ON THE CHARACTER DEGREE GRAPH OF SOLVABLE GROUPS

ZEINAB AKHLAGHI, CARLO CASOLO, SILVIO DOLFI, KHATOON KHEDRI, AND EMANUELE PACIFICI

Abstract. Let G be a finite solvable group, and let $\Delta(G)$ denote the *prime graph* built on the set of degrees of the irreducible complex characters of G. A fundamental result by P.P. Pálfy asserts that the complement $\bar{\Delta}(G)$ of the graph $\Delta(G)$ does not contain any cycle of length 3. In this paper we generalize Pálfy's result, showing that $\bar{\Delta}(G)$ does not contain any cycle of odd length, whence it is a bipartite graph. As an immediate consequence, the set of vertices of $\Delta(G)$ can be covered by two subsets, each inducing a complete subgraph. The latter property yields in turn that if n is the clique number of $\Delta(G)$, then $\Delta(G)$ has at most $2n$ vertices. This confirms a conjecture by Z. Akhlaghi and H.P. Tong-Viet, and provides some evidence for the famous *ρ-σ conjecture* by B. Huppert.

1. Introduction

Character Theory is one of the fundamental tools in the theory of finite groups, and, given a finite group G, the study of the set $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$, of all degrees of the irreducible complex characters of G, is a particularly intriguing aspect of this theory. One of the methods that have been devised to approach such degree-set is to consider the *prime graph* $\Delta(G)$ attached to it.

The character degree graph $\Delta(G)$ is thus defined as the (simple undirected) graph whose vertex set is the set $\mathrm{V}(G)$ of all the prime numbers that divide some $\chi(1) \in \mathrm{cd}(G)$, while a pair $\{p, q\}$ of distinct vertices p and q, belongs to the edge set $\mathrm{E}(G)$ if and only if pq divides an element in $\mathrm{cd}(G)$.

There is a well-developed literature on character degree graphs (see for instance the survey [6]). A large part of it is focussed on what is obviously one of the natural motivations behind the introduction of such graphs, that is studying to which extent specific properties of a group are reflected by graph theoretical features of its graph, or aimed at describing in detail the degree graph of interesting (classes of) groups.

As regards the investigation about general properties of $\Delta(G)$, the celebrated Ito-Michler Theorem may be regarded as the first crucial step: this fundamental result characterizes $\mathrm{V}(G)$ as the set of all primes p for which G does not have an abelian normal Sylow p-subgroup.

On the other hand, another fundamental result *in the context of solvable groups* is Pálfy's "Three-Vertex Theorem" ([8]): given any three distinct primes in $\mathrm{V}(G)$, at least two of them are adjacent in $\Delta(G)$. For instance, the bound of 3 for the diameter in the connected case, as well as the structure of the non-connected case as the union of two complete subgraphs, are straightforward consequences of Pálfy's

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theorem. A theorem that may be rephrased by saying that, for a finite solvable group G, the complement of $\Delta(G)$ (that we will denote by $\bar{\Delta}(G)$) does not contain any triangle.

The main result of this paper is the following extension of Pálfy's theorem. We stress that Palfy's theorem is actually proved here with an argument which embeds naturally into our analysis (specifically, in the last two paragraphs of the proof of Theorem A). This makes our treatment essentially self-contained.

Theorem A. *Let G be a finite solvable group. Then the graph $\bar{\Delta}(G)$ does not contain any cycle of odd length. $\bar{\Delta}(G)$ is therefore a bipartite graph.*

An immediate consequence is the following.

Corollary B. *Let G be a finite solvable group. Then $\mathrm{V}(G)$ is covered by two subsets, each inducing a clique (i.e. a complete subgraph) in $\Delta(G)$. In particular, for every subset $\mathcal{S}$ of $\mathrm{V}(G)$, at least half of the vertices in $\mathcal{S}$ are pairwise adjacent in $\Delta(G)$.*

The fact that, for a finite solvable group G, the set $\mathrm{V}(G)$ is covered by two subsets each inducing a clique was already known to be true in three special cases: when $\Delta(G)$ is disconnected (as already mentioned), for metanilpotent groups ([5, Theorem A]) and, in the connected case, when the diameter of $\Delta(G)$ attains the upper bound 3 (see Remark 4.4 in [2]). By Corollary B, this is indeed a feature of $\Delta(G)$ in full generality.

Again by Corollary B, if G is a finite solvable group and n is the maximum size of a clique in $\Delta(G)$, then $|\mathrm{V}(G)| \leq 2n$. This is precisely what is conjectured by Akhlaghi and Tong-Viet in [1] (where the authors prove the result for $n = 3$).

