Hilbert curves of 3-dimensional scrolls over surfaces

Antonio Lanteri

Abstract

Let (X, L) be a 3-dimensional scroll over a smooth surface Y. Its Hilbert curve is an affine plane cubic consisting of a given line and a conic. This conic turns out to be the Hilbert curve of the $\mathbb{Q}$-polarized surface $(Y, \frac{1}{2}\det\mathcal{E})$, where $\mathcal{E}$ is the rank-2 vector bundle obtained by pushing down L via the scroll proiection, if and only if $\mathcal{E}$ is properly semistable in the sense of Bogomolov.

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Introduction

The Hilbert curve of a polarized manifold was introduced in [5] and its study has been continued in [10], [11] and [4]. The natural expectation is that several properties of the polarized manifold are encoded by this object. In fact a relevant property of the Hilbert curve is its sensitivity with respect to fibrations that suitable adjoint linear systems to the polarizing line bundle may induce on the manifold [5, Theorem 6.1]. The case of projective bundles over a smooth curve, with special emphasis on scrolls, has been widely discussed in [10]. Other examples with special regard to threefolds are presented in [5]. However, the case of scrolls over a surface is not yet discussed in the literature, not even for dimension three. Filling this gap is exactly the aim of this paper. Moreover, confining to threefolds we get a precise parallel with the case of quadric fibrations over a smooth curve studied in [5, Proposition 4.8]. Recall that these two types of varieties play a similar role in adjunction theory. In particular, in the setting we consider, a precise answer is given to a problem raised in [5].

Here is a summary of the content. Let (X, L) be a 3-dimensional scroll over a smooth surface Y, and let $\mathcal{E} = \pi_* L$, where $\pi : X \to Y$ is the scroll projection. According to [5, Theorem 6.1], the Hilbert curve $\Gamma_{(X,L)}$ of (X, L) is reducible into a given line ℓ and a conic, say G. In Section 2 we determine explicitly its canonical equation. The problem whether the resulting conic G itself can in turn be the Hilbert curve of any $\mathbb{Q}$-polarized surface seems not affordable in the general case, due to a too large number of variables. In fact, in Section 3, we present some elementary examples illustrating a range of possibilities. This suggests to confine the problem to the case where the underlying surface is the base itself, Y, of the scroll. In this context, the Hodge index theorem provides a necessary condition: an upper bound expressed in terms of K_Y and of the ample rank-2 vector bundle $\mathcal{E}$, that the Bogomolov number of $\mathcal{E}$ has to satisfy. On the other hand, the base surface Y is endowed with a natural polarization, namely $\det\mathcal{E}$. Addressing the specific question raised in [5, Problem 6.6 (2)], we can then ask whether the conic G is the Hilbert curve of Y with some $\mathbb{Q}$-polarization related to $\det \mathcal{E}$. What we prove in Section 4 is that G is the Hilbert curve of $(Y, \frac{1}{2}\det \mathcal{E})$ up to HC-equivalence (see [11]), if and only if $\mathcal{E}$ is properly semistable (in the sense of Bogomolov).

1 Background material

Varieties considered in this paper are defined over the field $\mathbb{C}$ of complex numbers. We use the standard notation and terminology from algebraic geometry. A manifold is any smooth projective variety; a surface is a manifold of dimension 2. The symbol $\equiv$ will denote numerical equivalence. With a little abuse, we adopt the additive notation for the tensor products of line bundles. The pullback of a vector bundle $\mathcal{F}$ on a manifold X by an embedding $Y \hookrightarrow X$ is simply denoted by $\mathcal{F}_Y$. We denote by T_X and K_X the tangent bundle and the canonical bundle of a manifold X, respectively. A polarized manifold is a pair (X, L) consisting of a manifold X and an ample line bundle L on X. The word scroll has to be intended in the classical sense. We denote by $\mathbb{F}_e := \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)\big)$ the Segre–Hirzebruch surface of invariant e ($e \geq 0$), and C_0 and f will stand for the tautological section and a fiber respectively, as in [8, p. 373]. Clearly, $(\mathbb{F}_0, [aC_0 + bf]) = \big(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(a, b)\big)$.

For the notion and the general properties of the Hilbert curve associated to a polarized manifold we refer to [5], see also [10]. Here we just recall some basic facts. Let (X, L) be a polarized manifold of dimension $n \geq 2$: if $\mathrm{rk}\langle K_X, L\rangle = 2$ we can consider $\mathrm{N}(X) := \mathrm{Num}(X) \otimes_{\mathbb{Z}} \mathbb{C}$ as a complex affine space and inside it the plane $\mathbb{A}^2 = \mathbb{C}\langle K_X, L\rangle$, generated by the classes of K_X and L. For any line bundle D on X the Riemann–Roch theorem provides an expression for the Euler–Poincaré characteristic $\chi(D)$ in terms of D and the Chern classes of X. Let p denote the complexified polynomial of $\chi(D)$, when we set $D = xK_X + yL$, with x, y complex numbers, namely $p(x, y) = \chi(xK_X + yL)$. The Hilbert curve of (X, L) is the complex affine plane curve $\Gamma = \Gamma_{(X,L)} \subset \mathbb{A}^2$ of degree n defined by $p(x, y) = 0$ [5, Section 2]. Taking into account that $c := \frac{1}{2}K_X$ is the fixed point of the Serre involution $D \mapsto K_X - D$ acting on $\mathrm{N}(X)$, it is convenient to represent Γ in terms of affine coordinates $(u = x - \frac{1}{2}, v = y)$ centered at c instead of (x, y). In other words, rewrite our divisor as $D = \frac{1}{2}K_X + E$, where $E = uK_X + vL$. Then Γ can be represented with respect to these coordinates by $p(\frac{1}{2} + u, v) = 0$. An obvious advantage is that, due to Serre duality, Γ is symmetric with respect to c (the origin in the (u, v)-plane). We refer to $p(\frac{1}{2} + u, v) = 0$ as the *canonical equation* of Γ. Another consequence of Serre duality is that $c \in \Gamma$ if n is odd, while if n is even and $\Gamma \ni c$, then c is a singular point of Γ [5, Section 2].

According to the above, $\chi(D)$ can be re-expressed in terms of E and the Chern classes of X in a nice way. In particular, for $n = 2$ we get

$$\chi(D) = \frac{1}{2}E^2 + \big(\chi(\mathcal{O}_X) - \frac{1}{8}K_X^2\big). \tag{1}$$

If $n = 3$, recalling that $\chi(\mathcal{O}_X) = -\frac{1}{24}K_X \cdot c_2$, where $c_2 = c_2(X)$, the usual expression of the Riemann–Roch theorem (e. g., see [8, p. 437]) takes the more convenient form

$$\chi(D) = \frac{1}{6}E^3 + \frac{1}{24}E \cdot (2c_2 - K_X^2). \tag{2}$$

So, if L is an ample line bundle on X, letting $E = uK_X + vL$, the above expressions provide the canonical equation of the Hilbert curve Γ of (X, L). In particular the canonical equation of the Hilbert curve $\Gamma_{(S,\mathcal{L})}$ of a polarized surface $(S, \mathcal{L})$ is:

$$p_{(S,\mathcal{L})}\big(\frac{1}{2} + u, v\big) = \frac{1}{2}\Big[(uK_S + v\mathcal{L})^2 + 2\chi(\mathcal{O}_S) - \frac{1}{4}K_S^2\Big] = 0 \qquad (3)$$

[4, Section 3]. Note that the conic $\Gamma_{(S,\mathcal{L})}$ is of hyperbolic type since $K_S^2\mathcal{L}^2 < (K_S \cdot \mathcal{L})^2$ in view of the Hodge index theorem and the assumption on $\langle K_S, \mathcal{L}\rangle$. In particular, $\Gamma_{(S,\mathcal{L})}$ is reducible (into two distinct lines crossing at c) if and only if the underlying surface S satisfies the condition $K_S^2 = 8\chi(\mathcal{O}_S)$. In fact, the Hilbert curve can be defined also when the numerical classes of K_S and $\mathcal{L}$ are linearly dependent, but in this case the (u, v)-plane is only formal and $\Gamma_{(S,\mathcal{L})}$ (which now is of parabolic type and reducible) loses the meaning of a plane section of the Hilbert variety of S (see [5, Section 2], [4, Section 1]). This situation is referred to as the *degenerate case* in [5, 2.2]. We recall that, once S is fixed, the Hilbert curve of the polarized surface $(S, \mathcal{L})$ only encodes properties of the numerical equivalence class of the polarizing line bundle $\mathcal{L}$. However, it does not correspond bijectively to it as shown in [11], where the notion of HC-equivalence is introduced. We point out that numerical equivalence of two ample line bundles $\mathcal{L}$ and $\mathcal{M}$ implies HC-equivalence, and this, in turn, implies that $\mathcal{L}^2 = \mathcal{M}^2$ and $K_S \cdot \mathcal{L} = K_S \cdot \mathcal{M}$, provided that $\big(K_S^2, \chi(\mathcal{O}_S)\big) \neq (0, 0)$ [11, Proposition 2.1].

