TILINGS OF NORMED SPACES

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ABSTRACT. By a *tiling* of a topological linear space X we mean a covering of X by at least two closed convex sets, called *tiles*, whose nonempty interiors are pairwise disjoint. Study of tilings of infinite-dimensional spaces initiated in 1980's with pioneer papers by V. Klee. We prove some general properties of tilings of locally convex spaces, and then apply these results to study existence of tilings of normed and Banach spaces by tiles possessing certain smoothness or rotundity properties. For a Banach space X, our main results are the following.

(i) X admits no tiling by Fréchet smooth bounded tiles.

(ii) If X is locally uniformly rotund (LUR), it does not admit any tiling by balls.

(iii) On the other hand, some $\ell_1(\Gamma)$ spaces, Γ uncountable, do admit a tiling by pairwise disjoint LUR bounded tiles.

1. INTRODUCTION

Let X be a real topological vector space (t.v.s.). A set $B \subset X$ will be called a *body* if it is closed, convex and has nonempty interior. A *tiling* of X is a covering of X by at least two bodies whose interiors are pairwise disjoint. Members of a tiling are called *tiles*.

While there exists a wide theory of tilings (and their analogues with not necessarily convex "tiles") of finite dimensional spaces, especially the plane, the infinite dimensional theory is still much less developed, though the first "infinite dimensional" results appeared already in 1980's. The present paper is devoted to the question about existence, in infinite dimensional normed or Banach spaces, of tilings having some particular properties. Let us briefly recall main known results.

It is easy to see that *some* tilings always exist in any locally convex t.v.s. X. Indeed, if f is a nonzero continuous linear functional on X and $\mathcal{T}$ is a tiling of $\mathbb{R}$ then the family $\mathcal{B} = \{f^{-1}(T) : T \in \mathcal{T}\}$ is a tiling of X. Each tiling of this form will be called a *trivial tiling*.

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Notice that no countable tiling can be disjoint (i.e., consisting of pairwise disjoint tiles) since otherwise a line in X would admit a disjoint covering by at most countably many and at least two nonempty closed convex sets, which is known to be impossible (see Fact 3.2 or [3, Theorem 6.1.27]). It follows that *a separable normed space does not admit any disjoint tiling.* On the other hand, *some (nonseparable) $\ell_1(\Gamma)$ does admit a disjoint tiling by balls of the same positive radius* (see V. Klee [8]).

Tilings by bounded (convex) tiles always exist in normed spaces. In 1997, V.P. Fonf, A. Pezzotta and C. Zanco [6] proved that *every normed space X admits a tiling by bounded bodies each of which contains a ball of a certain fixed positive radius.* Subsequently, A. Marchese and C. Zanco [12] showed that *each normed space X admits a tiling by bounded bodies such that each point of X belongs to at most two of them.* Since this tiling is "point-finite", a natural question arises: can such a tiling be always taken locally finite, in the sense that each point of X has a neighborhood intersecting only finitely many tiles? An answer is contained in a paper by Fonf [5] who showed that *a separable Banach space admits a locally finite tiling by bounded bodies if and only if X is isomorphically polyhedral.* For nonseparable Banach spaces X it is only known that *if X contains an infinite dimensional closed subspace non containing c_0 then X does not admit any locally finite tiling by bounded bodies*; this follows from a generalization, due to Fonf and Zanco [7], of a theorem of Corson (1961).

A tiling $\mathcal{B}$ is said to be *equi-bounded below and above* if there exist two constants $0 < r < R < \infty$ such that each member of $\mathcal{B}$ contains a ball of radius r and is contained in a ball of radius R. Already in 1986, V. Klee [9] showed that *some nonseparable Hilbert spaces admit tilings that are equi-bounded below and above.* The case of the separable (infinite dimensional) Hilbert space was settled only in 2010 by D. Preiss [13] in the positive: *ℓ_2 admits a tiling which is equi-bounded below and above.* An interesting open problem is: which Banach spaces admit such tilings?

Our interest is devoted to the question of existence of (nontrivial) tilings by bodies that have a kind of smoothness or rotundity properties. The first such result, due to Klee and Tricot [11], affirms that a tiling of a Banach space by bounded tiles each of which is either smooth or rotund has to be uncountable. In particular, *a separable Banach space admits no tiling by smooth or rotund bounded bodies.* The only related known fact for nonseparable Banach spaces follows from results in [10] and says: *if X is a uniformly rotund or uniformly smooth Banach space then it does not admit any tiling by balls which would be either disjoint or bounded below* (i.e., having radii bounded away from zero).

In the present paper, after proving some general results on tilings in locally convex t.v.s. (Section 3), we show that *Banach spaces do not admit nontrivial tilings by Fréchet smooth bodies,* and *normed spaces do not admit disjoint*

tilings by Fréchet smooth bodies (see Theorem 4.2 and Corollary 4.3). Concerning existence of tilings by locally uniformly rotund (LUR) bodies, we show on the one hand that *a LUR Banach space does not admit any tiling by balls, and a LUR normed space does not admit any disjoint tiling by balls;* on the other hand, *some nonseparable $\ell_1(\Gamma)$ admits a disjoint tiling by bounded centrally symmetric LUR bodies* (see Section 5). This last construction is based on the above mentioned Klee's construction in [8]. We also show that some geometric conditions, concerning a single point of the unit sphere of a Banach space, imply nonexistence of tilings by balls whose radii are bounded away from zero; see Subsection 4.3.

2. Notations and preliminaries

Throughout this paper, all normed, Banach or topological vector spaces are real and at least two-dimensional.

Let X be a normed space with dual X^*. By B_X and S_X we denote the closed unit ball and the unit sphere of X, respectively. The polar set of a set $A \subset X$ is the set $A^\circ = \{f \in X^* : f(a) \leq 1 \text{ for each } a \in A\}$.

Let $B \subset X$ be a body, that is, a closed convex set with nonempty interior. A point $x \in \partial B$ is a *smooth point* of B if the set

$$D_B(x) = \{f \in S_{X^*} : f(x) = \sup f(B)\}$$

is a singleton. If $c \in \text{int } B$, $B_0 = B - c$ and $x_0 = x - c$, it is standard to see (and well-known for balls) that x is a smooth point of B if and only if the Minkowski functional p_{B_0} of B_0 is Gâteaux differentiable at x_0. We say that x is a *Fréchet smooth point* of B if p_{B_0} is Fréchet differentiable at x_0. (It is standard to see that this definition does not depend on the choice of c.)

We say that $x \in \partial B$ is a *LUR (locally uniformly rotund) point* of B if for each $\varepsilon > 0$ there exists $\delta > 0$ such that if $y \in \partial B$ and $\text{dist}(\partial B, (x + y)/2) < \delta$ then $\|x - y\| < \varepsilon$. If $B = B_X$, this definition coincides with the standard definition of local uniform rotundity of the norm at x.

We say that a body B is *smooth*, *Fréchet smooth*, or *LUR* if each boundary point of B is a smooth, Fréchet smooth, or LUR point of B, respectively.

For $f \in S_{X^*}$ and $\alpha \in [0, 1)$, we consider the closed convex cone

$$\text{C}(\alpha, f) = \{x \in X : f(x) \geq \alpha\|x\|\}.$$

Notice that $\text{C}(\alpha, f)$ has nonempty interior.

Observation 2.1. *Let $B \subset X$ be a body, $x \in \partial B$. If x is a Fréchet smooth point of B then there exists $f \in S_{X^*}$ with the following property: for each $\alpha > 0$ there exists $\varepsilon > 0$ such that*

(1) $$[x - \text{C}(\alpha, f)] \cap [x + \varepsilon B_X] \subset B\,.$$

(Also the converse is true, but we shall need only the above implication.)

Proof. Assume without any loss of generality that $0 \in \text{int } B$. Let g be the Fréchet derivative of p_B at x, and $f := \frac{g}{\|g\|}$. Then $p_B(x) = 1$ and $\lim_{\|h\| \to 0} \frac{p_B(x+h) - p_B(x) - g(h)}{\|h\|} = 0$. Consequently, for each $\alpha > 0$ there exists $\varepsilon > 0$ such that $p_B(x+h) - 1 - \|g\|f(h) \le \alpha\|g\|\|h\|$ whenever $h \in \varepsilon B_X$. Thus, for $h \in [-\text{C}(\alpha, f)] \cap \varepsilon B_X$, we obtain $p_B(x+h) \le 1 + \|g\|f(h) + \alpha\|g\|\|h\| \le 1$, and hence $x + h \in B$. This completes the proof. $\qquad\square$

Definition 2.2. Let $B \subset X$ be a body, $x \in \partial B$ and $\varepsilon > 0$. We say that x is an *ε-cone smooth* point of B if there exists $f \in S_{X^*}$ such that

$$[x - \text{C}(\tfrac{1}{7}, f)] \cap [x + \varepsilon B_X] \subset B\,,$$

that is, (1) holds for $\alpha = 1/7$. We denote by $\varepsilon\text{-CS}(B)$ the set of all *ε-cone smooth* points of B. We say that $x \in \partial B$ is a *cone smooth* point of B if it is an ε-cone smooth point of B for some $\varepsilon > 0$. Moreover, we say that B is *cone smooth* if each $x \in \partial B$ is a cone smooth point of B.

