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Height fluctuations in interacting dimers

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Abstract. We consider a non-integrable model for interacting dimers on the two-dimensional square lattice. Configurations are perfect matchings of $\mathbb{Z}^2$, i.e. subsets of edges such that each vertex is covered exactly once ("close-packing" condition). Dimer configurations are in bijection with discrete height functions, defined on faces $\boldsymbol{\xi}$ of $\mathbb{Z}^2$. The non-interacting model is "integrable" and solvable via Kasteleyn theory; it is known that all the moments of the height difference $h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}$ converge to those of the massless Gaussian Free Field (GFF), asymptotically as $|\boldsymbol{\xi} - \boldsymbol{\eta}| \to \infty$. We prove that the same holds for small non-zero interactions, as was conjectured in the theoretical physics literature. Remarkably, dimer-dimer correlation functions are instead not universal and decay with a critical exponent that depends on the interaction strength. Our proof is based on an exact representation of the model in terms of lattice interacting fermions, which are studied by constructive field theory methods. In the fermionic language, the height difference $h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}$ takes the form of a non-local operator, consisting of a sum of monomials along an *arbitrary* path connecting $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. As in the non-interacting case, this path-independence plays a crucial role in the proof.

Résumé. Nous étudions un modèle non integrable de dimères en interaction sur le réseau carré bidimensionnel. Les configurations sont des appariements parfaits de $\mathbb{Z}^2$, i.e. des sous-ensembles d'arêtes tels que tout sommet est contenu dans une et une seule arête (condition de "close-packing"). Les configurations de dimères sont en bijection avec une fonction de hauteur discrète, définie sur les faces $\boldsymbol{\xi}$ de $\mathbb{Z}^2$. Le modèle sans interaction est "integrable" et resoluble par la théorie de Kasteleyn; il est connu que tous les moments de la différence de hauteur $h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}$ convergent vers ceux du champ Gaussien libre (GFF), dans la limite où $|\boldsymbol{\xi} - \boldsymbol{\eta}| \to \infty$. Nous démontrons que le même résultat est valable quand le paramètre d'interaction est non nul et petit, comme il avait été conjecturé dans la littérature physique. Il est remarquable que, d'autre côté, les fonctions de correlation dimère-dimère ne sont pas universelles et décroissent avec un exposant critique qui dépend de la force de l'interaction. Notre preuve se base sur une representation exacte du modèle en termes de fermions en interaction su réseau, que nous étudions par des outils de la théorie constructive des champs. Dans le language fermionique, la différence de hauteur $h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}$ est un opérateur non local, qui s'écrit comme une somme de monômes le long d'un chemin *arbitraire* qui relie $\boldsymbol{\xi}$ et $\boldsymbol{\eta}$. Tout comme dans le cas sans interaction, l'indépendance par rapport au choix du chemin joue un rôle crucial dans la preuve.

MSC: 82B28; 82B20

Keywords: Dimer model; Constructive renormalization group; Gaussian free field; Universality

Contents

1. Introduction and main results

Two-dimensional dimer models were studied extensively in the 1960s for their equivalence with various statistical physics models such as the Ising model. At close packing, dimer models are critical (correlations decay polynomially with distance) and, as was later discovered, enjoy conformal invariance properties [42]. Their early study culminated in the *exact solution* of non-interacting dimers by Kasteleyn, Temperley and Fisher [27,40,61] and the related computation of the correlations [28]. However, even in the presence of a solution, a number of properties used in the physical literature were left for decades without a mathematical justification. In particular, the height field (see Section 1.1) was believed to be effectively described in terms of a continuum Gaussian field theory. The difficulty in substantiating mathematically such belief is due to the *ultraviolet divergences* that arise in the continuum limit. They produce ambiguities in the final formulas for the moments of the height function, which require ad hoc regularizations, see e.g. [3,21,64] for an analogous discussion in the context of the critical Ising model. It is fair to say that not only a mathematical proof, but even a solid, convincing, non-rigorous argument, proving the correctness of the scaling limit for the height function, was missing until very recent. The progress came from the mathematical community: in the last 15 years, radically new ideas and methods have been introduced [41–45], which provided a firm basis for the continuum field picture in the *non-interacting* dimer model. These works take advantage of the underlying discrete holomorphicity properties of the model, which arise from its integrability, and can be used to prove the emergence of conformal symmetry in the scaling limit [42,43]. Similar ideas also appeared and developed in the context of percolation and of the Ising model [59,60]. However, these methods fail as soon as integrability is lost, and the very natural question of whether the Gaussian Free Field (GFF) description survives for the interacting case requires radically new ideas. It was proposed in [24] to apply the methods of constructive Renormalization Group (RG) theory to interacting dimers, and in this way the large-distance asymptotics of the dimer-dimer correlations were derived, as well as certain universality relations between critical exponents. In this paper we extend the approach of [24] to the computation of all the moments of the height function, and we succeed in proving their convergence to those of the massless GFF. The control of the height fluctuations, as compared to that of the dimer correlations, poses new non-trivial problems, due to the non-local nature of the height function, as opposed to the local nature of single-dimer observables.

Constructive RG methods have proven, along the decades, to be an invaluable tool to control rigorously some non-integrable critical models and their universality properties, see references below. On the other hand, these methods seem to be very little known in the probability/combinatorics/discrete complex analysis communities, despite the fact that they are interested in very similar mathematical questions for the Ising model, percolation, etc. One of the aims of the present work is to make these methods accessible to a wider audience. For this reason, we make an effort to present the main ideas and steps in a pedagogical way (within reasonable limits: for the technical details of some constructive RG estimates we refer to the relevant literature), which (partly) explains the length of the article.

1.1. *The model*

To be definite, we study the model of interacting classical dimers proposed in [2] and [55]. We consider a periodic box Λ of side L (with L even), whose sites are labelled as follows: $\Lambda = \{\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2 : x_i = -L/2 + 1, \dots, L/2\}$. "Periodic," as usual, means that if $\hat{e}_i$ are the two unit coordinate vectors, then $(L/2, x_2) + \hat{e}_1$ should be identified with $(-L/2 + 1, x_2)$, and $(x_1, L/2) + \hat{e}_2$ with $(x_1, -L/2 + 1)$. The partition function of interest is

$$Z_\Lambda(\lambda, m) = \sum_{M \in \mathcal{M}_\Lambda} \Big[\prod_{b \in M} t_b^{(m)} \Big] e^{\lambda W_\Lambda(M)} \equiv \sum_{M \in \mathcal{M}_\Lambda} \mu_{\Lambda;\lambda,m}(M) : \tag{1.1}$$

- $\mathcal{M}_\Lambda$ is the set of dimer coverings (or perfect matchings) of Λ. We recall that a dimer covering is a subset of edges such that each vertex of Λ is contained in exactly one edge in M. We choose L even, otherwise $\mathcal{M}_\Lambda$ would be empty.
- $m > 0$ is the amplitude of a periodic modulation of the horizontal bond weights, playing the role of an "infrared regularization" (see later), to be eventually removed after performing the thermodynamic limit, by sending $m \to 0$. The modulation is defined as follows: $t^{(m)}_{(\mathbf{x},\mathbf{x}+\hat{e}_j)} = 1 + \delta_{j,1} m(-1)^{x_1}$. Note that $\lim_{m\to 0} t^{(m)}_b = 1$.
- $W_\Lambda(M) = \sum_{P\subset\Lambda} N_P(M)$, where P is a plaquette (face of $\mathbb{Z}^2$) and $N_P(M) = 1$ if the plaquette P is occupied by two parallel dimers in M, and $N_P(M) = 0$ otherwise.

