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FINITE GROUPS WITH NON-TRIVIAL INTERSECTIONS OF KERNELS OF ALL BUT ONE IRREDUCIBLE CHARACTERS

MARIAGRAZIA BIANCHI* AND MARCEL HERZOG

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Abstract. In this paper we consider finite groups G satisfying the following condition: G has two columns in its character table which differ by exactly one entry. It turns out that such groups exist and they are exactly the finite groups with a non-trivial intersection of the kernels of all but one irreducible characters or, equivalently, finite groups with an irreducible character vanishing on all but two conjugacy classes. We investigate such groups and in particular we characterize their subclass, which properly contains all finite groups with non-linear characters of distinct degrees, which were characterized by Berkovich, Chillag and Herzog in 1992.

1. Introduction

It is well known that the intersection of kernels of all irreducible characters of a finite group is trivial. This gives rise to the following question: which finite groups, if any, have an intersection of kernels of all but one irreducible characters which is non-trivial?

We were lead to this problem by considering an apparently more general question: which finite groups have two columns in their character table which differ by exactly one entry? This problem is more general, since if an intersection of kernels of all but one irreducible characters of a finite group is non-trivial and if $b \neq 1$ belongs to such an intersection, then clearly the column of the character degrees

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*Corresponding author.
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and the column of b differ by exactly one entry. The surprising fact is that these two families of finite groups coincide.

So in this paper we shall investigate finite groups with two columns in their character table differing by exactly one entry. Such groups will be called $CD1$-*groups* (Columns (of the character table) Differing (by) 1 (entry)). To eliminate trivialities, we shall assume that if $G \in CD1$, then $|G| > 2$. This problem was suggested to us by our late colleague David Chillag.

All groups in this paper are finite. The set of all irreducible characters of a group G will be denoted by $\mathrm{Irr}(G)$. We shall write

$$\mathrm{Irr}(G) = \{\chi_1, \chi_2, \ldots, \chi_k\},$$

where $\chi_1 = 1_G$, the principal character of G, and the degrees of the characters satisfy

$$\chi_1(1) = 1 \leq \chi_2(1) \leq\leq \chi_k(1).$$

If $x \in G$, we shall denote the (conjugacy) class of x in G by x^G.

By the orthogonality relations, two columns of the character table of a group G cannot be equal. But could a group G, satisfying $|G| > 2$, have two columns in its character table differing by exactly one entry, or, in other words, be a $CD1$-group? We shall show that such groups exist and we shall try to determine their structure.

Let G be a $CD1$-group and suppose that the two columns of the character table of G differing by one entry correspond to the classes a^G and b^G of G. A priori, these classes are not unique. However, we shall show in Section 2 that one of these classes, say a^G, must be the unit class $\{1\}$ (hence $a = 1$). Moreover, the two columns corresponding to a^G and b^G can differ only in the row of the character χ_k, and the degree of χ_k must be larger than that of any other χ_i. In particular, $\chi_k(1) > \chi_{k-1}(1)$.

Denote

$$\chi_i(1) = a_i \quad \text{and} \quad \chi_i(b) = b_i \quad \text{for } i = 1, 2, \ldots, k.$$

Then, by the results of Section 2 mentioned above, we have

$$b_i = a_i = \chi_i(1) \quad \text{for} \quad i = 1, 2, \ldots, k-1$$

and

$$b_k \neq a_k.$$

By the orthogonality relations, b_k is a negative integer satisfying

$$-a_k \leq b_k \leq -1.$$

On the other hand, we shall show that the k-th row of the character table of G is

$$(a_k, b_k, 0, 0, \ldots, 0).$$

Hence the class b^G is unique. Consequently, G is a $CD1$-group if and only if an intersection of kernels of all but one irreducible characters of G is non-trivial.

Let us denote

$$N = \bigcap_{i=1}^{k-1} \ker \chi_i.$$

Then N is a normal subgroup of G and $\{1\} \cup b^G \subseteq N$. The uniqueness of the class b^G implies that

$$N = \{1\} \cup b^G.$$

Therefore G is a $CD1$-group if and only if $N \neq \{1\}$. In particular, an intersection of kernels of all but one irreducible characters of a group G is trivial, except, possibly, if the missing character's degree is larger than those of the other characters.

If G is a $CD1$-group, then N is the union of two conjugacy classes of G. Hence N is a minimal normal subgroup of G, which is an elementary abelian p-group for some prime p. Thus

$$|N| = |\{1\} \cup b^G| = p^n$$

for some positive integer n and b is of order p.

We shall show also in Section 2 that

$$|C_G(b)| = |P|,$$

where P denotes a Sylow p-subgroup of G. Hence $C_G(b) = P$ for some Sylow p-subgroup P of G and

$$C_G(N) = \bigcap_{x \in G} C_G(b^x) = \bigcap_{x \in G} P^x = O_p(G).$$

We have seen above that a $CD1$-group has an irreducible character which vanishes on all but two conjugacy classes. We shall show in Section 2 that these two properties are equivalent: *an irreducible character of a group G vanishes on all but two conjugacy classes of G if and only if the columns coresponding to these classes in the character table of G differ by exactly one entry.* Moreover, if G is a $CD1$-group, then (G, N) is a Camina pair.

In his paper [6], Stephen Gagola investigated groups which have an irreducible character which vanishes on all but two conjugacy classes. Such groups of order greater than 2 will be called *Gagola groups*. The above observation implies that Gagola groups coincide with the $CD1$-groups. In particular, our notation applies also to Gagola groups.

Let G be a Gagola group. In his paper Gagola completely determined the structure of the quotient group

$$G/C_G(N) = G/O_p(G).$$

In particular, he proved that if G is solvable, then a Sylow p-subgroup of $G/O_p(G)$ is abelian. Moreover, if G is solvable, then $G/O_p(G)$ has a normal p-complement, which is isomorphic to the multiplicative group of a near-field. The multiplicative groups of finite near fields are in one-to-one correspondence with the class of doubly transitive Frobenius groups.

A Frobenius group G is called *doubly transitive* if

$$|H| = |F| - 1,$$

where F denotes the (Frobenius) kernel of G and H denotes a (Frobenius) complement of G. This implies that

$$|G| = (p^n - 1)p^n \quad \text{with} \quad |F| = p^n \text{ and } |H| = p^n - 1,$$

where p is a prime and n is a positive integer. Moreover, F is an elementary abelian p-group. We shall use this notation and this fact throughout this paper. The finite near-fields were classified by Hans Zassenhaus in [12].

