^\vee = \tilde{\eta}^\vee$, where $\tilde{\eta}$ and $\tilde{\eta}^\vee$ are the morphisms of Proposition 4.4.6. Then the $\lambda-$splitting $\psi : P(K) \to K$ constructed in Theorem 4.4.8 is compatible with the $L(K) \times L^\vee(K)-$linearization. In particular, it induces a $\lambda-$splitting of the biextension $Q_M(K)$ of $(M(K), M^\vee(K))$ by K^* if u and $u^\vee$ are injective.

Proof. We have to prove that the $\lambda-$splitting $\psi : P(K) \to K$ constructed in Theorem 4.4.8 is compatible with the $L(K) \times L^\vee(K)-$linearization. By Proposition 4.2.10, this happens if and only if $\psi \circ \tau_L = \psi \circ \tau_{L^\vee} = 0$.

Let $x \in L(K)$ and denote $\chi : T^\vee \to \mathbb{G}_m$ the homomorphism corresponding to it. We have the following diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & T^\vee & \longrightarrow & G^\vee & \longrightarrow & A^\vee & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle -\chi} & \ulcorner & \downarrow & & \| & & \\
0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & P_{v(x),A^\vee} & \longrightarrow & A^\vee & \longrightarrow & 0 \\
 & & \| & & \uparrow & \urcorner & \uparrow & & \\
0 & \longrightarrow & \mathbb{G}_m & \longrightarrow & P_{u(x)} & \longrightarrow & G^\vee & \longrightarrow & 0 \, ,
\end{array}$$

where $P_{v(x),A^\vee}$ denotes the pullback of P_A to $\{v(x)\} \times A^\vee$, which induces the following diagram whose rows are split exact sequences of Lie algebras

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Lie}\, T^\vee & \longrightarrow & \mathrm{Lie}\, G^\vee & \longrightarrow & \mathrm{Lie}\, A^\vee & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle -\mathrm{Lie}\,\chi} & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathrm{Lie}\, \mathbb{G}_m & \longrightarrow & \mathrm{Lie}\, P_{v(x),A^\vee} & \longrightarrow & \mathrm{Lie}\, A^\vee & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \mathrm{Lie}\, \mathbb{G}_m & \longrightarrow & \mathrm{Lie}\, P_{u(x)} & \longrightarrow & \mathrm{Lie}\, G^\vee & \longrightarrow & 0 \, .
\end{array}$$

By Proposition 1.3.10, $u^\natural(x) \in G^\natural(K)$ corresponds to the extension $[L^\vee \to P_{u(x)}]$ of $M^\vee$ by $\mathbb{G}_m$ induced by $-\chi$ endowed with a $\natural-$structure $\nabla_{u(x)}$. Notice that $\pi \circ u^\natural(x) \in \underline{\omega}_{T^\vee}(K)$ corresponds to $\mathrm{Lie}\,\chi \in \mathrm{Hom}_{\mathcal{O}_K}(\underline{\mathrm{Lie}} T^\vee, \mathbb{G}_a)$, since

$$\begin{aligned}
ev : \mathrm{Hom}_K(T^\vee, \mathbb{G}_m) \to \mathrm{Hom}_{\mathcal{O}_K}(\underline{\mathrm{Hom}}_K(L, \mathbb{G}_a), \mathcal{O}_K) = \mathrm{Hom}_{\mathcal{O}_K}(\underline{\mathrm{Lie}} T^\vee, \mathbb{G}_a) \\
\chi \mapsto (ev(x) : \underline{\mathrm{Hom}}_K(L, \mathbb{G}_a) \to \mathcal{O}_K) = \underline{\mathrm{Lie}}\chi.
\end{aligned}$$

On the other hand, by Proposition 1.3.7, $q^\natural \circ u^\natural(x) \in A^\#(K)$ corresponds to the extension $P_{v(x),A^\vee}$ of $A^\vee$ by $\mathbb{G}_m$ together with the normal invariant differential given by $pr_1 : \underline{\mathrm{Lie}} P_{v(x),A^\vee} \cong \underline{\mathrm{Lie}}\mathbb{G}_m \times \underline{\mathrm{Lie}} A^\vee \to \underline{\mathrm{Lie}}\mathbb{G}_m \cong \mathbb{G}_a$. We have by hypothesis that $\eta \circ u = u^\natural$, so

$$s^1_{u(x)} = \mathrm{Lie}\,\chi : \mathrm{Lie}\, T^\vee \to K,$$

since this is the morphism induced by $\pi \circ \eta(u(x)) = \pi \circ u^\natural(x)$, and

$$s^2_{u(x)} = pr_1 : \operatorname{Lie} P_{v(x),A^\vee} \cong K \times \operatorname{Lie} A^\vee \to K,$$

since this is the one induced by $q^\natural \circ \eta(u(x)) = q^\natural \circ u^\natural(x)$.

