A remark on generalized complete intersections

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Abstract

We observe that an interesting method to produce non-complete intersection subvarieties, the generalized complete intersections from L. Anderson and coworkers, can be understood and made explicit by using standard Cech cohomology machinery. We include a worked example of a generalized complete intersection Calabi–Yau threefold.

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0. Introduction

Calabi–Yau varieties, in particular those of dimension three, are of great interest in string theory. Since there are not many general results yet on their classification, but see [14], the explicit construction of CY threefolds is a quite important enterprise. For example, Kreuzer and Starke classified the toric fourfolds which have CY threefolds as (anticanonical) hypersurfaces [11], [3]. Besides generalizations to complete intersection CYs in certain ambient toric varieties, like products of projective spaces, there are various other examples of CY threefolds constructed with more sophisticated algebro-geometrical methods. Recent examples include [9], [7], [10].

In the recent paper [1], L. Anderson, F. Apruzzi, X. Gao, J. Gray and S-J. Lee found a very nice method to construct many more CY threefolds. The basic idea is to take a hypersurface $Y$ in an ambient variety $P$ and to consider hypersurfaces $X$ in $Y$. These hypersurfaces need not be complete intersections in $P$, that is, there need not exist two sections of two line bundles on $P$...
whose common zero locus is $X$. There are various generalizations of this method, but we will stick to this basis case. As in [1], we refer to these varieties as generalized complete intersections (gCIs).

A particularly interesting and accessible case that was found and studied by Anderson and coworkers is when the ambient variety is a product of two varieties, one of which is $\mathbb{P}^1$, so $P = P_2 \times \mathbb{P}^1$. The variety $P_2$ they consider is a product of projective spaces, but this is not essential, one could consider any toric variety or even more general cases. The factor $\mathbb{P}^1$ is important since there are line bundles on $\mathbb{P}^1$ with non-trivial first cohomology group and this is essential to find generalized complete intersections. We review this construction in Section 1.1.

We provide a proposition, proven with standard Cech cohomology methods, that allows one, under a certain hypothesis, to find three equations (more precisely, three sections of three line bundles on $P$) that define $X$. In Section 2 we work out a detailed example, with explicit equations, of a CY threefold which was already considered in [1]. The explicit example $X$ has an automorphism of order two and the quotient of $X$ by the involution provides, after desingularization, another CY threefold. More generally, we think that among the gCIs found in [1] one could find more examples of CY threefolds with non-trivial automorphisms. It might be hard though to implement a systematic search as was done in [6] for complete intersection CY threefolds in products of projective spaces. We did not find new CY threefolds with small Hodge numbers (see [5] for an update on these), but the gCICYs seem to be a promising class of CYs to search for these. The recent paper [4] by Berglund and Hübsch provides further techniques to deal with gCICYs whereas [2] explores string theoretical aspects of gCICYs.

1. The construction of generalized complete intersections

1.1. The general setting

Let $P_2$ be a projective variety of dimension $n$ and let $P := P_2 \times \mathbb{P}^1$. We denote the projections to the factors of $P$ by $\pi_1, \pi_2$ respectively. For a coherent sheaf $\mathcal{F}$ on $P_2$ and an integer $d$ we define a coherent sheaf on $P$ by:

$$\mathcal{F}[d] := \pi_1^* \mathcal{F} \otimes \pi_2^* \mathcal{O}_{\mathbb{P}^1}(d).$$

The Kähneth formula gives

$$H^r(P, \mathcal{F}[d]) = \bigoplus_{p+q=r} H^p(P_2, \mathcal{F}) \otimes H^q(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)).$$

Recall that the only non-zero cohomology of $\mathcal{O}_{\mathbb{P}^1}(d)$ is: $h^0(\mathcal{O}_{\mathbb{P}^1}(d)) = h^1(\mathcal{O}_{\mathbb{P}^1}(-2-d)) = d+1$ for $d \geq 0$ and a basis for $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ is given by the monomials $z_0^{d-i}z_1^i$, $i = 0, \ldots, d$, where $(z_0 : z_1)$ are the homogeneous coordinates on $\mathbb{P}^1$.

Let $L$ be a line bundle on $P_2$ and assume that $L[d]$, for some $d \geq 1$, has a non-trivial global section $F$. Using the Kähneth formula, we can write $F = \sum_i f_i z_0^{d-i}z_1^i$ for certain sections $f_i \in H^0(P_2, L)$. Let $Y = (F)$ be the zero locus of $F$ in $P$. We assume that $Y$ is a (reduced, irreducible) variety, although this will not be essential in this section.