Lemma 2.3. *Let $f, g \in S_{X^*}$, $\alpha, \beta, \delta > 0$, and $y \in \delta\beta B_X$. Suppose that*

$$\alpha + \beta + \alpha\beta < \tfrac{1}{2} \quad \text{and} \quad \inf g\big(y + [\text{C}(\alpha, f) \cap \delta(1 + \beta)B_X]\big) \ge 0\,.$$

Then $\|g - f\| \le 2(\alpha + \beta + \alpha\beta)$.

Proof. By homogeneity, we can (and do) suppose that $\delta = 1$. Put $\gamma = \alpha + \beta + \alpha\beta$. For each $x \in \text{C}(\gamma, f) \cap S_X$, we have $\|x - y\| \le 1 + \beta$ and $f(x - y) \ge \gamma - \beta = \alpha(1 + \beta) \ge \alpha\|x - y\|$. This shows that $\text{C}(\gamma, f) \cap S_X$ is contained in $y + [\text{C}(\alpha, f) \cap (1 + \beta)B_X]$. Hence by our assumption, $\inf g(\text{C}(\gamma, f) \cap S_X) \ge 0$. Since $\text{C}(\gamma, f)$ is a cone, we obtain that $\inf g(\text{C}(\gamma, f)) = 0$. Now, [2, Lemma 1.1] (based on the "Parallel Hyperplane Lemma") implies that $\|g - f\| \le 2\gamma$, which completes the proof. $\qquad\square$

3. General results on tilings

In the present section, if not specified otherwise, X denotes a real topological vector space (*t.v.s.*, for short).

Given a body $B \subset X$, a point $x \in \partial B$ is called a *conical* [resp. *flat*] point of B if the set (in fact, a convex cone) $\bigcup_{t>0} t(B - x)$ is closed [resp. is a closed halfspace]. The body B is *plump* [resp. *flat*] if each of its conical [resp. boundary] points is flat.

Let us specify that the terms *nonconical point* and *nonflat point* refer only to boundary points of the body. By the way, these two notions are equivalent for plump bodies.

Recall that a *hereditarily Baire* space is a topological space whose all non-empty closed subspaces are Baire spaces.

Let x be a boundary point of a body B. The following simple facts hold.

(α) x is a flat point of B if and only if there exists a (necessarily unique) supporting hyperplane H to B at x such that $x \in$ a-int$_H(B \cap H)$, where a-int$_H(B \cap H)$ is the relative algebraic interior (or "core") of $B \cap H$ in H.

(β) If X is hereditarily Baire then in (α) we can write int$_H(B \cap H)$, the topological relative interior of $B \cap H$ in H, instead of a-int$_H(B \cap H)$.

(γ) Hence if x is a flat point of B then x is a smooth point (Fréchet smooth point if X is a Banach space) of B.

(δ) Each smooth conical point is a flat point (see also [11, 1.3]); and, in normed spaces, each LUR point is a nonconical point. Hence if each boundary point of B is either a smooth or a LUR point then B is plump.

As usual, by "*arbitrarily near to x*" we mean "in any neighborhood of x". The following lemma is based on [11, 2.1].

Lemma 3.1. *Let x be a common boundary point of two bodies $B, B' \subset X$ whose interiors are disjoint.*

(i) *If x is a nonconical point of B, then arbitrarily near to x there exists a nonflat point of B that does not belong to B'.*

(ii) *If X is hereditarily Baire and x is a flat point of both B, B', then x belongs to $A := \text{int}(B \cup B')$ and all boundary points of B or B' that belong to A are flat.*

Proof. (i) Let H be a closed hyperplane through x that separates B and B'. By 2.1 in [11], arbitrarlily near to x there exist nonflat points of B that do not belong to H. This shows (i). Part (ii) follows easily from the observation (β) above. $\square$

A *tiling* of X is a covering of X by at least two bodies whose interiors are pairwise disjoint. A *disjoint tiling* is a tiling whose members are pairwise disjoint. A point $x \in X$ is a *singular point* for a tiling $\mathcal{B}$ if each neighborhood of x meets infinitely many members of $\mathcal{B}$. The points of $\bigcup_{B \in \mathcal{B}} \partial B$ that are not singular are called *regular* points.

A tiling $\mathcal{B}$ is *plump* if each of its members is plump. *Smooth* tiling, *Fréchet smooth* tiling, *LUR* tiling, etc. are defined in an analogous way. We say that $\mathcal{B}$ *has no common flat points* if no point of X is a flat point of two distinct members of $\mathcal{B}$. A *trivial tiling* has been defined in Introduction.

The following simple fact is well known; we sketch a proof of it for sake of completeness.

Fact 3.2. *Each disjoint covering of $\mathbb{R}$ by at least two nonempty closed convex subsets of $\mathbb{R}$ is uncountable. In particular, $\mathbb{R}$ admits no disjoint tiling.*

Proof. Suppose that $\mathcal{B}$ is such a covering and that $\mathcal{B}$ is countable. Then the set $S := \bigcup_{I \in \mathcal{B}} \partial I$ is a countable closed (nonempty) set. It is easy to see that S has no isolated points. On the other hand, a direct application of the Baire category theorem to the complete metric space $S = \bigcup_{s \in S} \{s\}$ gives existence of an isolated point of S. This contradiction completes the proof of the first part. The second part holds by separability. $\qquad\square$

Lemma 3.3. *Let $U \subset X$ be an open convex set, and $B \subset X$ a body whose boundary intersects U. Assume that each point of $\partial B \cap U$ is a flat point of B. Then each connected component of $\partial B \cap U$ is of the form $H \cap U$ where $H \subset X$ is a closed hyperplane.*

Proof. For $x \in \partial B \cap U$, let H_x be the unique supporting hyperplane to B at x. We claim that $H_x \cap U \subset \partial B$. Indeed, if this is not the case, there exists $y \in (H_x \cap U) \setminus B$; then $[x, y] \cap B = [x, z]$ for some $z \in (x, y) \subset H_x \cap U$, but this is impossible since clearly z is a non-flat point of B.

Assume without any loss of generality that $0 \in \operatorname{int} B$, and let us consider the function $\theta : \partial B \cap U \to X^*$ such that, for $x \in \partial B \cap U$, $\theta(x)$ is the unique functional of X^* such that

$$[\theta(x)](x) = 1 = \sup[\theta(x)](B).$$

By the claim above, the map θ is locally constant, and hence it is constant on each connected component of $\partial B \cap U$. $\qquad\square$

Lemma 3.4. *Let $U \subset X$ be an open convex set. Let $E_n \subset X$, $n \in \mathbb{N}$, be closed halfspaces intersecting U, such that $E_n \cap U \subset E_{n+1} \cap U$ for each n. Then either $C := \bigcup_{n \in \mathbb{N}} (E_n \cap U)$ contains U, or the set $\partial C \cap U$ is of the form $H \cap U$ where $H \subset X$ is a closed hyperplane.*

Proof. Suppose that $U \setminus C \neq \emptyset$. By our monotonicity assumption, the set C is convex. Notice that also the set $U \setminus C = \bigcap_{n \in \mathbb{N}} (U \setminus E_n)$ is convex. Since $\operatorname{int}(C \cap U) \neq \emptyset$, there exists a closed hyperplane $H \subset X$ that separates $C \cap U$ and $U \setminus C$. Since the union of the latter two sets is U and they belong to opposite halfspaces determined by H, either of the two sets must contain the relative interior in U of one of the two opposite halfspaces. This easily implies the assertion. $\qquad\square$

For simplicity of formulation, if E, F, U are sets and $E \cap U = F \cap U$, we shall say that *E coincides with F in U*.

Theorem 3.5. *Let $\mathcal{B}$ be a tiling of a locally convex metrizable t.v.s. X, B a member of $\mathcal{B}$, and $x \in \partial B$ a non-flat point of B. Then at least one of the following three conditions is satisfied.*

(a) *x belongs to at least three different members of $\mathcal{B}$.*

   (b) *x belongs to two different members B, B' of $\mathcal{B}$ and, in some neighborhood of x, $B \cup B'$ coincides with a closed halfspace.*

   (c) *Arbitrarily near to x there exist non-flat boundary points of members of $\mathcal{B}$ non containing x.*

Proof. Assume the contrary, that is, all (a),(b),(c) are false. In particular, $\mathcal{B}_x := \{C \in \mathcal{B} : x \in C\}$ contains at most two elements (one of which is B), and there exists an open convex neighborhood U of x in which the only non-flat boundary points of members of $\mathcal{B}$ belong to some element of $\mathcal{B}_x$.