If one sets $\lambda = m = 0$, one recovers the usual integrable, translation invariant, dimer model studied e.g. in [40,42,44].

Since Λ is bipartite we can paint white and black the sites of the two sublattices; with no loss of generality we can assume that the coordinates of the white sites are either (even, even) or (odd, odd). The expectation w.r.t. the measure corresponding to the partition function $Z_\Lambda(\lambda, m)$ will be denoted $\langle\cdot\rangle_{\Lambda;\lambda,m}$: if $O(M)$ is a function of the dimer configuration, we define

$$\langle O\rangle_{\Lambda;\lambda,m} = \frac{1}{Z_\Lambda(\lambda,m)} \sum_M \mu_{\Lambda;\lambda,m}(M) O(M). \tag{1.2}$$

Truncated expectations are denoted by a semicolon: e.g., $\langle O; O'\rangle_{\Lambda;\lambda,m} := \langle O\, O'\rangle_{\Lambda;\lambda,m} - \langle O\rangle_{\Lambda;\lambda,m}\langle O'\rangle_{\Lambda;\lambda,m}$. The massless infinite volume measure is defined via the following weak limit (existence of the limit for local observables is part of our results):

$$\langle\cdot\rangle_\lambda := \lim_{m\to 0} \lim_{\Lambda\nearrow\mathbb{Z}^2} \langle\cdot\rangle_{\Lambda;\lambda,m}. \tag{1.3}$$

The name "massless" refers to the fact that $\langle\cdot\rangle_\lambda$ exhibits algebraic decay of correlations, irrespective of the value of λ, see Theorem 2 below. If, instead of sending $m \to 0$ in (1.3), we keep $m > 0$ fixed in the thermodynamic limit, then the truncated correlations decay exponentially to zero at large distances, with rate proportional to m itself. In this sense, m plays the role of a mass (infrared regularization).

Given a dimer covering M and two faces of Λ centered at $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, one defines the *height* difference between $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ as

$$h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}} = \sum_{b\in C_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \big(\mathbb{1}_b(M) - \langle\mathbb{1}_b\rangle_{\Lambda;\lambda,m}\big)\sigma_b, \tag{1.4}$$

where $\mathbb{1}_b(M)$ denotes the dimer occupancy, i.e., the observable that is equal to 1 if b is occupied by a dimer in M, and 0 otherwise, while $C_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ is a nearest-neighbor path on the dual lattice of Λ (i.e. a path on faces of Λ). The sum runs over the edges crossed by the path and $\sigma_b = +1/-1$ depending on whether the oriented path $C_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ from $\boldsymbol{\xi}$ to $\boldsymbol{\eta}$ crosses b with the white site on the right/left. See Figure 1.

We have centered the height function to have gradients with zero average; remark that, for $m = 0$, $\langle\mathbb{1}_b\rangle_{\Lambda;\lambda,m} = 1/4$ by symmetry. A priori, the definition (1.4) depends on the choice of the path. The remarkable fact is that it is actually independent of it, provided the path "does not wind around the torus": more precisely, the r.h.s. of (1.4) computed along two different paths is the same, provided the loop obtained by taking the union of the two paths does not wind around the torus[1] [44]. We shall say that two such paths are equivalent. In particular, if $||\boldsymbol{\xi} - \boldsymbol{\eta}||_\infty < L/2$, then all the shortest lattice paths are equivalent, and we uniquely define the height difference between $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ as the r.h.s. of (1.4), computed along any path equivalent to one of the shortest lattice paths. In this way, given two faces with fixed (i.e., L-independent) coordinates $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, their height difference is uniquely defined, for sufficiently large L. If we arbitrarily assign height zero to the "central" plaquette (the one centered at $(1/2, 1/2)$), then the height profile is

[1] In general, if a path wraps n_1 times horizontally and n_2 times vertically over the torus, the r.h.s. of (1.4) picks up an additive term $n_1 T_1(M) + n_2 T_2(M)$, for suitable constants, called *periods*. In this sense, the height on the torus is additively multi-valued. The example in Figure 1 is special, in that $T_1(M) = T_2(M) = 0$ for the configuration M depicted there; however, it is easy to exhibit other configurations for which these periods are non-zero.

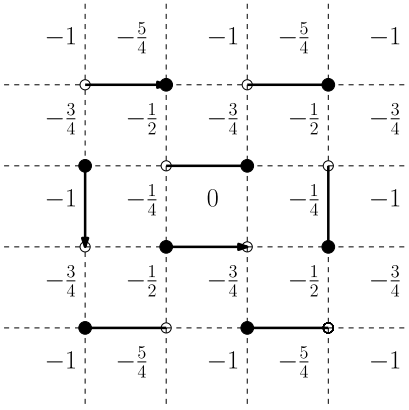

Fig. 1. A dimer configuration for $L = 4$ and the height function computed according to (1.4). In this picture we assume that $\langle \mathbb{1}_b \rangle = 1/4$ for every b, which is the case on the torus for $m = 0$. Moreover, we conventionally set the value of the height in the central plaquette equal to 0.

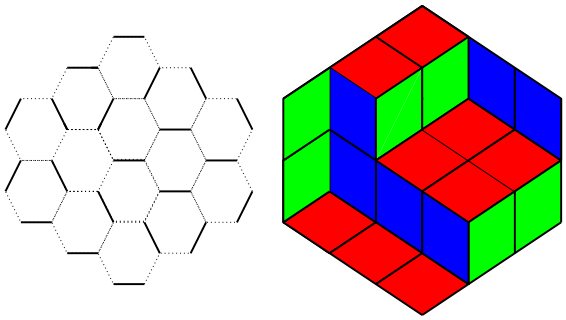

Fig. 2. A dimer covering of a domain of the honeycomb lattice and the corresponding discrete height function. The correspondence is established by drawing a segment along the main diagonal of each lozenge in the figure on the right: these segments are the same as the dimers in the figure on the left, and they make apparent the fact that dimer configurations are in one-to-one correspondence with lozenge tilings of planar domains. The vertices of the lozenges correspond to the centers of the hexagonal cells in the figure on the left.

uniquely determined everywhere, asymptotically as $L \to \infty$. In conclusion, each plaquette is associated with a value of the height function, and one can view each plaquette as the basis of a block which extends out of the page by an amount given by the height function. From this perspective, dimer covering may be viewed as a two-dimensional representation of the surface of a three-dimensional crystal.

Let us mention that the bijection between discrete interfaces and perfect matchings of planar bipartite graphs is a general fact: see for instance Figure 2 for the (visually more obvious) case of the honeycomb lattice.