In the non-solvable case, Gagola's result is much more complicated.

The structure of $O_p(G)$ for Gagola groups (or $CD1$-groups) G is **an open problem**. Gagola showed that there is no bound on the derived length or the nilpotence class of $O_p(G)$.

Concerning the structure of the Gagola groups themselves, Gagola proved the following theorem (see [6, Theorem 6.2]).

Theorem G. *If G is a Gagola group, then $N = C_G(N)$ if and only if G is a doubly transitive Frobenius group.*

At the end of his paper, Gagola constructs for $p = 2$ and $p = 3$ examples of Gagola groups which are not p-closed. These are important examples of non-p-closed Camina pairs.

Our research was concentrated on the structure of $CD1$-groups themselves, satisfying certain conditions with respect to the entries a_k and b_k.

A $CD1$-group will be called *extreme* if

$$\text{either} \quad b_k = -1 \quad \text{or} \quad b_k = -a_k.$$

Our main result in this paper is the following theorem, in which the extreme $CD1$-groups are completely determined. Recall that a group G is called *of central type* if there exists $\chi \in \mathrm{Irr}(G)$ such that

$$[G : Z(G)] = \chi(1)^2.$$

Theorem 1.1. *A group G is an extreme $CD1$-group if and only if one of the following holds:*

(1) *$b_k = -a_k$ and G is a 2-group of central type with $|Z(G)| = 2$;*
(2) *$b_k = -1$ and G is a a doubly transitive Frobenius group.*

Moreover, we shall prove the following proposition:

Proposition 1.2. *The class of extreme $CD1$-groups properly contains the class of groups with non-linear irreducible characters of distinct degrees.*

Groups with non-linear irreducible characters of distinct degrees (DD-groups in short) were determined by Berkovich, Chillag and Herzog in [1, Theorem 4.2]. Proposition 1.2 implies that Theorem 1.1 is a proper generalization of this theorem, which itself is a proper generalization of Seitz's theorem (see [11, Theorem 4.1]), in which groups with one non-linear irreducible character were determined.

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Other generalizations of the Berkovich, Chillag and Herzog theorem appeared in the literature. We mention here four such papers: [2] by Berkovich, Isaacs and Kazarin in 1999, [9] by Maria Loukaki in 2007, [3] by Dolfi, Navarro and Tiep in 2013 and [4] by Dolfi and Yadav in 2016.

It follows from the characterization of the DD groups in [1] that all such groups are solvable. On the other hand, we shall show that the set of extreme $CD1$-groups contains non-solvable groups, and in particular, it contains a perfect group. For details, see Section 4, where Proposition 1.2 is proved.

In Section 2 we shall also show that if G is a $CD1$-group, then it is of even order and either $|Z(G)| = 2$ or the center of G is trivial. Moreover, if $|Z(G)| = 2$, then $Z(G) = \{1, b\}$ and $b_k = -a_k$. Hence $CD1$-groups G with $|Z(G)| = 2$ are extreme $CD1$-groups and Theorem 1.1 implies the following general result:

Theorem 1.3. *A group G is a $CD1$-group with $Z(G) \neq 1$ if and only if it is a 2-group of central type with $|Z(G)| = 2$.*

However, the problem of characterizing $CD1$-groups with a trivial center is **still open**. There exist such groups which are non-extreme. For example, there exist two groups of order 54 which are non-Frobenius $DC1$-groups with trivial centers and with $a_k = 6$ and $b_k = -3$.

Theorem G and Theorem 1.1 immediately yield the following result.

Theorem 1.4. *If G is a $CD1$-group, then $N = C_G(N)$ if and only if $b_k = -1$.*

We were not able to prove this result directly, without using the results of Gagola.

Recalling that the element b has prime order p, in Section 3 we shall also characterize $CD1$-groups G satisfying the following conditions: $p \nmid a_k$ (see Proposition 3.2), $a_k = p^s$ for some positive integer s (see Proposition 3.3) and G is an r-group for some prime r (see Corollary 3.6).

Finally, in Section 5 we shall characterize $CD1$-groups with a_k being any prime power (see Theorem 5.1).

The structure of this paper is as follows. In Section 2 the basic properties of $CD1$-groups will be determined, including a classification result (see Proposition 2.7). In Section 3 we shall classify $CD1$-groups satisfying $b_k = -1$ and those satisfying $b_k = -a_k$ (see Propositions 3.2 and 3.3, respectively). These results will imply Theorem 1.1. Theorem 1.3 will be also proved in Section 3. The relation between extreme $CD1$-groups and DD-groups will be described in Section 4, including the proof of Proposition 1.2. This proposition implies that our Theorem 1.1 is a proper generalization of the Berkovich-Chillag-Herzog theorem. Finally, Section 5 will be devoted to the classification of $CD1$-groups with a_k being a power of a prime (see Theorem 5.1). In particular, $CD1$-groups with a_k being a prime are determined in Corollary 5.2.

2. Basic properties of $CD1$-groups

Recall that a finite group G is called a $CD1$-*group* if $|G| > 2$ and two columns of the character table of G ($CT(G)$ in short) differ by exactly one entry.

Suppose that these columns correspond to the two classes a^G and b^G for some $a, b \in G$ and they differ only in row j. Thus

$$\chi_i(a) = \chi_i(b) \quad \text{for } i \neq j \quad \text{and} \quad \chi_j(a) \neq \chi_j(b).$$

Two such classes will be called (temporarily) *special classes* of G.

We shall also use the following notation: $g = |G|$ and if $u \in G$, then the column in the $CT(G)$ corresponding to u^G will be denoted by U and $\chi_i(u)$ will be denoted by u_i for $1 \leq i \leq k$.

In particular, the column in the $CT(G)$ corresponding to a^G will be denoted by A and $a_i = \chi_i(a)$ for $1 \leq i \leq k$. Similarly, B will denote the column corresponding to b^G and $b_i = \chi_i(b)$ for all i.