Moreover, since the distinct prime divisors of a single irreducible character degree do of course induce a clique in $\Delta(G)$, Corollary B provides some more evidence for the famous (and still open) *ρ-σ conjecture* by Huppert, which predicts that any finite solvable group G has an irreducible character whose degree is divisible by at least half the primes in $\mathrm{V}(G)$.

Finally we just recall that, as shown for instance by $\mathrm{Alt}(5)$, our theorem (as well as Pálfy's) does not hold for non-solvable groups.

2. Notation and preliminary results

Throughout this paper, every group is tacitly assumed to be finite. As customary, for n a positive integer, $\pi(n)$ denotes the set of all prime divisors of n, and, if G is a group, $\pi(G) = \pi(|G|)$. For a given group G, we denote by $\Delta(G)$ the character degree graph as defined in the introduction, and write $\bar{\Delta}(G)$ for the complement of $\Delta(G)$; we emphasize that the set of vertices of $\Delta(G)$ (i.e. the set of all $p \in \pi(G)$ such that G does not have a normal abelian Sylow p-subgroup) is denoted by $\mathrm{V}(G)$. We also denote by $\mathrm{E}(G)$ and $\bar{\mathrm{E}}(G)$ the set of edges of $\Delta(G)$ and of $\bar{\Delta}(G)$, respectively. If N is a normal subgroup of G and $\lambda \in \mathrm{Irr}(N)$, then $\mathrm{Irr}(G \mid \lambda)$ denotes the set of irreducible characters of G whose restriction to N has λ as an irreducible constituent, and $\mathrm{cd}(G \mid \lambda)$ the set of degrees of the characters in $\mathrm{Irr}(G \mid \lambda)$.

We will freely use without references some basic facts of Character Theory such as Clifford Correspondence, Gallagher's Theorem, Ito-Michler's Theorem, properties of character extensions and coprime actions (see [4]).

Also, we will take into account the following well-known result concerning character degrees (in the following statement, $\mathbf{F}(G)$ and $\mathbf{F}_2(G)$ denote the Fitting subgroup and the second Fitting subgroup of G, respectively).

Lemma 2.1 ([7, Proposition 17.3]). *Let G be a solvable group. Then there exists $\chi \in \mathrm{Irr}(G)$ such that $\pi(\mathbf{F}_2(G)/\mathbf{F}(G)) \subseteq \pi(\chi(1))$.*

Let $\Gamma(p^n)$ be the semilinear group on the field $\mathrm{GF}(p^n)$ (see [7, Section 2]). If V is an n-dimensional vector space over $\mathrm{GF}(p)$, then V can be identified with the additive group of a field of order p^n, and in this sense we write $\Gamma(V)$ for $\Gamma(p^n)$.

The following result will be a key ingredient in our discussion. Before stating it, we also recall that a prime t is called a *primitive prime divisor* for (a, n) (where $a > 1$ and n are positive integers) if t divides $a^n - 1$ but t does not divide $a^j - 1$ for $1 \leq j < n$.

Lemma 2.2. *Let $G = AH$ be a solvable group, where $A = \mathbf{F}(G)$ is abelian and $(|A|, |H|) = 1$. Let $q \in \pi(\mathbf{F}(H))$, and let $s \in \pi(H)$ be distinct vertices such that $\{q, s\} \notin \mathrm{E}(G)$. Let $Q \in \mathrm{Syl}_q(\mathbf{F}(H))$, $S \in \mathrm{Syl}_s(H)$ and $L = (QS)^H$ (the normal closure of QS in H). Then the following conclusions hold.*

(a) *$A = U \times W$, where $U = [A, Q] = [A, L]$ is an elementary abelian p-group for a suitable prime p, and $W = \mathbf{C}_A(Q) = \mathbf{C}_A(L)$.*
(b) *$L = L_0 S \leq \Gamma(U)$, where $L_0 = \mathbf{F}(L)$ is a cyclic group acting fixed-point freely and irreducibly on U.*
(c) *Setting $|U| = p^n$, there exists a primitive prime divisor t of (p, n) such that t divides $|L_0|$.*
(d) *For every prime $r \in \pi(H)$ with $\{r, s\} \notin \mathrm{E}(G)$, H has an abelian normal Sylow r-subgroup. In particular, $Q \in \mathrm{Syl}_q(H)$.*

Proof. As concerns (a), (b) and (c), the statement is a special case of Lemma 3.9 in [2]. Thus we only have to prove (d).

Set $K = \mathbf{C}_H(U)$, and observe that U is a faithful H/K-module over $\mathrm{GF}(p)$. As clearly $L \cap K = 1$, claims (b) and (c) ensure that $\mathbf{F}(LK/K)$ has a characteristic subgroup T/K of order t, which is therefore characteristic in LK/K and hence normal in H/K. Since t is a primitive prime divisor of (p, n), the action of T/K on U is easily seen to be irreducible, and therefore H/K embeds in $\Gamma(U)$ (see Theorem 2.1 in [7], for instance). As a consequence, denoting by X/K the Fitting subgroup of H/K, the prime divisors of H/X constitute a clique in $\Delta(H/K)$, which is a subgraph of $\Delta(G)$; moreover, Lemma 3.7 of [2] yields that X/K is cyclic.