1.2. *Correlations and expected behavior*

Among the physically interesting correlations are the *dimer correlations* $\langle \mathbb{1}_{b_1}; \dots; \mathbb{1}_{b_n} \rangle_\lambda$, the *height moments* $\langle (h_\xi - h_\eta)^n \rangle_\lambda$ and the so-called *electric correlator*

$$\left\langle e^{i\alpha(h_\xi - h_\eta)} \right\rangle_\lambda. \tag{1.5}$$

For $\lambda = 0$ (non-interacting dimers) the partition function was exactly computed in [27,40,61], where it was shown that it can be expressed in terms of the Pfaffian of the Kasteleyn matrix (see below); such a Pfaffian can be rewritten exactly as a *Gaussian* Grassmann integral, so that the case $\lambda = 0$ is also called free fermion point (see e.g. [34,54] for a definition and an illustration of the basic properties of Grassmann integrals). The *dimer* correlations are easily computable from their Grassmann representation (the dimer occupancy $\mathbb{1}_b$ becomes a *local* quadratic monomial in

the language of Grassmann variables), by using the fermionic Wick theorem, see for instance [26]: one finds that if $m = 0$, the dimer correlations decay as a power law modulated by an oscillating factor; in particular, the decay of the two-point dimer correlation is proportional to the inverse distance squared (see Proposition 3 below).

The computation of the height or electric correlations is a completely different matter: the height and electric observables take the form of *non-local* expressions in the Grassmann variables (as can be guessed from (1.4)) and their computation is much harder. Indeed, the proof (for $\lambda = 0$) of GFF-like behavior for the height function [43, 45] and the computation of the large-distance behavior of the electric correlator [22,56] are very recent. The dimer model with $\lambda = 0$ strongly resembles the two-dimensional Ising model at the critical temperature, which admits a similar fermionic representation in terms of Gaussian Grassmann integrals [31,57]. The dimer correlations are the analogues of the Ising energy density correlations (i.e. the correlation between $\sigma_x\sigma_{x'}$ and $\sigma_y\sigma_{y'}$, if (x, x') and (y, y') are two lattice bonds) and the electric correlator at $\alpha = \pi$ is the analogue of the square of the spin-spin correlation at criticality. The analogy is not just formal, but also quantitative: it was recently shown in [23] that there is an exact identity, valid at the lattice level and at finite volume, between the energy correlations of the critical Ising model and the dimer correlations, as well as between the (square of the) two-point spin correlation of critical Ising and the electric correlator at $\alpha = \pi$. These identities play the role of *lattice bosonization* identities, see [23], and imply in particular that the critical exponents of the corresponding Ising and dimer observables are the same.

If $\lambda \neq 0$ the model is not solvable anymore. The Grassmann representation, reviewed below, shows that the interacting model can be expressed exactly in terms of a non-Gaussian Grassmann integral. That is, the interacting dimer model is equivalent to a model of interacting lattice fermions in two dimensions [24]. The critical exponents of the dimer observables change, as apparent from Theorem 2 below, where a non-trivial critical exponent 2κ appears. Nevertheless, a heuristic mapping of the theory into a sine-Gordon model [2] predicts that the height function, at least for small λ, still behaves in the continuum limit as a massless GFF:

$$h_u - h_v \sim \sqrt{\frac{K}{\pi}}(\varphi_u - \varphi_v), \quad u, v \in \mathbb{R}^2, \tag{1.6}$$

where φ is the massless GFF with covariance $-1/(2\pi)\log|u - v|$, and $K = K(\lambda)$ is an analytic function of λ such that $K(0) = 1$.

As already noticed, the identification (1.6) in the scaling limit has been rigorously proved in the *non-interacting* case only [42,43]. In the presence of interactions, (1.6) was until now a phenomenological assumption, not derived from the microscopic Hamiltonian but confirmed by numerical simulations, see [2] and [55]. From simulations, K appears to be a non-trivial function of λ, which suggest that the model should be in the same universality class of the Ashkin–Teller model, see [2,55]. On the basis of a universality relation [39,55], the amplitude K is expected to be computable in terms of the exponent κ of the two-point dimer correlation.

Our Theorems 1 and 3 are the first rigorous confirmations of (1.6) in the interacting case $\lambda \neq 0$. Let us mention that, in the same spirit, convergence of Ginzburg–Landau type $\nabla\phi$ interface models to a GFF was obtained for instance in [20,35,52,53].

1.3. *Results and perspectives*

In the last years methods based on constructive Renormalization Group (RG) have been applied to various classical and quantum statistical mechanics models, starting from [48]. In contrast with field theoretic RG, they can be applied in the presence of a lattice, they allow for a mathematically rigorous control of the effects of momentum cut-offs, of the irrelevant terms, and of the convergence of perturbation theory. These methods have already been successfully applied to the computation of the critical exponents associated with several different observables that, once re-expressed in the language of Grassmann variables, are local or quasi-local operators: examples include the energy and crossover observables in the eight vertex [7,48] and anisotropic Ashkin–Teller models [37], energy density correlations in non-integrable Ising models [38], the correlations of S^z (the z-component of the spin) in the XXZ model [12] and, more recently, the already mentioned dimer correlation of the interacting dimer model [24].

In this paper, we combine this approach with the methods used in the $\lambda = 0$ case in [45], thus applying for the first time constructive RG methods to the study of a *non-local* observable such as the height. Our main result is the following:

Theorem 1. *There exist*:

(1) *a positive constant* λ_0 *and a real analytic function* $K(\lambda)$ *on* $|\lambda| < \lambda_0$ *satisfying* $K(0) = 1$,
(2) *positive constants* C_n, *with* $n \geq 2$, *and a bounded function* $R(\boldsymbol{\xi})$ *satisfying* $|R(\boldsymbol{\xi})| \leq C_2$, $\forall \boldsymbol{\xi} \neq \mathbf{0}$,

such that the following is true: *if* $\boldsymbol{\xi} \neq \boldsymbol{\eta}$, *then*

$$\big\langle (h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}})^2 \big\rangle_\lambda = \frac{K(\lambda)}{\pi^2} \log|\boldsymbol{\xi} - \boldsymbol{\eta}| + R(\boldsymbol{\xi} - \boldsymbol{\eta}). \tag{1.7}$$

Moreover, if $n > 2$, *the* n*th cumulant of* $(h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}})$ *is bounded uniformly in* $|\boldsymbol{\xi} - \boldsymbol{\eta}|$ *as*

$$\big| \langle \underbrace{h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}; \ldots; h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}}_{n \text{ times}} \rangle_\lambda \big| \leq C_n. \tag{1.8}$$

In the non-interacting case $\lambda = m = 0$, the result is a refinement of previously known estimates: in fact, in that case (1.7) is proven in [41] and in [45, Theorem 4.5] (in a much more general setting of bipartite planar graphs), see also [47] for the height moments of order $n > 2$. Neither in [45] nor in [47] there is a sharp control of the error terms: for instance, for the variance the error term in [45] is $o(\log|\boldsymbol{\xi} - \boldsymbol{\eta}|)$ instead of $O(1)$.