Moreover, if also $v \in G$, then the product of the columns U and V corresponding to u^G and v^G, respectively, is defined as follows:

$$UV = \sum_{i=1}^{k} \chi_i(u)\chi_i(v) = \sum_{i=1}^{k} u_i v_i.$$

First we prove:

Lemma 2.1. *Let G be a $CD1$-group and let a^G, b^G be special classes of G with $a_j \neq b_j$. Then we may assume, without loss of generality, that $a^G = \{1\}$ is the trivial class and $a_i = \chi_i(1)$ for all i. Moreover,*

$$j = k, \quad a_k > a_{k-1} \geq 1$$

and b_k is an integer satisfying

$$-a_k \leq b_k \leq -1.$$

We also have

$$g = a_k^2 - b_k a_k,$$

$$a_k^2 + a_k \leq g \leq 2a_k^2$$

and

$$\{1\} \cup b^G \subseteq \bigcap_{i=1}^{k-1} \ker \chi_i.$$

Proof. If neither a nor b is equal to 1, then by the orthogonality relations (OR in short)

$$\sum_{i=1}^{k} \chi_i(a)\chi_i(1) = \sum_{i=1}^{k} \chi_i(b)\chi_i(1) = 0.$$

It follows, by our assumptions, that $\chi_j(a)\chi_j(1) = \chi_j(b)\chi_j(1)$ and hence $\chi_j(a) = \chi_j(b)$, a contradiction.

So we may assume, without loss of generality, that $a = 1$ and A is the column of the degrees $a_i = \chi_i(1)$ in the $CT(G)$. Thus $b_i = a_i = \chi_i(1)$ for all i except $i = j$ and consequently

$$\{1\} \cup b^G \subseteq \bigcap_{\chi_i \in \mathrm{Irr}(G) \setminus \chi_j} \ker \chi_i.$$

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Moreover, by the OR, we have

$$0 = BA = \sum_{i=1}^{k} a_i^2 - a_j^2 + b_j a_j = g - a_j^2 + b_j a_j,$$

so b_j is an integer and

$$g = a_j^2 - b_j a_j.$$

Since $g \neq 1$, it follows that $g > a_j^2$ and hence $-b_j \geq 1$. But $-a_j \leq b_j \leq a_j$, so

$$-a_j \leq b_j \leq -1.$$

Thus

$$a_j^2 + a_j \leq g \leq 2a_j^2$$

and $g > 2$ implies that $k > 2$ and $a_j > 1$. If $a_j \leq a_{k-1}$, then $g > a_k^2 + a_{k-1}^2 \geq 2a_j^2$, a contradiction. Hence

$$j = k \quad \text{and} \quad a_k > a_{k-1}.$$

By replacing j by k in the previous statements, we obtain the required results. □

This lemma implies that if G is a $CD1$-group, then an intersection of kernels of all but one irreducible characters of G is non-trivial. The converse of this statement was noticed in the Introduction. Hence the two families of groups coincide.

Moreover, as noticed in the Introduction, an intersection of kernels of all but one irreducible characters of a group G is trivial, except, possibly, if the missing character's degree is larger than those of the other characters.

We continue with other lemmas, using the notation of Lemma 2.1. We shall also assume that the first column of the $CT(G)$ corresponds to the class of $a = 1$, i.e. it is the column of the degrees of the characters, and the second column corresponds to the class of b. In the next lemma we shall also establish the equivalence between $CD1$-groups and groups with an irreducible character which vanishes on all but two conjugacy classes.

Lemma 2.2. *(a) Let G be a $CD1$-group. Then*

$$\{1\} \cup b^G \subseteq G',$$

so

$$|G'| \geq 1 + |b^G|.$$

Moreover, the k-th row of the $CT(G)$ is:

$$(a_k, b_k, 0, 0, \ldots, 0).$$

Hence the non-unit special class is unique.

(b) A group G is a $CD1$-group if and only if G has an irreducible character which vanishes on all but two conjugacy classes.

Proof. (a) Since $a_k > 1$, b^G lies in the kernel of all linear characters of G. Hence b^G, together with 1, lies in G' and $|G'| \geq 1 + |b^G|$.

Moreover, if $d \in G \setminus (\{1\} \cup b^G)$, then

$$DA = DB = 0 \Rightarrow d_k a_k = d_k b_k.$$

But $b_k \neq a_k$, so $d_k = 0$ and hence the k-th row of the $CT(G)$ is:

$$(a_k, b_k, 0, 0, \ldots, 0).$$

Therefore, by Lemma 2.1, no class other than b^G can generate together with $\{1\}$ a couple of special classes.

(b) By part (a), every $CD1$-group has an irreducible character which vanishes on all but two conjugacy classes.

Conversely, suppose that the group G has an irreducible character χ which vanishes on all but two conjugacy classes: a^G and b^G . Then we may assume, without loss of generality, that $a = 1$ and $\chi(a) \neq 0$. By the orthogonality relations for rows of the $CT(G)$ we have: $\chi(a) + \chi(b)|b^G| = 0$ and if ψ is any other irreducible character of G, then $\chi(a)\psi(a) + \chi(b)\psi(b)|b^G| = 0$. Thus $\chi(a)\psi(a) = \chi(a)\psi(b)$, which implies that $\psi(a) = \psi(b)$. Hence $\chi(a) \neq \chi(b)$ and the columns of a^G and b^G in the $CT(G)$ differ by exactly one entry, so G is a $CD1$-group. □

For a $CD1$-group G set

$$N = \bigcap_{i=1}^{k-1} \ker \chi_i.$$

The structure of N will now be determined.

Lemma 2.3. *Let G be a $CD1$-group. Then*

$$N = \{1\} \cup b^G = \bigcap_{i=1}^{k-1} \ker \chi_i.$$

Thus N is a minimal normal subgroup of G, the order of b equals a prime p and $|N| = p^n$ for some positive integer n. Hence

$$p^n = |N| = 1 + |b^G|$$

and

$$|b^G| = p^n - 1 \geq p - 1.$$

In particular, N is an elementary abelian p-group, $p \nmid |b^G|$ and

$$N \leq G'.$$

Proof. Since by Lemma 2.2 the class b^G is unique, it follows that

$$N = \{1\} \cup b^G = \bigcap_{i=1}^{k-1} \ker \chi_i.$$

Hence N is a normal subgroup of G. Since N is the union of the two classes $\{1\}$ and b^G, it follows that $o(b) = p$ for some prime p and N is a minimal normal subgroup of G of order

$$|N| = 1 + |b^G| = p^n \geq p$$

for some positive integer n. In particular, $|b^G| = p^n - 1 \geq p - 1$, $p \nmid |b^G|$ and N is an elementary abelian p-group. Since by Lemma 2.2

$$N = \{1\} \cup b^G \subseteq G',$$

it follows that N is a (normal) subgroup of G'. □

The next two lemmas provide important information concerning a_k, b_k and g.

