

# Regularity and Bernstein-type results for nonlocal minimal surfaces

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**Abstract.** We prove that, in every dimension, Lipschitz nonlocal minimal surfaces are smooth. Also, we extend to the nonlocal setting a famous theorem of De Giorgi [6] stating that the validity of Bernstein’s theorem in dimension  $n + 1$  is a consequence of the nonexistence of  $n$ -dimensional singular minimal cones in  $\mathbb{R}^n$ .

## 1. Introduction

Given  $n \geq 1$  and  $s \in (0, 1)$ , we investigate the regularity of nonlocal  $s$ -minimal surfaces in  $\mathbb{R}^{n+1}$ . To begin, we recall the notion of  $s$ -perimeter and  $s$ -minimal surface, as introduced in [3].

Given two disjoint measurable sets  $F, G \subseteq \mathbb{R}^{n+1}$ , we consider the  $s$ -interaction between them defined by

$$L(F, G) := \iint_{F \times G} \frac{dX dY}{|X - Y|^{n+1+s}}.$$

Given a measurable set  $E$  and a bounded set  $\Omega \subset \mathbb{R}^{n+1}$ , the “ $s$ -perimeter” of the set  $E$  inside  $\Omega$  is defined as

$$\text{Per}_s(E, \Omega) := L(E \cap \Omega, \mathbb{R}^{n+1} \setminus E) + L(E \setminus \Omega, \Omega \setminus E).$$

We say that  $E$  is an “ $s$ -minimal surface” in  $\Omega$  if  $\text{Per}_s(E, \Omega) < +\infty$  and for any measurable set  $F \subseteq \mathbb{R}^{n+1}$  with  $F \setminus \Omega = E \setminus \Omega$  we have that

$$\text{Per}_s(E, \Omega) \leq \text{Per}_s(F, \Omega).$$

If  $E$  is an  $s$ -minimal surface in any ball, we simply say that  $E$  is an  $s$ -minimal surface. Namely,  $s$ -minimal surfaces are local minimizers of the  $s$ -perimeter functional. The “ $s$ -mean curvature” of  $E$  at a point  $X \in \partial E$  is defined by

$$(1.1) \quad I[E](X) := \int_{\mathbb{R}^{n+1}} \frac{\chi_E(Y) - \chi_{E^c}(Y)}{|X - Y|^{n+1+s}} dY,$$

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where  $E^c := \mathbb{R}^{n+1} \setminus E$ . We remark that if  $\partial E$  is  $C^2$  in a neighborhood of  $X$ , then  $I[E](X)$  is well-defined in the principal value sense. On the other hand, while a priori an  $s$ -minimal surface  $E$  may not be smooth, it is shown in [3] that it satisfies the equation  $I[E](X) = 0$  for any  $X \in \partial E$  in a suitable viscosity sense (in particular, it satisfies the equation in the classical sense at every point where  $\partial E$  is  $C^2$ ).

With this notation,  $s$ -minimal surfaces have vanishing  $s$ -mean curvature, and the analogy with the classical perimeter case is evident. To make the analogy even stronger we recall that, as  $s \nearrow 1$ , the  $s$ -perimeter converges to the classical perimeter, with good geometric and functional analytic properties, see [1, 4].

From the results in [2, 3, 8] it is known that boundaries of  $s$ -minimal surfaces are  $C^\infty$  with the possible exception of a closed singular set of Hausdorff dimension at most  $n - 2$ .

The first result of this paper shows that Lipschitz  $s$ -minimal surfaces are smooth. Notice that, in the classical case, this result is a consequence of the De Giorgi–Nash Theorem on the Hölder regularity of solutions to uniformly elliptic equations in divergence form. However, in this nonlocal setting it does not seem possible to use the regularity theory for nonlocal equations to deduce this result and we need to employ geometric arguments instead.

**Theorem 1.1.** *Let  $n \geq 1$  and let  $E$  be an  $s$ -minimal surface in  $B_1 \subset \mathbb{R}^{n+1}$ . Suppose that  $\partial E \cap B_1$  is locally Lipschitz. Then  $\partial E \cap B_1$  is  $C^\infty$ .*

We say that an  $s$ -minimal surface  $E$  is an “ $s$ -minimal graph” if it can be written as a global graph in some direction (that is, up to a rotation,  $E = \{(x, \tau) \in \mathbb{R}^n \times \mathbb{R} : \tau < u(x)\}$  for some function  $u : \mathbb{R}^n \rightarrow \mathbb{R}$ ), and it is an “ $s$ -minimal cone” if it is a cone (that is, up to a translation,  $E = tE$  for any  $t > 0$ ). A variant of the techniques used in the proof of Theorem 1.1 allows us to show that the validity of Bernstein’s theorem in dimension  $n + 1$  is a consequence of the nonexistence of  $n$ -dimensional singular  $s$ -minimal cones in  $\mathbb{R}^n$ , thus extending to the fractional case a famous result of De Giorgi for minimal surfaces [6]:

**Theorem 1.2.** *Let  $E = \{(x, \tau) \in \mathbb{R}^n \times \mathbb{R} : \tau < u(x)\}$  be an  $s$ -minimal graph, and assume that there are no singular  $s$ -minimal cones in dimension  $n$  (that is, if  $\mathcal{C} \subset \mathbb{R}^n$  is an  $s$ -minimal cone, then  $\mathcal{C}$  is a half-space). Then  $u$  is an affine function (thus  $E$  is a half-space).*

The above result combined with the nonexistence of  $s$ -minimal cones in  $\mathbb{R}^n$  for  $n \leq 2$  (see [8]) implies the following:

**Corollary 1.3.** *Let  $E = \{(x, \tau) \in \mathbb{R}^n \times \mathbb{R} : \tau < u(x)\}$  be an  $s$ -minimal graph, and assume that  $n \in \{1, 2\}$ . Then  $u$  is an affine function.*

When  $n = 1$ , Corollary 1.3 is a particular case of the result in [8], but for  $n = 2$  the result is new.

The paper is organized as follows. Some preliminary results on Lipschitz functions are collected in Section 2, and a useful observation on the asymptotic behavior of the  $s$ -minimal cones at large scale is given in Section 3. Then, the proofs of Theorems 1.1 and 1.2 are given in Sections 4 and 5, respectively.

### 2. Technical lemmata on Lipschitz functions

This section contains some auxiliary results of elementary nature.

In the first lemma we show that Lipschitz functions whose gradient is almost constant in a suitably large set need to be uniformly close to an affine hyperplane:

**Lemma 2.1.** *Let  $M > 0$  and  $\omega \in \mathbb{R}^n$  with  $|\omega| \leq M$ . Given  $\varepsilon > 0$  there exists a constant  $\delta = \delta(n, \varepsilon, M) > 0$  such that the following holds: if  $u : B_1 \rightarrow \mathbb{R}$  is an  $M$ -Lipschitz function satisfying*

$$|\{x \in B_1 : |\nabla u(x) - \omega| \geq \delta\}| \leq \delta,$$

then  $|u(x) - u(0) - \omega \cdot x| \leq \varepsilon$  for any  $x \in B_1$ .

*Proof.* In this proof,  $C$  will denote a generic constant depending only  $M$ , which may change from line to line. Set  $w(x) := u(x) - \omega \cdot x$ . It is immediate to check that  $w$  is  $(2M)$ -Lipschitz and from our assumptions on  $u$  we get

$$\int_{B_1} |\nabla w(x)| \, dx = \int_{B_1 \cap \{|\nabla w| < \delta\}} |\nabla w(x)| \, dx + \int_{B_1 \cap \{|\nabla w| \in [\delta, 2M]\}} |\nabla w(x)| \, dx \leq C\delta.$$

Hence, by Hölder inequality,

$$\|\nabla w\|_{L^{n+1}(B_1)} \leq \|\nabla w\|_{L^1(B_1)}^{1/(n+1)} \|\nabla w\|_{L^\infty(B_1)}^{n/(n+1)} \leq C\delta^{1/(n+1)},$$

and applying Sobolev inequality in  $W^{1,n+1}(B_1)$  we deduce that there exists a constant  $\ell \in \mathbb{R}$  such that

$$\|w - \ell\|_{L^\infty(B_1)} \leq C\|\nabla w\|_{L^{n+1}(B_1)} \leq C\delta^{1/(n+1)}.$$

Since

$$\|w - w(0)\|_{L^\infty(B_1)} \leq \|w - \ell\|_{L^\infty(B_1)} + |\ell - w(0)| \leq 2\|w - \ell\|_{L^\infty(B_1)},$$

this concludes the proof with  $\varepsilon = C\delta^{1/(n+1)}$ . □

In the next result, we observe that if a Lipschitz function has local growth close to the maximal one at many points, then it needs to be uniformly close to an affine map:

**Proposition 2.2.** *Let  $M > 0$ . Then, for any  $\varepsilon > 0$  there exists a constant  $\mu = \mu(n, \varepsilon, M)$  with*

$$0 < \mu < \min\{1, M\}$$

such that the following holds: fix  $\sigma \in \partial B_1$ , and let  $u : B_1 \rightarrow \mathbb{R}$  be an  $M$ -Lipschitz function satisfying

$$(2.1) \quad |\{x \in B_1 : u(x + t_k \sigma) - u(x) \geq (M - \mu)t_k\}| \geq (1 - \mu)|B_1|$$

for some sequence  $t_k \searrow 0$ . Then  $|u(x) - u(0) - M\sigma \cdot x| \leq \varepsilon$  for all  $x \in B_1$ .

