Tesi di Dottorato di Ricerca

**Stickelberger Series and Iwasawa Main Conjecture for Function Fields**

Settore disciplinare: MAT/02 Algebra

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Introduction

The principal objects of study of Number Theory in positive characteristic are the global function fields, which are the natural analogues of the number fields. It is possible to associate to every function field some invariants like, for example, the genus or the class number, that play an important role in the study of the arithmetic properties. Even if these objects have an algebraic nature many analytic tools have been introduced and studied in order to have information on such invariants. For example we can mention the Weil Zeta function, the Artin $L$-functions and the Goss Zeta function which all have an analytic nature (both in the classical and in the $p$-adic setting). The goal of this thesis is to investigate some links between these families of objects and provide theorems that build a bridge between the analytic side and the algebraic side of the theory. In all these theorems an important role will be played by the Stickelberger Series $\Theta_S(X)$ which is an algebraic object, but it can be used to generate all sort of $L$-functions thus providing a kind of “universal series” from which all analytic functions originate. In the second part of this thesis we study the Iwasawa tower generated by the torsion of a Hayes module and use the Stickelberger series to prove a main conjecture.

Let $F$ be a global function field defined over a finite field $\mathbb{F}_q$ of characteristic $p$, fix a prime $\infty$ of $F$ and let $A$ be the ring of elements regular outside of $\infty$. We take a finite set of primes $S$ containing $\infty$ and we denote with $F_S$ the maximal abelian extension of $F$ unramified outside $S$ and with $G_S := \text{Gal}(F_S/F)$ its Galois group. The Stickelberger series of $S$ is an element $\Theta_S(X)$ in the power series algebra $\mathbb{Z}[G_S][X]$.

In Section 1.3 we study the relations between the Stickelberger series and the Artin $L$-function $L(s, \chi)$.

Let $F_\infty$ denote the completion of $F$ at the prime $\infty$, $\mathbb{C}_\infty$ the completion of a fixed algebraic closure of $F_\infty$ and $S_\infty$ the topological group $\mathbb{C}_\infty^\times \times \mathbb{Z}_p$. In [Gos2] the author defines a function $\zeta_A(s)$ over $S_\infty$ that takes values in $\mathbb{C}_\infty$, which represents the analogue in positive characteristic of the Riemann Zeta function, by

$$\zeta_A(s) = \sum a^{-s} \quad \text{for } s \in S_\infty,$$

where the sum is taken over all the non zero ideals $a$ of $A$.

For this function, which is called Goss Zeta function, we prove Theorem 1.6.4:

**Theorem.** For every $y \in \mathbb{Z}_p$, there exists a continuous ring homomorphism $\Psi_y : \mathbb{Z}[G_S][X] \rightarrow \mathbb{C}_\infty[X]$, such that

$$\Psi_y(\Theta_S(X))(x) = \zeta_A(-s) \prod_{p \in S, p \neq \infty} (1 - p^s) \quad \text{for every } x \in \mathbb{C}_\infty^\times,$$

where $s = (x, y) \in S_\infty$. 
There exists also a local version of the Goss Zeta function, which is called the \( \nu \)-adic Zeta function \( \zeta_\nu(s_\nu) \) and that takes value in the complete and algebraically closed field \( \mathbb{C}_\nu \) (here \( \nu \) is a prime of \( F \) different from \( \infty \)). For this function, Theorem 1.8.4 provides a connection with the Stickelberger series:

**Theorem.** Assume that \( \nu \in S \). Then for every \( y \in \mathbb{Z}_p \) and \( j \in \mathbb{Z}/|F_0^\times| \), there exists a continuous ring homomorphism \( \Psi_{\nu,j} : \mathbb{Z}[\pi_S][X] \rightarrow \mathbb{C}_\nu[X] \), such that

\[
\Psi_{\nu,j}(\Theta_S(X))(x) = \zeta_\nu(-s_\nu) \prod_{p \in S, \ p \neq \nu, \infty} (1 - p^{-s}) \quad \text{for every } x \in \mathbb{C}_\nu^\times,
\]

where \( s_\nu = (x, y, j) \in S_\nu := \mathbb{C}_\nu^N \times \mathbb{Z}_p \times \mathbb{Z}/|F_0^\times| \).

In [ABBL] the authors use a special case of the previous theorem, for the rational function field \( F = \mathbb{F}_q[T] \), to prove a main conjecture for (the \( p \)-parts of) the class groups in the \( p \)-cyclotomic extension generated over \( \mathbb{F}_q[T] \) by the \( p\infty \)-torsion of the Carlitz module (\( p \neq \infty \) a prime of \( \mathbb{F}_q[T] \)). In the second chapter of this thesis, for a general function field \( F \), we investigate the extension generated by the torsion of a Hayes module and we are able to prove the main conjecture for the \( \chi \) part of the Iwasawa module, when \( \chi \) is a character of type 1 or 2 (Definition 2.3.6). These results were achieved thanks to the work of Greither and Popescu on the Deligne’s Picard 1-motive ([GP1] and [GP2]).

We worked under two assumptions: the first one is that the degree of the prime \( \infty \) is 1. This assumption was needed to assure that all the field extensions we are dealing with are geometric, but the reader may observe that this assumption is not really restrictive since we can reduce to this case by extending the costant field of \( F \). The second assumption is that the class number of degree zero divisors \( h^0(F) \) is coprime with \( p \).

Let \( H_A \) be the Hilbert class field of \( A \) and \( \Psi : A \rightarrow H_A \{\tau\} \) a Hayes module. Fix a prime \( p \) of \( F \) with degree \( d_p \) and denote with \( F_n \) the extension of \( H_A \) generated by the \( p^{n+1} \)-torsion of \( \Psi \). The field \( F_n \) is an abelian Galois extension of \( F \) ramified only at the two primes \( p \) and \( \infty \). These fields form an Iwasawa tower

\[
F \subset H_A \subset F_0 \subset F_1 \subset \cdots \subset F_n \subset \cdots \subset \bigcup_{n \in \mathbb{N}} F_n =: F_\infty,
\]

indeed if we denote with \( \Gamma_n = \text{Gal}(F_n/F_0) \) we have that

\[
\Gamma_\infty := \text{Gal}(F_\infty/F_0) = \lim_{n} \text{Gal}(F_n/F_0) \simeq \mathbb{Z}_p^\infty.
\]

We denote with \( C_n := \text{Cl}_p(F_n) \{p\} \) the \( p \)-part of the class group of degree zero divisors of \( F_n \). There is a natural action of \( \Gamma_n := \text{Gal}(F_n/F_0) \) on \( C_n \), thus this group can be seen as a module over the ring \( \mathbb{Z}_p[\Gamma_n] \). These groups form a projective system with respect to the norm maps which allows us to define the limit

\[
C_\infty := \lim_{n} C_n,
\]

which is a module over the *Iwasawa algebra* \( \mathbb{Z}_p[\Gamma_\infty] \), but unlike the classical case, in this setting this algebra is not Noetherian, thus we do not have a structure theorem for the finitely generated torsion modules. The main goal of the second chapter is to understand the structure
of the module $C_\infty$ and compute its Fitting ideal. Instead of doing it directly we have to deal with the characters of the group $G_0 := \text{Gal}(F_0/F)$ which acts naturally on $C_n$ and $C_\infty$.

Let $\chi \in \text{Hom}(G_0, C^\times)$ be a complex character of the group $G_0$ and $W = \mathbb{Z}_p[\zeta]$ the Witt ring generated by a primitive root of unity $\zeta$ of order $|G_0|$. For each $G_0$-module $M$ we define its $\chi$-part as

$$M(\chi) := e_\chi (M \otimes_{\mathbb{Z}_p} W),$$

where $e_\chi \in W[G_0]$ is the idempotent of $\chi$.

Theorem 2.4.7 states:

**Theorem.** Let $\chi$ be a character of type 1 or 2. Then $C_\infty(\chi)$ is a finitely generated torsion module over the Iwasawa algebra $\Lambda := W[\Gamma_\infty]$.

Let $\Theta_\infty(X)$ be the projection of the Stickelberger series $\Theta_\infty(X)$ to $\mathbb{Z}[\Gamma_\infty \times G_0][X]$ (here we put $S = \{p, \infty\}$) and $\Theta_\infty(X, \chi) = e_\chi \Theta_\infty(X) \in W[\Gamma_\infty][X]$. Also put

$$\Theta^\sharp_\infty(X, \chi) = \begin{cases} \Theta_\infty(X, \chi) & \text{if } \chi \text{ is of type 1,} \\ \Theta_\infty(X, \chi) / (1 - X) & \text{if } \chi \text{ is of type 2,} \end{cases}$$

Our main results on the Iwasawa module $C_\infty(\chi)$ is Theorem 2.4.8:

**Theorem** (Iwasawa main conjecture). Let $\chi$ be a character of type 1 or 2. Then we have

$$\text{Fitt}_\Lambda (C_\infty(\chi)) = \left( \Theta^\sharp_\infty(1, \chi) \right).$$

For the characters of type 3 we have no direct information on the structure of $C_n(\chi)$ and $C_\infty(\chi)$, but, working on Pontrjagin duals, we are able to prove a result on the $\chi$-part of the Stickelberger series: for the characters of type 3 which are trivial on the decomposition group of $p$, the series $\Theta_n(X, \chi)$ and $\Theta_\infty(X, \chi)$ have a zero of order at least 2 at $X = 1$. 

Chapter 1

Stickelberger Series and $L$-functions

1.1 Setting and notations

In this first chapter we will introduce and study the properties of some analytic and algebraic objects that have been used to study the arithmetic of function fields. The main algebraic object we are dealing with is the Stickelberger series, while the analytic objects we will introduce are the Artin $L$-functions, the Goss Zeta function and the Goss $\nu$-adic Zeta function. The goal of this chapter is to investigate some links between these families of objects and provide theorems that build a bridge between the analytic side and the algebraic side of the theory.

- $F$ is a global function field of characteristic $p > 0$, i.e., a finite algebraic extension of a field of transcendence degree 1 over a finite field $\mathbb{F}_q := \mathbb{F}_q^r$ which we call the constant field of $F$.
- $\infty$ is a fixed place of $F$ and $A$ is the subring of $F$ of the elements regular outside of $\infty$;
- for any place $\nu$ of $F$ (including $\infty$), $F_{\nu}$ is the completion of $F$ at $\nu$. Its ring of integers will be denoted by $O_{\nu}$ and $U_1(\nu)$ will be the group of 1-units of $F_{\nu}$. The residue field $O_{\nu}/(\nu) := \mathbb{F}_q$ is a finite extension of $\mathbb{F}_q$ of degree $d_{\nu} := [F_{\nu} : \mathbb{F}_q]$ (also called the degree of $\nu$), its order will be denoted by $N_{\nu} := q^{d_{\nu}}$. The degree of a prime $\nu$ will be often denoted also by $\deg \nu$;
- $v_{\nu} : F_{\nu} \to \mathbb{Z}$ is the (canonical) valuation at $\nu$ and $\pi_{\nu}$ will denote a fixed uniformizer for $F_{\nu}$, i.e., an element with $v_{\nu}(\pi_{\nu}) = 1$;
- the degree of a fractional ideal $a = \prod_{\nu \neq \infty} \nu^{n_{\nu}}$ of $A$ is the quantity $\deg a = \sum_{\nu \neq \infty} n_{\nu} d_{\nu}$.

1.2 Stickelberger Series

Let $S$ be a finite set of places of $F$ that contains $\infty$ and we denote with $F_S$ the maximal abelian extension of $F$ unramified outside $S$ and with $G_S := \text{Gal}(F_S/F)$ its Galois group. For every place $\nu \notin S$ let $\phi_{\nu}$ be the Frobenius at $\nu$, i.e., the unique element of $G_S$ that satisfies

$$\phi_{\nu}(x) \equiv x^{N_{\nu}} \pmod{\tilde{\nu}}$$
for every $x \in F_S$, where $\tilde{\nu}$ is any place of $F_S$ lying above $\nu$. We observe that the extension $F_S/F$ is unramified at $\nu$, so the decomposition group of $\nu$ in $G_S$ is pro-cyclic and topologically generated by $\phi_\nu$.

**Definition 1.2.1.** The Stickelberger series of $S$ is defined by the Euler product

$$\Theta_S(X) = \prod_{\nu \not\in S} \left(1 - \phi_\nu^{-1}X^{\deg a}\right)^{-1}.$$  

Actually the product above provides a well defined element of $\mathbb{Z}[G_S][X]$: for every commutative unitary ring $R$, an element $f(X) \in R[X]$ is invertible if and only if $f(0)$ is invertible in $R$, thus every Euler factor $e_\nu(X) := 1 - \phi_\nu^{-1}X^{\deg a}$ is invertible in $\mathbb{Z}[G_S][X]$. Furthermore if we denote with $I_S$ the set of fractional ideals of $A$ with support outside of $S$ (remember that $\infty \in S$) and with $\phi_a$ the Artin symbol associated to $a \in I_S$, i.e.,

$$a = \prod_{\nu \not\in S} \nu^{n_\nu} \implies \phi_a = \prod \phi_\nu^{n_\nu},$$

(recall that $A$ is a Dedekind domain), then we can write

$$\Theta_S(X) = \sum_{a \in I_S \atop a \geq 0} \phi_a^{-1}X^{\deg a} = \sum_{n \geq 1} \sum_{a \in I_S \atop \deg a = n} \phi_a^{-1}X^n$$

(where we use the notation $a \geq 0$ to denote the integral ideals of $A$). Since for every positive integer $n$ there exists at most a finite number of primes $\nu$ with degree equal to $n$, the series on the right is clearly an element of $\mathbb{Z}[G_S][X]$.

1.3 **Artin L-functions**

Let $K/F$ be a finite subextension of $F_S$ whose Galois group is $G_K$ and let $S_K \subset S$ bet the set of ramified places together with $\infty$.

**Notation 1.3.1.** We remark that we shall always include $\infty$ in $S_K$ (i.e., if $K/F$ is not ramified at $\infty$ we put $S_K := \{\nu \text{ prime of } F \text{ ramified in } K/F\} \cup \{\infty\}$).

For every $\nu \not\in S_K$ let $\text{Frob}_K\nu \in G_K$ be the Artin symbol. In particular if $\nu \not\in S$, $\text{Frob}_K\nu$ is the image of $\phi_\nu$ through the canonical projection $G_S \to G_K$.

For every complex character $\chi$ of $G_K$, i.e., an element of $\text{Hom}(G_K, \mathbb{C}^\times)$, we put

$$\chi(\nu) = \begin{cases} 
\chi(\text{Frob}_K\nu) & \text{if } \nu \not\in S_K, \\
0 & \text{if } \nu \in S_K.
\end{cases}$$

**Definition 1.3.2.** The Artin $L$-function associated to $(K, \chi)$ is

$$L_K(s, \chi) = \prod_{\nu \not\in S_K} \left(1 - \chi(\nu)(\mathfrak{N}\nu)^{-s}\right)^{-1}, \quad \text{for } \Re(s) > 1$$

(where the condition $\Re(s) > 1$ guarantees convergence).
Our goal in this section is to provide a link between $L_K(s, \chi)$ and $\Theta_S(X)$. Let $\Psi : G_S \to \mathbb{C}^\times$ be a continuous character of $G_S$, i.e., a continuous homomorphism with respect to the natural topologies. We recall that the natural topology on a Galois group is the Krull topology which is generated by the left (or right) cosets of normal subgroups with finite index. With an abuse of notation we denote with $\Psi$ also the ring homomorphism $\mathbb{Z}[G_S][X] \to \mathbb{C}[X]$ induced in a natural way by $\Psi$.

**Theorem 1.3.3.**  (a) Let $K/F$ be a finite subextension of $F_S$ with Galois group $G_K$ and let $\chi$ be a complex character of $G_K$. Then there exists a continuous character $\Psi$ of $G_S$ such that

$$
\Psi (\Theta_S(X)) (q^{-s}) = L_K(s, \chi^{-1}) \prod_{\nu \in S-S_K} \left(1 - \chi^{-1}(\nu)q^{-s\nu}\right), \quad \text{for } \Re(s) > 1. \tag{1.1}
$$

(b) Let $\Psi$ be a continuous character of $G_S$. Then there exists a finite subextension $K$ of $F_S$ with Galois group $G_K$ and a complex character $\chi$ of $G_K$ such that equation (1.1) holds.

**Proof.** (a) Let $\pi_K$ be the canonical projection $G_S \to G_K$ and put $\Psi := \chi \circ \pi_K$. Clearly $\Psi$ is a continuous character of $G_S$. We have that

$$
\Psi (\Theta_S(X)) = \prod_{\nu \in \mathcal{S}} \left(1 - \chi^{-1}(\pi_K(\phi_{\nu}^{-1})) X^{d\nu}\right)^{-1}
$$

and so

$$
\Psi (\Theta_S(X)) (q^{-s}) = \prod_{\nu \in \mathcal{S}} \left(1 - \chi^{-1}(\text{Frob}_K \nu)q^{-s\nu}\right)^{-1}
$$

$$
= \prod_{\nu \in \mathcal{S}_K} \left(1 - \chi^{-1}(\nu)q^{-s\nu}\right)^{-1} \prod_{\nu \in \mathcal{S}-\mathcal{S}_K} \left(1 - \chi^{-1}(\nu)q^{-s\nu}\right)
$$

$$
= L_K(s, \chi^{-1}) \prod_{\nu \in \mathcal{S}-\mathcal{S}_K} \left(1 - \chi^{-1}(\nu)q^{-s\nu}\right).
$$

(b) The kernel of $\Psi$ has finite index, indeed $\Psi$ factors through the profinite group $G_S/Ker(\Psi)$, which is topologically isomorphic to $\Psi(G_S)$ due to the first homomorphism theorem. However the only profinite (hence compact) subgroups of $\mathbb{C}^\times$ are the finite subgroups.

We denote with $K$ the fixed field of $Ker(\Psi)$, whose Galois group $G_K$ is isomorphic to the quotient $G_S/Ker(\Psi)$, with $\chi$ the character induced by $\Psi$ on $G_K$ and with $\mathcal{S}_K$ the set of ramified primes in $K/F$ (recall that it always includes $\infty$ by our convention recalled in Notation 1.3.1).

Clearly for $\nu \notin \mathcal{S}$ the diagram

$$
\begin{array}{ccc}
G_S & \xrightarrow{\Psi} & \mathbb{C}^\times \\
\downarrow{\pi_K} & & \downarrow{\chi} \\
G_S/Ker(\Psi) & \simeq & G_K
\end{array}
$$
shows that \( \Psi(\phi_{\nu}) = \chi(\text{Frob}_K \nu) \) and
\[
\Psi (\Theta_S(X)) = \prod_{\nu \not\in S} \left( 1 - \Psi(\phi_{\nu}^{-1})X^{d_{\nu}} \right)^{-1} \\
= \prod_{\nu \not\in S} \left( 1 - \chi^{-1}(\text{Frob}_K \nu)X^{d_{\nu}} \right)^{-1} \\
= \prod_{\nu \not\in S_K} \left( 1 - \chi^{-1}(\nu)X^{d_{\nu}} \right)^{-1} \prod_{\nu \in S - S_K} \left( 1 - \chi^{-1}(\nu)X^{d_{\nu}} \right).
\]
From this equation (1.1) follows immediately if we observe that \( L_K(s, \chi^{-1}) \) is equal to the product \( \prod_{\nu \not\in S_K} \left( 1 - \chi^{-1}(\nu)X^{d_{\nu}} \right)^{-1} \) evaluated at \( X = q^{-s} \).

\[\square\]

In the final part of this section we will give an application of the previous theorem to prove that the Stickelberger series lies in the Tate algebra.

Let \( \mathcal{R} \) be any topological ring. The Tate algebra \( \mathcal{R} \langle X \rangle \) is the set of formal power series with coefficients in \( \mathcal{R} \), such that the coefficients tend to 0. Note, in particular, that the polynomial ring \( \mathcal{R}[X] \) is contained in the Tate algebra. Let \( W \) be the ring of integers of a finite extension of \( \mathbb{Q}_p \). For the purpose of this thesis and, in particular in Chapter 2, we will be mainly interested in rings of the form \( \mathcal{R} = W[G] \), where \( G \) is the Galois group of an infinite extension of function fields. We recall that the topology on this ring is the weakest such that the projection \( \pi : W[G] \to W[\text{Gal}(K/F)] \) is continuous for each finite subextension \( K/F \). A classical result on profinite groups tell that \( G \) admits a basis of neighbourhoods of \( 1_G \) consisting of open subgroup of finite index which correspond, by Galois Theory, to the finite subextensions. Thus, a sequence of elements \( a_n \) of \( W[G] \) tend to 0 if and only if the sequence of the projections \( \pi(a_n) \) is equal to 0 when \( n \) is big enough, for each finite subextension \( K/F \).

The coefficients of the Stickelberger series lie in \( \mathbb{Z}[G_S] \), but to show that \( \Theta_S(X) \) is an element of the Tate algebra we have to replace \( \mathbb{Z} \) with a non-archimedean complete ring, thus we will use the natural embedding \( \mathbb{Z} \to W \) to identify \( \Theta_S(X) \) with an element of \( W[G_S][X] \). (Note that we consider \( \mathbb{Z} \) with the discret topology, thus the previous embedding is continuous).

**Proposition 1.3.4.** Let \( W \) be the ring of integers of a finite extension of \( \mathbb{Q}_p \). Then \( \Theta_S(X) \) is an element of the Tate algebra \( W[G_S][X] \).

**Proof.** It is enough to show that for each finite subextension \( K/F \) the image of the Stickelberger series under the projection \( \pi : W[G_S][X] \to W[\text{Gal}(K/F)][X] \) is a polynomial. We will not work directly with \( \Theta_S(X) \) but we will consider \( f(X) := (1 - qX)\Theta_S(X) \). Since \( (1 - qX)^{-1} = \sum_{n \geq 0} q^n X^n \) is an element of the Tate algebra, i.e. \( (1 - qX) \) is a unit in \( W[G_S][X] \), the thesis will follow immediately if we prove that \( f(X) \) is in the Tate algebra. Here we want to underline the necessity of replacing \( \mathbb{Z} \) with \( W \): the series \( \sum_{n \geq 0} q^n X^n \) is not in the Tate algebra \( \mathbb{Z}[G_S][X] \).

Let \( \Psi : G_S \to \mathbb{C}^* \) a continuous character. Following part (b) of Theorem 1.3.3 let \( K \) be the fixed field of \( \ker \Psi \), \( G_K \) be the Galois group \( \text{Gal}(K/F) \), which is finite, and \( \chi \) be the character induced by \( \Psi \) on \( G_K \). We note that \( \chi \) is the trivial character if and only if \( \Psi \) is the trivial character on \( G_S \) and, in this case, \( K = F \) and \( S_K = \{ \infty \} \). For \( \Re(s) > 1 \) we have
\[
\Psi (\Theta_S(X))(q^{-s}) = L_K(s, \chi^{-1}) \prod_{\nu \in S - S_K} \left( 1 - \chi^{-1}(\nu)q^{-sd_{\nu}} \right).
\]
1.4 The $S_\infty$-power of an ideal

Now we introduce the full Artin $L$-function which is defined by the Euler product

$$L(s, \chi^{-1}) = \prod_{\nu} (1 - \chi^{-1}(\nu)(\mathbb{N}\nu)^{-s})^{-1}, \quad \text{for } \Re(s) > 1,$$

that differs from $L_K(s, \chi^{-1})$ only for the factors associated to the primes of $S_K$. Thus we have

$$\Psi(f(X))(q^{-s}) = (1 - q^{1-s})\Psi(\Theta_S(X))(q^{-s})$$

$$= (1 - q^{1-s})L(s, \chi^{-1}) \prod_{\nu \in S} \left(1 - \chi^{-1}(\nu)q^{-sd_{\nu}}\right).$$

A theorem of Weil [Wei, VII, Theorem 6] tells that if $\chi \neq \chi_0$ then $L(s, \chi^{-1})$ is a polynomial in $q^{-s}$, thus $\Psi(f(X)) \in W[\psi(G_S)][X]$. Another theorem of Weil [Wei, VII, Theorem 4] tells that for $\chi = \chi_0$ we have

$$L(s, \chi_0) = \frac{P(q^{-s})}{(1 - q^{-s})(1 - q^{1-s})},$$

where $P(X)$ is a polynomial of degree $2g$ ($g$ is the genus of $F$). Thus also in this case we have proved that $\Psi(f(X))$ is a polynomial, because the factor $1 - q^{1-s}$ in the denominator of $L(s, \chi_0)$ is killed by the same factor in $\Psi(f(X))(q^{-s})$, while the factor $1 - q^{-s}$ is killed by one of the terms in the product over the primes of $S$: here we are using that $S$ is not empty.

We have just proved that $\Psi(f(X))$ is a polynomial for each continuous character $\Psi$, thus for each finite subextension $K/F$, if we denote by $\pi$ the projection $G_S \to \text{Gal}(K/F)$ we have

$$\pi(f(X)) \in \mathbb{Z}[\text{Gal}(K/F)][X]$$

and so $f(X) \in \mathbb{Z}[G_S][X] \subset W[G_S][X]$, which is our thesis. \qed

In Chapter 2 we will need to evaluate the Stickelberger series $\Theta_S(X)$ at some point of $W[G_S]$, the previous proposition grants us that when we take $x$ in the unit disk $\{x \in W[G_S] : |x| \leq 1\}$, then the series $\Theta_S(x)$ converges.

1.4 The $S_\infty$-power of an ideal

Let $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure of $F_\infty$ and put $S_\infty := \mathbb{C}_\infty \times \mathbb{Z}_p$. The analogue of the Riemann Zeta function for $F$ (i.e., in positive characteristic) has been originally defined for some special values (the integers) by Carlitz in [Car] and later extended by Goss as a $\mathbb{C}_\infty$-valued function whose domain is $S_\infty$, in [Gos2]. We will see later how the integers embeds in this topological group. This work of Goss may be interpreted a sort of analytic continuation of the function defined by Carlitz.

