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On the stability of the perturbed central motion problem: a quasiconvexity and a Nekhoroshev type result

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Introduction

This thesis is devoted to the study of the dynamics of small perturbations of the spatial central motion.

In particular, we are interested in proving a Nekhoroshev type theorem for it.

The point is that Nekhoroshev's theorem applies to perturbations of integrable systems whose Hamiltonian when written in the action angle coordinates is steep. Now such a property, in its original form, is always violated in the spatial central motion since it is a superintegrable system and its Hamiltonian turns out to be always independent of one of the actions.

For degenerate systems, stability results have been obtained by Nekhoroshev in the papers ([Nek77, Nek79]) and, subsequently, by Niedermann (see [Nie96]), Guzzo and Morbidelli, see [GM96, Guz99] in which the authors apply exponential stability results in order to study the stability of the planetary problem.

A general Nekhoroshev theory for superintegrable systems has been developed also by Fassò [Fas95], [Fas05] (see also Blaom in [Bla01]), and the main known result is that a weaker version of Nekhoroshev's theorem ensuring almost conservation of the two actions on which the Hamiltonian depends holds provided the Hamiltonian is a convex function of these two actions. We remark that it is quite clear how to extend Fassò's theory to the case of steep dependence on the two actions.

The first goal of the thesis is to write a complete proof of Nekhoroshev's theorem for superintegrable systems under the assumption that the Hamiltonian is quasiconvex in the actions on which it actually depends. This is done by generalizing the proof by Lochak (see [Loc92]) which is much simpler then the original one by Nekhoroshev (see [Nek77, Nek79]) (the one extended by Fassò).

Then, we tackle the problem of proving that the Hamiltonian in action angle variables is quasiconvex. This is far from trivial since the expression of the Hamiltonian depends on the form of the potential and one expects that quasiconvexity holds under some conditions on the potential. The main technical result of the thesis is that actually there are only two central potentials corresponding to which the Hamiltonian is not quasiconvex, namely, the Harmonic and the Keplerian one.

We are now going to state in a precise way the main result of the thesis.

In Cartesian coordinates, the Hamiltonian of the spatial central motion is given by

$$H(\mathbf{x},\mathbf{p}) = \frac{|\mathbf{p}|^2}{2} + V(|\mathbf{x}|) \ , \tag{1}$$

$$\mathbf{p} \equiv (p_x, p_y, p_z) \ , \quad \mathbf{x} \equiv (x,y,z) \ , \quad |\mathbf{x}| := \sqrt{x^2+y^2+z^2} \ ,$$

where V is the potential that we assume to be analytic. Furthermore, we assume that it fulfills the following assumptions

(H0) $V : (0,+\infty) \to \mathbb{R}$ *is a real analytic function.*

(H1) $-\ell^* := \frac{1}{2} \lim_{r\to 0^+} r^2 V(r) > -\infty$

(H2) $\exists r > 0 \ : \ r^3 V'(r) > \max\{0, \ell^*\}$,

(H3) $\forall \ell > \max\{0,\ell^*\}$ the equation (in r) $r^3V'(r) = \ell$ has at most a finite number of solutions.

Remark 0.0.1. *(H1) ensures that there are no collision orbits provided the angular momentum is large enough; (H2) ensures that the effective potential has at least one strict minimum so that the domain of the actions is not empty; finally (H3) ensures that the domain of the actions is not too complicated.*

Remark 0.0.2. *For example any analytic potential of Schwartz class such that bounded orbits exist fulfills the assumptions.*

Define the total angular momentum $(L_1, L_2, L_3) \equiv \mathbf{L} := \mathbf{x} \times \mathbf{p}$ and denote by $L := \sqrt{L_1^2 + L_2^2 + L_3^2}$ its modulus.

Let $\mathcal{P}_A^{(3)} \subset \mathbb{R}^6$ be a compact subset of the phase space invariant under the dynamics of H,

Theorem 0.0.1. *Assume that V is neither Harmonic nor Keplerian; then there exists a set $\mathcal{K}^{(3)} \subset \mathcal{P}_A^{(3)}$, which is the union of finitely many analytic hypersurfaces, with the following property: let $P : \mathcal{P}_A^{(3)} \to \mathbb{R}$ be a real analytic function. Let $\mathcal{C}^{(3)} \subset \mathcal{P}_A^{(3)} \setminus \mathcal{K}^{(3)}$ be compact and invariant for the dynamics of H; then there exist positive ε_*, C_1, C_2, C_3, C_4 with the following property: for $|\varepsilon| < \varepsilon_*$, consider the dynamics of the Hamiltonian system*

$$H_\varepsilon := H + \varepsilon P$$

then, for any initial datum in $\mathcal{C}^{(3)}$ one has

$$|L(t) - L(0)| \le C_1 \varepsilon^{1/4} \ , \quad |H(t) - H(0)| \le C_2 \varepsilon^{1/4} \ , \tag{2}$$

for

$$|t| \le C_3 \exp(C_4 \varepsilon^{-1/4}) \ . \tag{3}$$

An immediate consequence of the above theorem is that the particle's orbits are confined between two spherical shells centered at the origin.

We now discuss the proof of the result. As anticipated above, the Hamiltonian system associated to the spatial central motion problem belongs to the class of superintegrable systems, namely, systems which admit a number of independent integrals of motion larger than the number of degrees of freedom. The main property of such systems is that, under some technical conditions, they admit generalized action angle coordinates and the Hamiltonian turns out to depend on a number of actions strictly smaller than the number of degrees of freedom.

Furthermore, in general, the level set of the actions is a nontrivial manifold which cannot be covered by only one system of coordinates. As pointed out by Fassò, this poses nontrivial problems for the development of the proof of Nekhoroshev's theorem. The geometrical idea introduced by Fassò in order to prove Nekhoroshev's theorem is that, even if normal form theory is classically developed using coordinates, in the framework of superintegrable systems, the expressions obtained in the chart, as well as the normalizing transformation, glue together and give a function and a normal form which are defined "semilocally". By this we mean on the manifold obtained by considering the union for I in a small open set of the level sets of I, where I are the actions of the system.

In Chapter 1 of this thesis, we show how to use these ideas in order to adapt Lochak's proof of Nekhoroshev's theorem to superintegrable systems. We also detail the proof for the case of quasiconvex systems, which, as far as we know, was not treated explicitly in literature. We remark that in order to get the proof only some of the ideas by Fassò are needed.

Then (Chapter 2), we come to a detailed study of the central motion problem. First, we apply the general geometric theory of superintegrable systems to the spatial central motion. To do so, we first analyze the planar central motion and show that we can reduce our analysis to this case. Thus, it turns out that the general Nekhoroshev's theorem proved in Chapter 1 applies if the Hamiltonian of the *planar* central motion is quasiconvex. Thus we study it.

In polar coordinates, the Hamiltonian of the planar central motion problem has the well known form

$$H(r, p_r, p_\theta) = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} + V(r) \ . \tag{4}$$

The main remark is that, in the case of systems with 2 degrees of freedom, the quasiconvexity condition turns out to be equivalent to the nonvanishing of the so called *Arnol'd determinant*, namely,

$$\mathcal{D} = \det \begin{pmatrix} \frac{\partial^2 h}{\partial I^2} & \left(\frac{\partial h}{\partial I}\right)^T \\ \frac{\partial h}{\partial I} & 0 \end{pmatrix} ,$$

where h is the unperturbed Hamiltonian written in the action variables. Moreover, in the analytic case, the Arnol'd determinant is an analytic function, thus only two possibilities occur: either it is a trivial analytic function, or it is always different from zero except on an analytic hypersurface.

Recall now that the two actions of the planar system are the angular momentum vector $I_2 := p_\theta$ and the action I_1 of the reduced system, that is, the system with Hamiltonian (4) where p_θ plays the role of a parameter. The action I_1 depends on the form of the effective potential, namely,

$$V_{eff}(r, p_\theta^2) := \frac{p_\theta^2}{2r^2} + V(r) .$$

We use the assumptions $(H0) - (H3)$ in order to study quite precisely the domain of I_1, I_2.

More precisely, we start by proving that, correspondingly to almost every value of I_2, the effective potential has only nondegenerate critical points. Then, we fix a value of the angular momentum I_2 and we proceed with the standard construction of the action I_1.

A simple analysis shows that the domain of definition of the action I_1 is the union of some open connected regions $\mathcal{E}_j$ of the phase-space. The regions $\mathcal{E}_j$ can be classified into two categories according to the nature of the critical points of V_{eff} contained in their closure. Precisely, we will distinguish between the regions $\mathcal{E}_j^{(1)}$ whose closure contains a minimum of the effective potential and the regions $\mathcal{E}_j^{(2)}$ whose closure, instead, does not contain a minimum but contains necessarily a maximum of the effective potential.

Then, the heart of the proof is based on the study of the asymptotic behavior of the Arnol'd determinant at circular orbits corresponding to the critical points of V_{eff} and goes differently in the two kinds of regions.

Specifically, we first consider the regions $\mathcal{E}_j^{(1)}$ and we compute the first terms of the expansion of the Hamiltonian at the minima by computing the first terms of the Birkhoff normal form of the effective system in terms of the derivatives of the potential. This has been done extending the procedure used by Féjoz in [FK04] who actually did the computation at order 4. Here we go at order 6. Then, we use such an expansion in order to compute the first terms of the Arnol'd determinant and to show that it is a nontrivial function, except in the Harmonic and Keplerian cases.

Precisely, we get that the first two terms of the expansion of the Arnol'd determinant vanish identically if the potential $V(r)$ fulfills a couple of differential equations. Then, we search for the common solutions of these two equations and we obtain that they are the Keplerian and the Harmonic potentials. This is a quite heavy computation and is done by the help of a symbolic manipulator (Mathematica^{TM}). The corresponding computation is reported in Appendix E.

Then, we consider the second regions $\mathcal{E}_j^{(2)}$. We prove that if such a region exists then, the Arnol'd determinant diverges at its boundary and thus it is a nontrivial function of the actions.

The result is obtained by exploiting the fact that the action I_1 at the maximum admits an asymptotic expansion of the form

$$I_1 = -\Lambda(E - V_0, I_2)\ln(E - V_0) + G_1(E - V_0, I_2) \ , \tag{5}$$

where we denote by V_0 the value of the effective potential at the maximum, by E the energy level while G_1 and Λ are two analytic functions. Moreover, Λ has a zero of order 1 in $(0, I_2)$. Secondly, from this expansion, we derive the asymptotic behavior of the Arnol'd determinant at the maximum and prove that it diverges at such a point. In the thesis we prove formula (5) exploiting a normal form result by Giorgilli [Gio01]. This formula also appears in [BC17]. In this work the authors apply a KAM type theorem to a nearly-integrable Hamiltonian system under suitable conditions. Previously, such a formula also appeared in the work by Neishtadt [Nei87].

To conclude we remark that our result, showing some peculiarities of the Harmonic and the Keplerian potentials, of course reminds Bertrand's theorem. Actually our analysis of the minima of the effective potential is a refinement of that used in the proof of Bertrand's theorem and can be used to get a new proof of such a theorem (see Section 2.7). In Appendix A, we also added a proof of such a result.

The results discussed here have been the object for two papers: [BF17] and [BFS17].

Chapter 1

Superintegrable Hamiltonian systems

The aim of this section is the development of a Hamiltonian perturbation theory for *superintegrable* sytems, that we are now going to define

Definition 1.0.1. *Let (H, M, ω) a Hamiltonian system, where $H : M \mapsto \mathbb{R}$ is the Hamiltonian function, M a $2d$-dimensional symplectic manifold and ω the symplectic form.*

Let us consider k functions $F_1, \dots, F_k : M \mapsto \mathbb{R}$. Define a map $F := (F_1, \dots, F_k) : M \mapsto \mathbb{R}^k$ and consider the function $F : M \mapsto F(M) := \mathcal{M} \subset \mathbb{R}^k$.

We say that the functions F_j constitute a maximal set of independent integrals *of motion if the following conditions are satisfied*

1. $\{H, F_j\} = 0 \quad , \forall j = 1, \dots, k \quad ,$
2. *$dF_1, \dots, dF_k$ are linearly independent at every point of M ,*
3. *for any other function G such that $\{H, G\}$, the differentials $dG, dF_1, \dots, dF_k$ are linearly dependent .*

Remark 1.0.3. *In a superintegrable system, the F_j's are the elements of a maximal set of independent integrals of motion, F is a surjective submersion since the rank of the Jacobian matrix associated is constant at every point of M^* and equal to $2d - n$. Furthermore, the fibers are the level sets $F^{-1}(c) := \{x \in M \ : \ F_l(x) = c_l, \quad l = 1, \dots, k\}$, $c_l \in \mathbb{R}$.*

Definition 1.0.2. *Let $F_1, \dots, F_k$ be a maximal set of independent integrals of motion and suppose that there exists real analytic functions $P_{i,j} : \mathcal{M} \mapsto \mathbb{R}$ such that*

$$\{F_i, F_j\} = P_{i,j} \circ F, \quad i, j = 1, \dots, k \ .$$

The $k \times k$ matrix whose entries are the functions $P_{i,j}$ is the Poisson matrix.

Definition 1.0.3. *Let us consider a Hamiltonian system and let* $F_1, \ldots, F_k$ *be a maximal set of independent integrals of motion admitting a Poisson matrix. The system is said to be* superintegrable *if* $k > d$ *and the Poisson matrix has constant rank equal to* $2k - 2d$ *at every point of* $\mathcal{M}$.

Definition 1.0.4. *If* $k = d$*, the Hamiltonian system is* completely integrable.

Superintegrable systems are characterized by the fact that the number of independent integrals of motion is greater than the number of degrees of freedom. They are also known as degenerate systems due to the fact that the corresponding Hamiltonian, when written in generalized action angle coordinates, does not depend on all the actions. A property that we will describe in detail below. Systems of that kind are frequent in literature: the Euler-Poinsot problem for the rigid body motion and the spatial central motion problem are two important examples.

Example 1. The central motion problem.

Let us consider the spatial central motion problem, that is, a particle in $\mathbb{R}^3$ *moving under a central potential. The Hamiltonian describing the system written in Cartesian coordinates is*

$$H(\mathbf{x}, \mathbf{p}) = \frac{|\mathbf{p}|^2}{2} + V(|\mathbf{x}|) \ ,$$
$$\mathbf{p} \equiv (p_x, p_y, p_z) \ , \quad \mathbf{x} \equiv (x, y, z) \ , \quad |\mathbf{x}| := \sqrt{x^2 + y^2 + z^2} \ .$$

Define the total angular momentum $(L_1, L_2, L_3) \equiv \mathbf{L} := \mathbf{x} \times \mathbf{p}$ *and denote by* $L := \sqrt{L_1^2 + L_2^2 + L_3^2}$ *its modulus.*

This system admits four integrals of motion, namely, the energy E *of the system and the three components* (L_1, L_2, L_3) *of the angular momentum vector. Provided we restrict to a subset* M^* *of the phase-space in which the modulus of the angular momentum vector is non zero and the motion is bounded, they constitute a maximal set of independent integrals of motion.*

Furthermore, if we compute the Poisson matrix associated, we obtain

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & L_3 & -L_2 \\ 0 & -L_3 & 0 & L_1 \\ 0 & L_2 & -L_1 & 0 \end{bmatrix} \ ,$$

which has rank 2 *at every point of* $F(M^*)$.

The main feature of these systems is that the variables corresponding to the degenerate directions do not contribute to the dynamics. Thus, the dimension of the tori filled by the flow is smaller than the number of degrees of freedom and it implies that the structure of the phase space is finer than the one which arises in the case of a complete integrable system.

In detail, in the first part of the current section, we introduce the *double fibration* structure of the phase-space by presenting first the locally trivial fibration in n-dimensional tori given by the adaptation of the Liouville-Arnol'd construction to the superintegrable case and, then, we will deal with a second fibration arising from a particular choice of a symplectic atlas adapted to the first fibration. The structure of the double fibration has been studied in detail in the works [Nek72], [KM12], [MF78], [Fas95] and [Fas05].

Secondly, we will present a stability result, a general Nekhoroshev type theorem for superintegrable Hamiltonian systems whose corresponding Hamiltonian is quasiconvex when written in the action angle variables.

1.1 The local geometry of a superintegrable system

We begin with a couple of definitions which are useful in the description of the phase space of a superintegrable system.

Definition 1.1.1. *Let M be a smooth manifold of dimension $2d$. A foliation of dimension n on M is an atlas $\{U_j, \phi_j\}_{j\in J}$ on M with the following properties*

1. $\forall p \in M$ *there exists a local chart* $\{U_j, \phi_j\}$ *such that* $\phi(U_j) = V' \times V''$, *with* $V' \subset \mathbb{R}^n$ *and* $V'' \subset \mathbb{R}^{2d-n}$ *open subsets.*

2. *if* $\{U_j, \phi_j\}$ *and* $\{U_k, \phi_k\}$ *are such that* $U_j \cap U_k \neq 0$ *then the transition functions* $\phi_k \circ \phi_j^{-1} : \phi_j(U_j \cap U_k) \mapsto \phi_k(U_j \cap U_k)$ *are of the form*

$$\phi_k \circ \phi_j^{-1}(x, y) = (f_1(x, y), f_2(y)) \ , (x, y) \in \mathbb{R}^n \times \mathbb{R}^{2d-n} \ .$$

The leaves of the foliation are locally described by sets of the form

$$\{y_{n+1} = c_{n+1}, \dots, y_m = c_m\} \ ,$$

with $c_k \in \mathbb{R}$ *which are* n*-dimensional submanifolds of* M.

Definition 1.1.2. *Let S be a manifold. A fibration (or fiber bundle) with fiber S on a manifold N is a C^∞ surjective map $f : M \mapsto N$ between a manifold M (the total space of the fibration) and the manifold N (the base space of the fibration) such that the following conditions are satisfied*

1. $\forall p \in N \ f^{-1}(p) := M_p \cong S$

2. *every fiber of f admits local trivializations, that is, $\forall p \in N$ there exist a neighborhood U of $p \in N$ and a diffeomorphism $\psi : f^{-1}(U) \mapsto U \times S$ such that the following diagram commutes*

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{\psi} & U \times S \\ & \searrow^{f} & \downarrow{\scriptstyle \pi^1} \\ & & U \end{array}$$

Let (H, M, ω) be a superintegrable Hamiltonian system where M is a $2d$-dimensional symplectic manifold. Let us consider a maximal set of independent integrals of motion

$$F_1, \ldots, F_{2d-n} : M^* \subset M \to \mathbb{R} \ ,$$

with $n < d$, defined on an open subset $M^* \subset M$. We remind the reader that we are considering a superintegrable system, thus, the number of constants of motion is greater than the number of degrees of freedom.
Let us consider the map $F := (F_1, \ldots, F_{2d-n}) : M^* \mapsto F(M^*) \subset \mathbb{R}^{2d-n}$. From Remark 1.0.3 we deduce that the map F is a *surjective submersion* whose fibers are the level sets $F^{-1}(c) = \{x \in M^* : F_k(x) = c_k \ , k = 1, \ldots, 2d-n\}$, $c_k \in \mathbb{R}$.

Thus, if we suppose that the fibers are compact and connected than it follows from *Ehresmann fibration lemma*[1] that the map F is a fibration in the sense of definition 1.1.2.

The main result which describes the local geometry of a superintegrable Hamiltonian system is a generalization of the Liouville-Arnol'd Theorem for complete integrable systems. Different versions of this theorem can be found in the works [Nek72], [MF78], [Fas95] and [Fas05]. The version we refer to in this work is the one given by Fassò in [Fas05] (or [Fas95]). The result is the following

Theorem 1.1.1. *Let (H, M, ω) be a $2d$-dimensional superintegrable Hamiltonian system. Let $F := (F_1, \ldots, F_{2d-n}) : M^* \subset M \mapsto F(M^*) := \mathcal{M} \subset \mathbb{R}^{2d-n}$ with $n < d$ be a map whose components belong to a maximal set of independent integrals of motion with the property that the rank of the Poisson matrix P is everywhere constant and equal to $2d - 2n$.*
Moreover, assume that the level sets, that is, the fibers of the map F are compact and connected.
Then,

1. *Every fiber of F is diffeomorphic to a n-dimensional torus $\mathbb{T}^n$*

[1] For details, see Appendix C

2. *Every fiber of F has a neighborhood $\mathrm{U} \subset M^*$ endowed with a diffeomorphism*

$$b \times \alpha : \mathrm{U} \to \mathcal{B} \times \mathbb{T}^n, \qquad \mathcal{B} = b(\mathrm{U}) \subset \mathbb{R}^{2d-n} \tag{1.1}$$

such that the level sets of F coincide with the level sets of b and, writing $b = (I_1, \cdots, I_n, p_1, \cdots, p_{d-n}, q_1, \cdots, q_{d-n})$, the symplectic form ω can be written as

$$\omega|_{\mathrm{U}} = \sum_{j=1}^{n} \mathrm{d}I_j \wedge \mathrm{d}\alpha_j + \sum_{k=1}^{d-n} \mathrm{d}p_k \wedge \mathrm{d}q_k \ .$$

This result states that the submanifold M^* presents a structure of a fibration whose fibers are diffeomorphic to n-dimensional invariant tori $\mathbb{T}^n$. Moreover, in a neighborhood of each torus, thus locally, there exists a set of generalized action-angle coordinates adapted to the fibration.

Definition 1.1.3. *The coordinates $b \times \alpha = (I, p, q, \alpha)$ are called a set of* generalized action-angle coordinates *since the variables (p, q) are not a couple of action angle coordinates.*

Lemma 1.1.1. *Let $H|_U$ be a local representative of the Hamiltonian H in a local system of generalized action angle coordinates*

$$b \times \alpha = (I_1, \cdots, I_n, p_1, \cdots, p_{d-n}, q_1, \cdots, q_{d-n}, \alpha_1, \ldots, \alpha_n) \ .$$

Then, $H|_U$ depends on the actions $(I_1, \ldots, I_n)$ only.

Proof. Let us consider the $2d - n$ integrals of motion F_j: they constitute a system of coordinates in U which is independent of the variables α.

Thus, since F_j depend only on (I, p, q) and, moreover, are in involution with the Hamiltonian, being integrals of motion, it follows

$$\{H|_U, F_j\} = 0 \Rightarrow \{H|_U, I\} = \{H|_U, p\} = \{H|_U, q\} = 0 \ ,$$

and, in particular,

$$\frac{\partial H|_U}{\partial \alpha} = \frac{\partial H|_U}{\partial q} = \frac{\partial H|_U}{\partial p} = 0 \ .$$

It implies that

$$H|_U = H|_U(I) \ .$$

□

The following lemma gives the connection between different sets of generalized action-angle coordinates.

Lemma 1.1.2. *Let us consider a symplectic atlas formed by generalized action angle coordinates and let* $\{U, b \times \alpha\}$ *and* $\{U', b' \times \alpha'\}$ *be two charts such that* $U \cap U' \neq 0$.
Then, the transition functions defined in each connected component of the intersection of the two chart domains have the following form,

$$I' = \mathrm{Z}I + z \tag{1.2}$$

$$(p', q') = \mathcal{G}(I, p, q) \tag{1.3}$$

$$\alpha' = \mathrm{Z}^{-T}\alpha + \mathcal{F}(I, p, q) \tag{1.4}$$

with $\mathcal{F}, \mathcal{G}$ *analytic functions,* $z \in \mathbb{R}^n$ *and* $Z \in SL_{\pm}(\mathbb{Z}, n)$.

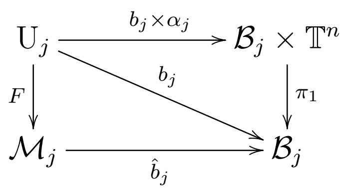
For the proof see for instance [Fas05].

1.2 The structure of the base space and the bi-fibration

As we have already underlined before, the difference between integrable and superintegrable system consists mainly of the fact that the tori on which we have quasi-periodic motion have dimension smaller than the dimension of the base space of the fibration.

We now describe this structure more in detail.

Let us consider the fibration $F : M^* \mapsto \mathcal{M}$ and a symplectic atlas $\{U_j, b_j \times \alpha_j\}_{j \in J}$ of generalized action angle coordinates. We can notice that this atlas induces an atlas for the base space manifold $\mathcal{M}$ with chart domains $B_j =: F(\mathrm{U}_j)$ and coordinates given by $\hat{b}_j = (\hat{I}_j, \hat{p}_j, \hat{q}_j) = \pi_1 \circ (b_j \times \alpha) \circ F^{-1} = b_j \circ F^{-1}$ where we have denoted as π_1 the projection onto the first coordinate, as showed in the diagram below



Thus, the family $\{B_j, \hat{b}_j\}_j$ represents an atlas for the base space $\mathcal{M}$. Let us now study this manifold in detail.

From Lemma 1.1.2, we notice that the transition functions for the actions involve only themselves meaning that we have a subset of the coordinate system which transforms independently from the other coordinates.

Due to the structure of $\mathcal{M}$ one can define a manifold $\mathcal{A}$ as follows: the range of its charts are given by the projection on the first factor of $\mathcal{B}_j$ and the transition functions are defined by (1.2). $\mathcal{A}$ is called the *action space.*

Then, of course, one can define a map

$$\tilde{F} : \mathcal{M} \to \mathcal{A}$$

which is a foliation (Def. 1.1.1) whose leaves are the set $\tilde{F}^{-1}(a)$ with $a \in \mathcal{A}$.

Remark 1.2.1. *Since the frequency of the quasi-periodic motion on the tori depends on the value of the actions, it follows that the tori based on the same leaf support motions with the same frequency.*

Hypothesis 1. *We assume that the map $\tilde{F} : \mathcal{M} \to \mathcal{A}$ defines a fibration.*

Thus, we have that M^* has the structure of a *bifibration*

$$M^* \xrightarrow{F} \mathcal{M} \xrightarrow{\tilde{F}} \mathcal{A} \ ,$$

and, furthermore, every fiber $\tilde{F}^{-1}(a)$ is isomorphic to a given manifold $\mathcal{Q}$. For example, in the situation of the spatial central motion problem, we will see that $\mathcal{Q} \cong S^2$.

We conclude this section with the following lemma which tell us that the transition functions can be given an easier form if we make a smart choice of the atlas of the fibration.

Lemma 1.2.1. *If the action space $\mathcal{A}$ is simply connected, then there exists an atlas with transition functions of the form*

$$\begin{aligned} I' &= I \\ (p', q') &= \mathcal{G}(I, p, q) \\ \alpha' &= \alpha + \mathcal{F}(I, p, q) \end{aligned}$$

For details see [Fas95].

In particular, we remark that, for any small enough open set $\mathcal{V} \subset \mathcal{A}$ one has

$$\tilde{F}^{-1}(\mathcal{V}) \cong \mathcal{V} \times \mathcal{Q} \ .$$

1.3 A Nekhoroshev type Theorem for superintegrable systems

The aim of this section is to develop a Nekhoroshev type theorem for a perturbation of a superintegrable Hamiltonian system with quasiconvex Hamiltonian. Precisely, we will show that the actions of such a system are approximately conserved for times which are exponentially long with the inverse of the perturbation parameter.

The main problem we have to tackle in order to get this result is related to the fact that, as we have seen in the previous sections, each system of action-angle coordinates is in general not *globally* defined. This is particularly evident when $\mathcal{Q}$ is compact.

This difficulty was solved by Fassò who adapted Nekhoroshev's proof of Nekhoroshev Theorem to this situation. Here we adapt Lochak's proof to this context: in particular, it simplifies considerably the result. A proof based on Lochak's method was already given in Blaom. In this work, the author produces an abstract version of the Nekhoroshev Theorem for perturbations of non-commutative integrable Hamiltonian systems. The result follows under the hypothesis that the unperturbed Hamiltonian satisfies certain properties of analyticity and convexity.

In particular, our proof is given for the case of quasiconvex systems in the sense that we will explain in a while.

Before stating the main result, we recall a couple of definitions.

Definition 1.3.1. *Let M^* be a $2d$-dimensional real manifold. An analytic structure on M^* is an atlas with the property that all the transition functions are real analytic. The pair $(M^*, \{U_j, \phi_j\})_{j\in J}$ is called a real analytic manifold.*

Let $M^* \xrightarrow{F} \mathcal{M} \xrightarrow{\tilde{F}} \mathcal{A}$ be the bifibration described previously with M^* a $2d$-dimensional real analytic manifold endowed with an atlas whose transition functions satisfy the hypothesis of Lemma 1.2.1 and let $H : M^* \mapsto \mathbb{R}$ be the unperturbed Hamiltonian of the corresponding superintegrable system.

Let us introduce a function $h : \mathcal{A} \mapsto \mathbb{R}$ defined on the action space such that $H = h \circ F_1$, where $F_1 = \tilde{F} \circ F$.

Lemma 1.3.1. *The function $h : \mathcal{A} \mapsto \mathbb{R}$ is a real analytic function on the whole $\mathcal{A}$.*

Proof. The result follows from the fact that the map which introduces the set of generalized action angle coordinates is an analytic diffeomorphism. □

Finally, we will make use of the following property of quasiconvexity.

Definition 1.3.2. *A function $h : \mathcal{A} \mapsto \mathbb{R}$ is said to be quasiconvex at a point I^* if the inequality*

$$\langle \eta, \frac{\partial^2 h}{\partial I^2}(I^*)\eta \rangle \geq c \,\|\eta\|^2$$

holds for any η such that $\langle \frac{\partial h}{\partial I}(I^*), \eta \rangle = 0$.

Our main result is stated in the following theorem

Theorem 1.3.1. *Let us consider a bifibration* $M^* \xrightarrow{F} \mathcal{M} \xrightarrow{\tilde{F}} \mathcal{A}$ *endowed with an atlas of generalized action-angle coordinates* $\{U_j, \phi_j\}_{j\in J}$ *of the form specified in Lemma 1.2.1. Let* $h : \mathcal{A} \mapsto \mathbb{R}$ *and* $f : M^* \mapsto \mathbb{R}$ *be two real analytic functions, define* $H := h \circ F_1$*, where* $F_1 = \tilde{F} \circ F$*, and assume that* h *is quasiconvex in* $\mathcal{A}$*. Let* $\mathcal{C} \subset M^*$ *be compact and invariant for the dynamics of* H*; then, there exist positive constants* $\varepsilon^*, \mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ *with the following property: for* $|\varepsilon| < \varepsilon^*$ *consider the dynamics of the Hamiltonian system*

$$H_\varepsilon := H + \varepsilon f$$

then, for any initial datum in $\mathcal{C}$ *one has*

$$\|I(t) - I(0)\| \leq \mathcal{C}_1 \varepsilon^{\frac{1}{2n}} \ ,$$

for all times t *satisfying*

$$|\, t \,| \leq \mathcal{C}_2 exp(\mathcal{C}_3 \varepsilon^{-\frac{1}{2n}}) \ .$$

1.4 Proof of the Theorem

1.4.1 Normal form Theorem

The first part of the proof concerns the construction of a semilocal normal form. We start by choosing an appropriate norm with which we measure the size of the functions and of their vector fields. Then, we will define and complexify the domain on which we will construct our normal form.

The methods used in the construction of the normal form are a slight modification of the ones of [Loc92]. The main point is that the normal form we provide here is *semilocal*, meaning that it is well defined on a neighborhood of a fiber of the action space, namely, using the notation of the previous sections, in

$$F^{-1}(\tilde{F}^{-1}(\mathcal{V})) \ ,$$

with $\mathcal{V} \subset \mathcal{A}$ a small open subset.

The result follows from the fact that the time averaging of a function and the function which generates the transformation which puts the Hamiltonian in normal form are defined semilocally.

However, the quantitative estimates have to be constructed locally by considering in each chart the local representative of the semilocal normal form.

Preliminaries and notations

Let us consider the bifibration $M^* \xrightarrow{F} \mathcal{M} \xrightarrow{\tilde{F}} \mathcal{A}$ where $(M^*, \{U_j, \phi_j\}_j)$ is the $2d$-dimensional analytic manifold with the atlas chosen as in Lemma 1.2.1 with

$$\phi_j : U_j \mapsto \mathcal{B}_j \times \mathbb{T}^n$$

given by $\phi(z) = (I_j, p_j, q_j, \alpha_j)$, $z \in M^*$.

We choose to work *semilocally* in the sense that we work close to a single value of the action. Thus, let us fix a value $I^* \in \mathcal{A}$. To be more precise, for $\rho > 0$, we define the ball

$$\mathcal{B}_\rho(I^*) = \{I \in \mathbb{R}^n : \|I - I^*\| < \rho\} \ .$$

Then, we define

$$M_{I^*,\rho} := F^{-1}(\tilde{F}^{-1}(\mathcal{B}_\rho(I^*))) \ ,$$

and we will construct a normal form in such a submanifold. Remark that $M_{I^*,\rho} \cong \mathcal{B}_\rho(I^*) \times \mathcal{P}$ where $\mathcal{P}$ is a suitable manifold.

At the end of the procedure, we will get a result valid over the whole of M^* by choosing a suitable collection of I_j^* and ρ_j such that

$$M^* = \cup_j M_{I_j^*,\rho_j} \ .$$

We fix an atlas of generalized action angle coordinates (I, p, q, α) in $M_{I^*,\rho}$. Denote by $\mathcal{U}_\mathrm{j} \subset \mathbb{R}^{2d}$ the range of the jth chart and remark that the I's coincide for all the charts while (p, q, α) are coordinates on $\mathcal{P}$. In order to measure distances on $\mathcal{U}_\mathrm{j}$, we will introduce two parameters $R > 0$ and $\sigma > 0$ and we introduce the norm

$$\|(I, p, q, \alpha)\| := \frac{1}{R}\sum_{i=1}^{n} |I_i| + \sup_j \frac{R|\alpha_j|}{\sigma} + \sqrt{\sum_{l=1}^{d-n} (|p_l|^2 + |q_l|^2)} \ . \qquad (1.5)$$

Let $f : M_{I^*,\rho} \mapsto \mathbb{R}$ be a function. We will say that $f \in C^\omega(\rho)$ if its local representative f_j in any chart is a real analytic function which extends to a bounded complex analytic function on

$$\mathcal{U}_\mathrm{j}^\rho := \cup_{z\in\mathcal{U}_\mathrm{j}} \mathcal{B}_\rho(z) \ ,$$

where $z = (I, p, q, \alpha)$.

Definition 1.4.1. *Let f_j be the local representative of a function $f \in C^\omega(\rho)$, we define its norm as follows*

$$\|f_j\|_\rho^* := \sup_{z\in\mathcal{U}_\mathrm{j}^\rho} |f_j(z)| \ . \qquad (1.6)$$

We will use the same notations for functions X taking values in $\mathbb{R}^n$, in particular for the Hamiltonian vector field X_f of a function f.

Definition 1.4.2. *Let $f \in C^\omega(\rho)$ with the further property that also its Hamiltonian vector field defines a complex analytic function (valued in $\mathbb{C}^{2d}$) on $C^\omega(\rho)$. We define*

$$\|f\|_\rho^* := \sup_j \|f_j\|_\rho^* \ ,$$

$$\|X_f\|_\rho^* := \sup_j \|X_{f_j}\|_\rho^* \ .$$

The normal form lemma close to a resonant torus
Take $I^* \in \mathcal{A}$ such that

$$\omega^* := \omega(I^*) = \frac{\partial h(I^*)}{\partial I}$$

is periodic of period T, that is, let us suppose that

$$\exists l \text{ such that } \frac{\omega_i^*}{\omega_l} \in \mathbb{Q},\ \forall i = 1, \dots, n\ .$$

By the assumptions of the main theorem, there exists $R > 0$ such that

(i) $h \in C^\omega(2R)$, that is, there exists a positive constant c_1 such that

$$\|h\|_{2R}^* \leq c_1\ .$$

(ii) h is quasiconvex at every point of the action space $\mathcal{A}$, that is, for every $I \in \mathcal{A}$ the following inequality

$$\langle \eta,\, \frac{\partial^2 h}{\partial I^2}(I)\eta \rangle \geq c\|\eta\|^2$$

holds for any η such that $\langle \frac{\partial h}{\partial I}(I),\, \eta \rangle = 0$.

(iii) there exists a positive constant $\mathcal{C}$ such that the following inequality,

$$\langle \eta,\, \frac{\partial^2 h}{\partial I^2}(I)\xi \rangle \leq \mathcal{C}\, \|\eta\|\, \|\xi\|\ ,$$

holds for any $\xi, \eta \in \mathbb{R}^n$ where $\mathcal{C} \geq c$ is the upper bound of the spectrum of the Hessian matrix.

We are now going to put the system in normal form in $M_{I^*,\rho}$ with ρ sufficiently small. First we Taylor expand at the third order $h(I)$ obtaining

$$h(I) = h(I^*) + \langle \omega^*,\, (I - I^*) \rangle + \hat{h}(I - I^*) + h_r(I - I^*) = h(I^*) + h_{\omega^*}(J) + \hat{h}(J) + h_r(J) \tag{1.7}$$

with $J := I - I^*$.

We underline that

(1) $h(I^*)$ is an unimportant constant

(2) h_{ω^*} is the linear part of the Hamiltonian and generates a periodic flow with frequency ω^* and period T, that is,

$$h_{\omega^*} = \langle \omega^*,\, J \rangle\ .$$

(3) $\hat{h}$ is the quadratic part of the unperturbed Hamiltonian, that is,

$$\hat{h}(J) = \frac{1}{2}\langle J, \frac{\partial^2 h}{\partial I^2}(I^*)J\rangle \ .$$

It is already in normal form with h_{ω^*}, namely, $\{\hat{h}, h_{\omega^*}\} = 0$.

(4) h_r is the remainder of the Taylor formula. It can be expressed as

$$h_r(J) = \frac{1}{2}\int_0^1 (1-t)^2 \sum_{i,j,k=1}^{n} \frac{\partial^3 h}{\partial I_i \partial I_j \partial I_k} J_i J_j J_k dt \ .$$

Remark that the quadratic term $\hat{h}$ and the remainder h_r are defined in the whole of $M_{I^*,R}$ and that they satisfy some estimates as stated in the lemmas below

Lemma 1.4.1. *Let $\hat{h}$ as above. The following estimates*

$$\|\hat{h}\|_R^* \leq \tilde{c}_1 R^4 \ , \tag{1.8}$$

$$\|X_{\hat{h}}\|_R^* \leq \tilde{c}_2 \frac{R^3}{\sigma} \tag{1.9}$$

hold.

Proof. To prove this lemma we use the Cauchy estimates (cf. Lemma C.2.2, Appendix C) to control the partial derivatives of the unperturbed Hamiltonian on the complex neighborhood $\mathcal{U}_{\mathrm{j}}^R$. Indeed, for $i, j = 1, \dots, n$, $0 < \delta < R$, we have

$$\left\| \frac{\partial^2 h}{\partial I^2} \right\|_{R+\delta}^* \leq \frac{2!}{\delta^2} \|h\|_{2R}^* \leq \frac{2!}{\delta^2} c_1 \tag{1.10}$$

Thus, from the definition of $\hat{h}$, we have that

$$\left|\hat{h}(J)\right| = \frac{1}{2}\langle J, \frac{\partial^2 h}{\partial I^2}(I^*)J\rangle \leq \frac{1}{2}\sum_{i,j=1}^{n} \left| \frac{\partial^2 h}{\partial I_i \partial I_j}(I^*) \right| |J_i||J_j| \ .$$

Then, passing to the supremum on the complex domain $\mathcal{U}_{\mathrm{j}}^R$ and using the estimates (1.10), we obtain

$$\left\|\hat{h}\right\|_R^* := \sup_{J\in\mathcal{B}_R(I^*)} \left|\hat{h}(J)\right| \leq \frac{1}{2}\frac{2!}{\delta^2} c_1 \underbrace{\sup_{\|J\|<R} \sum_{i=1}^{n} \left|J^i\right|}_{<R^2} \underbrace{\sup_{\|J\|<R} \sum_{j=1}^{n} \left|J^j\right|}_{<R^2} \leq \frac{1}{\delta^2} c_1 R^4 := \tilde{c}_1 R^4 \ ,$$

where $\tilde{c}_1 = \frac{c_1}{\delta^2}$.