Let us also mention that the logarithmic growth of the height variance (without sharp control of the constant in front of the log) for some discrete $(2+1)$-dimensional interface models (Solid-on-Solid and discrete Gaussian model) was obtained in [30]. Moreover, an asymptotic computation of the height variance in the six-vertex model was recently presented in [25].

For the proof of (1.7) the crucial estimate is provided by the following:

Theorem 2. *Let* $|\lambda| \leq \lambda_0$. *There exists* $K(\lambda)$ *as in Theorem 1, and two real analytic functions* $\tilde{K}(\lambda), \kappa(\lambda)$ *with* $\tilde{K}(0) = \kappa(0) = 1$ *such that the following holds. Given two bonds* $b = (\mathbf{x}, \mathbf{x} + \hat{e}_j)$ *and* $b' = (\mathbf{y}, \mathbf{y} + \hat{e}_{j'})$, *then*

$$\begin{aligned} \langle \mathbb{1}_b; \mathbb{1}_{b'} \rangle_\lambda = \mathbf{1}_{\mathbf{x} \neq \mathbf{y}} \Bigg[& -\frac{K(\lambda)}{2\pi^2} (-1)^{\mathbf{x}-\mathbf{y}} \operatorname{Re} \frac{(i)^{j+j'}}{((x_1 - y_1) + i(x_2 - y_2))^2} \\ & + \delta_{j,j'} \frac{\tilde{K}(\lambda)}{2\pi^2} (-1)^{x_j - y_j} \frac{1}{|\mathbf{x} - \mathbf{y}|^{2\kappa(\lambda)}} \Bigg] + R_{j,j'}(\mathbf{x} - \mathbf{y}), \end{aligned} \tag{1.9}$$

with $|R_{j,j'}(\mathbf{x} - \mathbf{y})| \leq C_\theta (1 + |\mathbf{x} - \mathbf{y}|)^{-2-\theta}$, *for some* $\frac{1}{2} \leq \theta < 1$ *and* $C_\theta > 0$.

This result appears in [24] together with a sketchy derivation, and its proof is reproduced in this paper, see Section 6.4 below (it builds on the tools introduced in Sections 6.1–6.3).

The estimates behind the proof of Theorem 1 are strong enough to actually prove that the height field converges in law, in the scaling limit, to the massless Gaussian Free Field X on the plane [58] with covariance

$$\big\langle X(x) X(y) \big\rangle = G_\lambda(x - y) := -\frac{K(\lambda)}{2\pi^2} \log|x - y|. \tag{1.10}$$

More precisely:

Theorem 3. *Recall that the height is set to zero at the central face,* $h(\mathbf{0}) = 0$. *For every* C^∞ *compactly supported test function* $\phi : \mathbb{R}^2 \mapsto \mathbb{R}$ *satisfying* $\int \phi(x)\, dx = 0$, *and* $\epsilon > 0$, *define*

$$h^\epsilon(\phi) = \epsilon^2 \sum_{\boldsymbol{\eta}} h_{\boldsymbol{\eta}} \phi(\epsilon \boldsymbol{\eta}), \tag{1.11}$$

where the sum runs over the faces of Λ. *Then, for every* $\alpha \in \mathbb{R}$,

$$\lim_{\epsilon \to 0} \big\langle e^{i\alpha h^\epsilon(\phi)} \big\rangle_\lambda = \exp\Big(-\frac{\alpha^2}{2} \int \phi(x) \phi(y) G_\lambda(x - y)\, dx\, dy \Big). \tag{1.12}$$

Remark 1. *Let us emphasise that the condition $\int \phi(x)\,dx = 0$ is not technical: if $\int \phi(x)\,dx \neq 0$ the variance of $h^\epsilon(\phi)$ diverges logarithmically as $\epsilon \to 0$. Recall also that the definition of the average $\langle\cdot\rangle_\lambda$ includes the thermodynamic and massless limit: in particular, the sum in (1.11) is unambiguously defined since the support of $\phi(\epsilon\cdot)$ is of order $1/\epsilon \ll L$. For the same reason, the limit GFF does not keep trace of the periodic boundary conditions.*

Remark 2 (Electric correlator). *Take χ a smooth, positive, compactly supported function on $\mathbb{R}^2$ centered around the origin and of average 1. Then, if $\chi_x(\cdot) := \chi(\cdot - x)$ for $x \in \mathbb{R}^2$, from Theorem 3 we obtain*

$$\lim_{\epsilon\to 0}\big\langle e^{i\alpha(h^\epsilon(\chi_x)-h^\epsilon(\chi_y))}\big\rangle_\lambda \sim const \times |x-y|^{-K\alpha^2/(2\pi^2)} \tag{1.13}$$

asymptotically as $|x-y| \to \infty$. This suggests that, at least for α small,

$$\big\langle e^{i\alpha(h_\xi - h_\eta)}\big\rangle_\lambda \sim const \times |\xi-\eta|^{-K\alpha^2/(2\pi^2)}, \tag{1.14}$$

asymptotically at large distances. Indeed, (1.13) can be seen as a coarse-grained version of (1.14) and actually (1.14) would follow from Theorem 1 if we could prove that $C_n = O(n!)$. We hope to come back to this issue in a future publication, possibly by combining the methods of constructive RG with the (strong) discrete holomorphicity used in [22], where (1.14) is proven for $\lambda = 0$ (see also [56]).

Remark 3 (Generalizations and extensions). *The above theorems can be straightforwardly extended to the case where the nearest neighbor interaction W_Λ in (1.1) is replaced by a general finite range interaction that respects the symmetries of the lattice. Another possible generalization (in the spirit of [45]), that we did not work out in detail but we believe would not entail new conceptual difficulties, is to work on different planar bipartite lattices, like the honeycomb lattice.*

The proof of Theorem 1 also implies analyticity of the free energy per unit volume for λ small, uniformly in the volume. Since the free energy can be seen as the Legendre transform of the large deviation functional of W_Λ, one can deduce a central limit theorem for the fluctuations of $W_\Lambda/\sqrt{|\Lambda|}$ around its mean, and similarly for space averages of other local observables.

In principle, the proof of Theorem 1 provides estimates on the convergence radius λ_0, as well as on the constants C_n. However, since we do not expect them to be optimal, we do not spell them explicitly here (e.g., our estimates on C_n grow proportionally to $(n!)^\beta$ with n, for some $\beta > 1$). The proof is based on precise asymptotics on multipoint dimer correlations, which requires the identification of remarkable cancellations in the (renormalized, convergent) expansion for the correlations, which follow from hidden Ward Identities [12] (i.e., asymptotic identities among correlation functions). The name "hidden" refers to the fact that these identities are not exact in the model at hand, while they are so in a continuum reference model (Section 6.3.2), which displays the same large-distance behavior as the interacting dimer model but on the other hand has more symmetries.

Note that in the above theorem no continuum limit is performed. Therefore, the $n \geq 3$ cumulants are not exactly vanishing, but are finite, while the 2-point function is log-divergent as $|\xi - \eta| \to \infty$.