Now we claim that arbitrarily near to x there are points of members of $\mathcal{B} \setminus \mathcal{B}_x$. This is clear if $\mathcal{B}_x = \{B\}$. Let $\mathcal{B}_x$ contain also some $B' \neq B$. If our claim is false, $B \cup B'$ contains a neighborhood V of x. But this easily implies that, in such a neighborhood V, the set B (and B', too) coincides with a closed halfspace, contradicting the assumption that x is a non-flat point of B.

Now, observe that if $C \in \mathcal{B} \setminus \mathcal{B}_x$ then, by Lemma 3.3, each component of $\partial C \cap U$ coincides in U with a closed hyperplane.

Fix a sequence $\{C_n\} \subset \mathcal{B} \setminus \mathcal{B}_x$ such that the distances $d(x, C_n)$ tend decreasingly to 0. By the above observation, for each $n \in \mathbb{N}$, there exists a closed hyperplane H_n such that $H_n \cap U \subset \partial C_n$, and H_n separates $C_n \cap U$ and $C_{n+1} \cap U$, as well as $C_n \cap U$ and $(\bigcup \mathcal{B}_x) \cap U$. Let E_n be the closed halfspace such that $\partial E_n = H_n$ and $C_n \cap U \subset E_n$. Since the sets $E_n \cap U$ form an increasing sequence which does not cover U, Lemma 3.4 implies that the relative boundary of $\bigcup_{n \in \mathbb{N}} (E_n \cap U)$ in U is of the form $H \cap U$ where H is a closed hyperplane. Clearly, H is a supporting hyperplane to B at x, and $(\bigcup \mathcal{B}_x) \cap U$ is contained in one of the two closed halfspaces, say G, determined by H.

If $\mathcal{B}_x = \{B\}$ then, in any neighborhood of x, B does not coincide with G, since x is a non-flat point. If $\mathcal{B}_x = \{B, B'\}$ (with $B' \neq B$) then, in any neighborhood of x, $B \cup B'$ does not coincide with G, since (b) is false. This means that, arbitrarily near to x, there exists a boundary point y of some $D \in \mathcal{B} \setminus \mathcal{B}_x$ such that $y \in \operatorname{int} G$. Notice that, for each such D, $D \cap U$ is contained in G since it is disjoint from each $C_n \cap U$ ($n \in \mathbb{N}$).

Proceeding as above, we can find a sequence $\{D_n\} \subset \mathcal{B} \setminus \mathcal{B}_x$ and closed halfspaces F_n, $n \in \mathbb{N}$, such that:

- $\partial F_n \cap U \subset \partial D_n$;
- ∂F_n separates $D_n \cap U$ and $D_{n+1} \cap U$, as well as $D_n \cap U$ and $(\bigcup \mathcal{B}_x) \cap U$;
- $D_n \cap U \subset F_n$;
- the sequence $\{F_n \cap U\}$ is increasing and contained in G.

Now, Lemma 3.4 again implies that the relative boundary of $\bigcup_{n \in \mathbb{N}} (F_n \cap U)$ in U is of the form $H' \cap U$ where H' is a closed hyperplane through x. Moreover, $H' \cap U \subset G$. But $x \in \partial G \cap H' \cap U$ easily implies that $H' = \partial G \ (= H)$. So we obtain that $B \cap U \subset G$ and also $B \cap U \subset \bigcap_{n \in \mathbb{N}} (U \setminus \overline{F_n}) \subset \overline{X \setminus G}$,

and consequently $B \cap U \subset H$, which is impossible since B is convex and has nonempty interior. This contradiction completes the proof. $\square$

Definition 3.6 ([1])**.** Let B be a body in a t.v.s. X. We say that x is a *QP-point* of B iff there exists a neighborhood V of x such that $[x, y] \subset \partial B$ whenever $y \in V \cap \partial B$.

We shall need the following simple lemma. (By the way, it is strictly connected with [11, Theorem 5.1] and with the easy implication in the following result of V.P. Fonf [5]: *a separable Banach space admits a locally finite tiling by bounded bodies if and only if it is isomorphically polyhedral.*)

Lemma 3.7. *Let $\mathcal{B}$ be a tiling of a t.v.s. X, and let $x \in \bigcup_{B \in \mathcal{B}} \partial B$ be a regular (i.e., non-singular) point for $\mathcal{B}$. Then x is a QP-point of each member of $\mathcal{B}$ that contains x. In particular, if x is contained in only two members of $\mathcal{B}$ then x is a flat QP-point for both of them.*

Proof. Assume that $x = 0$ and denote $\mathcal{B}_0 = \{B \in \mathcal{B} : 0 \in B\}$. Since $\mathcal{B}_0$ is finite, it is easy to see that $0 \in \mathrm{int}(\bigcup \mathcal{B}_0)$ (see also [9, Theorem 1.1]). Let $V \subset X$ be a balanced (i.e., starshaped w.r.t. 0) open neighborhood of 0 such that $V \subset \mathrm{int}(\bigcup \mathcal{B}_0)$. Notice that the sets $B \cap V$ ($B \in \mathcal{B}_0$) form a finite "tiling" of V. Now, fix arbitrary $B \in \mathcal{B}$ and $y \in \partial B \cap V$. Then necessarily $y \in B'$ for some $B' \in \mathcal{B}_0 \setminus \{B\}$, and hence $[0, y] \subset B \cap B' \subset \partial B$. This shows that 0 is a QP-point of B. Finally, if $\mathcal{B}_0 = \{B, B'\}$, let V be as above and let H be a closed hyperplane through x that separates B and B'. Notice that then necessarily $\partial B \cap V = \partial B' \cap V = H \cap V$. The proof is complete. $\square$

The following theorem is the main result of the present section. Given a set $E \subset X$ and a neighborhood V of the origin, we write

$$\mathrm{diam}(E) \prec V$$

to say that $x - y \in V$ whenever $x, y \in E$. Thus if X is a normed space then $\mathrm{diam}(E) \prec \varepsilon B_X$ is the same as $\mathrm{diam}(E) \leq \varepsilon$.

Theorem 3.8. *Let $\mathcal{B}$ be a tiling of a locally convex t.v.s. X. Assume that at least one of the following conditions is satisfied:*

   (i) *$\mathcal{B}$ is disjoint;*
   (ii) *X is completely metrizable, $\mathcal{B}$ is plump and nontrivial;*
   (iii) *X is completely metrizable, $\mathcal{B}$ has no common flat points.*

Then $\mathcal{B}$ is uncountable. Moreover, if

$$\mathcal{P} := \{(x, B) :\ B \in \mathcal{B},\ x \in \partial B\} = \bigcup_{n \in \mathbb{N}} \mathcal{P}_n \quad where \quad \mathcal{P}_1 \subset \mathcal{P}_2 \subset \dots ,$$

then there exists $m \in \mathbb{N}$ with the following property:

$(*)$ *for each neighborhood V of 0 there exist three couples $(x_i, B_i) \in \mathcal{P}_m$, $i = 1, 2, 3$, such that the bodies B_i are pairwise distinct and* $\operatorname{diam}\{x_1, x_2, x_3\} \prec V$.

Proof. (i) Let $\mathcal{B}$ be a disjoint tiling of X. Let $Y \subset X$ be a straight line intersecting at least two members of $\mathcal{B}$, and let $\mathcal{B}' = \{B \cap Y : B \in \mathcal{B}, B \cap Y \neq \emptyset\}$. Since $\mathcal{B}'$ is a disjoint cover of Y by nonempty closed convex sets, $\mathcal{B}'$ (and hence also $\mathcal{B}$) is uncountable by Fact 3.2. Since Y is separable, the family $\mathcal{B}'' = \{B \in \mathcal{B}' : \operatorname{int}_Y B \neq \emptyset\}$ is at most countable. Hence $\mathcal{C} := \{B \in \mathcal{B} : \#(B \cap Y) = 1\}$ is uncountable. For $n \in \mathbb{N}$, let $\mathcal{C}_n$ be the collection of all $B \in \mathcal{C}$ such that $B \cap Y = \{x\}$ and $(x, B) \in \mathcal{P}_n$. Since $\mathcal{C} = \bigcup_n \mathcal{C}_n$, there exists $m \in \mathbb{N}$ such that $\mathcal{C}_m$ is uncountable. The set $M := \bigcup_{B \in \mathcal{C}_m}(B \cap Y)$ is uncountable, and hence it has an accumulation point $w \in Y$. Then arbitrarily near to w we can find three distinct points $x_1, x_2, x_3 \in M$. Let $B_1, B_2, B_3 \in \mathcal{C}_m$ be such that $x_i \in \partial B_i$ $(i = 1, 2, 3)$. Then B_1, B_2, B_3 are pairwise distinct, $(x_i, B_i) \in \mathcal{P}_m$ for $i = 1, 2, 3$, and $\operatorname{diam}\{x_1, x_2, x_3\}$ "can be taken arbitrarily small". Thus (i) is proved.