Analogously, we can estimate the vector field $X_{\hat{h}}$ and obtain

$$\|X_{\hat{h}}\|_R^* \leq \tilde{c}_2 \frac{R^3}{\sigma} \ .$$

Indeed, from the definition of the Hamiltonian vector field we have $X_{\hat{h}} = J\nabla \hat{h} = (0, 0, 0, (X_{\hat{h}})_\alpha)$ where

$$(X_{\hat{h}})_{\alpha_k} = \frac{1}{2}\left(\sum_{i,j=1}^{n} \frac{\partial^2 h(I^*)}{\partial I_i \partial I_j} J_i + \sum_{i,j=1}^{n} \frac{\partial^2 h(I^*)}{\partial I_i \partial I_j} J_j\right) \ .$$

Thus,

$$|(X_{\hat{h}})_{\alpha_k}| \leq \frac{1}{2}\left(\sum_{i,j=1}^{n} \left|\frac{\partial^2 h(I^*)}{\partial I_i \partial I_j}\right| |J_i| + \sum_{i,j=1}^{n} \left|\frac{\partial^2 h(I^*)}{\partial I_i \partial I_j}\right| |J_j|\right) \ .$$

In $\mathcal{U}_{\mathrm{j}}^R$, we have

$$\begin{aligned} |(X_{\hat{h}})_{\alpha_k}| &\leq \frac{1}{2}\left(\frac{2!}{\delta^2} c_1 \sum_{i=1}^{n} |J_i| dt + \frac{2!}{\delta^2} c_1 \sum_{j=1}^{n} |J_j|\right) \\ &\leq \frac{2}{\delta^2} c_1 R^2 := \tilde{c}_2 R^2 \ , \end{aligned}$$

where $\tilde{c}_2 = \frac{2}{\delta^2} c_1$. So, we have that

$$|(X_{\hat{h}})_{\alpha_k}| \leq \tilde{c}_2 R^2 \ , \forall k \ .$$

And, from the definition of the norm, we obtain

$$\|X_{\hat{h}}\|_R^* := \sup_k \frac{R |(X_{\hat{h}})_{\alpha_k}|}{\sigma} \leq \tilde{c}_2 \frac{R^3}{\sigma} \ .$$

□

Lemma 1.4.2. *Let h_r as above. Then, the following estimates*

$$\|h_r\|_R^* \leq \mathbf{c}_1 R^6 \ , \tag{1.11}$$

$$\|X_{h_r}\|_R^* \leq \mathbf{c}_2 \frac{R^5}{\sigma} \tag{1.12}$$

hold.

Proof. By using the same strategy as in Lemma 1.4.1, we can prove similar estimates for the remainder h_r and its vector field. Indeed, from the definition of h_r, we have

$$|h_r(J)| \leq \frac{1}{2}\int_0^1 (1-t)^2 \sum_{i,j,k} \left| \frac{\partial^3 h}{\partial I_i \partial I_j \partial I_k}(I^* + tJ) \right| |J_i||J_j||J_k| dt \ ,$$

and, passing to the supremum on the complex domain $\mathcal{U}_{\mathrm{j}}^R$, thanks to the Cauchy estimates, one can find

$$\|h_r\|_R^* \leq \frac{1}{2}\int_0^1 \frac{6}{\delta^3} c_1 (1-t)^2 \underbrace{\sup_{\|J\|<R} \sum_{i=1}^n \left|J^i\right|}_{<R^2} \underbrace{\sup_{\|J\|<R} \sum_{j=1}^n \left|J^j\right|}_{<R^2} \underbrace{\sup_{\|J\|<R} \sum_{i=1}^n \left|J^k\right|}_{<R^2} dt \ ,$$

that is,

$$\|h_r\|_R^* \leq \frac{3}{\delta^3} c_1 R^6 \int_0^1 (1-t)^2 dt = \frac{1}{\delta^3} c_1 R^6 := \mathbf{c}_1 R^6 \ ,$$

where $\mathbf{c}_1 := \frac{c_1}{\delta^3}$.

Analogously, one can find an estimate for the vector field of the remainder h_r. Precisely, it is easy to prove that the following bound,

$$\|X_{h_r}\|_R^* \leq \mathbf{c}_2 \frac{R^5}{\sigma} \ ,$$

holds, where $\mathbf{c}_2$ is a positive constant. □

We go back now to the Hamiltonian (1.7) which, up to irrelevant constants, takes the form

$$h_\varepsilon = h_{\omega^*} + \hat{h} + h_r + \varepsilon f \ . \tag{1.13}$$

Let us assume that R is so small that the perturbation $f \in C^\omega(R)$ and therefore there exists a positive constant $\tilde{c}$ such that

$$\|f\|_R^* \leq \tilde{c} \ .$$

By redefining ε, we can put this constant equal to 1, namely,

$$\|\varepsilon f\|_R^* \leq \varepsilon \ . \tag{1.14}$$

In the norm (1.5) one can estimate $X_{\varepsilon f}$ by

Lemma 1.4.3. *Let $f \in C^\omega(R)$ which satisfies (1.14). Then, the following estimate*

$$\|X_{\varepsilon f}\|_R^* \leq \mathbf{C}_3 \frac{\varepsilon}{R} \tag{1.15}$$

holds.

Proof. Let f_j be the local representative of the function f. Let us compute the vector field of f_j at a point z. We have

$$\begin{aligned} X_{f_j} &:= \big((X_{f_j})_I, (X_{f_j})_p, (X_{f_j})_q, (X_{f_j})_\alpha\big) \\ &= \left(\frac{\partial f_j}{\partial \alpha}, \frac{\partial f_j}{\partial q}, \frac{\partial f_j}{\partial p}, \frac{\partial f_j}{\partial I}\right) . \end{aligned}$$

We compute the norm and obtain

$$\| X_{f_j} \|_R^* \leq \frac{\varepsilon}{R} \sum_{i=1}^{n} \left| \frac{\partial f_j}{\partial \alpha_i} \right| + \sup_j \frac{R}{\sigma} \left| \frac{\partial f_j}{\partial I_j} \right| + \sqrt{\sum_{l=1}^{d-n} \left(\left| \frac{\partial f_j}{\partial q_l} \right|^2 + \left| \frac{\partial f_j}{\partial p_l} \right|^2 \right)} .$$

From Lemma C.2.2 of Appendix C and from the fact that $R << 1$, we deduce

$$\| X_{f_j} \|_R^* \leq \mathbf{C}_3 \frac{\varepsilon}{R} .$$

Passing to the supremum over j, we obtain (1.15). □

First, we study the kind of average needed to solve the homological equation. The main point is that the domain on which the functions are constructed is $M_{I^*,\rho}$, on which everything is well defined and can be estimated there. Indeed,

Lemma 1.4.4. *Let f_j be the local representative of the function f in the chart domain $\mathcal{U}_{\mathrm{j}}^\rho$ and let*

$$< f_j > (I, p_j, q_j, \alpha_j) := \frac{1}{T} \int_0^T f_j(I, p_j, q_j, \alpha_j + \omega^* t) \mathrm{d}t$$

be its time averaging where $\alpha_j \mapsto \alpha_j + \omega^* t$ is the periodic flow over the family of resonant tori $I = I^*$; then, $< f_j >$ are the local representatives of a function $\langle f \rangle$. Moreover, let us consider the functions χ_j defined on each chart domain by

$$\chi_j(I, p_j, q_j, \alpha_j) = \frac{1}{\mathrm{T}} \int_0^{\mathrm{T}} t[f_j - < f_j >](I, p_j, q_j, \alpha_j + \omega^* t) \mathrm{d}t .$$

The functions χ_j are the local representatives of a function χ which is defined on the whole $M_{I^*,\rho}$.

Proof. Let us consider the local representative f_k of the map f in an other chart domain $\mathcal{U}_{\mathrm{k}}^\rho$ such that $\mathcal{U}_{\mathrm{j}}^\rho \cap \mathcal{U}_{\mathrm{k}}^\rho \neq 0$. Let $z \in \mathcal{U}_{\mathrm{j}}^\rho \cap \mathcal{U}_{\mathrm{k}}^\rho$ and let us consider the time averaging of f_k. Using the transition functions as specified in Lemma 1.2.1, we have

$$\begin{aligned} < f >_k (I, p_k, q_k, \alpha_k) :&=< f_j > (I, \mathcal{G}_1, \mathcal{G}_2, \alpha_k + \mathcal{F}) \\ &= \frac{1}{T} \int_0^T f_j(I, \mathcal{G}_1, \mathcal{G}_2, \alpha_k + \mathcal{F} + \omega^* t) \mathrm{d}t . \end{aligned} \tag{1.16}$$

If we construct directly the time averaging of the local representative f_k, then we obtain

$$\begin{aligned} < f_k > (I, p_k, q_k, \alpha_k) &= \frac{1}{T}\int_0^T f_k(I, p_k, q_k, \alpha_k + \omega^* t)\mathrm{d}t \\ &= \frac{1}{T}\int_0^T f_j(I, \mathcal{G}_1, \mathcal{G}_2, \alpha_k + \omega^* t + \mathcal{F})\mathrm{d}t \end{aligned} \tag{1.17}$$

that is completely equivalent to the expression (1.16).

We have proved that the time averaging $< f >$ of the function f is an intrinsic function on the subspace $M_{I^*,\rho}$ and $< f >_j$ are its local representatives.

At this point, let us consider the function χ_j defined in the chart domain $\mathcal{U}^\rho_{\mathrm{j}}$, that is

$$\chi_j(I, p_j, q_j, \alpha_j) := \frac{1}{T}\int_0^T t[f_j - < f_j >](I, p_j, q_j, \alpha_j + \omega^* t)\mathrm{d}t\ .$$

Let us now apply the transition functions as in Lemma 1.2.1 in order to pass from the chart $\mathcal{U}^\rho_{\mathrm{j}}$ to the chart $\mathcal{U}^\rho_{\mathrm{k}}$. We have

$$\begin{aligned} \chi_j(I, \mathcal{G}_1, \mathcal{G}_2, \alpha_k + \mathcal{F}) &= \frac{1}{T}\int_0^T t[f_j - < f_j >](I, \mathcal{G}_1, \mathcal{G}_2, \alpha_k + \mathcal{F} + \omega^* t)\mathrm{d}t \\ &= \frac{1}{T}\int_0^T t[f_k - < f >_k](I, p_k, q_k, \alpha_k + \omega^* t)\mathrm{d}t\ . \end{aligned} \tag{1.18}$$

Note that in the second equivalence we have used the results proved in the first part of this lemma.

Now, proceeding as before, we consider the function χ_k defined on a second chart, that is

$$\begin{aligned} \chi_k(I, p_k, q_k, \alpha_k) &= \frac{1}{T}\int_0^T t[f_k - < f_k >](I, p_k, q_k, \alpha_k + \omega^* t)\mathrm{d}t \\ &= \frac{1}{T}\int_0^T t[f_k - < f >_k](I, p_k, q_k, \alpha_k + \omega^* t)\mathrm{d}t \end{aligned} \tag{1.19}$$

Thus, from the equivalence of the expressions (1.18) and (1.19), it follows that we can construct globally a function χ on the subspace $M_{I^*,\rho}$ whose local representatives are the functions χ_j. □

Thus, we have the following lemma

Lemma 1.4.5. *Let f and h_{ω^*} as above. Then, the homological equation*

$$\{\chi, h_{\omega^*}\} + f = \langle f \rangle \tag{1.20}$$

can be solved by

$$\chi = \frac{1}{T}\int_0^T t(f - \langle f\rangle)(\Phi^t_{\omega^*})dt \ . \tag{1.21}$$

Moreover, $\langle f\rangle, \chi$ *and their symplectic gradient belong to* $C^\omega(\rho)$ *and satisfy the following estimates*

1. $\|\langle f\rangle\|^*_\rho \le \|f\|^*_\rho \ , \ \|\chi\|^*_\rho \le \ T\|f\|^*_\rho$
2. $\left\|X_{\langle f\rangle}\right\|^*_\rho \le \|X_f\|^*_\rho \ , \ \|X_\chi\|^*_\rho \le T\,\|X_f\|^*_\rho$

Proof. Let us denote by $\Phi^t_{\omega^*}$ the flow of the Hamiltonian h_{ω^*} at time t. It is continuous and differentiable on the whole domain $M_{I^*,\rho}$. Since the following equality holds,

$$\{h_{\omega^*}, \chi\} = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \chi(\Phi^t_{\omega^*}) \ ,$$

we have only to prove that the time derivative of the function χ satisfies the following identity

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \chi(\Phi^t_{\omega^*}) = f - \langle f\rangle \ .$$

Thus, let us compute

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \chi(\Phi^t_{\omega^*}) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \frac{1}{T}\int_0^T sg(\Phi^{t+s}_{\omega^*})\mathrm{d}s \\
&= \left[\frac{1}{T}\left(sg(\Phi^{t+s}_{\omega^*})\right|_{t=0}\right]_0^T - \frac{1}{T}\int_0^T g(\Phi^{t+s}_{\omega^*})\mathrm{d}s\Bigg|_{t=0} \\
&= g(\Phi^T_{\omega^*}) - \frac{1}{T}\int_0^T g(\Phi^s_{\omega^*})\mathrm{d}s \ ,
\end{aligned}$$

where $g = f - \langle f\rangle$. At this point, noticing that g is a function of zero average, we conclude

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \chi(\Phi^t_{\omega^*}) = g(\Phi^T_{\omega^*}) = g(\Phi^0_{\omega^*}) = g = f - \langle f\rangle \ .$$

Now, it remains to prove the estimates. To do so, we have to pass to the local representatives of the functions. Remark that in any canonical coordinate system one has $\Phi^t_{\omega^*}(I,p,q,\alpha) = (I,p,q,\alpha+\omega^*t)$ and, furthermore, the domain of the coordinate system is invariant under $\Phi^t_{\omega^*}$. Thus,

$$\left|\langle f\rangle_j\,(I,p,q,\alpha)\right| \le \frac{1}{T}\int_0^T |f_j(I,p,q,\alpha+\omega^*t)|\mathrm{d}t \ ,$$

that is,

$$\left\|\langle f\rangle_j\right\|_\rho^* \leq \|f_j\|_\rho^* \ ,$$

and, from the definition of the norm, it follows that

$$\|\langle f\rangle\|_\rho^* \leq \|f\|_\rho^* \ .$$

Similarly, one has

$$|\chi_j(z)| \leq \frac{1}{T}\int_0^T t\left|(f_j - \langle f\rangle_j)(\Phi^t_{\omega^*}(z))\right|\mathrm{d}t \ ,$$

and passing to the supremum, from the previous estimate, we obtain

$$\|\chi_j\|_\rho^* \leq \frac{2}{T}\int_0^T t\,\|f_j\|_\rho^*\,\mathrm{d}t \leq T\,\|f_j\|_\rho^* \ ,$$

and,

$$\|\chi\|_\rho^* \leq T\,\|f\|_\rho^* \ .$$

We conclude by proving the estimates on the vector fields. To do so, remark first that, for any canonical transformation $\mathcal{T}$ and any function g, one has

$$X_{g\circ\mathcal{T}} = \mathcal{T}^* X_g \ ,$$

where $\mathcal{T}^* X_g$ is the pull back of the vector field, so that, in any coordinate system one has

$$\left(X_{g_j\circ\mathcal{T}_j}\right)(z) = \mathrm{d}\mathcal{T}_j^{-1}(\mathcal{T}_j(z))X_{g_j}(\mathcal{T}_j(z)) \ , \tag{1.22}$$

from which

$$X_{g_j\circ\Phi^t_{\omega^*}} = X_{g_j}\circ\Phi^t_{\omega^*} \ ,$$

where we used that, in any system of generalized action angle coordinates, $d\Phi^t_{\omega^*} = \mathbb{I}$. In particular, the Hamiltonian vector field becomes

$$X_{\langle f\rangle_j}(z) = \frac{1}{T}\int_0^T X_{f_j\circ\Phi^t_{\omega^*}}(z)dt = \frac{1}{T}\int_0^T X_{f_j}(I,p,q,\alpha+\omega^* t)dt \ ,$$

from which

$$\left\|X_{\langle f\rangle_j}\right\|_\rho^* := \sup_{z\in\mathcal{U}_j^\rho}\left|X_{\langle f\rangle_j}(z)\right| \leq \frac{1}{T}\int_0^T \sup_{z\in\mathcal{U}_j^\rho}\left|X_{f_j}(z)\right|dt \leq \left\|X_{f_j}\right\|_\rho^* \ ,$$

and,

$$\left\|X_{\langle f\rangle}\right\|_\rho^* \leq \|X_f\|_\rho^* \ .$$

Similarly, one gets the estimate of X_χ. One has

$$X_{\chi_j}(z) = \frac{1}{T}\int_0^T tX_{g_j\circ\Phi^t_{\omega^*}}(z)dt = \frac{1}{T}\int_0^T tX_{g_j}(I,p,q,\alpha+\omega^* t)dt \ ,$$

where $g_j(z) := (f_j - \langle f\rangle_j)(z)$. Now, passing to the supremum over $\mathcal{U}^\rho_{\mathrm{j}}$, we have

$$\begin{aligned}\sup_{z\in\mathcal{U}^\rho_{\mathrm{j}}} \left|X_{\chi_j}(z)\right| &\leq \frac{1}{T}\int_0^T t \sup_{z\in\mathcal{U}^\rho_{\mathrm{j}}} \left|X_{g_j}(I,p,q,\alpha+\omega^* t)\right| dt \\ &\leq \frac{1}{T}\int_0^T t \sup_{z\in\mathcal{U}^\rho_{\mathrm{j}}} \left|X_{g_j}(z)\right| dt \\ &\leq \frac{T}{2}\left\|X_{g_j}\right\|^*_\rho \ ,\end{aligned}$$

Thus, since

$$\left\| X_{g_j} \right\|^*_\rho \leq 2 \left\| X_{f_j} \right\|^*_\rho \ ,$$

we conclude that

$$\left\| X_{\chi_j} \right\|^*_\rho \leq T \left\| X_{f_j} \right\|^*_\rho \ ,$$

and,

$$\|X_\chi\|^*_\rho \leq T\|X_f\|^*_\rho \ .$$

□

Thus, the generating function and the averages are defined *semilocally* and it is not necessary to work locally in each chart. Precisely, let us consider the Hamiltonian (1.13). In what follows we shall consider the term $\hat{h}$ and the remainder h_r together. Thus, let $\hat{h}_r := \hat{h} + h_r$. We can notice that, since $R << 1$, the following estimate

$$\left\|\hat{h}_r\right\|^*_R \leq \left\|\hat{h}\right\|^*_R + \|h_r\|^*_R \leq \tilde{c}_1 R^4 + \mathbf{c}_1 R^6 \leq \mathbf{C}_1 R^4 \ ,$$

with $\mathbf{C}_1 := \max\{\tilde{c}_1, \mathbf{c}_1\}$, holds. Analogously,

$$\left\| X_{\hat{h}_r} \right\|^*_R \leq \|X_{\hat{h}}\|^*_R + \|X_{h_r}\|^*_R \leq \tilde{c}_2\frac{R^3}{\sigma} + \mathbf{c}_2\frac{R^5}{\sigma} \leq \mathbf{C}_2\frac{R^3}{\sigma} \ ,$$

where $\mathbf{C}_2 := \max\{\tilde{c}_2, \mathbf{c}_2\}$. Thus, we are now going to work with the following Hamiltonian

$$h = h_{\omega^*} + \hat{h}_r + \varepsilon f \ ,$$

where $\hat{h}_r$ satisfies the estimates above.

Lemma 1.4.6. *For $k \geq 0$, consider the Hamiltonian*

$$h^k = h_{\omega^*} + \hat{h}_r + \mathcal{Z}^k + \mathcal{R}^k \ . \tag{1.23}$$

Let $\delta < \frac{R}{k+1}$ and let us assume that the functions $\mathcal{Z}^k$ and $\mathcal{R}^k$ belong to $C^\omega(R-k\delta)$ together with their vector fields and that they satisfy the following estimates

(i)

$$\left\|\mathcal{Z}^k\right\|^*_{R-k\delta} \leq \begin{cases} 0 & \text{if} \quad k=0 \\ \varepsilon & \text{if} \quad k=1 \\ \mathbf{C}_4\varepsilon\sum_{i=0}^{k-1}\mu^i & \text{if} \quad k\geq 2 \end{cases} , \|X_{\mathcal{Z}^k}\|^*_{R-k\delta} \leq \begin{cases} 0 & \text{if} \quad k=0 \\ \mathbf{C}_3\frac{\varepsilon}{R}\sum_{i=0}^{k-1}\mu^i & \text{if} \quad k\geq 1 \end{cases} \tag{1.24}$$

(ii)

$$\left\|\mathcal{R}^k\right\|^*_{R-k\delta} \leq \begin{cases} \varepsilon & \text{if} \quad k=0 \\ \mathbf{C}_4\varepsilon\mu^k & \text{if} \quad k\geq 1 \end{cases} , \quad \|X_{\mathcal{R}^k}\|^*_{R-k\delta} \leq \mathbf{C}_3\frac{\varepsilon\mu^k}{R} \tag{1.25}$$

where $\mu := \frac{T}{\delta}(\frac{18\varepsilon\mathbf{C}_3}{R}+\frac{5\mathbf{C}_2R^3}{\sigma})$ and $\mathbf{C}_4$ is a positive constant given by $\mathbf{C}_4 = \max\{c_4, 1\}$, where

$$c_4 = \frac{2\mathbf{C}_3\max\{3,\mathbf{C}_1\}}{\min\{10\mathbf{C}_3, 5\mathbf{C}_2/\sigma\}} \ .$$

If $\mu < \frac{1}{2}$, then there exists a canonical transformation $\mathcal{T}^k$ which is close to the identity, that is,

$$\left\|\mathcal{T}^k - \mathbb{I}\right\|^*_{R-(k+1)\delta} \leq \mathbf{C}_3\frac{T\varepsilon\mu^k}{R}$$

such that the function $h^k \circ \mathcal{T}^k$ has the form (1.23) where $\mathcal{Z}^{k+1} = \mathcal{Z}^k + \langle\mathcal{R}^k\rangle$ and satisfies the above estimates with $k+1$ in place of k.

Proof. This is essentially Lemma 7.1. of [Bam99]. The proof can be divided into two parts. In the first part, we will describe the successive-transformation scheme which permits us to normalize formally the Hamiltonian up to a certain order k while the second part concerns the quantitative estimates which make rigorous the procedures used in the first part.

The iterative procedure

As in the classical scheme, our aim is to choose as the canonical transformation normalizing the Hamiltonian h^k the time one flow of a generating function, which shall be the solution of the homological equation.

Precisely, let χ^k be the solution of the homological equation

$$\{\chi^k, h_{\omega^*}\} + \mathcal{R}^k = \langle\mathcal{R}^k\rangle \ , \tag{1.26}$$

which exists by Lemma 1.4.5, that is,

$$\chi^k = \frac{1}{T}\int_0^T t\left(\mathcal{R}^k - \langle\mathcal{R}^k\rangle\right)(\Phi^t_{\omega^*})\mathrm{d}t \ .$$

Then one has

$$\|X_{\chi^k}\|^*_{R-k\delta} \leq \ T\|X_{\mathcal{R}^k}\|^*_{R-k\delta} \leq \mathbf{C}_3\frac{T\varepsilon\mu^k}{R} \ . \tag{1.27}$$

Let us denote by $\mathcal{T}^k := \Phi_{\chi^k}$ the corresponding time one flow. From Lemma C.2.3 of Appendix C for $t = 1$, we obtain

$$\left\| \mathcal{T}^k - \mathbb{I} \right\|^*_{R-(k+1)\delta} \leq \left\| X_{\chi^k} \right\|^*_{R-k\delta} \ .$$

At this point, using (1.27), we have

$$\left\|\mathcal{T}^k - \mathbb{I}\right\|^*_{R-(k+1)\delta} \leq \mathbf{C}_3\frac{T\varepsilon\mu^k}{R} \ .$$

Thus, the map $\mathcal{T}^k : M_{I^*,R-(k+1)\delta} \mapsto M_{I^*,R-k\delta}$ is well defined and, moreover, it is a close to the identity canonical transformation.

By the composition with h^k, we obtain the new Hamiltonian

$$\begin{aligned} h^{k+1} &= h^k \circ \mathcal{T}^k \\ &= h_{\omega^*} + \hat{h}_r + \mathcal{Z}^k + \langle\mathcal{R}^k\rangle + \mathcal{R}^{k+1} \\ &= h_{\omega^*} + \hat{h}_r + \mathcal{Z}^{k+1} + \mathcal{R}^{k+1} \ , \end{aligned}$$

where $\mathcal{Z}^{k+1} = \mathcal{Z}^k + \langle\mathcal{R}^k\rangle$ is the term which is already in involution with h_{ω^*} while $\mathcal{R}^{k+1}$ is the remainder which is composed by the following terms

$$\begin{aligned} \mathcal{R}^{k+1} = {} & h_{\omega^*} \circ \mathcal{T}^k - h_{\omega^*} - \{\chi^k, h_{\omega^*}\} \\ & + \hat{h}_r \circ \mathcal{T}^k - \hat{h}_r \\ & + \mathcal{R}^k \circ \mathcal{T}^k - \mathcal{R}^k \\ & + \mathcal{Z}^k \circ \mathcal{T}^k - \mathcal{Z}^k \ . \end{aligned} \tag{1.28}$$

The quantitative estimates

It remains now to compute the estimates of the terms of the new Hamiltonian to make rigorous the procedure. The main point is that we will first produce estimates in a single chart and, then, thanks to the definition of the norm we have given on the whole space, we will construct semilocal estimates.

We compute the estimates for $\mathcal{Z}^{k+1}$ and the remainder $\mathcal{R}^{k+1}$ as well as for their vector fields.

Thus, from the definition of $\mathcal{Z}^{k+1}$, we have

$$\|\mathcal{Z}^{k+1}\|^*_{R-(k+1)\delta} \leq \|\mathcal{Z}^k\|^*_{R-k\delta} + \|\mathcal{R}^k\|^*_{R-k\delta} \ ,$$

where we have used (1) of Lemma 1.4.5. For $k=0$, we obtain

$$\|\mathcal{Z}^1\|^*_{R-\delta} \leq \|\mathcal{Z}^0\|^*_R + \|\mathcal{R}^0\|^*_R \leq \|\varepsilon f_j\|^*_R \leq \varepsilon \ ,$$

since the term $\mathcal{Z}^0$ is equal to zero.

Analogously, for $k \geq 1$, from the estimates (1.24) and (1.25), we obtain

$$\|\mathcal{Z}^{k+1}\|^*_{R-(k+1)\delta} \leq \mathbf{C}_4 \varepsilon \sum_{i=0}^{k} \mu^i \ .$$

We have also

$$\|X_{\mathcal{Z}^{k+1}}\|^*_{R-(k+1)\delta} \leq \|X_{\mathcal{Z}^k}\|^*_{R-k\delta} + \|X_{\mathcal{R}^k}\|^*_{R-k\delta} \ .$$

For $k=0$, we can compute the estimate

$$\|X_{\mathcal{Z}^1}\|^*_{R-\delta} \leq \|X_{\mathcal{Z}^0}\|^*_R + \|X_{\mathcal{R}^0}\|^*_R \leq \|X_{\varepsilon f}\|^*_R \leq \mathbf{C}_3 \frac{\varepsilon}{R} \ .$$

Analogously, for $k \geq 1$, from the estimates (1.24) and (1.25), we obtain

$$\|X_{\mathcal{Z}^{k+1}}\|^*_{R-(k+1)\delta} \leq \mathbf{C}_3 \frac{\varepsilon}{R} \sum_{i=0}^{k-1} \mu^i + \mathbf{C}_3 \mu^k \frac{\varepsilon}{R} = \mathbf{C}_3 \frac{\varepsilon}{R} \sum_{i=0}^{k} \mu^i \ .$$

At this point, it remains to compute the estimates for the remainder $\mathcal{R}^{k+1}$ and its vector filed. To do so, we first compute the norm of the fourth term which composes the remainder, that is,

$$r^4 := \mathcal{Z}^k \circ \mathcal{T}^k - \mathcal{Z}^k \ .$$

From Lemma C.2.4 of Appendix C, we obtain the following estimate

$$\left\| r^4 \right\|^*_{R-(k+1)\delta} \leq \frac{2}{\delta} \left\| X_{\chi^k} \right\|^*_{R-k\delta} \left\| \mathcal{Z}^k \right\|^*_{R-k\delta} \ .$$

Analogously, we can compute the estimates for the second and the third term in (1.28), namely,

$$r^2 := \hat{h}_r \circ \mathcal{T}^k - \hat{h}_r \ ,$$

and

$$r^3 := \mathcal{R}^k \circ \mathcal{T}^k - \mathcal{R}^k \ .$$

We have

$$\left\|r^2\right\|^*_{R-(k+1)\delta} \le \frac{2}{\delta}\left\|X_{\chi^k}\right\|^*_{R-k\delta}\left\|\hat{h}_r\right\|^*_{R-k\delta}\ ,$$
$$\left\|r^3\right\|^*_{R-(k+1)\delta} \le \frac{2}{\delta}\left\|X_{\chi^k}\right\|^*_{R-k\delta}\left\|\mathcal{R}^k\right\|^*_{R-k\delta}\ .$$

Now, it remains to estimate the first term of (1.28), that is,

$$r^1_j := \left(h_{\omega^*}\circ\mathcal{T}^k - h_{\omega^*} - \{\chi^k, h_{\omega^*}\}\right)\ .$$

From Lemma C.2.5 of Appendix C, we have

$$\left\|r^1\right\|^*_{R-(k+1)\delta} \le \frac{4}{\delta}\left\|X_{\chi^k}\right\|^*_{R-k\delta}\left\|\mathcal{R}^k\right\|^*_{R-k\delta}\ .$$

At this point, we can put together all the previous estimates and find out that the remainder (1.28) can be estimated as follows

$$\begin{aligned}\left\|\mathcal{R}^{k+1}\right\|^*_{R-(k+1)\delta} &\le \sum_{l=1}^{4}\left\|r^l\right\|^*_{R-(k+1)\delta}\\ &\le \frac{4}{\delta}\left\|X_{\chi^k}\right\|^*_{R-k\delta}\left\|\mathcal{R}^k\right\|^*_{R-k\delta} + \frac{2}{\delta}\left\|X_{\chi^k}\right\|^*_{R-k\delta}\left\|\hat{h}_r\right\|^*_{R-k\delta}\\ &+ \frac{2}{\delta}\left\|X_{\chi^k}\right\|^*_{R-k\delta}\left\|\mathcal{R}^k\right\|^*_{R-k\delta} + \frac{2}{\delta}\left\|X_{\chi^k}\right\|^*_{R-k\delta}\left\|\mathcal{Z}^k\right\|^*_{R-k\delta}\ .\end{aligned}$$

Thus,

$$\left\|\mathcal{R}^{k+1}\right\|^*_{R-(k+1)\delta} \le \left(\frac{6}{\delta}\left\|\mathcal{R}^k\right\|^*_{R-k\delta} + \frac{2}{\delta}\left\|\hat{h}_r\right\|^*_{R-k\delta} + \frac{2}{\delta}\left\|\mathcal{Z}^k\right\|^*_{R-k\delta}\right)\left\|X_{\chi^k}\right\|^*_{R-k\delta}\ .$$

Now, for $k = 0$, we have

$$\begin{aligned}\left\|\mathcal{R}^1\right\|^*_{R-\delta} &\le \left(\frac{6}{\delta}\left\|\mathcal{R}^0\right\|^*_{R} + \frac{2}{\delta}\left\|\hat{h}_r\right\|^*_{R} + \frac{2}{\delta}\left\|\mathcal{Z}^0\right\|^*_{R}\right)\left\|X_{\chi^0}\right\|^*_{R}\\ &\le \frac{2}{\delta}\left(3\varepsilon + \mathbf{C}_1 R^4\right)\mathbf{C}_3\frac{T\varepsilon}{R} := \tilde{c}_1\varepsilon\mu\ ,\end{aligned}$$

where

$$\tilde{c}_1 = \frac{2\mathbf{C}_3\left(3\varepsilon + \mathbf{C}_1 R^4\right)}{18\mathbf{C}_3\varepsilon + 5\mathbf{C}_2 R^4/\sigma} \le \frac{2\mathbf{C}_3\max\{3, \mathbf{C}_1\}(\varepsilon + R^4)}{\min\{10\mathbf{C}_3, 5\mathbf{C}_2/\sigma\}(\varepsilon + R^4)} \le \frac{2\mathbf{C}_3\max\{3, \mathbf{C}_1\}}{\min\{10\mathbf{C}_3, 5\mathbf{C}_2/\sigma\}} := c_4\ .$$

In particular, we have $c_4 \le \mathbf{C}_4$ and, thus,

$$\left\|\mathcal{R}^1\right\|^*_{R-\delta} \le \mathbf{C}_4\varepsilon\mu\ .$$

We can proceed analogously in order to find out the estimate for the remainder for $k \geq 1$. Indeed, from the estimates (1.24),(1.25) and (1.27) we obtain

$$\left\|\mathcal{R}^{k+1}\right\|^*_{R-(k+1)\delta} \leq \frac{2}{\delta}\left(3\mathbf{C}_4\varepsilon\mu^k + \mathbf{C}_1 R^4 + \mathbf{C}_4\varepsilon\sum_{i=0}^{k-1}\mu^i\right)\frac{\mathbf{C}_3 T\varepsilon\mu^k}{R} \ . \qquad (1.29)$$

At this point, if we choose μ small enough such that $\mu < \frac{1}{2}$, then it is easy to see that the quantity between the round brackets in (1.34) satisfies the following inequality

$$3\mathbf{C}_4\varepsilon\mu^k + \mathbf{C}_1 R^4 + \mathbf{C}_4\varepsilon\sum_{i=0}^{k-1}\mu^i \leq \frac{3\mathbf{C}_4\varepsilon}{2} + \mathbf{C}_1 R^4 + 2\mathbf{C}_4\varepsilon \leq 4\mathbf{C}_4\varepsilon + \mathbf{C}_1 R^4 \ .$$

Thus,

$$\left\|\mathcal{R}^{k+1}\right\|^*_{R-(k+1)\delta} \leq \frac{2}{\delta}\left(4\mathbf{C}_4\varepsilon + \mathbf{C}_1 R^4\right)\frac{\mathbf{C}_3 T\varepsilon\mu^k}{R} := \tilde{c}_2\varepsilon\mu^{k+1} \ ,$$

where

$$\tilde{c}_2 = \frac{2(4\mathbf{C}_4\varepsilon + \mathbf{C}_1 R^4)}{18\mathbf{C}_3\varepsilon + 5\mathbf{C}_2 R^4/\sigma} \leq \mathbf{C}_4 \ .$$

Thus,

$$\left\|\mathcal{R}^{k+1}\right\|^*_{R-(k+1)\delta} \leq \mathbf{C}_4\varepsilon\mu^{k+1} \ .$$

Let us conclude by computing the norm of the vector field of the remainder $\mathcal{R}^{k+1}$, that is,

$$\begin{aligned} X_{\mathcal{R}^{k+1}} := J\nabla\mathcal{R}^{k+1} &= X_{h_{\omega^*}\circ\mathcal{T}^k - h_{\omega^*} - \{\chi^k, h_{\omega^*}\}} \\ &+ X_{\hat{h}_r\circ\mathcal{T}^k - \hat{h}_r} \\ &+ X_{\mathcal{R}^k\circ\mathcal{T}^k - \mathcal{R}^k} \\ &+ X_{\mathcal{Z}^k\circ\mathcal{T}^k - \mathcal{Z}^k} \ . \end{aligned}$$

We proceed as in the previous computation. Thus, we estimate each term which appears in the definition of $X_{\mathcal{R}^{k+1}}$. Let us begin with the fourth term

$$\tilde{r}^4 := X_{\mathcal{Z}^k\circ\mathcal{T}^k - \mathcal{Z}^k} \ .$$

From Lemma C.2.4 of Appendix C, we obtain

$$\|\tilde{r}^4\|^*_{R-(k+1)\delta} \leq \frac{5}{\delta}\|X_{\chi^k}\|^*_{R-k\delta}\|X_{\mathcal{Z}^k}\|^*_{R-k\delta} \ .$$

Analogously, we can estimate the second and the third term, obtaining

$$\|\tilde{r}^2\|^*_{R-(k+1)\delta} \leq \frac{5}{\delta}\|X_{\chi^k}\|^*_{R-k\delta}\|X_{\hat{h}_r}\|^*_{R-k\delta} \ ,$$

$$\|\tilde{r}^3\|^*_{R-(k+1)\delta} \leq \frac{5}{\delta}\|X_{\chi^k}\|^*_{R-k\delta}\|X_{\mathcal{R}^k}\|^*_{R-k\delta} \ .$$

Let us now conclude with the first term, that is,

$$\tilde{r}^1 := X_{h_{\omega^*}\circ\mathcal{T}^k - h_{\omega^*} - \{\chi^k, h_{\omega^*}\}} \ .$$

From Lemma C.2.5 of Appendix C, we obtain the following estimate

$$\|\tilde{r}^1\|^*_{R-(k+1)\delta} \leq \frac{10}{\delta}\|X_{\chi^k}\|^*_{R-k\delta}\|X_{\mathcal{R}^k}\|^*_{R-k\delta} \ .$$

At this point, we can put together all the estimates and find out that the vector field can be estimated as follows

$$\begin{aligned}
\|X_{\mathcal{R}^{k+1}}\|^*_{R-(k+1)\delta} &\leq \sum_{i=1}^{4}\|\tilde{r}^i\| \\
&\leq \frac{10}{\delta}\|X_{\chi^k}\|^*_{R-k\delta}\|X_{\mathcal{R}^k}\|^*_{R-k\delta} + \frac{5}{\delta}\|X_{\chi^k}\|^*_{R-k\delta}\|X_{\hat{h}_r}\|^*_{R-k\delta} \\
&+ \frac{5}{\delta}\|X_{\chi^k}\|^*_{R-k\delta}\|X_{\mathcal{R}^k}\|^*_{R-k\delta} + \frac{5}{\delta}\|X_{\chi^k}\|^*_{R-k\delta}\|X_{\mathcal{Z}^k}\|^*_{R-k\delta} \ ,
\end{aligned}$$

that is,

$$\|X_{\mathcal{R}^{k+1}}\|^*_{R-(k+1)\delta} \leq \left(\frac{15}{\delta}\|X_{\mathcal{R}^k}\|^*_{R-k\delta} + \frac{5}{\delta}\|X_{\hat{h}_r}\|^*_{R-k\delta} + \frac{5}{\delta}\|X_{\mathcal{Z}^k}\|^*_{R-k\delta}\right)\|X_{\chi^k}\|^*_{R-k\delta} \ .$$

For $k = 0$, we obtain

$$\begin{aligned}
\|X_{\mathcal{R}^1}\|^*_{R-\delta} &\leq \left(\frac{15}{\delta}\|X_{\mathcal{R}^0}\|^*_{R} + \frac{5}{\delta}\|X_{\hat{h}_r}\|^*_{R} + \frac{5}{\delta}\|X_{\mathcal{Z}^0}\|^*_{R}\right)\|X_{\chi^0}\|^*_{R} \\
&\leq \left(\frac{15}{\delta}\frac{\mathbf{C}_3\varepsilon}{R} + \frac{5}{\delta}\frac{\mathbf{C}_2R^3}{\sigma}\right)\frac{T\varepsilon\mathbf{C}_3}{R} \\
&= \frac{\mathbf{C}_3\varepsilon}{R}\frac{T}{\delta}\left(\frac{15\mathbf{C}_3\varepsilon}{R} + \frac{5\mathbf{C}_2R^3}{\sigma}\right) \leq \frac{\mathbf{C}_3\varepsilon}{R}\mu \ ,
\end{aligned}$$

since

$$\frac{T}{\delta}\left(\frac{15\mathbf{C}_3\varepsilon}{R} + \frac{5\mathbf{C}_2R^3}{\sigma}\right) \leq \mu \ .$$

Analogously, for $k \geq 1$, we obtain

$$\begin{aligned}
\|X_{\mathcal{R}^{k+1}}\|^*_{R-(k+1)\delta} &\leq \left(\frac{15}{\delta}\|X_{\mathcal{R}^k}\|^*_{R-k\delta} + \frac{5}{\delta}\|X_{\hat{h}_r}\|^*_{R-k\delta} + \frac{5}{\delta}\|X_{\mathcal{Z}^k}\|^*_{R-k\delta}\right)\|X_{\chi^k}\|^*_{R-k\delta} \\
&\leq \left(\frac{15}{\delta}\frac{\varepsilon\mathbf{C}_3\mu^k}{R} + \frac{5}{\delta}\frac{\mathbf{C}_2R^3}{\sigma} + \frac{5}{\delta}\frac{\varepsilon\mathbf{C}_3}{R}\sum_{i=0}^{k-1}\mu^i\right)\frac{T\varepsilon\mathbf{C}_3\mu^k}{R} \ ,
\end{aligned}$$

where we have exploited the estimates (1.24),(1.25) and (1.27).