Let us also remark that our result is not just a corollary of the estimates on the dimer correlations, which can be inferred from (the methods of) [24]. In fact, a naive substitution of these estimates into the expression of the nth cumulant of $(h_\xi - h_\eta)$ obtained by plugging (1.4) into the l.h.s.s of (1.7)–(1.8) leads to very poor bounds, growing faster than $(\log|\xi-\eta|)^{n/2}$ at large distances. A key fact that we need to implement is the path-independence of the r.h.s. of (1.4), which is a (weak) instance of the underlying discrete holomorphicity of the model, and relies crucially on the presence of the oscillatory factor σ_b: these oscillatory factors produce remarkable cancellations in the perturbation series, which we keep track of within our constructive multi-scale computation of the height correlations.

Finally let us mention that, for $\lambda = 0$, height correlations in finite domains exhibit conformal covariance properties in the scaling limit where the lattice spacing tends to zero; this was proven for instance by Kenyon [42,43] for some suitably chosen boundary conditions. It would be extremely interesting to prove that conformal invariance survives for $\lambda \neq 0$, where integrability is lost. While we believe that constructive RG is again the right approach to attack this problem, new difficulties will need to be overcome with respect to the present work, notably due to the loss of translation invariance arising from non-periodic boundary conditions.

1.4. *Organization of the paper*

In Section 2 we show how to represent the partition function and the multi-dimer correlations for $\lambda = 0$ (resp. $\lambda \neq 0$) as a Gaussian (resp. non-Gaussian) Grassmann integral. In Section 3, as a warm-up, we prove Theorem 1 for $\lambda = 0$. In Section 4 we prove (1.7), conditionally on Theorem 2. In Sections 5 and 6 we discuss respectively formal perturbation expansion in λ and the "renormalized expansion." The latter is convergent for λ small and allows to get the large-distance behavior of multi-dimer correlations of the interacting model, thereby completing the proof of Theorem 2. Finally, in Section 7 we use the results of Section 6 to prove Theorems 1 and 3 for $\lambda \neq 0$.

2. Grassmann representation of partition function and correlations

In this section we explain how to derive a representation of the interacting partition function $Z_\Lambda(\lambda, m)$ and dimer correlation functions in terms of non-Gaussian Grassmann integrals. This representation is exact and valid as an algebraic identity for every finite lattice Λ. For the reader who is not used to Grassmann variables, we refer for instance to [34, Section 4] for some of their basic properties. The key points to keep in mind are the following: Grassmann variables anti-commute, in particular $\psi_{\mathbf{x}}^2 = 0$. Gaussian Grassmann integrals are just an alternative way of writing determinants (or Pfaffians); non-Gaussian Grassmann integrals are just an alternative, compact, way of writing certain series of determinants (or of Pfaffians); the rewriting of $Z_\Lambda(\lambda, m)$ in terms of a non-Gaussian Grassmann integral is very convenient for its subsequent computation via the methods of constructive field theory, which makes the analogy with the rigorous multi-scale analysis of perturbed Gaussian measures as apparent as possible.

2.1. *The non-interacting model*

2.1.1. *Partition function and dimer correlations*

Kasteleyn's theory [40] gives an explicit formula for the dimer partition function with bond-dependent activities $\mathbf{t} = \{t_b\}_{b \subset \Lambda}$,

$$Z_\Lambda(\mathbf{t}) = \sum_{M \in \mathcal{M}_\Lambda} \prod_{b \in M} t_b. \tag{2.1}$$

Introduce the Kasteleyn matrix[2] $K_{\mathbf{t}}$, which is a $|\Lambda| \times |\Lambda|$ antisymmetric matrix indexed by vertices in Λ, such that its elements $(K_{\mathbf{t}})_{\mathbf{x},\mathbf{y}}$ are non-zero if and only if $\mathbf{x}$ and $\mathbf{y}$ are nearest neighbors; in this case $(K_{\mathbf{t}})_{\mathbf{x},\mathbf{x}+\hat{e}_1} = -(K_{\mathbf{t}})_{\mathbf{x}+\hat{e}_1,\mathbf{x}} = t_{(\mathbf{x},\mathbf{x}+\hat{e}_1)}$, and $(K_{\mathbf{t}})_{\mathbf{x},\mathbf{x}+\hat{e}_2} = -(K_{\mathbf{t}})_{\mathbf{x}+\hat{e}_2,\mathbf{x}} = it_{(\mathbf{x},\mathbf{x}+\hat{e}_2)}$. Also, for $\theta, \tau \in \{0, 1\}$ let $K_{\mathbf{t}}^{(\theta\tau)}$ be the antisymmetric matrix obtained from $K_{\mathbf{t}}$ by multiplying the matrix elements $(K_{\mathbf{t}})_{\mathbf{x},\mathbf{x}+\hat{e}_1} = -(K_{\mathbf{t}})_{\mathbf{x}+\hat{e}_1,\mathbf{x}}$ by $(-1)^\theta$ if $\mathbf{x}$ belongs to the rightmost column of Λ and $(K_{\mathbf{t}})_{\mathbf{x},\mathbf{x}+\hat{e}_2} = -(K_{\mathbf{t}})_{\mathbf{x}+\hat{e}_2,\mathbf{x}}$ by $(-1)^\tau$ if $\mathbf{x}$ is in the top row of Λ. Of course, $K_{\mathbf{t}}^{(00)} = K_{\mathbf{t}}$. Then one has (cf. [44] and [45, Section 3.1.2])

$$\begin{aligned} Z_\Lambda(\mathbf{t}) &= \frac{1}{2}\big(-\operatorname{Pf} K_{\mathbf{t}}^{(00)} + \operatorname{Pf} K_{\mathbf{t}}^{(01)} + \operatorname{Pf} K_{\mathbf{t}}^{(10)} + \operatorname{Pf} K_{\mathbf{t}}^{(11)}\big) \\ &:= \frac{1}{2} \sum_{\theta,\tau=0,1} C_{\theta,\tau} \operatorname{Pf} K_{\mathbf{t}}^{(\theta\tau)}. \end{aligned} \tag{2.2}$$

Here, $\operatorname{Pf}(A)$ indicates the Pfaffian of A. [We recall that the Pfaffian of a $2n \times 2n$ antisymmetric matrix A is defined as

$$\operatorname{Pf} A := \frac{1}{2^n n!} \sum_\pi (-1)^\pi A_{\pi(1),\pi(2)} \cdots A_{\pi(2n-1),\pi(2n)}; \tag{2.3}$$

[2]There is a certain amount of freedom in choosing the Kasteleyn matrix. For instance, in [40] matrix elements are all chosen to be real. Two Kasteleyn matrices are gauge equivalent if there exists a function $c : \Lambda \mapsto \mathbb{C}$ such that $K'_{\mathbf{x},\mathbf{y}} = K_{\mathbf{x},\mathbf{y}} c_{\mathbf{x}} c_{\mathbf{y}}$. See [44, Section 3.3] for a discussion of this point.