We will also need to work with $\mathbb{Q}$-polarized surfaces. Suppose that $\mathcal{L}$ is an ample $\mathbb{Q}$-line bundle on a surface S. Then there exists a positive integer m such that $\mathcal{M} := m\mathcal{L} \in \mathrm{Pic}(S)$. Letting $p_{(S,\mathcal{L})}(\frac{1}{2} + u, v)$ denote the extension of the polynomial expression $\chi(\frac{1}{2}K_S + E)$ where $E = uK_S + v\mathcal{L}$, from the equality $E = uK_S + \frac{v}{m}\mathcal{M}$ we see that $p_{(S,\mathcal{L})}(\frac{1}{2} + u, v) = p_{(S,\mathcal{M})}(\frac{1}{2} + u, \frac{v}{m})$, the polynomial defining the canonical equation of the Hilbert curve $\Gamma_{(S,\mathcal{M})}$. Thus we can speak about the *Hilbert curve* $\Gamma_{(S,\mathcal{L})}$ *of the* $\mathbb{Q}$*-polarized surface* $(S, \mathcal{L})$, its *canonical equation* being

$$p_{(S,\mathcal{L})}(\frac{1}{2} + u, v) = \frac{1}{2}\ [u\ v\ 1]\ \begin{bmatrix} K_S^2 & K_S \cdot \mathcal{L} & 0 \\ K_S \cdot \mathcal{L} & \mathcal{L}^2 & 0 \\ 0 & 0 & 2\chi(\mathcal{O}_S) - \frac{K_S^2}{4} \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0,$$

formally the same equation as (3). Now let C be a conic in the (u, v)-plane of equation

$$[u\ v\ 1]\ \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0,$$

where $a_{ik} \in \mathbb{Q}$. Clearly C is the Hilbert curve of a $\mathbb{Q}$-polarized surface $(S, \mathcal{L})$ if and only if there exists a nonzero rational number ρ such that

$$\big(a_{11}, a_{12}, a_{22}, a_{33}\big) = \rho\ \big(K_S^2, K_S \cdot \mathcal{L}, \mathcal{L}^2, 2\chi(\mathcal{O}_S) - \frac{1}{4}K_S^2\big). \qquad (4)$$

We note that ρ has the same sign as a_{22}, since $\mathcal{L}$ is ample.

Lemma 1.1 *Let $C = \Gamma_{(S,\mathcal{L})}$ for some $\mathbb{Q}$-polarized surface $(S, \mathcal{L})$.*

1. *Then $a_{11}a_{22} - a_{12}^2 \leq 0$, equality implying that $K_S \equiv \lambda\mathcal{L}$ for some $\lambda \in \mathbb{Q}$.*

2. If $(S,\mathcal{L})$ is a polarized surface and $a_{22}>0$, then $a_{12}+a_{22}\geq 0$ unless $(S,\mathcal{L}) = \big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(a)\big)$ with $a=1,2$, or $\big(\mathbb{F}_e, [C_0+bf]\big)$ with $b>e$.

Proof. Suppose that $\mathcal{M}=m\mathcal{L}\in \mathrm{Pic}(S)$. By (4) we have

$$a_{11}a_{22}-a_{12}^2=\frac{\rho^2}{m^2}\big(K_S^2\mathcal{M}^2-(K_S\cdot\mathcal{M})^2\big),$$

hence the first assertion follows from the Hodge index theorem. Now let $\mathcal{L}\in\mathrm{Pic}(S)$; we know that $\rho>0$ since $a_{22}>0$. Thus (4) and the genus formula give

$$a_{12}+a_{22}=\rho\,\left(K_S\cdot\mathcal{L}+\mathcal{L}^2\right)=\rho\big(2g(S,\mathcal{L})-2\big),$$

and this term is non-negative unless $(S,\mathcal{L})$ is a polarized surface of sectional genus zero. Then the second assertion follows from [12, Corollary 2.3]. Q.E.D.

In particular, the first assertion of Lemma 1.1 says that in order to be the Hilbert curve of a $\mathbb{Q}$-polarized surface $(S,\mathcal{L})$, the conic C must be of hyperbolic type in general, and reducible and of parabolic type if and only if the classes of K_S and $\mathcal{L}$ in $\mathrm{Num}(S)$ are linearly dependent over $\mathbb{Q}$. Notice that the second assertion in Lemma 1.1 is no longer true if $(S,\mathcal{L})$ is simply a $\mathbb{Q}$-polarized surface, e. g., for $\big(\mathbb{F}_1, [2C_0+\frac{5}{2}f]\big)$ we have that $a_{12}+a_{22}=-\rho<0$.

Letting the $\mathbb{Q}$-polarization vary on a ray in the ample cone of a given surface and looking at the behavior of the corresponding Hilbert curve we can get a dynamic understanding of the situation. Let S be a surface with $\big(K_S^2,\chi(\mathcal{O}_S)\big)\neq(0,0)$, and let $\mathcal{L}$ be an ample $\mathbb{Q}$-line bundle on S such that $\mathrm{rk}\langle K_S,\mathcal{L}\rangle=2$. Set $\Gamma_n:=\Gamma_{(S,\frac{1}{n}\mathcal{L})}$ for any positive integer n. A straightforward verification shows that Γ_n is a hyperbola whose asymptotes have slopes

$$\frac{n}{\mathcal{L}^2}\Big(-K_S\cdot\mathcal{L}\pm\sqrt{(K_S\cdot\mathcal{L})^2-K_S^2\mathcal{L}^2}\;\Big).$$

Note that one of the asymptotes is the u-axis if $K_S^2=0$. Now, if $K_S^2\neq 0$, then the limit of Γ_n as $n\to\infty$ is the conic $K_S^2u^2+\big(2\chi(\mathcal{O}_S)-\frac{1}{4}K_S^2\big)=0$. From the real point of view, expressing the angle α_n between the two asymptotes of Γ_n in terms of the linear and the quadratic orthogonal invariants of Γ_n, we get

$$\tan\alpha_n=2\,\frac{\sqrt{\frac{1}{n^2}\Big((K_S\cdot\mathcal{L})^2-K_S^2\mathcal{L}^2\Big)}}{\Big(K_S^2+\frac{1}{n^2}\mathcal{L}^2\Big)}=\frac{2n}{n^2K_S^2+\mathcal{L}^2}\;\sqrt{(K_S\cdot\mathcal{L})^2-K_S^2\mathcal{L}^2}$$

which tends to zero as $n\to\infty$. This is in accordance with the above assertion about the limit of Γ_n. On the other hand, if $K_S^2=0$, then Γ_n does not admit a limit curve in the affine plane; however, its projective closure has the line at infinity with multiplicity two as limit curve, provided that $\chi(\mathcal{O}_S)\neq 0$.