Let $g^\vee \in G^\vee(K)$ and denote $y_A \in P_A(K)$ the image of $\tau_L(x, g^\vee) \in P(K)$. We have the following split exact sequence coming from the previous diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Lie} T^\vee & \longrightarrow & \operatorname{Lie} P_{u(x)} & \longrightarrow & \operatorname{Lie} P_{v(x),A^\vee} & \longrightarrow & 0 \\ & & z & \longleftarrow & \lambda_{P_{u(x)}}(\tau_L(x,g^\vee)) & \longmapsto & \lambda_{P_{v(x),A^\vee}}(y_A) & & . \end{array}$$

Notice that, under the isomorphism $\operatorname{Lie} P_{v(x),A^\vee} \cong K \times \operatorname{Lie} A^\vee$,

$$\lambda_{P_{v(x),A^\vee}}(y_A) = (-\operatorname{Lie}\chi(z), \lambda_{A^\vee}(a^\vee)).$$

Therefore, we get that $\psi \circ \tau_L(x, g^\vee)$ equals

$$\begin{aligned} \psi \circ \tau_L(x, g^\vee) &= s_{u(x)} \circ \lambda_{P_{u(x)}}(\tau_L(x, g^\vee)) \\ &= s^1_{u(x)}(z) + s^2_{u(x)}(\lambda_{P_{v(x),A^\vee}}(y_A)) \\ &= \operatorname{Lie}\chi(z) + pr_1(-\operatorname{Lie}\chi(z), \lambda_{A^\vee}(a^\vee)) \\ &= \operatorname{Lie}\chi(z) - \operatorname{Lie}\chi(z) \\ &= 0. \end{aligned}$$

The fact that $\psi \circ \tau_{L^\vee}(g, x^\vee) = 0$ is implied from the previous arguments, using the equivalent definition (4.18) of ψ. □

Corollary 4.4.10. *Let $\rho : K^* \to \mathbb{Q}_p$ be a ramified homomorphism and consider $r : \operatorname{Lie} G \to \operatorname{Lie} G^\natural$ and $r^\vee : \operatorname{Lie} G^\vee \to \operatorname{Lie} G^{\vee\natural}$ a pair of splittings of the exact sequences*

$$0 \to \underline{\omega}_{G^\vee}(K) \to \operatorname{Lie} G^\natural \to \operatorname{Lie} G \to 0$$

$$0 \to \underline{\omega}_{G}(K) \to \operatorname{Lie} G^{\vee\natural} \to \operatorname{Lie} G^\vee \to 0,$$

respectively, which are dual with respect to $(\cdot,\cdot)^{Del}_M$. Then:

i) There is a $\rho-$splitting $\psi : P(K) \to \mathbb{Q}_p$.

ii) Let $\eta : G(K) \to G^\natural(K)$ and $\eta^\vee : G^\vee(K) \to G^{\vee\natural}(K)$ be the morphisms such that $\operatorname{Lie}\eta = r$ and $\operatorname{Lie}\eta^\vee = r^\vee$, given by Lemma 4.4.5. If the following diagrams commute

$$\begin{array}{ccc} L(K) & = & L(K) \\ {\scriptstyle u(K)}\downarrow & & \downarrow{\scriptstyle u^\natural(K)} \\ G(K) & \xrightarrow{\eta} & G^\natural(K) \end{array} \qquad \begin{array}{ccc} L^\vee(K) & = & L^\vee(K) \\ {\scriptstyle u^\vee(K)}\downarrow & & \downarrow{\scriptstyle u^{\vee\natural}(K)} \\ G^\vee(K) & \xrightarrow{\eta^\vee} & G^{\vee\natural}(K) \end{array}$$

and $\eta = \tilde{\eta}$, $\eta^\vee = \tilde{\eta}^\vee$, where $\tilde{\eta}$ and $\tilde{\eta}^\vee$ are the morphisms of Proposition 4.4.6, then the $\rho-$splitting $\psi : P(K) \to \mathbb{Q}_p$ of i) is compatible with the $L(K) \times L^\vee(K)-$linearization. In particular, if u and $u^\vee$ are injective then it induces a $\rho-$splitting of the biextension $Q_M(K)$ of $(M(K), M^\vee(K))$ by K^*.

Proof. i) By [Zar90, p. 319], we know that ρ factors through a branch $\lambda : K^* \to K$ of the $p-$adic logarithm

$$\begin{array}{ccc} K^* & \xrightarrow{\rho} & \mathbb{Q}_p \\ & {\scriptstyle\lambda}\searrow \quad \nearrow{\scriptstyle\delta} & \\ & K & \end{array}.$$

Let $\psi : P(K) \to K$ be the $\lambda-$splitting constructed as in Theorem 4.4.8. Then $\psi_\rho := \delta \circ \psi : P(K) \to \mathbb{Q}_p$ is a $\rho-$splitting of $P(K)$.

ii) We have that

$$\psi_\rho \circ \tau_L = \delta \circ \psi \circ \tau_L = 0,$$

and similarly for $\tau_{L^\vee}$. By Proposition 4.2.10, ψ_ρ is compatible with the $L(K) \times L^\vee(K)-$linearization and thus induces a $\rho-$splitting of $Q_M(K)$, in the case that u and $u^\vee$ are injective.

□

4.5 Pairings on divisors and zero cycles

Let K be a finite extension of $\mathbb{Q}_p$. We fix a continuous homomorphism $\rho : K^* \to \mathbb{Q}_p$ and consider a $\rho-$splitting $\psi : P(K) \to \mathbb{Q}_p$ which is compatible with the canonical linearization, where P denotes in each of the cases below the Poincaré biextension of the 1-motives considered. We will use the notation $Z^n(X)$ (resp. $Z_n(X)$) for the group of cycles of codimension (resp. dimension) n on a variety X over K, and $Z^n(X)_0$ (resp. $Z_n(X)_0$) for its subgroup of cycles of degree 0. We can define pairings on Weil divisors and zero cycles with disjoint support as follows.