Before going into the details of the Goss Zeta function, we need to define the term $I^s$ for any nonzero fractional ideal $I$ of $F$ and any $s \in S_\infty$ (see also [Gos1, Chapter 8]).

**Definition 1.4.1.** A sign function on $F_\infty$ is any homomorphism $\text{sgn} : F_\infty^\times \to \mathbb{F}_\infty^\times$ such that its restriction to $\mathbb{F}_\infty^\times$ is the identity. We extend $\text{sgn}$ to all $F_\infty$ by defining $\text{sgn}(0) = 0$.

We fix a generator $\pi_\infty$ of the maximal ideal of $F_\infty$. We will say that the sign function $\text{sgn}$ is normalized if $\text{sgn}(\pi_\infty) = 1$. Since $U_1(\infty)$ is a pro-$p$-group and the image of $\text{sgn}$ has order prime
to $p$, every sign function is trivial on $U_1(\infty)$. From the decomposition $a = \pi_v^\infty(a)\zeta u$ given by the isomorphism
\[
F_\infty^\times \simeq \pi_\infty^\times \times \mathbb{Z}_p^\times \times U_1(\infty)
\] (1.2)
we deduce that for every normalized sign function we have $\text{sgn}(a) = \zeta$. The $1$-unit associated to $a \in F_\infty^\times$ is the element
\[
(a)_\infty := \frac{a}{\pi_\infty^\times(a)\; \text{sgn}(a)} \in U_1(\infty).
\]
From now on we will consider a fixed normalized sign function and the decomposition (1.2) will be written as
\[
a = \pi_v^\infty(a)\cdot \text{sgn}(a)\cdot (a)_\infty.
\]
For every $u \in U_1(\infty)$ and $y \in \mathbb{Z}_p$ the series $\sum_{n \geq 0} \binom{y}{n}(u - 1)^n$ converges in $U_1(\infty)$, so we put
\[
u_y = ((u - 1) + 1)^y := \sum_{n \geq 0} \binom{y}{n}(u - 1)^n.
\]
We say that an element $a \in F$ is positive if $\text{sgn}(a) = 1$ and we denote by $A_+$ the set of positive integers of $F$, i.e., the subset positive elements in $A$.
Let $I$ be the set of nonzero fractional ideals of $F$ and denote by $\mathcal{P}_+$ the principal fractional ideals with a positive generator. The group $I/\mathcal{P}_+$ is finite and we put
\[
h^+(A) := |I/\mathcal{P}_+|.
\]
We also denote with $p^d$ be the maximal power of $p$ that divides $h^+(A)$. We recall that $d_\infty = [F_\infty : F_q]$ and, for any $a \in F_\infty^\times$, define the degree of $a$ as $\deg(a) = -d_\infty v_\infty(a)$. Note that if $I = (i)$ is principal, then the definition of $\deg(i)$ coincides with the degree of the ideal $I$, i.e.,
\[
\deg(i) = \deg(I) := \log_q |A/I|.
\]

**Remark 1.4.2.** We have that $h^+(A) = h^0(F) \cdot d_\infty \cdot (q^{d_\infty} - 1)/(q - 1)$, where $h^0(F)$ denotes the cardinality of the class group of degree zero divisors of $F$: let $\mathcal{P}$ denote the full subgroup of principal ideals of $I$ and let $h(A)$ be the cardinality of $I/\mathcal{P}$. Applying ([Ros], Proposition 14.1, part (b)) to our setting we deduce that $h(A) = h^0(F) \cdot d_\infty$. Since $\mathcal{P}_+ \subset \mathcal{P}$, we have a surjective map $I/\mathcal{P}_+ \to I/\mathcal{P}$ whose kernel is isomorphic to $\mathcal{P}/\mathcal{P}_+$, thus $h^+(A) = h(A)|\mathcal{P}/\mathcal{P}_+| = h^0(F)d_\infty|\mathcal{P}/\mathcal{P}_+|.

To compute the cardinality of $\mathcal{P}/\mathcal{P}_+$ consider the following diagram
\[
\begin{array}{c}
F_+^\times \xrightarrow{\zeta} F^\times \xrightarrow{\text{sgn}} F_\infty^\times \\
\downarrow \quad \downarrow \quad \downarrow \\
\mathcal{P}_+^\times \xrightarrow{\zeta} \mathcal{P} \xrightarrow{\cdot} \mathcal{P}/\mathcal{P}_+
\end{array}
\]
where the vertical map on the right is induced by the central vertical map. Clearly the map $F^\times \to \mathcal{P}$ is surjective and so it is the map $F_\infty^\times \to \mathcal{P}/\mathcal{P}_+$. Note that the map $F_+ \to \mathcal{P}_+$ is an isomorphism (its kernel is $F_+ \cap F_\infty^\times = \{1\}$) thus we deduce, by the snake lemma, that the kernel of the right vertical map is isomorphic to the kernel of the central vertical map, which is $F_q^\times$. And so we have proved that $|\mathcal{P}/\mathcal{P}_+| = |F_\infty^\times|/|F_q^\times| = (q^{d_\infty} - 1)/(q - 1)$.
1.4 The $S_\infty$-power of an ideal

We fix a $d_\infty$-th root of $\pi_\infty$ and call it $\pi_\ast$: for every integer $j$ we put $s_j = (\pi_\ast^{-j}, j)$. The map $j \mapsto s_j$ gives us an embedding $\mathbb{Z} \hookrightarrow S_\infty$.

For every $s = (x, y) \in S_\infty$ our goal is to define the exponential of a fractional ideal $I^s$. We start by defining the exponential of a positive element: given $a \in F_\infty^\times$ with $\text{sgn}(a) = 1$ and $s \in S_\infty$ we put

$$a^s = x^{\text{deg}(a)} \langle a \rangle_\infty^y.$$ 

The following proposition sums up some fundamental properties which can be found in Section 8.1. of [Gos1]. We report here the proof to make the exposition more clear.

**Proposition 1.4.3.** For every $a, b \in F_\infty^\times$ with $\text{sgn}(a) = \text{sgn}(b) = 1$, one has

- $a^{s+t} = a^s a^t$ for every $s, t \in S_\infty$.
- $(ab)^s = a^s b^s$ for every $s \in S_\infty$.
- $(a^s)^j = a^{sj}$ for every $i, j \in \mathbb{Z}$.
- $a^{si} = a^i$ for every $i \in \mathbb{Z}$.

**Proof.** This is an easy exercise: let $s = (x_1, y_1)$ and $t = (x_2, y_2)$. Then

- $a^{s+t} = a^{(x_1x_2, y_1+y_2)} = (x_1x_2)^{\text{deg}(a)} \langle a \rangle_\infty^{y_1+y_2} = x_1^{\text{deg}(a)} \langle a \rangle_\infty^{y_1} x_2^{\text{deg}(a)} \langle a \rangle_\infty^{y_2} = a^s a^t$.
- $(ab)^s = x_1^{\text{deg}(ab)} \langle ab \rangle_\infty^y = x_1^{\text{deg}(a)+\text{deg}(b)} \langle a \rangle_\infty^{y_1} \langle b \rangle_\infty^{y_1} = x_1^{\text{deg}(a)} \langle a \rangle_\infty^{y_1} x_1^{\text{deg}(b)} \langle b \rangle_\infty^{y_1} = a^s b^s$.

- We recall that $a^s = \pi_\ast^{-i\text{deg}(a)} \langle a \rangle_\infty^i$ thus we have $\langle a^s \rangle_\infty = \langle a \rangle_\infty^i$ and

$$\text{deg}(a^s) = \text{deg} \left( \pi_\ast^{-i\text{deg}(a)} \right)$$

$$= -d_\infty v_\infty \left( \pi_\ast^{-id_\infty v_\infty(a)} \right)$$

$$= -d_\infty v_\infty \left( \pi_\ast^{-iv_\infty(a)} \right)$$

$$= -d_\infty iv_\infty(a)$$

$$= i \text{deg}(a).$$

And so $(a^s)^j = \pi_\ast^{-j(i\text{deg}(a))} \langle a \rangle_\infty^{ij} = \pi_\ast^{-ij\text{deg}(a)} \langle a \rangle_\infty^{ij} = a^{sj}$.

- $a^{si} = \pi_\ast^{-i\text{deg}(a)} \langle a \rangle_\infty^i = \pi_\ast^{id_\infty v_\infty(a)} \langle a \rangle_\infty^i = \left( \pi_\ast^{v_\infty(a)} \langle a \rangle_\infty \right)^i$ and, since $a$ is positive, this last term is equal to $a^i$.

To simplify notations put

$$e := h^+(A) = |I/P_+|.$$ 

Now we can define the exponential of a fractional ideal: given $I \in \mathcal{I}$, there exists a positive element $\alpha \in F^\times$ such that $I^e = (\alpha)$. We put

$$I^s = x^{\text{deg} I} \langle \alpha \rangle_\infty^{y/e},$$
where \( \langle \alpha \rangle_{\infty}^{1/e} \) denotes the unique e-th root of \( \langle \alpha \rangle_{\infty} \) that is a 1-unit. Furthermore we put
\[
(I)_{\infty} := \langle \alpha \rangle_{\infty}^{1/e}.
\]

We observe that in general \( (I)_{\infty} \) and \( I^s \) do not belong to \( F_{\infty} \), but to a suitable extension (see the following proposition).

**Proposition 1.4.4.**  (a) \( (I)_{\infty} \) and \( I^s \) are well defined.

(b) If \( I = (\alpha) \in \mathcal{P}_+ \) then \( I^s = \alpha^s \).

(c) For every \( j \in \mathbb{Z} : I^{s_j} \) is algebraic over \( F \).

(d) Let \( F_V \) be the extension of \( F \) obtained by adding every element of the form \( I^{s_1} \) with \( I \in \mathcal{I} \). Then \( F_V/F \) is a finite extension with degree at most \( e \).

(e) Let \( F_{\infty,V} \) be the extension of \( F_{\infty} \) obtained by adding every element of the form \( (I)_{\infty} \) with \( I \in \mathcal{I} \). Then \( F_{\infty,V}/F \) is a finite extension with degree that divides \( p^j \).

**Proof.**  (a) We suppose \( I^e = (\alpha) = (\beta) \) for some positive \( \alpha, \beta \). Then \( \alpha \) and \( \beta \) differ by a unit of \( A \), i.e., by an element \( a \in \mathbb{F}_q^\times \) (see, for example [Ros, Proposition 5.1]). Therefore \( \alpha = \beta a \) implies \( \text{sgn}(\alpha) = \text{sgn}(\beta)\text{sgn}(a) \). However \( \alpha \) and \( \beta \) are positive and so \( \text{sgn}(a) = 1 \). Finally, since \( a \in \mathbb{F}_q^\times \), we conclude that \( a = \text{sgn}(a) = 1 \) and \( \alpha = \beta \).

(b) From the equality \( I = (\alpha) \) it follows that \( \text{deg}(I) = \text{deg}(\alpha) \) and \( I^e = (\alpha^e) = (\alpha^e) \). Therefore
\[
I^s = x^{\text{deg}(I)}(\alpha^e)^{y/e} = x^{\text{deg}(\alpha)}(\alpha)^{y} = \alpha^s.
\]

(c) Let \( I \in \mathcal{I} \) and \( \alpha \in F \) be such that \( I^e = (\alpha) \). We prove that \( I^{s_j} \) is a root of the polynomial \( f(X) = X^e - \alpha^j \in F[X] \).

The equality \( I^e = (\alpha) \) implies that \( e \text{ deg}(I) = \text{deg}(\alpha) = -d_{\infty} v_{\infty}(\alpha) \). Therefore
\[
(I^{s_j})^e = \left( \pi^e_s \right)^{-j} \text{deg}(I)^{j/e} (\alpha)^{j/e} = \left( \pi^e_s \right)^{-j} \text{deg}(I)^{j/e} (\alpha)^{j/e} = \left( \pi^e_s (\alpha)^{j/e} \right).
\]

(d) We have already proved in the previous step that \( I^{s_1} \) is algebraic over \( F \), so it is enough to show that \( F_V \) can be generated by a finite number of elements of the form \( I^{s_1} \). Let \( I \) and \( J \) be representatives of the same equivalence class in \( \mathcal{I}/\mathcal{P}_+ \) and let \( \alpha, \beta \) and \( \gamma \) be positive elements such that \( I^e = (\alpha) \), \( J^e = (\beta) \) and \( I = \gamma J \). Clearly there exists a constant \( a \) such that the equality \( \alpha = \gamma^e \beta a \) holds, but the same considerations explained in (a), lead us to conclude that \( a = 1 \). Furthermore we have that \( \text{deg}(I) = \text{deg}(J) + \text{deg}(\gamma) \). Then
\[
I^{s_1} = \pi^{-\text{deg}(I)}(\alpha)^{1/e}_{\infty} = \pi^{-\text{deg}(J) - \text{deg}(\gamma)}(\beta)^{1/e}_{\infty} = J^{s_1} \pi^{-\text{deg}(\gamma)}(\gamma)^{1/e}_{\infty}.
\]
\[
= J^{s_1}\pi_{\infty}^{\nu_\infty(\gamma)}(\gamma) = J^{s_1} \gamma.
\]

Therefore to generate \( F_V \) it is enough to take one fractional ideal \( I^{s_1} \) for every equivalence class of \( I/\mathcal{P}_+ \).

To prove that the degree of \( F_V/F \) is at most \( e \) we proceed in the following way:

Step 1: If the equivalence class of \( J \) is a power of the class of \( I \) in \( I/\mathcal{P}_+ \) then \( F(J^{s_1}) \subseteq F(I^{s_1}) \), indeed if \( J = I^r \gamma \) for some positive element \( \gamma \) and some \( r \in \mathbb{N} \) we have that \( J^{s_1} = I^{rs_1} \gamma \) and so \( J^{s_1} \in F(I^{s_1}) \).

Step 2: If \( C \) is a cyclic component of \( I/\mathcal{P}_+ \) and \( I \) is a representative for a generator of \( C \), then from the previous step it follows that

\[
F(I^{s_1}) \subseteq \bigcup_{[J] \in C} F(J^{s_1}) \subseteq F(I^{s_1}).
\]

Step 3: We decompose \( I/\mathcal{P}_+ \) as the sum of its cyclic components

\[
I/\mathcal{P}_+ = \bigoplus_{i=1}^r C_i
\]

with \( n_i = |C_i| \), and for every cyclic component \( C_i \) let \( I_i \) be a representative of a generator. We observe that \( e = \prod_{i=1}^r n_i \).

In the proof of (c) we have proved that \( I^{s_j} \) is a root of the polynomial \( f(X) = X^e - \alpha^j \), when \( I^e = (\alpha) \). However in general \( f(X) \) is not the minimal polynomial of \( I^{s_j} \), indeed if there exists \( k \in \mathbb{N} \) and a positive \( \beta \) such that \( I^k = (\beta) \), the same argument of (c) shows that \( I^{s_j} \) is a root of \( g(X) = X^k - \beta^j \). Therefore the degree of the extension \( F(I^{s_1})/F \) is smaller than or equal to the order of the class of \( I \) in \( I/\mathcal{P}_+ \). From this it also follows that the minimal polynomial of \( I^{s_1} \) over \( F \) has degree at most \( n_i \).

From the previous step we deduce that \( F_V = \bigcup_{i=1}^r F(I_i^{s_1}) \) and so

\[
[F_V : F] \leq \prod_{i=1}^r [F(I_i^{s_1}) : F] \leq \prod_{i=1}^r n_i = e.
\]

(e) The proof of this point is similar to the previous one, it is enough to recall that \( F_{\infty,V} \) is obtained by adding to \( F_{\infty} \) the \( e \)-th roots of elements in \( U_1(\infty) \) and that, since \( U_1(\infty) \) is a multiplicative group isomorphic to \( \mathbb{Z}_{p^\infty}^\times \), \( F_{\infty} \) already contains the \( n \)-th roots of the elements of \( U_1(\infty) \) when \( n \) is coprime with \( p \) (by Hensel’s Lemma).

All the objects defined until now depend on the choice of the positive element \( \pi_{\infty} \) and on the choice of its \( d_{\infty} \)-th root \( \pi_* \). Before concluding this section we want to see what happens when we change these two elements. To do this we will follow the ideas of [Gos1, Section 8.2.]. Let \( \pi_{(i)}, i = 1, 2 \) be generators of the maximal ideal of \( F_{\infty} \), both positive with respect to the sign function and \( \pi_{*(i)}, i = 1, 2 \) two fixed \( d_{\infty} \)-th roots. We denote with \( (I)_{\infty,(i)} \) the 1-unit associated
to $I$, with respect to $\pi_{(1)}$ and with $I^e_{(i)}$ the corresponding exponential.

The elements $\pi_{(1)}$ and $\pi_{(2)}$ differ by a unit in $A^\times_{\infty} = \mathbb{F}^\times_{\infty} \times U_1(\infty)$. Let $u = \pi_{(1)}/\pi_{(2)}$, since $\pi_{(1)}$ and $\pi_{(2)}$ are both positive, $u$ is a 1-unit. We denote with $u^{1/d_{\infty}}$ the only $d_{\infty}$-th root of $u$ which is a 1-unit. We observe that it is uniquely determined by $\pi_{(1)}$ and $\pi_{(2)}$, i.e., it does not depend on $\pi_{*(1)}$ and $\pi_{*(2)}$.

**Lemma 1.4.5.** For every $I \in \mathcal{I}$ we have

$$\langle I \rangle_{\infty,(1)} = \left( u^{1/d_{\infty}} \right)^{\deg(I)} \langle I \rangle_{\infty,(2)}. $$

**Proof.** From the equality

$$\left( u^{1/d_{\infty}} \right)^{d_{\infty}} = \frac{\pi_{(1)}}{\pi_{(2)}} \left( \frac{\pi_{*(1)}}{\pi_{*(2)}} \right)^{d_{\infty}}$$

it follows that there exists a $d_{\infty}$-th root of unity $\zeta$ such that

$$u^{1/d_{\infty}} = \zeta \frac{\pi_{*(1)}}{\pi_{*(2)}}.$$  

Let $\alpha$ be such that $I^e = (\alpha)$. We have that $e \deg(I) = \deg(\alpha) = -d_{\infty} \nu_{\infty}(\alpha)$ and, for $i = 1, 2$

$$\langle I^e \rangle_{\infty,(i)} = (\alpha)_{\infty,(i)} = \frac{\alpha}{\pi_{\nu_{\infty}(\alpha)}} = \frac{\alpha}{\pi_{*(i)}}.$$  

Therefore recalling that, by Remark 1.4.2, $d_{\infty}$ divides $e$, we have

$$\left( \frac{\langle I \rangle_{\infty,(1)}}{\langle I \rangle_{\infty,(2)}} \right)^e = \left( u^{1/d_{\infty}} \zeta^{-1} \right)^e = \left( u^{1/d_{\infty}} \right)^e.$$  

Finally, we observe that inside the brackets in the first and in the last term of the equality there are 1-units, so we can extract the $e$-th root without ambiguity and obtain the thesis.  

**Corollary 1.4.6.** There exists a $d_{\infty}$-th root of unity $\zeta$ such that

$$I_s^{\zeta_{j \deg(I)}} = \zeta_{j \deg(I)} I_{s(2)}^{\zeta_{j \deg(I)}}.$$  

**Proof.** Let $\zeta$ be as in the previous lemma. Then

$$I_s^{\zeta_{j \deg(I)}} = \pi_{*(1)}^{-j \deg(I)} \langle I \rangle_{\infty,(1)}^{\zeta_{j \deg(I)}}$$

$$= \left( \pi_{*(2)}^{-j \deg(I)} \right)^{u^{1/d_{\infty}}} \left( u^{1/d_{\infty}} \right)^{j \deg(I)} \langle I \rangle_{\infty,(2)}^{\zeta_{j \deg(I)}}$$

$$= \zeta_{j \deg(I)} I_{s(2)}^{\zeta_{j \deg(I)}}.$$  

**Corollary 1.4.7.** If $F$ contains all the $d_{\infty}$-th roots of unity then $F_V$ does not depend on the choice of $\pi_{\infty}$ and $\pi_*$.  

We observe that the hypothesis of the corollary is equivalent saying that $d_{\infty}$ has the form $p^m$, where $m$ is any divisor of $q - 1$.  


1.5 The Goss Zeta Function

Recall that by \(a \geq 0\) we mean the integral ideals of \(A\).

**Definition 1.5.1.** The Goss Zeta function is defined by the sum

\[
\zeta_A(s) = \sum_{a \in I, a \geq 0} a^{-s},
\]

for \(s = (x, y) \in S_\infty\).

This sum is clearly convergent for \(|x|_\infty > 1\) and can also be rewritten as an Euler product just like the classical Riemann Zeta function

\[
\zeta_A(s) = \prod_{\nu \neq \infty} (1 - \nu^{-s})^{-1}.
\]

Since \(-s = (x^{-1}, -y)\), this product amounts to

\[
\zeta_A(s) = \prod_{\nu \neq \infty} \left(1 - \langle \nu \rangle_\infty^{-y} x^{-\deg(\nu)}\right)^{-1} = \prod_{\nu \neq \infty} \left(1 - \langle \alpha_\nu \rangle_\infty^{-y} x^{-\deg(\nu)}\right)^{-1}
\]

for some \(\alpha_\nu\) such that \(\nu^e = (\alpha_\nu)\).

We recall that for every integer \(n\) we have only a finite number of integral ideals \(a\) with degree equal to \(n\) and so the sum

\[
a_n(y) = \sum_{\substack{a \in I, a \geq 0 \\ deg(a) = n}} \langle a \rangle_\infty^{-y} \in \mathbb{C}_\infty
\]

is finite. This allows us to write the Goss Zeta function as a sum over the positive integers.

\[
\zeta_A(s) = \sum_{n \geq 0} a_n(y)x^{-n}. \tag{1.3}
\]

This last form is very important because we will prove it is convergent for every \((x, y) \in S_\infty\) and so can be interpreted as an analytic extension to the whole space \(S_\infty\) of the Goss Zeta function.

1.5.1 Convergence of the Goss Zeta Function

In this section we will prove that for every \(s = (x, y) \in S_\infty\) the sum (1.3) is convergent and, in particular, this provides the analytic extension of the Goss Zeta function to the whole space. More precisely we will prove that when \(n\) goes to infinity the number \(v_\infty(a_n(y))\) diverges faster than a linear polynomial in \(n\), uniformly with respect to \(y\) and, since \(v_\infty(x^{-n})\) is linear with respect to \(n\), the convergence of the series will readily follow. Indeed we shall prove that, when \(n\) is big enough, the number \(v_\infty(a_n(y))\) behaves like a polynomial in \(n\) of degree 2 (or is larger than that).
We fix $a_1, a_2, \ldots, a_e \in I$ representatives of the equivalence classes $C_1, C_2, \ldots, C_e$ of $I/P_+$. Every non null fractional ideal of $C_j$ can be written uniquely as the product $a = a a_j$, with $\alpha \in F_+$. Now we have

$$a_n(y) = \sum_{j=1}^{e} (a_j y)^{-\infty} \sum_{\substack{\alpha \in F_+ \\alpha a_j \geq 0 \\deg(\alpha a_j) = n}} \langle \alpha \rangle^{-y}.$$  \hspace{1cm} (1.4)

Fix an index $j$, let $n_j = n - \deg(a_j)$ and put

$$a_n(C_j, y) = \sum_{\substack{\alpha \in C_j \\alpha \geq 0 \\deg(\alpha) = n}} \langle \alpha \rangle^{-y} \quad \text{and} \quad b_{n,j}(y) = \sum_{\substack{\alpha \in F_+ \\alpha a_j \geq 0 \\deg(\alpha) = n_j}} \langle \alpha \rangle^{-y},$$

so that $a_n(C_j, y) = \langle a_j \rangle^{-y} \cdot b_{n,j}(y)$. From (1.4) we deduce that our thesis can be obtained just proving that the function $v_{\infty}(b_{n,j}(y))$ is bounded from below by a polynomial of degree 2 in $n$ for any $j$, since $|\langle a_j \rangle^{-y}|_{\infty} = 1$.

We can limit ourselves to integers $n$ such that $d_\infty$ divides $n_j$, because for the other values of $n$ we have $b_{n,j}(y) = 0$ (there are no $\alpha$ with degree $n_j$).

Let $\text{div}(x)$ be the divisor associated to $x$, i.e., the element

$$\text{div}(x) = \sum_{\nu} \text{ord}_\nu(x) \nu.$$ 

Without ambiguity we will use this notation for both the elements $\alpha \in F_+$ and for the fractional ideals $a \in I$. Be careful: the divisors associated to the element $\alpha$ and to the fractional ideal $(\alpha)$ may not coincide since the component at infinity may not be the same. For the same reason the degrees of $\alpha$ and of the divisor associated to $\alpha$ may be different (remember that principal divisors have always degree zero). Contrariwise the degrees of a fractional ideal and of its divisor always coincide. To make this more clear consider the following example.

**Example 1.5.2.** Let $F$ be the rational function field $F_q(T)$ and fix as a place at infinity the rational place corresponding to $T - 1$. In this case the ring of $\infty$-integers is just the polynomial ring $F_q[T]$. If we denote with $p$ the prime corresponding to the element $T$ we have that the divisor associated to the element $T$ is $p - \infty$ while the divisor associated to the ideal $(T)$ is just $p$.

Furthemore we have that $\deg(T - 1) = 1 = \deg(p)$ and $\deg(p - \infty) = 0$.