To put in perspective a notion we use in Section 4 let us recall the following facts. For any vector bundle $\mathcal{V}$ of rank 2 on a smooth surface S set

$$\delta(\mathcal{V}):=c_1(\mathcal{V})^2-4c_2(\mathcal{V}). \tag{5}$$

A celebrated result of Bogomolov [6, Theorem p. 500] states that if $\mathcal{V}$ is H-stable for an ample line bundle H on S, then $\delta(\mathcal{V})\leq 0$ (Bogomolov inequality). This provides a strong notion of instability: $\mathcal{V}$ is said *B-unstable* if $\delta(\mathcal{V}) > 0$. According to [6] (see also [14, Theorem 1]), this is equivalent to the existence of an exact sequence

$$0 \to L \to \mathcal{V} \to M \otimes \mathcal{I}_Z \to 0,$$

where L and M are line bundles on S, and Z is a 0-dimensional subscheme of S with sheaf of ideals $\mathcal{I}_Z$, such that $(L-M)^2 > 4 \deg Z$ and $(L-M) \cdot H > 0$ for any ample line bundle H on S. Note that if $\mathcal{V}$ is ample, then M is ample too [9, Remark 1.8]. In accordance with the usual terminology [13, p. 168, and comment at p. 190], we will say that $\mathcal{V}$ is *B-semistable* if $\delta(\mathcal{V}) \leq 0$, and *properly B-semistable* if equality occurs. For any S, if $\mathcal{V} = A \oplus B$, with A, B ample line bundles, we have $c_1(\mathcal{V}) = A + B$ and $c_2(\mathcal{V}) = A \cdot B$, hence $\delta(\mathcal{V}) = (A+B)^2 - 4A \cdot B = (A-B)^2$. We thus see that if $A \equiv B$ then $\mathcal{V}$ is properly B-semistable.

2 The Hilbert curve of a 3-dimensional scroll over a surface

Let (X, L) be a polarized threefold and set $E = uK_X + vL$, as in Section 1. Then, by (2), the polynomial $p\big(\frac{1}{2} + u, v\big)$ defining the canonical equation of the Hilbert curve $\Gamma_{(X,L)}$ of (X, L) in the (u, v) plane is

$$p\Big(\frac{1}{2} + u, v\Big) = \frac{1}{6}(uK_X + vL)^3 + \frac{1}{24} f_1(u, v), \tag{6}$$

where the linear term f_1 is given by

$$f_1(u, v) = (uK_X + vL) \cdot \big(2c_2(X) - K_X^2\big).$$

Now suppose that (X, L) is a scroll over a smooth surface Y, with projection $\pi : X \to Y$; in this case, $\chi(\mathcal{O}_X) = \chi(\mathcal{O}_Y)$, since X is a $\mathbb{P}^1$-bundle over Y. Let us fix some notation. First of all we can write $X = \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \pi_* L$ is an ample rank 2 vector bundle on Y with Chern classes $c_i(\mathcal{E})$, $i = 1, 2$, L being its tautological line bundle on X. We recall the Chern–Wu relation

$$L^2 - \pi^* c_1(\mathcal{E}) \cdot L + \pi^* c_2(\mathcal{E}) = 0$$

(see e.g., [8, p. 429]), which implies

$$L^3 = c_1(\mathcal{E})^2 - c_2(\mathcal{E}) \quad \text{and} \quad L^2 \cdot \pi^* M = c_1(\mathcal{E}) \cdot M, \tag{7}$$

for any line bundle M on Y. Recall also the exact sequence

$$0 \to T_{X/Y} \to T_X \to \pi^* T_Y \to 0$$

defining the relative tangent bundle $T_{X/Y}$, and the relative Euler sequence

$$0 \to \mathcal{O}_X \to \pi^* \mathcal{E}^\vee \otimes L \to T_{X/Y} \to 0.$$

They imply the following relation expressing the Chern polynomial $c(T_X; t)$ of T_X:

$$c(T_X; t) = c(T_{X/Y}; t)\ c(\pi^* T_Y; t) = c(\pi^* \mathcal{E}^\vee \otimes L; t)\ c(\pi^* T_Y; t). \tag{8}$$

Making (8) explicit in terms of Chern classes, we get the canonical bundle formula

$$K_X = -2L + \pi^*\big(K_Y + c_1(\mathcal{E})\big),$$

and, after standard computations,

$$\begin{aligned} c_2(X) &= c_2(T_X) \\ &= c_2(\pi^*\mathcal{E}^\vee \otimes L) + c_1(\pi^*\mathcal{E}^\vee \otimes L) \cdot c_1(\pi^* T_Y) + c_2(\pi^* T_Y) \\ &= \pi^* c_2(\mathcal{E}) - \pi^* c_1(\mathcal{E}) \cdot L + L^2 - 2\pi^* K_Y \cdot L + \pi^*(K_Y \cdot c_1(\mathcal{E})) + \pi^* c_2(Y). \end{aligned}$$

On the other hand, the canonical bundle formula and (7) give the expressions

$$\begin{aligned} K_X^3 &= -2c_1(\mathcal{E})^2 + 8c_2(\mathcal{E}) - 6K_Y^2, \\ K_X^2 \cdot L &= c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) - 2K_Y \cdot c_1(\mathcal{E}) + K_Y^2, \\ K_X \cdot L^2 &= -c_1(\mathcal{E})^2 + 2c_2(\mathcal{E}) + K_Y \cdot c_1(\mathcal{E}). \end{aligned}$$

Now consider the Hilbert curve $\Gamma_{(X,L)}$ of our scroll (X, L). According to [5, Theorem 6.1], $\Gamma_{(X,L)} = \ell + G$ is reducible into the line ℓ of equation $2u - v = 0$, when expressed in coordinates (u, v), plus a conic, say G. To determine the canonical equation of $\Gamma_{(X,L)}$ we need some computations. First of all, in view of the above formulas and Nother's formula on Y, the linear term in (6) becomes

$$\begin{aligned} f_1(u,v) &= -\Big[\big(48\chi(\mathcal{O}_X) + K_X^3\big)u + \big(c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) + K_Y^2 - 2e(Y)\big)v\Big] \\ &= (2u - v)\big(c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) + K_Y^2 - 2e(Y)\big), \end{aligned} \tag{9}$$

where $e(Y) = c_2(Y)$ stands for the topological Euler–Poincaré characteristic of Y.

Remark. Note that the expression provided by (9) is equivalent to that appearing in [5, formula (5)], assuming that $|L|$ contains a smooth surface S:

$$f_1(u,v) = -\Big[\big(48\chi(\mathcal{O}_X) + K_X^3\big)u + \big(K_X^2 \cdot L + 2K_X \cdot L^2 + 2L^3 - 2e(S)\big)v\Big],$$

To see this, it is enough to note that $\pi|_S : S \to Y$ is a birational morphism expressing S as Y blown-up at $c_2(\mathcal{E})$ points, hence $e(S) = e(Y) + c_2(\mathcal{E})$.

As to the degree three term in (6), it can be expressed as

$$\frac{1}{6}(uK_X + vL)^3 = \frac{1}{6}(2u - v)(k_1u^2 + k_2uv + k_3v^2), \tag{10}$$

where

$$k_1 = -3K_Y^2 + 4c_2(\mathcal{E}) - c_1(\mathcal{E})^2, \;\; k_2 = -3K_Y \cdot c_1(\mathcal{E}) + c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}), \;\; k_3 = c_2(\mathcal{E}) - c_1(\mathcal{E})^2.$$

Actually, computing, we find

$$\begin{aligned} E^3 &= (uK_X + vL)^3 \\ &= -3K_Y^2 u^2(2u - v) - 3K_Y \cdot c_1(\mathcal{E})uv(2u - v) - c_1(\mathcal{E})^2(2u^3 - 3u^2v + 3uv^2 - v^3) \\ &\quad + c_2(\mathcal{E})(2u - v)^3. \end{aligned}$$

Then relation (10) simply follows noting that

$$2u^3 - 3u^2v + 3uv^2 - v^3 = (u^2 - uv + v^2)(2u - v).$$

Summarizing the above discussion we obtain

$$\begin{aligned}
p\big(\frac{1}{2}+u, v\big) &= \frac{1}{6}(2u-v)(k_1u^2 + k_2uv + k_3v^2) \\
&\quad + \frac{1}{24}(2u-v)\big(c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) + K_Y^2 - 2e(Y)\big) \\
&= \frac{1}{3}(2u-v)\Big[\frac{1}{2}\Big(\big(-3K_Y^2 - c_1(\mathcal{E})^2 + 4c_2(\mathcal{E})\big)u^2 \\
&\quad + \big(c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) - 3K_Y\cdot c_1(\mathcal{E})\big)uv - \big(c_1(\mathcal{E})^2 - c_2(\mathcal{E})\big)v^2\Big) \\
&\quad + \frac{1}{8}\big(c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}) + K_Y^2 - 2e(Y)\big)\Big].
\end{aligned} \tag{11}$$

Recalling (5), we set

$$\delta := \delta(\mathcal{E}) = c_1(\mathcal{E})^2 - 4c_2(\mathcal{E}). \tag{12}$$