To define a codimension two subvariety of $P$, we consider another line bundle $M$ on $P_2$. The Kähneth formula shows that $M[-e]$ has no global sections if $e \geq 1$. But upon restricting to $Y$, the vector space $H^0(Y, M[-e]|_Y)$ could still be non-trivial. In fact, from the exact sequence

$$0 \rightarrow (L^{-1} \otimes M)[-d-e] \xrightarrow{F} M[-e] \rightarrow M[-e]|_Y \rightarrow 0$$

(1)
we deduce the exact sequence

\[
0 \longrightarrow H^0(Y, M[-e]|_Y) \xrightarrow{d^0} H^1(P, (L^{-1} \otimes M)[-d - e]) \xrightarrow{F_1} H^1(P, M[-e])
\]

(2)

thus \( H^0(Y, M[-e]|_Y) \cong \ker(F_1) \), where we denote by \( F_1 \) the map induced by multiplication by \( F \) on the first cohomology groups. Since now

\[
H^1(P, (L^{-1} \otimes M)[-d - e]) \cong H^0(P_2, L^{-1} \otimes M) \otimes H^1(P^1, O_{P^1}(-d - e))
\]

(d + e ≥ 2)

the domain of \( F_1 \) is non-trivial if and only if \( h^0(P_2, L^{-1} \otimes M) \neq 0 \). So for suitable choices of line bundles on \( P_2 \) we might find interesting, non-complete intersection, codimension two subvarieties of \( P \) in this way. In the proof of Proposition 1.4 we explain how to compute \( F_1 \).

1.2. Example

Let \( P_2 = P^n, L = O_{P^0}(k), M = O_{P^0}(k + l) \) with \( l \geq 0 \), let \( d \geq 1 \) and \( e = 1 \). Then \( h^0(P^n, L^{-1} \otimes M) = h^0(O_{P^0}(l)) \neq 0 \) so \( h^1(P, (L^{-1} \otimes M)[-d - e]) \neq 0 \), but \( h^1(P, M[-e]) = 0 \) since \( O_{P^0}(-1) \) has no cohomology. Thus \( H^0(Y, M[-e]|_Y) \cong H^1(P, (L^{-1} \otimes M)[-d - e]) \) is indeed non-trivial.

1.3. Generalized complete intersections

Given a variety \( Y \subset P \) that is the zero locus of \( F \in H^0(L[d]) \) as in Section 1.1, and given a global section \( \tau \in H^0(M[-e]|_Y) \), its zero locus \( X := (\tau) \subset Y \) is called a generalized complete intersection.

The scheme \( X \) may not be defined by two global sections \( \sigma_1, \sigma_2 \) of line bundles \( L_1, L_2 \) on \( P \). However in certain cases we can find three sections of line bundles on \( P \) which define \( X \):

1.4. Proposition

Let \( F \in H^0(P, L[d]) \), let \( Y = (F) \), let \( \tau \in H^0(Y, M[-e]|_Y) \) with \( d, e \geq 1 \) be as above and assume that \( H^1(P_2, L^{-1} \otimes M) = 0 \).

Then there are two global sections \( G, H \in H^0(P, M[d - 1]) \) such that the generalized complete intersection subscheme \( X \) of \( P \) defined by \( \tau \) in \( Y \) can also be defined as

\[
X = \{ x \in P : F(x) = G(x) = H(x) = 0 \}
\]

(the equality is of schemes). Moreover, there is a global section \( A \in H^0(P, (L^{-1} \otimes M)(d + e - 2)) \) such that \( AF = z_1^{d+e-1}G + z_0^{d+e-1}H \), so that on the open subset of \( P_2 \times P^1 \) where \( z_0 \neq 0 \) the subscheme \( X \) of \( P \) is defined by the two equations \( F = G = 0 \).