π is a permutation of $(1,\dots,2n)$, $(-1)^\pi$ is its signature. One of the properties of the Pfaffian is that $(\mathrm{Pf}\, A)^2 = \det A$.] Since the ordering of the labels $(1,\dots,2n)$ matters in the definition of Pfaffian (changing the ordering, the sign of the Pfaffian can change), in (2.2) we use the convention that the sites $\mathbf{x} \in \Lambda$ that label the elements of $K_{\mathbf{t}}$ are ordered from left to right on every row, starting from the bottom and going upwards to the top row. Using (2.2) we immediately obtain:

$$\sum_{M\in\mathcal{M}_\Lambda}\Big[\prod_{b\in M} t_b^{(m)}\Big]\mathbb{1}_{b_1}\cdots\mathbb{1}_{b_k} = \frac{1}{2}\sum_{\theta,\tau=0,1} C_{\theta,\tau}\, t_{b_1}\partial_{t_{b_1}}\cdots t_{b_k}\partial_{t_{b_k}}\,\mathrm{Pf}\, K_{\mathbf{t}}^{(\theta\tau)}\Big|_{\mathbf{t}=\mathbf{t}^{(m)}}, \tag{2.4}$$

where $\mathbf{t} = \mathbf{t}^{(m)}$ means $t_b = t_b^{(m)}$ for every b. The r.h.s. of (2.4) is itself a sum over Pfaffians, and can be conveniently represented in terms of Gaussian Grassmann integrals. In fact, given any $2n \times 2n$ antisymmetric matrix A,

$$\mathrm{Pf}\, A = \int\Big[\prod_{i=1}^{2n} d\psi_i\Big] e^{-1/2(\psi, A\psi)}, \tag{2.5}$$

where the Grassmann integration is normalized in such a way that

$$\int\Big[\prod_{i=1}^{2n} d\psi_i\Big]\psi_{2n}\cdots\psi_1 = 1.$$

For later purposes, it is also useful to recall that the averages of Grassmann monomials with respect to the Grassmann Gaussian integration can be computed in terms of the fermionic Wick rule:

$$\langle\psi_{k_1}\cdots\psi_{k_m}\rangle_A := \frac{1}{\mathrm{Pf}\, A}\int\Big[\prod_{i=1}^{2n} d\psi_i\Big]\psi_{k_1}\cdots\psi_{k_m} e^{-1/2(\psi, A\psi)} = \mathrm{Pf}\, G, \tag{2.6}$$

where, if m is even, G is the $m \times m$ matrix with entries

$$G_{ij} = \langle\psi_{k_i}\psi_{k_j}\rangle_A = \big[A^{-1}\big]_{k_i,k_j} \tag{2.7}$$

(if m is odd, the r.h.s. of (2.6) should be interpreted as 0).

Specializing these formulas to the case $A = K_{\mathbf{t}}^{(\theta\tau)}$ we find:

$$\mathrm{Pf}\, K_{\mathbf{t}}^{(\theta\tau)} = \int_{(\theta\tau)}\Big[\prod_{\mathbf{x}\in\Lambda} d\psi_{\mathbf{x}}\Big] e^{S_{\mathbf{t}}(\psi)}, \tag{2.8}$$

$$S_{\mathbf{t}}(\psi) = -\frac{1}{2}\sum_{\mathbf{x},\mathbf{y}\in\Lambda}\psi_{\mathbf{x}}\big(K_{\mathbf{t}}^{(\theta\tau)}\big)_{\mathbf{x},\mathbf{y}}\psi_{\mathbf{y}} \tag{2.9}$$

$$= -\sum_{\mathbf{x}\in\Lambda}[t_{(\mathbf{x},\mathbf{x}+\hat{e}_1)}E_{(\mathbf{x},\mathbf{x}+\hat{e}_1)} + t_{(\mathbf{x},\mathbf{x}+\hat{e}_2)}E_{(\mathbf{x},\mathbf{x}+\hat{e}_2)}], \tag{2.10}$$

where $E_{(\mathbf{x},\mathbf{x}+\hat{e}_1)} = \psi_{\mathbf{x}}\psi_{\mathbf{x}+\hat{e}_1}$ while $E_{(\mathbf{x},\mathbf{x}+\hat{e}_2)} = i\psi_{\mathbf{x}}\psi_{\mathbf{x}+\hat{e}_2}$ and the index $(\theta\tau)$ under the integral means that we have to identify $\psi_{(L/2+1,y)} \equiv \psi_{(-L/2+1,y)}(-1)^\theta$ and similarly $\psi_{(x,L/2+1)} \equiv \psi_{(x,-L/2+1)}(-1)^\tau$. The choice $\theta = 0$ (resp. $\theta = 1$) means periodic (resp. antiperiodic) boundary conditions for the Grassmann field in the horizontal direction, and similarly τ determines periodic/antiperiodic boundary conditions in the vertical direction.

Inserting (2.10) into (2.4) we find that, *if the bonds $b_1,\dots,b_k$ are all different* [here we say that two bonds are different if they are not identical; their geometrical supports may overlap], then

$$\sum_{M\in\mathcal{M}_\Lambda}\Big[\prod_{b\in M} t_b^{(m)}\Big]\mathbb{1}_{b_1}\cdots\mathbb{1}_{b_k} = \sum_{\theta\tau}\frac{C_{\theta,\tau}}{2}\int_{(\theta\tau)}\prod_{\mathbf{x}\in\Lambda} d\psi_{\mathbf{x}}(-1)^k E_{b_1}^{(m)}\cdots E_{b_k}^{(m)} e^{S(\psi)}, \tag{2.11}$$

where $S(\psi) = S_{\mathbf{t}^{(m)}}(\psi)$, see (2.10), $E_b^{(m)} = t_b^{(m)} E_b$ and the r.h.s. can be computed via (2.6).

2.1.2. *Gaussian Grassmann measures and the free propagator*

Definition 1. *Given an anti-symmetric matrix M we define the Gaussian Grassmann measure with "propagator" M, denoted $P_M(d\psi)$, which maps a polynomial f of the ψ variables into a complex number denoted*

$$\int P_M(d\psi) f(\psi) \quad or \quad \langle f \rangle_M. \tag{2.12}$$

To fix the map, we require:

- *linearity*: $\langle af_1 + bf_2 \rangle_M = a\langle f_1 \rangle_M + b\langle f_1 \rangle_M$ *if* $a, b \in \mathbb{C}$.
- $\langle 1 \rangle_M = 1$.
- $\langle \psi_{k_1} \cdots \psi_{k_m} \rangle_M = \mathrm{Pf}[M(k_i, k_j)_{i,j \le m}]$.

if M is invertible, then we can write more explicitly (cf. (2.6))

$$\int P_M(d\psi) f(\psi) = \frac{1}{\mathrm{Pf}(M^{-1})} \int \left[\prod_x d\psi_x \right] e^{-1/2(\psi, M^{-1}\psi)} f(\psi). \tag{2.13}$$

We emphasize that $P_M(d\psi)$ is not a measure in the usual probabilistic sense. We list two useful properties of Grassmann Gaussian measures, that are analogous to properties of usual Gaussian measures:

Proposition 1. *The following identities hold*:

(1) Addition formula: *If g_1, g_2 are two propagators and $g := g_1 + g_2$, then $P_g(d\psi) = P_{g_1}(d\psi_1) P_{g_2}(d\psi_2)$, in the sense that for every polynomial f*

$$\int P_g(d\psi) f(\psi) = \int P_{g_1}(d\psi_1) \int P_{g_2}(d\psi_2) f(\psi_1 + \psi_2). \tag{2.14}$$

(2) Change of measure: *Given anti-symmetric matrices M and V such that* $\det(1 - \mu MV) > 0$ *for every* $\mu \in [0, 1]$, *we have*

$$\int P_M(d\psi) e^{1/2(\psi, V\psi)} f(\psi) = \sqrt{\det(1 - MV)} \int P_{M'}(d\psi) f(\psi) \tag{2.15}$$

with $M' = (1 - MV)^{-1} M$.