According to (12), we can write $c_1(\mathcal{E})^2 - c_2(\mathcal{E}) = \frac{1}{4}\big(3c_1(\mathcal{E})^2 + \delta\big)$. Taking also into account Noether's formula and collecting -3 as common factor of all terms in the square brackets, the expression in (11) takes a more handleable form. In conclusion, we have

Proposition 2.1 *Let (X, L) be a threefold scroll over a smooth surface Y, let $\mathcal{E} := \pi_*L$, where $\pi : X \to Y$ is the scroll projection, and let δ be as in* (12)*. Then the Hilbert curve $\Gamma_{(X,L)}$ has the following canonical equation:*

$$\begin{aligned}
p\big(\frac{1}{2}+u, v\big) &= -\frac{1}{2}(2u-v)\Big[\big(K_Y^2 + \frac{\delta}{3}\big)u^2 + \big(K_Y\cdot c_1(\mathcal{E}) - \frac{\delta}{3}\big)uv + \big(\frac{c_1(\mathcal{E})^2}{4} + \frac{\delta}{12}\big)v^2 \\
&\quad + 2\chi(\mathcal{O}_Y) - \frac{K_Y^2}{4} - \frac{\delta}{12}\Big].
\end{aligned} \tag{13}$$

3 The conic G

Let (X, L) be as in Section 2. The fact that $\Gamma_{(X,L)}$ consists of the line ℓ of equation $2u - v = 0$ and a conic G was already known by [5, Theorem 6.1], but now Proposition 2.1 provides an explicit equation for G. Actually, dividing the polynomial $p\big(\frac{1}{2}+u, v\big)$ in (13) by $-(2u - v)$, we see that G can be represented by the equation

$$\frac{1}{2}\ [u\ v\ 1]\ A_\delta \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = 0,$$

where

$$A_\delta = \begin{bmatrix} K_Y^2 + \frac{\delta}{3} & K_Y\cdot\frac{c_1(\mathcal{E})}{2} - \frac{\delta}{6} & 0 \\ K_Y\cdot\frac{c_1(\mathcal{E})}{2} - \frac{\delta}{6} & \frac{c_1(\mathcal{E})^2}{4} + \frac{\delta}{12} & 0 \\ 0 & 0 & 2\chi(\mathcal{O}_Y) - \frac{K_Y^2}{4} - \frac{\delta}{12} \end{bmatrix}.$$

It seems natural to ask the following question, related to [5, Problem 6.6].

Is G the Hilbert curve of some polarized (or $\mathbb{Q}$-polarized) surface, say $(S, \mathcal{M})$? (14)

Of course, in view of what we said in Section 1, we have to include the possibility that $(S,\mathcal{M})$ is as in the degenerate case. Here are some examples.

Example 1. Let $(Y,\mathcal{E}) = \big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 2}\big)$. In this case, $(X,L) = \big(\mathbb{P}^2 \times \mathbb{P}^1, \mathcal{O}(1,1)\big)$, $\delta = 0$, and the canonical equation of $\Gamma_{(X,L)}$ is

$$p\big(\frac{1}{2}+u,v\big) = -(2u-v)\frac{1}{2}\Big[(3u-v)^2-\frac{1}{4}\Big] = -\frac{1}{2}(2u-v)\Big(3u-v-\frac{1}{2}\Big)\Big(3u-v+\frac{1}{2}\Big).$$

In particular, this shows that $\Gamma_{(X,L)}$ is reducible into three lines, with G consisting of two parallel lines. This is in accordance with [10, Corollary 4.1], because (X,L) can also be regarded as a scroll over $\mathbb{P}^1$, via the second projection. Since G has equation $9u^2-6uv+v^2-\frac{1}{4}=0$, it is immediate to check that $G=\Gamma_{(S,\mathcal{M})}$, where $(S,\mathcal{M}) = \big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)\big)$.

Example 2. Let $(Y,\mathcal{E}) = \big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)\oplus\mathcal{O}_{\mathbb{P}^2}(1)\big)$. Here (X,L) is the del Pezzo 3-fold of degree 7 [7, §8], and $\delta=1$. In this case, the canonical equation of $\Gamma_{(X,L)}$ is

$$p\big(\frac{1}{2}+u,v\big) = -\frac{1}{6}(2u-v)\big(7(2u-v)^2-1\big).$$

So the cubic curve $\Gamma_{(X,L)}$ consists of three distinct parallel lines. Note that $K_X=-2L$, hence we are in the degenerate case. Here G has equation $28u^2-28uv+7v^2-1=0$.

Example 3. Let $(Y,\mathcal{E}) = \big(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}(1,1)^{\oplus 2}\big)$. Thus $(X,L) = \big(\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}(1,1,1)\big)$ is a del Pezzo 3 fold of degree 6 and $\delta=0$. The canonical equation of $\Gamma_{(X,L)}$ is

$$p\big(\frac{1}{2}+u,v\big) = -(2u-v)^3;$$

so the cubic curve $\Gamma_{(X,L)}$ is a line with multiplicity three. Note that this is consistent with [5, p, 412, comment after Proposition 4.8]; actually (X,L) can also be regarded as a quadric bundle over $\mathbb{P}^1$, via any of the three projections, however, $K_X=-2L$, hence we fall in the degenerate case. Here G has equation $(2u-v)^2=0$, and it is immediate to check that $G=\Gamma_{(S,\mathcal{M})}$, where $(S,\mathcal{M}) = \big(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}(1,1)\big)$.

Example 4. Let $(Y,\mathcal{E}) = \big(\mathbb{P}^2, T_{\mathbb{P}^2}\big)$ (tangent bundle). Here (X,L) is another del Pezzo 3-fold of degree 6 and $\delta=-3$. The canonical equation of $\Gamma_{(X,L)}$ is

$$p\big(\frac{1}{2}+u,v\big) = -(2u-v)^3,$$

hence $\Gamma_{(X,L)}$ is a line with multiplicity three again. The conic G has equation $(2u-v)^2=0$ as in Example 3. However, there is a subtle difference between the two cases, as it will be more clear in Section 4.

As to the key property of G in the above examples, the situation is summarized as follows.

Proposition 3.1 *If G is as in Example 2, then there is no $\mathbb{Q}$-polarized surface $(S,\mathcal{L})$ whose Hilbert curve is G. If G is as in Example 1 and $G=\Gamma_{(S,\mathcal{L})}$ for some $\mathbb{Q}$-polarized surface $(S,\mathcal{L})$, then $(S,\mathcal{L}) = \big(\mathbb{P}^2,\mathcal{O}_{\mathbb{P}^2}(1)\big)$. If G is as in Example 3 (or 4), and $G=\Gamma_{(S,\mathcal{L})}$ for some $\mathbb{Q}$-polarized surface $(S,\mathcal{L})$, then $(S,\mathcal{L})$ is either*

(i) $\big(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)\big)$, *or*

(ii) $(\mathbb{F}_1, [C_0 + \frac{3}{2}f])$.

Note that $\mathcal{L} = -\frac{1}{2}K_S$ in both cases.

Proof. In Example 2, G has equation $28u^2-28uv+7v^2-1=0$. Suppose, by contradiction, that $G = \Gamma_{(S,\mathcal{L})}$ for some ample $\mathbb{Q}$-line bundle $\mathcal{L} \in \mathrm{Pic}(S) \otimes \mathbb{Q}$. Then, taking into account the discussion in Section 1, there exists a nonzero rational number ρ such that

$$\big(K_S^2, K_S \cdot \mathcal{L}, \mathcal{L}^2, 2\chi(\mathcal{O}_S) - \frac{1}{4}K_S^2\big) = \rho(28, -14, 7, -1).$$

Assume these four conditions are satisfied. The third one implies that $\rho > 0$, and then $K_S^2 > 0$, due to the first one. On the other hand, the second condition says that $K_S \cdot (m\mathcal{L}) < 0$, m being a positive integer such that $m\mathcal{L} \in \mathrm{Pic}(S)$. Since $m\mathcal{L}$ is ample this implies that all the plurigenera of S are zero, and then S has negative Kodaira dimension, by the Enriques ruledness criterion [2, Theorem VI.17]. Since $K_S^2 > 0$, this says that S is a rational surface: in particular, $\chi(\mathcal{O}_S) = 1$. Combining this with the first and the fourth conditions we get $1 = \chi(\mathcal{O}_S) = 3\rho$, whence $\rho = \frac{1}{3}$. But then the first condition would imply $K_S^2 = \frac{28}{3}$, which is not an integer, a contradiction. This proves the first assertion in the statement. Now, let G be as in Example 1, so that it has equation $9u^2 - 6uv + v^2 - \frac{1}{4} = 0$. If $G = \Gamma_{(S,\mathcal{L})}$, then we have to consider the following equalities

$$\big(K_S^2, K_S \cdot \mathcal{L}, \mathcal{L}^2, 2\chi(\mathcal{O}_S) - \frac{1}{4}K_S^2\big) = \rho(9, -3, 1, -\frac{1}{4}).$$