For $\mu < \frac{1}{2}$, the quantity between the round brackets becomes

$$\frac{15}{\delta}\frac{\varepsilon \mathbf{C}_3 \mu^k}{R}+\frac{5}{\delta}\frac{\mathbf{C}_2 R^3}{\sigma}+\frac{5}{\delta}\frac{\varepsilon \mathbf{C}_3}{R}\sum_{i=0}^{k-1}\mu^i \leq \frac{15}{2\delta}\frac{\varepsilon \mathbf{C}_3}{R}+\frac{5}{\delta}\frac{\mathbf{C}_2 R^3}{\sigma}+\frac{10}{\delta}\frac{\varepsilon \mathbf{C}_3}{R} = \frac{35}{2\delta}\frac{\varepsilon \mathbf{C}_3}{R}+\frac{5}{\delta}\frac{\mathbf{C}_2 R^3}{\sigma} \ .$$

Thus, from the definition of μ,

$$\begin{aligned} \|X_{\mathcal{R}^{k+1}}\|^*_{R-(k+1)\delta} &\leq \left(\frac{18}{\delta}\frac{\varepsilon \mathbf{C}_3}{R} + \frac{5}{\delta}\frac{\mathbf{C}_2 R^3}{\sigma}\right)\frac{T\varepsilon \mathbf{C}_3 \mu^k}{R} \\ &\leq \frac{T}{\delta}\left(\frac{18\varepsilon \mathbf{C}_3}{R} + \frac{5\mathbf{C}_2 R^3}{\sigma}\right)\frac{\varepsilon \mathbf{C}_3 \mu^k}{R} \\ &\leq \mathbf{C}_3 \frac{\varepsilon \mu^{k+1}}{R} \ . \end{aligned}$$

This concludes the proof. □

From the iterative Lemma 1.4.6, the following theorem follows directly

Theorem 1.4.1. *Consider a Hamiltonian of the form*

$$h_\varepsilon = h_{\omega^*} + \hat{h}_r + \varepsilon f \ , \tag{1.30}$$

satisfying

$$\left\|\hat{h}_r\right\|^*_R \leq \mathbf{C}_1 R^4 \ , \quad \|X_{\hat{h}_r}\|^*_R \leq \mathbf{C}_2 \frac{R^3}{\sigma} \ ,$$
$$\|\varepsilon f\|^*_R \leq \varepsilon \ , \quad \|X_{\varepsilon f}\|^*_R \leq \mathbf{C}_3 \frac{\varepsilon}{R} \ .$$

Define $\bar{\mu}$

$$\bar{\mu} := 57e\left(\frac{\varepsilon \mathbf{C}_3 T}{R^2} + \frac{\mathbf{C}_2 T R^2}{\sigma}\right) \ ,$$

and assume that $\bar{\mu} < \frac{1}{2}$*; then, there exists an analytic canonical transformation* $\mathcal{T} : M_{I^*,R/2} \mapsto M_{I^*,R}$ *with the following properties*

(1) $\mathcal{T}$ *is close to the identity, namely, it satisfies*

$$\|\mathcal{T} - \mathbb{I}\|^*_{R/2} \leq \mathbf{C}_5 \frac{\varepsilon \bar{\mu} T}{R} \ . \tag{1.31}$$

(2) $\mathcal{T}$ *puts the Hamiltonian in resonant normal form up to an exponentially small remainder, namely, one has*

$$h \circ \mathcal{T} = h_{\omega^*} + \hat{h}_r + \varepsilon \langle f \rangle + \mathcal{Z} + \mathcal{R} \ , \tag{1.32}$$

where

(i) $\mathcal{Z}$ is in normal form, namely, $\{\mathcal{Z}, h_{\omega^*}\} = 0$, and of order higher than $\langle f \rangle$, namely, it is estimated by

$$\|\mathcal{Z}\|^*_{R/2} \leq 2\mathbf{C}_6\varepsilon \ , \quad \|X_{\mathcal{Z}}\|^*_{R/2} \leq 2\mathbf{C}_3\frac{\varepsilon\bar{\mu}}{R} \ . \tag{1.33}$$

(ii) $\mathcal{R}$ is an exponentially small remainder estimated by

$$\|\mathcal{R}\|^*_{R/2} \leq \mathbf{C}_6\varepsilon e^{-\frac{1}{\bar{\mu}}} \ , \quad \|X_{\mathcal{R}}\|^*_{R/2} \leq \mathbf{C}_3\frac{\varepsilon\bar{\mu}}{R}e^{-\frac{1}{\bar{\mu}}} \ , \tag{1.34}$$

where $\mathbf{C}_6$ is a positive constant.

Proof. Firstly, we notice that the Hamiltonian (1.30) satisfies the hypothesis of the iterative lemma with $k = 0$ considering $\mathcal{Z}^0 = 0$ and $\mathcal{R}^0 = \varepsilon f$. Indeed, we have that $\mathcal{R}^0$ is analytic in the domain $M_{I^*,R}$ together with its vector field and that the following estimates are satisfied

$$\left\| \mathcal{R}^0 \right\|^*_R = \|\varepsilon f\|^*_R \leq \varepsilon \ , \quad \|X_{\mathcal{R}^0}\|^*_R = \|X_{\varepsilon f}\|^*_R \leq \mathbf{C}_3\frac{\varepsilon}{R} \ .$$

Moreover, if we us choose $\delta = \frac{R}{4} < R$, then

$$\begin{aligned}
\mu = \frac{4T}{R}\left(\frac{18\varepsilon\mathbf{C}_3}{R} + \frac{5\mathbf{C}_2R^3}{\sigma}\right) &= \left(\frac{72\varepsilon\mathbf{C}_3T}{R^2} + \frac{20\mathbf{C}_2TR^2}{\sigma}\right) \\
\leq 57e\left(\frac{\varepsilon\mathbf{C}_3T}{R^2} + \frac{\mathbf{C}_2TR^2}{\sigma}\right) &:= \bar{\mu} < \frac{1}{2} \ .
\end{aligned}$$

Thus, since μ is sufficiently small, we can apply the iterative lemma: there exists a canonical transformation close to the identity, we denote it by $\mathcal{T}^0 : M_{I^*,\frac{3R}{4}} \mapsto M_{I^*,R}$, such that

$$\left\| \mathcal{T}^0 - \mathbb{I} \right\|^*_{\frac{3R}{4}} \leq \frac{\varepsilon\mathbf{C}_3T}{R} \ ,$$

which puts the Hamiltonian in normal form

$$h^1_\varepsilon := h_{\omega^*} + \hat{h}_r + \mathcal{Z}^1 + \mathcal{R}^1 \ , \tag{1.35}$$

with $\mathcal{Z}^1 = \varepsilon \langle f \rangle$. Moreover, as proved in Lemma 1.4.6, we have the following estimates

$$\left\| \mathcal{Z}^1 \right\|^*_{\frac{3R}{4}} \leq \varepsilon \ , \quad \|X_{\mathcal{Z}^1}\|^*_{\frac{3R}{4}} \leq \mathbf{C}_3\frac{\varepsilon}{R} \ ,$$

$$\left\| \mathcal{R}^1 \right\|^*_{\frac{3R}{4}} \leq \mathbf{C}_4\varepsilon\mu \ , \quad \|X_{\mathcal{R}^1}\|^*_{\frac{3R}{4}} \leq \mathbf{C}_3\frac{\varepsilon\mu}{R} \ ,$$

with $\mathbf{C}_4$ a positive constant. Precisely, exploiting the fact that $\mu \le \bar{\mu}$, the last two estimates can be rewritten as

$$\left\| \mathcal{R}^1 \right\|^*_{\frac{3R}{4}} \le \mathbf{C}_4 \varepsilon \bar{\mu} \ , \quad \| X_{\mathcal{R}^1} \|^*_{\frac{3R}{4}} \le \mathbf{C}_3 \frac{\varepsilon \bar{\mu}}{R} \ .$$

At this point, if we rename $\hat{H} = \hat{h}_r + \mathcal{Z}^1$, $\mathbf{R}^0 := \mathcal{R}^1$, then we have that the Hamiltonian

$$h^1_\varepsilon = h_{\omega^*} + \hat{H} + \mathbf{R}^0 \tag{1.36}$$

satisfies again the hypothesis of the iterative lemma for $k = 0$ on the domain $M_{I^*, \frac{3R}{4}}$. Indeed, we have the following estimates

$$\begin{aligned} \left\| \hat{H} \right\|^*_{\frac{3R}{4}} &\le \left\| \hat{h}_r \right\|^*_{\frac{3R}{4}} + \left\| \mathcal{Z}^1 \right\|^*_{\frac{3R}{4}} \le \mathbf{C}_1 R^4 + \varepsilon \ , \\ \| X_{\hat{H}} \|^*_{\frac{3R}{4}} &= \left\| X_{\hat{h}_r} \right\|^*_{\frac{3R}{4}} + \| X_{\mathcal{Z}^1} \|^*_{\frac{3R}{4}} \le \mathbf{C}_2 \frac{R^3}{\sigma} + \mathbf{C}_3 \frac{\varepsilon}{R} \end{aligned}$$

and

$$\begin{aligned} \left\| \mathbf{R}^0 \right\|^*_{\frac{3R}{4}} &= \left\| \mathcal{R}^1 \right\|^*_{\frac{3R}{4}} \le \mathbf{C}_4 \varepsilon \bar{\mu} \ , \\ \| X_{\mathbf{R}^0} \|^*_{\frac{3R}{4}} &= \| X_{\mathcal{R}^1} \|^*_{\frac{3R}{4}} \le \mathbf{C}_3 \frac{\varepsilon \bar{\mu}}{R} \ . \end{aligned}$$

Now, let us fix $\delta = \frac{R}{4k}$ and let us apply the lemma k times with $\frac{3}{4}R$ in place of R. After k steps, we obtain that there exists a canonical transformation $\mathcal{T}$ close to the identity which puts the Hamiltonian (1.36) in normal form

$$h \circ \mathcal{T} = h_{\omega^*} + \hat{H} + \mathcal{Z} + \mathcal{R} = h_{\omega^*} + \hat{h}_r + \varepsilon \langle f \rangle + \mathcal{Z} + \mathcal{R} \ .$$

Moreover, we have that $\mathcal{Z}$ and $\mathcal{R}$ satisfy the following estimates

$$\begin{aligned} \| \mathcal{Z} \|^*_{R/2} &\le \mathbf{C}_6 \varepsilon \sum_{i=0}^{k-1} m^i \ , \\ \| \mathcal{R} \|^*_{R/2} &\le \mathbf{C}_6 \varepsilon m^k \end{aligned}$$

with

$$\begin{aligned} m &= \frac{4kT}{R} \left(\mathbf{C}_3 \frac{18 \varepsilon \bar{\mu}}{R} + \mathbf{C}_2 \frac{5R^3}{\sigma} + \mathbf{C}_3 \frac{5\varepsilon}{R} \right) \\ &= \left[e \frac{4T}{R} \left(\mathbf{C}_3 \frac{18 \varepsilon \bar{\mu}}{R} + \mathbf{C}_2 \frac{5R^3}{\sigma} + \mathbf{C}_3 \frac{5\varepsilon}{R} \right) \right] \frac{k}{e} \ . \end{aligned} \tag{1.37}$$

Moreover, we can prove that the vector fields $X_{\mathcal{Z}}$ and $X_{\mathcal{R}}$ satisfy the following estimates

$$\| X_{\mathcal{Z}} \|^*_{R/2} \le \mathbf{C}_3 \bar{\mu} \frac{\varepsilon}{R} \sum_{i=0}^{k-1} m^i$$

and

$$\|X_{\mathcal{R}}\|^*_{R/2} \le \mathbf{C}_3 \bar{\mu} \frac{\varepsilon}{R} m^k \ ,$$

while the canonical transformation $\mathcal{T}$ is close to the identity since from the estimates for the remainder it follows that

$$\|\mathcal{T} - \mathbb{I}\|^*_{R/2} \le \mathbf{C}_3 \frac{\varepsilon \bar{\mu} T}{R} m^{k-1} \ .$$

Furthermore, since we have that $m \le \bar{\mu}$,

$$\begin{aligned} m &\le \frac{4T}{R}\left(\mathbf{C}_3 \frac{9\varepsilon}{R} + \mathbf{C}_2 \frac{5R^3}{\sigma} + \mathbf{C}_3 \frac{5\varepsilon}{R}\right) \\ &\le \frac{4T}{R}\left(\mathbf{C}_3 \frac{14\varepsilon}{R} + \mathbf{C}_2 \frac{5R^3}{\sigma}\right) \\ &\le 56\left(\frac{\varepsilon \mathbf{C}_3 T}{R^2} + \frac{\mathbf{C}_2 T R^2}{\sigma}\right) \\ &\le 27e\left(\frac{\varepsilon \mathbf{C}_3 T}{R^2} + \frac{\mathbf{C}_2 T R^2}{\sigma}\right) := \bar{\mu} \le \frac{1}{2} \ , \end{aligned}$$

we can rewrite

$$\|\mathcal{Z}\|^*_{R/2} \le 2\mathbf{C}_6 \varepsilon \ ,$$

$$\|X_{\mathcal{Z}}\|^*_{R/2} \le 2\mathbf{C}_3 \frac{\varepsilon \bar{\mu}}{R}$$

and

$$\|\mathcal{T} - \mathbb{I}\|^*_{R/2} \le \mathbf{C}_5 \frac{\varepsilon \bar{\mu} T}{R} \ .$$

At this point, we would like to determine the number k of steps in order to minimize the reminder. For this purpose, let us denote by M the term in the square brackets in (1.37) and let us minimize the function $F(k) := \left(M\frac{k}{e}\right)^k$.

The minimum is assumed for $k = \left[\frac{1}{m}\right]$. Indeed,

$$F'(k) = \left(M\frac{k}{e}\right)^k \left[\ln\left(M\frac{k}{e}\right) + 1\right] \ge 0 \ \Rightarrow \ \ln\left(M\frac{k}{e}\right) \ge -1 \ .$$

Thus,

$$M\frac{k}{e} \ge \frac{1}{e} \ \Rightarrow \ k \ge \frac{1}{M} \ .$$

Therefore, the minimum is assumed for k equal to the integer part of $\frac{1}{M}$. Moreover, since

$$m^k = e^{-\frac{1}{M}} \ ,$$

the norm of the remainder $\mathcal{R}$ and of its vector field are exponentially small. Indeed,

$$\begin{aligned}\|\mathcal{R}\|^*_{R/2} &\leq \mathbf{C}_6 \varepsilon m^k = \mathbf{C}_6 \varepsilon e^{-\frac{1}{M}} \leq \mathbf{C}_6 \varepsilon e^{-\frac{1}{\bar\mu}} \ , \\ \|X_{\mathcal{R}}\|^*_{\frac{R}{2}} &\leq \mathbf{C}_3 \bar\mu \frac{\varepsilon}{R} m^k = \mathbf{C}_3 \bar\mu \frac{\varepsilon}{R} e^{-\frac{1}{M}} \leq \mathbf{C}_3 \frac{\varepsilon\bar\mu}{R} e^{-\frac{1}{\bar\mu}}\end{aligned}$$

where the last inequality follows from the fact that $M \leq \bar\mu$. Indeed,

$$\begin{aligned}M &= e\frac{4T}{R}\left(\mathbf{C}_3\frac{18\varepsilon\bar\mu}{R} + \mathbf{C}_2\frac{5R^3}{\sigma} + \mathbf{C}_3\frac{5\varepsilon}{R}\right) \\ &\leq 56e\left(\frac{\varepsilon\mathbf{C}_3 T}{R^2} + \frac{\mathbf{C}_2 T R^2}{\sigma}\right) \leq \bar\mu \ .\end{aligned}$$

This concludes the proof. □

1.4.2 Semilocal stability estimates in the neighborhood of a resonant torus

At this point, we have at our disposal a normal form theorem which permits us to prove a result of semilocal stability in the neighborhood of a resonant torus by exploiting the conservation of the energy and the quasiconvexity of the unperturbed Hamiltonian. Precisely, in this subsection, we will construct semilocal estimates near periodic solutions.

By means of Theorem 1.4.1, we can make use of a Hamiltonian in normal form of the kind

$$\tilde h := h \circ \mathcal{T} = h_{\omega^*} + \hat h_r + \varepsilon \langle f \rangle + \mathcal{Z} + \mathcal{R} \ , \tag{1.38}$$

where $\mathcal{T} : M_{I^*,R/2} \mapsto M_{I^*,R}$ is the canonical transformation used for the normalization. Let us choose the initial datum $z(0) \in \mathcal{M}_{I^*,R/4}$ and let us denote by $z = \mathcal{T}(z')$ the new variables introduced with the normalization procedure. We will prove the following result

Lemma 1.4.7. *Assume that $J'(t) \in \mathcal{B}^*_{R/4}(0)$ and let $\varepsilon < R^4$. Then there exists positive constants $\mathcal{C}_1, \mathcal{C}_2$ such that*

$$|h_{\omega^*}(J'(t))| \leq \mathcal{C}_1 R^2 \ , \tag{1.39}$$

and,

$$\left|\hat h(J'(t))\right| \leq \mathcal{C}_2 R^4 \tag{1.40}$$

hold

$$\forall t : |t| \leq R e^{\frac{1}{\bar\mu}} \ .$$

Proof. We begin by considering the term

$$h_1(J') := h_{\omega^*}(J') + \hat{h}_r(J') \ .$$

From the definition of $\hat{h}_r = \hat{h} + h_r$, we want to estimate

$$\begin{aligned}\left|\hat{h}(J'(t)) - \hat{h}(J'(0))\right| \leq & |h_1(J'(t)) - h_1(J'(0))| + |h_{\omega^*}(J'(t)) - h_{\omega^*}(J'(0))| \\ & + |h_r(J'(t)) - h_r(J'(0))| \ .\end{aligned} \tag{1.41}$$

We start with the estimate of the first term by exploiting the conservation of the energy, that is, from $\tilde{h}(z'(t)) = \tilde{h}(z'(0))$, we deduce

$$\begin{aligned}|h_1(J'(t)) - h_1(J'(0))| &\leq \left|\tilde{\mathcal{Z}}(z'(t)) - \tilde{\mathcal{Z}}(z'(0))\right| + |\mathcal{R}(z'(t)) - \mathcal{R}(z'(0))| \\ &\leq 2 \left\| \tilde{\mathcal{Z}} \right\|^*_{R/2} + 2 \left\|\mathcal{R}\right\|^*_{R/2} \ ,\end{aligned}$$

where we have denoted by $\tilde{\mathcal{Z}}(z') = \varepsilon \langle f \rangle (z') + \mathcal{Z}(z')$. Thus,

$$\begin{aligned}|h_1(J'(t)) - h_1(J'(0))| &\leq 2(\|\varepsilon f\|^*_{R/2} + \|\mathcal{Z}\|^*_{R/2}) + 2 \|\mathcal{R}\|^*_{R/2} \\ &\leq 2(\varepsilon + 2\mathbf{C}_6\varepsilon) + 2\mathbf{C}_6\varepsilon e^{-\frac{1}{\bar{\mu}}} \\ &\leq (2 + 4\mathbf{C}_6 + 2\mathbf{C}_6 e^{-\frac{1}{\bar{\mu}}})\varepsilon \\ &\leq (2 + 5\mathbf{C}_6)\varepsilon := \mathbf{C}_7\varepsilon \leq \mathbf{C}_7 R^4 \ ,\end{aligned} \tag{1.42}$$

where we used $\varepsilon < R^4$, the estimates (1.24) and (1.34) and the fact that $e^{-\frac{1}{\bar{\mu}}} \leq \bar{\mu} \leq \frac{1}{2}$.

We pass now to estimate the second term on the right-hand side of (1.41). Let us consider

$$|h_{\omega^*}(J'(t)) - h_{\omega^*}(J'(0))| \leq \int_0^t \left|\frac{\mathrm{d}h_{\omega^*}(J'(s))}{\mathrm{d}s}\right| ds \ .$$

We use now the fact that $\hat{h}$, $\varepsilon \langle f \rangle$ and $\mathcal{Z}$ are already in normal form and, thus, they commute with the Hamiltonian h_{ω^*}: we obtain

$$\frac{\mathrm{d}h_{\omega^*}(J'(s))}{\mathrm{d}s} = \{h_{\omega^*}, \tilde{h}\}(J'(s)) = \{h_{\omega^*}, \mathcal{R}\}(J'(s)) \ .$$

Therefore, by means of Lemma C.2.1 of Appendix C, we can find the following estimate

$$|h_{\omega^*}(J'(t)) - h_{\omega^*}(J'(0))| \leq |t| \left\| \frac{\mathrm{d}h_{\omega^*}}{\mathrm{d}s} \right\|^*_{R/4} \leq |t| \, \|\omega^*\| \, \|X_{\mathcal{R}}\|^*_{R/2} \ , \tag{1.43}$$

and, by means of the estimates (1.34), we find out that

$$|h_{\omega^*}(J'(t)) - h_{\omega^*}(J'(0))| \leq \mathbf{C}_3 |t| \, \|\omega^*\| \, \frac{\varepsilon\bar{\mu}}{R} e^{-\frac{1}{\bar{\mu}}} \ .$$

Let us denote by $\Omega^* := \|\omega^*\|$ the norm of the frequency vector ω^*, then

$$|h_{\omega^*}(J'(t)) - h_{\omega^*}(J'(0))| \leq \mathbf{C}_3 |t| \Omega^* \frac{\varepsilon\bar{\mu}}{R} e^{-\frac{1}{\bar{\mu}}} \leq \frac{\mathbf{C}_3}{2} \Omega^* |t| \frac{\varepsilon}{R} e^{-\frac{1}{\bar{\mu}}} \ .$$

If we assume

$$|t| \leq R e^{\frac{1}{\bar{\mu}}} \ ,$$

then,

$$|h_{\omega^*}(J'(t)) - h_{\omega^*}(J'(0))| \leq c_2 \varepsilon \ , \tag{1.44}$$

with $c_2 := \frac{\mathbf{C}_3}{2}\Omega^*$. Furthermore, since

$$|h_{\omega^*}(J'(0))| \leq \Omega^* |J'(0)| \leq \tilde{c_2} R^2 \ ,$$

we deduce

$$|h_{\omega^*}(J'(t))| \leq \tilde{c_2} R^2 + c_2 \varepsilon \leq \max\{\tilde{c_2}, c_2\}(\varepsilon + R^2) \leq \mathcal{C}_1 R^2 \ ,$$

where we have assumed that $\varepsilon < R^4$.

We conclude with the estimate of the third and last term on the right-hand side of (1.41). Thus,

$$|h_r(J'(t)) - h_r(J'(0))| \leq 2 \, \|h_r\|_R^* \leq 2 \left\| \hat{h} \right\|_R^* \leq 2 \mathbf{C}_1 R^4 \ . \tag{1.45}$$

Finally, putting together the estimates (1.42), (1.44) and (1.45), we obtain an estimate for (1.41)

$$\left| \hat{h}(J'(t)) - \hat{h}(J'(0)) \right| \leq \mathbf{C}_7 R^4 + \mathcal{C}_1 R^4 + 2\mathbf{C}_1 R^4 \leq (\mathbf{C}_7 + \mathcal{C}_1 + 2\mathbf{C}_1) R^4 := \mathbf{C}_8 R^4 \ ,$$

where we have assumed $\varepsilon < R^4$. Thus,

$$\left| \hat{h}(J'(t)) \right| \leq \left| \hat{h}(J'(0)) \right| + \mathbf{C}_8 R^4 \ .$$

We compute now the estimate for $\hat{h}(J'(0))$ by exploiting the definition of $\hat{h}$, namely,

$$\hat{h}(J'(0)) = \frac{1}{2} \langle J'(0), \, \frac{\partial^2 h}{\partial I^2}(I^*) J'(0) \rangle \ .$$

Thus, from the Cauchy estimates (cf. Lemma C.2.2, Appendix C), we obtain

$$\left|\hat{h}(J'(0))\right| \leq \mathbf{C}_9 R^4 \ ,$$

and, finally,

$$\left|\hat{h}(J'(t))\right| \leq \mathbf{C}_9 R^4 + \mathbf{C}_8 R^4 \leq (\mathbf{C}_9 + \mathbf{C}_8) R^4 := \mathcal{C}_2 R^4 \ .$$

This concludes the proof. □

At this point, we have to exploit the *quasiconvexity* condition to prove

Lemma 1.4.8. *Let J' such that*

$$|h_{\omega^*}(J')| \leq \mathcal{C}_1 R^2 \ , \tag{1.46}$$

and,

$$\left|\hat{h}(J')\right| \leq \mathcal{C}_2 R^4 \ . \tag{1.47}$$

Then, there exists a constant $\mathcal{C}_3$ such that $J' \in \mathcal{B}^*_{\mathcal{C}_3 R}(0)$.

Proof. Let us first rewrite the term on the left-hand side of (1.47) by using the definition of $\hat{h}$, that is,

$$\left|\hat{h}(J')\right| = \frac{1}{2}\langle J', \, \frac{\partial^2 h(I^*)}{\partial I^2} J'\rangle \ ,$$

and, then, we decompose the vector $v := J'$ into the sum of two components by means of the projector operator Π_* onto ω^*. Thus, we consider

$$v = \Pi_* v + \Pi_*^{\perp} v$$

and we pull this decomposition into the quadratic form

$$\mathcal{Q}(v) := \langle v, \, \frac{\partial^2 h}{\partial I^2}(I^*) v\rangle \ .$$

We obtain

$$\langle v, \, \frac{\partial^2 h}{\partial I^2}(I^*)(v)\rangle = \langle \Pi_* v, \, \frac{\partial^2 h}{\partial I^2}(I^*)\Pi_* v\rangle + \langle \Pi_*^{\perp} v, \, \frac{\partial^2 h}{\partial I^2}(I^*)\Pi_*^{\perp} v\rangle + 2\langle \Pi_* v, \, \frac{\partial^2 h}{\partial I^2}(I^*)\Pi_*^{\perp} v\rangle \ . \tag{1.48}$$

We use now the quasiconvexity condition (cf. Def. 1.3.2) in (1.48), thus, there exists a positive constant c such that the following inequality,

$$\langle \Pi_*^{\perp} v, \, \frac{\partial^2 h}{\partial I^2}(I^*)\Pi_*^{\perp} v\rangle \geq c \left\| \Pi_*^{\perp} v \right\|^2 \ ,$$

holds. Furthermore, we have the following estimates on the other two terms in the right-hand side of (1.48)

$$\langle \Pi_* v,\, \frac{\partial^2 h}{\partial I^2}(I^*)\Pi_* v\rangle \geq -\mathcal{C}\, \|\Pi_* v\|^2 \ ,$$
$$\langle \Pi_* v,\, \frac{\partial^2 h}{\partial I^2}(I^*)\Pi_*^{\perp} v\rangle \geq -\mathcal{C}\, \|\Pi_* v\| \, \left\|\Pi_*^{\perp} v\right\| \ .$$

At this point, we put all the estimates in (1.48) and obtain

$$\langle v,\, \frac{\partial^2 h}{\partial I^2}(I^*)(v)\rangle \geq c\, \left\|\Pi_*^{\perp} v\right\|^2 - \mathcal{C}\, \|\Pi_* v\|^2 - 2\mathcal{C}\, \|\Pi_* v\| \, \left\|\Pi_*^{\perp} v\right\| \ ,$$

that is,

$$c\, \left\|\Pi_*^{\perp} v\right\|^2 \leq \langle v,\, \frac{\partial^2 h}{\partial I^2}(I^*)(v)\rangle + \mathcal{C}\, \|\Pi_* v\|^2 + 2\mathcal{C}\, \|\Pi_* v\| \, \left\|\Pi_*^{\perp} v\right\| \ . \tag{1.49}$$

The size of the component $\Pi_* v$ can be estimated by using(1.46). Indeed, from the definition of h_{ω^*} and of the orthogonal component $\Pi_*^{\perp} v$, we have

$$h_{\omega^*}(J'(t)) = \langle \omega^*,\, v\rangle = \langle \omega^*,\, \Pi_* v\rangle \ .$$

Thus, by using (1.46), we obtain the following estimate

$$\|\Pi_* v\| \leq \mathbf{C}_{10} R^2 \ .$$

At this point, we can rewrite (1.49) as follows

$$c\, \left\|\Pi_*^{\perp} v\right\|^2 \leq \langle v,\, \frac{\partial^2 h}{\partial I^2}(I^*)(v)\rangle + \mathcal{C}\mathbf{C}_{10}^2 R^4 + 2\mathcal{C}\mathbf{C}_{10} R^2 \left\|\Pi_*^{\perp} v\right\| \ .$$

Moreover, from the definition of $\hat{h}$, we have that

$$\langle v,\, \frac{\partial^2 h}{\partial I^2}(I^*)(v)\rangle = 2\left|\hat{h}(J')\right| \ ,$$

and, thus, by exploiting the estimate (1.47), we obtain

$$\begin{aligned} c\, \left\|\Pi_*^{\perp} v\right\|^2 &\leq 2\left|\hat{h}(J')\right| + \mathcal{C}\mathbf{C}_{10}^2 R^4 + 2\mathcal{C}\mathbf{C}_{10} R^2 \left\|\Pi_*^{\perp} v\right\| \\ &\leq 2\mathcal{C}_2 R^4 + \mathcal{C}\mathbf{C}_{10}^2 R^4 + 2\mathcal{C}\mathbf{C}_{10} R^2 \left\|\Pi_*^{\perp} v\right\| \ . \end{aligned}$$

Thus, the inequality we have to solve takes the form

$$c\, \left\|\Pi_*^{\perp} v\right\|^2 - 2\mathcal{C}\mathbf{C}_{10} R^2 \left\|\Pi_*^{\perp} v\right\| - \left(2\mathcal{C}_2 + \mathcal{C}\mathbf{C}_{10}^2\right) R^4 \leq 0 \ .$$

We solve this inequality and find out that there exists a positive constant $\mathbf{C}_{11}$ such that

$$\left\| \Pi_*^{\perp} v \right\| \leq \mathbf{C}_{11} R^2 \ .$$

Therefore,

$$\|v\| = \|\Pi_* v\| + \left\| \Pi_*^{\perp} v \right\| \leq \mathbf{C}_{10} R^2 + \mathbf{C}_{11} R^2 := \mathcal{C}_3 R^2 \ ,$$

where $\mathcal{C}_3 = \mathbf{C}_{10} + \mathbf{C}_{11} > 0$. □

Corollary 1.4.1. *Assume that $J'(0) \in \mathcal{B}^*_{R/4\mathcal{C}_3}(0)$, then one has*

$$J'(t) \in \mathcal{B}^*_{R/4}(0) \ , \quad \forall t : |t| \leq Re^{-\frac{1}{\mu}} \ .$$

We go back now to the old variables. By exploiting the estimate on the deformation of the action variables, we have

Corollary 1.4.2. ***(Stability of resonant tori)***
There exists a constant $\mathcal{C}_4$ such that $J(0) \in \mathcal{B}^*_{R/\mathcal{C}_4}(0)$ implies

$$J(t) \in \mathcal{B}^*_{R}(0) \ , \quad \forall t : |t| \leq Re^{-\frac{1}{\mu}} \ .$$

Proof. Let us compute

$$\|J\| \leq \|J - J'\| + \|J'\| \leq \frac{\varepsilon \mathbf{C}_3 T}{R^2} + \mathcal{C}_3 R^2 := \mathcal{C}_4 R^2 \ .$$

Thus, if we assume that $J(0) \in \mathcal{B}^*_{R/\mathcal{C}_4}$, then one has

$$J(t) \in \mathcal{B}^*_{R}(0) \ , \quad \forall t : |t| \leq Re^{-\frac{1}{\mu}} \ .$$

□

1.4.3 Dirichlet Theorem and semilocal stability

In this subsection we present a useful tool for the proof of the semilocal stability which concludes Nekhoroshev's theorem: the so called *Dirichlet theorem* for simultaneous approximations.

Theorem 1.4.2. ***(Dirichlet theorem for simultaneous approximations)***
Let $\alpha_1, \dots, \alpha_n \in \mathbb{R}^+$. For any $Q > 1$ there exists an integer $q \ : \ 1 \leq q < Q$ and a vector $p = (p_1, \dots, p_n) \in \mathbb{N}^n$ such that

$$|\alpha_i q - p_i| \leq \frac{1}{Q^{\frac{1}{n}}} \ , \qquad i = 1, \dots, n \ .$$

The proof follows directly from an application of the Minkowski's convex body theorem that we report in Appendix B for the sake of completeness. As we have anticipated before, this theorem shall be applied in order to produce semilocal stability estimates.

Let I_0 be the initial value of the actions, denote $\omega = (\omega_1, \ldots, \omega_n) = \frac{\partial h}{\partial I}(I_0) \subset \mathbb{R}^n$. The Dirichlet Theorem applies: $\forall Q > 1$ there exists a resonant frequency vector ω^* of period $T = q$ such that $1 \leq T < Q$, whose components are rational numbers of the form $\frac{p_i}{q}$ such that

$$|\omega_i - \omega_i^*| \leq \frac{1}{TQ^{\frac{1}{n-1}}} \ , \quad i = 1, \ldots, n \ . \tag{1.50}$$

If Q is large enough, the frequency map $\omega : I \to \omega(I)$ is invertible, then we can invert the relation (1.50) and find out that the following inequality,

$$|I_i - I_i^*| \leq \tilde{C}\frac{1}{TQ^{\frac{1}{n-1}}} \ , \quad i = 1, \ldots, n$$

holds.

Now, we would like to apply the stability estimates in a neighborhood of I^* which corresponds to a resonant torus of frequency ω^*. To do so, we have to choose R such that $\|I - I^*\| \leq \frac{R}{4}$. Namely,

$$\tilde{C}\frac{1}{TQ^{\frac{1}{n-1}}} = \frac{R^2}{4} \ . \tag{1.51}$$

At this point, we can compute the parameter $\bar{\mu}$ and verify that it is sufficiently small. Thus, let us begin by the definition of $\bar{\mu}$, that is,

$$\bar{\mu} = 57e\left(\frac{\varepsilon \mathbf{C}_3 T}{R^2} + \frac{\mathbf{C}_2 T R^2}{\sigma}\right) \leq \mathcal{C}_5\left(\varepsilon Q^{\frac{2n-1}{n-1}} + \frac{1}{Q^{\frac{1}{n-1}}}\right) \ .$$

Choosing $Q = \varepsilon^{-\frac{n-1}{2n}}$, one gets

$$\bar{\mu} \leq 2\mathcal{C}_5 \varepsilon^{\frac{1}{2n}} \ .$$

Inserting in the other estimates one gets the thesis.

Chapter 2

The spatial central motion problem

In this section, we apply the theory of Chapter 1 to the spatial central motion problem, in particular we show that, when written in action angle coordinates, its Hamiltonian is quasiconvex for any potential but the Keplerian and the Harmonic ones.

2.1 Statement of the structure theorem

As in the Introduction, we consider the Hamiltonian of a particle of unitary mass moving in space under the action of a central potential. In Cartesian coordinates, it is given by

$$H(\mathbf{x}, \mathbf{p}) = \frac{|\mathbf{p}|^2}{2} + V(|\mathbf{x}|) \tag{2.1}$$

and we define the total angular momentum $(L_1, L_2, L_3) \equiv \mathbf{L} := \mathbf{x} \times \mathbf{p}$ and denote by $L := \sqrt{L_1^2 + L_2^2 + L_3^2}$ its modulus.

Let $\mathcal{P}_A^{(3)}$ be a compact subset of $\mathbb{R}^6$ invariant under the dynamics of H. Consider the effective Hamiltonian, namely,

$$H_{eff}(r, p_r, L^2) := \frac{p_r^2}{2} + V_{eff}(r, L^2) \ ,$$

where

$$V_{eff}(r, L^2) := \frac{L^2}{2r^2} + V(r) \ , \tag{2.2}$$

which will be considered as a function of (r, p_r) only and, thus, L plays the role of a parameter. Assume now that the central potential V satisfies the assumptions (H0)-(H3). We have the following result

Theorem 2.1.1. *There exists a finite number of open disjoint sets* $\mathcal{O}_j^{(3)} \subset \mathcal{P}_A^{(3)}$, $j = 1, ..., N$, *and a compact subset* $\mathcal{S}^{(3)} \subset \mathcal{P}_A^{(3)}$ *which is the union of a finite number of analytic hypersurfaces*[1], *with the following properties:*

(1) $\mathcal{S}^{(3)} \cap \mathcal{O}_j^{(3)} = \emptyset, \forall j$

(2) $\mathcal{S}^{(3)} \bigcup \left(\bigcup_j \mathcal{O}_j^{(3)} \right) = \mathcal{P}_A^{(3)}$

(3) *Each of the domains* $\mathcal{O}_j^{(3)}$ *has the structure of a bifibration*

$$\mathcal{O}_j^{(3)} \xrightarrow{F} \mathcal{M}_j \xrightarrow{\tilde{F}} \mathcal{A}_j \subset \mathbb{R}^2 \ , \tag{2.3}$$

with the following properties

(i) *Every fiber of* $\mathcal{O}_j^{(3)} \xrightarrow{F} \mathcal{M}_j$ *is diffeomorphic to* $\mathbb{T}^2$

(ii) *Every fiber of* $\mathcal{M}_j \xrightarrow{\tilde{F}} \mathcal{A}_j$ *is diffeomorphic to* S^2

(iii) *the bifibration is symplectic: precisely, every fiber of* $\mathcal{O}_j^{(3)} \xrightarrow{F} \mathcal{M}_j$ *has a neighborhood* U *endowed with an analytic diffeomorphism*

$$U \to b(U) \times \mathcal{A}_j \times \mathbb{T}^2 \tag{2.4}$$

such that the level sets of F^{-1} *coincide with the level sets of* $b \times I$ *and, writing* $b = (p, q)$, *the symplectic form becomes*

$$dp \wedge dq + dI_1 \wedge d\alpha_1 + dI_2 \wedge d\alpha_2 \ . \tag{2.5}$$

(iv) *In each of the domains* $\mathcal{O}_j^{(3)}$, $L \equiv I_2$ *varies in an open interval, say* $\mathcal{I}_j$ *and* (r, p_r) *vary in some level sets of* H_{eff}. *The infimum of the energy* H_{eff} *is either a* nondegenerate *maximum or a* nondegenerate *minimum of the effective potential.*

The proof consists essentially of two steps: first we give a detailed construction of the action angle coordinates in the planar case, and then we analyze the geometry of the three dimensional case and show how to use the result of the planar case for the construction of the generalized action angle coordinates.