For (2.14) see [34, Eq. (4.21)]; for (2.15) see the analogous [34, Eq. (4.29)] and use the property $\mathrm{Pf}(A)^2 = \det(A)$.

With this language, and recalling formulas (2.8)–(2.11), we see that dimer observables can be expressed as averages of suitable fermionic polynomials under the linear combination

$$\sum_{\theta,\tau} \frac{C_{\theta,\tau} \, \mathrm{Pf}(K^{(\theta\tau)}_{\mathbf{t}^{(m)}})}{\sum_{\theta,\tau} C_{\theta,\tau} \, \mathrm{Pf}(K^{(\theta\tau)}_{\mathbf{t}^{(m)}})} P^{(\theta\tau)}_\Lambda(d\psi)$$

of Grassmann Gaussian measures

$$P^{(\theta\tau)}_\Lambda(d\psi) := P_{[K^{(\theta\tau)}_{\mathbf{t}^{(m)}}]^{-1}}.$$

It is understood that boundary conditions on ψ are (θ, τ), as above. The propagator $[K^{(\theta\tau)}_{\mathbf{t}^{(m)}}]^{-1}$ can be computed exactly: we have (cf. Appendix A)

Lemma 1.

$$g^{(\theta\tau)}_\Lambda(\mathbf{x}, \mathbf{y}) := \int_{(\theta\tau)} P^{(\theta\tau)}_\Lambda(d\psi) \psi_{\mathbf{x}} \psi_{\mathbf{y}} = \left[\left(K^{(\theta\tau)}_{\mathbf{t}^{(m)}} \right)^{-1} \right]_{\mathbf{x},\mathbf{y}} = \frac{1}{L^2} \sum_{\mathbf{k} \in \mathcal{D}^{(\theta\tau)}_\Lambda} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \frac{N(\mathbf{k}, m, y_1)}{2D(\mathbf{k}, m)}, \tag{2.16}$$

where

$$N(\mathbf{k}, m, y_1) = i \sin k_1 + \sin k_2 + m(-1)^{y_1} \cos k_1,$$
$$D(\mathbf{k}, m) = m^2 + \big(1 - m^2\big)(\sin k_1)^2 + (\sin k_2)^2$$

and

$$\mathcal{D}_\Lambda^{(\theta\tau)} = \big\{(2\pi/L)\big(\mathbf{n} + (\theta, \tau)/2\big), \mathbf{n} \in \Lambda\big\}. \tag{2.17}$$

Note that $g_\Lambda^{(\theta\tau)}(\mathbf{x}, \mathbf{y})$ is zero whenever $\mathbf{x}$ and $\mathbf{y}$ have the same parity (this can be seen by observing that the ratio in (2.16) changes sign if $\mathbf{k}$ is changed to $\mathbf{k} + (\pi, \pi)$, while $e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}$ remains unchanged if $\mathbf{x}$ has the same parity as $\mathbf{y}$). The propagator is not translation invariant, but is invariant under translations in $2\mathbb{Z} \times \mathbb{Z}$ (because of the horizontal periodic modulation of the bond weights). Of course, when $m = 0$ full translation invariance is recovered.

In the following we will need to evaluate the propagator for fixed $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^2$, as $\Lambda \nearrow \mathbb{Z}^2$. In this limit, the propagator takes a particularly simple form, independent of $(\theta\tau)$:

$$g(\mathbf{x}, \mathbf{y}) = \lim_{\Lambda \nearrow \mathbb{Z}^2} g_\Lambda^{(\theta\tau)}(\mathbf{x}, \mathbf{y}) = \int_{\mathbb{T}^2} \frac{d\mathbf{k}}{(2\pi)^2} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \frac{N(\mathbf{k}, m, y_1)}{2D(\mathbf{k}, m)}, \tag{2.18}$$

where the torus $\mathbb{T}^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$ is also called the *Brillouin zone*. In analogy with its finite volume counterpart, $g(\mathbf{x}, \mathbf{y})$ is zero whenever $\mathbf{x}$ and $\mathbf{y}$ have the same parity. We will see in Appendix A.2 that the finite-volume corrections to $g_\Lambda^{(\theta\tau)}$ are exponentially small in L, if $m > 0$.

Remark 4. *At this point the role of the regularization parameter $m > 0$ should be apparent. If $m = 0$ then the integrand in $g(\mathbf{x}, \mathbf{y})$ has poles whenever $\sin k_1 = \sin k_2 = 0$. As we will see in next section, the propagator then decays slowly at large distances (like $1/|\mathbf{x} - \mathbf{y}|$), signalling that the system is critical (or massless). When instead $m > 0$ the integrand is analytic on the Brillouin zone and therefore $g(\mathbf{x}, \mathbf{y})$ (that is its Fourier transform) decays exponentially fast and the system is off-critical (or massive). The exponential decay however kicks in only when $|\mathbf{x} - \mathbf{y}| \gtrapprox 1/m$, and for $m \to 0$ the critical decay is recovered. In the language of* [45], *one says that the non-interacting ($\lambda = 0$) system is in the "liquid phase" when $m = 0$ and in the "gaseous phase" when $m > 0$.*

2.1.3. *Majorana fermions*

In this section we discuss the large-distance behavior of the non-interacting propagator $g(\mathbf{x}, \mathbf{y})$ introduced above. Similar estimates (with different notations) are obtained in [45]. The fall-off properties of g play a key role in the computation of the dimer correlations, as well as of the height fluctuations, to be discussed in the next sections. As we will see, it is convenient to split $\psi_\mathbf{x}$ as the sum of oscillating functions times four new Grassmann variables $\psi_{\mathbf{x},\gamma}$, $\gamma = 1, \dots, 4$, each of which has a propagator with well-defined limiting behavior for large distances,

$$\langle \psi_{\mathbf{x},\gamma}\, \psi_{\mathbf{y},\gamma} \rangle \sim \frac{1}{4\pi} \frac{1}{(x_1 - y_1) + i(-1)^{\gamma+1}(x_2 - y_2)}, \tag{2.19}$$

when $|\mathbf{x} - \mathbf{y}|$ is large (but $|\mathbf{x} - \mathbf{y}| \lessapprox 1/m$). For $m = 0$, the four fields ψ_γ are independent (i.e. their propagator is diagonal in the γ index) and are the lattice analogues of "real," massless, Majorana fermions, see [31, Section 2.3.1]. These lattice Majorana fields can be also combined in pairs, to form two "complex," massless, Dirac fields, $\psi_\omega^\pm$, $\omega = \pm 1$, see next section. Besides the terminology, which is borrowed from high energy physics, the transformations from the original Grassmann field, to the Majorana, and then the Dirac fields, are just restatements of a couple of simple, and convenient, algebraic manipulations of the propagator, which are discussed in the following.