Lemma 2.4. *Let G be a $CD1$-group. Then*

$$a_k = -b_k|b^G|$$

and

$$g = -a_k b_k(1 + |b^G|) = a_k^2 \left(\frac{1 + |b^G|}{|b^G|}\right) = b_k^2|b^G|(1 + |b^G|).$$

Hence $a_k \geq |b^G| \geq p - 1$ and G is of even order.

Proof. Since by Lemma 2.2 the k-th row of the $CT(G)$ is $(a_k, b_k, 0, 0, \dots, 0)$, it follows by the OR that $a_k \cdot 1 + b_k \cdot |b^G| = 0$. Hence $a_k = -b_k|b^G|$ and by Lemma 2.1 we get:

$$g = a_k^2 - b_k a_k = -a_k b_k(1 + |b^G|) = a_k^2 \left(\frac{1 + |b^G|}{|b^G|}\right) = b_k^2|b^G|(1 + |b^G|).$$

Since $-b_k \geq 1$ by Lemma 2.1 and $|b^G| \geq p - 1$ by Lemma 2.3, it follows that $a_k \geq |b^G| \geq p - 1$ and $g = b_k^2|b^G|(1 + |b^G|)$ implies that G is of even order. □

For the next lemma recall that b is an element of prime order p and $|b^G| = p^n - 1$ for a suitable positive integer n.

Lemma 2.5. *Let G be a $CD1$-group. Then*

$$b_k = -p^t$$

for some non-negative integer t, implying that

$$a_k = p^t(p^n - 1) \quad \textit{and} \quad g = p^{n+2t}(p^n - 1).$$

In particular, $-b_k$ is the p-part of a_k and the following statements hold: $b_k = -1$ if and only if $p \nmid a_k$ and $b_k = -a_k$ if and only if $a_k = p^r$ for some positive integer r. Moreover, $b_k = -1$ if and only if $g = p^n(p^n - 1)$ and $b_k = -a_k$ if and only if $g = 2^{1+2t}$

Proof. Since by Lemma 2.2 the k-th row of the $CT(G)$ is $(a_k, b_k, 0, 0, \dots, 0)$, it follows that if Q is a Sylow q-subgroup of G for some prime $q \neq p$, then χ_k vanishes on $Q \setminus \{1\}$ and consequently the restriction of χ_k to Q is a multiple of the regular character of Q by a positive integer. Thus $|Q|$ divides $a_k = \chi_k(1)$ and it follows that $\frac{g}{a_k}$ is a power of p. Since $\frac{g}{a_k} = -b_k(1 + |b^G|) = -b_k p^n$, we may conclude that

$$b_k = -p^t$$

for some non-negative integer t. Thus

$$a_k = -b_k|b^G| = p^t(p^n - 1),$$

which implies that $-b_k$ is the p-part of a_k. Hence $b_k = -1$ if and only if $p \nmid a_k$ and $b_k = -a_k$ if and only if $a_k = p^r$ for some positive integer r. Moreover,

$$g = -a_k b_k p^n = p^t(p^n - 1)p^t p^n = p^{n+2t}(p^n - 1),$$

so if $b_k = -1$ then $g = p^n(p^n - 1)$ and if $b_k = -a_k$ then $b_k = b_k(p^n - 1)$, implying that $p^n - 1 = 1$, $p^n = 2^1$ and $g = -a_k b_k p^n = 2^{2t}2 = 2^{1+2t}$. Conversely, if $g = p^n(p^n - 1)$, then $2t = 0$ and $b_k = -1$ and if $g = p^{n+2t}(p^n - 1) = 2^{1+2t}$, then $n \geq 1$ implies that $p = 2$, $p^n = 2$ and $b_k = -2^t = -a_k$. Hence $b_k = -1$ if and only if $g = p^n(p^n - 1)$ and $b_k = -a_k$ if and only if $g = 2^{1+2t}$. The proof of Lemma 2.5 is now complete. □

From now on, we shall use the notation $b_k = -p^t$. Recall that a $CD1$-group is called extreme if either $b_k = -1$ or $b_k = -a_k$. Therefore a $CD1$-group is extreme if either $p \nmid a_k$ or $a_k = p^r$ for some positive integer r

Lemma 2.5 implies the following two general results.

Proposition 2.6. *Let G be a $CD1$-group. Then the following statements hold:*

(1)
$$|C_G(b)| = p^{n+2t},$$
so the centralizer of each non-trivial element of N is a Sylow p-subgroup of G. In particular, each non-trivial element of N belongs to the center of a unique Sylow p-subgroup of G.

(2) *The center of each Sylow p-subgroup of G is contained in N.*

(3) *If N is not a Sylow p-subgroup of G, then the Sylow p-subgroups of G are non-abelian.*

(4)
$$N = \bigcup \{Z(P) \mid P \in Syl_p(G)\}.$$
Moreover, if $P, Q \in Syl_p(G)$ and $P \neq Q$, then $Z(P) \cap Z(Q) = \{1\}$.

(5) *If $P \in Syl_p(G)$, then $Z(P) = N$ if and only if $P \lhd G$.*

(6) $C_G(N) = O_p(G)$.

(7) $O_{p'}(G) = 1$.

(8) *If $x \in G \setminus N$, then $|C_G(x)| = |C_{G/N}(xN)|$. In particular, (G, N) is a Camina pair.*

(9) *Let $P \in Syl_p(G)$. Then $N_G(Z(P)) = N_G(P)$.*

Proof. (1) Since $g = p^{n+2t}(p^n - 1)$, it follows that $|C_G(b)| = \frac{g}{|b^G|} = \frac{g}{p^n-1} = p^{n+2t}$ and $C_G(b)$ is a Sylow p-subgroup of G. Since $N = \{1\} \cup b^G$, it follows that the centralizer of each non-trivial element of N is a Sylow p-subgroup of G. In particular, if $y \in N \setminus \{1\}$, then y belongs to the center of some Sylow p-subgroup P of G. Moreover, this Sylow p-subgroup P is unique, since if y belongs to the center of another Sylow p-subgroup Q of G, then $|C_G(y)| > |P| = p^{n+2t}$, a contradiction.