[1]namely level surfaces of analytic functions

2.2 Action angle coordinates for the planar case

In the planar case, the Hamiltonian in polar coordinates is given by

$$H(r, p_r, p_\theta) := \frac{p_r^2}{2} + V_{eff}(r, p_\theta^2) \ , \tag{2.6}$$

where the effective potential $V_{eff}(r, p_\theta^2)$ was defined in (2.2).

Remark 2.2.1. *Due to assumption (H2) there* do not exist *constants k_1, k_2 s.t. the potential has the form*

$$V(r) = \frac{k_1}{2r^2} + k_2 \ . \tag{2.7}$$

To fix ideas, one example of a possible effective potential (for fixed value of p_θ) is the one in Figure 2.1.

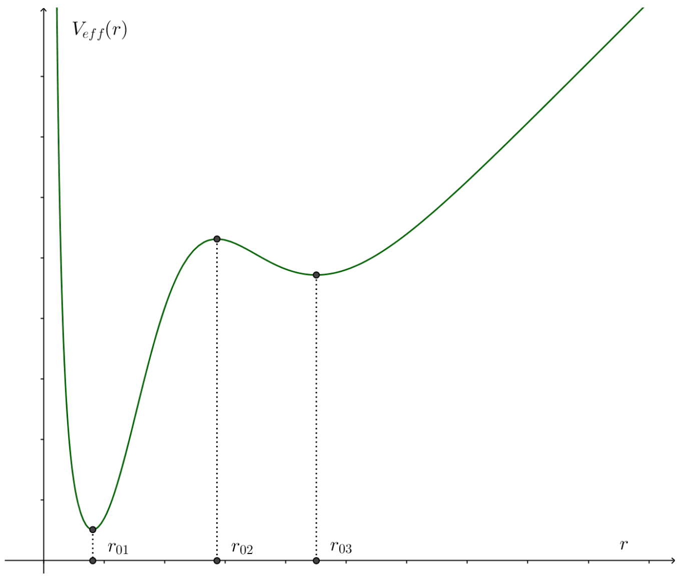


Figure 2.1: A possible shape for the effective potential for a fixed value of p_θ.

We now describe the domain in which the action angle variables can be introduced.

We begin with the set where the angular momentum varies. Define

- $L_m^2 := \min \{[\text{Range}(r^3 V'(r))] \cap [0, +\infty] \cap [\ell^*, +\infty]\}$,
- and
  - L_M to be an arbitrary (large) positive number, if $\sup r^3 V'(r) = +\infty$,

– $L_M := \sup \sqrt{r^3 V'(r)}$, if $\sup r^3 V'(r) < +\infty$.

Then, the angular momentum will be assumed to vary in

$$\mathcal{I} := (L_m, L_M) \ . \tag{2.8}$$

We define now the domain for (r, p_r). We fix $p_\theta \in \mathcal{I}$ and $E \in \mathbb{R}$ and consider the sublevels

$$\mathcal{S}_{p_\theta}(E) := \{(r, p_r) \ : \ H(r, p_r, p_\theta) < E\} \ . \tag{2.9}$$

As $E \in \mathbb{R}$ and $p_\theta \in \mathcal{I}$ vary, the sets $\mathcal{S}_{p_\theta}(E)$ can be empty or can have one or more connected components. We denote by $\mathcal{S}^{comp}_{p_\theta}(E)$ the union of the connected components of $\mathcal{S}_{p_\theta}(E)$ whose closure is compact. We underline that we take the union over the components whose closure is *compact* since we want to exclude the unbounded domains which cannot be covered by action angle variables.

In conclusion, collecting all the information together, we can state that the set of the phase space which will be covered using action angle systems of coordinates is essentially the following one

$$\mathcal{P}_A := \left\{ (r, p_r, \theta, p_\theta) \ : \ \theta \in \mathbb{T} \ , \ p_\theta \in \mathcal{I} \ , \ (r, p_r) \in \bigcup_{E \in \mathbb{R}} \mathcal{S}^{comp}_{p_\theta}(E) \right\} \ . \tag{2.10}$$

Of course, for fixed values of p_θ, the critical points of V_{eff} correspond to singular values of action angle variables, so in order to have well defined action angle variables, we have to eliminate some singular sets. Furthermore, in order to proceed in the verification of quasiconvexity, we will exclude values of p_θ corresponding to which V_{eff} has degenerate critical points. Precisely, we have the following theorem

Theorem 2.2.1. *There exists a finite number of open disjoint sets $\mathcal{O}_j \subset \mathcal{P}_A$, $j = 1, ..., N$, and a compact subset $\mathcal{S} \subset \mathcal{P}_A$ which is the union of a finite number of analytic hypersurfaces, with the following properties:*

(1) $\mathcal{S} \cap \mathcal{O}_j = \emptyset$, $\forall j$

(2) $\mathcal{S} \bigcup \left(\bigcup_j \mathcal{O}_j \right) = \mathcal{P}_A$

(3) On each of the domains $\mathcal{O}_j$ there exists an analytic diffeomorphism

$$\Phi_j : \mathcal{O}_j \to \mathcal{A}_j \times \mathbb{T}^2 \ , \qquad \mathcal{A}_j \subset \mathbb{R}^2 \tag{2.11}$$

$$(r, p_r, \theta, p_\theta) \mapsto (I_1, I_2, \alpha_1, \alpha_2) \tag{2.12}$$

which introduces action angle variables.

(4) For every j, the Hamiltonian in action angle variables is a real analytic function over the whole of $\mathcal{A}_j$

$$h_j : \mathcal{A}_j \mapsto \mathbb{R} \ .$$

(5) Each of the domains $\mathcal{O}_j$ is the union for I_2 in an open interval, say $\mathcal{I}_j$ of level sets of H considered as a function of (r, p_r) only. The infimum of the energy H is either a nondegenerate maximum or a nondegenerate minimum of the effective potential.

The main point in the proof of this result consists in showing that, except for at most a finite number of values of $p_\theta \in \mathcal{I}$, the effective potential has only *nondegenerate critical points.* We will also eliminate some values of p_θ in order to get that the critical levels of V_{eff} are distinct (see Lemma 2.2.3). This allows to classify completely the domains of the action I_1 for fixed value of the angular momentum.

Precisely, the construction of the sets $\mathcal{O}_j$ will be done by first eliminating from $\mathcal{I}$ a finite set $\mathcal{I}_s$ in such a way that, for $p_\theta \in \mathcal{I} \setminus \mathcal{I}_s$ the effective potential has only nondegenerate extrema at different levels. So one gets that $\mathcal{I} \setminus \mathcal{I}_s$ turns out to be the union of finitely many intervals $\mathcal{I}_l$. Then having fixed $p_\theta \in \mathcal{I}_l$ one considers the level sets of the effective Hamiltonian and takes the union of one of its compact connected components as E varies in an interval *not containing critical levels.* The set $\mathcal{O}_j$ is obtained by taking also the union over $p_\theta \in \mathcal{I}_l$. The construction is explained in detail in the next subsection.

2.2.1 The construction of the action angle variables

We begin the construction with two useful lemmas

Lemma 2.2.1. *Let $(\bar{r}, \bar{\ell})$ be such that $\bar{r}$ is an extremum of $V_{eff}(., \bar{\ell})$. Then there exists an odd n, a neighborhood $\mathbb{R} \supset \mathcal{U}$ of 0 and a function $r_0 = r_0((\ell - \bar{\ell})^{1/n})$ analytic in $\mathcal{U}$, s.t. $r_0((\ell - \bar{\ell})^{1/n})$ is an extremum of $V_{eff}(.;\ell)$. Furthermore $r_0(0) = \bar{r}$, and for any $\ell \neq \bar{\ell}$ the extremum is nondegenerate.*

Proof. The proof is standard, but we give it for the sake of completeness. To fix ideas assume that $\bar{r}$ is a maximum. Of course the theorem holds with $n = 1$ if the maximum is nondegenerate. So, assume it is degenerate. Then, since by the assumptions the function $V_{eff}(., \bar{\ell})$ is nontrivial, there exists an odd $n > 2$, s.t. $\partial_r^{n+1} V_{eff}(\bar{r}, \bar{\ell}) = a \neq 0$. Thus we look for $\delta = \delta(\xi)$ solving

$$F(\delta, \xi) := \partial_r V_{eff}(\bar{r} + \delta, \bar{\ell} + \xi^n) = \left[V'(\bar{r} + \delta) - \frac{\bar{\ell}}{(\bar{r} + \delta)^3} \right] - \frac{\xi^n}{(\bar{r} + \delta)^3} = 0 \ . \quad (2.13)$$

It is convenient to rewrite the square bracket as

$$\frac{a}{n!}\delta^n + R_0(\delta) \ ,$$

where R_0 is an analytic function with a zero of order at least $n+1$ at the origin. A short computation shows that we can rewrite (2.13) in the form

$$\delta\left[\left(\frac{a}{n!}+\frac{R_0(\delta)}{\delta^n}\right)(\bar{r}+\delta)^3\right]^{1/n} = \xi \ , \tag{2.14}$$

which is in a form suitable for the application of the implicit function theorem. Thus it admits a solution $\delta(\xi)$ which is analytic and which has the form

$$\delta(\xi) = \left(\frac{a}{n!}\bar{r}^3\right)^{-1/n}\xi + \mathcal{O}(\xi^2) \ . \tag{2.15}$$

It remains to show that for ξ different from zero (small) the critical point just constructed is nondegenerate. To this end we compute the derivative with respect to δ of F (cf. eq. (2.13)); we get

$$\partial_\delta F(\delta(\xi),\xi) = \frac{a}{(n-1)!}[\delta(\xi)]^{n-1} + R_0'(\delta(\xi)) + \frac{3\xi^n}{(\bar{r}+\delta)^4} \tag{2.16}$$

$$= \frac{a}{(n-1)!}\left(\frac{n!}{a\bar{r}^3}\right)^{\frac{n-1}{n}}\xi^{n-1} + \mathcal{O}(\xi^n) \ , \tag{2.17}$$

which for small ξ is nonvanishing. □

Lemma 2.2.2. *Let $\bar{r}$ be a degenerate critical point of $V_{eff}(.;\bar{\ell})$ which is neither a maximum nor a minimum. Then for ℓ in a neighborhood of $\bar{\ell}$, the effective potential $V_{eff}(.;\ell)$ either has no critical points in a neighborhood of $\bar{r}$, or it has a nondegenerate maximum and a nondegenerate minimum which depend smoothly on ℓ.*

Proof. A procedure similar to that used to deduce the equation (2.14) leads to the equation

$$\delta^n\left(\frac{a}{n!}+\frac{R_0(\delta)}{\delta^n}\right)(\bar{r}+\delta)^3 = \ell - \bar{\ell} \ , \tag{2.18}$$

where n *is now even* and the sign of a is arbitrary. It is thus clear that for $(\ell-\bar{\ell})/a$ negative the critical point disappears. When this quantity is positive then it is easy to see that two new critical points bifurcate from $\bar{r}$. Using a computation similar to that of eqs. (2.16), (2.17) one sees that they are a maximum and a minimum which are nondegenerate. □

Corollary 2.2.1. *There exists a finite set* $\mathcal{I}_{s1} \subset \mathcal{I}$ *such that,* $\forall L \in \mathcal{I} \setminus \mathcal{I}_{s1}$ *the effective potential* $V_{eff}(.;L^2)$ *has only critical points which are nondegenerate extrema.*

Proof. Just remark that the values of L for which $V_{eff}(.;L^2)$ has at least one degenerate critical point are isolated. Thus, due to the compactness of $\overline{\mathcal{I}}$ their number is finite. □

Remark 2.2.2. *The set* $\mathcal{I} \setminus \mathcal{I}_{s1}$ *is the union of finitely many open intervals. The critical points of* V_{eff} *are analytic functions of* L^2 *in such intervals; furthermore they do not cross (at crossing points their multiplicity would be greater then one, against nondegeneracy). Therefore the number, the order and the nature of the critical points is constant in each of the subintervals.*

The main structural result we need for the effective potential is the following lemma.

Lemma 2.2.3. *There exists a* finite *set* $\mathcal{I}_s \subset \mathcal{I}$ *such that,* $\forall L \in \mathcal{I} \setminus \mathcal{I}_s$ *the effective potential* $V_{eff}(.;L^2)$ *has only critical points which are nondegenerate extrema and the critical levels are all different. Furthermore each critical level does not coincide with*

$$V^{\infty} := \lim_{r\to\infty} V(r) \ . \tag{2.19}$$

Proof. First we restrict to $\mathcal{I} \setminus \mathcal{I}_{s1}$ (defined in Corollary 2.2.1), so that all the critical points of V_{eff} are nondegenerate. We concentrate on one of the open subintervals of $\mathcal{I} \setminus \mathcal{I}_{s1}$ (cf. Remark 2.2.2). Let $\ell := L^2$, and let $r(\ell)$ be a critical point of $V_{eff}(.;\ell)$. Consider the corresponding critical level $V_{eff}(r(\ell),\ell)$ and compute

$$\frac{d}{d\ell}V_{eff}(r(\ell),\ell) = \frac{\partial r}{\partial \ell}\frac{\partial V_{eff}}{\partial r}(r(\ell),\ell) + \frac{\partial V_{eff}}{\partial \ell} = \frac{1}{2r^2} \ , \tag{2.20}$$

where we used the fact that $r(\ell)$ is critical, so that $\frac{\partial V_{eff}}{\partial r}(r(\ell),\ell) = 0$ and the explicit expression of V_{eff} as a function of ℓ. Thus the derivative (2.20) depends on r only. It follows that if two critical levels coincide, then their derivatives with respect to ℓ are different, and therefore they become different when ℓ is changed. It follows that also the set of the values of ℓ for which some critical levels coincide is formed by isolated points, and therefore it is composed by at most a finite number of points in each subinterval. Of course a similar argument applies to the comparison with V^{∞}. □

Remark 2.2.3. *As* L *varies in one of the connected subintervals of* $\mathcal{I} \setminus \mathcal{I}_s$ *the critical levels and* V^{∞} *remain ordered in the same way, in the sense that they do not cross.*

We are now ready for the construction of action angle variables (and the proof of Theorem 2.2.1).

Consider one of the connected subintervals of $\mathcal{I} \setminus \mathcal{I}_s$ and denote it by $\widetilde{\mathcal{I}}$. We distinguish two cases: (1) the effective potential has no local maxima for $p_\theta \in \widetilde{\mathcal{I}}$ but it must present one local minimum; (2) the effective potential has at least one local maximum for $p_\theta \in \widetilde{\mathcal{I}}$.

We start by the case (1) and denote by r_0 the minimum of $V_{eff}(\cdot, p_\theta^2)$. The second action is $I_2 := p_\theta$, while the first one is the action of the one dimensional effective system with Hamiltonian $H(r, p_r, p_\theta)$ which is given by

$$I_1 = G(E, I_2) := \frac{1}{\pi} \int_{r_{min}}^{r_{max}} \sqrt{2(E - V_{eff}(r; I_2^2))} dr \tag{2.21}$$

$$E \in (V_{eff}(r_0; I_2^2), V^\infty) \ , \tag{2.22}$$

where r_{min} and r_{max} are the solutions of the equation $E = V_{eff}(r; I_2^2)$ and r_0 is the minimum of the potential.

We can notice that the set $\mathcal{S}_{p_\theta}(V^\infty)$ in the phase-space (r, p_r) is compact. Correspondingly the action I_1 varies in $(0, G(V^\infty, I_2))$. Thus, we first construct the domain $\widetilde{\mathcal{O}}$ as

$$\widetilde{\mathcal{O}} := \left\{ (r, p_r, \theta, p_\theta) \ : \ p_\theta \in \widetilde{\mathcal{I}} \ , (r, p_r) \in \mathcal{S}_{p_\theta}(V^\infty) \right\} \ , \tag{2.23}$$

The actions vary in

$$\widetilde{\mathcal{A}} := \left\{ (I_1, I_2) \ : \ I_2 \in \widetilde{\mathcal{I}} \ , \ I_1 \in (0, G(V^\infty, I_2)) \right\} \ . \tag{2.24}$$

In this domain the Hamiltonian is obtained by computing E as a function of I_1, I_2 by inverting the function G defined in (2.21).

Consider now the case (2) where the effective potential has at least one local maximum. In this case there are in general several different domains which are described by action angle coordinates. To fix ideas consider the case where $V_{eff}(.; p_\theta^2)$ has exactly two minima $r_1 > r_2$ and one maximum R_1 fulfilling $V_{eff}(R_1; p_\theta^2) < V^\infty$. Then, the sublevel $\mathcal{S}_{p_\theta}(V_{eff}(R_1; p_\theta^2))$ has two connected components, in each of which one can construct the action variables exactly by the formula (2.21) (with a suitable redefinition of r_{min} and r_{max}). The two corresponding domains are

$$\widetilde{\mathcal{A}}_i := \left\{ (I_1, I_2) \ : \ I_2 \in \widetilde{\mathcal{I}} \ , \ I_1 \in (0, G_i(V_{eff}(R_1; I_2^2), I_2)) \right\} \ , \quad i = 1, 2 \ , \tag{2.25}$$

with an obvious definition of G_i.

Then there is further domain in which the action I_1 can be defined; such a domain is above the local maximum of V_{eff} and is $\mathcal{S}_{p_\theta}(V^\infty) \setminus \overline{\mathcal{S}_{p_\theta}(V_{eff}(R_1; p_\theta^2)}$. In this domain the action is still given by the formula (2.21), and the corresponding domain of the actions is given by

$$\widetilde{\mathcal{A}}_3 := \left\{(I_1, I_2) : I_2 \in \widetilde{\mathcal{I}},\, I_1 \in (G_1(V_{eff}(R_1; I_2^2), I_2) + G_2(V_{eff}(R_1; I_2^2), I_2), G_3(V^\infty, I_2))\right\} . \tag{2.26}$$

In this case, the domains $\tilde{\mathcal{O}}_i$ can be thought of as the regions in which the phase-space (r, p_r) is divided by the separatrix.

It is clear that in more general situations only one more kind of domains of the phase space can exists: namely domains in which both the minimal energy and the maximal energy correspond to the energies of local maxima of the effective potential.

In order to conclude the construction of the domains, in both the situations (1) and (2) described previously, we take the union over $p_\theta \in \tilde{\mathcal{I}}$ and obtain the domains $\mathcal{O}_j$ as well as the action spaces $\mathcal{A}_j$ exploiting the definition of the action as the area under a curve in the phase-space.

Remark 2.2.4. *We make the reader notice that all the domains $\mathcal{O}_j$ are bounded below by critical points of the effective potential V_{eff}. In what follows, we are going to differentiate the techniques used in order to prove our result of quasiconvexity according to the different nature of the critical point considered.*

Summarizing we have that the following Lemma holds.

Lemma 2.2.4. *Each of the domains $\mathcal{O}_j$ in which a system of action angle variables is defined is the union for I_2 in an open interval, say $\mathcal{I}_j$ of level sets of H considered as a function of (r, p_r) only. The infimum of the energy H is either a nondegenerate maximum or a nondegenerate minimum of the effective potential. The corresponding value of the radius will be denoted by $r_{0j} = r_{0j}(I_2^2)$ and depends analytically on $I_2 \in \mathcal{I}_j$.*

In the following we will denote by

$$V_{0j}(I_2) := V_{eff}(r_{0j}(I_2^2), I_2^2) = V(r_{0j}) + \frac{r_{0j} V'(r_{0j})}{2} \tag{2.27}$$

the corresponding critical level.

Proof of Theorem 2.2.1. In order to prove this result, it remains only to construct the subset $\mathcal{S}$. Thus, simply define $\mathcal{S}$ to be the union of the following analytic hypersurfaces:

(1) $\{(r, p_r, \theta, p_\theta) \ : \ p_\theta = 0\}$

(2) $\{(r, p_r, \theta, p_\theta) \ : \ p_\theta \in \mathcal{I}_s\}$

(3) $\{(r, p_r, \theta, p_\theta) \ : \ V_{0j}(p_\theta) = H(r, p_r, p_\theta)\}$.

Having said so, the Arnol'd-Liouville theorem can be applied in each domain $\mathcal{O}_j$ and, thus, a system of action angle coordinates can be introduced in each $\mathcal{O}_j$. Moreover, since the maps Φ_j are analytic diffeomorphisms, then for every j, the Hamiltonian $h_j : \mathcal{A}_J \mapsto \mathbb{R}$ written in action variables is a real analytic function all over $\mathcal{A}_J$.

This concludes the proof. □

2.3 From the planar to the spatial case

We come to the proof of Theorem 2.1.1. To do so, as we have anticipated, we can reduce our analysis to the planar motion.

First, we remark that the whole phase-space can be covered using two systems of polar coordinates with z-axis ($\theta = 0$) not coinciding. Using any one of the two systems, one can introduce explicitly by the classical procedure action angle variables which turn out to be $I_2 = L$ and I_1 which is the action of the Hamiltonian system with 1 degree of freedom and Hamiltonian $H_{eff}(r, p_r, L^2)$. Thus, I_1 has exactly the same expression as in the planar case, but with p_θ^2 replaced by $L^2 = |\mathbf{x} \wedge \mathbf{p}|^2$.

Furthermore, the Hamiltonian as a function of I_1, I_2 has the same functional form as in the planar case.

We come to the construction of the set $\mathcal{O}_j^{(3)}$ and the description of the phase-space and of the fibration related to the superintegrable structure of the spatial case.

Then, we remark that any compact subset of the phase-space invariant under the dynamics can be constructed as follows.

For $L \in \mathcal{I}$ (cf. (2.8)), define

$$\mathcal{S}_L^{(3)}(E) := \left\{(\mathbf{x}, \mathbf{p}) \ : \ L^2(\mathbf{x}, \mathbf{p}) = L^2 \text{ and } H(\mathbf{x}, \mathbf{p}) < E\right\} \ , \qquad (2.28)$$

then the sets $\mathcal{S}_L^{(3)}(E)$ can be empty or can have one or more connected components. Denote again by $\mathcal{S}_L^{comp}(E)$ the union of the connected components of $\mathcal{S}_L^{(3)}(E)$ whose closure is compact. Define

$$\mathcal{P}_A^{(3)} := \bigcup_{L \in \mathcal{I}} \bigcup_{E \in \mathbb{R}} \mathcal{S}_L^{comp}(E) \ , \qquad (2.29)$$

so that the reference domain introduced at the beginning of Chapter 2 can be redefined according to (2.29).

We construct now the subsets $\mathcal{O}_j^{(3)}$ as follows. Let $\tilde{\mathcal{I}}$ be one of the intervals of Remark 2.2.3. For $L \in \tilde{\mathcal{I}}$, the structure of $V_{eff}(\cdot, L^2)$ and the construction of I_1 have been described after Remark 2.2.3.

To fix ideas, let us concentrate on the situation (1) in which $V_{eff}(\cdot, L^2)$ has only one nondegenerate minimum, $r_0(L)$. Define

$$\mathcal{I}_E := (V_{eff}(r_0(L), L^2), V^\infty)$$

and consider the set

$$\mathbb{R}^4 \supset \mathcal{M} := \left\{ (E, L_1, L_2, L_3) : \sqrt{L_1^2 + L_2^2 + L_3^2} =: L \in \tilde{\mathcal{I}} \ , E \in \mathcal{I}_E \right\} \ ,$$

and denote by F the map

$$F : \mathcal{P}_A^{(3)} \mapsto \mathcal{M} \ .$$

Then, in the previous section we constructed a domain $\mathcal{O}$ for the action angle variables of the planar case by

$$\mathcal{O} := \cup_{p_\theta \in \mathcal{I}} \cup_{E \in \mathcal{I}_E} \{(r, p_r, \theta, p_\theta) : p_\theta \in \tilde{\mathcal{I}} \ , H(r, p_r, \theta, p_\theta) = E\} \ .$$

Correspondingly, one has the set for the spatial case constructed as follows.

$$\mathcal{O}^{(3)} := F^{-1}(\mathcal{M}) = \cup_{L \in \mathcal{I}} \cup_{E \in \mathcal{I}_E} \{(\mathbf{x}, \mathbf{p}) : L^2(\mathbf{x}, \mathbf{p}) = L^2 \ , H(\mathbf{x}, \mathbf{p}) = E\} \ .$$

In situation (2), we are considering an effective potential that admits at least a nondegenerate maximum, thus, in this case different domains can be covered by the action angle variables. However, we can proceed analogously in order to construct the domain $\mathcal{O}^{(3)}$.

The map F restricted to the subset $\mathcal{O}^{(3)}$ is a surjective submersion since the components are a maximal set of independent integrals of motion. Furthermore, in Chapter 1, we have proved that their Poisson matrix satisfies the property of having constant rank equal to 2 at every point of $\mathcal{M}$.

Moreover, the set $\mathcal{M}$ is diffeomorphic to

$$\mathcal{M} \cong S^2 \times \tilde{\mathcal{I}} \times \mathcal{I}_E$$

and the subset $\mathcal{O}^{(3)}$ is a fiber bundle over $\mathcal{M}$ whose fibers are compact and connected, thus, Liouville-Arnol'd theorem assures that the fibers are diffeomorphic to 2-dimensional tori $\mathbb{T}^2$. Furthermore, every fiber of F posses a system of generalized action angle coordinates.

To conclude, we define the map

$$\tilde{F} : \mathcal{M} \mapsto \tilde{\mathcal{I}} \times \mathcal{I}_E := \mathcal{A}_j$$

which is a fibration whose fibers are diffeomorphic to 2-dimensional spheres S^2.

2.4 A Nekhoroshev type theorem

Our aim is to apply the abstract version of the Nekhoroshev type theorem (cf. Theorem 1.3.1, Chapter 1) to the spatial central motion problem. Precisely, we have the following result

Theorem 2.4.1. *Assume that V is neither Harmonic nor Keplerian; then there exists a set $\mathcal{K}^{(3)} \subset \mathcal{P}_A^{(3)}$, which is the union of finitely many analytic hypersurfaces, with the following property: let $P : \mathcal{P}_A^{(3)} \to \mathbb{R}$ be a real analytic function. Let $\mathcal{C}^{(3)} \subset \mathcal{P}_A^{(3)} \setminus \mathcal{K}^{(3)}$ be compact and invariant for the dynamics of H; then there exist positive ε_*, C_1, C_2, C_3, C_4 with the following property: for $|\varepsilon| < \varepsilon_*$, consider the dynamics of the Hamiltonian system*

$$H_\varepsilon := H + \varepsilon P$$

then, for any initial datum in $\mathcal{C}^{(3)}$ one has

$$|L(t) - L(0)| \le C_1 \varepsilon^{1/4} \ , \quad |H(t) - H(0)| \le C_2 \varepsilon^{1/4} \ , \tag{2.30}$$

for

$$|t| \le C_3 \exp(C_4 \varepsilon^{-1/4}) \ . \tag{2.31}$$

Remark 2.4.1. *In the case of the Harmonic and the Keplerian potentials the Hamiltonian depends only on one action, therefore neither the steep Nekhoroshev theorem applies (see e.g. [GCB16]).*

The main point in the proof of this result is the remark that the Hamiltonian of the spatial central motion problem, when written in the action variables, has the same functional form as the planar case due to degeneracy. Thus, we can reduce our analysis to the planar case.

Our main result for the planar case is the following theorem.

Theorem 2.4.2. *Consider the planar central motion problem. Assume (H0)-(H3), then one of the following two alternatives hold:*

(1) For every $j = 1, ..., N$ there exists at most one analytic hypersurface $\mathcal{K}_j \subset \mathcal{A}_j$, s.t. h_j is quasiconvex for all $(I_1, I_2) \in \mathcal{A}_j \setminus \mathcal{K}_j$.

(2) there exists $k > 0$ s.t. $V(r) = kr^2$ or $V(r) = -k/r$.

Corollary 2.4.1. *Assume that V is neither Harmonic nor Keplerian; then there exists a set $\mathcal{K} \subset \mathcal{P}_A$, which is the union of a finite number of analytic hypersurfaces s.t. a system of analytic action angle coordinates exists in an open neighborhood of any point of $\mathcal{P}_A \setminus \mathcal{K}$. Furthermore, the Hamiltonian H written in action angle coordinates is quasiconvex at all points of $\mathcal{P}_A \setminus \mathcal{K}$.*

Proof. (*Theorem 2.4.1)*
We apply the abstract Nekhoroshev's theorem for degenerate systems (Theorem 1.3.1, Chapter 1). First we describe the set $\mathcal{K}^{(3)}$. The first set it contains is the set $\mathcal{S}^{(3)}$. Then consider one of the sets $\mathcal{O}_j^{(3)}$ and the corresponding set $\mathcal{A}_j$. We consider the hypersurface $\mathcal{K}_j \subset \mathcal{A}_j$ on which the Hamiltonian h_j is not quasiconvex; when pulled back to $\mathcal{O}_j^{(3)}$ this is still an analytic hypersurface (it is the zero locus of the Arnol'd determinant, which is an analytic function which is defined on the whole of $\mathcal{O}_j^{(3)}$, since the actions are analytic on the whole of $\mathcal{O}_j^{(3)}$). We define $\mathcal{K}^{(3)}$ to be the union of such analytic hypersurfaces and of $\mathcal{S}^{(3)}$.

Then, it follows that the action angle coordinates exist, are analytic and the Hamiltonian is quasiconvex in $\mathcal{P}_A^{(3)} \setminus \mathcal{K}^{(3)}$. However, the action angle coordinates can have singularities at the boundary of such a set or the Hamiltonian can fail to be quasiconvex at such a boundary. Any compact invariant subset of $\mathcal{P}_A^{(3)} \setminus \mathcal{K}^{(3)}$ is the preimage in the phase space of $\bigcup_j \mathcal{C}_j^{(3)}$, where $\mathcal{C}_j^{(3)} \subset \mathcal{A}_j \setminus \mathcal{K}_j$ is compact. It follows that the maps introducing action angle coordinates extend to bounded analytic maps in a complex neighborhood of $\mathcal{C}_j^{(3)}$ for any j and furthermore the Hamiltonians h_j are quasiconvex on a neighborhood of $\mathcal{C}_j^{(3)}$ with uniform constants. Thus Theorem 2.4.1 follows. □

The proof of Theorem 2.4.2 will cover the next two sections: the strategy consists in studying the asymptotic behavior of the Arnol'd determinant at circular orbits and it goes differently according to the domains $\mathcal{O}_j$. Indeed, we will differentiate the techniques according to the nature of the critical point contained into the domains. Precisely, in the first part of the proof (see Section 2.6), we will concentrate on the domains which are bounded below by a minimum of the effective potential: we will first expand the Hamiltonian at the minimum by computing the Birkhoff normal form and, secondly, we will use this expansion to compute the first terms in the expansion of the Arnol'd determinant. We will show that the Arnol'd determinant is a non trivial function except for the Harmonic and the Keplerian potentials.

In Section 2.8, we will discuss the domains bounded below by a maximum: we will prove that the Arnol'd determinant diverges at the maximum and, thus, it is a non trivial function of the actions.

2.5 The condition of quasiconvexity

First we remark that the notion of quasiconvexity can be expressed in a couple of equivalent forms in the case of a Hamiltonian with two degrees of freedom.

Let us begin with the first one: let us fix one Hamiltonian $h : \mathcal{A} \to \mathbb{R}$ in two degrees of freedom in action variables and denote

$$\omega_1 = \frac{\partial h}{\partial I_1} \ , \quad \omega_2 = \frac{\partial h}{\partial I_2} \ .$$

Let us define by

$$\mathcal{D} = \det \begin{pmatrix} \dfrac{\partial^2 h}{\partial I^2} & \left(\dfrac{\partial h}{\partial I}\right)^T \\ \dfrac{\partial h}{\partial I} & 0 \end{pmatrix} \ , \tag{2.32}$$

the well known Arnol'd determinant. We are now going to show that, for systems with two degrees of freedom, quasiconvexity is equivalent to the nonvanishing of the Arnol'd determinant (cf. Proposition 2.5.1).

Definition 2.5.1. *Let h be a complete integrable Hamiltonian with n degrees of freedom and frequency ω. Then, h is said to satisfy the* Arnol'd condition *at I^* if the following map*

$$(I, \lambda) \to (\lambda\omega(I), h(I))$$

has maximal rank at $(I^*, 1)$.

Explicitly, this condition can be written in the form

$$\mathcal{D}(I^*) = \det \begin{pmatrix} \dfrac{\partial \omega(I^*)}{\partial I} & \left(\dfrac{\partial h(I^*)}{\partial I}\right)^T \\ \dfrac{\partial h(I^*)}{\partial I} & 0 \end{pmatrix} \neq 0 \ .$$

Proposition 2.5.1. *Let $h : \mathcal{A} \to \mathbb{R}$ with $\mathcal{A} \subset \mathbb{R}^2$ be a Hamiltonian with two degrees of freedom in action variables. Then, h is quasiconvex at $I^* \in \mathcal{A}$ if and only if $\mathcal{D}(I^*) \neq 0$.*

Proof. In the two dimensional case, $\mathcal{D} \neq 0$ takes the form

$$\omega_1 \left(\frac{\partial^2 h}{\partial I_1 \partial I_2}\omega_2 - \frac{\partial^2 h}{\partial I_2^2}\omega_1 \right) - \omega_2 \left(\frac{\partial^2 h}{\partial I_1^2}\omega_2 - \frac{\partial^2 h}{\partial I_1 \partial I_2}\omega_1 \right) \neq 0 \ ,$$

namely,

$$\frac{\partial^2 h}{\partial I_1^2}\omega_2^2 - 2\frac{\partial^2 h}{\partial I_1 \partial I_2}\omega_1\omega_2 + \frac{\partial^2 h}{\partial I_2^2}\omega_1^2 \neq 0 \ ,$$

where all the quantities are evaluated at the point I^*.

Moreover, this condition can be explicitly written as

$$\mathcal{Q}(\eta)(I^*) := \langle \eta, \frac{\partial^2 h}{\partial I^2}(I^*)\eta \rangle \neq 0 \ ,$$

where we denoted by $\eta = (\omega_2, -\omega_1)$.

Thus, we conclude that, in the case $n = 2$, the Arnol'd condition is equivalent to the request that the quadratic form $\mathcal{Q}$ is different from zero on the hyperplane generated by the vector η normal to the gradient $\nabla h(I^*)$, namely, quasiconvexity. □

Finally, we show that the condition of quasiconvexity can be written in a second form by means of the Burgers equation. Indeed, let us rewrite explicitly the condition $\mathcal{D} = 0$: we have

$$\mathcal{D} = -\frac{\partial^2 h}{\partial I_1^2}\omega_2^2 + 2\frac{\partial^2 h}{\partial I_1 \partial I_2}\omega_1\omega_2 - \frac{\partial^2 h}{\partial I_2^2}\omega_1^2 \ . \tag{2.33}$$

Thus, rearranging the terms appearing in $\mathcal{D}$, it is straightforward to see that, if ω_2 does not vanish, the condition $\mathcal{D} = 0$ can be written as a Burgers equation, precisely

$$\frac{\partial \nu}{\partial I_1} = \nu \frac{\partial \nu}{\partial I_2} \ , \quad \text{with } \nu = \frac{\omega_1}{\omega_2} \ . \tag{2.34}$$

The two forms in which we have expressed the quasiconvexity condition are absolutely equivalent and our main result can be obtained using both these forms. However, for simplicity, for the study of $\mathcal{D}$ close to a minimum of the effective potential, we will choose the latter one.

2.6 Domains bounded below by a minimum

In this section we concentrate on the domains $\mathcal{O}_j$ s.t. the infimum of the energy H at fixed I_2 is a minimum of the effective potential. Thus the point r_{0j}, of Lemma 2.2.4 is a nondegenerate minimum of the effective potential. In this section, since the domain is fixed we omit the index j from the various quantities. Thus $\mathcal{A}$ will be the domain of the actions, h the Hamiltonian written in action variables, r_0 the minimum of the effective potential and V_0 the corresponding value.

The main result of this section is the following Lemma.

Lemma 2.6.1. *Let $\mathcal{O}_j$ be a domain s.t. the infimum of the effective Hamiltonian at fixed I_2 is a nondegenerate minimum of the effective potential. Assume that the Arnol'd determinant vanishes in an open subset of $\mathcal{O}_j$, then the potential is either Keplerian or Harmonic.*

The rest of the section is devoted to the proof of such a lemma.

We exploit the remark that in one dimensional analytic systems Birkhoff normal form converges in a (complex) neighborhood of a nondegenerate minimum. This, together with the uniqueness of the action variables in one dimensional systems, implies that, for any I_2, the Hamiltonian h, as a function of I_1, extends to a complex analytic function in a neighborhood of $I_1 = 0$ and that the expansion constructed through the one dimensional Birkhoff normal form is actually the expansion of $h(I_1, I_2)$ at $I_1 = 0$. It follows that also $\mathcal{D}$ extends to a complex analytic function of I_1 in a neighborhood of 0. Thus one has an expansion

$$h(I_1, I_2) = h_0(I_2) + h_1(I_2)I_1 + \ldots + h_r(I_2)I_1^r + \ldots \ , \tag{2.35}$$

where the quantities h_r can be in principle computed as functions of the derivatives of V at $r_0(I_2)$ and of I_2.

Here we will proceed by an explicit construction using a symbolic manipulator.

Remark 2.6.1. *In $\mathcal{O}_j$ there is a 1-1 correspondence between I_2 and r_0, so each of the functions h_r can be considered just a function of r_0 and of the derivatives of V at r_0. Correspondingly the derivatives with respect to I_2 can be converted into derivatives with respect to r_0 through the rule*

$$\frac{\partial}{\partial I_2} = \frac{2}{(3 + g(r_0))\sqrt{r_0 V'(r_0)}} \frac{\partial}{\partial r_0} \tag{2.36}$$

where we have defined

$$g(r_0) := \frac{r_0 V''(r_0)}{V'(r_0)} \ . \tag{2.37}$$

Furthermore, it is convenient to define

$$R(r_0, V'(r_0), g(r_0)) := \frac{2}{(3 + g(r_0))\sqrt{r_0 V'(r_0)}} \ . \tag{2.38}$$

Remark 2.6.2. *g constant is equivalent to the fact that the potential is homogeneous or logarithmic , precisely, one has*

$$g(r_0) = c \quad \Longleftrightarrow \begin{cases} V(r) = \dfrac{k}{c+1} r^{c+1} \ , \quad k \in \mathbb{R} \ , \quad \mathit{for}\, c \geq 3 \ , c \neq -1 \\ V(r) = k \ln(r) \ , \quad k \in \mathbb{R} \ . \end{cases} \tag{2.39}$$

Thus, starting from the Birkhoff normal form, we compute the frequencies ω_1 and ω_2, expanded in power series of I_1, namely,

$$\begin{aligned} \omega_1(I_2) &= \omega_{1,0}(I_2) + \omega_{1,1}(I_2)I_1 + \omega_{1,2}(I_2)I_1^2 + \ldots \ , \\ \omega_2(I_2) &= \omega_{2,0}(I_2) + \omega_{2,1}(I_2)I_1 + \omega_{2,2}(I_2)I_1^2 + \ldots \ , \end{aligned}$$

and we use it to compute an expansion of $\nu \equiv \omega_1/\omega_2$ at the minimum

$$\nu(I_1, I_2) = \nu_0(I_2) + \nu_1(I_2)I_1 + \ldots + \nu_r(I_2)I_1^r + \ldots \ . \tag{2.40}$$

Indeed, we have

$$\begin{aligned}\nu = \frac{\omega_1}{\omega_2} &= \frac{\omega_{1,0} + \omega_{1,1} I_1 + \dots}{\omega_{2,0}\left(1 + \frac{\omega_{2,1}}{\omega_{2,0}} I_1 + \dots\right)} = \frac{1}{\omega_{2,0}} \left(\omega_{1,0} + \omega_{1,1} I_1 + \dots\right) \left(1 + \frac{1}{2}\frac{\omega_{2,1}}{\omega_{2,0}} I_1 + \dots\right) \\ &= \frac{\omega_{1,0}}{\omega_{2,0}} + \left(\frac{\omega_{1,1}}{\omega_{2,0}} + \frac{1}{2}\frac{\omega_{1,0}\omega_{2,1}}{\omega_{2,0}^2}\right) I_1 + \dots \\ &:= \nu_0 + \nu_1 I_1 + \dots \ .\end{aligned}$$

The idea is now to impose that the Burgers equation (2.34) is satisfied up to the first order in I_1 identically as function of I_2. Thus, let us consider the Burgers equation (2.34) and let us pull into it the expansion (2.40). We obtain

$$\frac{\partial}{\partial I_1}(\nu_0(I_2)+\nu_1(I_2)I_1+\nu_2(I_2)I_1^2+\dots) = (\nu_0(I_2)+\nu_1(I_2)I_1+\dots)\frac{\partial}{\partial I_2}(\nu_0(I_2)+\nu_1(I_2)I_1+\dots) \, ,$$

that is

$$\nu_1(I_2) + 2\nu_2(I_2)I_1 + \dots = \nu_0(I_2)\frac{\partial \nu_0(I_2)}{\partial I_2} + \left(\nu_0(I_2)\frac{\partial \nu_1(I_2)}{\partial I_2} + \nu_1(I_2)\frac{\partial \nu_0(I_2)}{\partial I_2}\right) I_1 + \dots \ .$$

At this point, we impose that the equation is satisfied up to the first order in I_1, that is, we impose

$$\begin{aligned}\nu_1 &= \nu_0 \frac{\partial \nu_0}{\partial I_2} \ , \\ \nu_2 &= \frac{1}{2}\left(\nu_0 \frac{\partial \nu_1}{\partial I_2} + \nu_1 \frac{\partial \nu_0}{\partial I_2}\right) \ ,\end{aligned} \tag{2.41}$$

and we consider such equations as equations that determine the degenerate potentials. We will show that such equations admit the only common solutions given by the Harmonic and the Keplerian potentials.