*Proof.* Up to a rotation we can assume that  $\sigma = e_1$ . Set

$$(2.2) \quad A_k := \{x \in B_1 : u(x + t_k e_1) - u(x) \geq (M - \mu)t_k\}$$

and

$$A_\star := \bigcap_{k=0}^{+\infty} \bigcup_{j=k}^{+\infty} A_j.$$

Notice that  $|A_k| \geq (1 - \mu) |B_1|$  (thanks to (2.1)) and

$$\bigcap_{k=0}^m \bigcup_{j=k}^{+\infty} A_j \supseteq A_m.$$

Therefore, by monotone convergence,

$$(2.3) \quad |A_\star| = \lim_{m \nearrow \infty} \left| \bigcap_{k=0}^m \bigcup_{j=k}^{+\infty} A_j \right| \geq \lim_{m \nearrow \infty} |A_m| \geq (1 - \mu) |B_1|.$$

Let  $D \subset B_1$  denote the set of differentiability points of  $u$  (recall that  $D$  has full measure). We claim that

$$(2.4) \quad \sup_{x \in A_\star \cap D} |\nabla u(x) - M e_1| < \mu^{1/4}.$$

For this, we take  $x \in A_\star \cap D$ . By the definition of  $A_\star$ , there exists a subsequence  $j_k \nearrow \infty$  such that  $x \in A_{j_k}$ , thus, by (2.2),

$$u(x + t_{j_k} e_1) - u(x) \geq (M - \mu) t_{j_k}.$$

Dividing by  $t_{j_k} > 0$  and letting  $k \nearrow \infty$ , we obtain (recall that  $t_{j_k} \searrow 0$  as  $k \nearrow \infty$ )

$$(2.5) \quad \partial_1 u(x) \geq (M - \mu).$$

As a consequence

$$M^2 \geq |\nabla u(x)|^2 \geq (M - \mu)^2 + \sum_{i=2}^n |\partial_i u(x)|^2,$$

which gives

$$\sum_{i=2}^n |\partial_i u(x)|^2 \leq 2M\mu.$$

Also (2.5) and the fact that  $\partial_1 u(x) \leq M$  imply that

$$-\mu \leq \partial_1 u(x) - M \leq 0,$$

hence  $|\partial_1 u(x) - M| \leq \mu$ . We conclude that

$$|\nabla u(x) - M e_1|^2 = |\partial_1 u(x) - M|^2 + \sum_{i=2}^n |\partial_i u(x)|^2 \leq \mu^2 + 2M\mu < \sqrt{\mu}$$

provided  $\mu$  is sufficiently small, proving (2.4).

By (2.3) and (2.4) we deduce that

$$|\{x \in B_1 : |\nabla u(x) - M e_1| \geq \mu^{1/4}\}| \leq |B_1 \setminus A_\star| \leq \mu |B_1| \leq \mu^{1/4}.$$

Hence, if  $\mu$  is small enough, we can apply Lemma 2.1 with  $\delta = \mu^{1/4}$  to obtain the desired result.  $\square$

### 3. A remark on flat blow-downs

First of all, we recall here the notion of blow-up and blow-down of a  $s$ -minimal surface  $E$ , which will be used in the proofs of Theorems 1.1 and 1.2.

Assume that  $0 \in \partial E$ , define the family of sets  $E_r := E/r$ , and let  $E_0$  (resp.  $E_\infty$ ) be a cluster point with respect to the  $L^1_{\text{loc}}$ -convergence for  $E_r$  as  $r \searrow 0$  (resp.  $r \nearrow \infty$ ).

With this notation, the set  $E_0$  is called a “blow-up” of  $E$  (at 0), while the set  $E_\infty$  is called a “blow-down”. By [3, Theorem 9.2], we know that both  $E_0$  and  $E_\infty$  are  $s$ -minimal cones passing through the origin.

In the proof of Theorem 1.2 we will use the following observation:

**Lemma 3.1.** *If  $E_\infty$  is a half-space, then  $E = E_\infty$ .*

*Proof.* Up to a rotation we can assume that

$$E_\infty = \{(x, \tau) \in \mathbb{R}^n \times \mathbb{R} : \tau \leq 0\}.$$

Let  $r_k \nearrow \infty$  be a sequence such that  $E_{r_k} \rightarrow E_\infty$ , and let  $\varepsilon_0$  be the universal flatness parameter in [3, Theorem 6.1]. By the uniform density estimates for  $s$ -minimal surfaces (see [3, Theorem 4.1]), we have that

$$\partial E_{r_k} \cap B_1 \rightarrow \partial E_\infty \cap B_1$$

in the Hausdorff distance<sup>1)</sup> as  $r_k \nearrow \infty$ .