**Proof.** We use Cech cohomology to make the isomorphism \( H^0(Y, M[-e]|_Y) \cong \ker(F_1) \), see exact sequence (2), explicit. Let \( U_i \subset P^1 \) be the open subset where \( z_i \neq 0 \). For a coherent sheaf \( \mathcal{G} \) on \( P^1 \) we have the exact sequence

\[
0 \longrightarrow H^0(P^1, \mathcal{G}) \longrightarrow \mathcal{G}(U_0) \oplus \mathcal{G}(U_1) \xrightarrow{\delta} \mathcal{G}(U_0 \cap U_1) \longrightarrow H^1(P^1, \mathcal{G}) \longrightarrow 0,
\]

where \( \delta(t_0, t_1) = t_0 - t_1 \). The cohomology groups we consider are computed with the Künneth formula. Note that after tensoring this exact sequence by a vector space \( W \), we obtain that \( W \otimes H^0(P^1, \mathcal{G}) = \ker(1_W \otimes \delta) \) and \( W \otimes H^1(P^1, \mathcal{G}) = \coker(1_W \otimes \delta) \).
For an affine open subset $V \subset \mathbb{P}^1$, the cohomology of the exact sequence (1) on $P_2 \times V$ gives the exact sequence, where we extend $M[-e]_{|Y}$ by zero to $P_2 \times V$,

$$H^0(P_2 \times V, M[-e]) \longrightarrow H^0(P_2 \times V, M[-e]_{|Y}) \longrightarrow H^1(P_2 \times V, (L^{-1} \otimes M)[-d-e]).$$

The Künneth formula, combined with the assumption $H^1(P_2, L^{-1} \otimes M) = 0$ and the fact that $H^1(V, \mathcal{F}) = 0$ for any coherent sheaf $\mathcal{F}$ since $V$ is affine, implies that the last group is zero.

Taking $V = U_0, U_1$, the exact sequence (1) on $P_2 \times V$ thus gives two exact sequences whose sum (term by term) is

$$0 \longrightarrow \bigoplus_{i=0}^1 H^0(L^{-1} \otimes M) \otimes (\mathcal{O}_{\mathbb{P}^1}(-d-e)(U_i)) \overset{F}{\longrightarrow} \bigoplus_{i=0}^1 H^0(M) \otimes (\mathcal{O}_{\mathbb{P}^1}(-e)(U_i)) \longrightarrow \bigoplus_{i=0}^1 (M[-e]_{|Y})(P_2 \times U_i) \longrightarrow 0.$$  \hspace{1cm} (3)

Similarly taking $V = U_0 \cap U_1$ one has the exact sequence:

$$0 \longrightarrow H^0(L^{-1} \otimes M) \otimes (\mathcal{O}_{\mathbb{P}^1}(-d-e)(U_0 \cap U_1)) \overset{F}{\longrightarrow} H^0(M) \otimes (\mathcal{O}_{\mathbb{P}^1}(-e)(U_0 \cap U_1)) \longrightarrow (M[-e]_{|Y})(P_2 \times (U_0 \cap U_1)) \longrightarrow 0.$$  \hspace{1cm} (4)

Next we use the Čech boundary map $\delta$ to map sequence (3) to sequence (4) and we obtain a commutative diagram with three sequences as columns. The first two columns are Čech complexes for the covering $\{U_i\}_{i=0,1}$ of $\mathbb{P}^1$, their cohomology groups are respectively

$$H^0(L^{-1} \otimes M) \otimes H^q(\mathcal{O}_{\mathbb{P}^1}(-d-e)) \cong H^q(P, (L^{-1} \otimes M)[-d-e]),$$

$$H^0(M) \otimes H^q(\mathcal{O}_{\mathbb{P}^1}(-e)) \cong H^q(P, M[-e]), \quad (q = 0, 1).$$

The zero-th cohomology group of the last column is $H^0(Y, M[-e]_{|Y})$. So we conclude that the map $F_1$ can be computed with the long exact cohomology sequence associated to this diagram.

We observe, but will not use, that the Künneth formula implies that $H^2(P, (L^{-1} \otimes M)[-d-e]) = 0$ and thus the cohomology sequence of (1) gives a six term exact sequence with the zero-th and first cohomology groups. The first 5 terms are the same as those of the long exact cohomology sequence associated to the diagram, so we conclude that the first cohomology group of the last column is $H^1(Y, M[-e]_{|Y})$.

Given $\tau \in H^0(Y, M[-e]_{|Y})$, let $q := d^0(\tau) \in \ker(F_1)$. Since the first row (3) of the complex is exact, the section $\tau$ is locally given by restricting sections $\tau_i \in M[-e](P_2 \times U_i)$ to $Y$. By the snake lemma, they satisfy $\tau_0 - \tau_1 = F q$ on $P_2 \times (U_0 \cap U_1)$, in particular $\tau_0 = \tau_1$ on $Y \cap (P_2 \times (U_0 \cap U_1))$ since $F = 0$ on $Y$.