According to Remark 2.6.1, we will consider all the functions ν_j as functions of r_0 instead of I_2 and convert all the derivatives with respect to I_2 into derivatives with respect to r_0 using (2.36).

Finally, it is convenient to use, as much as possible, g as an independent variable (see eq. (2.37)) instead of V. We remark that $V''(r_0) = \frac{g(r_0)V'(r_0)}{r_0}$, which implies that $\forall r \geq 2$ the r-th derivative of the potential can be expressed as a function of $r_0, V'(r_0), g(r_0), g'(r_0), \dots, g^{(r-2)}(r_0)$. We will systematically do this.

There is a remarkable fact: writing explicitly the equations (2.41), it turns out that they are independent of V', so that they are only differential equations for g. For a proof of this fact see Appendix D.

We report below the outline of the computations and the key formulæ. The complete calculations have been implemented in Mathematica™and are collected in Appendix E.

First we computed explicitly ν_0, ν_1, ν_2, (defined by (2.40)) getting formulæ of the form

$$\nu_0 = \sqrt{3 + g(r_0)}\ , \tag{2.42}$$

and

$$\nu_1 = \nu_1(r_0, V'(r_0), g(r_0), g'(r_0), g''(r_0))\ ,$$
$$\nu_2 = \nu_2(r_0, V'(r_0), g(r_0), g'(r_0), g''(r_0), g^{(3)}(r_0), g^{(4)}(r_0))\ .$$

The explicit forms of ν_1 and ν_2 are rather long and are reported in Appendix E.

Then one can use the explicit forms of the functions ν_0 and ν_1 to compute the r.h.s. of eq. (2.41), which will have the form

$$R\nu_0 \frac{\partial \nu_0}{\partial r_0} =: G_1(r_0, V'(r_0), g(r_0), g'(r_0))\ ,$$
$$\frac{1}{2} R \left(\nu_1 \frac{\partial \nu_0}{\partial r_0} + \nu_0 \frac{\partial \nu_1}{\partial r_0} \right) =: G_2(r_0, V'(r_0), g(r_0), g'(r_0), g''(r_0), g^{(3)}(r_0))\ ,$$

where R is the expression defined in eq. (2.38).

Then we have imposed $\nu_1 = G_1$ and $\nu_2 = G_2$, which are the couple of differential equations for g that we solved.

The strategy in order to find the common solutions is standard: it consists in taking derivatives of the equation of lower order until one gets two equations of the same order (fourth order in g, in our case), then one solves one of the equations for the higher order derivative and substitutes it in the other one, thus getting an equation of order smaller then the previous one. Then one iterates. In our case the final equation will be an algebraic equation for g, whose solutions are just constants. The value of such constants correspond to the Kepler and the Harmonic potentials, so the conclusion will hold.

So, we solve $\nu_1 = G_1$ for $g''(r_0)$ and $\nu_2 = G_2$ for $g^{(4)}(r_0)$, getting

$$\begin{aligned} g''(r_0) &= f_2(r_0, g(r_0), g'(r_0))\ , \\ g^{(4)}(r_0) &= f_4(r_0, g(r_0), g'(r_0), g^{(3)}(r_0))\ , \end{aligned} \tag{2.43}$$

where we have used the fact that the powers of $V'(r_0)$ can be factor out and, in the second one, we have also used $f_2(r_0, g(r_0), g'(r_0))$ to remove the dependence of $g^{(4)}$ on $g''(r_0)$. A similar procedure will be done systematically.

Starting from (2.43), we compute

$$\frac{\mathrm{d}^2 f_2}{\mathrm{d}r_0^2} = F_4(r_0, g(r_0), g'(r_0), g^{(3)}(r_0))\ ,$$

and solve the equation $F_4(r_0, g(r_0), g'(r_0), g^{(3)}(r_0)) = f_4(r_0, g(r_0), g'(r_0), g^{(3)}(r_0))$ for $g^{(3)}$, getting

$$g^{(3)} = f_3(r_0, g(r_0), g'(r_0))\ .$$

Starting again from (2.43), we compute

$$\frac{\mathrm{d}f_2}{\mathrm{d}r_0} = F_3(r_0, g(r_0), g'(r_0)) \ ,$$

and solve the equation $F_3(r_0, g(r_0), g'(r_0)) = f_3(r_0, g(r_0), g'(r_0))$ for g' getting

$$g' = f_1(r_0, g(r_0)) \ .$$

Finally we compute

$$\frac{\mathrm{d}f_1}{\mathrm{d}r_0} = F_2(r_0, g(r_0)) \ ,$$

and solve $F_2(r_0, g(r_0)) = f_2(r_0, g(r_0))$ for g. It is remarkable that such an equation turns out to be independent of r_0, so that the solutions for $g(r_0)$ are just isolated points, namely constants. In particular, it turns out that the only real constants solutions are $g = -3, g = -2$ and $g = 1$. The value -3 is excluded according to Remark 2.2.1, so that the only remaining potentials are the Keplerian and the Harmonic ones. This concludes the proof of Lemma 2.6.1.

2.7 A new proof of Bertrand's Theorem

Bertrand's Theorem. *Among all the central force potentials giving rise to bounded orbits, the are only two types for which all bounded orbits are closed: the Keplerian potential and the Harmonic potential.*

Proof. By the previous section, the ratio $\nu(I_1, I_2) \equiv \frac{\omega_1}{\omega_2}$ is a trivial function of the actions only in the Harmonic and the Keplerian case. Thus, in all the other cases, there exist I_1, I_2 such that ν is irrational and thus on the corresponding torus the motion is not periodic, against the assumption. □

2.8 Domains bounded below by a maximum

Consider now domains the $\mathcal{O}_j$ s.t. the infimum of the energy H at a fixed $I_2 \in \tilde{\mathcal{I}}$ is a nondegenerate maximum of the effective potential V_{eff}. Denote by $V_0 = V_0(I_2)$ the value of the effective potential at such a maximum delimiting from below the range of the energy in $\mathcal{O}_j$. The main result of this section is the following

Lemma 2.8.1. *Let $\mathcal{O}_j$ be a domain s.t. the infimum of the effective Hamiltonian at fixed I_2 is a nondegenerate maximum of the effective potential, then the Arnol'd determinant vanishes in $\mathcal{O}_j$ at most on an analytic hypersurface.*

The rest of the section is devoted to the proof of such a lemma. The main tool for studying the limiting behavior of the action close to the maximum V_0 is the following normal form theorem, which is a slight reformulation of a simplified version of the main result of [Gio01].

Theorem 2.8.1. *Let*

$$W(r) = W_0 - \frac{\lambda^2}{2} r^2 + \mathcal{O}(r^3)$$

be an analytic potential having a nondegenerate maximum at $r = 0$; consider the Hamiltonian system with Hamiltonian

$$H(r, p_r) = \frac{p_r^2}{2} + W(r)$$

then, there exists an open neighborhood $\mathcal{V}_0$ of 0 and a near to identity canonical transformation $\Phi : \mathcal{V}_0 \ni (x, y) \mapsto (r, p_r) \in \mathcal{U}_0 := \Phi(\mathcal{V}_0)$ of the form

$$\begin{cases} r = \dfrac{x}{\sqrt{\lambda}} + f_1(x, y) \\ p_r = \sqrt{\lambda} y + f_2(x, y) \end{cases} \tag{2.44}$$

with f_1, f_2 analytic functions which are at least quadratic in x, y and such that in the variables x, y, the Hamiltonian takes the form

$$h(x, y) = W_0 + \lambda J + \sum_{i \geq 1} \lambda_i J^{i+1} \tag{2.45}$$

where

$$J := \frac{y^2 - x^2}{2} \ . \tag{2.46}$$

Furthermore, the series is convergent in $\mathcal{V}_0$.

The behavior of the action variable close to the maximum of the effective potential is described by the following theorem.

Theorem 2.8.2. *There exist analytic functions $\Lambda(\bar{E}, I_2)$, $G_1\left(\bar{E}, I_2\right)$, where*

$$\bar{E} = E - V_0(I_2) \ ,$$

analytic and bounded in the domain

$$\left\{ (\bar{E}, I_2) \ : \ I_2 \in \tilde{\mathcal{I}} \ , \ \bar{E} \in [0, V_M(I_2) - V_0(I_2)) \right\} \ , \tag{2.47}$$

where $V_M(I_2)$ is the maximal value of the energy at fixed I_2 in $\mathcal{O}_j$ and s.t. the first action I_1 is given by

$$I_1 = G(\bar{E}, I_2) := -\Lambda(\bar{E}, I_2) \ln \bar{E} + G_1\left(\bar{E}, I_2\right) \tag{2.48}$$

Furthermore

$$\Lambda(\bar{E}, I_2) := \frac{\bar{E} + \mathcal{F}(\bar{E}, I_2)}{\pi\lambda(I_2)} \tag{2.49}$$

with $\mathcal{F}$ having a zero of order 2 in $(0, I_2)$ and $\lambda^2 = \lambda^2(I_2) := -\frac{d^2 V_{eff}}{dr^2}(r_0) > 0$.

Remark 2.8.1. *The main point is that the lower bound of the interval (2.47) for $\bar{E}$ is included, so that equation (2.48) describes the actions until the maximum. Furthermore, by (2.48) the following limit exists*

$$I_{1\,0} := \lim_{\bar{E} \to 0^+} G(\bar{E}, I_2) = G_1(0, I_2)$$

and is finite.

Remark 2.8.2. *Since $E \mapsto G(E - V_0(I_2), I_2)$ is a monotonically increasing function for $E \in (V_0(I_2), V_M(I_2))$, there exists a function $h(I_1, I_2)$ such that*

$$G(h(I_1, I_2) - V_0(I_2), I_2) \equiv I_1.$$

Furthermore, by the implicit function theorem, h is analytic in I_1, I_2 for $I_2 \in \tilde{\mathcal{I}}$ and $I_1 > I_{1\,0}$.

Proof of Theorem 2.8.2. Let $I_2 \in \tilde{I}$ and consider the Hamiltonian

$$H(r, p_r, p_\theta) = \frac{p_r^2}{2} + V_{eff}(r, p_\theta^2) \ ,$$

with $p_\theta = I_2$. In the whole construction I_2 will play the role of a parameter, so, until the end of the proof, we work in the space (r, p_r) and we omit the dependence on I_2.

We first make an expansion at r_0 and obtain

$$H(r, p_r) = \frac{p_r^2}{2} + V_0 - \frac{\lambda^2}{2}(r - r_0)^2 + \mathcal{O}\left((r - r_0)^3\right) \ .$$

Secondly, we make a change of variable to $r' := r - r_0$; omitting the primes, we obtain

$$H(r, p_r) = \frac{p_r^2}{2} + V_0 - \frac{\lambda^2}{2} r^2 + \mathcal{O}(r^3) \ . \tag{2.50}$$

Fix a value of the energy $E > V_0$, close enough to V_0 and denote by $\gamma(E)$ the level curve of H at level E. Then, $\gamma(E)$ is a closed curve in the phase-space whose normalized enclosed area is the action I_1 that we want to compute.

Thus, by definition, we have

$$I_1 = \frac{1}{2\pi} \int_{\gamma(E)} p_r dr = \frac{1}{\pi} \int_{\gamma^+(E)} p_r dr \ , \tag{2.51}$$

where $\gamma^+(E)$ is the upper part of the level curve, namely, the intersection of γ with $p_r > 0$.

We split the domain of integration into two regions, namely,

$$I_1 = \frac{1}{\pi}\left[\int_{\gamma^+(E)\cap\mathcal{U}} p_r dr + \int_{\gamma^+(E)\cap\mathcal{U}^c} p_r dr\right] . \tag{2.52}$$

where $\mathcal{U}$ is a neighborhood of the nondegenerate maximum that will be fixed in a while. First, we remark that the second integral does not see the critical point, so it is an analytic function of E until V_0. To analyze the first integral, we exploit Theorem 2.8.1.

Let us fix a small positive x_1 and let us consider the neighborhood $\mathcal{V}$ of 0

$$\mathbb{R}^2 \supset \mathcal{V} := (-x_1, x_1) \times \left(-\sqrt{\frac{4\bar{E}}{\lambda} + x_1^2}, \sqrt{\frac{4\bar{E}}{\lambda} + x_1^2}\right) .$$

Provided $\bar{E}$ and x_1 are small enough, one has $\mathcal{V} \subset \mathcal{V}_0$ (c.f. Theorem 2.8.1). Let us define $\mathcal{U} := \Phi(\mathcal{V})$.

We now write $\gamma_+(E) \cap \mathcal{U}$ in the variables (x, y) and parametrize it with $x \in (-x_1, x_1)$. To this end remark that, since the Hamiltonian H is a function of J only, namely

$$H = V_0 + \lambda J + \mathcal{G}(J) ,$$

where $\mathcal{G}(J) = \sum_{i\geq 1} \lambda_i J^{i+1}$, by the implicit function theorem, there exists an analytic function $\mathcal{F}(\bar{E})$ having a zero of order 2 at 0 and such that

$$J = \frac{\bar{E} + \mathcal{F}(\bar{E})}{\lambda}$$

and, therefore, $\gamma_+(E)$ can be written in the form $(x, y(x))$ with

$$y(x) := \sqrt{\frac{2(\bar{E} + \mathcal{F}(\bar{E}))}{\lambda} + x^2} . \tag{2.53}$$

To compute the first integral in (2.52), we remark that since Φ is canonical and analytic in a neighborhood of the origin, there exists a function $S(x, y)$ analytic in a neighborhood of the origin s.t.

$$p_r dr = y dx + dS ,$$

so, we have,

$$\int_{\gamma_+(E)\cap\mathcal{U}} p_r dr = \int_{-x_1}^{x_1} y dx + S(x_1, y(x_1)) - S(-x_1, y(-x_1)) .$$

Since x_1 is fixed, the terms involving S are analytic functions of $\bar{E}$. Thus, we only compute the first integral, namely,

$$\int_{-x_1}^{x_1} y dx = \int_{-x_1}^{x_1} \sqrt{\frac{2(\bar{E}+\mathcal{F}(\bar{E},I_2))}{\lambda} + x^2}\, dx$$

which takes the form

$$\begin{aligned}\int_{-x_1}^{x_1} y dx = &\frac{2(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda} \ln\left(x_1 + \sqrt{x_1^2 + \frac{2(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}}\right) + x_1\sqrt{x_1^2 + \frac{2(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}} \\ &- \frac{\bar{E}+\mathcal{F}(\bar{E})}{\lambda} \ln\left(\frac{2(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}\right) .\end{aligned} \tag{2.54}$$

It is easy to see that the first two terms are analytic in a neighborhood of 0. Rewriting the third term as

$$-\frac{\bar{E}+\mathcal{F}(\bar{E})}{\lambda} \ln \bar{E} - \ln\left(\frac{2}{\lambda} + \frac{2\mathcal{F}(\bar{E})}{\bar{E}}\right)$$

and remarking that the second function is analytic in a neighborhood of 0, we get the result. All the computations needed are collected in Appendix D.

The formula (2.48) is obtained by reinserting the dependence on I_2. □

We come to the Arnol'd determinant. We will work in the region $\bar{E} > 0$ so that the function G is regular and the implicit function theorem applies and allows to compute h and its derivatives. Then, we will study the limit $\bar{E} \to 0^+$.

By the implicit function theorem, the frequency ω_1 is given by

$$\omega_1 = \frac{\partial h}{\partial I_1} = \left(\frac{\partial G}{\partial \bar{E}}\right)^{-1} =: \mathcal{W}_1(\bar{E}, I_2) \ . \tag{2.55}$$

Lemma 2.8.2. *Let f be an analytic function of the form*

$$f = f(\bar{E}, I_2) = f(h(I_1,I_2) - V_0(I_2), I_2) \ ,$$

then,

$$\frac{\mathrm{d}f}{\mathrm{d}I_2} := \frac{\partial f}{\partial \bar{E}}\frac{\partial \bar{E}}{\partial I_2} + \frac{\partial f}{\partial I_2} = \frac{\partial f}{\partial \bar{E}}\left(\omega_2 - \frac{\partial V_0}{\partial I_2}\right) + \frac{\partial f}{\partial I_2}.$$

Proof. It follows directly from the definition of the function $\bar{E}$. □

It follows from Lemma 2.8.2 and the implicit function theorem that the frequency ω_2 is given by

$$\omega_2 = \frac{\partial h}{\partial I_2} = -\frac{\partial G}{\partial I_2}\mathcal{W}_1 + \frac{\partial V_0}{\partial I_2}.$$

Indeed, we have

$$0 = \frac{\mathrm{d}G}{\mathrm{d}I_2} = \frac{\partial G}{\partial \bar{E}}\left(\omega_2 - \frac{\partial V_0}{\partial I_2}\right) + \frac{\partial G}{\partial I_2} \ ,$$

that is,

$$\omega_2 = -\frac{\partial G_1}{\partial I_2}\left(\frac{\partial G}{\partial \bar{E}}\right)^{-1} + \frac{\partial V_0}{\partial I_2} \ .$$

Thus, it is worth introducing a function $\mathcal{W}_2$ defined by

$$\mathcal{W}_2(\bar{E}, I_2) := -\frac{\partial G}{\partial I_2}\mathcal{W}_1 + \frac{\partial V_0}{\partial I_2} \ . \tag{2.56}$$

Proposition 2.8.1. *Let $h : \mathcal{A} \to \mathbb{R}$ be the Hamiltonian in two degrees of freedom written in action angle coordinates, then the Arnol'd determinant can be rewritten in terms of G and $\mathcal{W}_1$ as*

$$\mathcal{D} = -\mathcal{W}_1\frac{\partial \mathcal{W}_1}{\partial \bar{E}}\left(\frac{\partial V_0}{\partial I_2}\right)^2 + 2\mathcal{W}_1\frac{\partial \mathcal{W}_1}{\partial I_2}\frac{\partial V_0}{\partial I_2} + \mathcal{W}_1^3\frac{\partial^2 G}{\partial I_2^2} - \mathcal{W}_1^2\frac{\partial^2 V_0}{\partial I_2^2} \ . \tag{2.57}$$

Proof. By exploiting the formulæ (2.55), (2.56) and Remark 2.8.2, we compute the second derivatives of the Hamiltonian h. We have

$$\begin{aligned}
\frac{\partial^2 h}{\partial I_1^2} &= \frac{\partial \mathcal{W}_1(\bar{E}, I_2)}{\partial I_1} = \frac{\partial \mathcal{W}_1}{\partial \bar{E}}\frac{\partial h}{\partial I_1} = \mathcal{W}_1\frac{\partial \mathcal{W}_1}{\partial \bar{E}} \ , \\
\frac{\partial^2 h}{\partial I_1 \partial I_2} &= \frac{\mathrm{d}\mathcal{W}_1(\bar{E}, I_2)}{\mathrm{d}I_2} \ , \\
\frac{\partial^2 h}{\partial I_2^2} &= \frac{\mathrm{d}\mathcal{W}_2(\bar{E}, I_2)}{\mathrm{d}I_2} = \frac{\mathrm{d}}{\mathrm{d}I_2}\left(-\mathcal{W}_1\frac{\partial G}{\partial I_2} + \frac{\partial V_0}{\partial I_2}\right) \\
&= -\frac{\mathrm{d}\mathcal{W}_1}{\mathrm{d}I_2}\frac{\partial G}{\partial I_2} - \mathcal{W}_1\frac{\mathrm{d}}{\mathrm{d}I_2}\left(\frac{\partial G}{\partial I_2}\right) + \frac{\partial^2 V_0}{\partial I_2^2} \ .
\end{aligned}$$

We report here the expression of the Arnol'd determinant, that is,

$$\mathcal{D} = -\frac{\partial^2 h}{\partial I_1^2}\omega_2^2 + 2\frac{\partial^2 h}{\partial I_1 \partial I_2}\omega_1\omega_2 - \frac{\partial^2 h}{\partial I_2^2}\omega_1^2 \ . \tag{2.58}$$

We can rewrite the three terms of the Arnol'd determinant (2.58) separately as

$$\mathcal{D}_1 = -\mathcal{W}_1\mathcal{W}_2^2\frac{\partial \mathcal{W}_1}{\partial \bar{E}} \ , \tag{2.59}$$

$$\mathcal{D}_2 = 2\mathcal{W}_1\mathcal{W}_2\frac{\mathrm{d}\mathcal{W}_1}{\mathrm{d}I_2} \ , \tag{2.60}$$

$$\mathcal{D}_3 = \mathcal{W}_1^2 \frac{\partial G}{\partial I_2} \frac{\mathrm{d}\mathcal{W}_1}{\mathrm{d}I_2} + \mathcal{W}_1^3 \frac{\mathrm{d}}{\mathrm{d}I_2}\left(\frac{\partial G}{\partial I_2}\right) - \mathcal{W}_1^2 \frac{\partial^2 V_0}{\partial I_2^2} \ . \tag{2.61}$$

And, gathering together the expressions (2.59), (2.60) and (2.61), after simple computation (for details see Appendix D), we obtain

$$\mathcal{D} = -\mathcal{W}_1 \frac{\partial \mathcal{W}_1}{\partial \bar{E}} \left(\frac{\partial V_0}{\partial I_2}\right)^2 + 2\mathcal{W}_1 \frac{\partial \mathcal{W}_1}{\partial I_2} \frac{\partial V_0}{\partial I_2} + \mathcal{W}_1^3 \frac{\partial^2 G}{\partial I_2^2} - \mathcal{W}_1^2 \frac{\partial^2 V_0}{\partial I_2^2}$$

This concludes the proof. □

Proposition 2.8.2. *The Arnol'd determinant diverges as $\bar{E}$ tends to zero.*

Proof. Due to the structure (2.48) of G, it is easy to see that $\frac{\partial^2 G}{\partial I_2^2}$ is bounded as $\bar{E} \to 0^+$ (remark that $\frac{\partial G}{\partial I_2}$ means derivative with respect to the second argument).

Therefore, since $\frac{\partial^2 V_0}{\partial I_2^2}$ is a regular function of I_2 and $\mathcal{W}_1 \to 0$ as $\bar{E}$ approaches zero, we have that

$$\lim_{\bar{E}\to 0^+} \left(\mathcal{W}_1^3 \frac{\partial^2 G}{\partial I_2^2} - \mathcal{W}_1^2 \frac{\partial^2 V_0}{\partial I_2^2}\right) = 0.$$

Let us now concentrate on the analysis of the remaining terms of (2.57). The asymptotic behavior of the function $\mathcal{W}_1$ is given by

$$\mathcal{W}_1 \sim -\frac{\pi\lambda}{\ln \bar{E}}.$$

Concerning the derivatives, we have

$$\frac{\partial \mathcal{W}_1}{\partial I_2} = -\left(\frac{\partial G}{\partial \bar{E}}\right)^{-2} \frac{\partial^2 G}{\partial I_2 \partial \bar{E}} \overset{\bar{E}\to 0^+}{\to} 0 \quad \Longrightarrow \quad \lim_{\bar{E}\to 0^+} \left(2\mathcal{W}_1 \frac{\partial \mathcal{W}_1}{\partial I_2} \frac{\partial V_0}{\partial I_2}\right) = 0 \ .$$

Concerning the first term, using

$$\frac{\partial \mathcal{W}_1}{\partial \bar{E}} \sim \frac{\pi\lambda}{\bar{E}\ln^2 \bar{E}} \ ,$$

we have that it behaves as

$$\frac{\pi^2\lambda^2}{\bar{E}\ln^3 \bar{E}} \left(\frac{\partial V_0}{\partial I_2}\right)^2$$

which diverges to infinity as $\bar{E} \to 0^+$. This concludes the proof. □

Lemma 2.8.1 is a consequence of the fact that $\mathcal{D}$ is a nontrivial analytic function in $\mathcal{A}_j$.

Appendix A

The Bertrand's Theorem

An interesting result concerning central force potentials is the following theorem due to Bertrand

Bertrand's Theorem. *Among all the central force potentials giving rise to bounded orbits, the are only two types for which all bounded orbits are closed: the Keplerian potential and the Harmonic potential.*

A.1 Classical proof of Bertrand's Theorem

We will present here a revisited version of the original proof given by Bertrand in [Ber73]. Precisely, we will follow the proof given by Arnol'd in [Arn91].

Consider the effective potential V_{eff}, fix a value L of the angular momentum and assume that it has a strict minimum at r_0. We impose this condition in order to consider only the family of central potentials which give rise to bounded orbits. Denote

$$\omega_1(r_0) := \frac{\partial^2 V_{eff}}{\partial r^2}(r_0; L^2(r_0)) \ ,$$

and $\omega_2(r_0)$ the frequency associated to the radial motion. Then, we have the following

Lemma A.1.1. *The only central potentials for which $\nu_0(r_0) := \frac{\omega_1}{\omega_2}$ is independent of r_0 are*

$$\begin{cases} V(r) = \dfrac{k}{\alpha} r^\alpha \quad \text{for} \quad \alpha > -2 \ , \alpha \neq 0 \ , \\ V(r) = k \ln r \ , \end{cases} \tag{A.1}$$

where k is a positive constant.

Proof. Just remark that the ratio ν_0 has been computed in Chapter 2 (cf. eq.(2.42)) and is given by

$$\nu_0(r_0) = \sqrt{3 + g(r_0)} \ ,$$

so that this is independent of r_0 only if g is equal to a constant and, thus, according to the definition of g

$$g(r_0) = \frac{r_0 V''(r_0)}{V'(r_0)} \ ,$$

we have that (A.1) holds. □

Lemma A.1.2. *Let $V(r)$ be a central potential of the form* (A.1). *Then,*

$$\lim_{E \mapsto +\infty} \frac{1}{2\pi} \int_{r_m}^{r_M} \frac{L}{r^2 \sqrt{2(E - V_{eff}(r))}} dr = \frac{1}{4} \ , \tag{A.2}$$

for $\alpha > 0$ and for the logarithmic potential and

$$\lim_{E \mapsto +\infty} \frac{1}{2\pi} \int_{r_m}^{r_M} \frac{L}{r^2 \sqrt{2(E - V_{eff}(r))}} dr = \frac{1}{2(2+\alpha)} \ , \tag{A.3}$$

for $-2 < \alpha < 0$.

Proof. Consider a potential of the form (A.1). Let us consider the integral

$$\frac{1}{2\pi} \int_{r_m}^{r_M} \frac{L}{r^2 \sqrt{2(E - V_{eff}(r))}} dr \ , \tag{A.4}$$

where r_m and r_M are the two solutions of $E = V_{eff}(r)$.

Making the change of variable

$$s = \frac{r_m}{r} \ ,$$

the integral is reduced to

$$\frac{1}{2\pi} \int_{\frac{r_m}{r_M}}^{1} \frac{L}{\sqrt{2\left(r_m^2 E - \frac{L^2 s^2}{2} - r_m^2 V\left(\frac{r_m}{s}\right)\right)}} ds \ . \tag{A.5}$$

We distinguish now the two cases in (A.1). In the case of the logarithmic potential (the associated effective potential is depicted in Figure A.1 for different values of the positive constant k), the integral becomes

$$\frac{1}{2\pi} \int_{\frac{r_m}{r_M}}^{1} \frac{L}{\sqrt{2\left(r_m^2 E - \frac{L^2 s^2}{2} - r_m^2 k \ln\left(\frac{r_m}{s}\right)\right)}} ds \ , \tag{A.6}$$

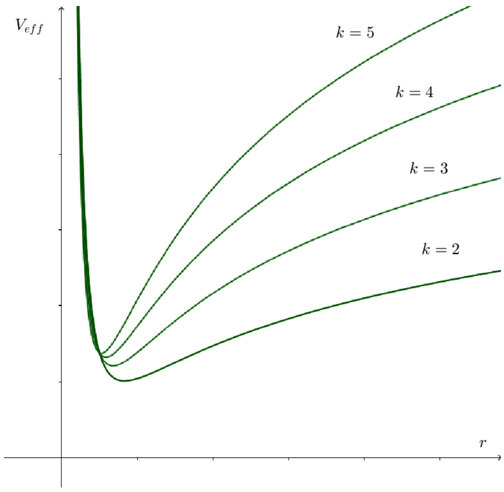


Figure A.1: The effective potential for the logarithmic potential $V(r) = k\ln(r)$ for different values of the positive constant k .

and the maximal value of the energy E_M for which we have bounded orbits can be taken as $E_M = +\infty$.
Let us now take the limit of the integral (A.6) for $E \mapsto E_M$. In this case, the extrema r_m and r_M tends to 0^+ and $+\infty$ respectively and the integral tends to

$$\frac{1}{2\pi}\int_0^1 \frac{L}{\sqrt{2\left(\frac{L^2}{2} - \frac{L^2 s^2}{2}\right)}}ds = \frac{1}{2\pi}\int_0^1 \frac{1}{\sqrt{1-s^2}}ds = \frac{1}{2\pi}\frac{\pi}{2} = \frac{1}{4} .$$

Let us now pass to consider the case of the power law potential. Following the reasoning above, the integral in this case becomes

$$\frac{1}{2\pi}\int_{\frac{r_m}{r_M}}^1 \frac{L}{\sqrt{2\left(r_m^2 E - \frac{L^2 s^2}{2} - \frac{k r_m^{\alpha+2}}{\alpha s^{\alpha}}\right)}}ds . \tag{A.7}$$

For $\alpha > 0$, the corresponding effective potential is depicted in Figure A.2.
We can notice that also in this case, the maximal energy E_M for which we have bounded orbits is $E_M = +\infty$. Thus, we can take the limit of the integral above for $E \mapsto +\infty$. Also in this limit, the extrema r_m and r_M tends to 0^+ and $+\infty$ respectively. Thus, the computation is the same as in the previous case and the integrals tends to $\frac{1}{4}$.

We analyze the last case. For $-2 < \alpha < 0$, as we can see from Figure A.3, the maximal energy in this case is equal to $E_M = 0^-$ while the extrema $r_m^{\alpha+2} = -\frac{2k}{\alpha L^2}$ and $r_M \mapsto +\infty$.

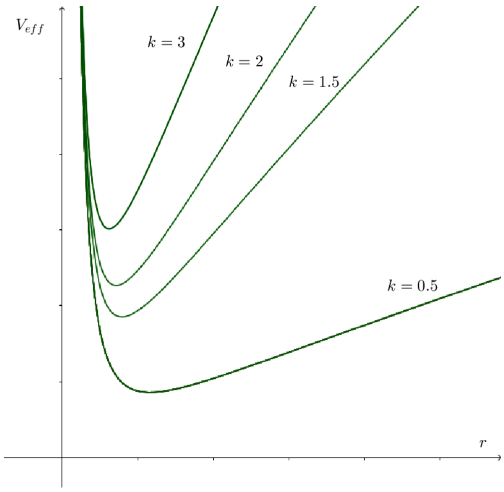


Figure A.2: The effective potential for the power law potential $V(r) = \frac{k}{\alpha} r^{\alpha}$, $\alpha > 0$ for different values of the positive constant k .

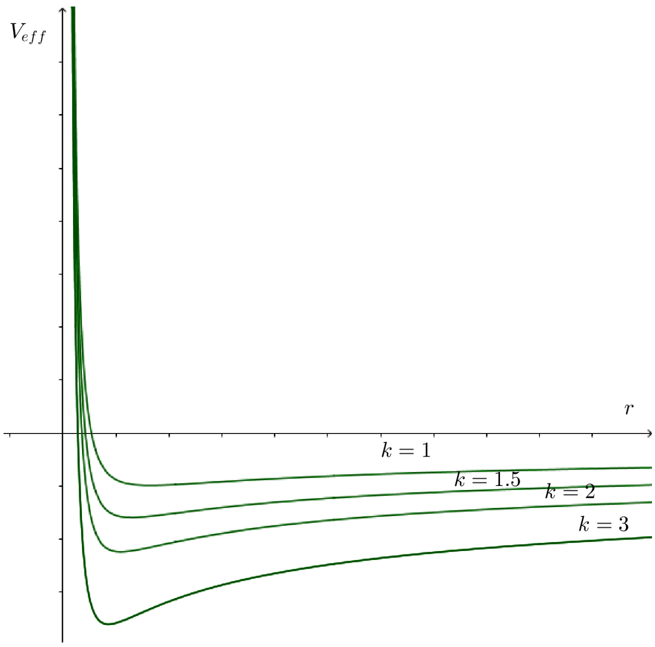


Figure A.3: The effective potential for the power law potential $V(r) = \frac{k}{\alpha} r^{\alpha}$, $-2 < \alpha < 0$ for different values of the positive constant k .

Substituting this into the formula (A.7) and taking the limit as $E \mapsto 0^-$ we obtain

$$\frac{1}{2\pi} \int_0^1 \frac{L}{\sqrt{2\left(-\frac{L^2 s^2}{2} + \frac{L^2}{2s^{\alpha}}\right)}} ds = \frac{1}{2\pi} \int_0^1 \frac{1}{\sqrt{s^{-\alpha} - s^2}} ds = \frac{1}{2\pi} \frac{\pi}{\alpha + 2} = \frac{1}{2(\alpha + 2)} \ .$$

This concludes the proof. □

The proof of Bertrand's Theorem follows from these two lemmas.

Proof. (Bertrand's Theorem)
It is well known that the orbit is closed if and only if the integral (A.4) belongs to $\mathbb{Q}$. In general, this integral will depend continuously on the initial values. Thus, let us compute the value of this integral when we are approaching the minimum of the effective potential, namely,

$$\lim_{E \mapsto E_{\min}} \frac{1}{2\pi} \int_{r_m}^{r_M} \frac{L}{r^2 \sqrt{2(E - V_{eff}(r))}} dr = \nu_0(r_0) \ .$$

where we denoted $E_{\min}$ the energy associated to the minimum. For continuity, we have to impose that ν_0 is constant when L varies.

Furthermore, by comparing the two integrals when $E \mapsto E_{\min}$ and when $E \mapsto E_M$, we deduce that $\alpha = -1, 2$ which correspond precisely to the Keplerian and the Harmonic potentials. □

Appendix B

Dirichlet Theorem

We start by presenting the one-dimensional version whose proof is much easier. For details on the results collected in this appendix see for instance [Sch96].

Theorem B.0.1. *(Dirichlet Theorem: one-dimensional version)*
Let α and Q be real numbers with $Q > 1$. Then, there exist integers p, q such that $1 \leq q < Q$ and $|\alpha q - p| \leq \frac{1}{Q}$.

Proof. Firstly, we prove the case for $Q \in \mathbb{N}$. Let us denote by $\{\alpha\}$ the fractional part of α, that is $\{\alpha\} = \alpha - [\alpha]$, where we have used $[\alpha]$ to indicate the integer part of α, and we observe that $\{\alpha\} \in [0, 1)$. Let us consider the following real numbers

$$0, 1, \{\alpha\}, \{2\alpha\}, \{3\alpha\}, \dots, \{(Q-1)\alpha\} .$$

They are $(Q + 1)$ real numbers in the unit interval $[0, 1]$. Therefore, let us divide the unit interval $[0, 1]$ into Q sub-intervals of the same length $\frac{1}{Q}$. Since we have $(Q + 1)$ real numbers in the unit interval which has been divided into Q parts, at least two real numbers $\{r_1\alpha\}$ and $\{r_2\alpha\}$ belong to the same sub-interval with $r_1, r_2 \in \mathbb{N}$ such that $0 \leq r_i \leq Q - 1,\ i = 1, 2$ and $r_1 \neq r_2$. Moreover, we have that

$$|\{r_1\alpha\} - \{r_2\alpha\}| \leq \frac{1}{Q} .$$

And, if we indicate by s_1 and s_2 the integer part of $r_1\alpha$ and $r_2\alpha$ respectively, that is, $\{r_1\alpha\} = r_1\alpha - s_1$ and $\{r_2\alpha\} = r_2\alpha - s_2$, then we have

$$|r_1\alpha - s_1 - (r_2\alpha - s_2)| \leq \frac{1}{Q} .$$

Let us suppose that $r_1 > r_2$, then

$$|q\alpha - p| \leq \frac{1}{Q} ,$$

where we denoted $q = r_1 - r_2 \in \mathbb{N}$ with $1 \le q \le Q-1$ and $p = s_1 - s_2 \in \mathbb{N}$. Let us consider now the case $Q \in \mathbb{R}$. If Q is not an integer, we can construct an integer by $Q' = [Q] + 1$. Then, we can apply the result above and we obtain that there exist two integers $q, p \in \mathbb{N}$ such that $1 \le q \le Q' - 1$ and

$$|q\alpha - p| \le \frac{1}{Q'} \,.$$

Since $Q' - 1 = [Q]$, then the inequality above is satisfied for $1 \le q \le [Q] < Q$, that is, the following inequality

$$|q\alpha - p| \le \frac{1}{Q}$$

is satisfied for $q \in \mathbb{N}$ such that $1 \le q < Q$. □

The theorem above has an obvious generalisation to the multi-dimensional case whose proof follows directly from the one-dimensional one which we report here for the sake of completeness.

Theorem B.0.2. *Suppose that $\alpha_1, \ldots, \alpha_n$ are n real numbers and that $Q > 1$ is an integer. Then, there exist integers $q, p_1, \ldots, p_n$ with $1 \le q < Q^n$ and $|q\alpha_i - p_i| \le \frac{1}{Q}$ with $i = 1, \ldots, n$.*

Theorem B.0.3. *(Minkowski's convex body theorem)*
Let $\Omega \subset \mathbb{R}^d$ be a non empty, bounded, centrally symmetric, convex subset of $\mathbb{R}^d$ with volume $Vol(\Omega) = \int_{\mathbb{R}^d} I_\Omega > 2^d$.
Then, there exist a vector of integers $x = (x_1, \ldots, x_d) \in \Omega$ such that x_i are not all equal to zero.