Hence, for  $r_k$  sufficiently large we have that  $E_{r_k} \cap B_1$  lies in an  $\varepsilon_0$ -neighborhood of  $E_\infty$ , and [3, Theorem 6.1] yields that  $\partial E_{r_k} \cap B_{1/2}$  is a  $C^{1,\alpha}$ -graph parameterized by a function  $u_{r_k} : B_1 \rightarrow \mathbb{R}$ , with  $\|u_{r_k}\|_{C^{1,\alpha}(B_{1/2})} \leq C$  for some universal constants  $\alpha \in (0, 1)$  and  $C > 0$  (to be precise, [3, Theorem 6.1] gives that  $u_{r_k} \in C^{1,\alpha}(B_{1/2})$ , and the fact that the  $C^{1,\alpha}$ -norm is universally bounded is pointed out for instance in [7, Corollary 3.5]).

Scaling back, we deduce that  $\partial E \cap B_{r_k/2}$  coincides with the graph of a function  $u$  which satisfies  $u(x) = r_k u_{r_k}(x/r_k)$  and  $u(0) = 0$  (since  $0 \in \partial E$ ). Hence

$$r_k^\alpha [\nabla u]_{C^\alpha(B_{r_k/2})} = [\nabla u_{r_k}]_{C^\alpha(B_{1/2})} \leq C,$$

and by letting  $r_k \nearrow \infty$  we see that  $\nabla u$  is constant. Thus  $u$  is a linear function, which implies that  $E$  is a half-space. Since  $0 \in \partial E$ , it is immediate to check that  $E = E_r$  for all  $r > 0$ , therefore (by letting  $r \nearrow \infty$ )  $E = E_\infty$  as desired.  $\square$

<sup>1)</sup> The fact that density estimates improve the  $L^1$ -convergence into Hausdorff convergence is a classical fact. For completeness, we sketch the argument. Let  $X_k \in \partial E_{r_k} \cap B_1$ , and consider  $X$  a cluster point. Taking the limit in the density estimate

$$\min\{|E_{r_k} \cap B_\rho(X_k)|, |B_\rho(X_k) \setminus E_{r_k}|\} \geq c\rho^n \quad \text{for all } \rho \in (0, 1),$$

we deduce that

$$(3.1) \quad \min\{|E_\infty \cap B_\rho(X)|, |B_\rho(X) \setminus E_\infty|\} \geq c\rho^n \quad \text{for all } \rho \in (0, 1),$$

which implies that  $X \in \partial E_\infty$ .

Viceversa, if  $X \in \partial E_\infty \cap B_1$ , then (3.1) and the  $L^1_{\text{loc}}$ -convergence of  $E_k$  to  $E_\infty$  imply that, for any  $\rho \in (0, 1)$ ,  $\partial E_{r_k} \cap B_\rho(X) \neq \emptyset$  for  $k \gg 1$ . Then, the arbitrariness of  $\rho$  yields the existence of a sequence  $X_k \in \partial E_{r_k} \cap B_1$  converging to  $X$ .

### 4. Proof of Theorem 1.1

The idea of the proof is that, in some cases, nonlocal equations give a measure theoretic estimate on the separation of approximate solutions which in turn is helpful to control the pointwise fluctuations of the solution.

By [3, Theorem 6.1] and [2, Theorem 5], there exists an  $\varepsilon_0 > 0$  such that, if  $B_r(X) \subset B_1$  and  $\partial E \cap B_r(X)$  lies in a slab of height  $2\varepsilon_0 r$ , then  $\partial E \cap B_{r/2}(X)$  is  $C^\infty$ . Hence we only need to show that, for any  $X \in B_1$ , there exists a radius  $r < 1 - |X|$  such that  $\partial E \cap B_r(X)$  lies in a slab of height  $2\varepsilon_0 r$ .

So, we fix  $X_0 \in B_1$ , we suppose (up to a change of coordinates) that  $\partial E$  is a Lipschitz graph in the  $e_{n+1}$ -direction in a neighborhood of  $X$ , and we assume by contradiction that, for any  $r > 0$  small,

$$(4.1) \quad \partial E \cap B_r(X_0) \text{ is never trapped inside a slab of height } 2\varepsilon_0 r.$$

After translating the system of coordinate, we can assume that  $X_0 = 0$ , and we consider a blow-up  $E_0$  of  $E$  (recall the notion of blow-up introduced in Section 3). By [3, Theorem 9.2], we know that  $E_0$  is a Lipschitz  $s$ -minimal cone passing through the origin.