By Lemma 2.1 and by our assumption of nonadjacency between q and s, the prime s is not in $\pi(\mathbf{F}(H))$. It follows that S is not normal in L, so SK/K is not normal in LK/K and H/K does not have a normal Sylow s-subgroup. In other words, s is a divisor of H/X. Now, taking into account the conclusion of the paragraph above, if $r \in \pi(H)$ is nonadjacent to s in $\Delta(G)$, then $r \nmid |H/X|$. As a consequence, H/K has a cyclic normal Sylow r-subgroup, thus $r \notin \mathrm{V}(H/K)$. Observe also that $r \notin \mathrm{V}(K)$ as well. In fact, certainly there exists $\theta \in \mathrm{Irr}(L)$ such that $\theta(1)$ is divisible by s; thus, if $\phi \in \mathrm{Irr}(K)$ has a degree divisible by r, then, for any $\xi \in \mathrm{Irr}(H)$ such that $\theta \times \phi \in \mathrm{Irr}(LK)$ is a constituent of ξ_{LK}, we would have $r \cdot s \mid \xi(1)$, against the nonadjacency between r and s in $\Delta(G)$. Now, let χ be in $\mathrm{Irr}(H)$, and let β be an irreducible constituent of χ_K: we have

$$\chi(1) = e \cdot \beta(1) \cdot |H : I_H(\beta)|,$$

where e is the degree of an irreducible projective representation of $I_H(\beta)/K$. If r divides $|H : I_H(\beta)|$, then it also divides $|H : I_H(\theta \times \beta)|$ (where $\theta \in \mathrm{Irr}(L)$ is as above), and therefore any $\xi \in \mathrm{Irr}(H \mid \theta \times \beta)$ would have a degree divisible by $r \cdot s$, a gain a contradiction. Also, e is the degree of an irreducible ordinary representation of a Schur covering Γ of $I_H(\beta)/K$; but, since $I_H(\beta)/K$ has a cyclic normal Sylow r-subgroup, we have that Γ has an abelian normal Sylow r-subgroup, therefore r does not divide e as well. Since we observed that $\beta(1)$ is not divisible by r, we conclude that $r \nmid \chi(1)$; as this holds for every $\chi \in \mathrm{Irr}(H)$, we get that H has an abelian normal Sylow r-subgroup, as claimed. ■

3. Proof of Theorem A

In this section we prove Theorem A, whereas Corollary B can be immediately deduced from Theorem A, and its proof is therefore omitted.

The next three lemmas will be the core of our argument. In the following statement, $\mathbf{\Phi}(G)$ denotes as customary the Frattini subgroup of the group G.

Lemma 3.1. *Let G be a solvable group such that, for every proper factor group $\overline{G}$ of G, we have $\mathrm{V}(\overline{G}) \neq \mathrm{V}(G)$. Let M be a minimal normal subgroup of G, let $p \in \mathrm{V}(G) \setminus \mathrm{V}(G/M)$ and $P \in \mathrm{Syl}_p(G)$. Also, let π_p be the set of vertices of the connected component of p in $\bar{\Delta}(G)$. Then there exists a normal subgroup W of G such that $\pi_p \subseteq \mathrm{V}(W)$, and either $\mathbf{F}(W) = P$ with $P' = M$, or $\mathbf{F}(W) = M$ with $M \cap \mathbf{\Phi}(G) = 1$; in particular, $\mathbf{F}(W)$ is complemented in G.*

Proof. Let $\mathcal{N}_0$ be the set of the minimal normal subgroups of G which are contained in $\mathbf{\Phi}(G)$. For $N_0 \in \mathcal{N}_0$, if $p \in \mathrm{V}(G) \setminus \mathrm{V}(G/N_0)$ and $P \in \mathrm{Syl}_p(G)$, then PN_0/N_0 is abelian and normal in G/N_0, so P is normal in G (as $\mathbf{F}(G/N_0) = \mathbf{F}(G)/N_0$) and $P' = N_0$. Note that p is therefore the unique prime in $\mathrm{V}(G) \setminus \mathrm{V}(G/N_0)$. In this situation, we define $N_0^{\#} = P$; obviously $N_0^{\#}$ is complemented in G and, setting $F = \mathbf{F}(G)$, there exists $K \trianglelefteq G$ such that $F = K \times N_0^{\#}$. On the other hand, let $\mathcal{N}_1$ be the set of the minimal normal subgroups of G that are *not* contained in $\mathbf{\Phi}(G)$. If $N_1 \in \mathcal{N}_1$, we define $N_1^{\#} = N_1$: since $N_1^{\#} \cap \mathbf{\Phi}(G) = 1$, also in this case we have that $N_1^{\#}$ is complemented in G (see 4.4 in [3, III]) and, as $N_1^{\#} \leq \mathbf{Z}(F)$, there exists $K \trianglelefteq G$ such that $F = K \times N_1^{\#}$.