The images of the $z_0^{-j} z_1^{-d-e+j} \in \mathcal{O}_{\mathbb{P}^1}(-d-e)(U_0 \cap U_1)$, $j = 1, \ldots, d + e - 1$, form a basis of $H^1(\mathbb{P}^1, \mathcal{O}(-d-e))$. A cohomology class $q \in H^1(P, (L^{-1} \otimes M)[-d-e]) \cong H^0(P_2, L^{-1} \otimes M) \otimes H^1(\mathbb{P}^1, \mathcal{O}(-d-e))$ can thus be represented by $q = \sum_j q_j z_0^{-j} z_1^{-d-e+j}$ with $q_j \in H^0(P_2, L^{-1} \otimes M)$. Let $F = \sum_i f_i z_0^{d-i} z_1^{-d-i}$, where $f_i \in H^0(P_2, L)$, then $F q$ is homogeneous of degree $d - (d + e) = -e$ and it is a sum of terms $r_k z_0^{k-e-k}$ with $r_k \in H^0(P_2, M)$.

Writing

$$F q = \sum_{k=-d-e+1}^{d-1} r_k z_0^k z_1^{-e-k},$$

$$= \left( \sum_{k=-d-e+1}^{-e} r_k z_0^k z_1^{-e-k} \right) + \left( \sum_{k=-e+1}^{d-1} r_k z_0^k z_1^{-e-k} \right) + \left( \sum_{k=0}^{d-1} r_k z_0^k z_1^{-e-k} \right),$$

we observe that the first term vanishes since $r_k = 0$ for $k < -d-e+1$ and the third term vanishes since $r_k = 0$ for $k > d-1$.
the first summand lies in $M[-e](P_2 \times U_0)$ (where $z_0 \neq 0$) and the last summand lies in $M[-e](P_2 \times U_1)$, we denote these summands by $\tau_0$ and $-\tau_1$ respectively. The middle summand has monomials $z_0^{d-e}d_0^e$ with both $a, b < 0$. Thus $Fq$ represents a class in $q' \in H^1(P, M[-e])$, which is the same as the class represented by the middle summand. By definition, one has $q' = F_1(q)$ and thus $q \in \ker(F_1)$ when all coefficients $r_k, k = -e + 1, \ldots, -1$, are zero.

Since $q \in \ker(F_1)$ this middle summand is zero, so that $Fq = \tau_0 - \tau_1$ as desired. Now we define $G := z_0^{d-e-1}r_0$ and $H := -z_0^{d+e-1}r_1$ so that all their monomials $z_0^{d+e}d_0^a$ have $a, b \geq 0$ and $a + b = d - 1$, thus both $G, H \in H^0(P, M[d-1])$. Then $(z_0z_1)^{d+e-1}Fq = z_1^{d+e-1}G + z_0^{d+e-1}H$ and with $A := (z_0z_1)^{d+e-1}q \in H^0(P, (L^{-1} \otimes M)(d + e - 2))$ we find the desired relation. □

1.5. Example

With the choices of $P_2, L, M$ as in Example 1.2, and if $X$ is a smooth variety (of dimension $n-1$), then $H^1(P_2, L^{-1} \otimes M) = H^1(P^e, O_{P^e}(l)) = 0$, for any $l$, if $n > 1$. The adjunction formula implies that $X$ has trivial canonical bundle if we choose $l = n + 1 - 2k$ and $d = 3$. In that case $P = P^e \times P^f$ and $F$ is homogeneous of bidegree $(k, 3)$ whereas $G, H$ have bidegree $(n + 1 - k, 2)$.

1.6. A fibration on X

Given $X$ as in the proposition, the projection $\pi_2 : P_2 \times P^f \to P^f$ restricts to $X$ to give a fibration denoted by $\pi_2 : X \to P^f$. For a point $p = (z_0 : z_1) \in P^f$, we denote by $F_p \in H^0(P_2, L)$, $H_p \in H^0(P_2, M)$ the restrictions of $F$ and $H$ to the fiber $X_p$. The equation $AF = z_1^{d+e-1}G + z_0^{d+e-1}H$ shows that if $z_1 \neq 0$ then $F_p$ and $H_p$ define the fiber $X_p$, which is thus a complete intersection in $P_2$.