We claim that

$$(4.2) \quad \partial E_0 \cap B_R \text{ is never trapped inside a slab of height } 2\varepsilon_0 R,$$

for any  $R > 0$ . Indeed, if by contradiction  $\partial E_0 \cap B_R$  is trapped inside a slab of height  $2\varepsilon_0 R$ , then [3, Theorem 6.1] and [2, Theorem 5] imply that  $\partial E_0$  is  $C^\infty$ , hence (since  $E_0$  is a cone)  $\partial E_0$  is a hyperplane. Then by the uniform density estimates for  $s$ -minimal surfaces (see [3, Theorem 4.1], or footnote 1 here) we obtain that  $\partial E_r \cap B_1$  lies in a  $\varepsilon_0$ -neighborhood of  $\partial E_0$ , and scaling back we obtain that  $\partial E \cap B_r$  lies in a slab of height  $2\varepsilon_0 r$ . This is in contradiction with (4.1), and the proof of (4.2) is complete.

Now, up to a standard “dimension reduction argument” (see [3, Theorem 10.3]) we can “remove” all the singular points of  $\partial E_0$  except the origin and we end up<sup>2)</sup> with the following situation:  $E_0$  is a Lipschitz cone passing through the origin,

$$(4.3) \quad \partial E_0 \cap B_1 \text{ does not lie in a slab of height } \varepsilon_0,$$

and  $\partial E_0$  is a Lipschitz graph in the  $e_{n+1}$ -direction which is smooth outside the origin, that is,

$$E_0 = \{X = (x, \tau) \in \mathbb{R}^n \times \mathbb{R} : \tau < u(x)\},$$

where

$$(4.4) \quad u(0) = 0, \quad u \in C^2(\mathbb{R}^n \setminus \{0\}), \quad |\nabla u(x)| \leq M \quad \text{for any } x \neq 0.$$

To be precise, the dimension reduction argument in [3] gives that  $u \in C^{1,\alpha}(\mathbb{R}^n \setminus \{0\})$ , and by [2, Theorem 1] we obtain that  $u \in C^\infty(\mathbb{R}^n \setminus \{0\})$ . Of course, we can take  $M > 0$  to be the smallest possible (i.e.,  $M$  is the optimal Lipschitz constant of  $u$ ).

<sup>2)</sup> We notice that, since  $\partial E_0$  is a Lipschitz graph, one can perform the dimension reduction argument without changing system of coordinates. Therefore, after a finite number of blow-ups, we still end up with a Lipschitz graph.

Take  $\mu_0 := \mu(n, \varepsilon_0/2, M)$  as in Proposition 2.2. Then it follows from (4.3) that (2.1) cannot hold true. Hence, for any  $\sigma \in \partial B_1$  there exists  $t_\sigma > 0$  such that

$$(4.5) \quad |\{x \in B_1 : u(x + t\sigma) - u(x) < (M - \mu_0)t\}| \geq \mu_0 |B_1| \quad \text{for all } t \in (0, t_\sigma).$$

Now we take  $w_0 \in C^\infty(\mathbb{R}, [0, 1])$ , with  $w_0(t) = 0$  for any  $t \in (-\infty, 1/4] \cup [3/4, +\infty)$  and  $w_0(t) = 1$  for any  $t \in [2/5, 3/5]$ . We set  $w(x) = w_0(|x|)$  and observe that

$$(4.6) \quad w(x) = 1 \quad \text{for any } x \in B_{3/5} \setminus B_{2/5}.$$

Our goal is to show that there exists a constant  $\vartheta > 0$  such that

$$(4.7) \quad u(x + t\sigma) \leq u(x) + Mt - \vartheta tw(x) \quad \text{for all } x \in B_1, t \in (0, t_\sigma), \sigma \in \partial B_1.$$

Before proving (4.7) we observe that, once (4.7) is established, we easily reach a contradiction and complete the proof of Theorem 1.1. Indeed, letting  $t \searrow 0$  in (4.7) and using (4.6) we deduce that

$$\nabla u(x) \cdot \sigma \leq M - \vartheta w(x) = M - \vartheta \quad \text{for all } x \in B_{3/5} \setminus B_{2/5}, \sigma \in \partial B_1,$$

hence

$$|\nabla u(x)| \leq M - \vartheta \quad \text{for all } x \in B_{3/5} \setminus B_{2/5}$$

by the arbitrariness of  $\sigma$ . Since  $\nabla u$  is homogeneous of degree zero, it follows that

$$|\nabla u(x)| \leq M - \vartheta$$

for any  $x \neq 0$ , which contradicts our assumption that  $M$  was the optimal Lipschitz constant of  $u$ . So, it only remains to prove (4.7).