Arguing exactly as in the previous case we obtain that S is rational, hence $\chi(\mathcal{O}_S) = 1$ and then the first and the fourth conditions imply $\rho = 1$ and $K_S^2 = 9$. Therefore $S = \mathbb{P}^2$. So we can write $m\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(a)$ for some positive integers m and a and thus the third equality gives $m = a$, i.e., $\mathcal{L} = \mathcal{O}_{\mathbb{P}^2}(1)$. This proves the second assertion in the statement. Finally, let G be as in Example 3 (or 4); then G has equation $4u^2 - 4uv + v^2 = 0$. If $G = \Gamma_{(S,\mathcal{L})}$, the equalities we have to consider now are

$$\big(K_S^2, K_S \cdot \mathcal{L}, \mathcal{L}^2, 2\chi(\mathcal{O}_S) - \frac{1}{4}K_S^2\big) = \rho(4, -2, 1, 0).$$

Arguing as in the previous cases we obtain that S is rational, hence $\chi(\mathcal{O}_S) = 1$. Then from $0 = 2\chi(\mathcal{O}_S) - \frac{1}{4}K_S^2 = 2 - \frac{1}{4}4\rho = 2 - \rho$, we get $\rho = 2$. Thus $K_S^2 = 8$, $\mathcal{L}^2 = 2$, and $\mathcal{L} \cdot K_S = -4$. Therefore $S = \mathbb{F}_e$ (for some $e \geq 0$), and we can write $m\mathcal{L} = [aC_0 + bf]$ for some positive inger m, where a, b satisfy the ampleness conditions $a > 0, b > ae$ [8, Corollary 2.18, p. 380]. Then

$$2 = \mathcal{L}^2 = \frac{1}{m^2}a(2b - ae) \quad \text{and} \quad -4 = \mathcal{L} \cdot K_S = \frac{1}{m}(ae - 2b - 2a).$$

We obtain

$$\frac{2m^2}{a} = 2b - ae = 4m - 2a,$$

which, in turn, gives $2(m-a)^2 = 0$, i.e., $m = a$. This implies $2b - ae = 2a$, hence $b = a + \frac{ae}{2}$. Combining this with the ampleness conditions mentioned above we see that $e < 2$, which leads to the following possibilities: either $e = 0$ with $b = a$, or $e = 1$ with $b = \frac{3}{2}a$. This proves the final assertion. Q.E.D.

Proposition 3.1 illustrates a range of possibilites in connection with question (14). Moreover, the argument in the proof seems to indicate that (14) is not affordable in full generality, since too many variables are involved. In spite of this we can add something more. Set $A_\delta = [a_{ij}]$. To claim that $G = \Gamma_{(S,\mathcal{L})}$, for some $\mathbb{Q}$-polarized surface $(S, \mathcal{L})$, according to Lemma 1.1 it must be

$$\mathcal{J} := a_{11}a_{22} - a_{12}^2 \leq 0, \tag{15}$$

equality holding if and only if $(S, \mathcal{L})$ is as in the degenerate case. We stress that (15) is only a necessary condition for G being the Hilbert curve of a $\mathbb{Q}$-polarized surface, as shown by Example 2. A straightforward computation gives:

$$\mathcal{J} = \frac{1}{4}\left(K_Y^2 c_1(\mathcal{E})^2 - \Big(K_Y \cdot c_1(\mathcal{E})\Big)^2\right) + \frac{\delta}{12}\Big(K_Y + c_1(\mathcal{E})\Big)^2. \tag{16}$$

Let us consider the two summands in (16) separately.

Lemma 3.2 *Let (X, L) be a threefold scroll over a smooth surface Y and let $\mathcal{E} = \pi_* L$, where $\pi : X \to Y$ is the scroll projection. Then*

$$K_Y^2 c_1(\mathcal{E})^2 - \big(K_Y \cdot c_1(\mathcal{E})\big)^2 \leq 0,$$

with equality if and only if $\mathrm{rk}\langle K_Y, c_1(\mathcal{E})\rangle < 2$. In particular, if Y has negative Kodaira dimension, equality holds if and only if $\big(Y, c_1(\mathcal{E})\big)$ is either $\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(e)\big)$ or $\big(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(e,e)\big)$, with $e \geq 2$ in both cases.

Proof. The inequality follows from the Hodge index theorem, equality occurring if and only if $\mathrm{rk}\langle K_Y, c_1(\mathcal{E})\rangle < 2$. If $\kappa(Y) < 0$, this means that Y is a del Pezzo surface, with $-K_Y = \frac{r}{s}\, c_1(\mathcal{E})$ for some positive integers r, s, due to the ampleness of $c_1(\mathcal{E})$. Note that $\big(Y, c_1(\mathcal{E})\big)$ cannot contain lines since $\mathcal{E}$ is an ample vector bundle of rank 2, hence, in view of the classification of del Pezzo surfaces, we conclude that $\big(Y, c_1(\mathcal{E})\big)$ can only be either $\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(e)\big)$, with $e \geq 2$, or $\big(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a,b)\big)$, with $a = b \geq 2$, since $c_1(\mathcal{E})$ is a rational multiple of the anticanonical bundle. Q.E.D.

Now let us look at the second summand.

Lemma 3.3 *Let (X, L) be a threefold scroll over a smooth surface Y and let $\mathcal{E} = \pi_* L$, where $\pi : X \to Y$ is the scroll projection. Then*

$$\big(K_Y + c_1(\mathcal{E})\big)^2 \geq 0.$$

Moreover, equality holds if and only if either:

1. *$(Y, \mathcal{E})$ is as in Examples 2 and 4, or*
2. *Y is a $\mathbb{P}^1$-bundle over a smooth curve and $\mathcal{E}_f = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2}$ for every fiber f of the bundle projection.*

Note that the pair $(Y, \mathcal{E})$ as in Example 3 fits into case 2.

Proof. If $(Y, \mathcal{E})$ is as in Example 1, then $\big(K_Y + c_1(\mathcal{E})\big)^2 = \big(\mathcal{O}_{\mathbb{P}^2}(-1)\big)^2 = 1$. On the other hand, if $(Y, \mathcal{E})$ is not as in Example 1, then $K_Y + c_1(\mathcal{E})$ is nef by [16, Theorem 2], which implies the inequality. If equality holds, i. e., $K_Y + c_1(\mathcal{E})$ is nef but not big, then either $\big(Y, c_1(\mathcal{E})\big)$ is a conic fibration over a smooth curve B, or Y is a del Pezzo surface and $c_1(\mathcal{E}) = -K_Y$ [3, Theorem 7.3.2]. Since $\big(Y, c_1(\mathcal{E})\big)$ cannot contain lines, the former case leads to 2 in view of the ampleness of $\mathcal{E}$, while, in the latter, $\big(Y, c_1(\mathcal{E})\big)$ can only be one of the following pairs: i) $\big(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(e)\big)$, with $e = 2, 3$, and ii) $\big(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2,2)\big)$. In case i) $\mathcal{E}$ is uniform; thus a well known theorem of Van de Ven [13, p. 211] implies that $(Y, \mathcal{E})$ is as in Examples 2 and 4, since it is not as in Example 1. Similarly, in case ii) $(Y, \mathcal{E})$ is as in Example 3, since $\mathcal{E}$ is uniform with respect to each of the two projections of $\mathbb{P}^1 \times \mathbb{P}^1$. Q.E.D.

As a consequence, apart from cases 1 and 2 in Lemma 3.3, condition (15) can be rephrased as follows.