1.7. Example

This example illustrates that $X$, as in Proposition 1.4, might be reducible, even if $h^0(Y, M[-e]|_Y)$ is rather large. The example is taken from [1, Table 4], third item (with $i = 2$) where it is in fact observed that no smooth varieties arise in that case. We take

$$P_2 := P^2 \times P^1 \times P^f, \quad L := O(0, 1, 1), \quad M := O(3, 1, 1), \quad d = 4, e = -2.$$ 

Notice that $H^1(P_2, L^{-1} \otimes M) = H^1(P^2 \times P^1 \times P^f, O(3, 0, 0)) = 0$ by the Künneth formula, so we can, but will not, apply Proposition 1.4. Since $h^1((L^{-1} \otimes M)[-d - e]) = h^1(O(3, 0, 0)[-6]) = 10 \cdot 1 \cdot 1 \cdot 5 = 50$ and $h^1(M[-e]) = 10 \cdot 2 \cdot 2 \cdot 1 = 40$, we find $h^0(M[-e]|_Y) \geq 10$. We will show that, for general $Y$, $h^0(M[-e]|_Y) = 10$ but that all sections of $M[-e]|_Y$ define reducible subvarieties of $Y$.

Due to the first zero in $L = O(0, 1, 1)$, the variety $Y$ is a product, $Y = P^2 \times S \subset P^f$, with $S \subset (P^1)^3$ the surface defined by a section of $O(1, 1, 4)$. Then we have $h^0(M[-e]|_Y) = h^0(P^2 \times S, \pi_3^*O_{F_2}(3) \otimes \pi_3^*O_{S}(1, 1, -2))$ and using the Künneth formula we find $h^0(M[-e]|_Y) = h^0(O_{F_2}(3))h^0(O_{S}(1, 1, -2)) = 10h^0(O_{S}(1, 1, -2))$. The exact sequence

$$0 \to O_{P^2}(0, 0, -6) \xrightarrow{f} O_{P^2}(1, 1, -2) \to O_{S}(1, 1, -2) \to 0,$$

where $f$ is the equation of $S$, shows that (with $f_1$ the map induced by $f$ on $H^1$):

$$h^0(O_{S}(1, 1, -2)) = \dim \ker \left( f_1 : H^1(O_{P^2}(0, 0, -6)) \to H^1(O_{P^2}(1, 1, -2)) \right).$$
Since these spaces have dimensions $1 \cdot 1 \cdot 5 = 5$ and $2 \cdot 2 \cdot 1 = 4$ respectively, one expects $h^0(\mathcal{O}_S(1, 1, -2)) = 1$. In that case any section $\tau \in H^0(M[-e]|_Y)$ would be the product $\tau = gs$ with $g \in H^0(\mathcal{O}_P(3))$ and $s \in H^0(\mathcal{O}_S(1, 1, -2))$ the unique (up to scalar multiple) section, hence $X$ would be reducible.

To see that indeed $h^0(\mathcal{O}_S(1, 1, -2)) = 1$ for a general equation $f$, take a smooth (genus one) curve $C$ of bidegree $(2, 2)$ in $\mathbb{P}^1 \times \mathbb{P}^1$ and choose eight distinct points on $C$ which are not cut out by another curve of bidegree $(2, 2)$. As curves of bidegree $(1, 4)$ depend on $2 \cdot 5 = 10$ parameters, we can find two polynomials $g_0, g_1$ of bidegree $(1, 4)$ such that $g_0 = g_1 = 0$ consists of these eight points on $C$. Take $f = x_0g_0 + x_1g_1$ with $(x_0 : x_1) \in \mathbb{P}^1$, the first copy of $\mathbb{P}^1$ in $(\mathbb{P}^1)^3$, and the $g_i$ on the last two copies of $\mathbb{P}^1$. The surface $S \subset (\mathbb{P}^1)^3$ defined by $f$ is thus the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ in the eight points where $g_0 = g_1 = 0$. The adjunction formula shows that the line bundle $\mathcal{O}_S(1, 1, -2)$ is the anticanonical bundle of $S$. The effective anticanonical divisors are the strict transforms of bidegree $(2, 2)$-curves on passing through these eight points. Hence the strict transform of $C$ in $S$ will be the unique effective anticanonical divisor on $S$ and therefore $h^0(\mathcal{O}_S(1, 1, -2)) = 1$.