For this we consider the sets

$$F := \{(x, \tau) \in \mathbb{R}^n \times \mathbb{R} : \tau < u(x + t\sigma)\}$$

and

$$G_{\vartheta, \alpha} := \{(x, \tau) \in \mathbb{R}^n \times \mathbb{R} : \tau < u(x) + (M - \vartheta w(x))t + \alpha\},$$

and notice that (4.7) is equivalent to

$$(4.8) \quad F \subseteq G_{\vartheta, 0}.$$

To prove (4.8), we first observe that

$$u(x + t\sigma) \leq u(x) + Mt \leq u(x) + (M - \vartheta w(x))t + Mt$$

provided  $\vartheta \leq M$ , thus  $F \subseteq G_{\vartheta, \alpha}$  for any  $\alpha > Mt$ . Now we reduce  $\alpha$  until we find a critical  $\alpha_0$  for which  $G_{\vartheta, \alpha_0}$  touches  $F$  from above. We claim that

$$(4.9) \quad \alpha_0 \leq 0.$$

Suppose by contradiction that  $\alpha_0 > 0$ . Since  $u$  is  $M$ -Lipschitz, we have that

$$u(x + t\sigma) \leq u(x) + Mt < u(x) + Mt + \alpha_0,$$

which implies that  $G_{\vartheta, \alpha_0}$  and  $F$  can only touch at a some point  $X_0 = (x_0, t_0) \in \mathbb{R}^n \times \mathbb{R}$  with  $x_0 \in \text{supp}(w) \subset B_{3/4} \setminus B_{1/4}$ . Hence, it is easy to see (by compactness) that a contact point  $X_0 = (x_0, t_0)$  exists, and since  $x_0 \in B_{3/4} \setminus B_{1/4}$  we have that both sets are uniformly  $C^2$  near  $X_0$ , so the  $s$ -mean curvature operators  $I[F]$  and  $I[G_{\vartheta, \alpha_0}]$  (recall (1.1)) may be computed at  $X_0$  in the classical sense.

Since  $F$  and  $G_{0, \alpha_0}$  are  $s$ -minimal surfaces, we have that

$$(4.10) \quad I[F](X_0) = 0 = I[G_{0, \alpha_0}](X_0).$$

Also, since  $G_{\vartheta, \alpha_0}$  is a  $C^2$ -diffeomorphism of  $G_{0, \alpha_0}$  of size  $\vartheta t$ , and  $G_{\vartheta, \alpha_0}$  is uniformly  $C^2$  in a neighborhood of  $X_0$ , we have that

$$(4.11) \quad |I[G_{\vartheta, \alpha_0}](X_0)| \leq C \vartheta t,$$

for some universal constant  $C > 0$  (see also [5] for detailed computations on the effect of a small diffeomorphism on the fractional mean curvature). Furthermore, since  $F \subseteq G_{\vartheta, \alpha_0}$ , we have that

$$(4.12) \quad \chi_{G_{\vartheta, \alpha_0}} - \chi_{G_{\vartheta, \alpha_0}^c} - \chi_F + \chi_{F^c} = 2\chi_{G_{\vartheta, \alpha_0} \setminus F}.$$

Now we define

$$Z := \{x \in B_1 : u(x + t\sigma) - u(x) < (M - \mu_0)t\} \subset \mathbb{R}^n$$

and

$$W := \left\{ (x, \tau) \in \mathbb{R}^n \times \mathbb{R} : x \in Z \text{ and } u(x + t\sigma) < \tau < u(x + t\sigma) + \frac{\mu_0 t}{2} \right\} \subset \mathbb{R}^{n+1}.$$

We remark that  $|Z| \geq \mu_0 |B_1|$  thanks to (4.5), therefore  $|W| \geq \mu_0^2 t |B_1|/2$ . (Notice that, by abuse of notation, we are using  $|\cdot|$  to denote both the Lebesgue measure in  $\mathbb{R}^n$  and  $\mathbb{R}^{n+1}$ .)

We claim that

$$(4.13) \quad (G_{\vartheta, \alpha_0} \setminus F) \supseteq W$$

provided  $\vartheta$  is sufficiently small. Indeed, let  $(x, \tau) \in W$ . Then  $x \in Z$  and

$$u(x + t\sigma) < \tau < u(x + t\sigma) + \frac{\mu_0 t}{2},$$

which implies that  $(x, \tau) \notin F$  and

$$\begin{aligned} \tau < u(x + t\sigma) + \frac{\mu_0 t}{2} &< u(x) + (M - \mu_0)t + \frac{\mu_0 t}{2} \\ &\leq u(x) + Mt - \vartheta t \\ &\leq u(x) + Mt - \vartheta w(x)t \end{aligned}$$

provided  $\vartheta \in (0, \mu_0/4)$ . This shows that  $(x, \tau) \in G_{\vartheta, \alpha_0}$  proving (4.13).