(ii) Let X be completely metrizable and let $\mathcal{B}$ be a nontrivial plump tiling of X. Let us use the following notations.

$$
\begin{aligned}
&\mathcal{B}_x := \{B \in \mathcal{B} : x \in B\} \qquad (x \in X), \\
(2) \quad &\eta B := \{x \in \partial B : \ x \text{ is a nonflat point of } B\} \qquad (B \in \mathcal{B}), \\
&\eta\mathcal{B} := \textstyle\bigcup_{B \in \mathcal{B}} \eta B, \qquad S := \textstyle\bigcup_{B \in \mathcal{B}} \partial B.
\end{aligned}
$$

Notice that S is closed in X. If $\eta\mathcal{B} = \emptyset$ then each element of $\mathcal{B}$ is flat; in this case, Lemma 3.3 (with $U = X$) easily implies that $\mathcal{B}$ is a trivial tiling. Hence we have that $\eta\mathcal{B} \neq \emptyset$, and the closure $\overline{\eta\mathcal{B}}$ is a Baire space.

Suppose that $\mathcal{B}$ is countable. Then $\overline{\eta\mathcal{B}} = \bigcup_{B \in \mathcal{B}}(\overline{\eta\mathcal{B}} \cap \partial B)$, and hence there exist $B_0 \in \mathcal{B}$, $x \in \overline{\eta\mathcal{B}} \cap \partial B_0$ and a convex open neighborhood U of x such that $\overline{\eta\mathcal{B}} \cap U \subset \partial B_0$. Clearly, x cannot be a flat point of B_0; indeed, otherwise we can suppose that, in U, the body B_0 coincides with a closed halfspace, and then $\overline{\eta\mathcal{B}} \cap U = \emptyset$, which is impossible. Thus $x \in \eta B_0$. In this case, one of the conditions (a),(b) in Theorem 3.5 must hold. It is easy to see that this implies existence of some $B_1 \in \mathcal{B}$ such that $x \in \eta B_1$. Notice that x is a nonconical point of B_1 since B_1 is plump. By Lemma 3.1(i), there exists a nonflat point of B_1 contained in $U \setminus B_0$. But this contradicts the choice of x. We have proved that $\mathcal{B}$ is uncountable.

Now, let us show the second part of (ii). If some $x \in S$ belongs to three distinct $B_1, B_2, B_3 \in \mathcal{B}$, there exists $m \in \mathbb{N}$ such that $(x, B_i) \in \mathcal{P}_m$ for all $i \in \{1, 2, 3\}$, and then $(*)$ trivially holds with $x_1 = x_2 = x_3 = x$.

So, let us suppose that each $x \in S$ belongs to at most two members of $\mathcal{B}$. Then $\overline{\eta \mathcal{B}} = \bigcup_{n \in \mathbb{N}} A_n$ where

$$A_n = \{x \in \overline{\eta \mathcal{B}} : \ (x, B) \in \mathcal{P}_n \text{ whenever } B \in \mathcal{B}_x\}.$$

Since $\overline{\eta \mathcal{B}}$ is a Baire space, there is $m \in \mathbb{N}$ such that $\overline{A_m}$ has nonempty relative interior in $\overline{\eta \mathcal{B}}$. That is, there exist $x_1 \in A_m$ and a convex open neighborhood U of x_1 such that $U \cap \overline{\eta \mathcal{B}} \subset \overline{A_m}$. By Lemma 3.1(ii), x_1 cannot be a common flat point of two distinct elements of $\mathcal{B}_{x_1}$. Thus either x_1 is a nonflat point of some $B_1 \in \mathcal{B}_{x_1}$, or $\mathcal{B}_{x_1} = \{B_1\}$ and x_1 is a flat point of B_1.

Case 1: $x_1 \in \eta B_1$. Since we must have either (b) or (c) in Theorem 3.5, arbitrarily near to x_1 there exists a nonflat point x of some $B \in \mathcal{B} \setminus \{B_1\}$. If $x \in B_1$, we can apply Lemma 3.1(i) to find, arbitrarily near to x, a point $x' \in \eta B \setminus B_1$. So we can always suppose that $x \in U \setminus B_1$, and hence $x \in \overline{A_m}$. Arbitrarily near to x there exists $x_2 \in A_m \cap (U \setminus B_1)$.

Case 2: $\mathcal{B}_{x_1} = \{B_1\}$ and $x_1 \notin \eta B_1$. Since $x_1 \in \overline{\eta \mathcal{B}}$ and all points sufficiently near to x_1 are flat for B_1, arbitrarily near to x_1 we can find a nonflat point x of some $B \in \mathcal{B} \setminus \{B_1\}$. Proceeding as in Case 1, we get that arbitrarily near to x there exists $x_2 \in A_m \cap (U \setminus B_1)$.

In both Case 1 and 2, fix $B_2 \in \mathcal{B}_{x_2}$. Now, considering an open convex neighborhood U' of x_2 such that $U' \subset U \setminus B_1$, we can proceed as above to find, arbitrarily near to x_2, a point $x_3 \in A_m \cap (U' \setminus B_2)$. Fix $B_3 \in \mathcal{B}_{x_3}$ and notice that the bodies B_1, B_2, B_3 are pairwise distinct, $(x_i, B_i) \in \mathcal{P}_m$ for $i = 1, 2, 3$, and $\operatorname{diam}\{x_1, x_2, x_3\}$ "can be taken arbitrarily small". This completes the proof of part (ii).

(iii) Let X be completely metrizable and let $\mathcal{B}$ be a tiling that has no common flat points. We shall use the above notations (2). Assume that $\mathcal{B}$ is countable. Then $S = \bigcup_{B \in \mathcal{B}} \partial B$ is a Baire space (since it is closed), and hence there exists $B_0 \in \mathcal{B}$ for which ∂B_0 has nonempty relative interior in S. That is, there exist $x \in \partial B_0$ and a convex open neighborhood U of x such that $U \cap S \subset \partial B_0$. We can (and do) suppose that $U \cap \partial B_0$ is connected, and so $U \setminus \partial B_0$ has two components. Hence there exists $B_1 \in \mathcal{B} \setminus \{B_0\}$ such that $U \subset B_0 \cup B_1$. It follows that $U \cap \partial B_0 = U \cap \partial B_1$ is contained in a closed hyperplane and hence x is a common flat point of B_0 and B_1. This contradiction shows that $\mathcal{B}$ must be uncountable.

Now, as in (ii), if some $x \in S$ belongs to three distinct members of $\mathcal{B}$, we are immediately done. So, let us assume that $\#\mathcal{B}_x \leq 2$ for each $x \in S$. For $n \in \mathbb{N}$, denote $A_n = \{x \in S : \ (x, B) \in \mathcal{P}_n \text{ whenever } B \in \mathcal{B}_x\}$, and notice that $S = \bigcup_{n \in \mathbb{N}} A_n$. Hence, for some $m \in \mathbb{N}$, $\overline{A_m}$ has nonempty relative interior in S, that is, there exist $x_1 \in A_m$ and a convex open neighborhood U of x_1 such that $U \cap S \subset \overline{A_m}$. We claim that x_1 is necessarily a singular point of $\mathcal{B}$. Indeed, if not, we can apply Lemma 3.7 to obtain that $\mathcal{B}_{x_1} = \{B_0, B_1\}$ with $B_0 \neq B_1$, and x_1 is a common flat point of B_0, B_1, which is impossible.

Thus x_1 is a singular point of $\mathcal{B}$. Fix $B_1 \in \mathcal{B}_{x_1}$. Then arbitrarily near to x_1 there exists $x \in \partial B \cap U$ with $B \in \mathcal{B} \setminus \{B_1\}$; and arbitrarily near to x there exists $x_2 \in A_m \setminus B_1$. Fix $B_2 \in \mathcal{B}_{x_2}$ and notice that $B_2 \neq B_1$. Now, arbitrarily near to x_1 find $x' \in \partial B'$ with $x' \in \mathcal{B} \setminus \{B_1, B_2\}$; and arbitrarily near to x' find $x_3 \in A_m \setminus (B_1 \cup B_2)$. Fix $B_3 \in \mathcal{B}_{x_3}$. Then B_1, B_2, B_3 are pairwise distinct, $(x_i, B_i) \in \mathcal{P}_m$ for $i = 1, 2, 3$, and $\operatorname{diam}\{x_1, x_2, x_3\}$ "can be taken arbitrarily small". This proves (iii), and we are done. $\qquad\square$

Corollary 3.9 (cf. [11]). *Let $\mathcal{B}$ be a countable plump tiling of a locally convex completely metrizable t.v.s. Then $\mathcal{B}$ is trivial.*

4. Applications to normed and Banach spaces

In what follows, X is a normed space.