For every $i \in \mathbb{N}$ let $D_i = \text{div}(a_j) + (n_j/d_\infty - i)\infty$ and $\mathcal{L}(D_i)$ be its Riemann-Roch space, which is defined by

$$\mathcal{L}(D_i) = \{ \alpha \in F^\times : \text{div}(\alpha) + D_i \geq 0 \}.$$ 

We observe that the divisors $D_i$ are sorted in descending order and so each space $\mathcal{L}(D_i)$ is contained in the previous one, i.e., $\mathcal{L}(D_i) \supseteq \mathcal{L}(D_{i+1})$ for any $i$. Furthermore the following equality is true

$$\mathcal{L}(D_i) = \left\{ \alpha \in \mathcal{L}(D_0) : v_\infty(\alpha) \geq i - \frac{n_j}{d_\infty} \right\}. \hspace{1cm} (1.5)$$
where we denote with \( g \) for \( i < n \) calculate it. Since we need to know the dimension of some of these spaces and we use the Riemann-Roch theorem to this reason we can always assume that there exists at least one index \( i \) such that the previous equality holds.

From equality (1.5) it follows that \( \alpha \in F^\times \) satisfies both the following conditions

- \( \alpha a_j \geq 0 \)
- \( \deg(\alpha) = n_j \)

if and only if \( \alpha \in \mathcal{L}(D_0) - \mathcal{L}(D_1) \) (note that they imply \( v_\infty(\alpha) = -\frac{n_j}{d_\infty} \)). If we put \( X = \{ \alpha \in \mathcal{L}(D_0) - \mathcal{L}(D_1) : \sgn(\alpha) = 1 \} \), we can rewrite \( b_{n,j}(y) \) as

\[
b_{n,j}(y) = \sum_{\alpha \in X} \langle \alpha \rangle^{-y}.
\]

**Lemma 1.5.3.** Let \( a \in \mathcal{L}(D_1) \) and \( b \in X \). Then \( a + b \in X \).

**Proof.** Both \( \mathcal{L}(D_1) \) and \( X \) are subsets of the vector space \( \mathcal{L}(D_0) \) and so \( a + b \) belongs to \( \mathcal{L}(D_0) \). We observe that \( v_\infty(a) \geq 1 - n_j/d_\infty \) while \( v_\infty(b) = -n_j/d_\infty \neq v_\infty(a) \), therefore

\[
v_\infty(a + b) = \min\{v_\infty(a), v_\infty(b)\} = -\frac{n_j}{d_\infty}
\]

and \( a + b \not\in \mathcal{L}(D_1) \).

To complete our proof it remains to show that \( a + b \) is positive. Let \( m = v_\infty(a) \geq 1 - n_j/d_\infty \) and let \( \hat{a} \in O_\infty^\times \) be such that \( a = \pi_\infty^m \hat{a} \).

For every \( \alpha \in F_\infty^\times \) we have that \( \sgn(\alpha) \equiv \alpha \pi_\infty^{-v_\infty(a)} \pmod{\pi_\infty} \) and so we can write \( b = \pi_\infty^{-n_j/d_\infty} (1 + \pi_\infty \hat{b}) \) with \( \hat{b} \in O_\infty \) (note that \( b \) is positive). Then

\[
a + b = \pi_\infty^{-n_j/d_\infty} (\pi_\infty^{m+n_j/d_\infty} \hat{a} + 1 + \pi_\infty \hat{b}) = \pi_\infty^{v_\infty(a+b)} (1 + \pi_\infty^{m+n_j/d_\infty} \hat{a} + \pi_\infty \hat{b}).
\]

Hence

\[
\sgn(a + b) \equiv (a + b)\pi_\infty^{-v_\infty(a+b)} \pmod{\pi_\infty} \quad (1.6)
\equiv 1 + \pi_\infty^{m+n_j/d_\infty} \hat{a} + \pi_\infty \hat{b} \pmod{\pi_\infty}
\equiv 1 \pmod{\pi_\infty}
\]

and the thesis is proved. \( \square \)
The previous lemma tells us that the sum defines an action of \( \mathcal{L}(D_1) \) (seen as an abelian group) on the set \( X \) by translation. It is obvious that this action is free, since the equality 
\[ g + x = h + x, \]
with \( g, h \in \mathcal{L}(D_1) \) and \( x \in X \) can be obtained only for \( g = h \). Now we can decompose \( X \) as the disjoint union of the orbits of its elements under this action. Let \( X_1, X_2, \ldots, X_t \) be the orbits of the elements of \( X \) (so that with \( X_i \cap X_j = \emptyset \) if \( i \neq j \) and \( X = \bigcup_{i=1}^t X_i \)) and fix an element \( x_l \in X_l \) for each orbit. Then

\[
\begin{align*}
\sum_{\alpha \in X} \langle \alpha \rangle_{\infty}^{-y} &= \sum_{l=1}^t \sum_{\alpha \in X_l} \langle \alpha \rangle_{\infty}^{-y} \\
&= \sum_{l=1}^t \sum_{u \in \mathcal{L}(D_1)} \langle u + x_l \rangle_{\infty}^{-y}.
\end{align*}
\]

We put

\[
h_n(y) = h_{n,j,l}(y) := \sum_{u \in \mathcal{L}(D_1)} \langle u + x_l \rangle_{\infty}^{-y}.
\]

Just like before note that our thesis on the growth of \( v_\infty(h_{n,j}(y)) \) may be obtained by proving that the growth of the function \( v_\infty(h_n(y)) \) (as a function of \( n \)) is greater than a linear polynomial in \( n \).

The following lemma puts together two results of [Tha, Chapter 5]. Here we give a detailed proof.

**Lemma 1.5.4.** Let \( K \) be a function field with constant field \( \mathbb{F}_q \), \( v \) any normalized valuation on \( K \) and \( W \subset K \) an \( \mathbb{F}_q \)-vector space with finite dimension. Assume that \( v(w) > 0 \) for every \( w \in W \).

(a) If \( i \) is an integer with \( 0 \leq i < (q-1)\dim_{\mathbb{F}_q} W \), for every \( x \in K \) we have

\[
\sum_{w \in W} (x + w)^i = 0.
\]

(b) For every \( j \in \mathbb{N}^+ \) we put \( W_j = \{ w \in W : v(w) \geq j \} \). Then for every \( y \in \mathbb{Z}_q \) we have

\[
v \left( \sum_{w \in W} (1 + w)^y \right) \geq (q-1)Q,
\]

where we put \( Q = \sum_j \dim_{\mathbb{F}_q} W_j \).

**Proof.** (a) Let \( d = \dim_{\mathbb{F}_q} W \) and we fix a basis \( \{ e_1, e_2, \ldots, e_d \} \) over \( \mathbb{F}_q \). For every \( j \in \mathbb{N} \) we have

\[
\delta_j = \sum_{c \in \mathbb{F}_q} c^j = \begin{cases} 
-1 & \text{if } q-1 \text{ divides } j \text{ and } j > 0, \\
0 & \text{otherwise}.
\end{cases}
\]

\(^1\)In the sum it may appear a term of the form \( 0^0 \). In this case we use the convention that \( 0^0 := 1 \).
Applying the multinomial theorem we have

\[ \sum_{w \in W} (x + w)^i = \sum_{c_1, c_2, \ldots, c_d \in \mathbb{F}_q} (x + c_1e_1 + \ldots + c_de_d)^i \]
\[ = \sum_{c_1, c_2, \ldots, c_d \in \mathbb{F}_q} \sum_{j_1, j_2, \ldots, j_d} \frac{i!}{j_1! \ldots j_d!(i - j_1 - \cdots - j_d)!} x^{i-j_1-\cdots-j_d} (c_1e_1)^{j_1} \ldots (c_de_d)^{j_d} \]
\[ = \sum_{j_1, j_2, \ldots, j_d} \frac{i!}{j_1! \ldots j_d!(i - j_1 - \cdots - j_d)!} x^{i-j_1-\cdots-j_d} e_1^{j_1} \ldots e_d^{j_d} \delta_{j_1} \ldots \delta_{j_d}. \]

In this sum each term is equal to zero, indeed we have that \( j_1 + \cdots + j_d \leq i < (q - 1)d \) and so at least one of the index \( j \) is smaller than \( q - 1 \). For that index we have \( \delta_j = 0 \).

(b) The first member of the inequality (seen as a function of \( y \)) is continuous and the second member does not depend on \( y \). This implies that it is enough to prove the statement when \( y \) is a positive integer because the set of positive integers \( \mathbb{N}^+ \) is dense in \( \mathbb{Z}_p \).

Let \( J \) be the greatest positive integer such that \( W_J \neq \{0\} \) and so we have

\[ \{0\} = W_{J+1} \subsetneq W_J \subseteq W_{J-1} \subseteq \cdots \subseteq W_2 \subseteq W_1 = W. \]

For every \( j = 1, \ldots, J \) we denote with \( d_j \) the dimension of \( W_j \) and with \( d = d_1 = \dim_{\mathbb{F}_q} W \) such that

\[ 0 < d_j \leq d_{j-1} \leq \cdots \leq d_{j+1} \leq d_j \leq \cdots \leq d_2 \leq d_1 = d. \]

We fix a basis \( \{e_1, e_2, \ldots, e_d\} \) of \( W \) such that for every \( j \) the set \( \{e_1, e_2, \ldots, e_{d_j}\} \) forms a basis for \( W_j \). Furthermore we put

\[ U_j = \text{Span}_{\mathbb{F}_q} \{e_{1+d_{j+1}}, \ldots, e_d\}. \]

We observe that for every index \( j \) we have:

- \( W_j = W_{j+1} \oplus U_j \).
- \( \dim_{\mathbb{F}_q} U_j = d_j - d_{j+1} \).
- For every \( u \in U_j, v(u) \geq j \) (in particular we have the equality for every \( u \neq 0 \)).

We will prove the following statement: for every index \( j \in \{1, 2, \ldots, J\} \) and for every positive integer \( y \in \mathbb{N}^+ \) we have the inequality

\[ v \left( \sum_{w \in W_j} (1 + w)^y \right) \geq (q - 1) \left( jd_j + \sum_{i=j+1}^J d_i \right). \tag{1.8} \]

Note that for \( j = 1 \) the previous statement is exactly the thesis of \( (b) \).

We will prove the inequality by an inductive process: we will first give a proof for \( j = J \) and then we will prove it for a generic index \( j \), by assuming it is true for the index \( j + 1 \).

Case \( j = J \): by the binomial theorem we have

\[ \sum_{w \in W_J} (1 + w)^y = \sum_{h=0}^y \binom{y}{h} \sum_{w \in W_J} w^h. \]
1.5 The Goss Zeta Function

We apply part (a) to the vector space $W_j$ to deduce that the sum $\sum_{w \in W_j} w^h$ is equal to zero when $h < (q - 1)d_j$, thus we have

$$\sum_{w \in W_j} (1 + w)^y = \sum_{h=(q-1)d_j}^y \binom{h}{y} \sum_{w \in W_j} w^h.$$

Since $v$ is a non-Archimedean valuation we have

$$v \left( \sum_{h=(q-1)d_j}^y \binom{h}{y} \sum_{w \in W_j} w^h \right) \geq \min_{h \geq (q-1)d_j} \{v(w^h)\} = \min_{h \geq (q-1)d_j} \{hJ\} = (q - 1)Jd_j,$$

thus we have proved

$$v \left( \sum_{w \in W_j} (1 + w)^y \right) \geq (q - 1)Jd_j.$$

Now we assume that the inequality (1.8) is true for the index $j + 1$ and we prove it for the index $j$: using the decomposition $W_j = W_{j+1} \oplus U_j$ and the binomial theorem we have

$$\sum_{w \in W_j} (1 + w)^y = \sum_{t \in W_{j+1}} (1 + t + u)^y = \sum_{h=0}^y \binom{h}{y} \sum_{t \in W_{j+1}} (1 + t)^h \sum_{u \in U_j} u^{y-h}.$$

We apply part (a) to the vector space $U_j$ to deduce that the sum $\sum_{u \in U_j} u^{y-h}$ is equal to zero when $y - h < (q - 1)\text{dim}_q U_j$, i.e., $h > y - (q - 1)(d_j - d_{j+1}) =: \hat{y}$, thus we have

$$\sum_{w \in W_j} (1 + w)^y = \sum_{h=0}^{\hat{y}} \binom{h}{y} \sum_{u \in U_j} u^{y-h} \sum_{t \in W_{j+1}} (1 + t)^h.$$

Proceeding like the case $j = J$ and using the assumption for the index $j + 1$ we have

$$v \left( \sum_{w \in W_j} (1 + w)^y \right) \geq \min_{h \leq \hat{y}} \left\{ v \left( u^{y-h} \sum_{t \in W_{j+1}} (1 + t)^h \right) \right\}$$

$$= \min_{h \leq \hat{y}} \left\{ (y - h)v(u) + v \left( \sum_{t \in W_{j+1}} (1 + t)^h \right) \right\}$$

$$\geq \min_{h \leq \hat{y}} \left\{ (y - h)v(u) + (q - 1) \left( (j + 1)d_{j+1} + \sum_{l=j+2}^J d_l \right) \right\}$$

$$= (q - 1)(d_j - d_{j+1}) j + (q - 1) \left( (j + 1)d_{j+1} + \sum_{l=j+2}^J d_l \right)$$

$$= (q - 1) \left( jd_j + \sum_{l=j+1}^J d_l \right).$$

$\square$
We use this lemma to give an estimate of $h_n(y)$:

**Lemma 1.5.5.** Let $h_n(y)$ be the function defined in (1.7). Then we have

$$v_\infty (h_n(y)) \geq \frac{(q-1)n^2}{2d_\infty} + O(n)$$

uniformly with respect to $y$.

**Proof.** We have

$$(u + x_l)_\infty = \pi_\infty^{-v_\infty(u+x_l)}(u+x_l) = \pi_\infty^{n_j/d_\infty} x_l \left(1 + \frac{u}{x_l}\right).$$

Recall that, by definition of $X$, $x_l$ is positive and $v_\infty(x_l) = -n_j/d_\infty$, thus $\pi_\infty^{n_j/d_\infty} x_l$ is a 1-unit and so we can elevate it to the power $-y$, furthermore $v_\infty(u/x_l) = v_\infty(u) - v_\infty(x_l) = v_\infty(u) + n_j/d_\infty > 0$. Therefore from (1.7) we deduce

$$h_n(y) = (\pi_\infty^{n_j/d_\infty} x_l)^{-y} \sum_{u \in L(D_1)} \left(1 + \frac{u}{x_l}\right)^{-y} \sum_{w \in x_l^{-1} L(D_1)} (1 + w)^{-y}.$$

The elements of $x_l^{-1} L(D_1)$ have positive valuation at $\infty$ and the vector space $x_l^{-1} L(D_1)$ satisfies every hypothesis of point (b) of Lemma 1.5.4. Hence we have

$$v_\infty (h_n(y)) = v_\infty \left( \sum_{w \in x_l^{-1} L(D_1)} (1 + w)^{-y} \right) \geq (q-1)Q,$$

where $Q = \sum_i \dim_{\mathbb{F}_q} W_i$ and $W_i = \{ w \in x_l^{-1} L(D_1) : v_\infty(w) \geq i \}$. Now we observe that the element $w$ is in $W_i$ if and only if $u = x_l w$ is in $L(D_1)$ and $v_\infty(u) \geq i + v_\infty(x_l) = i - n_j/d_\infty$, i.e., $u \in L(D_1)$. From this fact we can deduce that the map $W_i \to L(D_i)$, $w \mapsto x_l w$ is an isomorphism of $\mathbb{F}_q$-vector spaces and so

$$\dim_{\mathbb{F}_q} W_i = \dim_{\mathbb{F}_q} L(D_i) = \ell_i.$$

Finally, with some calculations we obtain that

$$Q = \sum_{i=1}^{\ell_i} \ell_i \geq \sum_{i=1}^{[(n-2g+2)/d_\infty]} \ell_i.$$ 

If $d_\infty$ does not divide $n - 2g + 2$ the last sum is equal to

$$\sum_{i=1}^{[(n-2g+2)/d_\infty]} (n - id_\infty - g + 1),$$

if $d_\infty$ divides $n - 2g + 2$ the last sum is equal or greater than

$$\sum_{i=1}^{(n-2g+2)/d_\infty - 1} (n - id_\infty - g + 1).$$
thus in both cases we have:
\[
Q \geq \sum_{i=1}^{\infty} \left( n - id_{\infty} + O(1) \right)
\]
\[
= (n + O(1)) \left( \frac{n}{d_{\infty}} + O(1) \right) - d_{\infty} \sum_{i=1}^{\infty} i
\]
\[
= \frac{n^2}{d_{\infty}} + O(n) - \frac{d_{\infty}}{2} \left( \frac{n}{d_{\infty}} + O(1) \right) \left( 1 + \frac{n}{d_{\infty}} + O(1) \right)
\]
\[
= \frac{n^2}{d_{\infty}} + O(n) - \frac{n^2}{2d_{\infty}} + O(n)
\]
\[
= \frac{n^2}{2d_{\infty}} + O(n).
\]

From this lemma and the previous remarks it follows that

**Theorem 1.5.6** (Analytic extension of the Goss Zeta Function). The series
\[
\sum_{n \geq 0} a_n(y)x^{-n}
\]
is absolutely convergent for every \((x, y) \in S_{\infty}\) and is also uniformly convergent on the compact subsets of \(S_{\infty}\).

1.6 Stickelberger series and Goss Zeta Function

Our goal is to show a link between the Zeta function and the Stickelberger series \(\Theta_S(X)\) as we did in Section 1.3 for the Artin L-functions.

Let \(W_S\) be the subgroup of \(G_S\) generated by all the Artin symbols \(\phi_{\nu}\) with \(\nu \notin S\) and let \(K\) be the fixed field of the topological closure of \(W_S\). In Section 1.2 we have already observed that the element \(\phi_{\nu}\) is a topological generator of the decomposition group of \(\nu\) in \(G_S\), therefore the extension \(K/F\) is totally split at every prime \(\nu \notin S\). From Tchebotarev density theorem we deduce that \(K = F\) and so \(G_S\) is the topological closure of \(W_S\).

**Remark 1.6.1.** We note that the group \(W_S\) is contained in the projection of the Weil group to \(G_S\).

**Lemma 1.6.2.** Let \(\lambda\) and \(\mu\) be two distinct primes outside \(S\). Then \(\phi_{\mu} \neq \phi_{\lambda}\).

**Proof.** Let \(\mathbb{I}_F\) be the idèle group of \(F\) and \(H\) be the subgroup
\[
F_\mu^\times \times \prod_{\nu \neq \mu, \nu \notin S} O_\nu^\times \times \prod_{\nu \in S} \{1\}.
\]
Furthermore let \(K\) be the class field of \(F^\times H\) (as usual \(F^\times\) is embedded diagonally in \(\mathbb{I}_F\), while \(F_\nu^\times\) is embedded via the map which sends the element \(x\) to the idèle whose coordinates
are all equal to 1, except for the one corresponding to the prime $\nu$ which is equal to $x$) and
$\operatorname{rec} : I_F \to \text{Gal}(K/F)$ be the map induced by the Artin map.
For every prime $\nu \notin S$ we have that $O^\times_\nu$ is contained in the kernel of $\operatorname{rec}$ and so the extension $K/F$ is unramified at every prime outside $S$, in particular we have that $K$ is contained is $F_S$. Moreover
$$\operatorname{Frob}_K \mu = \operatorname{rec}(\pi_\mu) = 1$$
$$\operatorname{Frob}_K \lambda = \operatorname{rec}(\pi_\lambda) \neq 1$$
since $\pi_\lambda \notin F^\times H$. Therefore the extension $K/F$ is totally split at the prime $\mu$, while $\lambda$ is inert in $K$. From this observation it follows that the decomposition groups of $\mu$ and $\lambda$ in $G_S$ do not coincide and so to distinct primes correspond distinct Artin symbols.

We denote with $f$ the degree of the extension $F_{\infty, V}/F_\infty$ (recall that $f$ divides $p^l$ by Proposition 1.4.4) and with $N : F^\times_{\infty, V} \to F^\times_\infty$ the norm map. For any $y \in \mathbb{Z}_p$ and for any $\nu \notin S$ we put
$$\Psi_y(\phi_\nu) = N \left( (\nu)^{-1} \right)^{y/f} .$$
Observe that the norm sends 1-units to 1-units and therefore it is possible to extract the $f$-th root without ambiguity. From the previous lemma it follows that the map $\Psi_y$ is well defined on Artin symbols.

**Lemma 1.6.3.** The map $\Psi_y$ extends to a group homomorphism $\Psi_y : W_S \to \mathbb{C}_\infty^\times$.

**Proof.** Since $W_S$ is generated by the Artin symbols every $\sigma \in W_S$ may be written as $\sigma = \prod \phi_\nu^{n_\nu}$ for some integers $n_\nu$. We put $\Psi_y(\sigma) = \prod \Psi_y(\nu)^{n_\nu}$. We have to check that this extension is well defined.

For every $a \in I$ we denote with $i_a$ the idèle whose $\nu$-coordinate is equal to $\pi_\nu^{\nu}(a)$; in particular it is equal to 1 if and only if $\nu$ does not belong to the support of $a$. We also observe that every idèle of this form is a finite idèle, i.e., it has component at $\infty$ equal to 1, since fractional ideals do not have $\infty$ in their support. Clearly the map $\varphi : I \to I_F$, $a \mapsto i_a$ is an injective homomorphism. Let $\operatorname{rec}_S : I_F \to G_S$ be the map induced by the Artin homomorphism and $i \in \text{Im}(\varphi) \cap \ker(\operatorname{rec}_S)$. Since $\ker(\operatorname{rec}_S) = F^\times \prod_{\nu \notin S} O^\times_\nu$ we have that $i$ can be written as a product of an element $x \in F^\times$ and an idèle $\mathfrak{o} \in \prod_{\nu \notin S} O^\times_\nu$. The component at infinity of $\mathfrak{o}$ is equal to 1 since $\infty \in S$, and the same is true for $i$ because of the definition of the function $\varphi$. This implies that $x$ must be equal to 1 and $i = \mathfrak{o}$. Now for every prime $\nu \in S$ different from $\infty$ the component $i_\nu$ of $\mathfrak{o}$ must be equal to 1 because it belongs to the kernel of $\operatorname{rec}_S$ and for every prime $\nu \notin S$ it must be equal to $\pi_\nu^n$ for some integer $n$ and belongs to $O^\times_\nu$. The only possibility is that $i$ is the unit.

We have proved that the composition of $\varphi$ and $\operatorname{rec}_S$ is injective. The image of a fractional ideal $a = \prod \nu^{n_\nu}$ under this map is the element $\sigma = \prod \phi_\nu^{n_\nu}$ and so we have proved that if $\sigma \in W_S$ may be written in two different ways as a product of Artin symbols, then $\Psi_y(\sigma)$ does not depend on the chosen one.

From the continuity of the norm map and of the root extraction map it follows immediately that $\Psi_y : W_S \to \mathbb{C}_\infty^\times$ is a continuous homomorphism. From what we have previously observed $G_S$ is the topological closure of $W_S$ and, since $\mathbb{C}_\infty$ is a complete topological space, $\Psi_y$ may be extended in a unique way to a continuous homomorphism defined on $G_S$ which we will still call $\Psi_y$ by a little abuse of notation.

Let $\Psi_y : \mathbb{Z}[G_S] \lbrack X \rbrack \to \mathbb{C}_\infty \lbrack X \rbrack$ be the natural map induced by $\Psi_y$. 


Theorem 1.6.4. Let $\Theta_S(X)$ be the Stickelberger series and $\zeta_A$ the Goss Zeta function. Then, for every $s = (x, y) \in S_\infty$ we have

$$\Psi_y(\Theta_S(X))(x) = \zeta_A(-s) \prod_{\nu \in S \setminus \nu_\infty}(1 - \nu^s).$$

Proof. Let $\nu$ be a prime not in $S$, $n$ be the order of $[\nu]$ in $\mathbb{I}/\mathbb{P}_+$ and $\alpha$ be a positive element such that $\nu^n = (\alpha)$. Due to what we have already observed $\langle \nu \rangle_\infty$ is the only 1-unit whose $n$-th power coincides with $\langle \alpha \rangle_\infty$. We write $n = p^h n'$ with $(p, n') = 1$ and let $u$ be the only 1-unit whose $n'$-th power coincides with $\langle \alpha \rangle_\infty$. Since $\nu^n$ is a root of the polynomial $a(X) = X^{p^h} - u \in F_\infty[X]$. Let $b(X)$ be the minimal polynomial of $\langle \nu \rangle_\infty$ over $F_\infty$. Since $\langle \nu \rangle_\infty$ is a root of $a(X)$ and is totally inseparable, it must be $a(X) = b(X)^{p^h}$ and $b(X) = X^{p^k} - v$, where $l, k$ and $v$ satisfy $h = k + l$ and $u = v^{p^h}$. If we denote with $K$ the extension of $F_\infty$ obtained by adding $\langle \nu \rangle_\infty$, we have that $K/F_\infty$ has degree $p^k$, while $F_\infty, V/K$ has degree $f/p^k$. Therefore

$$N(\langle \nu \rangle_\infty) = N_{K,F_\infty}(\langle \nu \rangle_\infty)^{f/p^k} = v^{f/p^k} = \langle \nu \rangle_\infty^f$$

(everything works for $p = 2$ as well since in that case $N_{K,F_\infty}(\langle \nu \rangle_\infty) = -v = v$).

From this it follows that $\Psi_y(\phi_\nu) = \langle \nu \rangle_\infty^{-y}$ and that

$$\Psi_y(\Theta_S(X)) = \prod_{\nu \in S \setminus \nu_\infty} \left(1 - \Psi_y(\phi_\nu^{-1})X^{d_\nu}\right)^{-1}$$

and so

$$\Psi_y(\Theta_S(X))(x) = \prod_{\nu \in S \setminus \nu_\infty} \left(1 - \langle \nu \rangle_\infty^{-y}x^{d_\nu}\right)^{-1}$$

$$= \prod_{\nu \not\in \infty} \left(1 - \langle \nu \rangle_\infty^{-y}x^{d_\nu}\right)^{-1} \prod_{\nu \in S \setminus \nu_\infty} \left(1 - \langle \nu \rangle_\infty^{-y}x^{d_\nu}\right)$$

$$= \zeta_A(-s) \prod_{\nu \in S \setminus \nu_\infty}(1 - \nu^s).$$

1.7 A special case

In this section we will consider a special case and provide a link between the map $\Psi_y$ of the previous section and the Artin reciprocity map. We assume that the class number of $F$ is equal to 1 and we choose a prime at infinity of degree 1. The particularity of this case is that we can give an explicit description of the idèle class group (Theorem 1.7.1). One example of function field that satisfies these properties is the rational function fields $\mathbb{F}_q(T)$ which have been
studied in [ABBL]. Theorem 1.7.2 at the end of this section will show that Theorem 1.6.4 is a generalization of [ABBL, Theorem 3.8].