4.5.1 Case of proper varieties

Suppose X is an irreducible proper variety over K, so that $\bar{X} = \tilde{X}$, $\bar{S} = \tilde{S}$ and $Y = \emptyset$. In this case we have

$$\mathrm{Pic}^-(X) = [u : \mathrm{Div}^0_{\tilde{S}/S}(\tilde{X}) \to \mathrm{Pic}^0(\tilde{X}, Y)]$$

and $\mathrm{Alb}^+(X) = \mathrm{Pic}^-(X)^\vee$ is the semi-abelian variety

$$0 \to T(\tilde{S}/S) \to \mathrm{Alb}^+(X) \to \mathrm{Alb}(\tilde{X}) \to 0.$$

$$\begin{array}{ccccc}
 & & K^* & \xrightarrow{\quad\rho\quad} & \\
 & & \downarrow & & \searrow \\
 & \overset{\tau_L}{\nearrow} & P(K) & \xrightarrow{\quad\psi\quad} & K \\
 & & \downarrow & & \\
\mathrm{Div}^0_{\tilde{S}/S}(\tilde{X}) \times \mathrm{Alb}^+(X) & \xrightarrow{u\times Id} & \mathrm{Pic}^0(\tilde{X}) \times \mathrm{Alb}^+(X) & &
\end{array}$$

Consider the Albanese mapping $a_X^+ : X_{\mathrm{reg}} \to \mathrm{Alb}^+(X)$ defined in [BS01, §3.2]. Define

$$\begin{aligned}
[\,\cdot\,,\,\cdot\,] : (\mathrm{Div}^0(\tilde{X}) \times Z_0(X_{\mathrm{reg}})_0)' &\to \mathbb{Q}_p \\
(D, \sum n_j x_j) &\mapsto \sum n_j \psi \circ s_D \circ a_X^+(x_j),
\end{aligned}$$

where s_D is a rational section associated to D. Since the trivialization τ_L of the Poincaré biextension P over $\{D\} \times \mathrm{Alb}^+(X)$, for $D \in \mathrm{Div}^0_{\tilde{S}/S}(\tilde{X})$, is given by the rational section s_D associated to D and $\psi \circ \tau_L = 0$, we have that $(D, \sum n_j x_j) = 0$ when $D \in \mathrm{Div}^0_{\tilde{S}/S}(\tilde{X})$.

$$\begin{array}{ccccc}
 & & K^* & \xrightarrow{\quad\rho\quad} & \\
 & & \downarrow & & \searrow \\
 & \overset{\tau_{L,D}=s_D}{\nearrow} & P(K)_D & \xrightarrow{\quad\psi\quad} & K \\
 & & \downarrow & & \\
\{D\} \times \mathrm{Alb}^+(X) & \xrightarrow{\cong} & \{\mathcal{O}(D)\} \times \mathrm{Alb}^+(X) & &
\end{array} \quad .$$

We have the following commutative diagram, where $f : \tilde{X} \to X$ denotes the resolution of singularities considered and $\mathrm{Div}^0(X)$ is defined as the pushout,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Div}^0_{\tilde{S}/S}(\tilde{X}) & = & \mathrm{Div}^0_{\tilde{S}/S}(\tilde{X}) & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \mathrm{Div}^0_{\tilde{S}}(\tilde{X}) & \longrightarrow & \mathrm{Div}^0(\tilde{X}) & \longrightarrow & \mathrm{Div}^0(X_{\mathrm{reg}}) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f_*} & \ulcorner & \downarrow{\scriptstyle f_*} & & \| & & \\
0 & \longrightarrow & \mathrm{Div}^0_S(X) & \longrightarrow & \mathrm{Div}^0(X) & \longrightarrow & \mathrm{Div}^0(X_{\mathrm{reg}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{Coker}(f_*) & = & \mathrm{Coker}(f_*) & & & &
\end{array} \quad .$$

Whenever $\mathrm{Coker}(f_*)$ is torsion we get an exact sequence

$$\mathrm{Div}^0_{\tilde{S}/S}(\tilde{X}) \otimes \mathbb{Q} \to \mathrm{Div}^0(\tilde{X}) \otimes \mathbb{Q} \to \mathrm{Div}^0(X) \otimes \mathbb{Q} \to 0.$$

Extending the coefficients of divisors on $\tilde{X}$ to $\mathbb{Q}$ we get a pairing on

$$(\mathrm{Div}^0(\tilde{X}) \otimes \mathbb{Q} \times Z_0(X_{\mathrm{reg}})_0)'$$

which is zero on $\mathrm{Div}^0_{\tilde{S}/S}(\tilde{X})$, yielding a pairing on

$$(\mathrm{Div}^0(X) \otimes \mathbb{Q} \times Z_0(X_{\mathrm{reg}})_0)'.$$

Finally, restricting to integer coefficients we get

$$[\,\cdot\,,\,\cdot\,]_X : \big(\mathrm{Div}^0(X) \times Z_0(X_{\mathrm{reg}})_0\big)' \to \mathbb{Q}_p.$$