Proof. Let Ω be as above and let us consider the dilate subset $\frac{1}{2}\Omega$. It is easy to see that the dilate is a convex body too. Moreover, it follows from the hypothesis that its volume is $Vol\left(\frac{1}{2}\Omega\right) = Vol(\Omega) \cdot \left(\frac{1}{2}\right)^d > 1$. We say that Ω contains a non zero vector of integers, namely $x \in \mathbb{Z}^d$, if and only if the dilate subset $\frac{1}{2}\Omega$ contains a non zero vector y such that $2y \in \mathbb{Z}^d$.
Therefore, it is sufficient to prove that a convex body Ω with $Vol(\Omega) > 1$ contains a non zero vector x s.t. $2x \in \mathbb{Z}^d$.
For this reason, let us consider two vectors p and q in Ω. Since Ω is centrally symmetric, $-q \in \Omega$ and for convexity we deduce that any convex combination of the form $\lambda_1 p - \lambda_2 q$ is contained in Ω. In particular, if we take $\lambda_1 = \lambda_2 = \frac{1}{2}$, then we have $\frac{1}{2}p - \frac{1}{2}q \in \Omega$.
At this point, let us define by $N(r)$ the number of non zero vectors $p \in \Omega$ such that $rp \in \mathbb{Z}^d$. We notice that as $r \to +\infty$, $N(r)$ will be asymptotically equal to $r^d Vol(\Omega)$, that is, $\lim_{r\to+\infty} \frac{N(r)}{r^d} = Vol(\Omega)$.
However, since $Vol(\Omega) > 1$, then $N(r) > r^d$ when r tends to infinity. Therefore,

there exist two unique integer points p and q such that $\frac{1}{r}p$ and $\frac{1}{r}q$ are contained in Ω and, if we denote by $p = (x_1, \ldots, x_d)$ and $q = (y_1, \ldots, y_d)$ their components, then we have $x_i = y_i (mod\, r)$ for all $i = 1, \ldots, d$.
Thus, the convex combination of these two vectors, that is, $z = \frac{1}{2}\left(\frac{1}{r}p\right) - \frac{1}{2}\left(\frac{1}{r}q\right)$ is contained in Ω. We can rewrite the above combination and we obtain that $z \in \Omega$ with $2z = \frac{1}{r}(p-q) = \left(\frac{x_1 - y_1}{r}, \ldots, \frac{x_d - y_d}{r}\right)$.
However, from the definition of the vectors p and q, we obtain that the difference between each of their coordinates is an integer multiple of r. Thus, the components of the new vector $2z$ belong to $\mathbb{Z}$. Thus, the theorem is proved. □

Theorem B.0.4. *(Dirichlet theorem for simultaneous approximations)*
Let $\alpha_1, \ldots, \alpha_n \in \mathbb{R}^+$. For any $Q > 1$ there exists an integer $q : 1 \le q < Q$ and a vector $p = (p_1, \ldots, p_n) \in \mathbb{N}^n$ such that

$$|\alpha_i q - p_i| \le \frac{1}{Q^{\frac{1}{n}}} \,, \qquad i = 1, \ldots, n \,.$$

Proof. We consider the subset $\Omega \subset \mathbb{R}^{n+1}$ given by

$$\Omega = \left\{ (q, p_1, \ldots, p_n) \in \mathbb{R}^{n+1} : -Q - \frac{1}{2} \le q \le Q + \frac{1}{2},\ |\alpha_i q - p_i| \le \frac{1}{Q^{1/n}} \right\} \,.$$

We can easily compute its volume and we obtain

$$Vol(\Omega) = (2Q+1) \cdot \prod_{i=1}^{n} \frac{2}{Q^{1/n}} = (2Q+1) \cdot \left(\frac{2}{Q^{1/n}}\right)^n = (2Q+1) \cdot \frac{2^n}{Q} > 2^{n+1} \,.$$

Therefore, since all the hypothesis are satisfied, we can apply Minkowski's theorem and we obtain that there exists a vector of integers $(q, p_1, \ldots, p_n) \in \mathbb{N}^{n+1}$ which is contained in Ω. It means that there exists an integer q such that $|q| \le Q + \frac{1}{2}$ and a vector $(p_1, \ldots, p_n) \in \mathbb{N}^n$ such that

$$|\alpha_i q - p_i| \le \frac{1}{Q^{1/n}} \,.$$

Thus, the theorem is proved. □

Appendix C

Technical lemmas and results

C.1 Ehresmann fibration lemma

A useful criterion which is needed in order to establish when a map is a fibration is the following lemma due to Ehresmann [Ehr48]

Lemma C.1.1. *Let $F : M \mapsto B$ be a surjective submersion between two differentiable manifolds. If $\forall x \in B$, $F^{-1}(x)$ is compact and connected, then the sets $F^{-1}(x)$ are the fibers of a fibration.*

C.2 Some useful lemmas

We report here some technical lemmas that will be used in the proof of the main theorem

Lemma C.2.1. *Given two functions f and g of class $C^\omega(\rho)$, one has for every $\delta > 0$ that $\{f, g\} \in C^\omega(\rho - \delta)$ and*

$$\|\{f, g\}\|^*_{\rho-\delta} \leq \frac{1}{\delta}\|X_f\|^*_\rho \|g\|^*_\rho \ , \tag{C.1}$$

$$\|X_{\{f,g\}}\|^*_{\rho-\delta} \leq \frac{2}{\delta}\|X_f\|^*_\rho \|X_g\|^*_\rho \ . \tag{C.2}$$

To prove this result, we need the following lemma about the Cauchy estimates

Lemma C.2.2. *Let us consider a function $f \in C^\omega(\rho)$. Then, for every chart and $\forall\delta > 0$ the k-th partial derivative of f_j is bounded by*

$$\left\| \frac{\partial^{|k|} f_j}{\partial z_1^{k_1} \dots \partial z_{2d}^{k_{2d}}} \right\|^*_{\rho-\delta} \leq \frac{|k|!}{\delta^k} \|f\|^*_\rho$$

where k is a multi-index and $|k| = k_1 + \cdots + k_{2d}$.

Proof of Lemma C.2.1. Let us begin with the first estimate. Let $z \in \mathcal{U}_{\mathrm{j}}^{\rho}$ and let us denote by f_j and g_j the local representatives of the functions f and g respectively. We compute the norm on a smaller domain. Indeed, let $0 < \delta < \rho$ and consider

$$\begin{aligned}\|\{f_j, g_j\}\|^*_{\rho-\delta} &= \sup_{z\in\mathcal{U}_{\mathrm{j}}^{\rho-\delta}} |\{f_j, g_j\}(z)| = \sup_{z\in\mathcal{U}_{\mathrm{j}}^{\rho-\delta}} \left|dg_j(z)X_{f_j}(z)\right| \\ &\leq \sup_{z\in\mathcal{U}_{\mathrm{j}}^{\rho-\delta}} \left|X_{f_j}(z)\right| \sup_{z\in\mathcal{U}_{\mathrm{j}}^{\rho-\delta}} |dg_j(z)| \ ,\end{aligned}$$

where we have used the definition of the Poisson brackets to obtain the second identity. From Lemma C.2.2, we obtain

$$\|\{f_j, g_j\}\|^*_{\rho-\delta} \leq \frac{1}{\delta} \sup_{z\in\mathcal{U}_{\mathrm{j}}^{\rho-\delta}} \left|X_{f_j}(z)\right| \sup_{z\in\mathcal{U}_{\mathrm{j}}^{\rho}} |g_j(z)| \ .$$

And, since $\mathcal{U}_{\mathrm{j}}^{\rho-\delta} \subset \mathcal{U}_{\mathrm{j}}^{\rho}$, we have

$$\|\{f_j, g_j\}\|^*_{\rho-\delta} \leq \frac{1}{\delta} \left\| X_{f_j} \right\|^*_{\rho} \|g_j\|^*_{\rho} \ .$$

Passing to the supremum over j, we obtain (C.1).

Let us now conclude with the second estimate. Let $z \in \mathcal{U}_{\mathrm{j}}^{\rho}$ and f_j and g_j as above. From the definition of the Poisson brackets the following identity

$$X_{\{f_j,g_j\}} = [X_{f_j}, X_{g_j}] = (dX_{f_j})X_{g_j} - (dX_{g_j})X_{f_j} \tag{C.3}$$

holds. Thus, if we compute the norm on a smaller region as before, we have

$$\begin{aligned}\left\| X_{\{f_j,g_j\}} \right\|^*_{\rho-\delta} &\leq \left\| (dX_{f_j})X_{g_j} \right\|^*_{\rho-\delta} + \left\| (dX_{g_j})X_{f_j} \right\|^*_{\rho-\delta} \\ &\leq \frac{1}{\delta} \left\| X_{f_j} \right\|^*_{\rho} \left\| X_{g_j} \right\|^*_{\rho-\delta} + \frac{1}{\delta} \left\| X_{g_j} \right\|^*_{\rho} \left\| X_{f_j} \right\|^*_{\rho-\delta} \ ,\end{aligned}$$

where we have used the Cauchy estimates of Lemma C.2.2. Proceeding as before, we have

$$\left\| X_{\{f_j,g_j\}} \right\|^*_{\rho-\delta} \leq \frac{2}{\delta} \left\| X_{f_j} \right\|^*_{\rho} \left\| X_{g_j} \right\|^*_{\rho} \ ,$$

and, passing to the supremum over j, we obtain (C.2). □

Lemma C.2.3. *Let $\chi \in C^{\omega}(\rho)$ together with its vector field and fix $0 < \delta < \rho$. Assume that $\|X_\chi\|^*_\rho < \delta$ and consider the time flow $\mathcal{T}^t$ of the corresponding Hamiltonian vector field. Then, for $|t| \leq 1$, one has*

$$\left\| \mathcal{T}^t - \mathbb{I} \right\|_{\rho-\delta} \leq \|X_\chi\|^*_\rho \ .$$

Proof. Let us consider $z \in \mathcal{U}_{\mathrm{j}}^{\rho}$ and denote by $\mathcal{T}_j^t$ the local representative of the time flow $\mathcal{T}$. We compute

$$\left|\mathcal{T}_j^t(z) - z\right| = \left|\mathcal{T}_j^t(z) - \mathcal{T}_j^0(z)\right| \le \int_0^t \left|\frac{\mathrm{d}\mathcal{T}_j^s(z)}{\mathrm{d}s}\right| ds \le \int_0^t X_{\chi_j}(\mathcal{T}_j^s(z)) ds \ .$$

Passing to the supremum on the chart domain $\mathcal{U}_{\mathrm{j}}^{\rho-\delta}$, we obtain

$$\begin{aligned} \sup_{z\in\mathcal{U}_{\mathrm{j}}^{\rho-\delta}} \left|\mathcal{T}_j^t(z) - z\right| &\le \int_0^t \sup_{z\in\mathcal{U}_{\mathrm{j}}^{\rho-\delta}} \left|X_{\chi_j}(\mathcal{T}_j^s(z))\right| ds \\ &\le \int_0^t \sup_{z\in\mathcal{U}_{\mathrm{j}}^{\rho}} \left|X_{\chi_j}(z)\right| ds \\ &\le t \sup_{z\in\mathcal{U}_{\mathrm{j}}^{\rho}} \left|X_{\chi_j}(z)\right| \ . \end{aligned}$$

Thus, since $|t| \le 1$, we have

$$\left\| \mathcal{T}^t - \mathbb{I} \right\|_{\rho-\delta}^* \le t \left\|X_\chi\right\|_\rho^* \ .$$

This concludes the proof. □

Lemma C.2.4. *Let $\chi \in C^\omega(\rho)$ and $\mathcal{T}^t$ as in the lemma above and consider a function $f \in C^\omega(\rho)$ together with its vector field. Let $0 < \delta < \rho$. Assume that $\left\|X_\chi\right\|_\rho^* \le \frac{\delta}{3}$. Then, for $|t| \le 1$, one has*

$$\left\| f \circ \mathcal{T}^t - f \right\|_{\rho-\delta}^* \le \frac{2}{\delta} \left\|X_f\right\|_\rho^* \left\|X_\chi\right\|_\rho^* \ , \tag{C.4}$$

and

$$\left\|X_{f\circ\mathcal{T}^t - f}\right\|_{\rho-\delta}^* \le \frac{5}{\delta} \left\|X_f\right\|_\rho^* \left\|X_\chi\right\|_\rho^* \ . \tag{C.5}$$

Proof. Let us denote by $\bar{\delta} := \frac{\delta}{2}$. Let $z \in \mathcal{U}_{\mathrm{j}}^{\rho}$ and let us denote by $\mathcal{T}_j^t$ and f_j the local representatives of the functions $\mathcal{T}^t$ and f respectively. We compute

$$\left|f_j(\mathcal{T}_j^t(z)) - f_j(z)\right| \le \int_0^t \left|\frac{\mathrm{d}}{\mathrm{d}s} f_j(\mathcal{T}_j^s(z))\right| ds = \int_0^t \left|\{\chi_j, f_j\}(\mathcal{T}_j^s(z))\right| ds \ .$$

At this point, we compute the norm on a smaller region, that is,

$$\begin{aligned} \sup_{z\in\mathcal{U}_{\mathrm{j}}^{\rho-2\bar{\delta}}} \left|f_j(\mathcal{T}_j^t(z)) - f_j(z)\right| &\le \int_0^t \sup_{z\in\mathcal{U}_{\mathrm{j}}^{\rho-2\bar{\delta}}} \left|\{\chi_j, f_j\}(\mathcal{T}_j^s(z))\right| ds \\ &\le \int_0^t \sup_{z\in\mathcal{U}_{\mathrm{j}}^{\rho-\bar{\delta}}} \left|\{\chi_j, f_j\}(z)\right| ds \\ &\le t \left\|\{\chi_j, f_j\}\right\|_{\rho-\bar{\delta}}^* \ . \end{aligned}$$

Thus, since $|t| \leq 1$ and $\mathcal{U}_{\mathrm{j}}^{\rho-\bar{\delta}} \subset \mathcal{U}_{\mathrm{j}}^{\rho}$, we have

$$\left\| f_j \circ \mathcal{T}^t - f_j \right\|_{\rho-2\bar{\delta}}^{*} \leq \|\{\chi_j, f_j\}\|_{\rho}^{*} \ ,$$

and, passing to the supremum over j, one has

$$\left\| f \circ \mathcal{T}^t - f \right\|_{\rho-2\bar{\delta}}^{*} \leq \|\{\chi, f\}\|_{\rho}^{*} \leq \frac{1}{\delta} \|X_\chi\|_{\rho}^{*} \|f\|_{\rho}^{*} \ ,$$

where we have used the estimates (C.1). Going back to δ, we obtain (C.4).

We come now to the second estimate. Let $z \in \mathcal{U}_{\mathrm{j}}^{\rho}$. From (1.22) of Chapter 1, we can write

$$X_{f_j \circ \mathcal{T}_j^t}(z) = d(\mathcal{T}_j^t)^{-1}(\mathcal{T}_j^t(z)) X_{g_j}(\mathcal{T}_j^t(z)) \ .$$

Therefore,

$$\left\| X_{f_j \circ \mathcal{T}_j^t - f_j} \right\|_{\rho-\delta}^{*} \leq \left\| (d(\mathcal{T}_j^t)^{-1} \circ \mathcal{T}_j^t - \mathbb{I})(X_{f_j} \circ \mathcal{T}_j^t) \right\|_{\rho-\delta}^{*} + \left\| X_{f_j} \circ \mathcal{T}_j^t - X_{f_j} \right\|_{\rho-\delta}^{*} \ . \tag{C.6}$$

Let us consider the first term in (C.6). Let $\tilde{\delta} := \frac{\delta}{3}$ and consider

$$\begin{aligned} \left\| (d(\mathcal{T}_j^t)^{-1} \circ \mathcal{T}_j^t - \mathbb{I})(X_{f_j} \circ \mathcal{T}_j^t) \right\|_{\rho-3\tilde{\delta}}^{*} &\leq \left\| d(\mathcal{T}_j^t)^{-1} - \mathbb{I} \right\|_{\rho-2\tilde{\delta}}^{*} \left\| X_{f_j} \right\|_{\rho-2\tilde{\delta}}^{*} \\ &\leq \frac{1}{\tilde{\delta}} \left\| (\mathcal{T}_j^t)^{-1} - \mathbb{I} \right\|_{\rho-\tilde{\delta}}^{*} \left\| X_{f_j} \right\|_{\rho-2\tilde{\delta}}^{*} \ , \end{aligned}$$

where we have used Lemma C.2.2. Furthermore, from Lemma C.2.3 we have

$$\left\| (d(\mathcal{T}_j^t)^{-1} \circ \mathcal{T}_j^t - \mathbb{I})(X_{f_j} \circ \mathcal{T}_j^t) \right\|_{\rho-3\tilde{\delta}}^{*} \leq \frac{1}{\tilde{\delta}} \left\| X_{\chi_j} \right\|_{\rho}^{*} \left\| X_{f_j} \right\|_{\rho}^{*} \ .$$

Going back to δ, one has

$$\left\| (d(\mathcal{T}_j^t)^{-1} \circ \mathcal{T}_j^t - \mathbb{I})(X_{f_j} \circ \mathcal{T}_j^t) \right\|_{\rho-\delta}^{*} \leq \frac{3}{\delta} \left\| X_{\chi_j} \right\|_{\rho}^{*} \left\| X_{f_j} \right\|_{\rho}^{*} \ . \tag{C.7}$$

We consider now the second term. Let $\bar{\delta} := \frac{\delta}{2}$ and consider

$$\begin{aligned} \left\| X_{f_j} \circ \mathcal{T}_j^t - X_{f_j} \right\|_{\rho-2\bar{\delta}}^{*} \leq \left\| \mathcal{T}_j^t - \mathbb{I} \right\|_{\rho-2\bar{\delta}}^{*} \left\| dX_{f_j} \right\|_{\rho-\bar{\delta}}^{*} &\leq \frac{1}{\bar{\delta}} \left\| \mathcal{T}_j^t - \mathbb{I} \right\|_{\rho-2\bar{\delta}}^{*} \left\| X_{f_j} \right\|_{\rho}^{*} \\ &\leq \frac{1}{\bar{\delta}} \left\| X_{\chi_j} \right\|_{\rho-2\bar{\delta}}^{*} \left\| X_{f_j} \right\|_{\rho}^{*} \ , \end{aligned}$$

that is,

$$\left\| X_{f_j} \circ \mathcal{T}_j^t - X_{f_j} \right\|_{\rho-2\bar{\delta}}^{*} \leq \frac{1}{\bar{\delta}} \left\| X_{\chi_j} \right\|_{\rho}^{*} \left\| X_{f_j} \right\|_{\rho}^{*} \ .$$

If we express $\bar{\delta}$ in terms of δ, we obtain

$$\| X_{f_j} \circ \mathcal{T}_j^t - X_{f_j} \|_{\rho-\delta}^* \leq \frac{2}{\delta} \| X_{\chi_j} \|_\rho^* \| X_{f_j} \|_\rho^* \ . \tag{C.8}$$

Let us now pull this estimates into expression (C.6). We have

$$\left\| X_{f_j \circ \mathcal{T}_j^t - f_j} \right\|_{\rho-\delta}^* \leq \frac{5}{\delta} \| X_{\chi_j} \|_\rho^* \| X_{f_j} \|_\rho^* \ .$$

Passing to the supremum over j we obtain the estimate (C.5).

This concludes the proof. □

Lemma C.2.5. *Let $\chi \in C^\omega(\rho)$ be the solution of the homological equation*

$$\{\chi, h_{\omega^*}\} + f = \langle f \rangle \ , \tag{C.9}$$

and consider the time one flow $\mathcal{T}$ of the corresponding vector field. Let $0 < \delta < \rho$. Then, one has

$$\| h_{\omega^*} \circ \mathcal{T} - h_{\omega^*} - \{\chi, h_{\omega^*}\} \|_{\rho-\delta}^* \leq \frac{4}{\delta} \|X_\chi\|_\rho^* \|f\|_\rho^* \ , \tag{C.10}$$

and,

$$\| X_{h_{\omega^*} \circ \mathcal{T} - h_{\omega^*} - \{\chi, h_{\omega^*}\}} \|_{\rho-\delta}^* \leq \frac{10}{\delta} \|X_\chi\|_\rho^* \|X_f\|_\rho^* \ . \tag{C.11}$$

Proof. Let us begin with the first estimate. Let $z \in \mathcal{U}_{\mathrm{j}}^\rho$. We can write

$$\begin{aligned} (h_{\omega^*,j} \circ \mathcal{T}_j - h_{\omega^*,j} - \{\chi_j, h_{\omega^*,j}\})(z) &= \int_0^1 \{\chi_j, h_{\omega^*,j}\}(\mathcal{T}_j^t(z)) dt - \int_0^1 \{\chi_j, h_{\omega^*,j}\}(z) dt \\ &= \int_0^1 (g_j(\mathcal{T}_j^t(z)) - g_j(z)) dt \ , \end{aligned} \tag{C.12}$$

where we denoted $g_j(z) := \{\chi_j, h_{\omega^*,j}\}(z)$.

Let us denote $\bar{\delta} := \frac{\delta}{2}$ and consider

$$\sup_{z \in \mathcal{U}_{\mathrm{j}}^{\rho-2\bar{\delta}}} |(h_{\omega^*,j} \circ \mathcal{T}_j - h_{\omega^*,j} - \{\chi_j, h_{\omega^*,j}\})(z)| \leq \int_0^1 \sup_{z \in \mathcal{U}_{\mathrm{j}}^{\rho-2\bar{\delta}}} \left| g_j(\mathcal{T}_j^t(z)) - g_j(z) \right| dt \ .$$

Passing to the supremum over j and exploiting the estimate (C.4) of the previous lemma, we obtain

$$\| h_{\omega^*} \circ \mathcal{T} - h_{\omega^*} - \{\chi, h_{\omega^*}\} \|_{\rho-2\bar{\delta}}^* \leq \frac{1}{\delta} \|\chi_j\|_\rho^* \|g_j\|_\rho^* \ .$$

and, from the definition of g_j and the homological equation (C.9), we have

$$g_j = \{\chi_j, h_{\omega^*,j}\} = \langle f \rangle_j - f_j \ ,$$

and

$$\|g_j\|^*_\rho \leq 2\,\|f_j\|^*_\rho \ .$$

Finally, expressing $\bar{\delta}$ in terms of δ, we have

$$\|h_{\omega^*,j} \circ \mathcal{T}_j - h_{\omega^*,j} - \{\chi_j, h_{\omega^*,j}\}\|^*_{\rho-\delta} \leq \frac{2}{\delta}\,\|\chi_j\|^*_\rho\,\|g_j\|^*_\rho \leq \frac{4}{\delta}\,\|\chi_j\|^*_\rho\,\|f_j\|^*_\rho \ .$$

And, passing to the supremum over j, we obtain (C.10).

We now pass to the second estimate. From (C.12), we obtain that

$$\left\| X_{h_{\omega^*,j}\circ\mathcal{T}_j - h_{\omega^*,j} - \{\chi_j, h_{\omega^*,j}\}} \right\|^*_{\rho-\delta} \leq \left\| X_{g_j \circ \mathcal{T}^t_j - g_j} \right\|^*_{\rho-\delta} \ .$$

By exploiting the previous estimate (C.5), we obtain

$$\left\| X_{h_{\omega^*,j}\circ\mathcal{T}_j - h_{\omega^*,j} - \{\chi_j, h_{\omega^*,j}\}} \right\|^*_{\rho-\delta} \leq \frac{5}{\delta}\,\left\| X_{g_j} \right\|^*_\rho \left\| X_{\chi_j} \right\|^*_\rho \ ,$$

and, since

$$\left\| X_{g_j} \right\|^*_\rho \leq 2\,\left\| X_{f_j} \right\|^*_\rho \ ,$$

we conclude

$$\left\| X_{h_{\omega^*,j}\circ\mathcal{T}_j - h_{\omega^*,j} - \{\chi_j, h_{\omega^*,j}\}} \right\|^*_{\rho-\delta} \leq \frac{10}{\delta}\,\left\| X_{\chi_j} \right\|^*_\rho \left\| X_{f_j} \right\|^*_\rho \ .$$

The estimate (C.11) is obtained passing to the supremum over j. □

Appendix D

Normal forms and other results

D.1 Birkhoff normal form

In Chapter 2, we have made use of a Birkhoff normal form for the Hamiltonian of the planar central motion problem in a neighborhood of a circular orbit and we have constructed this by means of a symbolic manipulator (Mathematica^{TM}). Here we report the first steps of the Birkhoff normal form procedure to get an idea of how the scheme goes.

Let us begin by considering the Hamiltonian of the planar central motion in polar coordinates, namely,

$$H(r,p_r,p_\theta) := \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} + V(r) = \frac{p_r^2}{2} + V_{eff}(r,p_\theta^2) \ . \tag{D.1}$$

We assume that there exists a minimum r_0 of the effective potential, that is,

$$\exists r_0 > 0 \ : \ V'_{eff}(r_0,p_\theta^2) = -\frac{p_\theta^2}{r_0^3} + V'(r_0) = 0 \ \Rightarrow \ p_\theta^2 = r_0^3 V'(r_0) \ . \tag{D.2}$$

By restricting our analysis to the region where p_θ is positive, we can notice a 1-1 correspondence between p_θ and r_0 so that we can just deal with r_0.

We begin by Taylor expanding the Hamiltonian (D.1) in a neighborhood of the circular orbit: we obtain

$$H(r,p_r) = V_{eff}(r_0) + \frac{p_r^2}{2} + \sum_{l\geq 2} \frac{V_{eff}^{(l)}(r_0)}{l!}(r-r_0)^l \ ,$$

where we have used the fact that the first derivative of the effective potential vanishes at the minimum.

For the sake of clarity, we report here the explicit formulæ for the derivatives of V_{eff}. Thus, we first compute the derivatives

$$\begin{aligned}
V'_{eff} &= -\frac{p_\theta^2}{r_0^3} + V'(r_0) \ , \\
V^{(2)}_{eff} &= \frac{3p_\theta^2}{r_0^4} + V^{(2)}(r_0) \ , \\
V^{(3)}_{eff} &= -\frac{12p_\theta^2}{r_0^5} + V^{(3)}(r_0) \ , \\
\ldots \\
V^{(l)}_{eff} &= (-1)^l \frac{(l+1)!}{2}\frac{p_\theta^2}{r_0^{l+2}} + V^{(l)}(r_0) \ .
\end{aligned}$$

Secondly, we use the equality (D.2) and obtain the explicit formulæ for the derivatives of the effective potentials

$$V^{(l)}_{eff} = (-1)^l \frac{(l+1)!}{2}\frac{V'(r_0)}{r_0^{l-1}} + V^{(l)}(r_0) \ . \tag{D.3}$$

These formulæ will be used later in order to highlight a key property of the Taylor expansion.

Now, we go back to the expanded Hamiltonian where we denote by $\rho := r - r_0$ the displacement and where we isolate the quadratic part, that is,

$$H(r, p_r) = V_{eff}(r_0) + \frac{p_r^2}{2} + \left(\frac{3V'(r_0)}{r_0} + V^{(2)}(r_0)\right)\rho^2 + \sum_{l\geq 3} \frac{V^{(l)}_{eff}(r_0)}{l!}\rho^l \ .$$

The coefficient of the second order term in ρ, namely,

$$\mathcal{A}(r_0) := \frac{3V'(r_0)}{r_0} + V''(r_0) > 0 \ ,$$

is positive since it is computed at the minimum of the effective potential, thus, we use it to make a canonical change of variables $(\rho, p_r) \mapsto (x, y)$ with

$$\begin{cases} x = \sqrt[4]{\mathcal{A}(r_0)}\rho \\ y = \dfrac{p_r}{\sqrt[4]{\mathcal{A}(r_0)}} \end{cases} ,$$

in order to diagonalize the quadratic part of the Hamiltonian. The rescaled Hamiltonian becomes

$$H(x, y) = V_{eff}(r_0) + \sqrt{\mathcal{A}(r_0)}\,\frac{y^2 + x^2}{2} + \sum_{l\geq 3} \frac{V^{(l)}_{eff}(r_0)}{l!}(\mathcal{A}(r_0))^{-\frac{l}{4}} x^l \ . \tag{D.4}$$

The Hamiltonian above is the classical form of the Hamiltonian in a neighborhood of an elliptic equilibrium and, therefore, we can introduce the action angle variables by performing a Birkhoff normal form. In Chapter 2, we have constructed the Birkhoff normal form at the sixth order by means of a symbolic manipulator. Here, we outline the general scheme, thus, to get some ideas of the computations needed, we perform only the first steps obtaining the normal form arrested at the fourth order.

To do so, let us consider the Hamiltonian expansion (D.4) arrested at the fourth order, namely,

$$H(x,y) = V_{eff}(r_0) + \sqrt{\mathcal{A}(r_0)}\,\frac{y^2+x^2}{2} + \frac{\sqrt{\mathcal{A}(r_0)}}{2}\left(\alpha(r_0)x^3 + \beta(r_0)x^4\right) + o(x^4)\ ,$$

where we have denoted by α, β the coefficients of the higher order terms of the Hamiltonian. Precisely,

$$\begin{aligned} \alpha(r_0) &:= \frac{V^{(3)}_{eff}(r_0)}{3}(\mathcal{A}(r_0))^{-\frac{5}{4}} = \frac{\mathcal{B}(r_0)}{3}(\mathcal{A}(r_0))^{-\frac{5}{4}}\ , \\ \beta(r_0) &:= \frac{V^{(4)}_{eff}(r_0)}{12}(\mathcal{A}(r_0))^{-\frac{3}{2}} = \frac{\mathcal{C}(r_0)}{12}(\mathcal{A}(r_0))^{-\frac{3}{2}}\ , \end{aligned}$$

where we have denoted $\mathcal{B}(r_0) := V^{(3)}_{eff}(r_0)$ and $\mathcal{C}(r_0) := V^{(4)}_{eff}(r_0)$.

At this point, we consider the complex variables

$$\begin{cases} \xi = \frac{1}{\sqrt{2}}(y+ix) \\ \eta = \frac{1}{\sqrt{2}}(y-ix) \end{cases}\ ,$$

in which the symplectic form turns out to be $dx \wedge dy = i d\xi \wedge d\eta$. By exploiting this change of variables and considering that a homogeneous polynomial of degree l in the (y, x) variables does not change degree when expressed in the new variables (η, ξ), the new Hamiltonian becomes

$$\begin{aligned} H(\xi,\eta) = V_{eff}(r_0) + \sqrt{\mathcal{A}}\xi\eta + \frac{\alpha\sqrt{2\mathcal{A}}}{8}(\xi^3 - 3\xi^2\eta + 3\xi\eta^2 - \eta^3)i \\ + \frac{\beta\sqrt{\mathcal{A}}}{8}(\xi^4 - 4\xi^3\eta + 6\xi^2\eta^2 - 4\xi\eta^3 + \eta^4) + o((\xi+\eta)^4)\ , \end{aligned}$$

where we have omitted the dependence on r_0.

Let us now consider the Hamiltonian

$$\begin{aligned} \tilde{H}(\xi,\eta) = \sqrt{\mathcal{A}}\xi\eta + \frac{\alpha\sqrt{2\mathcal{A}}}{8}(\xi^3 - 3\xi^2\eta + 3\xi\eta^2 - \eta^3)i \\ + \frac{\beta\sqrt{\mathcal{A}}}{8}(\xi^4 - 4\xi^3\eta + 6\xi^2\eta^2 - 4\xi\eta^3 + \eta^4)\ . \end{aligned}$$

We proceed with the normalization at the third order: let us rewrite the expression above in a more concise way, that is,

$$\tilde{H}(\xi,\eta) = h^{(2)}(\xi,\eta) + P^{(3)}(\xi,\eta) + P^{(4)}(\xi,\eta) \ , \tag{D.5}$$

underlining the degree of the homogeneous polynomials. We make the reader notice that we have omitted the terms of order higher than four since they do not contribute to the construction of the fourth order Birkhoff normal form.

Having said this, we begin the construction by searching for the coefficients $P_{J,L}$ with (J,L) such that $|J|+|L| = 3$, $J \neq L$ in order to construct the third order term of the generating function χ, that is

$$\chi^{(3)}(\xi,\eta) = \sum_{\substack{J\neq L \\ |J|+|L|=3}} \frac{P^{(3)}_{J,L}}{\sqrt{\mathcal{A}}(J-L)i}\xi^J\eta^L \ .$$

To do so, let us consider the third order homogeneous polynomial $P^{(3)}$, that is,

$$\begin{aligned} P^{(3)}(\xi,\eta) &= \frac{\alpha\sqrt{2\mathcal{A}}}{8}i\xi^3 - \frac{3\alpha\sqrt{2\mathcal{A}}}{8}i\xi^2\eta + \frac{3\alpha\sqrt{2\mathcal{A}}}{8}i\xi\eta^2 - \frac{\alpha\sqrt{2\mathcal{A}}}{8}i\eta^3 \\ &:= P^{(3)}_{3,0}\xi^3 + P^{(3)}_{2,1}\xi^2\eta + P^{(3)}_{1,2}\xi\eta^2 + P^{(3)}_{0,3}\eta^3 \ . \end{aligned} \tag{D.6}$$

The third order term of the generating function is

$$\chi^{(3)}(\xi,\eta) = \frac{\alpha\sqrt{2}}{24}\xi^3 - \frac{3\alpha\sqrt{2}}{8}\xi^2\eta - \frac{3\alpha\sqrt{2}}{8}\xi\eta^2 + \frac{\alpha\sqrt{2}}{24}\eta^3 \ . \tag{D.7}$$

Then, using the Lie transform method, we can apply the time one Hamiltonian flow generated by $\chi^{(3)}$ to the Hamiltonian (D.5) in order to normalize it at the third order. Indeed, let $\mathcal{T}^{(3)} := \Phi_{\chi^{(3)}}$ and compute

$$\begin{aligned} \tilde{H}_1(\xi,\eta) := \tilde{H}\circ\Phi_{\chi^{(3)}}(\xi,\eta) = h^{(2)}(\xi,\eta) &+ \left(P^{(3)}(\xi,\eta) + \{\chi^{(3)},h^{(2)}\}(\xi,\eta)\right) \\ &+ \left(h^{(2)}\circ\mathcal{T}^{(3)} - h^{(2)} - \{\chi^{(3)},h^{(2)}\}\right)(\xi,\eta) \\ &+ \left(P^{(3)}\circ\mathcal{T}^{(3)} - P^{(3)}\right)(\xi,\eta) \\ &+ (P^{(4)}\circ\mathcal{T}^{(3)})(\xi,\eta) \ , \end{aligned}$$

that is,

$$\tilde{H}_1(\xi,\eta) = h^{(2)}(\xi,\eta) + \mathcal{Z}^{(3)}_1(\xi,\eta) + \mathcal{R}_1(\xi,\eta) \ , \tag{D.8}$$

with $\mathcal{Z}^{(3)}_1(\xi,\eta)$ being the solution of the homological equation. Precisely,

$$Z^{(3)}_1(\xi,\eta) = \sum_{|J|+|L|=3,\ J=L} P^{(3)}_{J,L}\xi^J\eta^L = 0 \ ,$$

and it turns out to be identically zero. The remaining terms are collected into the remainder $\mathcal{R}_1$, that is,

$$\begin{aligned}\mathcal{R}_1(\xi,\eta) = &\left(h^{(2)} \circ \mathcal{T}^{(3)} - h^{(2)} - \{\chi^{(3)}, h^{(2)}\}\right)(\xi,\eta) \\ &+ \left(P^{(3)} \circ \mathcal{T}^{(3)} - P^{(3)}\right)(\xi,\eta) \\ &+ (P^{(4)} \circ \mathcal{T}^{(3)})(\xi,\eta) \ .\end{aligned}$$

Remark D.1.1. *From the properties of the Lie transform, it is known that the composition between an homogeneous polynomial g of degree i and the time one Hamiltonian flow generated by a function χ of degree j is given by*

$$g \circ \Phi_\chi = \sum_{l\geq 0} g_l \ ,$$

with

$$\begin{cases} g_l = \dfrac{1}{l!}\{\chi, g_{l-1}\}, & l > 0 \\ g_0 = g \end{cases} .$$

Moreover, g_l is an homogeneous polynomial of degree $i + l(j-2)$.

At this point, we develop the terms of the remainder $\mathcal{R}_1$ by exploiting the properties contained in Remark D.1.1. We obtain

$$\begin{aligned}\left(h^{(2)} \circ \mathcal{T}^{(3)} - h^{(2)} - \{\chi^{(3)}, h^{(2)}\}\right)(\xi,\eta) =& h^{(2)}(\xi,\eta) + h_1^{(2)}(\xi,\eta) + h_2^{(2)}(\xi,\eta) + h.o.t. \\ &- h^{(2)}(\xi,\eta) - h_1^{(2)}(\xi,\eta) \ ,\end{aligned}$$

where we have use the fact that

$$h_1^{(2)}(\xi,\eta) = \{\chi^{(3)}, h^{(2)}\} \ .$$

Thus,

$$\left(h^{(2)} \circ \mathcal{T}^{(3)} - h^{(2)} - \{\chi^{(3)}, h^{(2)}\}\right)(\xi,\eta) = h_2^{(2)}(\xi,\eta) + h.o.t. \ .$$

Analogously, we have

$$\left(P^{(3)} \circ \mathcal{T}^{(3)} - P^{(3)}\right)(\xi,\eta) = P_1^{(3)}(\xi,\eta) + h.o.t. \ ,$$

and

$$(P^{(4)} \circ \mathcal{T}^{(3)})(\xi,\eta) = P^{(4)}(\xi,\eta) + h.o.t. \ .$$

Therefore, we can rewrite the remainder by collecting together the polynomials having the same degree, that is

$$\mathcal{R}_1(\xi,\eta) = \mathcal{R}_1^{(4)}(\xi,\eta) + h.o.t. \ ,$$

where

$$\mathcal{R}_1^{(4)}(\xi,\eta) = h_2^{(2)}(\xi,\eta) + P_1^{(3)}(\xi,\eta) + P^{(4)}(\xi,\eta)\ .$$

Thus, the Hamiltonian (D.8) becomes

$$\tilde{H}_1(\xi,\eta) = h^{(2)}(\xi,\eta) + R_1^{(4)}(\xi,\eta) + h.o.t.\ . \tag{D.9}$$

At this point, we notice that we have three terms that contribute to the fourth order homogeneous polynomial of the remainder. We can adjust them so that we can express $\mathcal{R}_1^{(4)}$ in an explicit way. We start by noticing that we can rewrite the term $h_2^{(2)}$ as

$$h_2^{(2)}(\xi,\eta) = \frac{1}{2}\{\chi^{(3)},\{\chi^{(3)},h^{(2)}\}\}(\xi,\eta) = -\frac{1}{2}\{\chi^{(3)},P^{(3)}\}(\xi,\eta) = -\frac{1}{2}P_1^{(3)}(\xi,\eta)\ ,$$

where the second identity comes from the homological equation exploiting the fact that the normal term $Z^{(3)}$ is equal to zero. Thus, the remainder $\mathcal{R}_1^{(4)}$ can be simplified as

$$\mathcal{R}_1^{(4)}(\xi,\eta) = \frac{1}{2}P_1^{(3)}(\xi,\eta) + P^{(4)}(\xi,\eta)\ . \tag{D.10}$$

Let us now compute the term $P_1^{(3)}$: inserting the expressions (D.6) and (D.7) in the definition of $P_1^{(3)}(\xi,\eta) = \{\chi^{(3)},P^{(3)}\}(\xi,\eta)$, after some computations, we obtain

$$\begin{aligned} P_1^{(3)}(\xi,\eta) =& \frac{3\alpha^2\sqrt{\mathcal{A}}}{16}\xi^4 + \frac{3\alpha^2\sqrt{\mathcal{A}}}{4}\xi^3\eta - \frac{15\alpha^2\sqrt{\mathcal{A}}}{8}\xi^2\eta^2 \\ &+ \frac{3\alpha^2\sqrt{\mathcal{A}}}{4}\xi\eta^3 + \frac{3\alpha^2\sqrt{\mathcal{A}}}{16}\eta^4\ . \end{aligned} \tag{D.11}$$

We remind the expression of $P^{(4)}$, that is,

$$P^{(4)}(\xi,\eta) = \frac{\beta\sqrt{\mathcal{A}}}{8}(\xi^4 - 4\xi^3\eta + 6\xi^2\eta^2 - 4\xi\eta^3 + \eta^4)\ . \tag{D.12}$$

We substitute (D.11) and (D.12) in the expression (D.10) for the remainder and obtain

$$\begin{aligned} R_1^{(4)}(\xi,\eta) =& \left(\frac{3\alpha^2}{32} + \frac{\beta}{8}\right)\sqrt{\mathcal{A}}\xi^4 + \left(\frac{3\alpha^2}{8} - \frac{\beta}{2}\right)\sqrt{\mathcal{A}}\xi^3\eta + \left(-\frac{15\alpha^2}{16} + \frac{3\beta}{4}\right)\sqrt{\mathcal{A}}\xi^2\eta^2 \\ &+ \left(\frac{3\alpha^2}{8} - \frac{\beta}{2}\right)\sqrt{\mathcal{A}}\xi\eta^3 + \left(\frac{3\alpha^2}{32} + \frac{\beta}{8}\right)\sqrt{\mathcal{A}}\eta^4 \\ :=& R_{1_{4,0}}^{(4)}\xi^4 + R_{1_{3,1}}^{(4)}\xi^3\eta + R_{1_{2,2}}^{(4)}\xi^2\eta^2 + R_{1_{1,3}}^{(4)}\xi\eta^3 + R_{1_{0,4}}^{(4)}\eta^4\ . \end{aligned} \tag{D.13}$$

One can compute also the higher order terms of the remainder. However, for what concerns our work we stop at the fourth order.