Under the assumptions on the class number and the degree of \( \infty \) we have that \( A \) is a principal ideal domain and that \( h^+(A) = 1 \). We also have that the residue field of \( \mathbb{F}_\infty \) coincides with the constant field \( \mathbb{F}_q \) and so every element of \( F^\times \) can be written in a unique way as product of a constant and a positive element of \( \nu \) for every prime \( \nu \). Moreover the condition \( h^+(A) = 1 \) implies that for every prime \( \nu \neq \infty \) the corresponding prime ideal of \( A \) is principal and can be generated by a positive element \( \pi_\nu \in F \). We choose an uniformizer at \( \infty \) in the following way: first we fix a prime \( \mathfrak{p} \neq \infty \) of degree 1, let \( \pi_\mathfrak{p} \) be its unique positive generator and then put \( \pi_\infty := \pi^{-1}_\mathfrak{p} \). Note that this uniformizer is positive and is an element of \( F^2 \). Finally we observe that \( F_\mathfrak{p} = F \) and \( F_\infty, V = F_\infty \).

The Goss Zeta function is defined (as before) by

\[
\zeta_A(s) = \sum_{a \in A^+} a^{-s} = \prod_{\nu \neq \infty} (1 - \pi_\nu^{-s})^{-1}.
\]

The image of the character \( \Psi_y \) is contained in \( U_1(\infty) \subset F_\infty^\times \) and satisfies \( \Psi_y(\phi_\nu) = \langle \pi_\nu \rangle_\infty^{-y} \).

We want to provide a slightly more explicit link between the character \( \Psi_1 \) and the Artin map. Let \( \mathbb{I}_F \) be the idele group of \( F \).

**Theorem 1.7.1.** The idele class group \( \mathbb{I}_F/F^\times \) is isomorphic to the product

\[
F_\infty^\times \times \prod_{\nu \neq \infty} O_\nu^\times =: \mathcal{H},
\]

where we have denoted with \( F_\infty^\times \) the kernel of the map \( \text{sgn}: F_\infty^\times \to \mathbb{F}_q^\times \).

**Proof.** First we prove that every idele of \( \mathcal{H} \) identifies a different equivalence class in \( \mathbb{I}_F/F^\times \). Let \( \mathbf{i} = (i_\infty, i_\nu_1, i_\nu_2, \ldots) \) and \( \mathbf{j} = (j_\infty, j_\nu_1, j_\nu_2, \ldots) \) be two ideles in \( \mathcal{H} \) and assume that they belong to the same equivalence class. Let \( x \in F^\times \) be such that \( \mathbf{i} = x \mathbf{j} \). For every \( \nu \neq \infty \) we have that \( i_\nu = xj_\nu \), but since both \( i_\nu \) and \( j_\nu \) are units in \( F_\nu \), it follows that \( v_\nu(x) = 0 \). Moreover from the product formula we have \( v_\infty(x) = -\sum_{\nu \neq \infty} d_\nu v_\nu(x) = 0 \) and so \( x \) is a constant. Finally since \( i_\infty = xj_\infty \) and both \( i_\infty \) and \( j_\infty \) are positive, we deduce that \( x = 1 \).

To complete the proof (and provide an explicit isomorphism) we have to show that every equivalence class of \( \mathbb{I}_F/F^\times \) contains an idele of \( \mathcal{H} \). Let \( \mathbf{i} = (i_\infty, i_\nu_1, i_\nu_2, \ldots) \) be any idele and consider the element

\[
x_{\mathbf{i}} = \text{sgn}(i_\infty)^{-1} \prod_{\nu \neq \infty} \pi_{\nu}^{-v_\nu(i_\nu)},
\]

which is in \( F^\times \) because there are only finitely many \( \nu \) with \( v_\nu(i_\nu) \neq 0 \). It is easy to check that the idele \( x_{\mathbf{i}} \mathbf{i} \) is in \( \mathcal{H} \) and that the map

\[
\mathbb{I}_F/F^\times \longrightarrow \mathcal{H}, \quad [\mathbf{i}] \mapsto x_{\mathbf{i}} \mathbf{i}
\]

is an isomorphism. \( \square \)

---

2 If \( F \) is the rational function field \( \mathbb{F}_q(T) \), here we are simply taking as uniformizer the element \( \pi_\infty = 1/(T - \alpha) \), where \( \alpha \) is any element of \( \mathbb{F}_q \) and \( \mathfrak{p} = (T - \alpha) \).
Let \( C_F := \mathbb{I}_F/F^\times \) be the idèle class group of \( F \) and consider the composition of the Artin map with the projection \( G_F^{ab} := \text{Gal}(\overline{F}/F)^{ab} \to G_S \). The map we obtain in this way has the group \( O_S = \prod_{\nu \not\in S} O_{\nu}^\times \) as its kernel and, therefore, the Artin map induces a continuous embedding
\[
\text{rec}_S : C_F/O_S \hookrightarrow G_S.
\]
This embedding is not surjective since the group \( G_S \) is profinite, while the quotient \( C_F/O_S \) is not. If we denote with \( \widehat{C_F/O_S} \) the profinite completion \(^3\) of \( C_F/O_S \), the map \( \text{rec}_S \) extends in a unique way to an isomorphism of topological groups
\[
\widehat{\text{rec}_S} : \widehat{C_F/O_S} \xrightarrow{\sim} G_S.
\]
Using the isomorphism in Theorem 1.7.1, one has that the quotient \( C_F/O_S \) is isomorphic to
\[
\pi_{\infty}^Z \times U_1(\infty) \times \prod_{\nu \not\in \{\infty\}} O_{\nu}^\times.
\]
Its profinite completion is the group
\[
\widehat{(\pi_{\infty})} \times U_1(\infty) \times \prod_{\nu \not\in \{\infty\}} O_{\nu}^\times
\]
where \( \widehat{(\pi_{\infty})} \simeq \hat{\mathbb{Z}} \).

**Theorem 1.7.2.** We denote with \( \pi_S \) the canonical projection \( \widehat{C_F/O_S} \to U_1(\infty) \). Then the following diagram is commutative
\[
\begin{array}{ccc}
G_S & \xrightarrow{\Psi_1} & U_1(\infty) \\
\downarrow{\text{rec}_S^{-1}} & & \downarrow{\pi_S} \\
\widehat{C_F/O_S} & \xrightarrow{\Psi_1} & U_1(\infty)
\end{array}
\]
In general for every \( y \in \mathbb{Z}_p \) we have the following commutative diagram
\[
\begin{array}{ccc}
G_S & \xrightarrow{\Psi_y} & U_1(\infty) \\
\downarrow{\text{rec}_S^{-1}} & \downarrow{y} & \downarrow{y} \\
\widehat{C_F/O_S} & \xrightarrow{\pi_S} & U_1(\infty)
\end{array}
\]
where \( y \) denotes the raise-to-the-power \( y \) map.

\(^3\)For a topological group \( G \), the profinite completion \( \hat{G} \) is the inverse limit of its finite quotients with respect to the natural projection maps.
1.8 $\nu$-adic Zeta Function

Proof. We have already noted that $G_S$ is the topological closure of the group $W_S$, generated by the Artin symbols. Since all the maps in the theorem are continuous, it is enough to show that $\Psi_1(\phi_\nu) = \pi \circ \text{rec}^{-1}_S(\phi_\nu)$ for every $\nu \not\in S$.

We recall that the global Artin map is the product on each idèle component of the local Artin maps and that the local Artin map $\text{rec}_\nu$ sends $\pi_\nu$ to $\phi_\nu$, since the extension $F_S/F$ is unramified at $\nu$. Let $i_\nu$ be the idèle whose $\nu$-coordinate is equal to $\pi_\nu$ and whose $\mu$-coordinates for $\mu \neq \nu$ are all equal to 1. The image of the equivalence class $[i_\nu] \in \mathbb{I}_F/F^\times$ under the map $\text{rec}_S$ is equal to $\phi_\nu$. As noted above, the hypothesis $h^+(A) = 1$ implies that we have a positive generator $\pi_\nu \in F$ for $\nu$, hence

$$\phi_\nu = \text{rec}_S([i_\nu]) = \text{rec}_S([\pi_\nu^{-1}i_\nu])$$

and, recalling that the idèle $\pi_\nu^{-1}i_\nu$ belongs to $\mathcal{H}$, we obtain that

$$\pi_S \circ \text{rec}_S^{-1}(\phi_\nu) = \langle \pi_\nu^{-1} \rangle_\infty = \Psi_1(\phi_\nu).$$

For a general $y \in \mathbb{Z}_p$, the thesis is obtained simply by observing that for every Artin symbol $\phi_\nu$, we have $\Psi_y(\phi_\nu) = \Psi_1(\phi_\nu)y$ by definition.

1.8 $\nu$-adic Zeta Function

In this section we will define a $\nu$-adic analogue of the Goss Zeta function.

Fix a place $\nu$ different from $\infty$ and let $F_\nu, \mathbb{C}_\nu, F_\nu$ and $\pi_\nu$ be the $\nu$-adic versions of the objects defined for the place $\infty$. Fix an algebraic closure $\overline{F}$ of $F$ and let $\sigma : \overline{F} \to \mathbb{C}_\nu$ be an $F$-embedding. All the objects that we shall define later on depend on $\sigma$, but we will omit this dependency to simplify the notations.

Let $F_{\nu, \mathbb{V}} := \sigma(F_\nu)F_{\nu}$, which will play the role of the field $F_{\infty, \mathbb{V}}$, and let $\mathbb{S}_\nu = \mathbb{C}_\nu^\times \times \mathbb{Z}_p \times \mathbb{Z}/|\mathbb{F}_\nu^\times|$, which is a subgroup of the group of $\mathbb{C}_\nu^\times$-valued characters on $\mathbb{F}_\nu^\times$. As done in Proposition 1.4.4, we can prove that $F_{\nu, \mathbb{V}}$ is a totally inseparable extension of $F_\nu$ with degree less than or equal to $p^t$ (the maximal power of $p$ that divides $h^+(A)$). Since this extension is totally inseparable the residue field of $F_{\nu, \mathbb{V}}$ is still $F_\nu$ and so the cyclic group $\mathbb{Z}/|\mathbb{F}_\nu^\times|$ acts on the multiplicative group of this residue field by raise to the power. Now we take an element $s_1 = (x, y, j) \in \mathbb{S}_\nu$ and we define the exponential $I^{s_1} \in \mathbb{C}_\nu^\times$ for every fractional ideal $I \in \mathcal{I}$ coprime with $\nu$. We recall that the element $I^{s_1} \in F_{\nu, \mathbb{V}}$ is a root of the polynomial $X^\nu - \alpha$ where $\alpha$ is the unique positive generator of $I^\nu$ (here $s_1$ is the element $\langle \pi_\nu^{-1}, 1 \rangle \in \mathbb{S}_{\infty}$) and so the valuation at $\nu$ of $I^{s_1}$ is equal to zero. This implies that the element $\sigma(I^{s_1})$ is a unit in $F_{\nu, \mathbb{V}}$ and so can be written uniquely as a product

$$\sigma(I^{s_1}) = \omega(I)\langle I \rangle_\nu$$

for some $\omega(I) \in \mathbb{F}_\nu^\times$ and $\langle I \rangle_\nu$ a 1-unit of $F_{\nu, \mathbb{V}}$. With the notation above it is easy to check that the map which sends $I$ to $\omega(I)$ is a group homomorphism $\omega : \mathcal{I}_\nu \to \mathbb{F}_\nu^\times$ defined on the group $\mathcal{I}_\nu$ of the fractional ideals coprime with $\nu$.

Finally if we take $s_\nu = (x, y, j) \in \mathbb{S}_\nu$ and $I$ coprime with $\nu$ we can define

$$I^{s_\nu} = x^{\deg(I)}\omega(I)^y\langle I \rangle_\nu^y.$$

We can embed $\mathbb{Z}$ in $\mathbb{S}_\nu$ via the map $j \in \mathbb{Z} \mapsto s_{\nu, j} = (1, j, j) \in \mathbb{S}_\nu$. One can show that this $\nu$-adic exponential satisfies the following properties (analogous to the ones of the exponential defined is Section 1.4)
Proposition 1.8.1. For every $I, J \in \mathcal{I}$ coprime with $\nu$, one has

- $I^{s_\nu + t_\nu} = I^{s_\nu} I^{t_\nu}$ for every $s_\nu, t_\nu \in \mathbb{S}_\nu$.
- $(IJ)^{s_\nu} = I^{s_\nu} J^{s_\nu}$ for every $s_\nu \in \mathbb{S}_\nu$.
- $(I^{s_\nu})^{s_{\nu,i}} = I^{s_{\nu,ij}}$ for every $i, j \in \mathbb{Z}$.
- $I^{s_\nu,j} = \sigma(I^{s_\nu,j})$ for every $j \in \mathbb{Z}$. In particular $I^{s_\nu,j}$ is algebraic over $F$.

Proof. The proofs of the first 3 properties are similar to the analogous properties of the exponential defined in Section 1.4 and so are left to the reader. For the last one we have

\[ I^{s_\nu,j} = \omega(I)j(I)^j_{\nu} = \sigma(I^j)j = \sigma(I^j). \]

Definition 1.8.2. The $\nu$-adic Goss Zeta function is defined by

\[ \zeta_\nu(s_\nu) = \sum_{\substack{a \in \mathbb{S}_\nu \\ a \geq 0 \\ \nu \mid a}} a^{-s_\nu} = \prod_{p \neq \nu, \infty} (1 - p^{-s_\nu})^{-1}. \]

Now we want to obtain the function $\zeta_\nu(s_\nu)$ as the image of the Stickelberger series under an appropriate map. This will be done for all the primes $\nu \neq \infty$ which are in $S$, so for the rest of this section we will assume that $\nu \in S$. Let $f_\nu$ be the degree of the extension $F_{\nu,V}/F_\nu$ and denote by $N_\nu : F_{\nu,V}^\times \to F_\nu^\times$ the norm map. We fix $(y, j) \in \mathbb{Z}_p \times \mathbb{Z}/|F_\nu^\times|$ and for every $p \notin S$ we put

\[ \Psi_{y,j}(\phi_p) = N_\nu((p)^{-1})_{\nu}^{y/j} \omega(p)^{-j}. \]

In Lemma 1.6.2 we have proved that different primes correspond to different Artin symbols and so $\Psi_{y,j}(\phi_p)$ is well defined.

Lemma 1.8.3. For every $(y, j) \in \mathbb{Z}_p \times \mathbb{Z}/|F_\nu^\times|$ the map $\Psi_{y,j}$ extends to a continuous ring homomorphism $\mathbb{Z}[[G_S]][X] \to \mathbb{C}_\nu[X]$.

Proof. Let $\tau = \prod \phi_p^{e_p} \in W_S$. By the proof of Lemma 1.6.3 we have that the map that sends $\prod p^{e_p}$ to $\prod \phi_p^{e_p}$ is injective on the set of fractional ideals with support outside of $S$. Then we can put $\Psi_{y,j}(\tau) := \prod \Psi_{y,j}(\phi_p)^{e_p}$ without ambiguity. Now since $W_S$ is dense in $G_S$ we have that $\Psi_{y,j}$ extends in a unique way as a continuous map defined over $G_S$ because $\mathbb{C}_\nu$ is complete.

We have the following

Theorem 1.8.4. Let $\Theta_S(X)$ be the Stickelberger series and $\zeta_\nu$ the $\nu$-adic Goss Zeta function. If we assume $\nu \in S$ then for every $s_\nu = (x, y, j) \in \mathbb{S}_\nu$ we have

\[ \Psi_{y,j}(\Theta_S(X))(x) = \zeta_\nu(-s_\nu) \prod_{p \in S} (1 - p^{-s_\nu}). \]
Proof. Let \( p \) be a prime not in \( S \), \( n \) the exact order of \( [p] \) in \( \mathcal{I}/\mathcal{P}_r \) and \( \alpha \) a positive element such that \( p^n = (\alpha) \). We have that \( (p^n)^{s_1} = (\alpha)^{s_1} = \alpha \) and so
\[
\omega(p)^n(p)^n = \sigma(p^{s_1})^n = \sigma(\alpha)^{s_1} = \alpha,
\]
which implies that \( (p)^n = (\alpha)_v \).

From now on with proceed like in Theorem 1.6.4. Write \( n = p^hn' \) with \( (p, n') = 1 \) and let \( u \) be the unique 1-unit whose \( n' \) power coincide with \( (\alpha)_v \). Since \( n' \) is coprime with \( p \), we have that \( u \) is in \( F_v \) and that \( (p)_v \) is a root of the polynomial \( a(X) = X^{p^n} - u \in F_v[X] \). Let \( b(X) \) be the minimal polynomial of \( (p)_v \) over \( F_v \). Since \( (p)_v \) is a root of \( a(X) \) and it is a totally inseparable element, it must be \( a(X) = b(X)^p \) and \( b(X) = X^{p^n} - v \), where \( l, k \) and \( v \) satisfy \( h = k + l \) and \( u = v^d \). If we denote with \( K \) the extension of \( F_v \) obtained by adding \( (p)_v \), we have that \( K/F_v \) is an extension of degree \( p^k \) and \( F_v, V/K \) has degree \( f_v/p^k \). Therefore
\[
N_v((p)_v) = N_{K,F_v}((p)_v)^{f_v/p^k} = v^{f_v/p^k} = (p)_v^{f_v}.
\]

From this we obtain that \( \Psi_{y,j}(\phi_p) = (p)_v^{-y\omega(p)^{-j}} \) and
\[
\Psi_{y,j}(\Theta_S(X)) = \prod_{p \notin S} \left(1 - \Psi_{y,j}(\phi_p)^{-1}X^{dp}\right)^{-1}
= \prod_{p \notin S} \left(1 - (p)_v^{-y\omega(p)^jX^{dp}}\right)^{-1}
\]
and so
\[
\Psi_{y,j}(\Theta_S(X))(x) = \prod_{p \notin \mathcal{I} \cup \mathcal{P}_r} \left(1 - (p)_v^{-y\omega(p)^jX^{dp}}\right)^{-1}
= \prod_{p \notin \mathcal{I} \cup \mathcal{P}_r} \left(1 - (p)_v^{-y\omega(p)^jX^{dp}}\right)^{-1}
\]
\[
= \zeta_v(-s_v) \prod_{p \notin \mathcal{I} \cup \mathcal{P}_r} \left(1 - p^{s_v}\right).
\]

\( \quad \)

1.9 Properties of \( \nu \)-adic Zeta Function

In this section we will investigate some properties of the \( \nu \)-adic Zeta function and in particular its values at integers.

For each pair of non negative integers \( j \) and \( n \), we define
\[
S_n(j) = \sum_{a \geq 0 \atop \deg a = n} a^{s_j}
\]
and the power series
\[
Z(X, j) = \sum_{n \geq 0} S_n(j)X^n
\]
whose coefficients lie in \( F_v \).
Lemma 1.9.1. The series \( Z(X, j) \) is actually a polynomial of degree less than or equal to \( d_\infty + g - 1 + \lfloor \frac{j}{q-1} \rfloor \).

Proof. Fix a non negative integer \( j \). We prove that when \( n \) is bigger than \( d_\infty + g - 1 + \lfloor \frac{j}{q-1} \rfloor \), \( S_n(j) \) is equal to 0. To prove this we shall apply the same tools used to prove the convergence of the Goss Zeta function.

Let \( C_h \) for \( h = 1, \ldots, e \) be the classes of \( \mathcal{I}/\mathcal{P}_+ \) and, for each \( h \), fix a representative \( a_h \in C_h \). Now define

\[
S_n(C_h, j) = \sum_{\deg(a) = n, a \in C_h} a_h^{s_j} \cdot \sum_{\alpha \in \mathcal{P}_+, \alpha a_h \geq 0, \deg(a) = n - \deg(a_h)} \alpha^j.
\]

Clearly \( S_n(j) \) is the sum of the \( S_n(C_h, j) \) (as \( h \) varies) so it is enough to show that each of them is equal to 0. Fix an index \( h \) and let \( n_h = n - \deg(a_h) \). We will consider only indices \( h \) such that \( d_\infty \) divides \( n_h \), otherwise the thesis is trivial. Just like in Section 1.5.1 we denote with \( X \) the set of positive elements \( h \) such that \( \alpha a_h \) is an integral ideal and whose degree is equal to \( n_h \), and with \( \mathcal{L}(D_1) \) the Riemann-Roch space associated to the divisor \( D_1 = \operatorname{div}(a_h) + (n_h/d_\infty - 1)\infty \).

The additive group of \( \mathcal{L}(D_1) \) acts by translation on \( X \) and the action is free, so we can decompose \( X \) as the union of its orbits under this action: \( X = \bigcup_{l=1}^t X_l \). Fix a representative \( x_l \in X_l \) for each orbit. Then we have

\[
S_n(C_h, j) = a_h^{s_j} \sum_{l=1}^t \sum_{u \in \mathcal{L}(D_1)} (u + x_l)^j = a_h^{s_j} \sum_{l=1}^t x_l^j \sum_{w \in x_l^{-1}\mathcal{L}(D_1)} (w + 1)^j.
\]

Observe that \( v_\infty(x_l) = -n_h/d_\infty \) and \( v_\infty(u) \geq 1 - n_h/d_\infty \). This implies that \( v_\infty(w) \) is positive for every \( w \in x_l^{-1}\mathcal{L}(D_1) \) and that the vector space \( x_l^{-1}\mathcal{L}(D_1) \) satisfies the hypothesis of Lemma 1.5.4 part (a). Therefore the inner sum is zero when \( j < (q-1)\ell_1 = (q-1)(n-g+1) \).

The polynomials \( Z(X, j) \) are strictly related to the special values of the Goss Zeta function since we have that \( Z(1, j) = \zeta_a(-s_j) \) for any \( j \in \mathbb{N} \).

We shall also need some \( \nu \)-adic version of this polynomials, which will be used to interpolate the special values of the \( \nu \)-adic Goss Zeta function.

Definition 1.9.2. The \( \nu \)-adic \( L \)-series is defined by

\[
L_\nu(X, y, \omega^i) = \sum_{n \geq 0} \left( \sum_{a \geq 0, \nu^n a, \deg(a) = n} \omega(a)^i (a)_\nu^y \right) X^n,
\]

for any \( y \in \mathbb{Z}_\nu \) and \( i \in \mathbb{Z}/|\mathbb{F}_\nu^*| \).

From the definition one immediately has

Proposition 1.9.3. For every \( s_\nu = (x, y, i) \in S_\nu \) we have

\[
L_\nu(x, y, \omega^i) = \zeta_\nu(-s_\nu).
\]
1.9 Properties of \( \nu \)-adic Zeta Function

Proof. We recall that for a fractional ideal \( a \) coprime with \( \nu \) we have \( a^{s_\nu} = \omega(a)^{i} (a)^{y_n} x^n \) and so

\[
L_{\nu}(x, y, \omega^i) = \sum_{n \geq 0} \left( \sum_{a \geq 0, \nu \nmid a, \deg(a) = n} \omega(a)^{i} (a)^{y_n} x^n \right)
\]

\[
= \sum_{n \geq 0} \left( \sum_{a \geq 0, \nu \nmid a, \deg(a) = n} a^{s_\nu} \right)
\]

\[
= \sum_{a \in \mathbb{Z}_p} a^{s_\nu}
\]

\[
= \zeta_{\nu}(-s_\nu).
\]

The following theorem provides a link between this power series and the polynomial \( Z(X, j) \) for some particular values of \( i \) and \( j \). It also shows that for these values of \( i \) and \( j \) the series \( L_{\nu}(X, y, \omega^i) \) is actually a polynomial.

**Theorem 1.9.4.** Assume that \( \nu \in S \).

(a) Let \( i \) and \( j \) be two non negative integers, such that \( i \equiv j \) (mod \( q^{d_\nu} - 1 \)). Then

\[
L_{\nu}(X, j, \omega^i) = Z(X, j)(1 - \nu^{s_j} X^{d_\nu}).
\]

In particular \( L_{\nu}(X, j, \omega^i) \) in a polynomial.

(b) For every \( y \in \mathbb{Z}_p \) we have that

\[
L_{\nu}(X, y, \omega^i) \equiv Z(X, i) \pmod{\mathfrak{p}},
\]

where \( \mathfrak{p} \) denotes any prime of \( F_{\mathbb{V}} \) which lies above \( \nu \).

Proof. (a) Fix a prime \( p \) different from \( \nu \) and \( \infty \) and consider \( f_p(X) = 1 - \omega(p)^{i} (p)^{y_n} X^{d_p} \) as an element of \( F_{\nu, \mathbb{V}}[X] \). Clearly \( f_p(X) \) is invertible since \( f_p(0) = 1 \) and its inverse is given by

\[
f_p(X)^{-1} = \sum_{n \geq 0} \left( \omega(p)^{i} (p)^{y_n} X^{d_p} \right)^n.
\]

Let

\[
f_n^{-1}(X) := \prod_{\deg(p) = n} f_p(X)^{-1}
\]

then the product

\[
\prod_{n \geq 0} f_n^{-1}(X)
\]
is convergent with respect to the $X$-adic topology.