4.1. **Smoothness-type prohibitive conditions.**
For the definition of ε-$CS(B)$ see Section 2.

Lemma 4.1. *There exists a constant $H \in (0, 1)$ such that if $\varepsilon > 0$, B_1, B_2, B_3 are convex bodies in X with pairwise disjoint interiors and $x_i \in \varepsilon\text{-CS}(B_i)$ $(i = 1, 2, 3)$, then $\operatorname{diam}(\{x_1, x_2, x_3\}) > H\varepsilon$.*

Proof. We claim that our lemma holds with $H = 1/51$. Put $\alpha = 1/7$ and $\beta = 1/50$, and notice that

$$\gamma := \alpha + \beta + \alpha\beta = \tfrac{29}{175} < \tfrac{1}{6}.$$

By homogeneity, we can (and do) suppose that $\varepsilon = 1 + \beta = 51/50$. Then $H\varepsilon = 1/50$.

Now, assume that the assertion is false, i.e., $\operatorname{diam}(\{x_1, x_2, x_3\}) \leq H\varepsilon = \beta$. Let $f_i \in S_{X^*}$ $(i = 1, 2, 3)$ be such that

$$K_i := x_i - [\mathrm{C}(\alpha, f_i) \cap (1 + \beta)B_X] \subset B_i \,.$$

Since the bodies K_1, K_2, K_3 have pairwise disjoint interiors, for each $i, j \in \{1, 2, 3\}$, $i < j$, there exists $g_{ij} \in S_{X^*}$ such that $\sup g_{ij}(-K_j) \leq \inf g_{ij}(-K_i)$, that is,

$$\inf g_{ij}\big((x_j - x_i) + [\mathrm{C}(\alpha, f_i) \cap (1 + \beta)B_X]\big) \geq \sup g_{ij}\big([\mathrm{C}(\alpha, f_j) \cap (1 + \beta)B_X]\big) \geq 0.$$

By Lemma 2.3, $\|g_{ij} - f_i\| \leq 2\gamma$. In a similar way we obtain $\|g_{ij} + f_j\| \leq 2\gamma$. But now, since $2f_1 = (f_1 - g_{12}) + (f_2 + g_{12}) - (f_2 - g_{23}) - (f_3 + g_{23}) + (f_3 + g_{13}) + (f_1 - g_{13})$, we obtain $2 = \|2f_1\| \leq 6 \cdot 2\gamma < 2$, a contradiction. $\qquad\square$

Theorem 4.2. *Let $\mathcal{B}$ be a tiling of a normed space X. Assume that at least one of the following conditions is satisfied:*

(i) *$\mathcal{B}$ is disjoint;*

(ii) *X is complete, $\mathcal{B}$ is plump and nontrivial;*

(iii) *X is complete, $\mathcal{B}$ has no common flat points.*

Then $\mathcal{B}$ is not cone smooth.

Proof. Let us suppose on the contrary that $\mathcal{B}$ is cone smooth. Let $\mathcal{P}$ be as in Theorem 3.8 and, for each $n \in \mathbb{N}$, let

$$\mathcal{P}_n := \{(x, B) : B \in \mathcal{B}, \ x \ \in (1/n)\text{-CS}(B)\}.$$

Then $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots$, moreover, since $\mathcal{B}$ is cone smooth, $\mathcal{P} = \bigcup_n \mathcal{P}_n$.

Now, let H be the constant from Lemma 4.1. By Theorem 3.8, there exist $m \in \mathbb{N}$ and three couples $(x_i, B_i) \in \mathcal{P}_m$, $i = 1, 2, 3$, such that the bodies B_i are pairwise distinct and $\text{diam}\{x_1, x_2, x_3\} < H/m$. But this contradicts Lemma 4.1. $\qquad\square$

Now, since Fréchet smooth bodies are cone smooth (see Section 2) and plump (see (δ) in Section 3), we immediately obtain the following

Corollary 4.3. *Let $\mathcal{B}$ be a Fréchet smooth tiling of a Banach space X. Then $\mathcal{B}$ is trivial. In particular, X does not admit any Fréchet smooth tiling by bounded bodies.*

4.2. **Rotundity-type prohibitive conditions for ball tilings.**

We shall need the following elementary observation. As usual, if $c, y \in X$ are two distinct points then (c, y) denotes the relatively open line segment with endpoints c, y, that is, $(c, y) = \{(1 - t)c + ty : 0 < t < 1\}$.

Observation 4.4. *Given $B = B(c, r)$, $y \in \partial B$, and $0 < \rho < r$, there exists $B' = B(c', \rho)$ such that $c' \in (c, y)$, $B' \subset B$ and $y \in \partial B'$.*

Lemma 4.5. *Let X be a normed space, $\alpha > 0$, $\varepsilon \geq 0$, and $B_0, B_1, B_2 \subset X$ three balls of radius at least α, whose interiors are pairwise disjoint. Consider three points $y_i \in \partial B_i$, $i = 0, 1, 2$, and denote $x_0 = \frac{y_0 - c_0}{\|y_0 - c_0\|}$ where c_0 is the center of B_0. If $\text{diam}\{y_0, y_1, y_2\} \leq \alpha\varepsilon$ then*

$$(3) \qquad\qquad \text{diam}\{y \in S_X : \|x_0 + y\| \geq 2 - \varepsilon\} \geq 2 - 2\varepsilon.$$

Proof. By homogeneity and by Observation 4.4, we can assume that $\alpha = 1$ and the three balls have unit radius and that $\text{diam}\{y_0, y_1, y_2\} \leq \varepsilon$. We can also assume that $c_0 = 0$, so that $x_0 = y_0$. For $i = 1, 2$, let c_i be the center of B_i, and denote $z_i = c_i - x_0$ and $\bar{z}_i = \frac{z_i}{\|z_i\|}$. Since $z_i = (c_i - y_i) + (y_i - y_0)$ and $\|c_i - y_i\| = 1$, we have $1 - \varepsilon \leq \|z_i\| \leq 1 + \varepsilon$, and hence $\|\bar{z}_i - z_i\| \leq \varepsilon$. Since $x_0 + \bar{z}_i = (c_i - c_0) + (\bar{z}_i - z_i)$ and $\|c_i - c_0\| \geq 2$, we have $\|\frac{x_0 + \bar{z}_i}{2}\| \geq 1 - \frac{\varepsilon}{2}$, and hence the points $\bar{z}_1, \bar{z}_2$ belong to the set from (3). Finally, since $\bar{z}_1 - \bar{z}_2 = (c_1 - c_2) + (\bar{z}_1 - z_1) + (z_2 - \bar{z}_2)$, we obtain that $\|\bar{z}_1 - \bar{z}_2\| \geq \|c_1 - c_2\| + \|\bar{z}_1 - z_1\| + \|z_2 - \bar{z}_2\| \geq 2 - 2\varepsilon$, completing the proof. $\qquad\square$

Definition 4.6. We shall say that $x \in S_X$ is a *locally non-D2* (or *LND2*) point of B_X if there exists $\delta > 0$ such that

$$(4) \qquad\qquad \text{diam}\{y \in S_X : \|\tfrac{x+y}{2}\| \geq 1 - \delta\} < 2.$$

In the above definition, "D2" stands for "diameter 2". Notice that each LUR point of B_X is clearly a LND2 point.

Theorem 4.7. *Let $\mathcal{B}$ be a tiling by balls of a normed space X. Assume that at least one of the following conditions is satisfied:*

(i) *$\mathcal{B}$ is disjoint;*
(ii) *X is a Banach space and $\mathcal{B}$ is plump;*
(iii) *X is a Banach space and $\mathcal{B}$ has no common flat points.*

Then there exists $x \in S_X$ such that x is neither a cone smooth point nor a LND2 point of B_X.

The proof is similar to the proof of Theorem 4.2.

Proof. Let us suppose on the contrary that this is not the case, that is, each $x \in S_X$ is either a cone smooth point or a LND2 point of B_X. If $B \in \mathcal{B}$ let us denote by r_B and c_B the radius and the center of B, respectively. Let $\mathcal{P}$ be as in Theorem 3.8 and, for each $n \in \mathbb{N}$, let $\mathcal{P}_n \subset \mathcal{P}$ be the set of all couples (x, B) such that $r_B \geq 1/n$ and at least one of the following two conditions holds:

(a) $\operatorname{diam}\big\{y \in S_X : \big\|\frac{x-c_B}{\|x-c_B\|} + y\big\| \geq 2 - \frac{1}{n}\big\} < 2 - \frac{2}{n}$;
(b) $x \in \frac{1}{n}\text{-CS}(B)$.