4.5.2 Case of smooth varieties

Suppose X is a smooth and connected equidimensional variety, so that $S = 0$. In this case, $\mathrm{Pic}^-(X)$ is the semi-abelian variety

$$0 \to T(\bar{X}, Y) \to \mathrm{Pic}^0(\bar{X}, Y) \to A(\bar{X}, Y) \to 0.$$

Since X is connected then so is $\bar{X}$. Therefore, with the notations of Section 1.4, we have $\mathbb{Z}^{\bar{X}} = \mathbb{Z}$ and so $\mathbb{Z}^{(\bar{X},Y)} := \mathrm{Ker}(\gamma : \mathbb{Z}^Y \to \mathbb{Z})$ is generated by classes $[Y_I] - [Y_J]$, where Y_I and Y_J are connected components of Y (in the case that the connected components are the same, this is just the cycle 0). The Albanese 1-motive of X is

$$\mathrm{Alb}^+(X) = \mathrm{Pic}^-(X)^\vee = \left[u^\vee : \mathbb{Z}^{(\bar{X},Y)} \to \frac{\mathrm{Alb}(\bar{X})}{\mathrm{Im}(\oplus\,\mathrm{Alb}(Y_i))}\right],$$

where $u^\vee$ is induced by the Albanese mapping $a_{\bar{X}} : Z_0(\bar{X})_0 \to \mathrm{Alb}(\bar{X})$. Define

$$\begin{aligned}[\,\cdot\,,\,\cdot\,] : \big(\mathrm{Div}^0(\bar{X}, Y) \times Z_0(\bar{X})_0\big)' &\to \mathbb{Q}_p \\ (D, \textstyle\sum n_j x_j) &\mapsto \textstyle\sum n_j \psi \circ s_D \circ a_{\bar{X}}(x_j),\end{aligned}$$

where s_D is a rational section associated to D. Since the trivialization $\tau_{L^\vee}$ of the Poincaré biextension P restricted to $\{\mathcal{O}(D)\} \times \mathbb{Z}^{(\bar{X},Y)}$ is given by the rational section s_D and $\psi \circ \tau_{L^\vee} = 0$ then $[\,\cdot\,,\,\cdot\,]$ is zero on the image of $\mathbb{Z}^{(\bar{X},Y)}$.

Denote $\iota : Y \hookrightarrow \bar{X}$ the closed immersion. We have the following exact sequence

$$Z_0(Y)_0 \xrightarrow{\iota_*} Z_0(\bar{X})_0 \to Z_0(X)_0 \to 0,$$

where the first map is pushforward by ι. Notice that $Z_0(Y)_0$ is generated by cycles of the form $y - y'$, for $y, y \in Y$. If Y_I, Y_J are the connected components of Y such that $y \in Y_I, y' \in Y_J$, then

$$\begin{aligned} \mathrm{Alb}(\bar{X}) &\twoheadrightarrow \frac{\mathrm{Alb}(\bar{X})}{\mathrm{Im}(\oplus \mathrm{Alb}(Y_i))} \\ a_{\bar{X}} \circ \iota_*(y - y') &\mapsto u_X([Y_I] - [Y_J]). \end{aligned}$$

Since $[\cdot\,,\,\cdot]$ is zero on the image of $\mathbb{Z}^{(\bar{X},Y)}$ this yields a pairing

$$[\cdot\,,\,\cdot]_X : (\mathrm{Div}^0(\bar{X}, Y) \times Z_0(X)_0)' \to \mathbb{Q}_p.$$

4.5.3 Case of curves

Let $X = C$ be a semi-normal irreducible curve over K. We have the following commutative diagram

$$\begin{array}{ccc} C' & \xhookrightarrow{j'} & \bar{C}' \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\bar{\pi}} \\ C & \xhookrightarrow{j} & \bar{C}\ , \end{array}$$

where C' is the normalization of C, $\bar{C}'$ is a smooth compactification of C' and $\bar{C}$ is a compactification of C. Denote S the set of singular points of C, $S' := \pi^{-1}(S)$ and $F := \bar{C}' - C' = \bar{C} - C$. In this case, we have

$$\mathrm{Pic}^-(C) = [u : \mathrm{Div}^0_{S'/S}(\bar{C}', F) \to \mathrm{Pic}^0(\bar{C}', F)]$$

and

$$\mathrm{Alb}^+(C) = \mathrm{Pic}^-(C)^\vee = [u^\vee : \mathrm{Div}^0_F(\bar{C}') \to \mathrm{Pic}^0(\bar{C})].$$

Consider the Albanese mapping $a^+_C : Z_0(\bar{C}_{\mathrm{reg}})_0 \to \mathrm{Pic}^0(\bar{C})$. Define

$$\begin{aligned} [\cdot\,,\,\cdot] : (\mathrm{Div}^0(\bar{C}', F) \times Z_0(\bar{C}_{\mathrm{reg}})_0)' &\to \mathbb{Q}_p \\ (D, \textstyle\sum n_j x_j) &\mapsto \textstyle\sum n_j \psi \circ s_D \circ a^+_C(x_j), \end{aligned}$$

where s_D is a rational section associated to D. Since $\tau_L \circ \psi = \tau_{L^\vee} \circ \psi = 0$ then $[\cdot\,,\,\cdot]$ is zero on the image of $\mathrm{Div}^0_{S'/S}(\bar{C}', F)$ and $\mathrm{Div}^0_F(\bar{C}')$.