Observe that F is a direct product of subgroups $N^{\#}$, where N varies in $\mathcal{N}_0$ and in a suitable subset of $\mathcal{N}_1$. Moreover, if σ is the set of prime divisors of $|N_0|$ for $N_0 \in \mathcal{N}_0$, we have $\mathbf{O}_{\sigma'}(F) \cap \mathbf{\Phi}(G) = 1$, and it is easily seen that F itself has a complement L in G.

Let now M be a minimal normal subgroup of G. Take again $p \in \mathrm{V}(G) \setminus \mathrm{V}(G/M)$, $P \in \mathrm{Syl}_p(G)$, and let $M^{\#}$ be as above. As mentioned, we can write $F = K \times M^{\#}$ where K is a suitable normal subgroup of G, thus $H = LM^{\#}$ is a complement for K in G.

Define $W = \mathbf{C}_H(K)$. We have $W \trianglelefteq G$, $W \cap F = M^{\#}$ and hence $\mathbf{F}(W) = M^{\#}$. Note also that, as P commutes with K modulo M, we get $[P, K] \leq M \cap K = 1$; in particular, W contains the Sylow p-subgroups of H, and $p \in \mathrm{V}(W)$.

Let now $q \in \pi_p$ be a vertex of the connected component of p in $\bar{\Delta}(G)$, and let $Q \in \mathrm{Syl}_q(L)$. We prove, by induction on the distance $d = d_{\bar{\Delta}(G)}(p, q)$ that $1 \neq Q \leq W$. From this it follows immediately $q \in \mathrm{V}(W)$, since $\mathbf{F}(W) = M^{\#}$.

We consider first the case $d = 1$. Given a character $\lambda \in \mathrm{Irr}(M^{\#})$ such that $\lambda_M \neq 1_M$, we have that p divides $\chi(1)$ for every $\chi \in \mathrm{Irr}(G|\lambda)$. Observe also that, both in the case $M^{\#} = P$ as also $M^{\#} = M$, λ extends to $I_G(\lambda)$. Hence, Gallagher's Theorem and Clifford Correspondence, with the nonadjacency between q and p in $\Delta(G)$, imply that $I_{KL}(\lambda) \simeq I_G(\lambda)/M^{\#}$ contains a Sylow q-subgroup Q_0 of LK as a normal subgroup, and that Q_0 is abelian. Let now Q be a Sylow q-subgroup of L; by a suitable choice of λ, we can assume that $Q = Q_0 \cap L$. Since K centralizes $M^{\#}$, K is a normal subgroup of $I_{KL}(\lambda)$ as well. Hence Q centralizes $K = (K \cap Q_0) \times \mathbf{O}_{q'}(K)$, i.e. $Q \leq W$. Observe that $Q \neq 1$, as otherwise $(K \cap Q_0) \times M$ would contain an abelian normal Sylow q-subgroup of G.

Assume now $d \geq 2$ and let t be the vertex adjacent to q in a path of length d from p to q in $\bar{\Delta}(G)$. Let $T \in \mathrm{Syl}_t(L)$. Working by induction on d, we have $1 \neq T \leq W$. Observe that, since $t \neq p$ and $M^{\#} = \mathbf{F}(W)$, T is certainly not normal in W. Let $X = T^G = T^W$ (the normal closure of T in W) and $U = \mathbf{O}^t(X)$; then $1 \neq U = [U, T]$. Let $\overline{U} = U/V$ be a chief factor of G. If $\overline{U} \leq \mathbf{\Phi}(\overline{G})$, then $\overline{T}$ is normal in $\overline{G}$ and so $[\overline{U}, \overline{T}] = 1$; in particular $[U, T] \leq V < U$, a contradiction. Thus $\overline{U} \cap \mathbf{\Phi}(\overline{G}) = 1$ and hence $\overline{U}$ is complemented in $\overline{G}$. Now, for $1 \neq \lambda \in \mathrm{Irr}(\overline{U})$, the prime t divides the index $|\overline{G} : I_{\overline{G}}(\lambda)|$ and hence the fact that t is not adjacent to q in $\Delta(\overline{G})$ forces $I_{\overline{G}}(\lambda)$ to contain $\overline{Q}_0 = Q_0V/V$ as an abelian normal subgroup, where Q_0 is a Sylow q-subgroup of G. Note that, by a suitable choice of λ, we can assume $Q = Q_0 \cap L$.