(2) On the other hand, if $c \in G \setminus N$, then by Lemma 2.2

$$|C_G(c)| \le g - a_k^2 = p^{n+2t}(p^n - 1) - p^{2t}(p^n - 1)^2 = p^{2t}(p^n - 1)(p^n - p^n + 1) = p^{2t}(p^n - 1)$$

and $|C_G(c)| < p^{n+2t}$. Hence the center of each Sylow p-subgroup of G is contained in N.

(3) Since $N \lhd G$, it follows that $N \subseteq P$ for each $P \in Syl_p(G)$. If $N \subsetneqq P$ for some $P \in Syl_p(G)$, then $N \subsetneqq P$ for all $P \in Syl_p(G)$ and by (2) $Z(P) \subsetneqq P$. Hence if N is not a Sylow p-subgroup of G, then the Sylow p-subgroups of G are non-abelian.

(4) By (1) and (2) $N = \bigcup\{Z(P) \mid P \in Syl_p(G)\}$. If $P, Q \in Syl_p(G)$, $P \neq Q$ and $x \in Z(P) \cap Z(Q)$, then $x \in N$ and by (1) $x = 1$. Hence $Z(P) \cap Z(Q) = \{1\}$.

(5) Let $P \in Syl_p(G)$. If $Z(P) = N$, then by (4) $Syl_p(G) = \{P\}$ and $P \lhd G$. Conversely, if $P \lhd G$ then by (4) $N = Z(P)$.

(6) By (1), $C_G(b) = P$ for some $P \in Syl_p(G)$. Hence

$$C_G(N) = \bigcap_{x \in G} C_G(b^x) = \bigcap_{x \in G} P^x = O_p(G).$$

(7) Since N is a normal p-subgroup of G, it follows that $O_{p'}(G) \le C_G(N) = O_p(G)$. Hence $O_{p'}(G) = 1$.

(8) Let $x \in G \setminus N$. Since $N = \bigcap_{i=1}^{k-1} \ker \chi_i$ and $\chi_k(x) = 0$, it follows that $|C_G(x)| = |C_{G/N}(xN)|$. Thus (G, N) satisfies one of the definitions of a Camina pair.

(9) By (4), if $P, Q \in Syl_p(G)$ and $P \neq Q$, then $Z(P) \cap Z(Q) = \{1\}$. Hence $[G : N_G(Z(P))] = [G : N_G(P)]$, yielding $|N_G(Z(P))| = |N_G(P)|$. But $N_G(P) \subseteq N_G(Z(P))$, so $N_G(Z(P)) = N_G(P)$. □

Notice that if G is an extreme $CD1$-group, then by Theorem 1.1 and Proposition 3.4 the Sylow p-subgroup of G is normal in G. However, for example, the $CD1$-groups of order 54 mentioned in the Introduction are $CD1$-groups which satisfy the condition $P \lhd G$, but they are non-extreme ($a_k = 6$ and $b_k = -3$). Thus the classification of $CD1$-groups with $P \lhd G$ is still **an open problem**.

By Lemma 2.3, if G is a $CD1$-group, then $N \le G'$. In the following proposition we shall classify $CD1$-groups which satisfy the equality $N = G'$.

Proposition 2.7. *The group G is a $CD1$-group with*

$$N = G'$$

if and only if one of the following statements holds:

(a) G is an extra-special 2-group of order 2^{2m+1}. The degree pattern of G is $(1^{(2^{2m})}, 2^m)$.

(b) G is a doubly transitive Frobenius groups of order $(p^n - 1)p^n$ with a cyclic complement. The degree pattern of G is $(1^{(p^n-1)}, p^n - 1)$.

Proof. Suppose, first, that G is a $CD1$-group with $N = G'$. Then $|G'| = p^n$ and by Lemma 2.5

$$[G : G'] = \frac{p^{n+2t}(p^n - 1)}{p^n} = p^{2t}(p^n - 1).$$

Therefore G has $p^{2t}(p^n - 1)$ linear characters and

$$a_k^2 + p^{2t}(p^n - 1) = p^{2t}(p^n - 1)^2 + p^{2t}(p^n - 1) = p^{2t}(p^n - 1)(p^n - 1 + 1) = p^{n+2t}(p^n - 1) = g.$$

Thus G has only one non-linear irreducible character and by Seitz's theorem (see Theorem 4.1 in Section 4) G satisfies either (a) or (b).

Conversely, suppose that G is of type (a) or (b). Then, as shown in the proof of Proposition 1.2 in Section 4, G is a $CD1$-group. Moreover, if G is of type (a), then G is the Sylow p-subgroup of itself and by Proposition 2.6.(5) $N = Z(G)$. But in extra-special groups $G' = Z(G)$, so $G' = N$, as required. Finally, if G is of type (b), then $F = G'$ and it follows by Proposition 3.4 that $N = F = G'$, again as required. The proof of Proposition 2.7 is now complete. □

We continue with information concerning the center $Z(G)$ of a $CD1$-group G.

Lemma 2.8. *Let G be a $CD1$-group. Then*

$$Z(G) \leq N$$

and either

$$Z(G) = 1,\ |b^G| \geq 2,\ -b_k \leq \frac{a_k}{2} \text{ and } g = a_k^2(\frac{1 + |b^G|}{|b^G|}) \leq \frac{3}{2}a_k^2,$$

or

$$|Z(G)| = 2,\ Z(G) = \{1, b\},\ |b^G| = 1,\ p = 2,\ n = 1,\ b_k = -a_k \text{ and } g = 2a_k^2.$$

Proof. If $c \in Z(G)$, then $|\chi_k(c)| = a_k$. Since by Lemma 2.2 the k-th row of the $CT(G)$ is $(a_k, b_k, 0, 0, \ldots, 0)$, it follows that $c \in \{1\} \cup b^G = N$. Hence $Z(G) \leq N$.

If $b \in Z(G)$, then $|b^G| = 1$, $Z(G) = \{1, b\} = N$, $|Z(G)| = 2 = p^n$, $a_k = -b_k|b^G| = -b_k$ and $g = a_k^2(\frac{1+|b^G|}{|b^G|}) = 2a_k^2$.