2. An example: a generalized complete intersection Calabi–Yau threefold

2.1. Introduction

We illustrate the use of Proposition 1.4 (and its proof) for the generalized complete intersection Calabi Yau discussed in [1, Section 2.2.2]. We also consider an explicit example which has a non-trivial involution and we compute the Hodge numbers of a desingularization of the quotient threefold which is again a CY.

2.2. The varieties $P_2$ and $Y$

We consider the case that $P_2 = \mathbb{P}^4$, we choose the line bundle $L := \mathcal{O}_{\mathbb{P}^4}(2)$ and we let $d = 3$. Then the line bundle $L(d) = \mathcal{O}_{\mathbb{P}(2, 3)}$ is very ample on $P = \mathbb{P}^4 \times \mathbb{P}^1$ and thus a general section $F$ will define a smooth fourfold $Y$ of $P$. To obtain a CY threefold in $Y$, we consider global sections of the anticanonical bundle of $Y$. By adjunction, $\omega_Y = (\mathcal{O}_P(-5, -2) \otimes \mathcal{O}_P(2, 3))|_Y = \mathcal{O}_Y(-3, 1)$. Thus we take $M = \mathcal{O}_{\mathbb{P}^4}(3)$ and $\epsilon = 1$, so that $M[-\epsilon]|_Y = \mathcal{O}_Y(3, -1) = \omega_Y^{-1}$. As the $H^1$ of any line bundle on $\mathbb{P}^4$ is trivial, we can use (the proof of) Proposition 1.4 to find polynomials $G, H \in H^0(P, \mathcal{O}_P(3, 2))$ which together with $F$ define a generalized complete intersection $X$.

As in Example 1.2, we get

$$H^0(\mathcal{O}_Y(3, -1)) \xrightarrow{\sim} H^1(\mathcal{O}_P(1, -4)) .$$

To find explicit elements of $H^0(\mathcal{O}_Y(3, -1))$, we write the defining equation of $Y$ as

$$F = P_0z_0^3 + P_1z_0^2z_1 + P_2z_0^2 + P_3z_1^3 \quad (\in H^0(P, \mathcal{O}_P(2, 3))) ,$$

with $P_i \in H^0(\mathbb{P}^4, \mathcal{O}(2))$ homogeneous polynomials of degree two in $y = (y_0 : \ldots : y_4)$. As $H^1(\mathcal{O}_P(1, -4)) \cong H^0(\mathcal{O}_P(1)) \otimes H^1(\mathcal{O}_P(-4))$, a basis of this $5 \cdot 3 = 15$ dimensional vector space are the products of one of $y_0, \ldots, y_4$ with one of $z_0^{-3}z_1^{-1}, z_0^{-2}z_1^{-2}, z_0^{-1}z_1^{-3}$. Thus any class $q \in H^1(\mathcal{O}_P(1, -4))$ has a representative

$$q = Q_0z_0^{-3}z_1^{-1} + Q_1z_0^{-2}z_1^{-2} + Q_2z_0^{-1}z_1^{-3} \quad (\in H^1(\mathcal{O}_P(1, -4))) ,$$
with linear forms $Q_i \in H^0(\mathbb{P}^4, \mathcal{O}(1))$. As in the proof of Proposition 1.4 we must write:

$$F q = \tau_0 - \tau_1, \quad G := z_0^2 \tau_0, \quad H := -z_1^3 \tau_1,$$

with $\tau_i \in \mathcal{O}_P(3, -1)(\mathbb{P}^4 \times U_i)$. So we find

$$G = z_0^2 (P_1 Q_0 + P_2 Q_1 + P_3 Q_2) + z_0 z_1 (P_2 Q_0 + P_3 Q_1) + z_1^2 P_3 Q_0,$$

$$H = z_0^2 P_0 Q_2 + z_0 z_1 (P_0 Q_1 + P_1 Q_2) + z_1^2 (P_0 Q_0 + P_1 Q_1 + P_2 Q_2).$$