The map that sends every ideal $I$ coprime with $\nu$ to the 1-unit $(I)_\nu$ is multiplicative and the same is true for the map $\omega$ defined on this set of ideals. This fact, together with the unique factorization of ideals in $A$, allows us to conclude that the limit of the product above is equal to $L_\nu(X,j,\omega^i)$, i.e.,

$$L_\nu(X,j,\omega^i) = \prod_{p \neq \nu, \infty} \left( 1 - \omega(p)^i(p)_\nu^j X^{dp} \right)^{-1}. \quad (1.10)$$

In the proof of Theorem 1.8.4 we have seen that this product is equal to

$$\Psi_{j,i}(\Theta_S(X)) \cdot \prod_{p \in S \atop p \neq \infty, \nu} \left( 1 - \omega(p)^i(p)_\nu^j X^{dp} \right)^{-1}.$$  

Since $i \equiv j \pmod{q^d \nu - 1}$, we have that $\omega(p)^i = \omega(p)^j$ and so

$$\Psi_{j,i}(\phi_p^{-1}) = \omega(p)^i(p)_\nu^j \omega(p)^j(p)_\nu^j = \sigma(p^{s_1})^i.$$

We recall that $p^{s_1}$ lies in $F_V$ and that $\sigma$ is the identity on this field, so

$$\Psi_{j,i}(\phi_p^{-1}) = p^{s_1} = \pi^{-jd_p} (p)_\infty^j = \Psi_j(\phi_p^{-1}) \pi^{-jd_p}.$$

From this equality it follows that

$$\Psi_{j,i}(\Theta_S(X)) = \prod_{p \in S} \left( 1 - \Psi_{j,i}(\phi_p^{-1}) X^{dp} \right)^{-1}$$

$$= \prod_{p \in S} \left( 1 - \Psi_j(\phi_p^{-1}) \pi^{-jd_p} X^{dp} \right)^{-1} \quad (1.11)$$

$$= \Psi_j(\Theta_S(\pi^{-j}X)).$$

We have proved that

$$L_\nu(X,j,\omega^i) = \Psi_j(\Theta_S(\pi^{-j}X)) \cdot \prod_{p \in S \atop p \neq \infty, \nu} \left( 1 - p^{s_1} X^{dp} \right)^{-1}. \quad (1.12)$$

Since

$$Z(X,j) = \sum_{a \in I \atop \deg a \geq 0} a^j X^{\deg a} = \sum_{n \geq 0} \left( \sum_{a \geq 0 \atop \deg a = n} \langle a \rangle_\infty^j \right) (\pi^{-j}X)^n,$$

the same arguments used to obtain (1.10), applied now to $Z(X,j)$, allow us to write

$$Z(X,j) = \prod_{p \neq \infty} \left( 1 - (p)_\infty^j (\pi^{-j}X)^{dp} \right)^{-1}. \quad (1.13)$$
The product (1.13) can be written as
\[
\Psi_j \left( \Theta_S(\pi_s^{-j} X) \right) \cdot \prod_{\substack{p \in S \\ p \neq \infty}} \left( 1 - p^{s_j} X^{d_p} \right)^{-1}.
\]

Thus, joining together (1.12) and (1.13), we obtain that
\[
L_\nu(X, j, \omega^i) = Z(X, j) (1 - \nu^{s_j} X^{d_\nu}).
\]

(b) For every ideal \( a \) we have that
\[
\langle a \rangle^\nu \equiv 1 \equiv \langle a \rangle^i \pmod{\nu}.
\]

Hence
\[
L_\nu(X, y, \omega^i) = \sum_{n \geq 0} \left( \sum_{\substack{a \geq 0, \nu \mid a \\ \deg a = n}} \omega(a)^i \langle a \rangle^\nu \right) X^n
\]
\[
\equiv \sum_{n \geq 0} \left( \sum_{\substack{a \geq 0, \nu \mid a \\ \deg a = n}} \omega(a)^i \langle a \rangle^i \right) X^n \pmod{\nu}
\]
\[
= L_\nu(X, i, \omega^i) \pmod{\nu}
\]
\[
= Z(X, i) (1 - \nu^{s_i} X^{d_\nu}) \pmod{\nu}
\]
\[
\equiv Z(X, i) \pmod{\nu}.
\]
Chapter 2

Stickelberger series and class groups

2.1 Introduction

In this chapter we will apply the results of Chapter 1 to the study of class group of degree zero divisors. We will first introduce the Hayes modules and use them to build an Iwasawa tower. Then we will study the behaviour of the $p$-part of the class groups of degree zero divisors of the fields in the tower and of their inverse limit taken with respect to the norm maps. The study of these objects is one of principal part of Iwasawa Theory. The Iwasawa Main Conjecture relates the characteristic ideals of Iwasawa modules to the $\nu$-adic $L$-functions, thus providing a link between the algebraic and the analytic side of the theory. A version of the Main Conjecture for $\mathbb{Z}_p^d$-extensions of global function fields was proved by R. Crew in [Cre] using mainly geometric tools and, later, with a different approach by Burns in [Bur], with the contribute of Kueh, Lai and Tan ([KLT]). The Iwasawa extensions we will consider in this chapter are $\mathbb{Z}_p^\infty$-extensions, thus the inverse limit of the $p$-part of the class groups turns out to be a module over a non-Noetherian Iwasawa algebra and, unlike the classical case, we do not have a structure theorem for finitely generated torsion modules like [Was, Theorem 13.12] and so we are forced to study its Fitting ideal instead of the characteristic ideal which cannot be defined in this setting. For an alternative approach to this problem the reader may look, for example, at [BBL], where the authors study $\mathbb{Z}_p^\infty$-extensions of global function fields using $\mathbb{Z}_p^d$-filtrations.

Through this chapter we will assume that $d_\infty = 1$ and that $p$ does not divide $h_0(F)$. Under these assumptions we have the following simplifications:

- The residue field $F_\infty$ coincides with the field of costants $\mathbb{F}_q$.
- Every principal ideal admits a positive generator.
- The class number of the ring of integers $A$ is equal to $h^0(F)$.
- The field $F_{\infty,V}$ coincides with the field $F_\infty$.
- For every $a \in F$: $\deg(a) = -v_\infty(a)$.
- $\pi_* = \pi_\infty$. 
2.2 Hayes extensions

Let $H_A$ be the Hilbert class field of $A$, i.e., the maximal abelian extension of $F$ which is unramified at every prime and totally split at $\infty$. Obviously we have that $H_A$ is a subfield of $F_S$ for any choice of the set $S$. Since the prime $\infty$ has degree 1 we have that the constant field of $H_A$ is $\mathbb{F}_q$. It is a well known fact from class field theory that $\text{Pic}(A) \simeq \text{Gal}(H_A/F)$ and the isomorphism is provided by the Artin reciprocity map. In particular the class of a fractional ideal $a \in \mathcal{O}_A$ is sent to its Frobenius in $\text{Gal}(H_A/F)$ and, in case the support of $a$ is disjoint from $S$, this is simply the restriction of its Artin symbol $\phi_a \in G_S$.

Definition 2.2.1. We denote with $H_A\{\tau\}$ the ring of skew-polynomials in the variable $\tau$ with coefficient in the field $H_A$. A Hayes module is an homomorphism of $\mathbb{F}_q$-algebras $\Phi : A \to H_A\{\tau\}$, such that:

(a) the image of $A$ is not contained in $H_A$;
(b) for every $a \in A$ the coefficient of degree 0 of $\Phi_a := \Phi(a)$ is equal to $a$;
(c) for every $a \in A$ the degree of $\Phi_a$, seen as a polynomial in $\tau$, is equal to $\text{deg}(a)$ (i.e., $\Phi$ has rank 1);
(d) for every $a \in A$, the leading coefficient of $\Phi_a$ is $\text{sgn}(a)$ (i.e., $\Phi$ is sgn-normalized).

For details on the Hayes module the reader may refer to ([Gos1] Chapter 7, [Hay] and [Shu]).

Remark 2.2.2. In the general theory a Hayes module is defined over the narrow class field $H_A^+$, i.e., the abelian extension of $F$ that is naturally isomorphic to $\text{Pic}^+(A) := I/P_+$ by class field theory. In our context we have a simplification because the fields $H_A$ and $H_A^+$ coincide since the degree of the prime at infinity is equal to 1.

We use a Hayes module to define an action of $A$ on the algebraic closure of $F$. For every $a \in A$ and $x \in \overline{F}$ we put $a \cdot x := \Phi_a(x)$. This action defines a structure of $A$-module on $\overline{F}$. If we take an integral ideal $\mathfrak{a}$ of $A$ we can consider the left ideal of $H_A\{\tau\}$ generated by all the elements $\Phi_a$ with $a \in \mathfrak{a}$. Since $H_A\{\tau\}$ is right-euclidean, we have that every left ideal is principal. We denote by $\Phi_\mathfrak{a}$ the unique monic generator of this ideal.

Definition 2.2.3. An element $x \in \overline{F}$ is said to be of $\mathfrak{a}$-torsion if $\Phi_\mathfrak{a}(x) = 0$. Since $\Phi_\mathfrak{a}$ is a nonzero polynomial (for $\mathfrak{a} \neq 0$) we have that the set $\Phi[\mathfrak{a}]$ of all elements of $\mathfrak{a}$-torsion is finite.

Proposition 2.2.4. Let $\mathfrak{a} \neq 0$ be an integral ideal and let $\Phi[\mathfrak{a}]$ be the $\mathfrak{a}$-torsion of $\Phi$. Then

(a) $\Phi[\mathfrak{a}]$ is an $A/\mathfrak{a}$-module;
(b) $\Phi[\mathfrak{a}] \simeq A/\mathfrak{a}$;
(c) if $\lambda_\mathfrak{a}$ is a generator of $\Phi[\mathfrak{a}]$ as $A/\mathfrak{a}$-module, then every other generator is of the form $\Phi_b(\lambda_\mathfrak{a})$ for some $b \in (A/\mathfrak{a})^\times$.

We use the $\mathfrak{a}$-torsion of the Hayes module $\Phi$ to define extensions of the field $F$ analogous to the cyclotomic extension of $\mathbb{Q}$. We denote by $F(\mathfrak{a})$ the extension of $H_A$ generated by $\Phi[\mathfrak{a}]$, i.e., $F(\mathfrak{a}) := H_A(\Phi[\mathfrak{a}])$. Equivalently this extension can be obtained simply by adding an $A/\mathfrak{a}$-generator of $\Phi[\mathfrak{a}]$ to $H_A$.

The following theorem summarizes the properties of the Hayes extension contained in [Gos1] Proposition 7.5.4, Corollary 7.5.6, Proposition 7.5.8 and Proposition 7.5.18.
2.2 Hayes extensions

Theorem 2.2.5. The following hold

(a) $F(a)$ is a geometric, abelian Galois extension of $F$;

(b) $\operatorname{Gal}(F(a)/H_A) \simeq (A/a)^{\times}$ via an isomorphism which sends $a \in A$ to the element $\sigma_a \in \operatorname{Gal}(F(a)/H_A)$ which verifies $\sigma_a(\lambda) = \Phi_a(\lambda)$ for every $\lambda \in \Phi[a]$;

(c) the only ramified primes in $F(a)/H_A$ are the primes of $H_A$ dividing $a$ and $\infty$;

(d) the inertia group of $\infty$ coincides with its decomposition group and is isomorphic to to $\mathbb{F}_q^{\times}$ via the isomorphism in (b);

(e) if $p^n$ is the exact power of $p$ dividing $a$, then the inertia group of $p$ is isomorphic to $(A/p^n)^{\times}$ via the isomorphism in (b);

(f) if $I$ is an ideal of $A$ coprime with $a$ and $\sigma_I \in \operatorname{Gal}(F(a)/F)$ is its Artin symbol, then we have that $\sigma_I(\lambda) = \Phi_I(\lambda)$ for every $\lambda \in \Phi[a]$.

Now we fix a prime $p$ for the rest of the section and we set $S = \{p, \infty\}$. For each non-negative integer $n$ we put $F_n := F(p^{n+1})$ and $G_n = \operatorname{Gal}(F_n/F)$. From point (b) of Theorem 2.2.5 we deduce that $F_n/F_0$ is a $p$-extension and that $F_0/F$ is an extension of order $h^0(F)(q^{ph} - 1)$. Since we have assumed $h^0(F)$ coprime with $p$ we can decompose $G_n$ as the product $G_0 \times \Gamma_n$ where $\Gamma_n = \operatorname{Gal}(F_n/F_0)$ is a $p$-group and $G_0$ is the part of order coprime with $p$.

The fields $F_n$ form an Iwasawa tower: if we denote with $F_\infty$ the union of all the fields $F_n$, with

$$G_\infty := \operatorname{Gal}(F_\infty/F) = \lim_{n} G_n$$

and with

$$\Gamma_\infty := \operatorname{Gal}(F_\infty/F_0) = \lim_{n} \Gamma_n$$

we have that $\Gamma_\infty \simeq \mathbb{Z}_p^\infty$. Observe that the only primes ramified in $F_\infty/F$ are $p$ and $\infty$ and so $F_\infty$ is a subextension of $F_S$. The following diagram gives a recap of the fields and Galois groups introduced above.

![Diagram](image)

2.2.1 Primes in Hayes extensions

Now we focus on the behaviour of the primes in these extensions: as we have already observed any prime different from $p$ and $\infty$ is unramified in all the extensions.

The prime $p$ is unramified in $H_A/F$ and totally ramified in $F_\infty/H_A$, in the sense that each prime
of $H_A$ lying above $p$ is totally ramified. Since $F_{\infty}/F$ is an abelian extension we have that all the primes of $H_A$ lying above $p$ have the same inertia group (isomorphic to $Gal(F_{\infty}/H_A)$) in $G_{\infty}$. The prime $\infty$ is totally split in $H_A/F$, then it ramifies (not totally in general) in $F_0/H_A$, with inertia group isomorphic to $F_{\infty}^\times$, and then it is again totally split in the remaining extensions.

### 2.3 Fitting ideals for Tate modules

Fix an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$ and denote with $\gamma$ the arithmetic Frobenius, which is a topological generator of the pro-cyclic Galois group $G_{\overline{F}_q} := Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)$. For every field $L$, we denote by $L^{ar}$ the compositum of $L$ with $\overline{\mathbb{F}}_q$. In the case of the field $F_n$ (resp. $F_{\infty}$) we have that $F_n^{ar}$ (resp. $F_{\infty}^{ar}$) is a Galois extension of $F$, whose Galois group is isomorphic to $G_n \times G_{\overline{F}_q}$ (resp. $G_{\infty} \times G_{\overline{F}_q}$) since the constant field of $F_n$ (resp. $F_{\infty}$) is $\mathbb{F}_q$.

#### 2.3.1 The modules $H_n(\nu)$

For every prime $\nu$ of $F$ there exists only a finite number of primes of $F_n^{ar}$ that lie above $\nu$, because the extension $F_n/F$ is finite and $F_n^{ar}/F_n$ is arithmetic (we recall that in an arithmetic extension every prime splits in finitely many places). The following proposition tells us the exact number of primes lying above $\nu$.

**Proposition 2.3.1.** The number of primes of $F_n^{ar}$ lying above $\nu$ in equal to

- $d_\nu \cdot [F_n : F]$ if $\nu \neq p, \infty$;
- $d_p \cdot h^0(F)$ if $\nu$ is equal to $p$;
- $h^0(F) \cdot q^{nd_p} \cdot (q^{d_p} - 1)/(q - 1)$ if $\nu$ is equal to $\infty$.

**Proof.** For every field $K$ and every positive integer $t$ we denote with $K(t)$ the compositum of $K$ with $\mathbb{F}_q^t$. One can see that $F_n(t)$ is the compositum of the two fields $F_n$ and $F(t)$ which are disjoint over $F$, since $F_n/F$ is a geometric extension, $F(t)/F$ is arithmetic and $\mathbb{F}_q^t$ is the constant field of $F_n(t)$. We also observe that $F_n^{ar}$ is the union of all the fields $F_n(t)$ when $t$ varies over all positive integers.

Fix a prime $\overline{\nu}$ of $F_n(t)$ which lies above $\nu$ and let $e$ be the ramification index of $\overline{\nu}$ over $\nu$ and $f$ be its inertia degree. Clearly these two numbers do not depend on the choice of $\overline{\nu}$ since $F_n(t)/F$ is an abelian extension. We have also that $e$ does not depend on $t$ since $F_n(t)/F_n$ is arithmetic. Now we have

$$f = [\mathbb{F}_{\overline{\nu}} : \mathbb{F}_\nu] = \frac{[\mathbb{F}_{\overline{\nu}} : \mathbb{F}_q^t][\mathbb{F}_q^t : \mathbb{F}_q]}{[\mathbb{F}_\nu : \mathbb{F}_q]} = t \cdot \frac{\deg(\overline{\nu})}{d_\nu}.$$ 

The degree of the extension $F_n(t)/F$ is equal to $[F_n : F] \cdot t$ so, if we denote with $r$ the number of primes of $F_n(t)$ lying above $\nu$, from the Kummer formula we have

$$[F_n : F] \cdot t = r \cdot e \cdot f = r \cdot e \cdot t \cdot \frac{\deg(\overline{\nu})}{d_\nu},$$

which can be re-arranged to obtain

$$r = \frac{d_\nu \cdot [F_n : F]}{e \cdot \deg(\overline{\nu})}.$$
2.3 Fitting ideals for Tate modules

When the integer \( t \) is big enough we have that the degree of \( \mathfrak{p} \) is equal to one. So to obtain the thesis it is enough to recall that every prime different from \( p \) and \( \infty \) is unramified if \( F_n/F \), that for the prime \( \mathfrak{p} \) the ramification index is equal to \([F_n : H_A] = [F_n : F]/h^0(F)\) and that for the prime \( \infty \) the ramification index is equal to \([\mathbb{F}_q^\times] = q - 1\). \( \square \)

Before going on we recall here the definition of Fitting ideal of a finitely generated module. For an in-depth discussion the reader may refer to [Nor, Chapter 3] or the appendix of [MW]. Let \( R \) be any commutative and unitary ring. For our purpose \( R \) will be one of the rings \( W[\Gamma_\bullet] \) or \( W[\Gamma_\bullet][G_F] \) where \( W \) is an appropriate local ring, but we give here the general definition. Let \( M \) be a finitely generated module over \( M \) and fix a set of generators \( \{e_1, \ldots, e_r\} \). A relation vector between the generators is an element \( \mathbf{a} = (a_1, \ldots, a_r) \in R^r \) such that \( \sum a_i e_i = 0 \). A matrix of relations is any \( q \times r \) matrix, with \( q \geq r \), whose entries are in \( R \) and whose rows are relation vectors.

**Definition 2.3.2.** The Fitting ideal of the finitely generated \( R \)-module \( M \) is the ideal generated by the determinants of the \( r \times r \) minors of all the matrices of relations of \( M \). It will be denoted by \( \text{Fitt}_R M \).

We want to point out that here we have called Fitting ideals what the author of [Nor] refers to as 0-Fitting invariant of \( M \).

**Remark 2.3.3.**

- It may appear that the definition above depends on the choice of the set of generators \( \{e_i\} \), but it can be proved that the Fitting ideal does not change when we take a different set of generators.
- It is easy to check that we can limit ourselves to determinants of the \( r \times r \) matrices of relations.
- One can prove that the Fitting ideal is contained if the annihilator of the module. Thus the definition appears interesting only for torsion modules.

We denote with \( H_n(\nu) \) the \( \mathbb{Z}_p \)-free module generated by the set of primes of \( F_{ar} \) lying above \( \nu \). Let \( I_n(\nu) \subset G_n \) be the inertia group of \( \nu \): we have that \( H_n(\nu) \) is also a free \( \mathbb{Z}_p[G_n/I_n(\nu)] \)-module of rank \( d_\nu \). Moreover there is a natural action of the group \( G_F \) on \( H_n(\nu) \) and we are interested in studying the structure of \( H_n(\nu) \) as a \( \mathbb{Z}_p[G_n][G_F] \)-module.

For \( \nu \in \{p, \infty\} \) we will denote by \( \text{Fr}_\nu \) any lift to \( G_n \) of the Frobenius map that belongs to \( G_n/I_n(\nu) \). For the prime \( \infty \) the inertia group \( I_n(\infty) \) is contained in \( G_0 \) and does not depend on \( n \), so we shall simply denote it with \( I_\infty \). Since the decomposition and inertia groups of \( \infty \) coincide, we can choose \( \text{Fr}_\infty = 1 \). The same choice can be done for the prime \( p \) if and only if the prime \( p \) is totally split in \( H_A \). For all the other primes we simply denote with \( \text{Fr}_\nu \) the Frobenius map in \( G_n \).

**Definition 2.3.4.** The Euler factor at \( \nu \) is

\[
e_\nu(X) := 1 - \text{Fr}_\nu^{-1}X^{d_\nu} \in \mathbb{Z}_p[G_n][[X]].
\]

Since we will also need to specialize the variable \( X \) at \( \gamma^{-1} \), we put

\[
e_\nu(\gamma^{-1}) := 1 - \text{Fr}_\nu^{-1}\gamma^{-d_\nu} \in \mathbb{Z}_p[G_n][[G_F]].
\]
The statements of [GP2, Lemmas 2.1 and 2.2] adapted to our setting translate into the following

**Lemma 2.3.5.** For \( \nu \in \{p, \infty\} \) we denote with \( \text{Aug}_{\nu,n} \) the augmentation ideal of \( \mathbb{Z}_p[G_n][G_F] \) associated to the inertia group \( I_n(\nu) \), i.e., the ideal generated by the elements of the form \( \tau - 1 \), with \( \tau \in I_n(\nu) \). Then

(a) if \( \nu \neq p, \infty \), then \( \text{Fitt}_{\mathbb{Z}_p[G_n][G_F]}(H_n(\nu)) = (e_\nu) \);

(b) \( \text{Fitt}_{\mathbb{Z}_p[G_n][G_F]}(H_n(\infty)) = (e_\infty, \text{Aug}_{\infty,n}) \);

(c) \( \text{Fitt}_{\mathbb{Z}_p[G_n][G_F]}(H_n(p)) = (e_p, \text{Aug}_{p,n}) \).

From this lemma and the fact that the \( H_n(\nu) \) are free \( \mathbb{Z}_p[G_n/I_n(\nu)] \)-modules, we have the isomorphisms

- if \( \nu \neq p, \infty \)
  \[ H_n(\nu) \cong \mathbb{Z}_p[G_n][G_F] / (e_\nu) ; \]
- if \( \nu = \infty \)
  \[ H_n(\infty) \cong \mathbb{Z}_p[G_0/[\mathbb{F}_q^\times \times G_n][G_F] / (e_\infty) ; \]
- if \( \nu = p \)
  \[ H_n(p) \cong \mathbb{Z}_p[\text{Pic}(A)][G_F] / (e_p) . \]

### 2.3.2 Complex characters

Let \( \chi \in \text{Hom}(G_0, \mathbb{C}^\times) \) be a complex character for \( G_0 \). The character \( \chi \) takes values in the set of roots of unity of order \( |G_0| = h^0(F)(q^d p^\nu - 1) \) and so we need to consider modules over the Witt ring \( W = \mathbb{Z}_p[\zeta] \), where \( \zeta \) denotes any primitive root of unity of order \( |G_0| \). Recall that we are assuming \(|G_0|, p = 1\), we put

\[ e_\chi := \frac{1}{|G_0|} \sum_{g \in G_0} \chi(g^{-1}) g \in W[G_0] \]

for the idempotent associated to \( \chi \). For any \( W[G_0] \)-module \( M \), we denote its \( \chi \)-part by \( M(\chi) := e_\chi M \). If \( M \) is a \( \mathbb{Z}_p[G_0] \)-module we first turn it into a \( W[G_0] \)-module by tensoring with \( W \) over \( \mathbb{Z}_p \) and then we consider the \( \chi \)-part of this tensor product (all this will often be tacitly assumed and forgotten in the notations). We also recall that \( W \) is a flat \( \mathbb{Z}_p \)-module and so the functor \( W \otimes_{\mathbb{Z}_p} - \) is exact in the category of \( \mathbb{Z}_p \)-modules. Finally note that if the action of \( G_0 \) is trivial on \( M \), then

\[ M(\chi) = \begin{cases} W \otimes_{\mathbb{Z}_p} M & \text{if } \chi = \chi_0, \\ 0 & \text{if } \chi \neq \chi_0. \end{cases} \]

**Definition 2.3.6.** Let \( \chi \) be a character of \( G_0 \). We will distinguish 3 types of characters:

- \( \chi \) is said to be of **type 1** if \( \chi(I_\infty) \neq 1 \);
- \( \chi \) is said to be of **type 2** if \( \chi(I_\infty) = 1 \) and \( \chi(Gal(F_0/H_A)) \neq 1 \);
- \( \chi \) is said to be of **type 3** if \( \chi(Gal(F_0/H_A)) = 1 \).