If $b \notin Z(G)$, then $Z(G) = 1$, $|b^G| \geq 2$, $-b_k = a_k/|b^G| \leq a_k/2$ and $g = a_k^2(\frac{1+|b^G|}{|b^G|}) \leq (3/2)a_k^2$. □

Lemma 2.8 immediately implies the following characterization of $CD1$-groups G with $|Z(G)| = 2$.

Lemma 2.9. *Let G be a $CD1$-group. Then $|Z(G)| = 2$ if and only if $b_k = -a_k$.*

Finally, we shall consider $CD1$-groups satisfyng $g = \frac{3}{2}a_k^2$.

Lemma 2.10. *Let G be a $CD1$-group satisfying $g = \frac{3}{2}a_k^2$. Then $|N| = p = 3$, $b_k = -3^t$, $a_k = 2 \cdot 3^t$ and $g = 2 \cdot 3^{2t+1}$. In particular, such G are p-closed.*

Proof. By Lemma 2.8, $g = \frac{3}{2}a_k^2$ implies that $|b^G| = p^n - 1 = 2$ and $a_k = p^t(p^n - 1) = 2p^t$. Hence $|N| = p = 3$, $a_k = 2 \cdot 3^t$, $b_k = -3^t$ and $g = 6p^{2t} = 2 \cdot 3^{2t+1}$. Since the index of the Sylow 3-subgroup in G equals 2, it follows that G is p-closed. □

Notice that if $t = 1$, then $g = 2 \cdot 27 = 54$ and we get the groups of order 54 mentioned in the Introduction.

3. The basic result and applications

Our main aim in this section is to prove Theorem 1.1, which will be restated as Theorem 3.1.

Theorem 3.1. *A group G is an extreme $CD1$-group if and only if one of the following holds:*

(1) $-b_k = 1$ *and G is a doubly transitive Frobenius group.;*

(2) $-b_k = a_k$ *and G is a 2-group of central type with* $|Z(G)| = 2$.

We shall use the notation of the previous section. Recall that a $CD1$-group is called *extreme* if either $b_k = -1$ or $b_k = -a_k$. Hence the theorem is a consequence of the following two propositions:

Proposition 3.2. *The following statements are equivalent:*

(1) *G is a $CD1$-group with* $b_k = -1$,

(2) *G is a $CD1$-group with* $p \nmid a_k$,

(3) *G is a doubly transitive Frobenius group with the Frobenius kernel F,*

where $\{1\} \cup b^G = F$.

and

Proposition 3.3. *The following statements are equivalent:*

(1) *G is a $CD1$-group with* $b_k = -a_k$,

(2) *G is a $CD1$-group with $a_k = p^s$ for some positive integer s,*

(3) *G is a 2-group of central type with* $|Z(G)| = 2$,

with $\{1\} \cup b^G = Z(G)$.

We first prove the following preliminary proposition.

Proposition 3.4. *Let G be a doubly transitive Frobenius group of order $(p^n - 1)p^n$. Then G is a $CD1$-group with $a_k = p^n - 1 = |b^G|$, $F = N$ and $b_k = -1$.*

Proof. Since G is a Frobenius group of order $(p^n - 1)p^n$ with the Frobenius kernel F of order p^n, it follows by [8, Theorem 18.7], that all the irreducible characters of G, except one of degree $p^n - 1$, contain F in their kernel. Hence G is a $DC1$-group with $N = F$, $a_k = p^n - 1$, $|b^G| = p^n - 1 = a_k$ and by Lemma 2.5 $b_k = -1$. The proof of the proposition is complete. □

We proceed now with proofs of the main propositions.

Proof of Proposition 3.2. (1) and (2) are equivalent by Lemma 2.5. We proceed with proving that (1) and (3) are equivalent.

Suppose, first, that G is a $CD1$-group with $b_k = -1$. Then by Lemmas 2.3 and 2.5 $g = (p^n - 1)p^n$ and N is a minimal normal subgroup of G of order p^n. Moreover, if $x \in N \setminus \{1\} = b^G$, then $|C_G(x)| =$

$\frac{|G|}{|b^G|} = p^n = |N|$ and it follows that $N = C_G(x)$ for each $x \in N \setminus \{1\}$. Thus G is a Frobenius group with N as its kernel, as required (see Feit's book [5, p. 133]).

Conversely, suppose that G is a doubly transitive Frobenius group of order $(p^n - 1)p^n$. Then by Proposition 3.4 G is a $CD1$-group with $b_k = -1$ and $F = N$, as required. □

Proof of Proposition 3.3. (1) and (2) are equivalent by Lemma 2.5. We proceed with proving that (1) and (3) are equivalent.

Suppose, first, that G is a $CD1$-group with $b_k = -a_k$. Then by Lemma 2.8 $|Z(G)| = 2$ and $g = 2a_k^2$. Hence

$$[G : Z(G)] = g/2 = a_k^2,$$

so G is a group of central type. Finally, by Lemma 2.5, G is a 2-group, as required.

Conversely, suppose that G is a 2-group of central type with $|Z(G)| = 2$. Then $[G : Z(G)] = \chi(1)^2$ for some $\chi \in \mathrm{Irr}(G)$ and $|G| = 2\chi(1)^2$. Since $|G| = \sum_{\psi \in \mathrm{Irr}(G)} \psi(1)^2$, it follows that

$$\chi(1)^2 = \sum_{\xi \in \mathrm{Irr}(G) \setminus \chi} \xi(1)^2.$$

Let $Z(G) = \{1, b\}$. Then $|b| = 2$ and for each $\psi \in \mathrm{Irr}(G)$, $\psi(b)$ is an integer and $|\psi(b)| = \psi(1)$. Hence $\psi(b) = \pm\psi(1)$.