Since by construction  $Z \subseteq B_1 \subset \mathbb{R}^n$  and  $u$  is  $M$ -Lipschitz with  $u(0) = 0$ , we deduce that  $W \subset B_1 \times [-1 - 2M, 1 + 2M]$ , which implies that

$$\sup_{Y \in W} |X_0 - Y| \leq C_M$$

for some  $C_M > 0$ , and (by (4.13)) that

$$|(G_{\vartheta, \alpha_0} \setminus F) \cap W| = |W| \geq \frac{\mu_0^2 t |B_1|}{2}.$$



From this, (1.1), and (4.12), we conclude that

$$\begin{aligned} I[G_{\vartheta, \alpha_0}](X_0) - I[F](X_0) &= \int_{\mathbb{R}^{n+1}} \frac{2\chi_{G_{\vartheta, \alpha_0} \setminus F}(Y)}{|X_0 - Y|^{n+1+s}} dy \\ &\geq \int_W \frac{2\chi_{G_{\vartheta, \alpha_0} \setminus F}(Y)}{|X_0 - Y|^{n+1+s}} dy \\ &\geq 2C_M^{-n-1-s} \int_W \chi_{G_{\vartheta, \alpha_0} \setminus F}(Y) dy \\ &= 2C_M^{-n-1-s} |W| \\ &\geq C_M^{-n-1-s} \mu_0^2 t |B_1|. \end{aligned}$$

Hence, combining (4.10) and (4.11) we get

$$(4.14) \quad C\vartheta t \geq I[G_{\vartheta, \alpha_0}](X_0) = I[G_{\vartheta, \alpha_0}](X_0) - I[F](X_0) \geq C_M^{-n-1-s} \mu_0^2 t |B_1|,$$

which is a contradiction if  $\vartheta$  is sufficiently small. This contradiction proves (4.9), that in turn implies (4.8) and so (4.7). This concludes the proof of Theorem 1.1.

### 5. Proof of Theorem 1.2

Let  $E_\infty$  be a blow-down of  $E$ , that is, a cluster point for  $E_r := E/r$  as  $r \nearrow \infty$ . In this way, we get an  $s$ -minimal cone, and the assumption that no singular  $s$ -minimal cones exist in dimension  $n$  combined with a standard dimension reduction argument implies that  $E_\infty$  can only be singular at the origin.

Also, because  $\partial E$  was a graph,  $E_\infty$  is an hypograph in  $\mathbb{R}^{n+1}$ , that is,

$$(5.1) \quad (x, \tau) \in E_\infty \implies (x, \tau - t) \in E_\infty \text{ for all } t \geq 0.$$

To check (5.1), one can observe that, as  $\partial E_r$  is a graph, the above property holds with  $E_r$  in place of  $E_\infty$ . Then, since  $E_r \rightarrow E_\infty$  in the Hausdorff distance (see for instance footnote 1 in the proof of Lemma 3.1), the claim in (5.1) follows.

Now we show that  $E_\infty$  is in fact a graph (and not only an hypograph). For this, suppose by contradiction that there exists  $\tau_\infty > 0$  such that  $\partial E_\infty$  touches  $\partial E_\infty + \tau_\infty e_{n+1}$  at some point. Then, by the strong maximum principle,<sup>3)</sup> we get  $E_\infty = E_\infty + \tau_\infty e_{n+1}$ , from which (iterating this equality) we get  $E_\infty = E_\infty + k\tau_\infty e_{n+1}$  for any  $k \in \mathbb{N}$ . This fact combined with (5.1) implies that

$$E_\infty = \mathcal{C} \times \mathbb{R},$$

where  $\mathcal{C}$  is an  $s$ -minimal cone in  $\mathbb{R}^n$ . Hence it follows by our assumption that  $\mathcal{C}$  is a half-space, and Lemma 3.1 gives that  $E = \mathcal{C} \times \mathbb{R}$  which is in contradiction with the fact that  $E$  was a graph.

<sup>3)</sup> A simple and direct way to see the strong maximum principle is to use that  $E_\infty$  and  $E_\infty + \tau_\infty e_{n+1}$  are smooth cones outside the origin. So, if they touch, we can find a contact point  $X_0 \neq 0$ , and by computing the operator  $I$  defined in (1.1) at  $X_0$  for both surfaces, since both  $E_\infty$  and  $E_\infty + \tau_\infty e_{n+1}$  are  $s$ -minimal and  $E_\infty \subset E_\infty + \tau_\infty e_{n+1}$ , we get

$$0 = I[E_\infty + \tau_\infty e_{n+1}](X_0) - I[E_\infty](X_0) = \int_{\mathbb{R}^{n+1}} \frac{2\chi_{E_\infty + \tau_\infty e_{n+1} \setminus E_\infty}(Y)}{|X_0 - Y|^{n+1+s}} dy,$$

which implies that  $E_\infty = E_\infty + \tau_\infty e_{n+1}$ , as desired.