Proposition 3.4 *Let (X, L) be a threefold scroll over a smooth surface Y, let $\mathcal{E} = \pi_* L$, where $\pi : X \to Y$ is the scroll projection. Let G be the residual component of $\Gamma_{(X,L)}$ with respect to the line ℓ of equation $2u - v = 0$. If $G = \Gamma_{(S,\mathcal{L})}$ is the Hilbert curve of some $\mathbb{Q}$-polarized surface $(S, \mathcal{L})$, and $\big(K_Y + c_1(\mathcal{E})\big)^2 > 0$, then*

$$\delta \leq 3 \, \frac{\big(K_Y \cdot c_1(\mathcal{E})\big)^2 - K_Y^2 c_1(\mathcal{E})^2}{\big(K_Y + c_1(\mathcal{E})\big)^2},$$

and equality occurs if and only if $(S, \mathcal{L})$ is as in the degenerate case.

Proof. The inequality follows from (16), since $\mathcal{J}$ has to be ≤ 0. The assertion about the equality follows from the following chain of equivalences: equality holds $\Leftrightarrow \mathcal{J} = 0 \Leftrightarrow G$ is of parabolic type $\Leftrightarrow (S, \mathcal{L})$ is as in the degenerate case. Q.E.D.

Let us look at pairs arisen in Lemma 3.3 more closely. Discussing Examples $1-4$ we noted that $\delta = 0$ except for Example 4. As to case 2 of Lemma 3.3 we have

Proposition 3.5 *If $(Y, \mathcal{E})$ is as in case 2 of Lemma 3.3 then $\mathcal{E}$ is properly B-semistable.*

Proof. We can write $Y = \mathbb{P}(\mathcal{V})$ for some vector bundle $\mathcal{V}$ of rank 2 on the base curve B. Let ξ be the tautological line bundle on Y. Since $c_1(\mathcal{E})_f = 2\xi_f$ for every fiber f of the bundle projection $\theta : Y \to B$, we have that $(\mathcal{E} \otimes \xi^{-1})_f = \mathcal{O}_f^{\oplus 2}$, hence $\mathcal{E} = \xi \otimes \theta^* \mathcal{G}$ for some vector bundle $\mathcal{G}$ of rank 2 on B. Thus $c_1(\mathcal{E}) = 2\xi + \theta^* c_1(\mathcal{G})$ and $c_2(\mathcal{E}) = \xi^2 + \xi \cdot \theta^* c_1(\mathcal{G})$. Recalling that $\xi^2 = \deg(\mathcal{V})$ and $\xi \cdot f = 1$, this gives $c_1(\mathcal{E})^2 = \big(2\xi + \deg(\mathcal{G}) f\big)^2 = 4\big(\deg(\mathcal{V}) + \deg(\mathcal{G})\big) = 4c_2(\mathcal{E})$, hence $\delta = 0$. Q.E.D.

Clearly, adding this to Lemma 3.3 shows that if $\big(K_Y + c_1(\mathcal{E})\big)^2 = 0$, then $\delta = 0$ except for Example 4; furthermore, recalling (16), Lemmas 3.2 and 3.3, we have the following implication.

Corollary 3.6 *Let (X, L) be a threefold scroll over a smooth surface Y and let $\mathcal{E} = \pi_* L$, where $\pi : X \to Y$ is the scroll projection. If $\mathcal{E}$ is B-semistable, then condition (15) is always satisfied. Moreover, if $\mathcal{E}$ is properly B-semistable then (15) is an equality if and only if $\mathrm{rk}\langle K_Y, c_1(\mathcal{E})\rangle < 2$.*

In fact, we can say more about case 2 in Lemma 3.3. With the same notation as in the proof of Proposition 3.5 we have $K_Y = -2\xi + \theta^*\big(K_B + c_1(\mathcal{V})\big)$ by the canonical bundle formula. So, letting q denote the genus of the base curve B, we get $K_Y^2 = 8(1-q)$ and $K_Y \cdot c_1(\mathcal{E}) = -4\xi^2 + 2\xi \cdot \theta^*\big(K_B + c_1(\mathcal{V}) - c_1(\mathcal{G})\big) = 2(2q-2-\deg\mathcal{V} - \deg\mathcal{G})$. Therefore, if $(Y, \mathcal{E})$ is as in case 2 of Lemma 3.3, G is the reducible conic of equation

$$\begin{aligned} & 8(1-q)u^2 + 2\big(2q-2-\deg(\mathcal{V}) - \deg(\mathcal{G})\big)uv + \big(\deg(\mathcal{V}) + \deg(\mathcal{G})\big)v^2 \\ = \; & \big(4(1-q)u - (\deg\mathcal{V} + \deg\mathcal{G})v\big)\ \big(2u - v\big) = 0. \end{aligned} \tag{17}$$

Then it is immediate to check that $G = \Gamma_{(Y,\mathcal{L})}$ for the ample $\mathbb{Q}$-line bundle $\mathcal{L} = \xi + \frac{1}{2}\theta^* c_1(\mathcal{G}) = \frac{1}{2}c_1(\mathcal{E})$ (for a more precise assertion, see Corollary 4.5). Note also that

$$\mathcal{J} = -\big(2(q-1) + \deg(\mathcal{V}) + \deg(\mathcal{G})\big)^2 \leq 0$$

and equality occurs if and only if $q = 0$ and $\deg\mathcal{V} + \deg\mathcal{G} = 2$, since $\deg\mathcal{V} + \deg\mathcal{G} = c_2(\mathcal{E}) > 0$. In particular, $Y = \mathbb{F}_e$, for some e, and $c_1(\mathcal{E})^2 = 8$. It thus follows from [9, Theorem 2.5] that $\mathcal{J} = 0$ if and only if the pair $(Y, \mathcal{E})$ is as in Example 3.

In conclusion, for (X, L) as in Example 2, G cannot be any Hilbert curve; for pairs (X, L) in Examples 1, 3, 4, G is the Hilbert curve of a polarized surface, while for those in case 2 of Lemma 3.3, it is the Hilbert curve of a $\mathbb{Q}$-polarized surface.

4 Characterizing $\mathcal{E}$ being properly B-semistable

Now let $\delta = 0$ and set $A = A_0$. Then

$$A = \begin{bmatrix} K_Y^2 & K_Y \cdot \frac{c_1(\mathcal{E})}{2} & 0 \\ K_Y \cdot \frac{c_1(\mathcal{E})}{2} & \frac{c_1(\mathcal{E})^2}{4} & 0 \\ 0 & 0 & 2\chi(\mathcal{O}_Y) - \frac{K_Y^2}{4} \end{bmatrix}.$$

We thus see that for $\delta = 0$ the equation of G in Section 2 is exactly the canonical equation of $\Gamma_{(Y, \frac{1}{2}c_1(\mathcal{E}))}$, the Hilbert curve of Y with the *average polarization* $\frac{1}{2}c_1(\mathcal{E}) \in \operatorname{Pic}(Y) \otimes \mathbb{Q}$ This expression, suggested by the fact that $\mathcal{E}$ has rank 2, will be used for short to label any line bundle numerically equivalent to $\frac{1}{2}c_1(\mathcal{E})$. So we have

Proposition 4.1 *Let (X, L) be a threefold scroll over a smooth surface Y and let $\mathcal{E} = \pi_* L$, where $\pi : X \to Y$ is the scroll projection. If $\mathcal{E}$ is properly B-semistable, then the conic G is the Hilbert curve of the surface Y polarized by any ample $\mathbb{Q}$-line bundle $\mathcal{L} \in \operatorname{Pic}(Y) \otimes \mathbb{Q}$,* HC-*equivalent to $\frac{1}{2}c_1(\mathcal{E})$ (in particular, by any $\mathbb{Q}$-line bundle $\mathcal{L} \equiv \frac{1}{2}c_1(\mathcal{E})$).*

Remarks. i) Let $\mathcal{E} = \mathcal{L}^{\oplus 2}$, $\mathcal{L}$ being any ample line bundle on Y. Then $X = \mathbb{P}(\mathcal{E}) = Y \times \mathbb{P}^1$, and $\mathcal{E} = \pi_* L$, where $\pi : X \to Y$ is the first projection and L is the tautological line bundle. Here $c_1(\mathcal{E}) = 2\mathcal{L}$ and $\delta = 0$, hence the matrix above shows that $G = \Gamma_{(Y,\mathcal{L})}$, $\mathcal{L}$ being the average polarization of $\mathcal{E}$; clearly this includes Examples 1 and 3.

ii) Consider (13) with $\delta = 0$ again. It is clear that if $\chi(\mathcal{O}_Y) = \frac{1}{8}K_Y^2$ (this happens e.g., if Y is a $\mathbb{P}^1$-bundle over a smooth curve), then $\Gamma_{(X,L)}$ consists of three lines through the origin. This shows that having a Hilbert curve consisting of n lines through the center of the Serre involution does not imply at all that a polarized n-fold (X, L) must be the product of n curves (see also [4, p. 289]).