Clearly $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \ldots$, and $\mathcal{P} = \bigcup_n \mathcal{P}_n$.

Let H be the constant from Lemma 4.1. By Theorem 3.8, there exists $m \in \mathbb{N}$ and three couples $(x_i, B_i) \in \mathcal{P}_m$, $i = 1, 2, 3$, such that $\operatorname{diam}\{x_1, x_2, x_3\} < H/m^2$ and the balls B_i $(i = 1, 2, 3)$ are pairwise distinct and with radii at least $1/m$. Now, it suffices to consider the following two cases.

Case 1: $x_i \in \frac{1}{m}\text{-CS}(B_i)$ for each $i = 1, 2, 3$. This case is impossible by Lemma 4.1 since $\operatorname{diam}\{x_1, x_2, x_3\} < H/m^2 \leq H/m$.

Case 2: $\operatorname{diam}\big\{y \in S_X : \big\|\frac{x_1-c_{B_1}}{\|x_1-c_{B_1}\|} + y\big\| \geq 2 - \frac{1}{m}\big\} < 2 - \frac{2}{m}$. Since

$$\operatorname{diam}\{x_1, x_2, x_3\} < H/m^2 < 1/m^2 \, ,$$

Lemma 4.5 (with $\alpha = \frac{1}{m}$) gives that $\operatorname{diam}\big\{y \in S_X : \big\|\frac{x_1-c_{B_1}}{\|x_1-c_{B_1}\|} + y\big\| \geq 2 - \frac{1}{m}\big\} \geq 2 - \frac{2}{m}$, which is a contradiction. $\square$

Since each LUR point is a LND2 point, and each Fréchet smooth point is a cone smooth point, we obtain the following corollary.

Corollary 4.8. *Let $\mathcal{B}$ be a tiling by balls of a normed space X. Assume that either $\mathcal{B}$ is disjoint or X is complete. Then there exists $x \in S_X$ such that x is neither a Fréchet smooth point nor a LUR point of B_X. In particular, a LUR Banach space admits no tiling by balls, and a normed LUR space admits no disjoint tiling by balls.*

4.3. **One-point prohibitive conditions for large-ball tilings.**

Now we are going to consider tilings by balls whose radii are bounded away from 0. Let us recall two well known geometrical notions. Given a norm-one functional $f \in X^*$, a *slice of B_X, determined by f,* is any set of the form

$$S(f, \delta) = \{y \in B_X : f(y) > 1 - \delta\} \qquad \text{where } \delta \in (0, 1).$$

A point $x \in S_X$ is said to be a *strongly exposed point of B_X* if there exists a norm-one $f \in X^*$ such that $f(x) = 1$ and $\text{diam}(S(f, \delta)) \to 0$ as $\delta \to 0^+$.

Theorem 4.9. *Let X be a normed space, suppose that at least one of the following conditions holds:*

(a) *there exists $x \in S_X$ such that X is LND2 at x, and x is not a QP-point for B_X;*

(b) *there exists a LUR point in S_X;*

(c) *there exists a Fréchet smooth point $x \in S_X$, which is not a QP-point for B_X, and the unique norm-one functional $f_x \in X^*$ that supports B_X at x determines a slice Σ of B_X with $\text{diam}(\Sigma) < 2$;*

(d) *there exists a Fréchet smooth point $x \in S_X$ which is a strongly exposed point of B_X.*

Then X admits no tiling by balls with radii bounded away from 0.

Proof. Let $\mathcal{B}$ be a tiling of X by balls with radii bounded away from 0. We can (and do) suppose that $B_X \in \mathcal{B}$.

First assume (a). Let $x \in S_X$ be such that X is LND2 at x and such that x is not a QP-point for B_X. By Lemma 3.7, x is a singular point for $\mathcal{B}$. Thus there exist a sequence $\{B_n\}$ of pairwise distinct elements of $\mathcal{B} \setminus \{B_X\}$, and points $y_n \in \partial B_n$ such that $y_n \to x$. Fix an arbitrary $\delta > 0$. Since $\text{diam}\{x, y_n, y_{n+1}\} \to 0$, we can apply Lemma 4.5 to deduce that, for each $\varepsilon \in (0, 2\delta)$, we have

$$\text{diam}\{y \in S_X : \|x + y\| \geq 2 - 2\delta\} \geq \text{diam}\{y \in S_X : \|x + y\| \geq 2 - \varepsilon\} \geq 2 - 2\varepsilon.$$

It follows that $\text{diam}\{y \in S_X : \|\frac{x+y}{2}\| \geq 1 - \delta\} = 2$ for each $\delta > 0$. But this contradicts the fact that x is a LND2 point for B_X, and part (a) is proved. Moreover, since each LUR point $x \in S_X$ satisfies (a), part (b) immediately follows.

Now assume (c). By the first part of the proof of part (a), $\text{diam}\{y \in S_X : \|\frac{x+y}{2}\| \geq 1 - \delta\} = 2$ for each $\delta > 0$. This easily implies existence of a sequence $\{y_n\} \subset S_X$ such that $\|\frac{x+y_n}{2}\| \to 1$, and $\text{diam}(\{y_n\}_{n \geq n_0}) = 2$ for each $n_0 \in \mathbb{N}$. By convexity of the norm, for each $n \in \mathbb{N}$ there exists $z_n \in (x, y_n)$ such that $\|z_n\| = \min\{\|z\| : z \in [x, y_n]\}$. It is not difficult to see that

$$\|z_n\| \geq \|x + y_n\| - 1$$

(indeed, if $z'_n \in (x, y_n)$ is such that $\frac{z_n + z'_n}{2} = \frac{x + y_n}{2}$, then $\|x + y_n\| = \|z_n + z'_n\| \leq \|z_n\| + 1$). For each $n \in \mathbb{N}$, let $f_n \in X^*$ be a norm-one functional that separates $\|z_n\| B_X$ and $[x, y_n]$; clearly,

$$f_n(z_n) = \|z_n\| = f_n(x) = f_n(y_n).$$

Notice that $\|z_n\| \to 1$, that is, $f_n(x) \to 1$. Since x is a Fréchet smooth point of B_X, we have that $f_n \to f_x$ in the norm topology (see, e.g., [4, Corollary 7.22]). It follows that $f_x(y_n) \to 1$. In particular, y_n belongs to Σ for each sufficiently large n, and hence $\text{diam}(\Sigma) \geq 2$. This contradiction proves (c). Finally, part (d) follows easily from part (c). Our theorem is proved. □

5. A LUR DISJOINT TILING OF $\ell_1(\Gamma)$

In this section we show that, for some suitable uncountable set Γ, the Banach space $\ell_1(\Gamma)$ admits a disjoint tiling by bounded LUR bodies. Our construction is based on a construction by V. Klee [8] of a disjoint tiling of $\ell_1(\Gamma)$ by translates of the unit ball. Let us start with some preliminary work.

Let Γ be a nonempty set and let us denote by $\|\cdot\|_i$ the canonical norm of $\ell_i(\Gamma)$ ($i = 1, 2$). For $x \in \ell_1(\Gamma)$, let us define $\|x\| = (\|x\|_1^2 + \|x\|_2^2)^{1/2}$. It is known that $\|\cdot\|$ is an equivalent LUR norm on $\ell_1(\Gamma)$ ([4, Lemma 13.26]).

In what follows, let $M > 4\sqrt{2}$ be a fixed constant. For $\gamma \in \Gamma$ and $x \in \ell_1(\Gamma)$, let us define

$$\|x\|_1^\gamma = \sum_{\beta \in \Gamma \setminus \{\gamma\}} |x(\beta)| + \frac{1}{M}|x(\gamma)|\,,$$

$$\|x\|_2^\gamma = \left(\sum_{\beta \in \Gamma \setminus \{\gamma\}} |x(\beta)|^2 + \frac{1}{M^2}|x(\gamma)|^2 \right)^{1/2},$$

$$\|x\|^\gamma = \left[\left(\|x\|_1^\gamma \right)^2 + \left(\|x\|_2^\gamma \right)^2 \right]^{1/2}.$$

It is easy to see that $(\ell_1(\Gamma), \|\cdot\|^\gamma)$ and $(\ell_1(\Gamma), \|\cdot\|)$ are linearly isometric, and hence $\|\cdot\|^\gamma$ is an equivalent LUR norm on $\ell_1(\Gamma)$.

As usual, $B_{\|\cdot\|}$ and $S_{\|\cdot\|}$ denote the closed unit ball and the unit sphere of a norm $\|\cdot\|$. Notice that the polar set $(B_{\|\cdot\|})^\circ$ is the corresponding dual unit ball. We shall need the following fact, the easy proof of which is left to the reader.