Since $F \subset \bar{C}_{\mathrm{reg}}$ is closed we have the following exact sequence

$$Z_0(F)_0 \to Z_0(\bar{C}_{\mathrm{reg}})_0 \to Z_0(C_{\mathrm{reg}})_0 \to 0.$$

Notice that $\mathrm{Div}^0_F(\bar{C}') = Z_0(F)_0$, because C is irreducible. So we have that $[\cdot\,,\,\cdot]$ is zero on $Z_0(F)_0$, giving us a pairing

$$[\cdot\,,\,\cdot]' : (\mathrm{Div}^0(\bar{C}', F) \times Z_0(C_{\mathrm{reg}})_0)' \to \mathbb{Q}_p.$$

We have the following commutative diagram

$$
\begin{array}{ccccccc}
 & & \mathrm{Ker}(\bar{\pi}_*) \otimes \mathbb{Q} & = & \mathrm{Ker}(\pi_*) \otimes \mathbb{Q} & & \\
 & & \downarrow & & \downarrow & & \\
Z_0(F)_0 \otimes \mathbb{Q} & \longrightarrow & Z_0(\bar{C}')_0 \otimes \mathbb{Q} & \longrightarrow & Z_0(C')_0 \otimes \mathbb{Q} & \longrightarrow & 0 \\
\| & & \downarrow{\scriptstyle \bar{\pi}_*} & & \downarrow{\scriptstyle \pi_*} & & \\
Z_0(F)_0 \otimes \mathbb{Q} & \longrightarrow & Z_0(\bar{C})_0 \otimes \mathbb{Q} & \longrightarrow & Z_0(C)_0 \otimes \mathbb{Q} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & .
\end{array}
$$

Since every closed point in C' is also closed in $\bar{C}'$, we have a map $Z_0(C')_0 \to \mathrm{Div}^0(\bar{C}', F)$ which maps $\mathrm{Ker}(\pi_*)$ to $\mathrm{Div}^0_{S'/S}(\bar{C}', F)$

$$
\begin{array}{ccc}
\mathrm{Ker}(\pi_*) & \longrightarrow & \mathrm{Div}^0_{S'/S}(\bar{C}', F) \\
\big\downarrow & & \big\downarrow \\
Z_0(C')_0 & \longrightarrow & \mathrm{Div}^0(\bar{C}', F)\ .
\end{array}
$$

By composing $[\,\cdot\,,\,\cdot\,]'$ with $Z_0(C')_0 \to \mathrm{Div}^0(\bar{C}', F)$ and extending coefficients to $\mathbb{Q}$ we get a pairing on

$$(Z_0(C')_0 \otimes \mathbb{Q} \times Z_0(C_{\mathrm{reg}})_0)'.$$

Since it must be zero on $\mathrm{Ker}(\pi_*) \otimes \mathbb{Q}$, this yields

$$[\,\cdot\,,\,\cdot\,]_C : (Z_0(C)_0 \times Z_0(C_{\mathrm{reg}})_0)' \to \mathbb{Q}_p$$

by restricting to integer coefficients.

4.6 Global pairing

Let F be a global field endowed with a set of places $\mathcal{V}$ which are either archimedean or discrete satisfying that $|c|_v = 1$, for every $c \in F^*$ and almost all places v. For each place v let F_v denote the completion of F with respect to v; for v discrete denote R_v the ring of integers of F_v and π_v a uniformizer. Consider a homomorphism $\rho = (\rho_v) : \mathbb{A}_F^* \to Y$ from the invertible elements of the ring of adeles $\mathbb{A}_F$ of F to a a torsion-free divisible group Y (this is equivalent to Y being a $\mathbb{Q}-$vector space) which annihilates the image of R_v^*, for almost all discrete places v, as well as the image of F^* under the canonical homomorphisms, and satisfies the "sum formula" $\sum_v \rho_v(c) = 0$ for all $c \in F^*$. Denote $\mathcal{V}_{un}$ the set of places v with ρ_v unramified; note that $\mathcal{V}_{un}$ consists only of discrete places and that $\mathcal{V} - \mathcal{V}_{un}$ is finite.

Let $M_F = [L_F \xrightarrow{u_F} G_F]$ be a 1-motive over F such that M_{F_v} is strict for every $v \in \mathcal{V}_{un}$ (see Definition 1.5.2). In particular, this implies that A_{F_v} has potentially good reduction for all $v \in \mathcal{V}_{un}$. Notice that we can always find a finite extension F' of F such that $A_{F'_w}$ has good reduction for all valuations w of F' extending a valuation in $\mathcal{V}_{un}$; we will assume that we are in this situation, *i.e.* that A_{F_v} has good reduction for all $v \in \mathcal{V}_{un}$. We are also assuming that L_F and T_F are split; this always happens over a finite extension of F as well.