Since $\sum_{\psi \in \mathrm{Irr}(G)} \psi(1)\psi(b) = 0$, it follows that

$$\chi(1)\chi(b) = -\sum_{\xi \in \mathrm{Irr}(G) \setminus \chi} \xi(1)\xi(b).$$

If $\chi(b) = \chi(1)$, then

$$\chi(1)^2 = -\sum_{\xi \in \mathrm{Irr}(G) \setminus \{\chi \cup 1\}} \xi(1)\xi(b) - 1 \leq \sum_{\xi \in \mathrm{Irr}(G) \setminus \{\chi \cup 1\}} \xi(1)^2 - 1 = \sum_{\xi \in \mathrm{Irr}(G) \setminus \chi} \xi(1)^2 - 2,$$

a contradiction. So $\chi(b) = -\chi(1)$ and

$$\chi(1)^2 = \sum_{\xi \in \mathrm{Irr}(G) \setminus \chi} \xi(1)\xi(b) = \sum_{\xi \in \mathrm{Irr}(G) \setminus \chi} \xi(1)^2.$$

Hence $\xi(b) = \xi(1)$ for all $\xi \in \mathrm{Irr}(G) \setminus \chi$ and G is a $CD1$-group with $\chi = \chi_k$, $Z(G) = \{1\} \cup b^G$ and $b_k = \chi(b) = -\chi(1) = -a_k$, as required. □

As mentioned above, Theorem 3.1 follows from Propositions 3.2 and 3.3.

We conclude this section with three applications of Propositions 3.2 and 3.3.

By Proposition 3.4 doubly transitive Frobenius groups are $CD1$-groups with $a_k = p^n - 1$. Therefore Proposition 3.2 immediately yields

Corollary 3.5. *G is a $CD1$-group with $a_k < p$ if and only if $a_k = p - 1$ and G is a doubly transitive Frobenius group of order $(p - 1)p$.*

If the $CD1$-group G is an r-group for some prime r, then $p = r$ and $a_k = r^s = p^s$ for some positive integer s. Therefore Proposition 3.3 immediately yields

Corollary 3.6. *Let G be an r-group for some prime r. Then G is a $CD1$-group if and only if $r = 2$ and G is a 2-group of central type with $|Z(G)| = 2$.*

The final result is also an application of Proposition 3.3.

proof of Theorem1.3. Suppose, first, that G is a $CD1$-group with $Z(G) \neq 1$. Then by Lemma 2.8 $|Z(G)| = 2$ and by Lemma 2.9 $b_k = -a_k$. Thus, by Proposition 3.3, G is a 2-group of central type with $|Z(G)| = 2$.

Conversely, if G is a 2-group of central type with $|Z(G)| = 2$, then by Proposition 3.3 G is a $CD1$-group with $|Z(G)| \neq 1$, as required. □

4. DD-groups vis-a-vis extreme $CD1$-groups

In [11], Seitz proved the following theorem:

Theorem 4.1. *The group G has only one non-linear irreducible character if and only if G is of one of the following two types:*

(1) *G is an extra-special 2-group of order 2^{2m+1}. The degree pattern of G is $(1^{(2^{2m})}, 2^m)$.*
(2) *G is a doubly transitive Frobenius group of order $(p^n - 1)p^n$ with a cyclic complement. The degree pattern of G is $(1^{(p^n-1)}, p^n - 1)$.*

In [1], Berkovich, Chillag and Herzog generalized Seitz's theorem. They proved

Theorem 4.2. *The non-linear irreducible characters of the group G are of distinct degrees if and only if either G is of one of the two types* (a) *and* (b) *of Theorem 4.1 or G is the following group:*
(c) *G is the Frobenius group of order $2^3 3^2$, with the Frobenius kernel of order 3^2 and with a quaternion Frobenius complement of order 2^3. The degree pattern of G is $(1^{(4)}, 2, 8)$.*

Groups with non-linear irreducible characters of distinct degrees were called DD-groups. It follows immediately from Theorem 4.2 that DD-groups are solvable.

The extreme $CD1$-groups were determined in our Theorem 1.1. Our aim now is to prove Proposition 1.2, which implies that the set of the extreme $CD1$-groups properly contains the set of DD-groups. Thus our Theorem 1.1 is a proper generalization of Theorem 4.2.

proof of Proposition 1.2. First we shall show that the set of DD-groups is contained in the set of extreme $CD1$-groups. Indeed, if G is a DD-group, then by Theorem 4.2, G is either a doubly transitive Frobenius group, or it is an extra-special 2-group. In the former case G is an extreme $CD1$-group by Proposition 3.4. So it remains only to show that extra-special 2-groups are 2-groups of central type with $|Z(G)| = 2$. It is well known that an extra-special 2-group G is of order 2^{2m+1} for some positive integer m and satisfies the following two properties: $|Z(G)| = 2$ and G contains an irreducible character of degree 2^m (see, for example, [8, Example 7.6(b)]). Hence G is a 2-group of central type with $|Z(G)| = 2$, as required. The proof of our claim is complete.

It remains only to show that the set of extreme $CD1$-groups properly contains the set of DD-groups. We shall show that extreme $CD1$-groups of each type (1) and (2), as described in Theorem 1.1, contain groups which are not DD-groups.

As an example of type (1), consider the group $G = Q_{16} : C_2$ of order 32. The center of G is of order 2 and it has an irreducible character of degree 4, so G is a 2-group of central type with $|Z(G)| = 2$. By Theorem 1.1, G is an extreme $CD1$-group. But G is not extra-special, since $|G'| = 4$, so in view of Theorem 4.2 G is not a DD-group. There are four other groups of order 32 with the same property.

The examples of type (2) are more interesting. By the results of Zassenhaus in [12] (see also Section 20.7 in [7]) there are the following three non-solvable Frobenius groups of order $(p^2-1)p^2$ for $p = 11, 29, 59$:

(i) $G_1 = FH$, where the Frobenius kernel F is elementary abelian of order 11^2 and the Frobenius complement H of order 120 is isomorphic to $SL(2,5)$.

(ii) $G_2 = FH$, where the Frobenius kernel F is elementary abelian of order 29^2 and the Frobenius complement H of order 840 is isomorphic to $H = SL(2,5) \times C_7$.

(iii) $G_3 = FH$, where the Frobenius kernel F is elementary abelian of order 59^2 and the Frobenius complement H of order 3480 is isomorphic to $H = SL(2,5) \times C_{29}$.

These groups are non-solvable extreme $CD1$-groups by Theorem 1.1 and G_1 is even perfect. In view of Theorem 4.2, they are not DD-groups.