Proposition 4.1 provides a partial answer to [5, Problem 6.6 (1)] and to question (14). Now suppose that δ is not necessarily zero. We can ask again whether the conic G is the Hilbert curve of Y endowed with some $\mathbb{Q}$-polarization $\mathcal{L}$: this is a special case of question (14), since here $S = Y$. In view of (3), the Hilbert curve $\Gamma_{(Y,\mathcal{L})}$ is the conic whose canonical equation is associated (up to the factor $\frac{1}{2}$) to the following matrix:

$$A' = \begin{bmatrix} K_Y^2 & K_Y \cdot \mathcal{L} & 0 \\ K_Y \cdot \mathcal{L} & \mathcal{L}^2 & 0 \\ 0 & 0 & 2\chi(\mathcal{O}_Y) - \frac{K_Y^2}{4} \end{bmatrix}. \tag{18}$$

So, G is the Hilbert curve of $(Y, \mathcal{L})$ if and only if there exists a nonzero constant factor $\rho \in \mathbb{Q}$ such that $A_\delta = \rho A'$. This translates into the following conditions:

$$K_Y^2 + \frac{\delta}{3} = \rho\ K_Y^2, \tag{19}$$

$$K_Y \cdot \frac{c_1(\mathcal{E})}{2} - \frac{\delta}{6} = \rho\ K_Y \cdot \mathcal{L}, \tag{20}$$

$$\frac{c_1(\mathcal{E})^2}{4} + \frac{\delta}{12} = \rho\ \mathcal{L}^2, \tag{21}$$

$$2\chi(\mathcal{O}_Y) - \frac{K_Y^2}{4} - \frac{\delta}{12} = \rho\ \big(2\chi(\mathcal{O}_Y) - \frac{K_Y^2}{4}\big). \tag{22}$$

First of all note that $\frac{c_1(\mathcal{E})^2}{4} + \frac{\delta}{12} = \frac{1}{3}L^3$, by (7). Hence (21) gives $\rho = \frac{L^3}{3\mathcal{L}^2} > 0$, since both L and $\mathcal{L}$ are ample. Coming to the other equations, (19) can be rewritten as

$$3(\rho - 1)K_Y^2 = \delta. \tag{23}$$

This shows that

$$\delta = 0 \ \text{ if and only if either } \ \rho = 1 \ \text{ or } \ K_Y^2 = 0.$$

On the other hand, in view of (19), condition (22) turns out to be equivalent to

$$(\rho - 1)\chi(\mathcal{O}_Y) = 0.$$

Thus, if $\rho \neq 1$, in order (19) and (22) to be satisfied, it must be

$$\chi(\mathcal{O}_Y) = 0. \tag{24}$$

By the Enriques–Kodaira classification [2], condition (24) implies that Y is birational to one of the following minimal surfaces:

a) a $\mathbb{P}^1$-bundle over a smooth curve of genus one;

b) an abelian or a bielliptic surface;

c) an elliptic quasi–bundle in the sense of Serrano [15, Definition 1.2].

Note that in all these cases we have $K_Y^2 \leq 0$, with equality if and only if Y is minimal.

In view of Proposition 4.1 we can now suppose that $\delta \neq 0$. Then $K_Y^2 \neq 0$ by (23). By combining this with the above discussion we get

Lemma 4.2 *Let $\delta \neq 0$ and suppose that $G = \Gamma_{(Y,\mathcal{L})}$ for some ample $\mathbb{Q}$-line bundle $\mathcal{L}$ on Y. Then*

$$\rho \neq 1, \quad \chi(\mathcal{O}_Y) = 0, \quad \textit{and} \quad K_Y^2 < 0.$$

Note that, for $\delta \neq 0$, Lemma 4.2 together with $\rho > 0$ gives only necessary conditions for G being the Hilbert curve of $(Y, \mathcal{L})$ for some $\mathbb{Q}$-polarization $\mathcal{L}$.

Referring to Example 4 of Section 3, Lemma 4.2 immediately shows that there is no $\mathbb{Q}$-polarization of Y having G as Hilbert curve, since $\delta = -3$ while $\chi(\mathcal{O}_Y) = 1$: this is in accordance with Proposition 3.1.

Lemma 4.3 *Let (X, L) be as at the beginning of Section 2, and suppose that $(Y, \mathcal{E})$ is not as in Example 1. Then*

$$c_1(\mathcal{E}) \cdot \big(K_Y + c_1(\mathcal{E})\big) \geq 0,$$

equality implying that $(Y, \mathcal{E})$ is as in Examples $2-4$. In particular, if Y is neither $\mathbb{P}^2$ nor $\mathbb{P}^1 \times \mathbb{P}^1$, then the above inequality is strict.

Proof. Actually $K_Y + c_1(\mathcal{E})$ is nef by [16, Theorem 2], since $(Y, \mathcal{E})$ is not as in Example 1. Hence the ampleness of $c_1(\mathcal{E})$ implies the inequality. Suppose it is an equality. Then the Hodge index theorem implies that $K_Y + c_1(\mathcal{E}) \equiv 0$, because $\big(K_Y + c_1(\mathcal{E})\big)^2 \geq 0$, due to the nefness. It turns out that $-K_Y \equiv c_1(\mathcal{E})$ is ample, hence Y is a del Pezzo surface, and then $\mathrm{Pic}(Y)$ has no torsion, which implies $-K_Y = c_1(\mathcal{E})$. Then the assertion follows arguing as in the proof of Lemma 3.3. Q.E.D.

This allows us to prove the following result.

Theorem 4.4 *Let (X, L) be a threefold scroll over a smooth surface Y and let $\mathcal{E} = \pi_* L$, where $\pi : X \to Y$ is the scroll projection. The conic G, residual part of the line ℓ in $\Gamma_{(X,L)}$, is the Hilbert curve $\Gamma_{(Y,\mathcal{L})}$ of Y endowed with an ample $\mathbb{Q}$-line bundle $\mathcal{L} \in \mathrm{Pic}(Y) \otimes \mathbb{Q}$,* HC*-equivalent to an average polarization of $\mathcal{E}$ if and only if the vector bundle $\mathcal{E}$ is properly B-semistable.*

Proof. The "if part" is given by Proposition 4.1. To prove the converse, suppose, by contradiction, that $\delta \neq 0$. Then $\rho \neq 1$ and $\chi(\mathcal{O}_Y) = 0$ by Lemma 4.2; in particular, Y cannot be rational, hence $c_1(\mathcal{E}) \cdot \big(K_Y + c_1(\mathcal{E})\big) > 0$ by Lemma 4.3. On the other hand, since $\mathcal{L}$ is HC-equivalent to $\frac{1}{2}c_1(\mathcal{E})$, we have $\mathcal{L}^2 = \frac{1}{4}c_1(\mathcal{E})^2$ and $\mathcal{L} \cdot K_Y = \frac{1}{2}c_1(\mathcal{E}) \cdot K_Y$, hence equations (21) and (20) become

$$(\rho - 1)\ c_1(\mathcal{E})^2 = \frac{\delta}{3} \quad \text{and} \quad (\rho - 1)\ K_Y \cdot c_1(\mathcal{E}) = -\frac{\delta}{3},$$

respectively. Summing them up we get

$$(\rho - 1)\ c_1(\mathcal{E}) \cdot \big(K_Y + c_1(\mathcal{E})\big) = 0,$$

which is clearly impossible. Q.E.D.

Taking into account Proposition 3.5, we get the following consequence.

Corollary 4.5 *If $(Y, \mathcal{E})$ is as in case 2 of Lemma 3.3, then $G = \Gamma_{(Y,\mathcal{L})}$, for $\mathcal{L}$ HC-equivalent to an average polarization of $\mathcal{E}$.*

Here is an example where HC-equivalence can be replaced with numerical equivalence in the statement of Theorem 4.4.