Now, let $Y = KU$; since $K \cap U = 1$, we have $\overline{Y} = Y/V = \overline{K} \times \overline{U}$, where $\overline{K} = KV/V \simeq K$. Let $1 \neq \lambda \in \mathrm{Irr}(\overline{U})$ and Q_0 be as above. For every $\phi \in \mathrm{Irr}(\overline{K})$ we have $I_{\overline{G}}(\phi \times \lambda) \leq I_{\overline{G}}(\lambda)$, hence t divides the index $|\overline{G} : I_{\overline{G}}(\phi \times \lambda)|$; since t and q are not adjacent in $\Delta(\overline{G})$, this yields $\overline{Q}_0 \leq I_{\overline{G}}(\phi \times \lambda) \leq I_{\overline{G}}(\phi)$, and this holds for every $\phi \in \mathrm{Irr}(\overline{K})$. Thus $[\overline{K}, \overline{Q}_0] = 1$, that is $[K, Q_0] \leq K \cap V = 1$. Hence, in particular, $Q \leq \mathbf{C}_H(K) = W$. Moreover, $Q_0 \cap K$ is abelian; so Q cannot be trivial, for otherwise $Q_0 = Q_0 \cap F$ would be abelian and normal in G. The proof is complete. ∎

Lemma 3.2. *Let p be a prime, E an elementary abelian p-group, and H a p'-group acting faithfully on E. Let $G = EH$, and write $\pi_0 = \pi(\mathbf{F}(H))$.*

(a) *Let q, r, s be distinct prime divisors of $|H|$ such that $\{q, s\}, \{s, r\} \in \bar{\mathrm{E}}(G)$, and $q \in \pi_0$. Then $s \notin \pi_0$ and $r \in \pi_0$. Moreover, H has both a normal abelian Sylow q-subgroup and Sylow r-subgroup.*

(b) *Let p_1, p_2, p_3, p_4 be distinct prime divisors of $|H|$ such that $\{p_i, p_{i+1}\} \in \bar{\mathrm{E}}(G)$ for $i \in \{1, 2, 3\}$, and $p_1 \in \pi_0$. Then $\{p_1, p_4\} \in \bar{\mathrm{E}}(G)$.*

Proof. As it regards (a), by Lemma 2.1 the subgraph of $\Delta(G)$ induced by π_0 is complete, so the claim $s \notin \pi_0$ is immediate from the assumption of nonadjacency between q and s. The remaining claims of (a) follow from Lemma 2.2, therefore we only have to prove (b).

Let p_1, p_2, p_3, p_4 be as in the assumptions. Then (a) yields $\{p_1, p_3\} \subseteq \pi_0$, whereas $\{p_2, p_4\} \cap \pi_0 = \emptyset$; actually, setting $P_i \in \mathrm{Syl}_{p_i}(H)$ for $i = 1, 2, 3, 4$, we have that P_1 and P_3 are abelian and normal in H.

Write $L_1 = (P_1P_2)^H$, $L_3 = (P_3P_4)^H$, $L = L_1L_3$, $M = \mathbf{C}_E(L)$ and $V = E/M$. Also, let $U_1 = [V, P_1]$ and $U_3 = [V, P_3]$. Then, clearly, $V = U_1U_3$, and, again by Lemma 2.2, we have that $U_1 = [V, L_1]$ and $U_3 = [V, L_3]$ are irreducible L-modules, hence irreducible H-modules.

Let us assume $U_1 \neq U_3$; then $V = U_1 \times U_3$, $U_1 = \mathbf{C}_V(L_3)$ and $U_3 = \mathbf{C}_V(L_1)$. Thus we get $L_1 \cap L_3 = 1$, whence $L = L_1 \times L_3$ and $VL = U_1L_1 \times U_3L_3$. As p_2 and

p_3 divide, respectively, the degree of a character of U_1L_1 and of U_3L_3, we get that p_2 and p_3 are adjacent in the graph $\Delta(VL)$, which is a contradiction because VL is isomorphic to a section of G.