The proof of Proposition 1.2 is now complete. □

5. Charaterizations of $CD1$-groups with a_k a power of a prime

In this section we shall prove the following classification theorem:

Theorem 5.1. *Let r be a prime. Then G is a $CD1$-group with $a_k = r^s$ for some positive integer s if and only if one of the following cases holds:*

1. *G is a 2-group of central type with $|Z(G)| = 2$ and $|G| = 2^{2s+1}$ for some positive integer s;*
2. *G is a doubly transitive Frobenius group of order $(2^n-1)2^n$, where $2^n - 1 = r$ is a Mersenne prime;*
3. *G is a doubly transitive Frobenius group of order $(3^2-1)3^2 = 72$;*
4. *G is a doubly transitive Frobenius group of order $(p-1)p$, where $p = 2^n + 1$ is a Fermat prime.*

Proof. Suppose, first, that r is a prime and G is a $CD1$-group with $a_k = r^s$ for some positive integer s. If $r = p$, then (1) holds by Proposition 3.3.

So suppose that $r \neq p$. Then $p \nmid a_k$ and, by Proposition 3.2, G is a doubly transitive Frobenius group of order $(p^n-1)p^n$. By Proposition 3.4, $a_k = |b^G| = p^n - 1$ and since $a_k = r^s$, it follows that

$$p^n - r^s = 1.$$

Since p and r are primes, [10, Lemma 19.3] implies that one of the following three cases holds:

(i) $s = 1$, $p = 2$ and $r = 2^n - 1$ is a Mersenne prime;

(ii) $s = 3$, $r = 2$, $p = 3$ and $n = 2$;

(iii) $r = 2$, $n = 1$ and $p = 2^s + 1$ is a Fermat prime.

If (i) holds, then $p = 2$, $a_k = r = 2^n - 1$ is a Mersenne prime and G is a doubly transitive Frobenius group of order $(2^n - 1)2^n = r(r + 1)$, as claimed in (2).

If (ii) holds, then $r = 2$, $a_k = 2^3$, $p = 3$, $n = 2$ and G is a doubly transitive Frobenius group of order $(3^2 - 1)3^2 = 72$, as claimed in (3).

Finally, if (iii) holds, then $r = 2$, $n = 1$, $p = 2^s + 1$ is a Fermat prime and G is a doubly transitive Frobenius group of order $(p - 1)p$, as claimed in (4).

Conversely, suppose that G is a group for which one of the cases (1), (2), (3) or (4) holds. If case (1) holds, then by Proposition 3.3 G is a $CD1$-group with $p = 2$ and $a_k = 2^s$ for some positive integer s, as required.

In the other three cases, G is a doubly transitive Frobenius group of order $(p^n - 1)p^n$, where p is a prime and $p^n - 1 = r^s$ for some prime r and some positive integer s. Thus it follows by Proposition 3.4 that that G is a $DC1$-group and $a_k = p^n - 1 = r^s$, as required.

The proof of the theorem is complete. □

The next corollary of Theorem 5.1 is the last result of this paper. We shall denote by S_3, D_8 and Q_8 the following groups: the symmetric group on 3 letters, the dihedral group of order 8 and the quaternion group of order 8, respectively.

Corollary 5.2. *The group G is a $CD1$-group with*

$$a_k = r$$

for some prime r if and only if either $r = 2$ and G is isomorphic to one of the groups: S_3, D_8 and Q_8, or $r = 2^n - 1$ is a Mersenne prime and G is a doubly transitive Frobenius group of order $(2^n - 1)2^n$.

Proof. We need to determine all groups with $s = 1$ in Theorem 5.1.

In case (1) of Theorem 5.1, $r = 2$ and the assumption that $s = 1$ implies that G is a 2-group of central type of order 8. Hence G is either D_8 or Q_8.

In case (2), it follows by Proposition 3.4 that $a_k = 2^n - 1 = r$, a Mersenne prime. Therefore all groups in (2) satisfy our assumption.

In case (3), it follows by Proposition 3.4 that $a_k = 3^2 - 1 = 8$, so G does not satisfy our assumption.

Finally, in case (4), it follows by Proposition 3.4 that $a_k = p - 1 = 2^n$. Therefore $r = 2$ and $s = 1$ implies that $n = 1$. Hence $p = 3$ is a Fermat prime and G is a Frobenius group of order 6. Thus G is S_3.

The proof of the corollary is complete. □

References

[1] Ya. Berkovich, D. Chillag and M. Herzog, Finite groups in which the degrees of the nonlinear irreducible characters are distinct, *Proc. Amer. Math. Soc.*, **115** (1992) 955–959.

[2] Ya. Berkovich, I. M. Isaacs and L. Kazarin, Groups with distinct monolithic character degrees, *J. Algebra*, **216** (1999) 448–480.

[3] S. Dolfi, G. Navarro and P. H. Tiep, Finite groups whose same degree characters are Galois conjugate, *Israel J. Math.*, **198** (2013) 283–331.

[4] S. Dolfi and M. K. Yadav, Finite groups whose non-linear irreducible characters of the same degree are Galois conjugate, *J. Algebra*, **452** (2016) 1–16.

[5] W. Feit, *Characters of finite groups*, W. A. Benjamin Inc., New York-Amsterdam, 1967.

[6] S. S. Gagola, Characters vanishing on all but two conjugacy classes, *Pacific J. Math.*, **109** (1983) 363–385.

[7] M. Hall, *The theory of groups*, AMS Chelsea Publishing, Providence, Rhode Island, 1999.

[8] B. Huppert, *Character theory of finite groups*, Walter de Gruyter, Berlin, 1998.

[9] M. Loukaki, On distinct character degrees, *Israel J. Math.*, **159** (2007) 93–107.

[10] D. Passman, *Permutation groups*, W. A. Benjamin Inc., New York-Amsterdam, 1968.

[11] G. Seitz, Finite groups having only one irreducible representation of degree greater than one, *Proc. Amer. Math. Soc.*, **19** (1968) 459–461.

[12] H. Zassenhaus, Kennzeichnung endlicher linear Gruppen als Permutationsgruppen, *Hamburg Abh.*, **11** (1936) 17–40.

Mariagrazia Bianchi

Dipartimento di Matematica F. Enriques, Università degli Studi di Milano, via Saldini 50, 20133 Milano, Italy

Email: mariagrazia.bianchi@unimi.it

Marcel Herzog

Email: herzogm@post.tau.ac.il