### 2.3. The base locus of $| - K_Y |$

In Section 2.2 we showed how to find the global sections of $\omega_Y^{-1} = \mathcal{O}_Y(3, -1)$ explicitly, locally such a section is given by the polynomials $G$ and $H$. From the formula for $F$ we see that if $x \in \mathbb{P}^4$ and $P_0(x) = \ldots = P_3(x) = 0$, then the curve $\{x\} \times \mathbb{P}^1$ lies in $Y$. This curve also lies in the zero loci of $G$ and $H$, for any choice of $Q_0, Q_1, Q_2 \in H^0(\mathcal{O}_{\mathbb{P}^1}(1))$, hence it lies in the base locus of anticanonical system $| - K_Y |$. Since the four quadrics $P_i = 0$ in $\mathbb{P}^4$ intersect in at least 24 points, counted with multiplicity, we see that this base locus is non-empty. Thus we cannot use Bertini’s theorem to guarantee that there are smooth CY threefolds $X \subset Y$, but we resort to an explicit example, see below.

### 2.4. The CY threefold $X$

To obtain an explicit example, we choose

$$P_0 := y_0^2 + y_1^2 + y_2^2 + y_3^2 + y_4^2, \quad P_1 := y_0^2 + y_2^2,$$

$$P_2 := \gamma_1^2 + y_3^2, \quad P_3 := y_0^2 + y_1^2 - y_2^2 - y_3^2 - y_4^2,$$

and

$$Q_0 := y_0, \quad Q_1 := y_1, \quad Q_2 := y_2.$$

Using a computer algebra system (we used Magma [12]), one can verify that $Y := (F = 0)$ and $X := (F = G = H = 0)$ are smooth varieties in $P$. The variety $X$ is a Calabi–Yau threefold since it is an anticanonical divisor on $Y$. In [1, (2.27), (2.28)] one finds that the Hodge numbers of $X$ are $(h^{1,1}(X), h^{2,1}(X)) = (2, 46)$, in particular, $h^2(X) = 2$, $h^3(X) = 94$.

### 2.5. Parameters

The CY threefold $X$ is defined by a section $F \in H^0(P, \mathcal{O}_P(2, 3))$ and a section $\tau \in H^0(Y, \mathcal{O}_Y(3, -1))$. The first is a vector space of dimension

$$h^0(P, \mathcal{O}_P(2, 3)) = h^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^1}(2)) \cdot h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3)) = 15 \cdot 4 = 60,$$

whereas the second has dimension 15. The group $GL(5, \mathbb{C}) \times GL(2, \mathbb{C})$ acts on $H^0(\mathcal{O}_P(2, 3))$ and has dimension $5^2 + 2^2 = 29$. The subgroup of elements $(\lambda I_5, \mu I_2)$ with $\lambda^2 \mu^3 = 1$ acts trivially, so we get $60 - 28 = 32$ parameters for $P$ and next $15 - 1 = 14$ parameters for $\tau$, so we do get $32 + 14 = 46 = h^{2,1}(X)$ parameters for $X$. So the general deformation of $X$ seems to be again a gCICY of the same type as $X$. (In [1], just below (2.28), the dependence of $X$ on $P$, which gives 32 parameters, seems to have been overlooked.)
2.6. A CY quotient

A well-known method to obtain Calabi–Yau threefolds is to consider desingularizations of quotients of such threefolds by finite groups, see for example [6]. In the example above, we see that $X \subset \mathbf{P}^4 \times \mathbf{P}^1$ has a subgroup $(\mathbf{Z}/2\mathbf{Z})^2 \subset \text{Aut}(X)$ given by the sign changes of $y_3$ and $y_4$. We consider the involution

$$\iota : X \rightarrow X, \quad \left( (y_0 : \ldots : y_4), (z_0 : z_1) \right) \mapsto \left( (y_0 : y_1 : y_2 : -y_3 : -y_4), (z_0 : z_1) \right).$$

Its fixed point locus has two components, one defined by $y_3 = y_4 = 0$ and the other by $y_0 = y_1 = y_2 = 0$ in $X$. The first is a curve in $\mathbf{P}^2 \times \mathbf{P}^1 \subset P$, which is smooth, irreducible and reduced of genus 8 according to Magma. Similarly, the other component is a genus 2 curve in $\mathbf{P}^1_{(y_3:y_4)} \times \mathbf{P}^1_{(z_0;z_1)} \subset P$. In fact, only $F = 0$ provides a non-trivial equation for this curve since $y_0 = y_1 = y_2 = 0$ implies $Q_0 = Q_1 = Q_2 = 0$ and hence $G = H = 0$ on this $\mathbf{P}^1 \times \mathbf{P}^1$. As $F = 0$ defines a smooth curve of bidegree $(2, 3)$ in $\mathbf{P}^1 \times \mathbf{P}^1$, this curve has genus $(2 - 1)(3 - 1) = 2$.