Hence we have shown that  $\partial E_\infty$  and  $\partial E_\infty + \tau e_{n+1}$  never touch for any  $\tau > 0$ , which implies that  $\partial E_\infty$  is the graph of a function  $u_\infty : \mathbb{R}^n \rightarrow \mathbb{R}$ . In addition, since  $E_\infty$  is smooth outside the origin, we can compute  $|\nabla u_\infty(x)|$  at any point  $x \neq 0$  (though, in principle, it can be infinite at points with vertical tangent hyperplanes).

Now, as in the proof of Theorem 1.1, we consider a bump function  $w_0 \in C^\infty(\mathbb{R}, [0, 1])$ , with  $w_0(t) = 0$  for any  $t \in (-\infty, 1/4] \cup [3/4, +\infty)$  and  $w_0(t) = 1$  for any  $t \in [2/5, 3/5]$ , and we define  $w(x) = w_0(|x|)$ . Then, we fix  $\sigma \in \partial B_1$  and consider the family of sets

$$F_t := \{(x, \tau) : \tau \leq u_\infty(x + t\vartheta w(x)\sigma) - t\},$$

where  $t \in [0, 1]$  and  $\vartheta > 0$ . By compactness, we see that, if  $\vartheta$  is sufficiently small, then  $F_1 \subseteq E_\infty$ . Let  $t_0 \in [0, 1]$  be the smallest  $t$  for which  $F_t \subseteq E_\infty$ , and assume by contradiction that  $t_0 > 0$ . Since  $E_\infty$  is a graph, we see that  $F_{t_0}$  can only touch  $E_\infty$  from below at some point  $X_0 = (x_0, t_0)$  with  $x_0 \in \text{supp}(w) \subset B_{3/4} \setminus B_{1/4}$ . Hence, it is easy to see (by compactness) that a contact point  $X_0 = (x_0, t_0)$  exists, and since  $x_0 \in B_{3/4} \setminus B_{1/4}$ , we have that both sets are smooth near  $X_0$ .

Therefore, we can easily adapt the arguments provided in (4.10)–(4.14) as follows: First, by the  $s$ -minimality of  $E_\infty$  we have that  $I[E_\infty](X_0) = 0 = I[F_0](X_0)$ . Also, since  $F_{t_0}$  is a  $C^2$ -diffeomorphism of  $F_0$  of size  $\vartheta t_0$  and  $F_0$  is uniformly  $C^2$  in a neighborhood of  $X_0$ , we have that

$$(5.2) \quad |I[F_{t_0}](X_0)| \leq C \vartheta t_0.$$

On the other hand, since the graph of  $u_\infty$  is uniformly Lipschitz in a nontrivial fraction of points (just pick a point where the tangent space to  $\partial E_\infty$  is not vertical and consider a small neighborhood of this point), we see that  $\partial F_{t_0}$  and  $\partial E_\infty$  lie at distance  $\geq c t_0$  on a nontrivial fraction of points, therefore

$$|(E_\infty \setminus F_{t_0}) \cap B_1| \geq c_0 t_0$$

for some  $c_0 > 0$ . Hence, arguing as in the proof of Theorem 1.1, we get

$$|I[F_{t_0}](X_0)| = |I[F_{t_0}](X_0) - I[E_\infty](X_0)| \geq c' t_0$$

for some  $c' > 0$ , which is in contradiction with (5.2) if  $\vartheta$  was chosen sufficiently small.

This proves that  $t_0 = 0$ , which implies that  $F_t \subseteq E_\infty$  for any  $t \in (0, 1)$ , or equivalently

$$\frac{u_\infty(x + t\vartheta w(x)\sigma) - u_\infty(x)}{t} \leq 1 \quad \text{for all } t \in (0, 1).$$

Hence, letting  $t \searrow 0$ , we obtain

$$\vartheta w(x) \nabla u_\infty(x) \cdot \sigma \leq 1 \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}, \sigma \in \partial B_1,$$

which combined with the fact that  $w = 1$  in  $B_{3/5} \setminus B_{2/5}$  and  $\sigma \in \partial B_1$  is arbitrary implies

$$|\nabla u_\infty(x)| \leq 1/\vartheta \quad \text{for all } x \in B_{3/5} \setminus B_{2/5}.$$

Since  $u_\infty$  is 1-homogeneous, we deduce that  $u_\infty$  is globally Lipschitz. So by Theorem 1.1 it is smooth also at the origin, hence (being a cone)  $E_\infty$  a half-space. Using again Lemma 3.1, we deduce that  $E$  is a half-space as well, concluding the proof of Theorem 1.2.

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