Assume that the canonical ρ_v–splitting $\psi_{A,v} : P_{A_{F_v}}(F_v) \to Y$ of $P_{A_{F_v}}$ given in Theorem 2.1.6 exists for all v and denote by ψ_v the composition

$$\psi_v : P_K(K) \to P_{A_K}(K) \xrightarrow{\psi_{A,v}} Y.$$

We have the following

Proposition 4.6.1. *For every $x^\vee \in L_F^\vee(F)$ and $g \in G_F(F)$ there exists $t \in T_F(F)$ such that*

$$\sum_v \psi_v \circ \tau_{L_v^\vee}(g, x^\vee) = \sum_{v \in \mathcal{V}-\mathcal{V}_{un}} \psi_v \circ \tau_{L_v^\vee}(t^{-1}g, x^\vee),$$

and similarly for every $x \in L_F(F)$ and $g^\vee \in G_F^\vee(F)$.

Proof. Suppose $L_F^\vee \cong \mathbb{Z}_F^r$ and let $x^\vee = (m_1, \ldots, m_r) \in L_F^\vee(F)$. Notice that this implies that $T_F \cong \mathbb{G}_{m,F}^r$. Consider a finite place v in $\mathcal{V}_{un}$. Since G_{F_v} has good reduction then $A_{F_v}(F_v) = A(R_v)$ and we have isomorphisms

$$\frac{T_{F_v}(F_v)}{T(R_v)} \cong \frac{G_{F_v}(F_v)}{G(R_v)}.$$

Let $g \in G_F(F)$. Any representative $t_v \in T_{F_v}(F_v)$ corresponding to the class of $g \in G_{F_v}(F_v)$ satisfies

$$\psi_v \circ \tau_{L_v^\vee}(t_v, x^\vee) = \psi_v \circ \tau_{L_v^\vee}(g, x^\vee).$$

We may choose as representative an element of the form $t_v := (\pi_v^{n_{1,v}}, \ldots, \pi_v^{n_{r,v}})$. In this way, t_v belongs to $T_F(F)$ and for every place $w \neq v$ in $\mathcal{V}_{un}$ we have

$$\psi_w \circ \tau_{L_w^\vee}(t_v, x^\vee) = \rho_w(\pi_v^{n_1 m_1} \ldots \pi_v^{n_r m_r}) = 0,$$

since $w(\pi_v^{n_1 m_1} \ldots \pi_v^{n_r m_r}) = 0$. Defining

$$t := \prod_{v \in \mathcal{V}_{un}} t_v = (\prod_{v \in \mathcal{V}_{un}} \pi_v^{n_{1,v}}, \ldots, \prod_{v \in \mathcal{V}_{un}} \pi_v^{n_{r,v}}) \in T_F(F)$$

we get from the previous equalities that t satisfies

$$\psi_v \circ \tau_{L_v^\vee}(g, x^\vee) = \psi_v \circ \tau_{L_v^\vee}(t, x^\vee),$$

for every $v \in \mathcal{V}_{un}$. So we obtain

$$\begin{aligned}\sum_v \psi_v \circ \tau_{L_v^\vee}(g, x^\vee) &= \sum_{v \in \mathcal{V}-\mathcal{V}_{un}} \psi_v \circ \tau_{L_v^\vee}(g, x^\vee) + \sum_{v \in \mathcal{V}_{un}} \psi_v \circ \tau_{L_v^\vee}(g, x^\vee) \\ &= \sum_{v \in \mathcal{V}-\mathcal{V}_{un}} \psi_v \circ \tau_{L_v^\vee}(g, x^\vee) + \sum_{v \in \mathcal{V}_{un}} \psi_v \circ \tau_{L_v^\vee}(t, x^\vee) \\ &= \sum_{v \in \mathcal{V}-\mathcal{V}_{un}} \psi_v \circ \tau_{L_v^\vee}(g, x^\vee) - \sum_{v \in \mathcal{V}-\mathcal{V}_{un}} \psi_v \circ \tau_{L_v^\vee}(t, x^\vee) \\ &= \sum_{v \in \mathcal{V}-\mathcal{V}_{un}} \psi_v \circ \tau_{L_v^\vee}(t^{-1}g, x^\vee).\end{aligned}$$

□

Corollary 4.6.2. *Suppose that u_F and $u_F^\vee$ are injective. If the ρ_v-splittings ψ_v are compatible with the $L_{F_v} \times L_{F_v}^\vee-$linearization of P_{F_v}, for every place $v \in \mathcal{V} - \mathcal{V}_{un}$, then the pairing $\langle \cdot , \cdot \rangle$ of Lemma 2.3.1 induces a pairing on $M_F(F) \times M_F^\vee(F)$.*

Proof. For $g \in G_F(F)$ and $x^\vee \in L_F^\vee(F)$ let $t \in T_F(F)$ be the element constructed in Lemma 4.6.1. We have

$$\sum_v \psi_v \circ \tau_{L_v^\vee}(g, x^\vee) = \sum_{v \in \mathcal{V}-\mathcal{V}_{un}} \psi_v \circ \tau_{L_v^\vee}(t^{-1}g, x^\vee) = 0.$$

Since we also have an analogous equality for every $x \in L_F(F)$ and $g^\vee \in G_F^\vee(F)$ then the composition

$$G_F(F) \times G_F^\vee(F) \to A_F(F) \times A_F^\vee(F) \xrightarrow{\langle \cdot , \cdot \rangle} Y$$

induces a pairing

$$\langle \cdot , \cdot \rangle_M : M_F(F) \times M_F^\vee(F) \to Y.$$

□

Bibliography

[AB05] Fabrizio Andreatta and Luca Barbieri-Viale. "Crystalline realizations of 1-motives". *Mathematische Annalen* 331.1 (2005), pp. 111–172.