Thus we have $U_1 = U_3$. Setting $U = [E, P_1]$, we now have $E = U \times M$. So, $[E, P_1] = [E, P_3]$ and $p_1 \cdot p_3$ divides $|H : I_H(\lambda)|$ for every nonprincipal $\lambda \in \mathrm{Irr}(U)$. Observe also that U is complemented in G by MH. Let χ be an irreducible character of G whose degree is divisible by p_1. Since P_1 is an abelian normal Sylow subgroup of H, then MP_1 is an abelian normal subgroup of $MH \simeq G/U$, and hence the kernel of χ cannot contain U. Therefore, denoting by λ a (nonprincipal) irreducible constituent of χ_U, the degree of χ is divisible by $|G : I_G(\lambda)| = |H : I_H(\lambda)|$, thus by $p_1 \cdot p_3$. As p_4 is not adjacent to p_3 in $\Delta(G)$, we conclude that p_4 does not divide $\chi(1)$. But this holds for every $\chi \in \mathrm{Irr}(G)$ such that $p_1 \mid \chi(1)$, hence $\{p_1, p_4\} \in \bar{\mathrm{E}}(G)$, as desired. ■

Lemma 3.3. *Let G be a solvable group, and p a prime in $\mathrm{V}(G)$. Assume that $P = \mathbf{F}(G)$ is a Sylow p-subgroup of G with $P' \neq 1$ minimal normal in G and, denoting by H a complement for P in G, define $\pi_0 = \pi(\mathbf{F}(H))$. Assume further that $p_0 = p$, p_1, p_2 and p_3 are distinct primes in $\mathrm{V}(G)$ (except possibly $p_0 = p_3$) such that, for $i \in \{0, 1, 2\}$, $\{p_i, p_{i+1}\} \in \bar{\mathrm{E}}(G)$. If $\{p_1, p_2\} \cap \pi_0 = \emptyset$, then $p \neq p_3$ and $\{p, p_3\} \in \bar{\mathrm{E}}(G)$.*

Proof. Assuming that p_1 is not in π_0, and that either $p = p_3$ or p is adjacent to p_3 in $\Delta(G)$, we will show $p_2 \in \pi_0$.

Setting $M = P'$, let $\widehat{M}$ denote the dual group $\mathrm{Irr}(M)$, and let λ be a nontrivial element in $\widehat{M}$. For any $\tau \in \mathrm{Irr}(P \mid \lambda)$ we get $I_H(\tau) \leq I_H(\lambda)$, because $M \leq \mathbf{Z}(P)$ and so τ_M is a multiple of λ; but, by [4, 13.28], there exists $\theta \in \mathrm{Irr}(P \mid \lambda)$ for which in fact equality holds. Now,

$$\mathrm{cd}(G \mid \theta) = \{|H : I_H(\lambda)| \cdot \theta(1) \cdot \xi(1) \ : \ \xi \in \mathrm{Irr}(I_G(\lambda)/P)\},$$

thus, the nonadjacency between p and p_1 forces $I_H(\lambda) \simeq I_G(\lambda)/P$ to contain a unique abelian Sylow p_1-subgroup of H. Set $K = \mathbf{C}_H(M)$ and note that, for $\lambda \in \widehat{M} \setminus \{1_M\}$, K is a normal subgroup of $I_H(\lambda)$. Denoting by P_1 the unique Sylow p_1-subgroup of H contained in $I_H(\lambda)$, we have that $P_1 \cap K$ is a normal Sylow p_1-subgroup of K, whence it lies in $\mathbf{F}(H)$; but we are assuming $p_1 \notin \pi_0$, thus $p_1 \nmid |K|$ and $P_1 \leq \mathbf{C}_H(K)$. As then p_1 divides $|H/K|$, an application of Lemma 3.6 in [2] yields that H/K can be identified with a subgroup of the semilinear group $\Gamma(\widehat{M})$. Finally, setting $X/K = \mathbf{F}(H/K)$, the prime p_1 divides the order of the cyclic group $|H/X|$; in fact, if we assume the contrary, then $P_1K = P_1 \times K$ is normal in H, and therefore $P_1 \leq \mathbf{F}(H)$ against our assumptions. Observe that the prime divisors of $|H/X|$ constitute a clique in $\Delta(G)$; in particular, $p_2 \nmid |H/X|$.

Assume now that p_2 is a divisor of $|X/K|$. Hence, if P_2 is a Sylow p_2-subgroup of H, we get $K < P_2K \trianglelefteq H$ and $[M, P_2K] = M$. It follows that, for every $\lambda \in \widehat{M} \setminus \{1_M\}$, p_2 divides $|H : I_H(\lambda)|$. In particular, p_2 divides the degree of every irreducible character of G whose kernel does not contain M. Now, there exists $\chi \in \mathrm{Irr}(G)$ such that $\{p, p_3\} \subseteq \pi(\chi(1))$, and the kernel of such a χ clearly does not contain M. As a consequence, $p_2 \cdot p_3$ divides $\chi(1)$, a contradiction. Our conclusion so far is that $P_2 \leq K$.