Fact 5.1. *Let $x \in S_{\|\cdot\|^\gamma}$, and $f \in \ell_\infty(\Gamma)$. Assume that $f = a_1 f_1 + a_2 f_2$, where $f_i \in (B_{\|\cdot\|_i^\gamma})^\circ$ ($i = 1, 2$), and $a_1, a_2 \in \mathbb{R}$ are such that $a_1^2 + a_2^2 \leq 1$. Then:*

    (a) *$f \in (B_{\|\cdot\|^\gamma})^\circ$;*
    (b) *if $a_i = f_i(x) = \|x\|_i^\gamma$ ($i = 1, 2$) then $f(x) = 1 = \sup f(B_{\|\cdot\|^\gamma})$.*

Let $\{e_\beta\}_{\beta \in \Gamma}$ denote the canonical basis of $\ell_1(\Gamma)$. Let us define

$$z_\gamma := \big(M/\sqrt{2}\big)\, e_\gamma \qquad (\gamma \in \Gamma).$$

It is elementary to see that

$$\|z_\gamma\|_1^\gamma = \|z_\gamma\|_2^\gamma = 1/\sqrt{2} \quad \text{and} \quad \|z_\gamma\|^\gamma = 1 \qquad (\gamma \in \Gamma).$$

Lemma 5.2. *Let γ_0 and γ_1 be two distinct elements of Γ. Let $x \in S_{\|\cdot\|^{\gamma_0}}$ be such that $x(\gamma_1) = 0$. Then there exists $f \in \ell_\infty(\Gamma) \setminus \{0\}$ such that $f(x) = \sup f(B_{\|\cdot\|^{\gamma_0}})$ and $f(z_{\gamma_1}) = \sup f(B_{\|\cdot\|^{\gamma_1}})$.*

Proof. Let us define $f_1, f_2, f \in \ell_\infty(\Gamma)$ as follows.

$$f_1(\gamma) = \begin{cases} \operatorname{sign}[x(\gamma)]/M & \text{if } \gamma = \gamma_0, \\ 1 & \text{if } \gamma = \gamma_1, \\ \operatorname{sign}[x(\gamma)] & \text{if } \gamma \neq \gamma_0, \gamma_1; \end{cases}$$

$$f_2(\gamma) = \begin{cases} x(\gamma)/[M^2 \|x\|_2^{\gamma_0}] & \text{if } \gamma = \gamma_0, \\ 0 & \text{if } \gamma = \gamma_1, \\ x(\gamma)/\|x\|_2^{\gamma_0} & \text{if } \gamma \neq \gamma_0, \gamma_1; \end{cases}$$

$$f = \|x\|_1^{\gamma_0} f_1 + \|x\|_2^{\gamma_0} f_2\,.$$

An easy calculation shows that:

- $f_1 \in (B_{\|\cdot\|_1^{\gamma_0}})^\circ$ and $f_1(x) = \|x\|_1^{\gamma_0}$;
- $f_2 \in (B_{\|\cdot\|_2^{\gamma_0}})^\circ$ and $f_2(x) = \|x\|_2^{\gamma_0}$;
- $f(x) = 1 = \sup f(B_{\|\cdot\|^{\gamma_0}})$ by Fact 5.1.

It remains to show that $\sup f(B_{\|\cdot\|^{\gamma_1}}) = f(z_{\gamma_1})$. Let us define $g_1, g_2, g \in \ell_\infty(\Gamma)$ as follows.

$$g_1(\gamma) = \begin{cases} 1/M & \text{if } \gamma = \gamma_1, \\ 2f(\gamma)/[\|x\|_1^{\gamma_0} M] & \text{if } \gamma \neq \gamma_1; \end{cases}$$

$$g_2(\gamma) = \begin{cases} 1/M & \text{if } \gamma = \gamma_1, \\ 0 & \text{if } \gamma \neq \gamma_1; \end{cases}$$

$$g = \|z_{\gamma_1}\|_1^{\gamma_1} g_1 + \|z_{\gamma_1}\|_2^{\gamma_1} g_2 = \big(1/\sqrt{2}\big)(g_1 + g_2).$$

Since $\|\cdot\|_1^{\gamma_0} \geq \|\cdot\|_2^{\gamma_0}$, we easily obtain $\|\cdot\|_1^{\gamma_0} \geq \|\cdot\|^{\gamma_0}/\sqrt{2}$. Moreover, since $|f(\gamma)| \leq 2 \ (\gamma \in \Gamma)$, we have $\frac{2|f(\gamma)|}{\|x\|_1^{\gamma_0} M} \leq \frac{2\sqrt{2}}{M}|f(\gamma)| \leq \frac{4\sqrt{2}}{M} \leq 1$. This easily implies that $g_1 \in (B_{\|\cdot\|_1^{\gamma_1}})^\circ$.

It is also easy to verify that $g_2 \in (B_{\|\cdot\|_2^{\gamma_1}})^\circ$, and $g_i(z_{\gamma_1}) = 1/\sqrt{2} = \|z_{\gamma_1}\|_i^{\gamma_1}$ $(i = 1, 2)$. Using Fact 5.1, we obtain

$$g(z_{\gamma_1}) = 1 = \sup g(B_{\|\cdot\|^{\gamma_1}}).$$

To finish the proof, notice that $f(\gamma_1) = \|x\|_1^{\gamma_0}$ and $g(\gamma_1) = \frac{\sqrt{2}}{M}$. Now it is easy to verify that $f = \frac{M}{\sqrt{2}}\|x\|_1^{\gamma_0}g$, and hence $f(z_{\gamma_1}) = \sup f(B_{\|\cdot\|^{\gamma_1}})$. $\qquad\square$

Lemma 5.3. *Let γ_0 and γ_1 be two distinct elements of Γ. Let x_{γ_1} be an element of $\ell_1(\Gamma) \setminus B_{\|\cdot\|^{\gamma_0}}$ such that $x_{\gamma_1}(\gamma_1) = 0$. Then the sets $B_0 := B_{\|\cdot\|^{\gamma_0}}$ and $B_1 := x_{\gamma_1} + z_{\gamma_1} + B_{\|\cdot\|^{\gamma_1}}$ are disjoint.*

Proof. Since $x_{\gamma_1}(\gamma_1) = 0$, by Lemma 5.2, there exists $f \in \ell_\infty(\Gamma) \setminus \{0\}$ such that

$$f(z_{\gamma_1}) = \sup f(B_{\|\cdot\|^{\gamma_1}}) = -\inf f(B_{\|\cdot\|^{\gamma_1}}) \quad \text{and} \quad \sup f(B_0) = f\big(\tfrac{x_{\gamma_1}}{\|x_{\gamma_1}\|^{\gamma_0}}\big).$$

Since $\|x_{\gamma_1}\|^{\gamma_0} > 1$ and clearly $f(x_{\gamma_1}) > 0$, we obtain

$$\sup f(B_0) = \inf f\big(\tfrac{x_{\gamma_1}}{\|x_{\gamma_1}\|^{\gamma_0}} + z_{\gamma_1} + B_{\|\cdot\|^{\gamma_1}}\big) < \inf f(B_1),$$

which proves that B_0 and B_1 are disjoint. $\qquad\square$

Let us recall that Γ is a *regular cardinal* if Γ is a cardinal such that each cofinal subset of Γ has the same cardinality as Γ. (A set $A \subset \Gamma$ is called *cofinal* with Γ if each element of Γ is less or equal to some element of A.) It is known that there exist arbitrarily large infinite regular cardinals Γ such that $\Gamma^{\aleph_0} = \Gamma$ (see [8]); such cardinals are clearly uncountable. Given a set M, we denote by $|M|$ its cardinality.

Theorem 5.4. *Let I be a cardinal such that $\mathrm{I}^{\aleph_0} = \mathrm{I}$. Then $\ell_1(\mathrm{I})$ admits a disjoint tiling $\mathcal{C}$ by centrally symmetric bounded LUR bodies.*

Proof. Let us proceed in two steps.

STEP1. Let Γ be a regular cardinal such that $\Gamma^{\aleph_0} = \Gamma$. Put $X := \ell_1(\Gamma)$. We shall show that X admits a disjoint tiling $\mathcal{B}$ by centrally symmetric bounded LUR bodies.

Proceeding as in the proof of [8, Theorem 1.2], it is not difficult to show that if K is an infinite cardinal then $|\ell_1(\mathrm{K})| = \mathrm{K}^{\aleph_0}$. Then $|X| = \Gamma$ and hence we may assume that X is well-ordered by an antireflexive relation $\prec$ as to be order-isomorphic with Γ.