Example 5. Let $Y := C^{(2)}$ be the second symmetric product of a general smooth curve of genus 2. The Abel–Jacobi map $\alpha : Y \to J(C)$ expresses Y as the Jacobian $J(C)$ blown up at the point p corresponding to the canonical series of C. Let $E \subset Y$ be the corresponding exceptional curve. Thus $K_Y = E$, since $J(C)$ is an abelian surface, hence $K_Y^2 = -1$. Consider $C \hookrightarrow J(C)$ embedded in its Jacobian and let $A := \alpha^* C - E$. Then A is an ample divisor on Y, and $A^2 = C^2 - 1 = 1$, $A \cdot K_Y = A \cdot E = 1$. Let a and b be two positive integers and consider the rank-2 vector bundle $\mathcal{E}_{a,b} := \mathcal{O}_Y(aA) \oplus \mathcal{O}_Y(bA)$. Clearly $\mathcal{E}_{a,b}$ is ample, so being both summands, and we can suppose $a \geq b$. We have: $c_1(\mathcal{E}_{a,b})^2 = \big((a+b)A\big)^2 = (a+b)^2$ and $c_2(\mathcal{E}_{a,b}) = abA^2 = ab$; hence

$$\delta = c_1(\mathcal{E}_{a,b})^2 - 4c_2(\mathcal{E}_{a,b}) = (a+b)^2 - 4ab = (a-b)^2. \tag{25}$$

Therefore $\mathcal{E}_{a,b}$ is B-unstable unless $a = b$, in which case $\frac{1}{2}c_1(\mathcal{E}_{a,a}) = \mathcal{O}_Y(aA)$ and $\mathcal{E}$ is properly B-semistable. Thus for $a = b$, according to Theorem 4.4, $G = \Gamma_{(Y,\mathcal{L})}$ for any ample line bundle $\mathcal{L}$, HC equivalent to $\mathcal{O}_Y(aA)$. Note, however, that Y has Picard number 2, since $\mathrm{rk}\big(\mathrm{NS}(J(C))\big) = 1$, C being general. So, this situation fits into [11, Corollary 3.3] because $K_Y^2 < 0$, and then we can replace the condition that $\mathcal{L}$ is HC-equivalent to $\frac{1}{2}c_1(\mathcal{E}_{a,a})$ with that of being an average polarization of $(Y, \mathcal{E}_{a,a})$.

Moreover, we have

Proposition 4.6 *Let Y and $\mathcal{E} = \mathcal{E}_{a,b}$ be as in Example 5, and let G be the residual part of the line ℓ in $\Gamma_{(\mathbb{P}_Y(\mathcal{E}),L)}$, where L is the tautological line bundle. The only possibility for being $G = \Gamma_{(Y,\mathcal{L})}$ for some ample $\mathbb{Q}$-line bundle $\mathcal{L}$ on Y is that $\delta = 0$.*

Proof. Assume that $\delta \neq 0$. Then $\delta > 0$, as we have seen in Example 5. Set $\epsilon := a - b$, $s := a + b = 2b + \epsilon$. As we said, we can suppose that $\epsilon \geq 0$; but $\delta = \epsilon^2$ by (25), hence $\epsilon > 0$. Recalling (16), an easy computation shows that

$$\mathcal{J} = \frac{1}{12}\left(-6s^2 + \delta\big((s+1)^2 - 2\big)\right).$$

Thus the necessary condition for Γ being a Hilbert curve is that

$$\delta \leq \frac{6s^2}{(s+1)^2 - 2}.$$

This implies $\delta < 6$ which means that $\delta = 1$ or 4, since δ is the square of an integer. In other words, either $\epsilon = 1$ ($\delta = 1$), or $\epsilon = 2$ ($\delta = 4$). The latter case, however, cannot occur. Actually, if $G = \Gamma_{(Y,\mathcal{L})}$ for some ample $\mathbb{Q}$-line bundle $\mathcal{L}$, by applying the usual argument, (19) would give $\rho = -\frac{1}{3}$, while ρ has to be positive in view of (21). Therefore, if $G = \Gamma_{(Y,\mathcal{L})}$ and $\delta \neq 0$, then $\epsilon = 1$. Thus, by applying the usual argument again, (19) says that $\rho = \frac{2}{3}$ and then (20) and (21) give the equations

$$K_Y \cdot \mathcal{L} = \frac{1}{4}(3s-1) \qquad \text{and} \qquad \mathcal{L}^2 = \frac{1}{8}(3s^2+1).$$

On the other hand, $\mathrm{NS}(Y) \cong \mathbb{Z}^2$, generated by A and E, since the curve C is general; so, up to numerical equivalence, we can write $\mathcal{L} = xA - yE$ for some $x, y \in \mathbb{Q}$, with $x > 0$ and $x - y > 0$ due to the ampleness conditions coming from $\mathcal{L} \cdot A = x - y > 0$ and $\mathcal{L} \cdot E = x + y > 0$. Thus, the above relations give

$$x + y = \frac{1}{4}(3s - 1) \qquad \text{and} \qquad x^2 - 2xy - y^2 = \frac{1}{8}(3s^2 + 1),$$

which in turn leads to the equation

$$32x^2 - 15s^2 + 6s - 3 = 0.$$

Now, letting $x = \frac{p}{q}$, with p, q integers, one can directly check, e. g., with Maple, that the corresponding equation has no solution in integers. Therefore, $\delta = 0$. Q.E.D.

Of course it may also happen that there are infinitely many not numerically equivalent ample $\mathbb{Q}$-line bundles $\mathcal{L}$ such that $G = \Gamma_{(Y,\mathcal{L})}$. Here is an enlightening example.

Example 6. Let C be a smooth curve of genus one, let $\mathcal{V}$ and $\mathcal{U}$ be two indecomposable rank-2 vector bundles on C of degree 1, consider the elliptic $\mathbb{P}^1$-bundle $Y := \mathbb{P}_C(\mathcal{V})$, with projection $\theta : Y \to C$, and denote by ξ and f the tautological line bundle and a fiber, respectively. Recall that $\xi^2 = \xi \cdot f = 1$. For any positive integer a, set $\mathcal{E}_a := \theta^*\mathcal{U} \otimes [a\xi]$. Note that $\mathcal{E}_a$ is ample (in fact ample and spanned by [1]). We have $c_1(\mathcal{E}_a) \equiv 2a\xi + f$ and $c_2(\mathcal{E}_a) = \theta^* c_1(\mathcal{U}) \cdot a\xi + (a\xi)^2 = a(a+1)$. So, $c_1(\mathcal{E}_a)^2 = 4a(a+1)$, and therefore

$$\delta = c_1(\mathcal{E}_a)^2 - 4c_2(\mathcal{E}_a) = 0$$

for every a. On the other hand, $K_Y \equiv -2\xi + f$; so $K_Y \cdot c_1(\mathcal{E}_a) = -2(a+1)$ and then (16) gives $\mathcal{J} = -(a+1)^2 < 0$. Furthermore, G has equation $(a+1)(av - 2u)v = 0$. Now suppose that $G = \Gamma_{(Y,\mathcal{L})}$ for some ample $\mathbb{Q}$-line bundle $\mathcal{L}$ on Y. Since the classes of ξ and f generate $\mathrm{Num}(Y)$, we can write $\mathcal{L} \equiv x\xi + yf$ for some $x, y \in \mathbb{Q}$, and the ampleness conditions applied to a suitable multiple of $\mathcal{L}$ imply that $x > 0$ and $y > -\frac{1}{2}x$. We have $K_Y \cdot \mathcal{L} = -(x + 2y)$ and $\mathcal{L}^2 = x(x + 2y)$. Moreover, $K_Y^2 = \chi(\mathcal{O}_Y) = 0$. Hence the system of (19)–(22), taking into account that the first and the last equations are trivial, reduces to the following:

$$\begin{cases} -(a+1) &= & -\rho(x + 2y) \\ a(a+1) &= & \rho x(x + 2y) \ . \end{cases}$$

This gives $x = a$ and then for any rational number $y > \frac{-a}{2}$ all our conditions are satisfied with $\rho = \frac{a+1}{a+2y}$. In particular there are infinitely many $\mathcal{L} \in \mathrm{Num}(Y) \otimes \mathbb{Q}$ such that $G = \Gamma_{(Y,\mathcal{L})}$ (see also [11, Proposition 2.3]).

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A. Lanteri
Dipartimento di Matematica "F. Enriques"
Università degli Studi di Milano
Via C. Saldini, 50
I-20133 Milano, Italy.
e-mail `antonio.lanteri@unimi.it`