Consider now the normal closure P_1^H of P_1 in H, set $Z = \mathbf{Z}(K)$ and $L = P_1^H Z \trianglelefteq H$. Recalling that $P_1 \leq \mathbf{C}_H(K) \trianglelefteq H$, we get $L \leq \mathbf{C}_H(K)$ as well, and so $K \cap L = Z$. In other words, $KL/Z = (K/Z) \times (L/Z)$. Observe that p_1 lies in $\mathrm{V}(L/Z)$, as otherwise we would have $P_1Z = P_1 \times Z \trianglelefteq L$, whence $P_1 \leq \mathbf{F}(H)$ against our assumptions. Now, the nonadjacency between p_1 and p_2 in $\Delta(G)$ forces $p_2 \notin \mathrm{V}(K/Z)$. Therefore we get $P_2Z/Z \trianglelefteq H/Z$. But P_2Z is nilpotent, and we conclude that $P_2 \leq \mathbf{F}(H)$, as desired. ∎

We are now ready to prove Theorem A, that we state again.

Theorem 3.4. *Let G be a solvable group. Then the graph $\bar{\Delta}(G)$ does not have any cycle of odd length.*

Proof. Let G be a counterexample of minimal order to the statement, and let ℓ be the smallest odd number for which a cycle $\mathcal{C}$ of length ℓ can be found in $\bar{\Delta}(G)$. Let $\{p_0, p_1, ..., p_{\ell-1}\}$ be the set of distinct vertices lying in $\mathcal{C}$, where we assume $\{p_{\ell-1}, p_0\}, \{p_i, p_{i+1}\} \in \bar{\mathrm{E}}(G)$ for every $i \in \{0, ..., \ell-2\}$; of course, by our choice of ℓ, these are the only edges in $\bar{\Delta}(G)$ between vertices in $\mathcal{C}$.

We start by observing that, for every proper factor group $\overline{G}$ of G, some vertex of $\mathcal{C}$ does not belong to $\mathrm{V}(\overline{G})$, for otherwise $\mathcal{C}$ would be a cycle of odd length in $\bar{\Delta}(\overline{G})$, against the minimality of G. Let M be a minimal normal subgroup of G and let $p = p_0 \notin \mathrm{V}(G/M)$.

By Lemma 3.1 and the minimality of G, either $F = \mathbf{F}(G)$ is a nonabelian Sylow p-subgroup of G with F' minimal normal in G, or F is itself minimal normal in G. In any case, F has a complement H in G; in what follows, we will denote by π_0 the set of prime divisors of $\mathbf{F}(H)$.

Let us first consider the case when F is a minimal normal subgroup of G. We clearly have $\{p_0, ..., p_{\ell-1}\} \subseteq \pi(H)$. Also, if P is the abelian normal Sylow p-subgroup of H, we get $\mathbf{C}_F(P) = 1$. In other words, p divides $|H : I_H(\lambda)|$ for every $\lambda \in \mathrm{Irr}(F) \setminus \{1_F\}$. As a consequence, we have that $I_H(\lambda)$ contains a unique Sylow p_1-subgroup of H for every $\lambda \in \mathrm{Irr}(F) \setminus \{1_F\}$. Now, by [2, Lemma 3.6], H can be identified with a group of semilinear maps on the dual group of F (note that, as $|\pi(H)| > 2$, $H \not\leq \mathrm{GL}_2(3)$). In particular, $H/\mathbf{F}(H)$ is abelian, and therefore both π_0 and $\pi(H) \setminus \pi_0$ induce complete subgraphs in $\Delta(G)$. It follows that two consecutive vertices of the cycle $\mathcal{C}$ cannot lie both in π_0 nor both in $\pi(H) \setminus \pi_0$, and this is clearly impossible because ℓ is odd.

Our conclusion so far is that F is a nonabelian Sylow p-subgroup of G, hence we are in the setting of Lemma 3.3. Let us assume first $\ell \geq 5$. Since $\{p, p_3\} \notin \bar{\mathrm{E}}(G)$, we conclude that one among p_1 and p_2 lies in π_0. Now, adopting the bar convention for $\overline{G} = G/\mathbf{\Phi}(G)$, we have that $\overline{F}$ is an elementary abelian p-group, and $\overline{H} \simeq H$ is a p'-group acting faithfully (by conjugation) on $\overline{F}$. Therefore, we can apply Lemma 3.2. Observe that $p_1, ..., p_{\ell-1}$ all lie in $\mathrm{V}(\overline{G}) \cap \pi(\overline{H})$, and they are of course consecutive vertices of a path in $\bar{\Delta}(\overline{G})$. If $p_1 \in \pi_0$, then Lemma 3.2(b) yields that $\{p_1, p_4\} \in \bar{\mathrm{E}}(G)$, a contradiction. On the other hand, if p_1 does not lie in π_0, then we have seen that p_2 does, and so does $p_{\ell-1}$ by an iterated application of Lemma 3.2(a). But again Lemma 3.2(b) yields now $\{p_{\ell-4}, p_{\ell-1}\} \in \bar{\mathrm{E}}(G)$, the final contradiction for the case $\ell \geq 5$.