[Bar07] Luca Barbieri-Viale. "On the theory of 1-motives". *Algebraic cycles and motives, Vol. 1.* Ed. by J. Nagel and C. Peters. London Mathematical Society Lecture Note Series 343. Cambridge University Press, Cambridge, 2007, pp. 55–101.

[BB] Luca Barbieri-Viale and Massimo Bertolini. "Values of 1-motivic L-functions". In preparation.

[BK16] Luca Barbieri-Viale and Bruno Kahn. *On the derived category of 1-motives.* Astérisque 381. 2016, pp. xi+254.

[BS01] Luca Barbieri-Viale and Vasudevan Srinivas. *Albanese and Picard 1-motives.* Mémoires de la Société Mathématique de France. Nouvelle Série 87. 2001, pp. vi+104.

[Ber09] Alessandra Bertapelle. "Deligne's duality for de Rham realizations of 1-motives". *Mathematische Nachrichten* 282.12 (2009), pp. 1637–1655.

[BCC04] Alessandra Bertapelle, Maurizio Candilera, and Valentino Cristante. "Monodromy of logarithmic Barsotti-Tate groups attached to 1-motives". *Journal für die Reine und Angewandte Mathematik [Crelle's Journal]* 573 (2004), pp. 211–234.

[Blo84] Spencer Bloch. "Height pairings for algebraic cycles". *Journal of Pure and Applied Algebra* 34.2 (1984), pp. 119–145.

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models.* Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge [A Series of Modern Surveys in Mathematics] 21. Springer-Verlag, Berlin, 1990, pp. x+325.

[Bre83] Lawrence Breen. *Fonctions thêta et théorème du cube.* Lecture Notes in Mathematics 980. Springer-Verlag, Berlin, 1983, pp. xiii+115.

[Col89] Robert F. Coleman. "Reciprocity laws on curves". *Compositio Mathematica* 72.2 (1989), pp. 205–235.

[Col91] Robert F. Coleman. "The universal vectorial bi-extension and p-adic heights". *Inventiones Mathematicae* 103.3 (1991), pp. 631–650.

[Col98] Robert F. Coleman. “Duality for the de Rham cohomology of an abelian scheme”. *Université de Grenoble. Annales de l'Institut Fourier* 48.5 (1998), pp. 1379–1393.

[CG89] Robert F. Coleman and Benedict H. Gross. “p-adic heights on curves”. *Algebraic Number Theory – in honor of K. Iwasawa*. Ed. by J. Coates et al. Advanced Studies in Pure Mathematics 17. Academic Press, Boston, MA, 1989, pp. 73–81.

[CI99] Robert F. Coleman and Adrian Iovita. “The Frobenius and monodromy operators for curves and abelian varieties”. *Duke Mathematical Journal* 97.1 (1999), pp. 171–215.

[Del71a] Pierre Deligne. “Théorie de Hodge. I”. *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*. Gauthier-Villars, Paris, 1971, pp. 425–430.

[Del71b] Pierre Deligne. “Théorie de Hodge. II”. *Institut des Hautes Études Scientifiques. Publications Mathématiques* 40 (1971), pp. 5–57.

[Del74] Pierre Deligne. “Théorie de Hodge. III”. *Institut des Hautes Études Scientifiques. Publications Mathématiques* 44 (1974), pp. 5–77.

[Del79] Pierre Deligne. “Valeurs de fonctions L et périodes d'intégrales”. *Automorphic Forms, Representations and L-functions (Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*. Ed. by A. Borel and W. Casselman. Proceedings of Symposia in Pure Mathematics 33. With an appendix by N. Koblitz and A. Ogus. American Mathematical Society, Providence, R.I., 1979, pp. 313–346.

[DG80] Michel Demazure and Peter Gabriel. *Introduction to algebraic geometry and algebraic groups*. North-Holland Mathematics Studies 39. Translated from the French by J. Bell. North-Holland Publishing Co., Amsterdam-New York, 1980, pp. xiv+357.

[FC90] Gerd Faltings and Ching-Li Chai. *Degeneration of abelian varieties*. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge [A Series of Modem Surveys in Mathematics] 22. With an appendix by David Mumford. Springer-Verlag, Berlin, 1990, pp. xii+316.

[Ful98] William Fulton. *Intersection theory*. Second edition. Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge [A Series of Modern Surveys in Mathematics] 2. Springer-Verlag, Berlin, 1998, pp. xiv+470.

[Gir71] Jean Giraud. *Cohomologie non abélienne*. Grundlehren der mathematischen Wissenschaften [A Series of Comprehensive Studies in Mathematics] 179. Springer-Verlag, Berlin-New York, 1971, pp. ix+467.

[Gro86] Benedict H. Gross. “Local Heights on Curves”. *Arithmetic Geometry (Storrs, Conn., 1984)*. Ed. by G. Cornell and J. H. Silverman. Springer, New York, 1986, pp. 327–339.

[SGA7-I] *Groupes de monodromie en géométrie algébrique I.* Lecture Notes in Mathematics 288. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I). Dirigé par A. Grothendieck, avec la collaboration de M. Raynaud et D. S. Rim. Springer-Verlag, Berlin-New York, 1972, pp. viii+523.