For $\beta \in \Gamma$, denote $X_\beta = \{x \in X : x(\gamma) = 0 \,\, \forall \gamma \geq \beta\}$. Let us inductively construct families $\mathcal{B}_\beta$ ($\beta \in \Gamma$). For $\alpha = \min \Gamma$, we define

$$x_\alpha = 0, \quad y_\alpha = z_\alpha, \quad \mathcal{B}_\alpha := \{y_\alpha + B_{\|\cdot\|^\alpha}\}$$

(recall that $z_\alpha = Me_\alpha/\sqrt{2}$).

Now, let $\beta \in \Gamma \setminus \{\alpha\}$, and assume that the families $\mathcal{B}_\gamma$ have been already defined for all $\gamma < \beta$. Put $Y_\beta = \bigcup_{\gamma < \beta}(\bigcup \mathcal{B}_\gamma)$.

(i) If $X_\beta \subset Y_\beta$, define $\mathcal{B}_\beta = \bigcup_{\gamma < \beta} \mathcal{B}_\gamma$;

(ii) if else, let x_β be the first point of $X_\beta \setminus Y_\beta$, and define

$$\mathcal{B}_\beta = \Big(\bigcup_{\gamma < \beta} \mathcal{B}_\gamma\Big) \cup \{y_\beta + B_{\|\cdot\|^\beta}\} \quad \text{where} \quad y_\beta = x_\beta + z_\beta.$$

We shall show that the family $\mathcal{B} = \bigcup_{\beta \in \Gamma} \mathcal{B}_\beta$ has the desired properties.

First of all, let us show that the elements of $\mathcal{B}$ are pairwise disjoint. Indeed, let $B_0 = y_{\beta_0} + B_{\|\cdot\|^{\beta_0}}$ and $B_1 = y_{\beta_1} + B_{\|\cdot\|^{\beta_1}}$ be two distinct elements of $\mathcal{B}$ (with $\beta_0, \beta_1 \in \Gamma$). By our construction, we may suppose that $\beta_0 < \beta_1$, $x_{\beta_1} \notin y_{\beta_0} + B_{\|\cdot\|^{\beta_0}}$ and $x_{\beta_1}(\beta_1) = 0$. Since $x_{\beta_1}(\beta_1) = x_{\beta_0}(\beta_1) = z_{\beta_0}(\beta_1) = 0$ and $x_{\beta_1} - x_{\beta_0} - z_{\beta_0} \notin B_{\|\cdot\|^{\beta_0}}$, we can apply Lemma 5.3 to conclude that $B_{\|\cdot\|^{\beta_0}}$ and $(x_{\beta_1} - x_{\beta_0} - z_{\beta_0}) + B_{\|\cdot\|^{\beta_1}}$ are disjoint. It follows that B_1 and B_2 are disjoint.

It remains to show that $\mathcal{B}$ covers X. Suppose that this is not the case and let w be the first element of $X \setminus \bigcup \mathcal{B}$. Put $L = \{x \in X : x \prec w\}$. Since L is covered by $\mathcal{B}$, for each $v \in L$ there exists $\beta_v \in \Gamma$ such that $v \in y_{\beta_v} + B_{\|\cdot\|^{\beta_v}}$. Let us consider the set $A \subset \Gamma$ defined by

$$A := \mathrm{supp}(w) \cup \bigcup_{v \in L} \big[\mathrm{supp}(v) \cup \mathrm{supp}(y_{\beta_v})\big]$$

(here $\mathrm{supp}(u)$ denotes the support of $u \in X$). Notice that

$$|A| \leq (2|L| + 1) \cdot \aleph_0 = \max\{|L|, \aleph_0\} < \Gamma.$$

Since Γ is regular and $|A| < \Gamma$, A is not cofinal in Γ. Thus there exists $\delta \in \Gamma$ such that $\beta < \delta$ for each $\beta \in A$. Clearly, $w \in X_\delta \setminus (\bigcup \mathcal{B}) \subset X_\delta \setminus Y_\delta$.

We claim that w is the first element of $X_\delta \setminus Y_\delta$. Indeed, if $v \in L$, then $\beta_v \in A$ and $\beta_v < \delta$, which implies that $v \in y_{\beta_v} + B_{\|\cdot\|^{\beta_v}} \subset \bigcup \mathcal{B}_{\beta_v} \subset Y_\delta$.

But this means that $w = x_\delta \in (y_\delta + B_{\|\cdot\|^\delta}) \in \mathcal{B}$, which is a contradiction that concludes the proof of the first step.

STEP2. To prove the general case, we proceed as in the proof of [9, Theorem 3.2]. Given our cardinal I, consider $\Gamma := (2^{\mathrm{I}})^+$, the successor of the cardinal 2^{I}. By [8, Remark 1.1], Γ is regular and $\Gamma^{\aleph_0} = \Gamma$. By Step 1, $\ell_1(\Gamma)$ admits a disjoint tiling $\mathcal{B}$ by centrally symmetric bounded LUR bodies. For each $B \in \mathcal{B}$ let c_B be its center of symmetry.

Let $\mathrm{I}_0 = \mathrm{I} \subset \Gamma$ and $\mathcal{C}_0 = \emptyset$. Let us inductively construct $\mathrm{I}_\beta \subset \Gamma$ and $\mathcal{C}_\beta \subset \mathcal{B}$ for $\beta < \omega_1$. Assume that I_γ and $\mathcal{C}_\gamma$ have been already defined for all $\gamma < \beta$. If β is a limit ordinal, define $\mathcal{C}_\beta = \bigcup_{\gamma < \beta} \mathcal{C}_\gamma$ and $\mathrm{I}_\beta = \bigcup_{\gamma < \beta} \mathrm{I}_\gamma$. If $\beta = \alpha + 1$, define

$$\mathcal{C}_\beta = \{B \in \mathcal{B};\ B \cap \ell_1(\mathrm{I}_\alpha) \neq \emptyset\}, \quad \mathrm{I}_\beta = \mathrm{I}_\alpha \cup \bigcup_{B \in \mathcal{C}_\beta} \mathrm{supp}(c_B).$$

We claim that $|\mathrm{I}_\beta| = \mathrm{I}$ and $|\mathcal{C}_\beta| \leq \mathrm{I}$ for each $\beta < \omega_1$. Let us prove our claim by induction. Clearly it holds for $\beta = 0$. Now, assume that $|\mathrm{I}_\alpha| = \mathrm{I} = \mathrm{I}^{\aleph_0}$ and $|\mathcal{C}_\alpha| \leq \mathrm{I}$ hold for each $\alpha < \beta$. If β is a limit ordinal then we have $\mathrm{I} \leq |\mathrm{I}_\beta| \leq$

$\aleph_0 I = I$ and $|\mathcal{C}_\beta| \leq \aleph_0 I = I$. For $\beta = \alpha + 1$, we have $|\mathcal{C}_\beta| \leq |\ell_1(I_\alpha)| = |I_\alpha|^{\aleph_0} = I$, and hence also $|I_\beta \setminus I_\alpha| \leq \aleph_0 |\mathcal{C}_\beta| \leq I$ which implies that $|I_\beta| = I$. The claim is proved.

Now, define $J = \bigcup_{\beta < \omega_1} I_\beta$. Clearly, $|J| \leq \aleph_1 I = I$ and hence $|J| = I$. Let $\mathcal{C}' = \bigcup_{\beta < \omega_1} \mathcal{C}_\beta$ and $\mathcal{C} = \{B \cap \ell_1(J) : B \in \mathcal{C}'\}$. Then $\mathcal{C}$ is a disjoint tiling of $\ell_1(J)$ by centrally symmetric bounded LUR bodies. We are done since $\ell_1(J)$ and $\ell_1(I)$ are isometrically isomorphic. $\qquad\square$

Remark 5.5. (a) Let us remark that, under the generalized continuum hypothesis, our assumption $I^{\aleph_0} = I$ holds for every uncountable cardinal I.

To see this, first suppose that $I = K^+$ (the successor of a cardinal K). Then $I = 2^K$ and hence $I^{\aleph_0} = 2^{K \cdot \aleph_0} = 2^K = I$. Now, assume that I is not a successor cardinal; hence $2^K < I$ whenever $K < I$. As usual, we can identify any ordinal K with the smallest ordinal of cardinality K, and hence also with the interval $[0, K)$ of ordinals. Since I is uncountable, every sequence in $[0, I)$ is contained in $[0, K)$ for some infinite cardinal $K < I$. Hence

$$I \leq I^{\aleph_0} \leq \sum_{\omega_0 \leq K < I} K^{\aleph_0} \leq \sum_{\omega_0 \leq K < I} (2^K)^{\aleph_0} = \sum_{\omega_0 \leq K < I} 2^K \leq I \cdot I = I\,,$$

and we are done.

(b) On the other hand, it follows by [8, Proposition 3.5] that if $\ell_1(\aleph_1)$ admits a disjoint tiling then we have $\aleph_1 = 2^{\aleph_0}$, the continuum hypothesis.

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