Among the characters of type 3 there is the trivial one which will be denoted by \( \chi_0 \).
2.3.3 The theorem of Greither and Popescu

Now we can start our computations: since $d_\infty = 1$ and we have chosen $Fr_\infty = 1$ we have that $e_\infty = 1 - \gamma^{-1}$. From Lemma 2.3.5 we have that $H_n(\infty) \simeq \mathbb{Z}_p[G_0/I_\infty \times \Gamma_n]$ and so

$$H_n(\infty)(\chi) \simeq \begin{cases} 0 & \text{if } \chi \text{ is of type 1,} \\ W[\Gamma_n] & \text{otherwise.} \end{cases}$$

(2.1)

For the prime $p$ we have $H_n(p) \simeq \mathbb{Z}_p[Pic(A)][G_\mathbb{F}] / (1 - Fr_p^{-1}\gamma^{-dp})$ and so we have the following exact sequence

$$(1 - Fr_p^{-1}\gamma^{-dp})\mathbb{Z}_p[Pic(A)][G_\mathbb{F}] \hookrightarrow \mathbb{Z}_p[Pic(A)][G_\mathbb{F}] \twoheadrightarrow H_n(p).$$

(2.2)

We can tensor the previous exact sequence with $W$ and multiply by $e_\chi$ to obtain

$$e_\chi(1 - Fr_p^{-1}\gamma^{-dp})W[Pic(A)][G_\mathbb{F}] \hookrightarrow e_\chi W[Pic(A)][G_\mathbb{F}] \twoheadrightarrow H_n(p)(\chi).$$

(2.3)

If we observe that $e_\chi(1 - Fr_p^{-1}\gamma^{-dp}) = (1 - \chi(Fr_p^{-1}\gamma^{-dp}) e_\chi$ and $e_\chi W[Pic(A)][G_\mathbb{F}] = W[G_\mathbb{F}]$, we can conclude that

$$H_n(p)(\chi) \simeq \begin{cases} 0 & \text{if } \chi \text{ is of type 1 or 2,} \\ W[G_\mathbb{F}] / (1 - \chi(Fr_p^{-1}\gamma^{-dp}) & \text{otherwise.} \end{cases}$$

(2.4)

Now consider the group of divisors $H_n(\infty) \oplus H_n(p)$ which is the free $\mathbb{Z}_p$-module of divisors of $F_n^{ar}$ whose support is contained in $S$. We denote with $D_n$ its subgroup of divisors of degree zero, i.e., the kernel of the map

$$\deg : H_n(\infty) \oplus H_n(p) \rightarrow \mathbb{Z}_p,$$

and with $D_n(\chi)$ its $\chi$-part. Since there is no action of $G_0$ on $\mathbb{Z}_p$ we have that when $\chi$ is not the trivial character, $D_n(\chi)$ is simply the sum $H_n(\infty)(\chi) \oplus H_n(p)(\chi)$, i.e.,

$$D_n(\chi) \simeq \begin{cases} 0 & \text{if } \chi \text{ is of type 1,} \\ W[\Gamma_n] & \text{if } \chi \text{ is of type 2,} \\ W[\Gamma_n] \oplus W[G_\mathbb{F}] / (1 - \chi(Fr_p^{-1}\gamma^{-dp}) & \text{if } \chi \text{ is of type 3 and } \chi \neq \chi_0. \end{cases}$$

(2.5)

Let $X_n$ be the projective curve defined over $\mathbb{F}_q$ and associated with $F_n$ and $Jac(X_n)(\overline{\mathbb{F}_q})$ the set of $\mathbb{F}_q$-rational points of its Jacobian. We recall that for each abelian group $M$, the multiplication by $p$ defines a surjective map $M[p^{n+1}] \rightarrow M[p^n]$ from the $p^{n+1}$-torsion subgroup of $M$ to the $p^n$-torsion subgroup. One can use this maps to define a projective limit which is called the $p$-adic Tate module of $M$ and we will denote it with $T_p(M)$ (note that the Tate module depends only on the $p$-part of $M$). For more details the reader may refer to the classical work of Tate [Tat] or [Mum, Chapter IV].

We will denote $T_p(F_n) := T_p(Jac(X_n)(\overline{\mathbb{F}_q}))$ the $p$-adic Tate module of the Jacobian of $X_n$, defined over the algebraically closed field $\mathbb{F}_q$. Our task is to study the structure of $T_p(F_n)$ as a
Galois module over $\mathbb{Z}_p[\Gamma_n][G_F]$. More precisely we will study its $\chi$-part when $\chi$ is of type 1 or 2.

Following the definitions and the properties given in [GP1, Section 2] we consider the Deligne’s Picard 1-motive $M_{S,\emptyset}$ associated to the field $F_n^{ar}$ and to the set of primes of $F_n^{ar}$ which lie above the primes of $S$. We can take the empty set as auxiliary set of primes in the definition of $M_{S,\emptyset}$ because we are only interested in the study of the $p$-adic Tate module of the Jacobian (see [GP1, Remark 2.7]). In what follows we will simply denote the Deligne’s Picard 1-motive with $M_n$.

The multiplication by $p$ induces a surjective map on the $p^m$-torsion of $M_n$ for every positive integer $m$ and so we can define the $p$-adic Tate module of $M_n$ as $T_p(M_n) = \lim_{\leftarrow m} M_n[p^m]$.

We denote with $\Theta_n(X)$ (resp. $\Theta_\infty(X)$) the projection of the Stickelberger series $\Theta_S(X)$ to $\mathbb{Z}[G_n][X]$ (resp. $\mathbb{Z}[G_\infty][X]$), which is easily seen to be the Stickelberger series associated to the extension $F_n/F$ (resp. $F_\infty/F$) since the set of ramified primes in this subextension is exactly $S$.

In [GP1, Theorem 4.3] the authors prove the following

**Theorem 2.3.7.** One has

$$Fitt_{\mathbb{Z}_p[G_n][G_F]}(T_p(M_n)) = (\Theta_n(\gamma^{-1}))$$

Note that evaluating the Stickelberger series $\Theta_n(X)$ at $X = \gamma^{-1}$ makes sense because of Proposition 1.3.4.

### 2.3.4 Fitting ideals for Tate modules: finite level

In [GP1, after Definition 2.6] the authors provide the following exact sequence

$$0 \rightarrow T_p(F_n) \rightarrow T_p(M_n) \rightarrow D_n \rightarrow 0. \quad (2.6)$$

For every character $\chi$ we denote by $\Theta_n(X,\chi)$ the only element of $W[\Gamma_n][X]$ that satisfies $\Theta_n(X,\chi)e_\chi = e_\chi \Theta_n(X)$.

**Theorem 2.3.8.** Let $\chi \in \widehat{G}_0$ be a character not of type 3. Then we have

$$Fitt_{W[\Gamma_n][G_F]}(T_p(F_n)(\chi)) = \left(\Theta_n^\chi(\gamma^{-1},\chi)\right),$$

where we put

$$\Theta_n^\chi(\gamma^{-1},\chi) = \begin{cases} 
\Theta_n(\gamma^{-1},\chi) & \text{if } \chi \text{ is of type 1,} \\
\Theta_n(\gamma^{-1},\chi) / (1 - \gamma^{-1}) & \text{if } \chi \text{ is of type 2.}
\end{cases}$$

**Proof.** From Theorem 2.3.7 we have that the Fitting ideal of $T_p(M_n)$ over the ring $\mathbb{Z}_p[G_n][G_F]$ is the ideal generated by $\Theta_n(\gamma^{-1})$. We want to use this information to determine the Fitting ideal of $T_p(M_n)(\chi)$ over the ring $W[\Gamma_n][G_F]$. Since $\mathbb{Z}_p[G_n][G_F]$ is a Noetherian ring and $T_p(M_n)$ is finitely generated we have a presentation

$$\mathbb{Z}_p[G_n][G_F]^{\oplus r} \xrightarrow{\alpha} \mathbb{Z}_p[G_n][G_F]^{\oplus s} \rightarrow T_p(M_n),$$
2.3 Fitting ideals for Tate modules

with \( r \geq s \) because \( T_p(M_n) \) is a torsion module. Note that we cannot assume, in general, that \( \alpha \) is injective. The map \( \alpha \) can be identified with a \( r \times s \) matrix whose entries are in \( \mathbb{Z}_p[G_n][G_p] \) and the ideal generated by the determinants of the \( s \times s \) minors of \( \alpha \) is \( (\Theta_n(\gamma^{-1})) \). Now we take the \( \chi \)-part in the previous presentation to obtain

\[ W[\Gamma_n][G_p] \xrightarrow{\alpha} W[\Gamma_n][G_p] \xrightarrow{\oplus s} T_p(M_n)(\chi), \]

i.e., a presentation for \( T_p(M_n)(\chi) \) as a module over \( W[\Gamma_n][G_p] \). Clearly the ideal generated by the determinants of the \( s \times s \) minors of \( \alpha^\chi \) is generated by \( e^\chi \Theta_n(\gamma^{-1}) = \Theta_n(\gamma^{-1}, \chi) \) and it is the Fitting ideal of \( T_p(M_n)(\chi) \) over \( W[\Gamma_n][G_p] \).

We have already observed that tensor product \( W \otimes_{\mathbb{Z}_p} - \) is an exact functor and so it preserves exact sequences. The same happens when we take the \( \chi \)-part and so, from (2.6), we obtain

\[ 0 \rightarrow T_p(F_n)(\chi) \rightarrow T_p(M_n)(\chi) \rightarrow D_n(\chi) \rightarrow 0. \tag{2.7} \]

- If \( \chi \) is of type 1, from (2.5) we have that \( D_n(\chi) = 0 \), thus
  \[ T_p(F_n)(\chi) \simeq T_p(M_n)(\chi) \]
  and so the thesis is proved because in this case \( \Theta_n^0(\gamma^{-1}, \chi) = \Theta_n(\gamma^{-1}, \chi) \).

- If \( \chi \) is of type 2, from (2.5) we have that \( D_n(\chi) = W[\Gamma_n] \simeq W[\Gamma_n]/[G_p]/(1-\gamma^{-1}) \) which is a cyclic \( W[\Gamma_n][G_p] \)-module. This allows us to apply [CG, Lemma 3] to the previous exact sequence and to obtain
  \[ (1-\gamma^{-1})\text{Fitt}_{W[\Gamma_n][G_p]}(T_p(F_n)(\chi)) = \text{Fitt}_{W[\Gamma_n][G_p]}(T_p(M_n)(\chi)) = (\Theta_n(\gamma^{-1}, \chi)) \].

\[ \square \]

When \( \chi \) is a character of type 3 things get more complicated, since \( D_n(\chi) \) is not cyclic and so we cannot apply [CG, Lemma 3] as we did in the previous theorem. We shall consider those characters in Section 2.5.

2.3.5 Fitting ideals for Tate modules: infinite level

Until now we have studied the Galois module \( T_p(F_n) \) for a fixed integer \( n \). Now we will consider two indices \( n > m \geq 0 \) and study the relation between \( T_p(F_n) \) and \( T_p(F_m) \).

We put \( \Gamma_n = \text{Gal}(F_n/F_m) \) and we recall that this extension of fields is totally ramified at every prime that lies above \( p \) and unramified at every other prime (moreover the number of primes in \( F_m \) above \( p \) is the same for any \( m \) and coincides with the number of primes of \( H_A \) lying above \( p \)). We denote with \( C_n \) the \( p \)-part of the class group of degree zero divisors of \( F_n^{ar} \) and we recall that \( T_p(F_n) = \text{Hom}(Q_p/Z_p, C_n) \). Thus the norm map from \( C_n \) to \( C_m \) induces a map \( \text{N}^n_m : T_p(F_n) \rightarrow T_p(F_m) \) and likewise the inclusion map induces a map \( i^m_n : T_p(F_m) \rightarrow T_p(F_n) \).

We define

\[ T_p(F_\infty)(\chi) = \lim_n T_p(F_n)(\chi), \]

where the limit is taken with respect to the norm maps. The limit \( T_p(F_\infty)(\chi) \) is a module over the profinite (non noetherian) algebra \( \Lambda_\infty := W[\Gamma_\infty][G_p] \). Our next goals are to prove that, for
2.3 Fitting ideals for Tate modules

characters of type 1 and 2, this module is finitely generated and torsion over \( \Lambda_p \) and to compute its Fitting ideal.

There is a natural Galois action of \( \Gamma^n_m \) on \( T_p(\mathcal{M}_n) \) and an inclusion map \( \iota^n_m : T_p(\mathcal{M}_m) \to T_p(\mathcal{M}_n) \) that satisfies \( T_p(\mathcal{M}_n)^{\Gamma^n_m} = \iota^n_m(T_p(\mathcal{M}_m)) \) (see [GP1, Theorem 3.1]). There is also a norm map \( N^n_m : T_p(\mathcal{M}_n) \to T_p(\mathcal{M}_m) \) such that the compositum \( N^n_m \circ \iota^n_m \) is the multiplication by \([F_n : F_m]\).

We have one last norm map defined from \( D_n \) to \( D_m \) and inclusion map from \( D_m \) to \( D_n \) whose compositum in again the multiplication by \([F_n : F_m]\).

All this norm and inclusion maps defined on \( T_p(F_\bullet), T_p(\mathcal{M}_\bullet) \) and \( D_\bullet \) are compatible with the exact sequence (2.6) in the sense that the following diagram is commutative for every index \( m \) and \( n \):

\[
\begin{array}{cccccc}
0 & \to & T_p(F_n) & \to & T_p(\mathcal{M}_n) & \to & D_n & \to 0 \\
N^n_m & \downarrow & \iota^n_m & \downarrow & N^n_m & \iota^n_m & \downarrow & N^n_m & \iota^n_m & \downarrow & 0 \\
0 & \to & T_p(F_m) & \to & T_p(\mathcal{M}_m) & \to & D_n & \to 0
\end{array}
\]

This diagram suggests to investigate the behaviour of the norm map on \( T_p(\mathcal{M}_n) \) first and then move to \( T_p(F_n) \). We denote with \( I^n_m \) the augmentation ideal of \( W[\Gamma^n_m] \) associated to the subgroup \( \Gamma^n_m \). The next lemma follows immediately from Theorem 3.9 of [GP1].

**Lemma 2.3.9.** The norm map \( N^n_m : T_p(\mathcal{M}_n) \to T_p(\mathcal{M}_m) \) is surjective and its kernel is \( I^n_m \circ T_p(\mathcal{M}_n) \).

**Proof.** From [GP1, Theorem 3.9 part (2)] we have that \( T_p(\mathcal{M}_n) \) is free over \( \mathbb{Z}_p[\Gamma^n_m] \) (because \( \Gamma^n_m \) is a \( p \)-group). Thus

\[
\hat{H}^i(\Gamma^n_m, T_p(\mathcal{M}_n)) = 0 \quad \text{for every integer } i,
\]

where we denoted by \( \hat{H}^i(\Gamma^n_m, \bullet) \) the \( i \)-th group of Tate cohomology.

Specializing the previous equality at \( i = 0 \) we obtain

\[
N^n_m(T_p(\mathcal{M}_n)) = T_p(\mathcal{M}_n)^{\Gamma^n_m} = \iota^n_m(T_p(\mathcal{M}_m)).
\]

In a similar way we obtain the second part of the lemma by specializing at \( i = -1 \). \( \square \)

Now we can study the norm map on \( T_p(F_n) \).

**Proposition 2.3.10.** Let \( \chi \) be a character of type 1 or 2. Then the norm map \( N^n_m : T_p(F_n)(\chi) \to T_p(F_m)(\chi) \) is surjective and its kernel is \( I^n_m \circ T_p(F_n)(\chi) \).

**Proof.** Consider the exact sequence (2.7). If \( \chi \) is a character of type 1 we have that \( T_p(F_n)(\chi) \simeq T_p(\mathcal{M}_n)(\chi) \) and so the thesis is simply a restatement of the previous lemma (recalling that taking the \( \chi \)-part of a module is an exact functor).

Now assume \( \chi \) is of type 2 so that \( D_n(\chi) \simeq W[\Gamma_n] \) and note that this is a \( \Gamma^n_m \)-cohomologically trivial module. Since \( T_p(\mathcal{M}_n)(\chi) \) is also cohomologically trivial by (the proof of) [GP1, Theorem 3.9], we have that

\[
\hat{H}^i(\Gamma^n_m, T_p(F_n)(\chi)) = 0 \quad \text{for every } i \in \mathbb{Z}.
\]
Specializing at $i = -1$ we obtain

$$\ker\{N^m_n : T_p(F_n)(\chi) \to T_p(F_m)(\chi)\} = I^m_n T_p(F_n)(\chi).$$

To prove surjectivity of the norm map we take $\Gamma_n$-invariants in (2.7) to obtain

$$0 \to T_p(F_n)(\chi)^{\Gamma_n} \to T_p(M_n)(\chi) \to D_n(\chi)^{\Gamma_n} \to \hat{H}^1(\Gamma_n, T_p(F_n)(\chi)) = 0.$$

Note that $D_n(\chi)^{\Gamma_n} \simeq W[\Gamma_n]^{\Gamma_n} \simeq W[\Gamma_n]$ and compare the previous exact sequence with (2.7) with $m$ in place of $n$. We readily get that $T_p(F_n)(\chi)^{\Gamma_n} \simeq T_p(F_m)(\chi)$. Now specializing (2.8) at $i = 0$ we obtain

$$N^m_n(T_p(F_n)(\chi)) = T_p(F_n)(\chi)^{\Gamma_n} \simeq T_p(F_m)(\chi).$$

We are now ready to state our first main theorem.

**Theorem 2.3.11.** Let $\chi$ be a character of type 1 or 2. Then $T_p(F_\infty)(\chi)$ is a finitely generated torsion $\Lambda_F$-module.

**Proof.** Fix an index $m$, and denote with $\mathcal{J}_m$ the augmentation ideal of $W[\Gamma_\infty]$ associated to the subgroup $\text{Gal}(F_\infty/F_m)$. In particular we have

$$\mathcal{J}_m = \lim_{\longrightarrow} I^n_m.$$

We also put

$$\tilde{\mathcal{J}}_m = \Lambda_F \otimes_{W[\Gamma_\infty]} \mathcal{J}_m$$

for the corresponding ideal of $\Lambda_F$. Now Proposition 2.3.10 yields $T_p(F_m)(\chi) = N^m_n(T_p(F_n)(\chi))$. Applying the first homomorphism theorem and again Proposition 2.3.10 we also have that

$$N^m_n(T_p(F_n)(\chi)) \simeq T_p(F_n)(\chi)/\ker N^m_n = T_p(F_n)(\chi)/I^m_n T_p(F_n)(\chi),$$

thus

$$T_p(F_m)(\chi) \simeq T_p(F_n)(\chi)/I^m_n T_p(F_n)(\chi).$$

The previous equality holds for every $n > m$ and so we have

$$T_p(F_m)(\chi) \simeq T_p(F_\infty)(\chi)/\mathcal{J}_m T_p(F_\infty)(\chi) = T_p(F_\infty)(\chi)/\tilde{\mathcal{J}}_m T_p(F_\infty)(\chi).$$

The module on the left is finitely generated over

$$W[\Gamma_m][G_F] = \Lambda_F/\tilde{\mathcal{J}}_m$$

and, since the ideals $\tilde{\mathcal{J}}_m$ form an open filtration of the profinite algebra $\Lambda_F$, we can apply the generalized Nakayama Lemma (see [BH]) and obtain that $T_p(F_\infty)(\chi)$ is a finitely generated module over $\Lambda_F$.

Now we define the element $\Theta_{\infty}(\gamma^{-1}, \chi) \in \Lambda_F$ as

$$\Theta_{\infty}(\gamma^{-1}, \chi) = \begin{cases} 
\Theta_{\infty}(\gamma^{-1}, \chi) & \text{if } \chi \text{ is of type 1,} \\
\Theta_{\infty}(\gamma^{-1}, \chi) / 1 - \gamma^{-1} & \text{if } \chi \text{ is of type 2,}
\end{cases}$$
which is simply the inverse limit of the generators of the Fitting ideals of \( T_p(F_m)(\chi) \) over \( W[\Gamma_m][G_\varphi] \) (computed in Theorem 2.3.8). At the finite level we have \( \Theta^T_\infty(\gamma^{-1}, \chi)T_p(F_n)(\chi) = 0 \) because the Fitting ideal of a module is contained in the annihilator. Taking the inverse limit of the previous equalities we obtain \( \Theta^T_\infty(\gamma^{-1}, \chi)T_p(F_\infty)(\chi) = 0 \) and so \( T_p(F_\infty)(\chi) \) is torsion.

Now that we have proved that the module \( T_p(F_\infty)(\chi) \) is a finitely generated torsion module over \( \Lambda_\varphi \), we know that its Fitting ideal is well defined. With our second main theorem we compute a generator of that ideal via an inverse limit process.

**Theorem 2.3.12.** Let \( \chi \) be a character of type 1 or 2. Then we have

\[
\text{Fitt}_{\Lambda_\varphi}(T_p(F_\infty)(\chi)) = \left( \Theta^T_\infty(\gamma^{-1}, \chi) \right).
\]

**Proof.** From the equality

\[
\left( \Theta^T_\infty(\gamma^{-1}, \chi) \right) = \lim_n \left( \Theta^T_n(\gamma^{-1}, \chi) \right) = \lim_n \text{Fitt}_{W[\Gamma_n][G_\varphi]}(T_p(F_n)(\chi))
\]

we reduce the proof to showing the equality

\[
\text{Fitt}_{\Lambda_\varphi}(T_p(F_\infty)(\chi)) = \lim_n \text{Fitt}_{W[\Gamma_n][G_\varphi]}(T_p(F_n)(\chi))
\]

We denote with \( N^\infty_m \) the projection \( T_p(F_\infty)(\chi) \to T_p(F_m)(\chi) \). These maps are obviously compatible with the norm maps, in the sense that \( N^\infty_m = N^\infty_n \circ N^\infty_m \). Let \( t_1, \ldots, t_r \) be generators of \( T_p(F_\infty)(\chi) \) over \( \Lambda_\varphi \), then we have that \( N^\infty_m(t_1), \ldots, N^\infty_m(t_r) \) generate \( T_p(F_m)(\chi) \) over \( W[\Gamma_m][G_\varphi] \), since \( T_p(F_m)(\chi) = T_p(F_\infty)(\chi)/\tilde{J}_mT_p(F_\infty)(\chi) \).

Then for every integer \( n \) we have the following exact sequence

\[
0 \to K_n \to W[\Gamma_n][G_\varphi]^{\oplus r} \to T_p(F_n)(\chi) \to 0
\]

where the map on the right is given by \( (w_1, \ldots, w_r) \mapsto \sum_i w_i N^\infty_n(t_i) \). We also have the exact sequence at the infinite level

\[
0 \to K_\infty \to \Lambda_\varphi^{\oplus r} \to T_p(F_\infty)(\chi) \to 0.
\]

The previous exact sequences fits into the diagram

\[
\begin{array}{cccc}
0 & \to & K_n & \to & W[\Gamma_n][G_\varphi]^{\oplus r} & \to & T_p(F_n)(\chi) & \to & 0 \\
\downarrow k_m^n & & \downarrow \pi_m^n & & & & \downarrow N_m^n & & \\
0 & \to & K_m & \to & W[\Gamma_m][G_\varphi]^{\oplus r} & \to & T_p(F_m)(\chi) & \to & 0
\end{array}
\]

where \( k_m^n \) denotes the restriction of the projection \( \pi_m^n \) to the kernel \( K_n \). The kernel of \( \pi_m^n \) is \( \left( \Gamma_m W[\Gamma_n][G_\varphi] \right)^{\oplus r} \), while the kernel of \( N_m^n \) is \( \Gamma_m T_p(F_n)(\chi) \) due to Proposition 2.3.10 and so the map between these two kernels is surjective. The map \( \pi_m^n \) is clearly surjective thus, by the snake lemma, we have that \( k_m^n \) is also surjective and so the previous diagram satisfies the
Mittag-Leffler condition which allows us to take the inverse limit. Comparing this limit with the exact sequence at level infinity we obtain that

$$K_\infty = \varprojlim K_n.$$ 

We denote with $\mathcal{M}_{r \times r}(K_m)$ the set of $r \times r$ matrices whose entries are in $W[\Gamma_m][G_F]$ and such that each row, seen as a vector in $W[\Gamma_m][G_F]^{\oplus r}$, is in $K_m$ and with $k_m^n$ the natural extension of the map $K_n \rightarrow K_m$ to $\mathcal{M}_{r \times r}(K_n) \rightarrow \mathcal{M}_{r \times r}(K_m)$. From the surjectivity of the first map we clearly have that also the extension is surjective. From the definition we have that $\text{Fitt}_{W[\Gamma_m][G_F]}(\chi) = \{ \text{det } M_m \}$. Since this is true for each index $m$ we have that

$$\pi_m^n(\text{det } M_n) = \text{det } (k_m^n(M_n)).$$

In what follows we will use the same notation also for the infinite level.

Now if we take $M_\infty \in \mathcal{M}_{r \times r}(K_\infty)$ we have that $\pi_\infty(\text{det } M_\infty) \in \text{Fitt}_{W[\Gamma_m][G_F]}(\chi)$ and so $\pi_m^n(\text{Fitt}_{\chi}(T_p(F_\infty)(\chi))) \subseteq \text{Fitt}_{W[\Gamma_m][G_F]}(\chi)$. Since this is true for each index $m$ we have that

$$\text{Fitt}_{\chi}(T_p(F_\infty)(\chi)) \subseteq \varprojlim_m \text{Fitt}_{W[\Gamma_m][G_F]}(T_p(F_m)(\chi)).$$

For the other inclusion we need a little bit more work and, in particular, we have to deal with some topological properties of the rings $W[\Gamma_m][G_F]$. Essentially we will follow the arguments of [GK, Theorem 2.1].

Each element of $\text{Fitt}_{W[\Gamma_m][G_F]}(T_p(F_m)(\chi))$ may be written as a linear combination

$$x_m = \sum_{i=1}^{s} \lambda_i \det M_m^{(i)}$$

with $\lambda_i \in W[\Gamma_m][G_F]$ and $M_m^{(i)} \in \mathcal{M}_{r \times r}(K_m)$. If we denote with $\overline{M}_m^{(i)}$ the matrix whose first row is equal to the first row of $M_m^{(i)}$ multiplied by $\lambda_i$ and whose other rows are all equal to the corresponding row of $M_m^{(i)}$ we have that $\overline{M}_m^{(i)}$ is in $\mathcal{M}_{r \times r}(K_m)$ and that $\det \overline{M}_m^{(i)} = \lambda_i \det M_m^{(i)}$ and so it is not restrictive to assume that all the coefficients $\lambda_i$ in the linear combination (2.9) are equal to 1. One can also show that the number $s$ of the terms in that sum may be chosen independently from $m$: this follows easily from the fact that also the number of elements needed to generate $T_p(F_m)(\chi)$ (and $T_p(F_\infty)(\chi)$) can be chosen independently from $m$.