[SGA7-II] *Groupes de monodromie en géométrie algébrique II.* Lecture Notes in Mathematics 340. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 II). Dirigé par P. Deligne et N. Katz. Springer-Verlag, Berlin-New York, 1973, pp. x+438.

[Har77] Robin Hartshorne. *Algebraic Geometry.* Graduate Texts in Mathematics 52. Springer-Verlag, New York, 1977, pp. xvi+496.

[Iov00] Adrian Iovita. "Formal sections and de Rham cohomology of semistable abelian varieties". *Israel Journal of Mathematics* 120.2 (2000), pp. 429–447.

[IW03] Adrian Iovita and Annette Werner. "p-adic height pairings on abelian varieties with semistable ordinary reduction". *Journal für die Reine und Angewandte Mathematik [Crelle's Journal]* 564 (2003), pp. 181–203.

[Jos09] Peter Jossen. "On the arithmetic of 1-motives". PhD thesis. Central European University, Department of mathematics and its applications, 2009.

[Kat70] Nicholas M. Katz. "Nilpotent connections and the monodromy theorem: Applications of a result of Turrittin". *Institut des Hautes Études Scientifiques. Publications Mathématiques* 39 (1970), pp. 175–232.

[Liu02] Qing Liu. *Algebraic Geometry and Arithmetic Curves.* Oxford Graduate Texts in Mathematics 6. Translated from the French by Reinie Erné, Oxford Science Publications. Oxford University Press, Oxford, 2002, pp. xvi+576.

[MM74] Barry Mazur and William Messing. *Universal extensions and one dimensional crystalline cohomology.* Lecture Notes in Mathematics 370. Springer-Verlag, Berlin-New York, 1974, pp. vii+134.

[MT83] Barry Mazur and John Tate. "Canonical Height Pairings via Biextensions". *Arithmetic and Geometry: Papers Dedicated to I.R. Shafarevich on the Occasion of His Sixtieth Birthday, Volume I: Arithmetic.* Ed. by M. Artin and J. Tate. Progress in Mathematics 35. Birkhäuser Boston, Boston, MA, 1983, pp. 195–237.

[Mil80] James S. Milne. *Étale cohomology.* Princeton Mathematical Series 33. Princeton University Press, Princeton, N.J., 1980, pp. xiii+323.

[Mil86] James S. Milne. "Jacobian Varieties". *Arithmetic Geometry (Storrs, Conn., 1984).* Ed. by G. Cornell and J. H. Silverman. Springer, New York, 1986, pp. 167–212.

[Mum69] David Mumford. "Bi-extensions of formal groups". *Algebraic Geometry (Internat. Colloq., Tata Inst. Fund. Res., Bombay, 1968).* Oxford Univ. Press, London, 1969, pp. 307–322.

[MFK94] David Mumford, John Fogarty, and Frances Kirwan. *Geometric Invariant Theory*. Third edition. Ergebnisse der Mathematik und ihrer Grenzgebiete, 2. Folge [A Series of Modem Surveys in Mathematics] 34. Springer-Verlag, Berlin, 1994, pp. xiv+292.

[Nér82] André Néron. "Fonctions thêta p-adiques et hauteurs p-adiques". *Séminaire de Théorie des Nombres, Paris 1980-81*. Ed. by M.-J. Bertin. Progress in Mathematics 22. Birkhäuser Boston, Boston, MA, 1982, pp. 149–174.

[Oor66] Frans Oort. *Commutative Group Schemes*. Lecture Notes in Mathematics 15. Springer-Verlag, Berlin-New York, 1966, vi+133 pp. (not consecutively paged).

[Org04] Fabrice Orgogozo. "Isomotifs de dimension inférieure ou égale à un". *Manuscripta Mathematica* 115.3 (2004), pp. 339–360.

[Ray94] Michel Raynaud. "1-motifs et monodromie géométrique". *Périodes p-adiques (Bures-sur-Yvette, 1988)*. Astérisque 223. 1994, pp. 295–319.

[Ros58] Maxwell Rosenlicht. "Extensions of vector groups by abelian varieties". *American Journal of Mathematics* 80 (1958), pp. 685–714.

[Ser88] Jean-Pierre Serre. *Algebraic groups and class fields*. Graduate Texts in Mathematics 117. Translated from the French. Springer-Verlag, New York, 1988, pp. x+207.

[ST68] Jean-Pierre Serre and John Tate. "Good Reduction of Abelian Varieties". *Annals of Mathematics. Second Series* 88.3 (1968), pp. 492–517.

[Wer98] Annette Werner. "Local heights on abelian varieties and rigid analytic uniformization". *Documenta Mathematica* 3 (1998), pp. 301–319.

[Zar90] Yuri G. Zarhin. "p-adic heights on abelian varieties". *Séminaire de Théorie des Nombres, Paris 1987–88*. Ed. by C. Goldstein. Progress in Mathematics 81. Birkhäuser Boston, Boston, MA, 1990, pp. 317–341.

[Zar96] Yuri G. Zarhin. "p-adic abelian integrals and commutative Lie groups". *Journal of Mathematical Sciences* 81.3 (1996), pp. 2744–2750.