In particular, the singular locus of the quotient $X/\iota$ consists of two curves of $A_1$-singularities. Since the fixed point locus $X'$ consists of two curves, we conclude that locally on $X$ the involution is given by $(t_1, t_2, t_3) \mapsto (-t_1, -t_2, t_3)$ in suitable coordinates. Hence $\iota$ acts trivially on the nowhere vanishing holomorphic 3-form on the CY threefold $X$. Thus the blow up $\tilde{Z}$ of $X/\iota$ in the singular locus will again be a CY threefold.

We determine the Hodge numbers of $\tilde{Z}$. To do so, it is more convenient to consider the blow up $\tilde{X}$ of $X$ in the fixed point locus $X'$. The involution extends to an involution $\tilde{\iota}$ on $\tilde{X}$, the fixed point set of $\tilde{\iota}$ consists of the two exceptional divisors and the quotient $\tilde{X}/\tilde{\iota}$ is the same $Z$. Moreover, $H^i(Z, \mathbb{Q}) \cong H^i(\tilde{X}, \mathbb{Q})^{\tilde{\iota}}$, the $\tilde{\iota}$-invariant subspace.

Standard results on the blow up of smooth varieties in smooth subvarieties (cf. [13, Thm 7.31]) show that $h^2(\tilde{X}) = h^2(X) + 2 = 4$ (due to the two exceptional divisors over the two fixed curves) and $h^3(\tilde{X}) = h^3(X) + 2 \cdot 8 + 2 \cdot 2 = 114$ (the contribution of the $H^1$ of the fixed curves to $H^3$ of the blow up). The Lefschetz fixed point formula for $\tilde{\iota}$ gives

$$\chi(\tilde{X}^{\tilde{\iota}}) = \sum_{i=0}^{6} (-1)^i \text{tr}(\tilde{\iota}^*|H^i(\tilde{X}, \mathbb{Q})) = 4.$$

Notice that $\tilde{\iota}^*$ is the identity on $H^0, H^2, H^4, H^6$, in particular $h^2(Z) = \dim H^2(\tilde{X}, \mathbb{Q})^{\tilde{\iota}} = 4$. The fixed points of $\tilde{\iota}$ are the two exceptional divisors, these are $\mathbf{P}^1$-bundles over the exceptional curves hence

$$2(2 - 2 \cdot 2) + 2(2 - 2 \cdot 8) = 1 - 0 + 4 - t_3 + 4 - 0 + 1 \implies t_3 = 42.$$

If the $+$, $-$ eigenspaces of $\tilde{\iota}$ on $H^3(\tilde{X}, \mathbb{Q})$ have dimensions $a, b$ respectively, then $a + b = 114$ and $a - b = 42$, thus $a = 78$ and $a = \dim H^3(\tilde{X}, \mathbb{Q})^{\tilde{\iota}} = h^3(Z)$. As $Z$ is a CY threefold it has $h^{3,0}(Z) = 1$ and thus $h^{2,1}(Z) = (78 - 2)/2 = 38$. Other examples of CY threefolds with $(h^{1,1}, h^{2,1}) = (4, 38)$ are already known.

2.7. A (singular) projective model of $Z$

The fibers of $\pi_2 : X \rightarrow \mathbf{P}^1$ are K3 surfaces, complete intersections of a quadric and a cubic hypersurface in $\mathbf{P}^4$. The involution $\iota$ on $X$ restricts to a Nikulin involution on each smooth fiber. The quotient of such a fiber by the involution will in general be isomorphic to a K3 surface in
$\mathbb{P}^2 \times \mathbb{P}^1$, defined by an equation of bidegree $(3, 2)$ (see [8, Section 3.3]). Using the same method as in that reference, we found that the rational map

$$\mathbb{P}^4 \times \mathbb{P}^1 \to \mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^1,$$

$$\left( (y_0 : \ldots : y_4), (z_0 : z_1) \right) \mapsto \left( (y_0 : y_1 : y_2), (y_3 : y_4), (z_0 : z_1) \right)$$

factors over $X/\iota$ and the image, defined by an equation of multidegree $(3, 2, 2)$, is birational with $Z$. Using the explicit equation for the image and Magma, we found that the image has 38 singular points.

References