We go back now to the Hamiltonian (D.9) and we search now for another canonical transformation in order to normalize the Hamiltonian at the fourth order. Thus, following the steps we have done previously to obtain the third order normalization, let us search for the generating function $\chi^{(4)}$ having the form

$$\chi^{(4)}(\xi,\eta) = \sum_{|J|+|L|=4,\ J\neq L} \frac{R^{(4)}_{1_{J,L}}}{\sqrt{\mathcal{A}}(J-L)i}\xi^J\eta^L \ .$$

By exploiting the coefficients of (D.13), we can construct the generating function $\chi^{(4)}$ which takes the form

$$\chi^{(4)}(\xi,\eta) = \left(-\frac{3\alpha^2}{128}-\frac{\beta}{32}\right)i\xi^4+\left(-\frac{3\alpha^2}{16}+\frac{\beta}{4}\right)i\xi^3\eta+\left(\frac{3\alpha^2}{16}-\frac{\beta}{4}\right)i\xi\eta^3+\left(\frac{3\alpha^2}{128}+\frac{\beta}{32}\right)i\eta^4 \ .$$

At last, applying the canonical transformation to the Hamiltonian (D.9), we obtain the Hamiltonian below, normalized at the fourth order

$$\tilde{H}_2(\xi,\eta) = h^{(2)}(\xi,\eta) + Z^{(4)}_2(\xi,\eta) + R_2(\xi,\eta) \ ,$$

where

$$Z^{(4)}_2(\xi,\eta) = \sum_{|J|+|L|=4,\ J=L} R^{(4)}_{1_{J,L}}\xi^J\eta^L = \left(-\frac{15\alpha^2}{16}+\frac{3\beta}{4}\right)\sqrt{\mathcal{A}}\xi^2\eta^2 \ .$$

At this point, substituting the coefficients $\alpha = \frac{1}{3}\mathcal{B}(\mathcal{A})^{-\frac{5}{4}}$ and $\beta = \frac{1}{12}\mathcal{C}(\mathcal{A})^{-\frac{3}{2}}$ in the Hamiltonian above, we obtain

$$\tilde{H}_2(\xi,\eta) = \sqrt{\mathcal{A}}\xi\eta + \left(\frac{-5\mathcal{B}^2+3\mathcal{C}\mathcal{A}}{48\mathcal{A}^2}\right)\xi^2\eta^2 + o\left((\xi\eta)^2\right) \ .$$

Finally, we can introduce the action angle variables. To do so, let us first consider the normalized Hamiltonian

$$H_2(\xi,\eta) = V_{eff}(r_0) + \sqrt{\mathcal{A}}\xi\eta + \left(\frac{-5\mathcal{B}^2+3\mathcal{C}\mathcal{A}}{48\mathcal{A}^2}\right)\xi^2\eta^2 + o\left((\xi\eta)^2\right) \ ,$$

and the new variables (I_1,ϕ_1) introduced by $(\xi,\eta)\mapsto(I_1,\phi_1)$, where

$$\begin{cases} \xi = \sqrt{I}_1 e^{i\phi_1} \\ \eta = \sqrt{I}_1 e^{-i\phi_1} \end{cases} .$$

Then, the Hamiltonian (D.1) normalized at the fourth order becomes

$$H = V_{eff}(r_0) + \sqrt{\mathcal{A}} I_1 + \left(\frac{-5\mathcal{B}^2 + 3\mathcal{C}\mathcal{A}}{48\mathcal{A}^2} \right) I_1^2 + o(I_1^3) \ .$$

Let us denote by $I_2 := p_\theta$ the second action and by ϕ_2 the corresponding angle, then, by exploiting the 1-1 correspondence between r_0 and p_θ (and, thus, I_2), we conclude with the following expression for the Hamiltonian (D.1) normalized at the fourth order

$$\begin{aligned} h(I_1, I_2) &= V_{eff}(I_2) + \sqrt{\mathcal{A}(I_2)} I_1 + \left(\frac{-5\mathcal{B}^2(I_2) + 3\mathcal{C}(I_2)\mathcal{A}(I_2)}{48\mathcal{A}(I_2)^2} \right) I_1^2 + o(I_1^3) \\ &:= h_0(I_2) + h_1(I_2) I_1 + h_2(I_2) I_1^2 + o(I_1^3) \ . \end{aligned}$$

D.2 Burgers equation

We report here the proof of a useful property satisfied by the coefficients of the Birkhoff normal form which turns out to be powerful when writing the expansion of the Burgers equation.

Thus, let us consider the Hamiltonian (D.4), that is,

$$H(x, y) = V_{eff}(r_0) + \sqrt{\mathcal{A}(r_0)}\, \frac{y^2 + x^2}{2} + \sum_{l \geq 3} \frac{V_{eff}^{(l)}(r_0)}{l!} (\mathcal{A}(r_0))^{-\frac{l}{4}} x^l \ , \tag{D.14}$$

and, before the normalization procedure by means of the Birkhoff normal form, let us focus on the first order approximation of the Hamiltonian, that is,

$$H(x, y) \sim V_{eff}(r_0) + \sqrt{\mathcal{A}(r_0)} \frac{y^2 + x^2}{2} \ .$$

Before introducing the action angle variables, we can notice that the first order approximation of the action I_1 is given by

$$I_1 = \frac{y^2 + x^2}{2} \ ,$$

thus, we can compute the frequencies ω_1 and ω_2 at order zero in I_1, namely,

$$\omega_{1,0} = \sqrt{\frac{3V'(r_0)}{r_0} + V''(r_0)} \ , \quad \omega_{2,0} = \sqrt{\frac{V'(r_0)}{r_0}} \ .$$

The key quantity that we can compute now is the ratio of the frequency which at order zero can be expressed as

$$\nu_0 = \frac{\omega_{1,0}}{\omega_{2,0}} = \sqrt{3 + \frac{V''(r_0) r_0}{V'(r_0)}} \ .$$

We make the reader notice that if we introduce the function

$$g(r_0) = \frac{V''(r_0)r_0}{V'(r_0)} ,$$

then, the ratio can be expressed as $\nu_0 = \sqrt{3 + g(r_0)}$ and all the computations hugely simplify: it follows that the derivatives of the potential can be rewritten as

$$V^{(l)}(r_0) = \mathcal{F}_l(r_0, g(r_0), \ldots, g^{(l-2)}(r_0))\, V'(r_0),\ l \geq 3 .$$

Let us prove this fact by induction. For $l = 3$, we have

$$V^{(3)}(r_0) = \frac{\mathrm{d}}{\mathrm{d}r_0}\left(\frac{g(r_0)V'(r_0)}{r_0}\right) = \frac{g'(r_0)V'(r_0)}{r_0} - \frac{g(r_0)V'(r_0)}{r_0^2} + \frac{g(r_0)V''(r_0)}{r_0} .$$

Let us now pull the definition of V'' in the last term. We obtain

$$V^{(3)}(r_0) = \left(\frac{g'(r_0)}{r_0} - \frac{g(r_0)}{r_0^2} + \frac{g^2(r_0)}{r_0}\right) V'(r_0) := \mathcal{F}_3(r_0, g(r_0), g'(r_0))\, V'(r_0) .$$

Now, let us suppose that the thesis is verified for $l - 1$ and let us prove it for l. Thus, we consider

$$\begin{aligned} V^{(l)}(r_0) =& \frac{\mathrm{d}}{\mathrm{d}r_0}\left(\mathcal{F}_{l-1}(r_0, g(r_0), g'(r_0), \ldots, g^{(l-3)}(r_0))\, V'(r_0)\right) \\ =& \frac{\mathrm{d}}{\mathrm{d}r_0}(\mathcal{F}_{l-1}(r_0, g(r_0), g'(r_0), \ldots, g^{(l-3)}(r_0)))\, V'(r_0) \\ &+ \mathcal{F}_{l-1}(r_0, g(r_0), g'(r_0), \ldots, g^{(l-3)}(r_0))V''(r_0) , \end{aligned}$$

and, from the definition of V'', we obtain the thesis

$$V^{(l)}(r_0) = \mathcal{F}_l(r_0, g(r_0), g'(r_0), \ldots, g^{(l-2)}(r_0))\, V'(r_0),\ l \geq 3 . \tag{D.15}$$

We can use this fact to rewrite the derivatives of the effective potential. Indeed, by inserting (D.15) in (D.3), we obtain

$$V_{eff}^{(l)}(r_0) = \mathcal{G}_l(r_0, g(r_0), g'(r_0), \ldots, g^{(l-2)}(r_0))\, V'(r_0),\ l \geq 3 .$$

It follows that the Hamiltonian (D.14) takes the form

$$\begin{aligned} H(x, y) = V_{eff}(r_0) &+ \mathcal{G}_2(r_0, g(r_0))\, (V'(r_0))^{\frac{1}{2}}\, \frac{y^2 + x^2}{2} \\ &+ \sum_{l \geq 3} \mathcal{G}_l(r_0, g(r_0), g'(r_0), \ldots, g^{(l-2)}(r_0))\, V'(r_0)^{1-\frac{l}{4}}\, x^l . \end{aligned}$$

We are now ready to perform a Birkhoff normal form in order to get a normalized Hamiltonian $h(I_1, I_2)$ which will depend only on the actions I_1 and I_2 of the form

$$h(I_1, I_2) = h_0(I_2) + h_1(I_2)I_1 + h_2(I_2)I_1^2 + \dots \ .$$

The main point is that the Birkhoff normal form procedure does not change the form of the coefficients $h_0, h_1, \dots$: it contributes only to shift the index in the powers of $V'(r_0)$. Thus, we have

$$\begin{aligned} h_0 &= V_{eff}(r_0) \\ h_1 &= \mathcal{H}_1(r_0, g(r_0))V'(r_0)^{\frac{1}{2}} \\ h_l &= \mathcal{H}_l(r_0, g(r_0), g'(r_0), \dots, g^{2(l-1)}(r_0))V'(r_0)^{1-\frac{l}{2}}, \ l \geq 2 \ . \end{aligned}$$

Starting from the Birkhoff normal form, one can compute the expansion of the frequencies ω_1 and ω_2 in I_1 as a function of the radius r_0. Therefore, we have

$$\omega_1 = \frac{\partial h}{\partial I_1} = \mathcal{H}_1(r_0, g(r_0))V'(r_0)^{\frac{1}{2}} + \sum_{l\geq 2} l\mathcal{H}_l(r_0, g(r_0), g'(r_0), \dots, g^{2(l-1)}(r_0))V'(r_0)^{1-\frac{l}{2}} I_1^{l-1} \ ,$$

thus,

$$\omega_1 = \sum_{l\geq 1} \mathcal{W}_l^1(r_0, g(r_0), g'(r_0), \dots, g^{2(l-1)}(r_0))V'(r_0)^{1-\frac{l}{2}} I_1^{l-1} \ .$$

Analogously, we can prove

$$\omega_2 = \frac{\partial h}{\partial I_2} = \sqrt{\frac{V'(r_0)}{r_0}} + \sum_{l\geq 2} \mathcal{W}_l^2(r_0, g(r_0), g'(r_0), \dots, g^{2l-3}(r_0))V'(r_0)^{1-\frac{l}{2}} I_1^{l-1} \ .$$

Then, one can check that the ratio of the frequency $\nu = \frac{\omega_1}{\omega_2}$ can be expressed as

$$\nu = \sqrt{3 + g(r_0)} + \sum_{l\geq 1} \mathcal{V}_l(r_0, g(r_0), g'(r_0), \dots, g^{2l}(r_0))V'(r_0)^{-\frac{l}{2}} I_1^l \ .$$

By putting this expansion into the Burgers equation, after some trivial computations, one can check that at each order the powers of $V'(r_0)$ can be factor out so that the Burgers equation turns out to be independent from $V'(r_0)$ and it can be solved order-by-order for $g(r_0)$ and its derivatives.

D.3 Domains bounded below by a maximum of the effective potential

We report here some standard computation needed in order to prove Theorem 2.8.2 of Chapter 2.

Let us begin with the following lemma

Lemma D.3.1. *Let us fix $x_1 > 0$ and $\bar{E} > 0$ small enough and consider the curve $\gamma : [-x_1, x_1] \mapsto \mathbb{R}^2$ given by*

$$\gamma = (x, y(x)) \ ,$$

with $y(x) = \sqrt{\frac{2(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda} + x^2}$, where $\mathcal{F}(\bar{E})$ is an analytic function having a zero of order 2 at the origin and $\lambda > 0$ is a positive parameter.
Then,

$$\int_{-x_1}^{x_1} y dx = -\frac{\bar{E} + \mathcal{F}(\bar{E})}{\lambda} \ln \bar{E} + G_1(\bar{E}) \ ,$$

where G_1 is a bounded analytic function for $\bar{E} > 0$.

Proof. Let us rewrite the integral as

$$\mathcal{I} := \int_{-x_1}^{x_1} \sqrt{\frac{2(\bar{E} + \mathcal{F}(\bar{E}))}{\lambda} + x^2 dx} = \int_{-x_1}^{x_1} \sqrt{F(\bar{E})^2 + x^2}\, dx \ , \tag{D.16}$$

where we denote by F the function

$$F(\bar{E}) := \sqrt{\frac{2(\bar{E} + \mathcal{F}(\bar{E}))}{\lambda}}.$$

In order to compute the integral (D.16), we first change the variable of integration to $s := \frac{x}{F(\bar{E})}$ such that the integral takes now the form

$$\mathcal{I} = F(\bar{E}) \int_{x_1}^{x_1} \sqrt{1 + \left(\frac{x}{F(\bar{E})}\right)^2} dx = F(\bar{E})^2 \int_{-s_1}^{s_1} \sqrt{1 + s^2}\, ds \ ,$$

where we denote $s_1 = \frac{x_1}{F(\bar{E})}$.

At this point, we perform standard computation. Indeed, let us begin by considering the integral

$$\mathcal{I}_0 := \int_{-s_1}^{s_1} \sqrt{1 + s^2}\, ds \ .$$

We perform a new change of variables, that is, we introduce a variable ϑ such that $s = \sinh \vartheta$. The integral becomes

$$\mathcal{I}_0 = \int_{-s_1}^{s_1} \sqrt{1 + s^2}\, ds = \int_{-\vartheta_1}^{\vartheta_1} \cosh^2 \vartheta \, d\vartheta = \int_{-\vartheta_1}^{\vartheta_1} \frac{\cosh(2\vartheta) + 1}{2} d\vartheta \ ,$$

where we used the formula $\cosh(2\vartheta) = \cosh^2 \vartheta - 1$ and we denoted $\vartheta_1 = \text{arcsinh} s_1$.

Thus,

$$\mathcal{I}_0 = \frac{\sinh \vartheta_1 \cosh \vartheta_1}{2} - \frac{\sinh(-\vartheta_1) \cosh(-\vartheta_1)}{2} + \vartheta_1 = \sinh \vartheta_1 \cosh \vartheta_1 + \vartheta_1 \ ,$$

where we used the relations $\sinh(-\vartheta) = \sinh\vartheta$ and $\cosh(-\vartheta) = -\cosh\vartheta$. We go back to the original variables and obtain

$$\mathcal{I}_0 = s_1\sqrt{1+s_1^2} + \mathrm{arcsinh} s_1 \ .$$

Thus, we have

$$\begin{aligned}\mathcal{I} &= F(\bar{E})^2\left(\frac{x_1}{F(\bar{E})}\sqrt{1+\left(\frac{x_1}{F(\bar{E})}\right)^2} + \mathrm{arcsinh}\left(\frac{x_1}{F(\bar{E})}\right)\right) \\ &= x_1\sqrt{F(\bar{E})^2 + x_1^2} + F(\bar{E})^2 \ln\left(\frac{x_1}{F(\bar{E})} + \sqrt{1+\left(\frac{x_1}{F(\bar{E})}\right)^2}\right) \\ &= x_1\sqrt{F(\bar{E})^2 + x_1^2} + F(\bar{E})^2 \ln\left(x_1 + \sqrt{F(\bar{E})^2 + x_1^2}\right) - F(\bar{E})^2\ln(F(\bar{E})) \ ,\end{aligned}$$

where we used the identity $\mathrm{arcsinh}x = \ln(x+\sqrt{1+x^2})$.

Finally, from the definition of the function $F(\bar{E})$, we obtain

$$\begin{aligned}\mathcal{I} =& x_1\sqrt{x_1^2 + \frac{2(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}} + \frac{2(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}\ln\left(x_1 + \sqrt{x_1^2+\frac{2(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}}\right) \\ &- \frac{(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}\ln\left(\frac{2(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}\right) \ .\end{aligned} \tag{D.17}$$

We begin to analyze the first term of (D.17).

$$\mathcal{I}_1 := x_1\sqrt{x_1^2 + \frac{2(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}} = x_1^2\sqrt{1 + \frac{2(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda x_1^2}} = x_1^2\sqrt{1+\tilde{f}(\bar{E})} \ ,$$

where we denoted $\tilde{f}(\bar{E}) = \frac{2(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda x_1^2}$. At this point, since $\tilde{f}(\bar{E}) \mapsto 0$ for $\bar{E} \mapsto 0$ and x_1 is fixed, we can develop this term obtaining

$$\mathcal{I}_1 = x_1^2\sum_{n\geq 0}\binom{1/2}{n}\tilde{f}(\bar{E})^n = x_1^2\sum_{n\geq 0} c_n(\bar{E}+\mathcal{F}(\bar{E}))^n \ ,$$

where $c_n := \binom{1/2}{n}\frac{2^n}{\lambda^n x_1^{2n}}$.

Furthermore, since $\mathcal{F}(\bar{E})$ is an analytic function having a zero of order 2 at the origin, we can exploit its expansion and obtain that the integral $\mathcal{I}_1$ can be expressed as

$$\mathcal{I}_1 = x_1^2\sum_{n\geq 0} b_n\bar{E}^n \ ,$$

with suitable coefficients b_n.

Analogously, we analyze the second term of (D.17), that is,

$$\begin{aligned}\mathcal{I}_2 &= \frac{2(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}\ln\left(x_1+\sqrt{x_1^2+\frac{2(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}}\right)\\ &= \frac{2(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}\ln\left(2x_1+\sum_{n\geq 1}b_n\bar{E}^n\right)\\ &= \frac{2(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}\ln(2x_1)+\frac{2(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}\ln\left(1+\sum_{n\geq 1}b_n\bar{E}^n\right)\ .\end{aligned}$$

By exploiting the analyticity of $\mathcal{F}(\bar{E})$, we can rewrite it as

$$\mathcal{I}_2 = \sum_{n\geq 2} a_n\bar{E}^n\ ,$$

where a_n are suitable coefficients. Thus,

$$\mathcal{I}_1+\mathcal{I}_2 = x_1^2+\sum_{n\geq 1}C_n\bar{E}^n\ .$$

We conclude by analyzing the third term. Let us begin by rewriting it as follows

$$\mathcal{I}_3 = -\frac{(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}\ln\left(\bar{E}\left(\frac{2}{\lambda}+\frac{\mathcal{F}(\bar{E})}{\lambda\bar{E}}\right)\right)\ ,$$

that is,

$$\mathcal{I}_3 = -\frac{(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}\ln\bar{E}-\frac{(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}\ln\left(\frac{2}{\lambda}+\frac{\mathcal{F}(\bar{E})}{\lambda\bar{E}}\right)\ .$$

We analyze the second term

$$\mathcal{I}_{3,2} = -\frac{(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}\ln\left(\frac{2}{\lambda}\right)-\frac{(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}\ln\left(1+\frac{\mathcal{F}(\bar{E})}{2\bar{E}}\right)\ .$$

The first term is an analytic function of $\bar{E}$, thus it can be expanded at the origin

$$\mathcal{I}_{3,2} = \sum_{n\geq 1}d_n\bar{E}^n-\frac{(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}\sum_{n\geq 0}\frac{(-1)^{n-1}}{n}\left(\frac{\mathcal{F}(\bar{E})}{2\bar{E}}\right)^n\ .$$

Since $\frac{\mathcal{F}(\bar{E})}{2\bar{E}}$ is an analytic function having a zero of order 1 at the origin, then it can be expanded at zero in order to obtain

$$\mathcal{I}_{3,2} = \sum_{n\geq 1}d_n\bar{E}^n-\frac{(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}\sum_{n\geq 1}\tilde{d}_n\bar{E}^n\ ,$$

that is,

$$\mathcal{I}_{3,2} = \sum_{n\geq 1} D_n \bar{E}^n \ .$$

Finally we have that the integral $\mathcal{I}$ takes the form

$$\begin{aligned}\mathcal{I} &= -\frac{(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}\ln\bar{E} + x_1^2 + \sum_{n\geq 1}\tilde{c}_n\bar{E}^n \\ &:= -\frac{(\bar{E}+\mathcal{F}(\bar{E}))}{\lambda}\ln(\bar{E}) + G_1(\bar{E}) \ ,\end{aligned}$$

where $G_1(\bar{E}) := x_1^2 + \sum_{n\geq 1}\tilde{c}_n\bar{E}^n$, with $\tilde{c}_n := C_n + D_n$, is a bounded analytic function for $\bar{E} > 0$.

This concludes the proof. □

Proof of formula (2.57) *of Chapter 2.* We remind the reader the expression for the Arnol'd determinant, that is,

$$\mathcal{D} = -\frac{\partial^2 h}{\partial I_1^2}\omega_2^2 + 2\frac{\partial^2 h}{\partial I_1 \partial I_2}\omega_1\omega_2 - \frac{\partial^2 h}{\partial I_2^2}\omega_1^2 \ . \tag{D.18}$$

After having rewritten the three terms of the Arnol'd determinant as functions of $\mathcal{W}_1 = \left(\frac{\partial G}{\partial \bar{E}}\right)^{-1}$, $\mathcal{W}_2 = -\frac{\partial G}{\partial I_2}\mathcal{W}_1 + \frac{\partial V_0}{\partial I_2}$, $\bar{E}$ and I_2, namely, after having computed

$$\mathcal{D}_1 = -\mathcal{W}_1\mathcal{W}_2^2\frac{\partial \mathcal{W}_1}{\partial \bar{E}} \ , \tag{D.19}$$

$$\mathcal{D}_2 = 2\mathcal{W}_1\mathcal{W}_2\frac{\mathrm{d}\mathcal{W}_1}{\mathrm{d}I_2} \ , \tag{D.20}$$

$$\mathcal{D}_3 = \mathcal{W}_1^2\frac{\partial G}{\partial I_2}\frac{\mathrm{d}\mathcal{W}_1}{\mathrm{d}I_2} + \mathcal{W}_1^3\frac{\mathrm{d}}{\mathrm{d}I_2}\left(\frac{\partial G}{\partial I_2}\right) - \mathcal{W}_1^2\frac{\partial^2 V_0}{\partial I_2^2} \ , \tag{D.21}$$

we can collect them together in (D.18): we obtain

$$\mathcal{D} = -\mathcal{W}_1\mathcal{W}_2^2\frac{\partial \mathcal{W}_1}{\partial \bar{E}} + \mathcal{W}_1\left(2\mathcal{W}_2 + \mathcal{W}_1\frac{\partial G}{\partial I_2}\right)\frac{\mathrm{d}\mathcal{W}_1}{\mathrm{d}I_2} + \mathcal{W}_1^3\frac{\mathrm{d}}{\mathrm{d}I_2}\left(\frac{\partial G}{\partial I_2}\right) - \mathcal{W}_1^2\frac{\partial^2 V_0}{\partial I_2^2} \ .$$

Furthermore, from Lemma 2.8.2 of Chapter 2, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}I_2}\left(\frac{\partial G}{\partial I_2}\right) = \frac{\partial^2 G}{\partial I_2 \partial \bar{E}}\left(\mathcal{W}_2 - \frac{\partial V_0}{\partial I_2}\right) + \frac{\partial^2 G}{\partial I_2^2} \ ,$$

and, from the definition of $\mathcal{W}_1$, we have

$$\frac{\mathrm{d}}{\mathrm{d}I_2}\left(\frac{\partial G}{\partial I_2}\right) = -\frac{1}{\mathcal{W}_1^2}\frac{\partial \mathcal{W}_1}{\partial I_2}\left(\mathcal{W}_2 - \frac{\partial V_0}{\partial I_2}\right) + \frac{\partial^2 G}{\partial I_2^2} \ .$$

Thus, the determinant can be rewritten as

$$\mathcal{D} = -\mathcal{W}_1\mathcal{W}_2^2\frac{\partial \mathcal{W}_1}{\partial \bar{E}} + \mathcal{W}_1\left(\mathcal{W}_2 + \frac{\partial V_0}{\partial I_2}\right)\frac{\mathrm{d}\mathcal{W}_1}{\mathrm{d}I_2} - \mathcal{W}_1\frac{\partial \mathcal{W}_1}{\partial I_2}\left(\mathcal{W}_2 - \frac{\partial V_0}{\partial I_2}\right) + \mathcal{W}_1^3\frac{\partial^2 G}{\partial I_2^2} - \mathcal{W}_1^2\frac{\partial^2 V_0}{\partial I_2^2} \ .$$

Moreover, by exploiting again Lemma 2.8.2 of Chapter 2, we have

$$\frac{\mathrm{d}\mathcal{W}_1}{\mathrm{d}I_2} = \frac{\partial \mathcal{W}_1}{\partial \bar{E}}\left(\mathcal{W}_2 - \frac{\partial V_0}{\partial I_2}\right) + \frac{\partial \mathcal{W}_1}{\partial I_2} \ ,$$

and,

$$\mathcal{D} = -\mathcal{W}_1\mathcal{W}_2^2\frac{\partial \mathcal{W}_1}{\partial \bar{E}} + \mathcal{W}_1\left(\mathcal{W}_2 + \frac{\partial V_0}{\partial I_2}\right)\left(\frac{\partial \mathcal{W}_1}{\partial \bar{E}}\left(\mathcal{W}_2 - \frac{\partial V_0}{\partial I_2}\right) + \frac{\partial \mathcal{W}_1}{\partial I_2}\right) + - \mathcal{W}_1\frac{\partial \mathcal{W}_1}{\partial I_2}\left(\mathcal{W}_2 - \frac{\partial V_0}{\partial I_2}\right) + \mathcal{W}_1^3\frac{\partial^2 G}{\partial I_2^2} - \mathcal{W}_1^2\frac{\partial^2 V_0}{\partial I_2^2} \ ,$$

that is,

$$\mathcal{D} = -\mathcal{W}_1\mathcal{W}_2^2\frac{\partial \mathcal{W}_1}{\partial \bar{E}} + \mathcal{W}_1\frac{\partial \mathcal{W}_1}{\partial \bar{E}}\left(\mathcal{W}_2^2 - \left(\frac{\partial V_0}{\partial I_2}\right)^2\right) + \mathcal{W}_1\frac{\partial \mathcal{W}_1}{\partial I_2}\left(\mathcal{W}_2 + \frac{\partial V_0}{\partial I_2}\right) - \mathcal{W}_1\frac{\partial \mathcal{W}_1}{\partial I_2}\left(\mathcal{W}_2 - \frac{\partial V_0}{\partial I_2}\right) + \mathcal{W}_1^3\frac{\partial^2 G}{\partial I_2^2} - \mathcal{W}_1^2\frac{\partial^2 V_0}{\partial I_2^2} \ .$$

We conclude that the Arnol'd determinant takes the form

$$\mathcal{D} = -\mathcal{W}_1\frac{\partial \mathcal{W}_1}{\partial \bar{E}}\left(\frac{\partial V_0}{\partial I_2}\right)^2 + 2\mathcal{W}_1\frac{\partial \mathcal{W}_1}{\partial I_2}\frac{\partial V_0}{\partial I_2} + \mathcal{W}_1^3\frac{\partial^2 G}{\partial I_2^2} - \mathcal{W}_1^2\frac{\partial^2 V_0}{\partial I_2^2} \ .$$

□

Appendix E

Mathematica^{TM} computation

Some useful functions

```
mySeries[f_, r0_, Nord_] := Module[{htmp, i, β},
  htmp = Expand[Normal[Series[f, {r, r0, Nord}]] /. r - r0 → r /.
      {r → r β, pr → pr β} /. β^k :→ 0 /; k > Nord];
  For[i = 0, i ≤ Nord, i++,
   H[0, i] = Coefficient[htmp, β, i]
  ];
  For[i = 0, i ≤ Nord, i++,
   Print["H[0,", i, "] = ", H[0, i]];
  ]
 ]
```

```
myPoissonBracket[f_, g_] :=
 Module[{}, Expand[Factor[D[f, ξ] D[g, η] - D[f, η] D[g, ξ]]]]
```

```
myKernelRange[f_] :=
  Module[{Zf, Rf, htmp, listamonomi, ntermini, i, x, den, m, n },
   htmp = Expand[Factor[f]];
   listamonomi = CoefficientRules[htmp, {ξ, η}];
   ntermini = Length[listamonomi];
   Zf = 0;
   Rf = 0;
   For[i = 1, i ≤ ntermini, i++,
    m = listamonomi[[i]][[1]][[1]];
    n = listamonomi[[i]][[1]][[2]];
    den = m - n;
    If[den == 0,
     Zf += FromCoefficientRules[{listamonomi[[i]]}, {ξ, η}],
     Rf += FromCoefficientRules[{listamonomi[[i]]}, {ξ, η}],
     Print["Houston, we've had a problem here!"]
    ]];
   {Expand[Factor[Zf]], Expand[Factor[Rf]]}
  ];
```

```
myHomologicalEquation[ω_, f_] :=
  Module[{htmp, listamonomi, ntermini, i, χ, den, m, n},
   htmp = Expand[Factor[f]];
   listamonomi = CoefficientRules[htmp, {ξ, η}];
   ntermini = Length[listamonomi];
   χ = 0;
   For[i = 1, i ≤ ntermini, i++,
    m = listamonomi[[i]][[1]][[1]];
    n = listamonomi[[i]][[1]][[2]];
    den = m - n;
    If[den == 0,
     Print["Houston, we've had a problem here!"],
     χ += (1 / (I den ω)) FromCoefficientRules[{listamonomi[[i]]}, {ξ, η}],
     Print["Houston, we've had a problem here!"]
    ]];
   Expand[Factor[χ]]
];
```

Birkhoff normal form for the Hamiltonian of the planar central motion at a minimum

```
ORDMAX = 6;
ORDBIRK = ORDMAX;
```

```
Hiniz = 1 / 2 (pr^2 + pθ^2 / r^2) + V[r]
```

$$\frac{1}{2}\left(pr^2 + \frac{p\theta^2}{r^2}\right) + V[r]$$

- **Expansion of the Hamiltonian Hiniz at the minimum r0**

```
htmp = mySeries[Hiniz, r0, ORDMAX];
```

$$H[0,0] = \frac{p\theta^2}{2\, r0^2} + V[r0]$$

$$H[0,1] = -\frac{p\theta^2\, r}{r0^3} + r\, V'[r0]$$

$$H[0,2] = \frac{pr^2}{2} + \frac{3\, p\theta^2\, r^2}{2\, r0^4} + \frac{1}{2}\, r^2\, V''[r0]$$

$$H[0,3] = -\frac{2\, p\theta^2\, r^3}{r0^5} + \frac{1}{6}\, r^3\, V^{(3)}[r0]$$

$$H[0,4] = \frac{5\, p\theta^2\, r^4}{2\, r0^6} + \frac{1}{24}\, r^4\, V^{(4)}[r0]$$

$$H[0,5] = -\frac{3\, p\theta^2\, r^5}{r0^7} + \frac{1}{120}\, r^5\, V^{(5)}[r0]$$

$$H[0,6] = \frac{7\, p\theta^2\, r^6}{2\, r0^8} + \frac{1}{720}\, r^6\, V^{(6)}[r0]$$

```
Hsvil =
 Sum[1 / i! * Simplify[Coefficient[i! * H[0, i], r, i]] * r^i, { i, 0, 6}] + pr^2/2
```

$$\frac{pr^2}{2} + \frac{p\theta^2}{2\,r0^2} + V[r0] + r\left(-\frac{p\theta^2}{r0^3} + V'[r0]\right) + \frac{1}{2}r^2\left(\frac{3\,p\theta^2}{r0^4} + V''[r0]\right) + \frac{1}{6}r^3\left(-\frac{12\,p\theta^2}{r0^5} + V^{(3)}[r0]\right) + \frac{1}{24}r^4\left(\frac{60\,p\theta^2}{r0^6} + V^{(4)}[r0]\right) + \frac{1}{120}r^5\left(-\frac{360\,p\theta^2}{r0^7} + V^{(5)}[r0]\right) + \frac{1}{720}r^6\left(\frac{2520\,p\theta^2}{r0^8} + V^{(6)}[r0]\right)$$

- **Assumptions on the central potential**

```
H[0, 1] = 0;
Veff2 = Simplify[2 * Coefficient[H[0, 2], r, 2]];
$Assumptions = {Veff2 > 0, r0 > 0, pθ == Sqrt[r0^3 V'[r0]], pθ > 0, 3 + g[r0] > 0};
```

- **Introduction of the complex variables ξ, η**

```
cfdiag = Power[Veff2, 1 / 4];

rulediag = {pr → cfdiag PR, r → R / cfdiag};
For[i = 0, i ≤ ORDMAX, i++,
 HD[0, i] = Factor[H[0, i] /. rulediag];
]

rulecomplex = {PR → 1 / Sqrt[2] (ξ + I η), R → I / Sqrt[2] (ξ - I η)};
For[i = 0, i ≤ ORDMAX, i++,
 HC[0, i] = Expand[Factor[HD[0, i] /. rulecomplex]]
]

ω = CoefficientRules[HC[0, 2], {ξ, η}][[1]][[2]] / I;
```

- **Birkhoff normal form algorithm**

```
For[i = ORDMAX + 1, i ≤ ORDBIRK, i++, HC[0, i] = 0];
```

```
For[i = 3, i ≤ ORDBIRK, i++,
 {ZHC[i - 2, i], RHC[i - 2, i]} = myKernelRange[HC[i - 3, i]];
 X[i] = myHomologicalEquation[ω, RHC[i - 2, i]];

 For[j = 0, j ≤ ORDBIRK, j++,
  HC[i - 2, j] = Expand[Factor[HC[i - 3, j]]]
 ];

 For[j = 0, j ≤ ORDBIRK, j++,
  tmp = HC[i - 3, j];
  For[l = 1, l ≤ IntegerPart[(ORDBIRK - j) / (i - 2)], l++,
   tmp = myPoissonBracket[tmp, X[i]];
   HC[i - 2, j + l (i - 2)] += tmp;
   HC[i - 2, j + l (i - 2)] = Expand[Factor[HC[i - 2, j + l ( i - 2)]]];
   tmp = Expand[Factor[tmp / (l + 1)]];
  ]
 ]
]

pteta = Sqrt[r0^3 V'[r0]];
d1 = 1 / D[pteta, r0];

rulact = {ξ → Sqrt[I1], η → -I Sqrt[I1]};
Haz[0] = HC[ORDBIRK - 2, 0];
Haz[1] = HC[ORDBIRK - 2, 2] /. rulact;
Haz[2] = HC[ORDBIRK - 2, 4] /. rulact;
Haz[3] = HC[ORDBIRK - 2, 6] /. rulact;

Ham = Haz[0] + Haz[1] + Haz[2] + Haz[3];

F[r0_] := r0 V''[r0] / V'[r0]
g0 = F[r0];
g1 = D[F[r0], {r0, 1}];
g2 = D[F[r0], {r0, 2}];
g3 = D[F[r0], {r0, 3}];
g4 = D[F[r0], {r0, 4}];
```

We express the function V[r0] and its derivatives in terms of g and its derivatives

```
s0 = Flatten[Solve[g[r0] == g0, D[V[r0], {r0, 2}]]]
```

$$\left\{V''[r0] \to \frac{g[r0]\, V'[r0]}{r0}\right\}$$

```
s1 = Flatten[Solve[D[g[r0], {r0, 1}] == g1, D[V[r0], {r0, 3}]]]];
S1 = Flatten[Simplify[s1 /. s0]]
```

$$\left\{V^{(3)}[r0] \to \frac{\left(-g[r0] + g[r0]^2 + r0\, g'[r0]\right) V'[r0]}{r0^2}\right\}$$

```
s2 = Flatten[Solve[D[g[r0], {r0, 2}] == g2, D[V[r0], {r0, 4}]]];
S2 = Flatten[Simplify[s2 /. s1 /. s0]]
```

$$\left\{V^{(4)}[r0] \to \frac{1}{r0^3} V'[r0] \left(-3\, g[r0]^2 + g[r0]^3 + g[r0]\,(2 + 3\, r0\, g'[r0]) + r0\,(-2\, g'[r0] + r0\, g''[r0])\right)\right\}$$

```
s3 = Flatten[Solve[D[g[r0], {r0, 3}] == g3, D[V[r0], {r0, 5}]]];
S3 = Flatten[Simplify[s3 /. s2 /. s1 /. s0]]
```

$$\left\{V^{(5)}[r0] \to \frac{1}{r0^4} V'[r0] \left(-6\, g[r0]^3 + g[r0]^4 + g[r0]^2\,(11 + 6\, r0\, g'[r0]) + 2\, g[r0] \left(-3 - 7\, r0\, g'[r0] + 2\, r0^2\, g''[r0]\right) + r0 \left(6\, g'[r0] + 3\, r0\, g'[r0]^2 + r0 \left(-3\, g''[r0] + r0\, g^{(3)}[r0]\right)\right)\right)\right\}$$

```
s4 = Flatten[Solve[D[g[r0], {r0, 4}] == g4, D[V[r0], {r0, 6}]]];
S4 = Flatten[Simplify[s4 /. s3 /. s2 /. s1 /. s0]]
```

$$\left\{V^{(6)}[r0] \to \frac{1}{r0^5} V'[r0] \left(-10\, g[r0]^4 + g[r0]^5 + 5\, g[r0]^3\,(7 + 2\, r0\, g'[r0]) + 10\, g[r0]^2 \left(-5 - 5\, r0\, g'[r0] + r0^2\, g''[r0]\right) + g[r0] \left(24 + 70\, r0\, g'[r0] + 15\, r0^2\, g'[r0]^2 - 25\, r0^2\, g''[r0] + 5\, r0^3\, g^{(3)}[r0]\right) + r0 \left(-20\, r0\, g'[r0]^2 + 2\, g'[r0] \left(-12 + 5\, r0^2\, g''[r0]\right) + r0 \left(12\, g''[r0] - 4\, r0\, g^{(3)}[r0] + r0^2\, g^{(4)}[r0]\right)\right)\right)\right\}$$

```
d2 = Simplify[d1 /. s0];
```