Now we put $B_m := \mathcal{M}_{r \times r}(K_m)^{\oplus s}$ and define the non-linear operator $\phi_m : B_m \rightarrow W[\Gamma_m][G_F]$, by $\phi_m(M_m^{(1)}, \ldots, M_m^{(s)}) = \sum_i \det M_m^{(i)}$ (as usual we define also an operator $\phi_\infty$ for the infinite level). This operator is clearly continuous because of the continuity of the determinant (on $B_m$ we put the natural topology induced from $W[\Gamma_m][G_F]$). From what we have observed we have that the image of $\phi_m$ is exactly the Fitting ideal of $T_p(F_m)(\chi)$. The map $k_m^n$ which we have previously extended to $\mathcal{M}_{r \times r}(K_m)$ can also be extended to $B_m$ and this extension fits into the diagram

$$\begin{array}{ccc}
B_n & \xrightarrow{\phi_n} & W[\Gamma_n][G_F] \\
\downarrow{k_m^n} & & \downarrow{\pi_m^n} \\
B_m & \xrightarrow{\phi_m} & W[\Gamma_m][G_F].
\end{array}$$
Now take a sequence \((x_m)_{m \in \mathbb{N}} \subseteq \lim_{\leftarrow k} \text{Fitt}_{W[\Gamma_m][G_F]}(T_p(F_m)(\chi))\), to prove the thesis we have to show that there exists an element \(b_\infty \in \mathcal{B}_\infty\) such that \(\phi_\infty(b_\infty) = (x_m)_{m \in \mathbb{N}}\).

For each integer \(m\) we put \(\Omega_m = \phi_\infty^{-1}(x_m)\). The topological ring \(W[\Gamma_m][G_F]\) is a Hausdorff space, thus finite sets are closed and, from the continuity of \(\phi_m\), we have that also \(\Omega_m\) is closed. Furthermore, since \(\mathcal{B}_m\) is compact (because \(W[\Gamma_m][G_F]\) is compact) and \(\Omega_m\) is contained in the direct sum of \(s \cdot r^2\) copies of \(W[\Gamma_m][G_F]\), we get that also \(\Omega_m\) is compact. For each \(\omega_n \in \Omega_n\) we have that

\[
\phi_m \left( k^n_m(\omega_n) \right) = \pi_m \left( \phi_n(\omega_n) \right) = \pi_m(x_n) = x_m
\]

thus the image of \(\Omega_n\) under \(k^n_m\) is contained in \(\Omega_m\). This allows us to define a new set

\[
\bar{\Omega}_m = \bigcap_{n>m} k^n_m(\Omega_n) \subseteq \Omega_m.
\]

From the compactness of \(\Omega_n\) and the continuity of \(k^n_m\) we deduce that the set \(\bar{\Omega}_m\) is not empty and it is also compact. Clearly the image of \(\bar{\Omega}_m\) under the map \(k^n_m\) is contained in \(\bar{\Omega}_m\), but we will now prove that it is exactly \(\bar{\Omega}_m\). Take an element \(\bar{\omega}_m \in \bar{\Omega}_m\). From the definition we have that there exists, for each \(n > m\), an element \(\omega_n \in \Omega_n\) such that \(k^n_m(\omega_n) = \bar{\omega}_m\). Now fix a positive index \(h\) and for \(n > m + h\) consider \(k^n_{m+h}(\omega_n) \in \Omega_{m+h}\) as a sequence in \(\bar{\omega}_m\). Since \(\Omega_{m+h}\) is compact we have that it admits a convergent subsequence. We denote with \(\bar{\omega}_{m+h}\) the limit of this subsequence, thus for each integer \(n > m\) we have defined an element \(\bar{\omega}_n \in \bar{\Omega}_m\). Then we have

\[
k^{n}_{m+h}(\bar{\omega}_n) = k^{n}_{m+h} \left( \lim_{t \rightarrow \infty} k^{t}_{n}(\omega_t) \right) = \lim_{t \rightarrow \infty} \left( k^{n}_{m+h} \circ k^{t}_{n}(\omega_t) \right)
\]

The previous equality shows us that \(\bar{\omega}_{m+h}\) is in \(\bar{\Omega}_{m+h}\) and, since \(\bar{\omega}_m = k^{m+h}_{m}(\bar{\omega}_{m+h})\), we have proved the surjectivity of the map \(k^{m+h}_{m}\). But we have also proved that the sequence \(\bar{\omega}_m\) is coherent and, since \(\phi_m(\bar{\omega}_m) = x_m\) for each integer \(m\) we have that \(\bar{\omega}_\infty := \lim \bar{\omega}_m\) is in \(\mathcal{B}_\infty\) and \(\phi_\infty(\bar{\omega}_\infty) = (x_m)_{m \in \mathbb{N}}\).

### 2.4 Fitting ideals for the class groups

In this section we will investigate the \(p\)-part of the class groups of degree zero divisors of the fields \(F_n\) in the Iwasawa tower and their inverse limit. As in the previous section we shall see the inverse limit as a module over the non-Noetherian Iwasawa algebra \(\Lambda := W[\Gamma]\) (which is the quotient of the algebra \(\Lambda_F\) with respect to the augmentation ideal of \(G_F\)), we will prove that it is finitely generated and torsion and we will compute its Fitting ideal as a limit of the Fitting ideals at the finite levels.
2.4 Fitting ideals for the class groups

2.4.1 Fitting ideals for the class groups: finite level

Let \(C_n := \text{Cl}^0(F_n)\{p\}\) be the \(p\)-torsion of the class groups of degree zero divisors of \(F_n\). In particular the natural action of \(\Gamma_n\) makes it a module over the commutative ring \(\mathbb{Z}_p[\Gamma_n]\). Since \(C_n\) is finite its Fitting ideal over \(\mathbb{Z}_p[\Gamma_n]\) is well defined. Using the computation done for \(T_p(F_n)(\chi)\) we will be able to compute also the \(\chi\)-part of the Fitting ideal of \(W \otimes_{\mathbb{Z}_p} C_n\) over \(W[\Gamma_n]\).

Denote with \(T_p(F_n)_{G_F}\) the quotient \(T_p(F_n)/(1 - \gamma^{-1})T_p(F_n)\). Note that the ideal \((1 - \gamma^{-1})\) of \(\mathbb{Z}_p[\Gamma_n][G_F]\) is the augmentation ideal of \(G_F\) (recall that \(G_F\) is pro-cyclic and \(\gamma\) is the arithmetic Frobenius, which is a topological generator) and so the action of \(G_F\) on \(T_p(F_n)_{G_F}\) is trivial.

**Lemma 2.4.1.** The module \(T_p(F_n)_{G_F}\) is isomorphic to \(C_n\) as module over \(\mathbb{Z}_p[G_n]\).

**Proof.** The proof of this lemma will be achieved by applying certain functors to some well known exact sequences.

We start with the classical sequence

\[
0 \to \mathbb{Z}_p \to \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to 0
\]

and apply the contravariant functor \(\text{Hom}(\ast, C_n)\) to obtain

\[
\to \text{Hom}(\mathbb{Q}_p, C_n) \to \text{Hom}(\mathbb{Z}_p, C_n) \to \text{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, C_n) \to \text{Ext}^1(\mathbb{Q}_p, C_n) \to .
\]

Since \(\mathbb{Q}_p\) is a field and \(C_n\) is finite we have that \(\text{Hom}(\mathbb{Q}_p, C_n) = \text{Ext}^1(\mathbb{Q}_p, C_n) = 0\) and so

\[
C_n \simeq \text{Hom}(\mathbb{Z}_p, C_n) \simeq \text{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, C_n).
\] (2.10)

We recall that \(\overline{C}_n\) is the \(p\)-part of the class group of degree zero divisors of \(F_n^{ar}\), which satisfies the equality \(T_p(F_n) = \text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \overline{C}_n)\). If we consider the multiplication by \(1 - \gamma^{-1}\) on \(\overline{C}_n\) we have the following exact sequence (where surjectivity on the right is due to Lang’s Theorem, see, for example, [Ser, Chapter VI, §4]):

\[
0 \to C_n \to \overline{C}_n \xrightarrow{1 - \gamma^{-1}} \overline{C}_n \to 0.
\]

If we apply the functor \(\text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \ast)\) to the previous sequence we get

\[
\xrightarrow{\text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, C_n)} \xrightarrow{T_p(F_n)} \xrightarrow{1 - \gamma^{-1}} \xrightarrow{T_p(F_n)} \xrightarrow{\text{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, C_n)}.
\]

The group \(C_n\) is a finite \(p\)-group thus \(\text{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, C_n) = 0\) and since the constant field of \(F_n^{ar}\) is algebraically closed we have that \(\overline{C}_n\) is divisible hence \(\text{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, \overline{C}_n) = 0\). From the last sequence we get

\[
\text{Ext}^1(\mathbb{Q}_p/\mathbb{Z}_p, C_n) \simeq T_p(F_n)/(1 - \gamma^{-1})T_p(F_n)
\]

and combining this with (2.10) we obtain the thesis. 

As we have previously anticipated we want to use our computation of the Fitting ideal of \(T_p(F_n)(\chi)\), which is a module over \(W[\Gamma_n][G_F]\), to compute the Fitting ideal of \(C_n(\chi)\), which is a module over \(W[\Gamma_n]\). To do this we need the following general result on Fitting ideals.
2.4 Fitting ideals for the class groups

**Lemma 2.4.2.** Let $R$ be any unitary ring and $M$ a finitely generated torsion module over $R$. Let $I$ be any non trivial ideal of $R$ and denote with $\pi$ the canonical projection $R \to R/I$. Then we have

$$ \text{Fitt}_{R/I}(M/IM) = \pi(\text{Fitt}_R(M)). $$

**Proof.** Let $m_1, m_2, \ldots, m_r$ be a set of generators of $M$. Clearly the image $\overline{m}_h$ of the elements $m_h$ under the canonical projection $M \to M/IM$ is a set of generators of $M/IM$ as a $R/I$-module and so any relation between the generators of $M$ determines a relation between the generators of $M/IM$. Thus

$$ \pi(\text{Fitt}_R(M)) \subseteq \text{Fitt}_{R/I}(M/IM). $$

To prove the other inclusion we fix a relation between the generators of $M/IM$ and prove that it comes from a relation between the generators of $M$, i.e., let $a_1, a_2, \ldots, a_r \in R$ be such that

$$ \sum_{h=1}^r \pi(a_h)\overline{m}_h = 0 \quad (\text{in } M/IM), $$

then $\sum_h a_h m_h \in IM$ and so there exist $i_1, i_2, \ldots, i_r \in I$ such that

$$ \sum_{h=1}^r a_h m_h = \sum_{h=1}^r i_h m_h. $$

From the previous equality we deduce that the coefficients $b_h := a_h - i_h$ forms a relation between the generators $m_h$ of $M$ and we also have that this relation induces the original one between the generators of $M/IM$ since $\pi(b_h) = \pi(a_h)$. Thus we have proved the inclusion

$$ \text{Fitt}_{R/I}(M/IM) \subseteq \pi(\text{Fitt}_R(M)). $$

Now we are ready to compute the Fitting ideal of $C_n(\chi)$. Let $\pi : W[\Gamma_n][G_F] \to W[\Gamma_n]$ be the map that sends $\gamma \mapsto 1$, since $W[\Gamma_n] = W[\Gamma_n][G_F]/IG$ we have that this map is the canonical projection. Thus, combining Lemma 2.4.1, Lemma 2.4.2 and the computations of Theorem 2.3.8 we obtain

**Theorem 2.4.3.** Let $\chi \in \widehat{G}_0$ be a character not of type 3. Then we have

$$ \text{Fitt}_{W[\Gamma_n]}(C_n(\chi)) = \left( \Theta^\sharp_n(1, \chi) \right), $$

where

$$ \Theta^\sharp_n(1, \chi) = \begin{cases} 
\Theta_n(1, \chi) & \text{if } \chi \text{ is of type } 1, \\
\Theta_n(\gamma^{-1}, \chi) / (1 - \gamma^{-1}) & \text{if } \chi \text{ is of type } 2.
\end{cases} $$

2.4.2 Fitting ideals for the class groups: infinite level

Now we consider the fields $F_n$ for different indices $n$. Let

$$C_\infty := \lim_{n \to \infty} C_n$$

where the limit is made with respect to the norm maps $N_{m/n}^n : C_n \to C_m$ with $n > m \geq 0$. Since every group $C_n$ has an action of the Galois group $\Gamma_n$ it turns out that $C_\infty$ has a structure of module over

$$\mathbb{Z}_p[\Gamma] := \lim_{n \to \infty} \mathbb{Z}_p[\Gamma_n].$$

We will also need to consider the maps $i_n^m : C_n \to C_m$ which are the maps induced by the immersions $i_n^m : \text{Div}(F_m) \hookrightarrow \text{Div}(F_n)$ with $n > m \geq 0$. We recall that for a divisor $D = \sum \nu \cdot n\nu$ of $F_m$ we have $i_n^m(D) := \sum \nu \cdot \sum w|\nu \cdot e(w|\nu)w$, where $e(w|\nu)$ is the ramification index of $w$ over $\nu$.

In particular one can see from the Kummer formula that $\text{deg}(i_n^m(D)) = [F_n : F_m] \cdot \text{deg}(D)$, thus the image of a degree zero divisor by the map $i_n^m$ still has degree zero. We also observe that the image of $\text{Div}(F_m)$ by $i_n^m$ is contained in the subset of $\Gamma_m$-invariant elements of $\text{Div}(F_n)$.

The following proposition gives us some information about the injectivity and surjectivity of the maps $N$ and $i$ on the class groups.

**Proposition 2.4.4.** Let $F_0 \subseteq K \subseteq E \subseteq F_\infty$ with $E/F_0$ a finite extension.

(a) The norm map $N_K^E : \text{Cl}^0(E) \to \text{Cl}^0(K)$ is surjective.

(b) The map $i_K^E : \text{Cl}^0(K) \to \text{Cl}^0(E)$ is injective.

**Proof.**

(a) Fix a prime $\nu$ of $K$ which lies above $\infty$ and let $B$ be the ring of elements regular outside $\nu$. As we have observed in Section 2.2.1 the prime at infinity does not have inertia in $F_\infty/F$ and so the degree of $\nu$ is equal to 1. This implies that $\text{Cl}(B) \simeq \text{Cl}^0(K)$.

Let $C$ be the integral closure of $B$ in $E$: there is a natural map $\text{Cl}^0(E) \to \text{Cl}(C)$ which is surjective because the extension $E/K$ is totally split at $\nu$ and so, again, every prime of $E$ which lies above $\nu$ has degree equal to one. So we are in the following setting:

$$\text{Cl}^0(E) \to \text{Cl}(C) \to \text{Cl}(B) \simeq \text{Cl}^0(K),$$

where the map $\text{Cl}(C) \to \text{Cl}(B)$ is the natural norm map. To complete the proof we will have to show that this map is surjective.

Let $H(B)$ (resp. $H(C)$) be the Hilbert class field, i.e. the maximal abelian extension of $K$ (resp. $E$) which is unramified everywhere and totally split at $\nu$ (resp. at the prime of $E$ which lies above $\nu$). By class field theory we have the isomorphisms $\text{Gal}(H(B)/K) \simeq \text{Cl}(B)$ and $\text{Gal}(H(C)/E) \simeq \text{Cl}(C)$ induced by the Artin map. Now we observe that the two fields $E$ and $H(B)$ are disjoint over $K$ because $E/K$ is totally ramified at every prime of $K$ which lies above $p$ and $H(B)/K$ is unramified at every prime, thus we have the canonical isomorphism $\text{Gal}(E \cdot H(B)/E) \simeq \text{Gal}(H(B)/K)$.

The restriction map

$$\text{Res} : \text{Gal}(H(C)/E) \to \text{Gal}(E \cdot H(B)/E) \simeq \text{Gal}(H(B)/K)$$
fits into the following commutative diagram (from class field theory)

\[
\begin{array}{ccc}
Cl(C) & \xrightarrow{\sim} & Gal(H(C)/E) \\
N & \downarrow & Res \\
Cl(B) & \xrightarrow{\sim} & Gal(H(B)/K)
\end{array}
\]

and so the surjectivity is proved.

(b) Let \( G := Gal(E/K) \) and for every field \( L \) we denote with \( P_L \) the set of principal divisors of \( L \). Taking the \( G \)-cohomology in the exact sequence

\[
0 \longrightarrow \mathbb{F}_q^\times \longrightarrow E^\times \longrightarrow \mathcal{P}_E \longrightarrow 0 \tag{2.11}
\]

we obtain

\[
0 \longrightarrow \mathbb{F}_q^\times \longrightarrow K^\times \longrightarrow \mathcal{P}_E^G \longrightarrow H^1(G, \mathbb{F}_q^\times) \longrightarrow H^1(G, E^\times) \longrightarrow \]

\[
\longrightarrow H^1(G, \mathcal{P}_E) \longrightarrow H^2(G, \mathbb{F}_q^\times).
\]

From Hilbert 90 we have that \( H^1(G, E^\times) = 0 \) and, since \( G \) is a \( p \)-group we also have that \( H^i(G, \mathbb{F}_q^\times) \) for every \( i \geq 1 \), thus \( H^1(G, \mathcal{P}_E) = 0 \) and the following exact sequence holds

\[
0 \longrightarrow \mathbb{F}_q^\times \longrightarrow K^\times \longrightarrow \mathcal{P}_E^G \longrightarrow 0.
\]

If we compare this last sequence with the analogue of (2.11) for the field \( K \) we have that \( \mathcal{P}_E^G = \mathcal{P}_K \).

Taking the \( G \)-cohomology in

\[
0 \longrightarrow \mathcal{P}_E \longrightarrow \text{Div}^0(E) \longrightarrow Cl^0(E) \longrightarrow 0,
\]

we obtain

\[
0 \longrightarrow \mathcal{P}_E^G = \mathcal{P}_K \longrightarrow \text{Div}^0(E)^G \longrightarrow Cl^0(E)^G \longrightarrow H^1(G, \mathcal{P}_E) = 0,
\]

which fits into the following commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{P}_K \\
\text{Div}^0(K) & \xrightarrow{\text{Id}} & Cl^0(K) \\
\downarrow & & \downarrow \text{i}_K^E \\
0 & \longrightarrow & \text{Div}^0(E)^G \longrightarrow Cl^0(E)^G \longrightarrow 0 \tag{2.12}
\end{array}
\]

Applying the snake lemma we obtain the thesis.

From diagram (2.12) we also deduce that

\[
Cl^0(E)^G / \text{i}_K^E (Cl^0(K)) \simeq \text{Div}^0(E)^G / \text{i}_K^E (\text{Div}^0(K)) \tag{2.13}
\]

Now we take a character \( \chi \) of \( G_0 \). Before computing the limit of the class groups we have to study the kernel of the norm map \( N_{\mathcal{K}}^E(\chi) : Cl^0(E)(\chi) \rightarrow Cl^0(K)(\chi) \) for the intermediate finite extensions \( E/K \).
2.4 Fitting ideals for the class groups

Lemma 2.4.5. Let \( F_0 \subseteq K \subseteq E \subseteq F_\infty \) with \( E/F_0 \) a finite extension and let \( G := \text{Gal}(E/K) \). Assume that \(|G| = p\). Then the group \( \Delta := \text{Gal}(F_0/H_A) \) acts trivially on \( \text{Cl}^0(E)^G/i_K^E (\text{Cl}^0(K)) \).

Proof. As we have already observed in (2.13) we have the isomorphism
\[
\text{Cl}^0(E)^G/i_K^E (\text{Cl}^0(K)) \simeq \text{Div}^0(E)^G/i_K^E (\text{Div}^0(K)) .
\]

If we consider the two maps
\[
\text{Div}^0(E)^G \hookrightarrow \text{Div}(E)^G \twoheadrightarrow \text{Div}(E)^G/i_K^E (\text{Div}(K))
\]
we have that the kernel of the composition of this two maps is \( \text{Div}^0(E)^G \cap i_K^E (\text{Div}(K)) = i_K^E (\text{Div}^0(K)) \) and so there is an injection
\[
\text{Div}^0(E)^G/i_K^E (\text{Div}^0(K)) \hookrightarrow \text{Div}(E)^G/i_K^E (\text{Div}(K)) .
\]

Thus it is enough to show that \( \Delta \) acts trivially on \( \text{Div}(E)^G/i_K^E (\text{Div}(K)) \).

Let \( p_1, \ldots, p_s \) be the set of primes of \( K \) which lie above \( p \) and \( \mathfrak{P}_1, \ldots, \mathfrak{P}_s \) the set of primes of \( E \) which lie above \( p \). We recall that the extension \( E/K \) is totally ramified at \( p \) and so we can assume for every \( j = 1, \ldots, s \) that \( \mathfrak{P}_j \) is the unique prime of \( E \) which lies over \( p_j \), moreover the only extension where the prime \( p \) (may) split is \( H_A/F \) and so \( s \) divides \( h^0(F) \), i.e., is coprime with \( p \). In particular we have that
\[
i_K^E(p_j) = p_j \mathfrak{P}_j .
\]

Now we prove that the group \( \text{Div}(E)^G/i_K^E (\text{Div}(K)) \) is isomorphic to the sum of \( s \) copies of \( \mathbb{Z}/p \) and that a set of generators is \( \{ \mathfrak{P}_j + i_K^E (\text{Div}(K)) \} \). We can write \( \text{Div}(K) = \bigoplus_{\nu} \mathbb{Z}_\nu \) (where \( \nu \) runs through all the primes of \( K \)) and \( \text{Div}(E) = \bigoplus_{\nu} H_{\nu} \), with \( H_{\nu} = \bigoplus_{w \mid \nu} \mathbb{Z}_w \) (where \( w \) runs through the set of primes of \( E \) which lie above \( \nu \)). Now for the ramified primes we have \( H_{\nu} = \mathbb{Z} \mathfrak{P}_j = H_{\nu}^G \) and for the unramified primes, if we denote with \( G_{\nu} \) the decomposition group of \( \nu \) in \( G \), we have that \( H_{\nu} = \mathbb{Z}[G_{\nu}]w \) and so \( H_{\nu}^G = i_K^E (\mathbb{Z}_\nu) \). We have proved that
\[
\text{Div}(E)^G = \bigoplus_{j=1}^s \mathbb{Z} \mathfrak{P}_j + \bigoplus_{\nu \neq p_j} i_K^E (\mathbb{Z}_\nu)
\]
and
\[
i_K^E (\text{Div}(K)) = \bigoplus_{j=1}^s p \mathbb{Z} \mathfrak{P}_j + \bigoplus_{\nu \neq p_j} i_K^E (\mathbb{Z}_\nu)
\]
thus
\[
\text{Div}(E)^G/i_K^E (\text{Div}(K)) = \bigoplus_{j=1}^s \left( \mathbb{Z}/p \right) \mathfrak{P}_j .
\]

Now we observe that if we take a set of integers \( \alpha_1, \ldots, \alpha_s \) each of them coprime with \( p \), we have that also the set \( \{ \alpha_j \mathfrak{P}_j + i_K^E (\text{Div}(K)) \} \) generates \( \text{Div}(E)^G/i_K^E (\text{Div}(K)) \): indeed for each index \( j \), by the Bezout identity, we have that there exist two integers \( m \) and \( n \) such that \( n\alpha_j = 1 + mp \) and so
\[
n (\alpha_j \mathfrak{P}_j + i_K^E (\text{Div}(K))) = \mathfrak{P}_j + mp \mathfrak{P}_j + i_K^E (\text{Div}(K))
\]
\[
= \mathfrak{P}_j + mi_K^E(p_j) + i_K^E (\text{Div}(K))
\]
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\[ = \mathfrak{p}_j + i^E_K (\text{Div}(K)) \, . \]

Now we consider the field \( E^\Delta \) (resp. \( K^\Delta \)) which is the subfield of \( E \) (resp. \( K \)) fixed by the elements of \( \Delta \). Since \( \Delta \) is a subgroup of \( G_0 \) (with cardinality coprime with \( p \)) there is a canonical isomorphism \( G^\Delta := \text{Gal}(E^\Delta/K^\Delta) \cong G \) and since the extension \( F_0/H_A \) is totally ramified at the primes above \( p \) we have that there are exactly \( s \) primes in \( E^\Delta \) (resp. \( K^\Delta \)) above \( p \).

We denote with \( \mathfrak{p}_j^\Delta \) (resp. \( p_j^\Delta \)) the unique prime of \( E^\Delta \) (resp. \( K^\Delta \)) which lies below \( \mathfrak{p}_j \) (resp. \( p_j \)).

As above we can prove that the group \( \text{Div}(p) \) of primes above \( p \) we have that there are exactly \( s \) primes in \( E^\Delta \) (resp. \( K^\Delta \)) above \( p \). For the inductive step we can now assume \( I \) is of the groups in the two sequences we deduce that \( |\mathfrak{p}_j^\Delta| = |\mathfrak{p}_j| \) for \( j = 1, \ldots, s \). To conclude we observe that \( i^E_{\Delta} (\mathfrak{p}_j^\Delta) = |\Delta| \mathfrak{p}_j \) and, since \( |\Delta| \) is coprime with \( p \), the image of these classes under the map \( i^E_{\Delta} \) is a set of generators for \( \text{Div}(E^\Delta) \) and clearly the action of \( \Delta \) on this classes is trivial.

We use the previous lemma to prove:

**Proposition 2.4.6.** Let \( F_0 \subseteq K \subseteq E \subset F_\infty \) with \( E/F_0 \) a finite extension and let \( G := \text{Gal}(E/K) \). Then for \( \chi \) of type 1 or 2 we have

\[ \ker (N^E_K (\chi)) = I_G C^0(E)(\chi) \]

where \( I_G \) denotes the augmentation of \( G \).

**Proof.** We will proceed by induction on \( |G| \). We need to consider two basic cases: for \( |G| = 1 \) both members of the equality are zero and so there is nothing to prove.