It remains to treat the situation when $\ell = 3$. In view of Lemma 3.3, we may assume that p_1 is a divisor of $|\mathbf{F}(H)|$ (thus p_2 is not). Set $P_1 = \mathbf{O}_{p_1}(H)$, $P_2 \in \mathrm{Syl}_{p_2}(H)$, and define $L = (P_1P_2)^H$. We first observe that P_1 centralizes F' and, for every $\lambda \in \mathrm{Irr}(F') \setminus \{1_{F'}\}$, a unique H-conjugate of P_2 lies in $I_H(\lambda)$. In fact, arguing as in the second paragraph of the proof of Lemma 3.3, we see that $|H : I_H(\lambda)|$

is coprime with $p_1 \cdot p_2$ and $p_2 \notin \mathrm{V}(I_H(\lambda))$, for every $\lambda \in \mathrm{Irr}(F') \setminus \{1_{F'}\}$. Note also that, for similar reasons, P_1 fixes all the nonlinear irreducible characters of F; therefore, setting $U = [F, P_1]$, Theorem 19.3(a) in [7] yields $U' = F'$. Now, by Lemma 2.2, we get $U = [F, L]$, and U/F' is a faithful irreducible L-module. Also, since $U = \mathbf{F}(UL)$ is nonabelian, we get $\{p_0, p_1, p_2\} \subseteq \mathrm{V}(UL)$; but UL is normal in G, thus the minimality of G forces $F = U$ (so $F' = \mathbf{C}_F(P_1)$) and $L = H$. Finally, observe that no nonprincipal irreducible character of $F/F' = [F/F', P_1]$ is fixed by P_1; therefore, in view of the nonadjacency between p_1 and p_2 in $\Delta(G)$, for every nonprincipal $\mu \in \mathrm{Irr}(F/F')$, a unique H-conjugate of P_2 lies in $I_H(\mu)$.

We are then in a position to apply Lemma 3.8 in [2], obtaining that $|F/F'| = |F'|$. Let x be an element in $F \setminus F'$, and consider the map $f \mapsto [f, x]$ from F to F'; recalling that $F' \leq \mathbf{Z}(F)$, this defines a group homomorphism whose kernel strictily contains F'. As a consequence, $|[F, x]|$ is strictly smaller than $|F/F'| = |F'|$ and we conclude that $[F, x]$ is properly contained in F'. Let K be a maximal subgroup of F' containing $[F, x]$, so $K \trianglelefteq FP_1$, and take $\lambda \in \mathrm{Irr}(F')$ with $K = \ker(\lambda)$. Since P_1 fixes no nontrivial element in F/F', we can choose an element $y \in P_1$ such that, setting $z = [x, y]$, we have $z \notin K$. As $\ker(\lambda^F) = K$, there exists $\theta \in \mathrm{Irr}(F \mid \lambda)$ such that $z \notin \ker(\theta)$. As already mentioned, θ is P_1-invariant, and hence it extends to a character $\psi \in \mathrm{Irr}(FP_1)$. If Ψ is a representation affording ψ, then $\Psi(x)$ is a scalar matrix; in fact, Ψ_F affords θ, and $x \in \mathbf{Z}(\theta)$ because $xK \in \mathbf{Z}(F/K)$. In particular, $\Psi(x)$ commutes with $\Psi(y)$. But now $\Psi(z)$ is the identity matrix and z lies in $\ker(\psi_F) = \ker(\theta)$, the final contradiction that completes the proof. ■

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Zeinab Akhlaghi, Faculty of Math. and Computer Sci.,
Amirkabir University of Technology (Tehran Polytechnic), 15914 Tehran, Iran.
E-mail address: z.akhlaghi@aut.ac.ir

Carlo Casolo, Dipartimento di Matematica e Informatica U. Dini,
Università degli Studi di Firenze, viale Morgagni 67/a, 50134 Firenze, Italy.
E-mail address: carlo.casolo@unifi.it

Silvio Dolfi, Dipartimento di Matematica e Informatica U. Dini,
Università degli Studi di Firenze, viale Morgagni 67/a, 50134 Firenze, Italy.
E-mail address: dolfi@math.unifi.it

Khatoon Khedri, Department of Mathematical Sciences,
Isfahan University of Technology, 84156-83111 Isfahan, Iran.
E-mail address: k.khedri@math.iut.ac.ir

Emanuele Pacifici, Dipartimento di Matematica F. Enriques,
Università degli Studi di Milano, via Saldini 50, 20133 Milano, Italy.
E-mail address: emanuele.pacifici@unimi.it