We rewrite the Hamiltonian in BNF in terms of V′[r0], g[r0] and its derivatives

```
Hbirk = Factor[Expand[Ham /. S4 /. S3 /. S2 /. S1 /. s0 /. pθ → pteta]];
CoeffHbirk = CoefficientList[Hbirk, I1];
Hbirk0 = Simplify[CoeffHbirk[[1]]]
```

$$V[r0] + \frac{1}{2}\, r0\, V'[r0]$$

```
Hbirk1 = Simplify[CoeffHbirk[[2]]]
```

$$\sqrt{\frac{(3 + g[r0])\, V'[r0]}{r0}}$$

```
Hbirk2 = Simplify[CoeffHbirk[[3]]]
```

$$\left(-180 + 10\, g[r0]^3 - 2\, g[r0]^4 + 102\, r0\, g'[r0] - 5\, r0^2\, g'[r0]^2 + g[r0]^2\,(94 - r0\, g'[r0]) + 9\, r0^2\, g''[r0] + g[r0] \left(78 + 31\, r0\, g'[r0] + 3\, r0^2\, g''[r0]\right)\right) \Big/ \left(48\, r0^2\, (3 + g[r0])^2\right)$$

```
Hbirk3 = Simplify[CoeffHbirk[[4]]]
```

$$\frac{1}{6912\, r0^{7/2}\, (3+g[r0])^{9/2}\, \sqrt{V'[r0]}}$$

$$\begin{aligned}
&\Big(184\, g[r0]^7 + 20\, g[r0]^8 + 4044\, r0^3\, g'[r0]^3 - 235\, r0^4\, g'[r0]^4 + \\
&\quad 4\, g[r0]^6\, (279 + 17\, r0\, g'[r0]) + g[r0]^5\, \big(4000 - 72\, r0\, g'[r0] - 84\, r0^2\, g''[r0]\big) + \\
&\quad 54\, r0^2\, g'[r0]^2\, \big(-494 + 25\, r0^2\, g''[r0]\big) - \\
&\quad 108\, r0\, g'[r0]\, \big(-364 + 181\, r0^2\, g''[r0] + 14\, r0^3\, g^{(3)}[r0]\big) - \\
&\quad g[r0]^4\, \big(6820 + 3504\, r0\, g'[r0] - 129\, r0^2\, g'[r0]^2 + 1152\, r0^2\, g''[r0] + 48\, r0^3\, g^{(3)}[r0]\big) + \\
&\quad 27\, \big(8880 + 3432\, r0^2\, g''[r0] - 17\, r0^4\, g''[r0]^2 + 576\, r0^3\, g^{(3)}[r0] + 24\, r0^4\, g^{(4)}[r0]\big) - \\
&\quad g[r0]^2\, \big(1515\, r0^2\, g'[r0]^2 + 94\, r0^3\, g'[r0]^3 + \\
&\qquad 12\, r0\, g'[r0]\, \big(-1389 + 100\, r0^2\, g''[r0] + 14\, r0^3\, g^{(3)}[r0]\big) + 3\, \big(58356 - \\
&\qquad\quad 5976\, r0^2\, g''[r0] + 17\, r0^4\, g''[r0]^2 - 1296\, r0^3\, g^{(3)}[r0] - 72\, r0^4\, g^{(4)}[r0]\big)\big) + \\
&\quad 2\, g[r0]^3\, \big(411\, r0^2\, g'[r0]^2 + r0\, g'[r0]\, \big(-3916 + 81\, r0^2\, g''[r0]\big) + \\
&\qquad 12\, \big(-3695 - 100\, r0^2\, g''[r0] + 6\, r0^3\, g^{(3)}[r0] + r0^4\, g^{(4)}[r0]\big)\big) + \\
&\quad 2\, g[r0]\, \big(533\, r0^3\, g'[r0]^3 + 9\, r0^2\, g'[r0]^2\, \big(-964 + 25\, r0^2\, g''[r0]\big) - \\
&\qquad 9\, r0\, g'[r0]\, \big(-3408 + 643\, r0^2\, g''[r0] + 56\, r0^3\, g^{(3)}[r0]\big) + \\
&\qquad 9\, \big(1416 + 4554\, r0^2\, g''[r0] - 17\, r0^4\, g''[r0]^2 + 792\, r0^3\, g^{(3)}[r0] + 36\, r0^4\, g^{(4)}[r0]\big)\big)\Big)
\end{aligned}$$

```
HBIRK = Hbirk0 + Hbirk1 * I1 + Hbirk2 * I1^2 + Hbirk3 * I1^3;
```

The Burgers equation

- **The expansion of the two frequencies $\omega 1$ and $\omega 2$**

```
$Assumptions = {V'[r0] > 0, r0 > 0};
```

The frequency $\omega 1$

```
ω1 = D[HBIRK, I1];
```

```
Coeffω1 = CoefficientList[ω1, I1];
```

```
ω10 = Simplify[Coeffω1[[1]]]
```

$$\sqrt{\frac{(3+g[r0])\, V'[r0]}{r0}}$$

```
ω11 = Simplify[Coeffω1[[2]]]
```

$$\begin{aligned}
&\big(-180 + 10\, g[r0]^3 - 2\, g[r0]^4 + 102\, r0\, g'[r0] - 5\, r0^2\, g'[r0]^2 + g[r0]^2\, (94 - r0\, g'[r0]) + \\
&\quad 9\, r0^2\, g''[r0] + g[r0]\, \big(78 + 31\, r0\, g'[r0] + 3\, r0^2\, g''[r0]\big)\big) \Big/ \big(24\, r0^2\, (3 + g[r0])^2\big)
\end{aligned}$$

```
ω12 = Simplify[Coeffω1[[3]]]
```

$$\begin{aligned}&\frac{1}{2304\, r0^{7/2}\, (3+g[r0])^{9/2} \sqrt{V'[r0]}}\\
&\Big(184\, g[r0]^7 + 20\, g[r0]^8 + 4044\, r0^3\, g'[r0]^3 - 235\, r0^4\, g'[r0]^4 +\\
&\quad 4\, g[r0]^6\, (279 + 17\, r0\, g'[r0]) + g[r0]^5\, \big(4000 - 72\, r0\, g'[r0] - 84\, r0^2\, g''[r0]\big) +\\
&\quad 54\, r0^2\, g'[r0]^2\, \big(-494 + 25\, r0^2\, g''[r0]\big) -\\
&\quad 108\, r0\, g'[r0]\, \big(-364 + 181\, r0^2\, g''[r0] + 14\, r0^3\, g^{(3)}[r0]\big) -\\
&\quad g[r0]^4\, \big(6820 + 3504\, r0\, g'[r0] - 129\, r0^2\, g'[r0]^2 + 1152\, r0^2\, g''[r0] + 48\, r0^3\, g^{(3)}[r0]\big) +\\
&\quad 27\, \big(8880 + 3432\, r0^2\, g''[r0] - 17\, r0^4\, g''[r0]^2 + 576\, r0^3\, g^{(3)}[r0] + 24\, r0^4\, g^{(4)}[r0]\big) -\\
&\quad g[r0]^2\, \big(1515\, r0^2\, g'[r0]^2 + 94\, r0^3\, g'[r0]^3 +\\
&\qquad 12\, r0\, g'[r0]\, \big(-1389 + 100\, r0^2\, g''[r0] + 14\, r0^3\, g^{(3)}[r0]\big) + 3\, \big(58356 -\\
&\qquad\quad 5976\, r0^2\, g''[r0] + 17\, r0^4\, g''[r0]^2 - 1296\, r0^3\, g^{(3)}[r0] - 72\, r0^4\, g^{(4)}[r0]\big)\big) +\\
&\quad 2\, g[r0]^3\, \big(411\, r0^2\, g'[r0]^2 + r0\, g'[r0]\, \big(-3916 + 81\, r0^2\, g''[r0]\big) +\\
&\qquad 12\, \big(-3695 - 100\, r0^2\, g''[r0] + 6\, r0^3\, g^{(3)}[r0] + r0^4\, g^{(4)}[r0]\big)\big) +\\
&\quad 2\, g[r0]\, \big(533\, r0^3\, g'[r0]^3 + 9\, r0^2\, g'[r0]^2\, \big(-964 + 25\, r0^2\, g''[r0]\big) -\\
&\qquad 9\, r0\, g'[r0]\, \big(-3408 + 643\, r0^2\, g''[r0] + 56\, r0^3\, g^{(3)}[r0]\big) +\\
&\qquad 9\, \big(1416 + 4554\, r0^2\, g''[r0] - 17\, r0^4\, g''[r0]^2 + 792\, r0^3\, g^{(3)}[r0] + 36\, r0^4\, g^{(4)}[r0]\big)\big)\Big)\end{aligned}$$

The frequency $\omega 2$

```
D2 = FullSimplify[d2]
```

$$\frac{2}{(3+g[r0])\sqrt{r0\, V'[r0]}}$$

```
ω2 = Simplify[D[HBIRK, r0] * D2 /. s0];
```

```
Coeffω2 = CoefficientList[ω2, I1];
```

```
ω20 = Simplify[Coeffω2[[1]]]
```

$$\sqrt{\frac{V'[r0]}{r0}}$$

```
ω21 = Simplify[Coeffω2[[2]]]
```

$$\frac{-3 + 2\, g[r0] + g[r0]^2 + r0\, g'[r0]}{r0^2\, (3+g[r0])^{3/2}}$$

```
ω22 = Simplify[Coeffω2[[3]]]
```

$$\begin{aligned}&\frac{1}{24\, r0^{7/2}\, (3+g[r0])^4 \sqrt{V'[r0]}}\\
&\Big(4\, g[r0]^5 - 111\, r0^2\, g'[r0]^2 + 10\, r0^3\, g'[r0]^3 - 4\, g[r0]^4\, (2 + r0\, g'[r0]) -\\
&\quad g[r0]^3\, \big(248 + 13\, r0\, g'[r0] + r0^2\, g''[r0]\big) + g'[r0]\, \big(288\, r0 - 39\, r0^3\, g''[r0]\big) +\\
&\quad g[r0]^2\, \big(-720 + 62\, r0\, g'[r0] + 28\, r0^2\, g''[r0] + 3\, r0^3\, g^{(3)}[r0]\big) +\\
&\quad 9\, \big(120 + 34\, r0^2\, g''[r0] + 3\, r0^3\, g^{(3)}[r0]\big) + g[r0]\, \big(-37\, r0^2\, g'[r0]^2 +\\
&\qquad g'[r0]\, \big(291\, r0 - 13\, r0^3\, g''[r0]\big) + 3\, \big(-36 + 65\, r0^2\, g''[r0] + 6\, r0^3\, g^{(3)}[r0]\big)\big)\Big)\end{aligned}$$

```
ω23 = Simplify[Coeffω2[[4]]];
```

- **The expansion of the ratio of the frequencies ν=ω1/ω2**

```
Invω20 = Simplify[1 / ω20];

W1 = Simplify[ω1 * Invω20];

w20 = 1;

w21 = Simplify[ω21 * Invω20];

w21
```

$$\frac{-3+2\,g[r0]+g[r0]^2+r0\,g'[r0]}{(r0\,(3+g[r0]))^{3/2}\sqrt{V'[r0]}}$$

```
w22 = Simplify[ω22 * Invω20];

w22
```

$$\frac{1}{24\,r0^3\,(3+g[r0])^4\,V'[r0]}\Big(4\,g[r0]^5-111\,r0^2\,g'[r0]^2+10\,r0^3\,g'[r0]^3-4\,g[r0]^4\,(2+r0\,g'[r0])-g[r0]^3\left(248+13\,r0\,g'[r0]+r0^2\,g''[r0]\right)+g'[r0]\left(288\,r0-39\,r0^3\,g''[r0]\right)+g[r0]^2\left(-720+62\,r0\,g'[r0]+28\,r0^2\,g''[r0]+3\,r0^3\,g^{(3)}[r0]\right)+9\left(120+34\,r0^2\,g''[r0]+3\,r0^3\,g^{(3)}[r0]\right)+g[r0]\left(-37\,r0^2\,g'[r0]^2+g'[r0]\left(291\,r0-13\,r0^3\,g''[r0]\right)+3\left(-36+65\,r0^2\,g''[r0]+6\,r0^3\,g^{(3)}[r0]\right)\right)\Big)$$

```
w23 = Simplify[ω23 * Invω20];

Omega2 = Simplify[w20 + w21 * I1^1 + w22 * I1^2 + w23 * I1^3];

W2 = Series[1 / Omega2, {I1, 0, 3}];

ν = Simplify[W1 * W2];
```

- **The Burgers equation**

```
νSecon = Simplify[ν * D[ν, r0] * D2];

νPrim = Simplify[D[ν, I1]];

Burg = νPrim - νSecon;
```

We extract the order 0 (burg0) and the order 1 (burg1) of the Burgers equation

```
CoeffBurg = CoefficientList[Burg, I1];

burg0 = Simplify[Expand[CoeffBurg[[1]]]]
```

$$-\left(14\,g[r0]^3+2\,g[r0]^4+42\,r0\,g'[r0]+5\,r0^2\,g'[r0]^2+g[r0]^2\,(26+r0\,g'[r0])+g[r0]\left(-6+17\,r0\,g'[r0]-3\,r0^2\,g''[r0]\right)-9\left(4+r0^2\,g''[r0]\right)\right)\Big/\left(24\,r0^{3/2}\,(3+g[r0])^2\sqrt{V'[r0]}\right)$$

```
burg1 = Simplify[CoeffBurg[[2]]] /. s0
```

```
               1
------------------------------------ (-48 g[r0] (3 + g[r0])^2 V'[r0]
1152 r0^3 (3 + g[r0])^(9/2) V'[r0]^2
      (14 g[r0]^3 + 2 g[r0]^4 - 30 r0 g'[r0] + 5 r0^2 g'[r0]^2 + g[r0]^2 (26 + r0 g'[r0]) -
        9 (4 + r0^2 g''[r0]) - g[r0] (6 + 7 r0 g'[r0] + 3 r0^2 g''[r0])) +
    V'[r0] (376 g[r0]^7 + 20 g[r0]^8 + 444 r0^3 g'[r0]^3 - 235 r0^4 g'[r0]^4 +
        4 g[r0]^6 (687 + 17 r0 g'[r0]) + g[r0]^5 (9664 + 1080 r0 g'[r0] - 84 r0^2 g''[r0]) +
        54 r0^2 g'[r0]^2 (26 + 25 r0^2 g''[r0]) + g[r0]^4
         (15068 + 6336 r0 g'[r0] + 129 r0^2 g'[r0]^2 - 1248 r0^2 g''[r0] - 48 r0^3 g^(3)[r0]) -
        108 r0 g'[r0] (44 + 9 r0^2 g''[r0] + 14 r0^3 g^(3)[r0]) -
        27 (-240 + 648 r0^2 g''[r0] + 17 r0^4 g''[r0]^2 - 24 r0^4 g^(4)[r0]) -
        g[r0]^2 (-5349 r0^2 g'[r0]^2 + 94 r0^3 g'[r0]^3 +
           12 r0 g'[r0] (-1569 - 72 r0^2 g''[r0] + 14 r0^3 g^(3)[r0]) + 3 (7380 +
              7128 r0^2 g''[r0] + 17 r0^4 g''[r0]^2 + 432 r0^3 g^(3)[r0] - 72 r0^4 g^(4)[r0])) +
        2 g[r0] (-67 r0^3 g'[r0]^3 + 9 r0^2 g'[r0]^2 (388 + 25 r0^2 g''[r0]) -
           9 r0 g'[r0] (-168 - 45 r0^2 g''[r0] + 56 r0^3 g^(3)[r0]) -
           9 (744 + 1710 r0^2 g''[r0] + 17 r0^4 g''[r0]^2 + 72 r0^3 g^(3)[r0] - 36 r0^4 g^(4)[r0])) +
        2 g[r0]^3 (723 r0^2 g'[r0]^2 + r0 g'[r0] (8444 + 81 r0^2 g''[r0]) +
           12 (49 - 306 r0^2 g''[r0] - 18 r0^3 g^(3)[r0] + r0^4 g^(4)[r0]))))
```

- **The iterative procedure**

We put burg0 and burg1 equal to 0 and solve with respect to `g''[r0]` and `g^(4)[r0]` respectively

```
equa1 = Flatten[Simplify[Solve[burg0 == 0, D[g[r0], {r0, 2}]]]]
```

```
{g''[r0] → (-36 + 14 g[r0]^3 + 2 g[r0]^4 + 42 r0 g'[r0] + 5 r0^2 g'[r0]^2 +
      g[r0]^2 (26 + r0 g'[r0]) + g[r0] (-6 + 17 r0 g'[r0])) / (3 r0^2 (3 + g[r0]))}
```

```
equazione2 = Simplify[equa1[[1]][[2]]]
```

```
(-36 + 14 g[r0]^3 + 2 g[r0]^4 + 42 r0 g'[r0] + 5 r0^2 g'[r0]^2 +
    g[r0]^2 (26 + r0 g'[r0]) + g[r0] (-6 + 17 r0 g'[r0])) / (3 r0^2 (3 + g[r0]))
```

```
equa2 = Flatten[Simplify[Solve[burg1 == 0, D[g[r0], {r0, 4}]]]]
```

```
{g^(4)[r0] ->
   1/(24 r0^4 (3 + g[r0])^3) (-280 g[r0]^7 - 20 g[r0]^8 - 444 r0^3 g'[r0]^3 + 235 r0^4 g'[r0]^4 -
     4 g[r0]^6 (375 + 17 r0 g'[r0]) + 4 g[r0]^5 (-880 - 258 r0 g'[r0] + 21 r0^2 g''[r0]) -
     54 r0^2 g'[r0]^2 (26 + 25 r0^2 g''[r0]) + 27 (-240 + 648 r0^2 g''[r0] + 17 r0^4 g''[r0]^2) +
     108 r0 g'[r0] (44 + 9 r0^2 g''[r0] + 14 r0^3 g^(3)[r0]) + g[r0]^4
      (-1820 - 6384 r0 g'[r0] - 129 r0^2 g'[r0]^2 + 1104 r0^2 g''[r0] + 48 r0^3 g^(3)[r0]) -
     2 g[r0]^3 (603 r0^2 g'[r0]^2 + r0 g'[r0] (9956 + 81 r0^2 g''[r0]) -
        12 (275 + 252 r0^2 g''[r0] + 18 r0^3 g^(3)[r0])) -
     2 g[r0] (-67 r0^3 g'[r0]^3 + 9 r0^2 g'[r0]^2 (268 + 25 r0^2 g''[r0]) -
        9 r0 g'[r0] (-888 - 45 r0^2 g''[r0] + 56 r0^3 g^(3)[r0]) -
        9 (-120 + 1494 r0^2 g''[r0] + 17 r0^4 g''[r0]^2 + 72 r0^3 g^(3)[r0])) +
     g[r0]^2 (-3909 r0^2 g'[r0]^2 + 94 r0^3 g'[r0]^3 +
        12 r0 g'[r0] (-2541 - 72 r0^2 g''[r0] + 14 r0^3 g^(3)[r0]) +
        3 (3060 + 5832 r0^2 g''[r0] + 17 r0^4 g''[r0]^2 + 432 r0^3 g^(3)[r0])))}
```

```
equazione4 = Simplify[equa2[[1]][[2]]]
```

```
1/(24 r0^4 (3 + g[r0])^3) (-280 g[r0]^7 - 20 g[r0]^8 - 444 r0^3 g'[r0]^3 + 235 r0^4 g'[r0]^4 -
   4 g[r0]^6 (375 + 17 r0 g'[r0]) + 4 g[r0]^5 (-880 - 258 r0 g'[r0] + 21 r0^2 g''[r0]) -
   54 r0^2 g'[r0]^2 (26 + 25 r0^2 g''[r0]) + 27 (-240 + 648 r0^2 g''[r0] + 17 r0^4 g''[r0]^2) +
   108 r0 g'[r0] (44 + 9 r0^2 g''[r0] + 14 r0^3 g^(3)[r0]) +
   g[r0]^4 (-1820 - 6384 r0 g'[r0] - 129 r0^2 g'[r0]^2 + 1104 r0^2 g''[r0] + 48 r0^3 g^(3)[r0]) -
   2 g[r0]^3 (603 r0^2 g'[r0]^2 + r0 g'[r0] (9956 + 81 r0^2 g''[r0]) -
      12 (275 + 252 r0^2 g''[r0] + 18 r0^3 g^(3)[r0])) -
   2 g[r0] (-67 r0^3 g'[r0]^3 + 9 r0^2 g'[r0]^2 (268 + 25 r0^2 g''[r0]) -
      9 r0 g'[r0] (-888 - 45 r0^2 g''[r0] + 56 r0^3 g^(3)[r0]) -
      9 (-120 + 1494 r0^2 g''[r0] + 17 r0^4 g''[r0]^2 + 72 r0^3 g^(3)[r0])) +
   g[r0]^2 (-3909 r0^2 g'[r0]^2 + 94 r0^3 g'[r0]^3 +
      12 r0 g'[r0] (-2541 - 72 r0^2 g''[r0] + 14 r0^3 g^(3)[r0]) +
      3 (3060 + 5832 r0^2 g''[r0] + 17 r0^4 g''[r0]^2 + 432 r0^3 g^(3)[r0])))
```

We insert `g''[r0]` in equazione4

```
equazione4new = Simplify[equazione4 /. equa1]
```

$$
\begin{aligned}
&\frac{1}{9\,r0^4\,(3+g[r0])^3}\\
&\Big(-25920+374\,g[r0]^7+22\,g[r0]^8+9828\,r0^2\,g'[r0]^2-1434\,r0^3\,g'[r0]^3-\\
&\quad 140\,r0^4\,g'[r0]^4+g[r0]^6\,(2712-47\,r0\,g'[r0])-4\,g[r0]^5\,(-2687+69\,r0\,g'[r0])+\\
&\quad 2\,g[r0]^4\,\big(12083+333\,r0\,g'[r0]-42\,r0^2\,g'[r0]^2+9\,r0^3\,g^{(3)}[r0]\big)+\\
&\quad 27\,r0\,g'[r0]\,\big(908+21\,r0^3\,g^{(3)}[r0]\big)+\\
&\quad g[r0]^3\,\big(26238+9116\,r0\,g'[r0]-729\,r0^2\,g'[r0]^2+162\,r0^3\,g^{(3)}[r0]\big)+\\
&\quad g[r0]\,\big(4527\,r0^2\,g'[r0]^2-781\,r0^3\,g'[r0]^3+54\,r0\,g'[r0]\,\big(760+7\,r0^3\,g^{(3)}[r0]\big)+\\
&\qquad 54\,\big(-664+9\,r0^3\,g^{(3)}[r0]\big)\big)+g[r0]^2\,\big(-1014\,r0^2\,g'[r0]^2-101\,r0^3\,g'[r0]^3+\\
&\qquad 63\,r0\,g'[r0]\,\big(455+r0^3\,g^{(3)}[r0]\big)+54\,\big(-46+9\,r0^3\,g^{(3)}[r0]\big)\big)\Big)
\end{aligned}
$$

We compute the second order derivative of equazione2 , we solve for $g^{(4)}[r0]$ and we equal this new expression to equazione4new. We solve for $g^{(3)}[r0]$

```
equazione2Der2 = Simplify[D[equazione2, {r0, 2}]]
```

$$
\begin{aligned}
&\frac{1}{3\,r0^4\,(3+g[r0])^3}\Big(12\,g[r0]^6+10\,r0^4\,g'[r0]^4+\\
&\quad 6\,g[r0]^5\,\big(26-4\,r0\,g'[r0]+r0^2\,g''[r0]\big)+g'[r0]^2\,\big(378\,r0^2-75\,r0^4\,g''[r0]\big)+\\
&\quad g[r0]^4\,\big(768-278\,r0\,g'[r0]+12\,r0^2\,g'[r0]^2+68\,r0^2\,g''[r0]+r0^3\,g^{(3)}[r0]\big)+\\
&\quad 9\,r0\,g'[r0]\,\big(60+9\,r0^2\,g''[r0]+10\,r0^3\,g^{(3)}[r0]\big)+\\
&\quad 18\,\big(-108-39\,r0^2\,g''[r0]+5\,r0^4\,g''[r0]^2+21\,r0^3\,g^{(3)}[r0]\big)+g[r0]^3\,\big(122\,r0^2\,g'[r0]^2+\\
&\qquad 262\,r0^2\,g''[r0]+r0\,g'[r0]\,\big(-1186+3\,r0^2\,g''[r0]\big)+23\,\big(72+r0^3\,g^{(3)}[r0]\big)\big)+\\
&\quad g[r0]^2\,\big(972+450\,r0^2\,g'[r0]^2+306\,r0^2\,g''[r0]+10\,r0^4\,g''[r0]^2+\\
&\qquad 153\,r0^3\,g^{(3)}[r0]+r0\,g'[r0]\,\big(-2142+27\,r0^2\,g''[r0]+10\,r0^3\,g^{(3)}[r0]\big)\big)+\\
&\quad g[r0]\,\big(-324\,r0^2\,g''[r0]+60\,r0^4\,g''[r0]^2+g'[r0]^2\,\big(702\,r0^2-25\,r0^4\,g''[r0]\big)+\\
&\qquad 405\,\big(-4+r0^3\,g^{(3)}[r0]\big)+3\,r0\,g'[r0]\,\big(-378+27\,r0^2\,g''[r0]+20\,r0^3\,g^{(3)}[r0]\big)\big)\Big)
\end{aligned}
$$

```
equa3 = Flatten[Simplify[
   Solve[equazione2Der2 - equazione4new == 0, D[g[r0], {r0, 3}]] /. equa1]]
```

$$
\begin{aligned}
&\Big\{g^{(3)}[r0]\to\Big(98496-10\,g[r0]^7+10\,g[r0]^8-131328\,r0\,g'[r0]-9450\,r0^2\,g'[r0]^2+\\
&\qquad 5757\,r0^3\,g'[r0]^3+385\,r0^4\,g'[r0]^4+g[r0]^6\,(-1788+217\,r0\,g'[r0])+2\,g[r0]^5\\
&\qquad\quad (-7742+1131\,r0\,g'[r0])+g[r0]^4\,\big(-57422+7254\,r0\,g'[r0]+519\,r0^2\,g'[r0]^2\big)+\\
&\qquad g[r0]^3\,\big(-96594-3136\,r0\,g'[r0]+4932\,r0^2\,g'[r0]^2\big)+\\
&\qquad g[r0]^2\,\big(-31968-72855\,r0\,g'[r0]+14529\,r0^2\,g'[r0]^2+373\,r0^3\,g'[r0]^3\big)+\\
&\qquad 2\,g[r0]\,\big(52380-84375\,r0\,g'[r0]+5031\,r0^2\,g'[r0]^2+1519\,r0^3\,g'[r0]^3\big)\Big)\Big/\\
&\quad\Big(9\,r0^3\,(3+g[r0])^2\,\big(-42+g[r0]+5\,g[r0]^2+11\,r0\,g'[r0]\big)\Big)\Big\}
\end{aligned}
$$

```
equazione3 = Simplify[equa3[[1]][[2]]]
```

$$\Big(98496 - 10\, g[r0]^7 + 10\, g[r0]^8 - 131328\, r0\, g'[r0] - 9450\, r0^2\, g'[r0]^2 + 5757\, r0^3\, g'[r0]^3 + 385\, r0^4\, g'[r0]^4 + g[r0]^6\, (-1788 + 217\, r0\, g'[r0]) + 2\, g[r0]^5\, (-7742 + 1131\, r0\, g'[r0]) + g[r0]^4\, \big(-57422 + 7254\, r0\, g'[r0] + 519\, r0^2\, g'[r0]^2\big) + g[r0]^3\, \big(-96594 - 3136\, r0\, g'[r0] + 4932\, r0^2\, g'[r0]^2\big) + g[r0]^2\, \big(-31968 - 72855\, r0\, g'[r0] + 14529\, r0^2\, g'[r0]^2 + 373\, r0^3\, g'[r0]^3\big) + 2\, g[r0]\, \big(52380 - 84375\, r0\, g'[r0] + 5031\, r0^2\, g'[r0]^2 + 1519\, r0^3\, g'[r0]^3\big)\Big) \Big/ \Big(9\, r0^3\, (3 + g[r0])^2\, \big(-42 + g[r0] + 5\, g[r0]^2 + 11\, r0\, g'[r0]\big)\Big)$$

We compute the first order derivative of the equazione2 and we solve for $g^{(3)}[r0]$

```
equazione2Der1 = Simplify[D[equazione2, {r0, 1}] /. equa1]
```

$$\frac{1}{9\, r0^3\, (3 + g[r0])^2} \Big(-864 + 36\, g[r0]^5 + 2\, g[r0]^6 + 1080\, r0\, g'[r0] + 657\, r0^2\, g'[r0]^2 + 35\, r0^3\, g'[r0]^3 + 3\, g[r0]^4\, (76 + 13\, r0\, g'[r0]) + g[r0]^3\, (616 + 327\, r0\, g'[r0]) + 3\, g[r0]^2\, \big(174 + 343\, r0\, g'[r0] + 6\, r0^2\, g'[r0]^2\big) + 3\, g[r0]\, \big(-180 + 519\, r0\, g'[r0] + 91\, r0^2\, g'[r0]^2\big)\Big)$$

We equal the two expressions for $g^{(3)}[r0]$ obtaining two equations for $g'[r0]$

```
Eq1 = Simplify[equazione2Der1 - equazione3];
```

```
equa1New = Simplify[Solve[Eq1 == 0, D[g[r0], {r0, 1}]]]
```

$$\Big\{\Big\{g'[r0] \to \Big(177 + 110\, g[r0] + 5\, g[r0]^2 - 4\, g[r0]^3 + \sqrt{3}\, \sqrt{\big((3 + g[r0])^2\, \big(851 + 690\, g[r0] + 171\, g[r0]^2 + 16\, g[r0]^3\big)\big)}\Big) \Big/ (29\, r0 - 2\, r0\, g[r0])\Big\}, \Big\{g'[r0] \to \Big(177 + 110\, g[r0] + 5\, g[r0]^2 - 4\, g[r0]^3 - \sqrt{3}\, \sqrt{\big((3 + g[r0])^2\, \big(851 + 690\, g[r0] + 171\, g[r0]^2 + 16\, g[r0]^3\big)\big)}\Big) \Big/ (29\, r0 - 2\, r0\, g[r0])\Big\}\Big\}$$

```
equazione11 = equa1New[[1]][[1]][[2]]
```

$$\Big(177 + 110\, g[r0] + 5\, g[r0]^2 - 4\, g[r0]^3 + \sqrt{3}\, \sqrt{(3 + g[r0])^2\, \big(851 + 690\, g[r0] + 171\, g[r0]^2 + 16\, g[r0]^3\big)}\Big) \Big/ (29\, r0 - 2\, r0\, g[r0])$$

```
equazione12 = Simplify[equa1New[[2]][[1]][[2]]]
```

$$\Big(177 + 110\, g[r0] + 5\, g[r0]^2 - 4\, g[r0]^3 - \sqrt{3}\, \sqrt{(3 + g[r0])^2\, \big(851 + 690\, g[r0] + 171\, g[r0]^2 + 16\, g[r0]^3\big)}\Big) \Big/ (29\, r0 - 2\, r0\, g[r0])$$

We consider the first one, compute the first oder derivative, susbtitute the expression for $g'[r0]$ and solve for $g''[r0]$

```
equa1New[[1]][[1]][[2]]
```

$$\Big(177 + 110\, g[r0] + 5\, g[r0]^2 - 4\, g[r0]^3 + \sqrt{3}\, \sqrt{(3 + g[r0])^2\, \big(851 + 690\, g[r0] + 171\, g[r0]^2 + 16\, g[r0]^3\big)}\Big) \Big/ (29\, r0 - 2\, r0\, g[r0])$$

```
Eq1Der = Simplify[D[equazione11, {r0, 1}] /. equa1New[[1]][[1]]]
```

```
-((177 + 110 g[r0] + 5 g[r0]^2 - 4 g[r0]^3 +
      √3 √((3 + g[r0])^2 (851 + 690 g[r0] + 171 g[r0]^2 + 16 g[r0]^3)))
    (179400 √3 + 200686 √3 g[r0] + 626 √3 g[r0]^4 - 48 √3 g[r0]^5 +
      2703 √((3 + g[r0])^2 (851 + 690 g[r0] + 171 g[r0]^2 + 16 g[r0]^3)) +
      406 g[r0] √((3 + g[r0])^2 (851 + 690 g[r0] + 171 g[r0]^2 + 16 g[r0]^3)) +
      2 g[r0]^2 (40455 √3 -
         181 √((3 + g[r0])^2 (851 + 690 g[r0] + 171 g[r0]^2 + 16 g[r0]^3))) + 2 g[r0]^3
       (6813 √3 + 8 √((3 + g[r0])^2 (851 + 690 g[r0] + 171 g[r0]^2 + 16 g[r0]^3))))) /
  (r0^2 (-29 + 2 g[r0])^3 √((3 + g[r0])^2 (851 + 690 g[r0] + 171 g[r0]^2 + 16 g[r0]^3)))
```

```
equazione2New = Simplify[equazione2 /. equa1New[[1]]]
```

```
1/(r0^2 (29 - 2 g[r0])^2) (50760 - 249 g[r0]^3 - 138 g[r0]^4 + 32 g[r0]^5 +
   332 √3 √((3 + g[r0])^2 (851 + 690 g[r0] + 171 g[r0]^2 + 16 g[r0]^3)) -
   2 g[r0]^2 (-5594 + 7 √3 √((3 + g[r0])^2 (851 + 690 g[r0] + 171 g[r0]^2 + 16 g[r0]^3))) +
   3 g[r0] (15469 + 19 √3 √((3 + g[r0])^2 (851 + 690 g[r0] + 171 g[r0]^2 + 16 g[r0]^3))))
```

We equal the two expressions for `g''[r0]` and obtain that the solutions are `g[r0] = constant`. Precisely,

```
Num = Numerator[Simplify[Eq1Der - equazione2New]];
```

```
Soluzioni1 = Flatten[Simplify[Solve[Num == 0]]]
```

```
{g[r0] → -3, g[r0] → 29/2, g[r0] → Root[
   179147 + 458229 #1 + 394875 #1^2 + 162491 #1^3 + 35862 #1^4 + 4140 #1^5 + 200 #1^6 &, 1],
 g[r0] → Root[179147 + 458229 #1 + 394875 #1^2 + 162491 #1^3 +
     35862 #1^4 + 4140 #1^5 + 200 #1^6 &, 5], g[r0] → Root[
   179147 + 458229 #1 + 394875 #1^2 + 162491 #1^3 + 35862 #1^4 + 4140 #1^5 + 200 #1^6 &, 6]}
```

```
NSolve[Num == 0]
```

```
{{g[r0] → -3.85584 + 2.12369 i}, {g[r0] → -3.85584 - 2.12369 i},
 {g[r0] → -3.}, {g[r0] → -3.}, {g[r0] → -3.}, {g[r0] → -2.14151}}
```

```
{{g[r0] → -3.85584 + 2.12369 i}, {g[r0] → -3.85584 - 2.12369 i},
 {g[r0] → -3.}, {g[r0] → -3.}, {g[r0] → -3.}, {g[r0] → -2.14151}}
```

Analogously, we consider the second equation for `g'[r0]` and we do the same

```
Eq2Der = Simplify[D[equazione12, {r0, 1}] /. equa1New[[2]]];
```

```
equazione2New2 = Simplify[equazione2 /. equa1New[[2]]];
```

```
Num2 = Numerator[Simplify[Eq2Der - equazione2New2]];
```

```
Soluzioni2 = Flatten[Simplify[Solve[Num2 == 0]]]
```

```
{g[r0] → -3, g[r0] → -2, g[r0] → 1, g[r0] → Root[
    179147 + 458229 #1 + 394875 #1^2 + 162491 #1^3 + 35862 #1^4 + 4140 #1^5 + 200 #1^6 &, 2],
  g[r0] → Root[179147 + 458229 #1 + 394875 #1^2 + 162491 #1^3 +
      35862 #1^4 + 4140 #1^5 + 200 #1^6 &, 3], g[r0] → Root[
    179147 + 458229 #1 + 394875 #1^2 + 162491 #1^3 + 35862 #1^4 + 4140 #1^5 + 200 #1^6 &, 4]}
```

```
NSolve[Num2 == 0]
```

```
{{g[r0] → -3.}, {g[r0] → -3.}, {g[r0] → -3.},
 {g[r0] → -2.}, {g[r0] → 1.}, {g[r0] → -0.743563}}
```

- **Check of the solutions**

```
eq = Numerator[equazione2 /. g'[r0] → 0]
```

```
-36 - 6 g[r0] + 26 g[r0]^2 + 14 g[r0]^3 + 2 g[r0]^4
```

Check for the first group of solutions

```
For[i = 1, i ≤ 5, i++,
   Solut1[i] = FullSimplify[eq /. Soluzioni1[[i]]]
  ];
```

```
For[i = 1, i ≤ 5, i++,
  If[Simplify[Solut1[i] == 0],
   Print["g[r0]=", Soluzioni1[[i]][[2]], " is a solution" ],
   Print["g[r0]=", Soluzioni1[[i]][[2]], " is not a solution" ]
  ]]
```

```
g[r0]=-3 is a solution
g[r0]=29/2 is not a solution
g[r0]=
 Root[179147 + 458229 #1 + 394875 #1^2 + 162491 #1^3 + 35862 #1^4 + 4140 #1^5 + 200 #1^6 &, 1]
  is not a solution
g[r0]=
 Root[179147 + 458229 #1 + 394875 #1^2 + 162491 #1^3 + 35862 #1^4 + 4140 #1^5 + 200 #1^6 &, 5]
  is not a solution
g[r0]=
 Root[179147 + 458229 #1 + 394875 #1^2 + 162491 #1^3 + 35862 #1^4 + 4140 #1^5 + 200 #1^6 &, 6]
  is not a solution
```

Check for the second group of solutions

```
For[i = 1, i ≤ 6, i++,
   Solut2[i] = FullSimplify[eq /. Soluzioni2[[i]]]
  ];
```

```
For[i = 1, i ≤ 6, i++,
 If[Simplify[Solut2[i] == 0],
  Print["g[r0]=", Soluzioni2[[i]][[2]], " is a solution" ],
  Print["g[r0]=", Soluzioni2[[i]][[2]], " is not a solution" ]
 ]]
```

```
g[r0]=-3 is a solution

g[r0]=-2 is a solution

g[r0]=1 is a solution

g[r0]=
 Root[179147 + 458229 #1 + 394875 #1^2 + 162491 #1^3 + 35862 #1^4 + 4140 #1^5 + 200 #1^6 &, 2]
  is not a solution

g[r0]=
 Root[179147 + 458229 #1 + 394875 #1^2 + 162491 #1^3 + 35862 #1^4 + 4140 #1^5 + 200 #1^6 &, 3]
  is not a solution

g[r0]=
 Root[179147 + 458229 #1 + 394875 #1^2 + 162491 #1^3 + 35862 #1^4 + 4140 #1^5 + 200 #1^6 &, 4]
  is not a solution
```

We obtain that g[r0]=-3, g[r0]=-2 and g[r0]=1 are solutions. We exclude the case g[r0]=-3 since it does not satisfy the hypothesis on the effective potential

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