For \( |G| = p \) we have that \( \chi \) may be seen as a non trivial character of \( \Delta = \text{Gal}(F_0/H_A) \) because it is of type 1 or 2 and, by the previous lemma, we have that \( \Delta \) acts trivially on \( C^0(E)^G/i^E_K (C^0(K)) \), thus we have

\[ (C^0(E)^G/i^E_K (C^0(K))) (\chi) = 0, \]

i.e., \( C^0(E)^G(\chi) = i^E_K (C^0(K))(\chi) \). The group \( G \) is cyclic and we denote with \( g \) a fixed generator, thus \( I_G C^0(E)(\chi) = (g - 1) C^0(E)(\chi) \). We also recall the well known isomorphism \( I_G C^0(E)(\chi) \cong C^0(E)(\chi)/C^0(E)^G(\chi) \). Then we have the following two sequences:

\[
\begin{array}{c}
0 \rightarrow \ker (N^E_K (\chi)) \rightarrow C^0(E)(\chi) \xrightarrow{N^E_K} C^0(K)(\chi) \rightarrow 0,
\end{array}
\]

\[
\begin{array}{c}
0 \rightarrow C^0(K)(\chi) \xrightarrow{i^E_K} C^0(E)(\chi) \xrightarrow{1-g} I_G C^0(E)(\chi) \rightarrow 0.
\end{array}
\]

The first one is exact because of the surjectivity of the norm (Proposition 2.4.4, part (a)), the second one is exact because of what we have previously observed. Comparing the cardinalities of the groups in the two sequences we deduce that \( |\ker (N^E_K (\chi))| = |I_G C^0(E)(\chi)| \), but since \( I_G C^0(E)(\chi) \subseteq \ker (N^E_K (\chi)) \), we have the equality between them.

For the inductive step we can now assume \( |G| = p' > p \), thus we can take an intermediate field \( K \subsetneq E' \subsetneq E \) and let \( G_1 = \text{Gal}(E/E') \) and \( G_2 = \text{Gal}(E'/K) \) with both \( G_1 \) and \( G_2 \) with cardinality strictly smaller than \( G \). Then by the inductive hypothesis we have

\[ \ker (N^E_{E'} (\chi)) = I_{G_1} C^0(E)(\chi) \]
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and

\[
\ker \left( N_K^E \right) (\chi) = I_{G_2} \Cl(E') (\chi).
\]

From Proposition 2.4.4, part (a), we have that \( N_K^E : \Cl(E)(\chi) \rightarrow \Cl(E')(\chi) \) is surjective and so

\[
N_K^E \left( I_{G_2} \Cl(E)(\chi) \right) = I_{G_2} \Cl(E')(\chi).
\]

If we take \( x \in \ker \left( N_K^E \right) (\chi) \) since \( N_K^E = N_E^E \circ N_K^E \) we have that \( N_K^E(x) \in \ker \left( N_K^E \right) = I_{G_2} \Cl(E')(\chi) \), thus there exists \( g \in I_G \) and \( y \in \Cl(E)(\chi) \) such that \( N_K^E(x) = N_K^E(gy) \). This is equivalent to \( x - gy \in \ker \left( N_K^E \right) (\chi) \) and so

\[
x \in \ker \left( N_K^E \right) (\chi) + I_G \Cl(E)(\chi) = I_G \Cl(E)(\chi),
\]

since \( \ker \left( N_K^E \right) (\chi) = I_G \Cl(E)(\chi) \subseteq I_G \Cl(E)(\chi) \). We have proved that

\[
\ker \left( N_K^E \right) (\chi) \subseteq I_G \Cl(E)(\chi).
\]

The other inclusion is trivial. \( \square \)

Now following the proof of Theorem 2.3.11 we can finally prove:

**Theorem 2.4.7.** Let \( \chi \) be a character of type 1 or 2. Then \( C_\infty(\chi) \) is a finitely generated torsion \( \Lambda \)-module.

**Proof.** We recall that for every index \( m \) we have denoted with \( \mathcal{I}_m \) the augmentation ideal of \( W[\Gamma_m] = \Lambda/\mathcal{I}_m \) associated to the subgroup \( \Gal(F_\infty/F_m) \) which is the inverse limit of the augmentation ideals \( I_{\Gamma_m} \). From Proposition 2.4.4 part (a) we have \( C_m(\chi) = N_m^n (C_n(\chi)) \) and from Proposition 2.4.6 we deduce

\[
N_m^n (C_n(\chi)) \simeq C_n(\chi)/\ker N_m^n = C_n(\chi)/I_{\Gamma_m} C_n(\chi),
\]

thus

\[
C_m(\chi) \simeq C_n(\chi)/I_{\Gamma_m} C_n(\chi).
\]

The previous equality holds for every \( n > m \) and so we have

\[
C_m(\chi) \simeq C_\infty(\chi)/\mathcal{I}_m C_\infty(\chi).
\]

The module on the left is finite and, in particular, finitely generated over

\[
W[\Gamma_m] = \Lambda/\mathcal{I}_m.
\]

The ideals \( \mathcal{I}_m \) form an open filtration of the Iwasawa algebra \( \Lambda \), thus we can deduce that \( C_\infty(\chi) \) is a finitely generated module over \( \Lambda \) because of the generalized Nakayama Lemma ([BH]).

To show that \( C_\infty(\chi) \) is also torsion we define the element \( \Theta_\infty(1, \chi) \in \Lambda \) as

\[
\Theta_\infty(1, \chi) = \begin{cases} 
\Theta_\infty(1, \chi) & \text{if } \chi \text{ is of type 1}, \\
\Theta_\infty(\gamma^{-1}, \chi) & \text{if } \chi \text{ is of type 2}, \\
\frac{1}{1 - \gamma^{-1}} |_{\gamma = 1} & \text{if } \chi \text{ is of type 2},
\end{cases}
\]
which is simply the inverse limit of the elements $\Theta^\sharp_\infty(1, \chi)$ which generate the Fitting ideals of the various modules $C_n(\chi)$ over $W[\Gamma_n]$ (Theorem 2.4.3).

Now it is easy to see that $\Theta^\sharp_\infty(1, \chi)C_\infty(\chi) = 0$, since $\Theta^\sharp_n(1, \chi)C_n(\chi) = 0$ for every $n$, and so $C_\infty(\chi)$ is a torsion $\Lambda$-module.

Now we can prove the main conjecture. The reader may see that the arguments of the proof are similar to the ones of Theorem 2.3.12, indeed this is a consequence of the surjectivity of the norm maps and of the computations on the kernels.

**Theorem 2.4.8 (Main Conjecture).** Let $\chi$ be a character of type 1 or 2. Then we have

$$Fitt_\Lambda(C_\infty(\chi)) = \left(\Theta^\sharp_\infty(1, \chi)\right).$$

**Proof.** The equality

$$\left(\Theta^\sharp_\infty(1, \chi)\right) = \lim_{\leftarrow n} \left(\Theta^\sharp_n(1, \chi)\right) = \lim_{\leftarrow n} Fitt_{W[\Gamma_n]}(C_n(\chi))$$

reduce the proof to proving that

$$Fitt_\Lambda(C_\infty(\chi)) = \lim_{\leftarrow n} Fitt_{W[\Gamma_n]}(C_n(\chi)).$$

We denote with $N_m^\infty$ the projection $C_\infty(\chi) \rightarrow C_m(\chi)$ and with $t_1, \ldots, t_r$ a fixed set of generators of $C_\infty(\chi)$ over $\Lambda$. From the equality $C_m(\chi) = C_\infty(\chi)/I_mC_\infty(\chi)$ we deduce that $N_m^\infty(t_1), \ldots, N_m^\infty(t_r)$ generate $C_m(\chi)$ over $W[\Gamma_m]$.

For every integer $n$ we have the following exact sequence

$$0 \rightarrow K_n \rightarrow W[\Gamma_n]|^\oplus r \rightarrow C_n(\chi) \rightarrow 0$$

where the map on the right is given by $(w_1, \ldots, w_r) \mapsto \sum_i w_i N_n^\infty(t_i)$, which holds also at the infinite level

$$0 \rightarrow K_\infty \rightarrow \Lambda|^\oplus r \rightarrow C_\infty(\chi) \rightarrow 0$$

and which fits into the diagram

$$
\begin{array}{ccc}
0 & \rightarrow & K_n \\
\downarrow k^n_m & & \downarrow \pi^n_m \\
0 & \rightarrow & K_m \\
\downarrow N^n_m & & \downarrow \\
 & \rightarrow & W[\Gamma_m]|^\oplus r \\
\end{array}
\rightarrow C_m(\chi) \rightarrow 0.
$$

As usual $k^n_m$ denotes the restriction of the projection $\pi^n_m$ to the kernel $K_n$. The kernel of $\pi^n_m$ is $(I_{\Gamma_m}W[\Gamma_n]|^\oplus r$, while the kernel of $N^n_m$ is $I_{\Gamma_m}C_n(\chi)$ due to Proposition 2.4.6 and so the map between these two kernels is surjective. The map $\pi^n_m$ is surjective thus, by the snake lemma, we have that $k^n_m$ is also surjective and so the previous diagram satisfies the Mittag-Leffler condition which allows us to take the inverse limit. Comparing this limit with the exact sequence at level infinity we obtain that

$$K_\infty = \lim_{\leftarrow n} K_n.$$

Now, to conclude the proof, the reader may follow the same technical arguments of the second part of the proof of Theorem 2.3.12. □
2.5 The characters of type 3

In this section we give a look at the \( \chi \)-part of the modules \( C_n \) and \( T_p(F_n) \) when \( \chi \) is a character of type 3. Note that the trivial character \( \chi_0 \) is one of these characters. Unlike the case of characters of type 1 and 2 we will not be able to obtain results directly on these modules, but we have to deal with their duals (\( \mathbb{Z}_p \)-duals and Pontrjagin duals). At the end of the section we will also focus on the problems that occur when we try to define the projective limits of these modules.

Throughout this section we will denote with \( \chi \) a character of type 3, which may be identified canonically with a character of the Picard group \( \Delta = Gal(H_A/F) \cong Pic(A) \). For every \( G_0 \)-module \( M \) we denote with \( M^* := \text{Hom}_{\mathbb{Z}_p}(M, \mathbb{Z}_p) \) the \( \mathbb{Z}_p \)-dual of \( M \). Note that when we consider the scalar extension we have \( M^* \otimes_{\mathbb{Z}_p} W = \text{Hom}_W(M \otimes_{\mathbb{Z}_p} W, W) \).

From our computations (2.1) and (2.4) we have that for type 3 characters:

\[
H_n(\infty)(\chi) = W[\Gamma_n] \quad \text{and} \quad H_n(p)(\chi) = W[\mathbb{G}_F]/\left(1 - \chi(F_{\mathbb{F}_p}^{-1})\gamma^{-d_p}\right) \tag{2.14}
\]

where \( F_{\mathbb{F}_p} \in \Delta \) is the Artin symbol of the prime \( p \).

We denote, as usual, with \( D_n \) the kernel of the degree map

\[
\deg : H_n(\infty) \oplus H_n(p) \to \mathbb{Z}_p.
\]

Then we have that if \( \chi \) is different from the trivial character \( \chi_0 \)

\[
D_n(\chi) \simeq W[\Gamma_n] \oplus W[\mathbb{G}_F]/\left(1 - \chi(F_{\mathbb{F}_p}^{-1})\gamma^{-d_p}\right), \tag{2.15}
\]

since there is no action of \( \Delta \) on \( \mathbb{Z}_p \) and so \( \mathbb{Z}_p(\chi) = 0 \).

For the trivial character \( \chi = \chi_0 \) we first need to consider the degree map on the submodule \( H_n(p)(\chi_0) \). We denote with \( D_{n,p}(\chi_0) \) the kernel of this map and define

\[
D_{n,p}(\chi_0) \oplus H_n(\infty)(\chi_0) \to H_n(p)(\chi_0) \oplus H_n(\infty)(\chi_0)
\]

\[
(x, y) \mapsto (x - \deg y \cdot 1, y),
\]

where \( 1 \) is the element which corresponds to the unity under the isomorphism \( H_n(p)(\chi_0) \simeq W[\mathbb{G}_F]/(1 - \gamma^{-d_p}) \). Clearly this map is injective and its image is contained in \( D_n(\chi_0) \), but one can easily see that this map is also onto \( D_n(\chi_0) \) with inverse the map

\[
D_n(\chi_0) \to D_{n,p}(\chi_0) \oplus H_n(\infty)(\chi_0)
\]

\[
(x, y) \mapsto (x + \deg y \cdot 1, y).
\]

We have shown that

\[
D_n(\chi_0) \simeq D_{n,p}(\chi_0) \oplus H_n(\infty)(\chi_0).
\]

Now we compute \( D_{n,p}(\chi_0) \): since the primes of \( F^m_{\mathbb{F}_p} \) which lie above \( p \) have all degree 1 and the action of \( \gamma \) (and also \( \gamma^{-1} \)) simply permutes these primes, we have that \( \gamma \) does not change the
degree of an element, i.e., \( \deg(\gamma^{-1}x) = \deg x \) for each element \( x \in H_n(p)(\chi_0) \). This implies that \((1 - \gamma^{-1})H_n(p)(\chi_0)\) is contained in \( D_n,p(\chi_0)\), but from the isomorphism \( H_n(p)(\chi_0) \simeq W[G_F]/(1 - \gamma^{-d_p})\) we deduce that \((1 - \gamma^{-1})H_n(p)(\chi_0)\) is exactly \( D_n,p(\chi_0)\).

Thus we have proved that

\[
D_n(\chi) \simeq \begin{cases} 
W[\Gamma_n] \oplus W[G_F]/(1 - \chi(F_{p^{-1}})\gamma^{-d_p}) & \text{if } \chi \neq \chi_0, \\
W[\Gamma_n] \oplus W[G_F]/\left(\frac{1-\gamma^{-d_p}}{1-\gamma}\right) & \text{if } \chi = \chi_0.
\end{cases}
\] (2.16)

Now we want to give a resolution of \( D_n(\chi) \) as a module over \( W[\Gamma_n][G_F] \). First we start considering the case \( \chi \neq \chi_0 \). We denote with \( n(\Gamma_n) \in W[\Gamma_n][G_F] \) the element

\[
n(\Gamma_n) = \sum_{\sigma \in \Gamma_n} \sigma.
\]

It is easy to see that for each \( \sigma \in \Gamma_n \) we have \((\sigma - 1)n(\Gamma_n) = 0\). Now we denote with \( I_{\Gamma_n} \) the augmentation ideal of \( W[\Gamma_n][G_F] \) associated to \( \Gamma_n \), i.e., the ideal generated by the elements \( \sigma - 1 \), with \( \sigma \in \Gamma_n \). From what we have just observed we have that the multiplication by \( n(\Gamma_n) \) on the quotient \( W[\Gamma_n][G_F]/(1 - \chi(F_{p^{-1}})\gamma^{-d_p}) \) induces a map

\[
H_n(p)(\chi) = W[\Gamma_n][G_F]/\left(I_{\Gamma_n}, 1 - \chi(F_{p^{-1}})\gamma^{-d_p}\right) \xrightarrow{n(\Gamma_n)} W[\Gamma_n][G_F]/\left(1 - \gamma(F_{p^{-1}})\gamma^{-d_p}\right)
\]

whose cokernel is \( W[\Gamma_n][G_F]/(n(\Gamma_n), 1 - \gamma(F_{p^{-1}})\gamma^{-d_p}) \). So we are in the following situation:

\[
H_n(p)(\chi) \xrightarrow{n(\Gamma_n)} W[\Gamma_n][G_F]/\left(1 - \gamma(F_{p^{-1}})\gamma^{-d_p}\right) \rightarrow W[\Gamma_n][G_F]/\left(n(\Gamma_n), 1 - \chi(F_{p^{-1}})\gamma^{-d_p}\right).
\] (2.17)

We will show now that this sequence is exact, i.e., that the map of the left is injective. To do this we observe that all the modules in the sequence are free over \( W \) and we count the ranks. From (2.14) we have that the module on the left has rank \( d_p \). For the central module we have

\[
W[\Gamma_n][G_F]/\left(1 - \gamma(F_{p^{-1}})\gamma^{-d_p}\right) \simeq \left(W[G_F]/\left(1 - \gamma(F_{p^{-1}})\gamma^{-d_p}\right)\right)[\Gamma_n]
\]

and so the rank is \( d_p|\Gamma_n| \). For the module on the right we have

\[
W[\Gamma_n][G_F]/\left(n(\Gamma_n), 1 - \chi(F_{p^{-1}})\gamma^{-d_p}\right) \simeq \left((W[\Gamma_n]/n(\Gamma_n)) [G_F]\right)/\left(1 - \gamma(F_{p^{-1}})\gamma^{-d_p}\right)
\]

whose rank is

\[
d_p(|\Gamma_n| - |\Gamma_n/\Gamma_n|) = d_p|\Gamma_n| - d_p.
\]

Thus we have proved that (2.17) is exact.

Now simply recalling that

\[
H_n(\infty)(\chi) = W[\Gamma_n] \simeq W[\Gamma_n][G_F]/\left(1 - \gamma^{-1}\right)
\]

we obtain the following resolution for \( D_n(\chi) \)

\[
\begin{array}{cccccccccc}
0 & \xrightarrow{} & D_n(\chi) & \xrightarrow{} & W[\Gamma_n][G_F]/\left(1 - \gamma^{-1}\right) & \oplus & W[\Gamma_n][G_F]/\left(1 - \gamma(F_{p^{-1}})\gamma^{-d_p}\right) & \xrightarrow{} & W[\Gamma_n][G_F]/\left(n(\Gamma_n), 1 - \chi(F_{p^{-1}})\gamma^{-d_p}\right) & \xrightarrow{} & 0.
\end{array}
\]

(2.18)
Proceeding in a similar way for the trivial character, and using the computation done for \( D_n(\chi_0) \) one obtains the resolution

\[
D_n(\chi_0) \hookrightarrow W[\Gamma_n][G_p]/(1 - \gamma^{-1}) \oplus W[\Gamma_n][G_p]/(n) \rightarrow W[\Gamma_n][G_p]/(n(\Gamma_n), v_p),
\]

(2.19)

where we have put \( v_p := (1 - \gamma^{-d_p})/(1 - \gamma^{-1}) \).

Taking the \( \chi \)-part in the exact sequence [GP1, after Definition 2.6] one obtains

\[
T_p(F_n)(\chi) \hookrightarrow T_p(M_n)(\chi) \rightarrow D_n(\chi)
\]

which can be joined with (2.18), to obtain

\[
T_p(F_n)(\chi) \hookrightarrow T_p(M_n)(\chi) \rightarrow W[\Gamma_n][G_p]/(1 - \gamma^{-1}) \oplus W[\Gamma_n][G_p]/(1 - \chi(F_p^{-1})\gamma^{-d_p}) \rightarrow W[\Gamma_n][G_p]/(n(\Gamma_n), 1 - \chi(F_p^{-1})\gamma^{-d_p})
\]

for \( \chi \neq \chi_0 \). All the four modules in the previous sequence are free and finitely generated over \( W \) (and so also over \( \mathbb{Z}_p \)). The module \( T_p(M_n)(\chi) \) has projective dimension 1 over \( W[\Gamma_n][G_p] \) because of [GP1, Theorem 3.9] and the same is trivially true also for the third module. This allows us to apply [GP2, Lemma 2.4] and obtain

\[
(1 - \gamma^{-1}) \left( 1 - \chi(F_p^{-1})\gamma^{-d_p} \right) \text{Fitt}_{W[\Gamma_n][G_p]} T_p(F_n)(\chi)^* = \Theta_n(\gamma^{-1}, \chi) \cdot \left( n(\Gamma_n), 1 - \chi(F_p^{-1})\gamma^{-d_p} \right).
\]

(2.20)

Proceeding in a similar way for the trivial character and using the resolution of \( D_n(\chi_0) \) above, we obtain

\[
(1 - \gamma^{-d_p})\text{Fitt}_{W[\Gamma_n][G_p]} T_p(F_n)(\chi_0)^* = \Theta_n(\gamma^{-1}, \chi_0) \cdot (n(\Gamma_n), v_p).
\]

(2.21)

Now we focus a little bit more on equation (2.20): consider the natural projection \( \pi : W[\Gamma_n][G_p] \rightarrow W[\Gamma_n] \) which maps \( \gamma \) to 1. If \( F_p \) lies in the kernel of the character \( \chi \), i.e., the character is trivial on the decomposition group of \( p \), the left hand side of the equality has a zero of order (at least) 2 when evaluated at \( \gamma = 1 \) and the ideal \((n(\Gamma_n), 1 - \chi(F_p^{-1})\gamma^{-d_p})\) becomes principal and generated by the non zero element \( n(\Gamma_n) \), thus the Stickelberger series \( \Theta_n(X, \chi) \) has a zero of order at least 2 in \( X = 1 \). If \( F_p \) does not belong to the kernel of \( \chi \), then the ideal \((n(\Gamma_n), 1 - \chi(F_p^{-1})\gamma^{-d_p})\) does not become principal (in general) when projected to \( W[\Gamma_n] \). Similar consideration allows us to conclude that the Stickelberger series has a zero at \( X = 1 \), but in this case we cannot say that the order is greater than one.

The same consideration applied to equation (2.21) lead us to conclude that also the Stickelberger series \( \Theta_n(X, \chi_0) \) has a zero at \( X = 1 \), of order at least 1. Thus if we put

\[
\Theta_n^\chi(\gamma^{-1}, \chi) = \begin{cases}
\frac{\Theta_n(\gamma^{-1}, \chi)}{(1 - \gamma^{-1})^2} & \text{if } F_p \in \ker \chi \text{ and } \chi \neq \chi_0, \\
\frac{\Theta_n(\gamma^{-1}, \chi)}{1 - \gamma^{-1}} & \text{otherwise},
\end{cases}
\]

we have
Theorem 2.5.1. Let $\chi \in \hat{G}_0$ be a character of type 3. Then we have

$$\text{Fitt}_{W[G_p]} T_p(F_n)(\chi)^* = \begin{cases} 
\Theta_n^\sharp(\gamma^{-1}, \chi) \cdot \left( \frac{n(\Gamma_n)}{1-\chi(F_p^-1)\gamma^{-p}}, 1 \right) & \text{if } F_p \notin \ker \chi, \\
\Theta_n^\sharp(\gamma^{-1}, \chi) \cdot \left( \frac{n(\Gamma_n)}{1+\gamma^{-1}, 1 + \gamma^{-1}}, 1 - \gamma^{-1} \right) & \text{if } F_p \in \ker \chi \text{ and } \chi \neq \chi_0, \\
\Theta_n^\sharp(\gamma^{-1}, \chi_0) \cdot \left( \frac{n(\Gamma_n)}{1+\gamma^{-1}, 1 + \gamma^{-1}}, 1 \right) & \text{if } \chi = \chi_0.
\end{cases}$$

Now we move to the study of the class group of $F_n$ as a module over $W[\Gamma_n]$. In the beginning of [GP2, Section 3] it is shown that

$$C_n^\vee \simeq T_p(F_n)^*/(1 - \gamma^{-1})T_p(F_n)^*,$$

where $C_n^\vee = \text{Hom}(C_n, \mathbb{Q}_p/\mathbb{Z}_p)$ is the Pontrjagin dual of $C_n$. Thus we can now apply Lemma 2.4.2 to obtain

$$\text{Fitt}_{W[\Gamma_n]} C_n(\chi)^\vee = \pi \left( \text{Fitt}_{W[\Gamma_n][G_p]} T_p(F_n)(\chi)^* \right),$$

where $\pi : W[\Gamma_n][G_p] \to W[\Gamma_n]$ is the map which sends $\gamma$ to $1$. Then we have

Theorem 2.5.2. Let $\chi \in G_0$ be a character of type 3. Then we have

$$\text{Fitt}_{W[\Gamma_n]} C_n(\chi)^\vee = \begin{cases} 
\Theta_n^\sharp(1, \chi) \cdot \left( \frac{n(\Gamma_n)}{1-\chi(F_p^-1)}, 1 \right) & \text{if } F_p \notin \ker \chi, \\
\Theta_n^\sharp(1, \chi) \cdot \left( \frac{n(\Gamma_n)}{d_p}, 1 \right) & \text{if } F_p \in \ker \chi \text{ and } \chi \neq \chi_0, \\
\Theta_n^\sharp(1, \chi_0) \cdot \left( \frac{n(\Gamma_n)}{d_p}, 1 \right) & \text{if } \chi = \chi_0.
\end{cases}$$

Remark 2.5.3. We would like to point out some particular cases: if the decomposition group of the prime $p$ is contained is the kernel of the non trivial character $\chi$, then the Fitting ideal of $C_n(\chi)^\vee$ is principal as we see from the previous theorem.

Another interesting case is when the degree of the prime $p$ is coprime with $p$. In this case the degree $d_p$ is invertible in $\mathbb{Z}_p$ and so the fractional ideal $\left( \frac{n(\Gamma_n)}{d_p}, 1 \right)$ actually is an integral ideal, which has $1$ as one of its generators. Thus in this case the Fitting ideal of $C_n(\chi_0)^\vee$ is principal and it is generated by $\Theta_n^\sharp(1, \chi_0)$.

Now it would be interesting to proceed like for characters of type 1 and 2 and make some kind of projective limits of the modules $T_p(F_n)(\chi)^*$ and $C_n(\chi)^\vee$ and study the Fitting ideal of this limit. However at this point it is not clear which are the maps we have to consider to make this limit: note that the natural norm maps $N_{n+m}^n$ from $T_p(F_{n+m})(\chi)$ to $T_p(F_n)(\chi)$ and from $C_{n+m}(\chi)$ to $C_n(\chi)$ induce maps on the dual with the opposite direction, so it is not possible to use them to make a projective limit. Another problem is that when considering the natural projection $\pi_{n+m}^n : W[\Gamma_{n+m}] \to W[\Gamma_n]$ we have that $\pi_{n+m}^n(n(\Gamma_{n+m})) = |A/p|^m \cdot n(\Gamma_n)$, thus the generators of the Fitting ideals of $C_n(\chi)^\vee$ are not compatible with respect of this projection map, and cannot be used to define an element in the algebra $W[\Gamma_\infty]$. 
Bibliography