Consider (2.18). The large distance asymptotics of $g(\mathbf{x}, \mathbf{y})$ is dominated by the contributions from the momenta close to the singularity points where the denominator is small (for m small), which are $\mathbf{p}_1 = (0, 0)$, $\mathbf{p}_2 = (\pi, 0)$, $\mathbf{p}_3 = (\pi, \pi)$, $\mathbf{p}_4 = (0, \pi)$. Therefore, $g(\mathbf{x}, \mathbf{y})$ can be naturally written as the superposition of four terms:

$$g(\mathbf{x}, \mathbf{y}) = \sum_{\gamma=1}^{4} \int_{\mathbb{T}^2} \frac{d\mathbf{k}}{(2\pi)^2} \chi_\gamma(\mathbf{k}) e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \frac{N(\mathbf{k}, m, y_1)}{2D(\mathbf{k}, m)}, \tag{2.20}$$

where $\chi_\gamma(\mathbf{k})$ are suitable smooth (say, C^∞) functions over the torus, centered at $\mathbf{p}_\gamma$, and defining a partition of the identity: $\sum_{\gamma=1}^4 \chi_\gamma(\mathbf{k}) = 1$. We assume that the functions $\chi_\gamma(\mathbf{k})$ satisfy the following: first of all,

$$\chi_\gamma(\mathbf{k}) = \bar{\chi}(\mathbf{k} - \mathbf{p}_\gamma), \tag{2.21}$$

for a nonnegative compactly supported smooth function $\bar{\chi}(\mathbf{k})$, centered at the origin and even in $\mathbf{k}$. We also require that the support of $\bar{\chi}(\cdot)$ does not include $\mathbf{p}_2, \mathbf{p}_3, \mathbf{p}_4$. For definiteness, one should think of $\bar{\chi}(\cdot)$ as a suitably smoothed version of $\mathbf{1}_{\|\cdot\|_\infty \le \pi/2}$. In Appendix C we make an explicit choice for $\bar{\chi}(\cdot)$, satisfying further smoothness properties.

The decomposition (2.20) with $\chi_\gamma(\mathbf{k})$ as in (2.21) induces the following decomposition on the Grassmann fields:

$$\psi_{\mathbf{x}} = e^{i\mathbf{p}_1\mathbf{x}}\psi_{\mathbf{x},1} - ie^{i\mathbf{p}_2\mathbf{x}}\psi_{\mathbf{x},2} + ie^{i\mathbf{p}_3\mathbf{x}}\psi_{\mathbf{x},3} + e^{i\mathbf{p}_4\mathbf{x}}\psi_{\mathbf{x},4}, \tag{2.22}$$

with $\psi_{\mathbf{x},\gamma}$ Grassmann variables with propagator

$$\langle \psi_{\mathbf{x},\gamma}\psi_{\mathbf{y},\gamma'}\rangle = \int P(d\psi)\psi_{\mathbf{x},\gamma}\psi_{\mathbf{y},\gamma'} = \begin{pmatrix} G(\mathbf{x}-\mathbf{y}) & 0 \\ 0 & G(\mathbf{x}-\mathbf{y}) \end{pmatrix}_{\gamma,\gamma'}, \tag{2.23}$$

where $G(\mathbf{x}) = \{G(\mathbf{x})_{\omega\omega'}\}$, $\omega, \omega' = \pm 1$, is the 2×2 matrix

$$G(\mathbf{x}) = \int_{\mathbb{T}^2} \frac{d\mathbf{k}}{(2\pi)^2}\bar{\chi}(\mathbf{k})\frac{e^{-i\mathbf{k}\mathbf{x}}}{2D(\mathbf{k},m)}\begin{pmatrix} i\sin k_1 + \sin k_2 & im\cos k_1 \\ -im\cos k_1 & i\sin k_1 - \sin k_2 \end{pmatrix}. \tag{2.24}$$

[The reader should simply check that with this definition the field $\psi_{\mathbf{x}}$ has the correct propagator $g(\mathbf{x},\mathbf{y})$ as in (2.20). Keep in mind that for x integer one has $\exp(i\pi x) = \exp(-i\pi x)$.] Note the symmetry properties

$$G_{++}(\mathbf{x}) = G^*_{--}(\mathbf{x}), \qquad G_{+-}(\mathbf{x}) = G^*_{-+}(\mathbf{x}) = -G_{-+}(\mathbf{x}), \tag{2.25}$$

$$G_{\omega\omega'}(\mathbf{x}) = -\omega\omega' G_{\omega\omega'}(-\mathbf{x}). \tag{2.26}$$

At $m = 0$, the large-distance behavior of $G(\mathbf{x})$ is given by (cf. Appendix A.1):

Proposition 2. *If* $\mathbf{x} \neq \mathbf{0}$ *and* $m = 0$,

$$G(\mathbf{x}) = \mathfrak{g}(\mathbf{x}) + R(\mathbf{x}), \tag{2.27}$$

where both $\mathfrak{g}$ *and* R *are diagonal matrices. The diagonal elements of* $\mathfrak{g}$ *are*:

$$\mathfrak{g}_{\omega\omega}(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^2}\frac{e^{-i\mathbf{k}\mathbf{x}}}{2(-ik_1 + \omega k_2)} = \frac{1}{4\pi}\frac{1}{x_1 + i\omega x_2}, \quad \omega = \pm. \tag{2.28}$$

The matrix R *is a remainder such that*

$$\big|R_{\omega\omega}(\mathbf{x})\big| \le \frac{C}{|\mathbf{x}|^2}, \tag{2.29}$$

for a suitable $C > 0$.

Remark 5. *Using* (2.27)*, we see that* $\langle \psi_{\mathbf{x},\gamma}\psi_{\mathbf{y},\gamma}\rangle$ *behaves asymptotically as the propagator of a real massless Majorana field, in the sense of* [31, *Section* 2.3.1]*. Similarly, if* $m \neq 0$, G *behaves asymptotically as a* massive *Majorana field. Therefore, the fields* $\psi_{\mathbf{x},\gamma}$ *are referred to as lattice Majorana fields.*

From the above discussion we see that the propagator G decays as the inverse of the distance, without any oscillating factor. The discrete derivatives of G decay as the inverse distance squared, while the same is not true for g, due to oscillatory factors in (2.22).

The decomposition of the field ψ in terms of four Majorana fields ψ_γ can be done analogously in finite volume and for the boundary conditions $(\theta\tau)$. In this case, one should simply interpret, e.g. in (2.24), integrals as sums for $\mathbf{k} \in \mathcal{D}_\Lambda^{(\theta\tau)}$, the estimates in Proposition 2 still hold and the Grassmann integration w.r.t. ψ_γ will be denoted $P_\Lambda^{(\theta\tau)}(d\psi_\gamma)$.

2.1.4. *Dirac fermions*
Since the propagator of ψ_γ depends only on the parity of γ, it can be convenient to group the two pairs of so-called "real fields" (ψ_1, ψ_3) and (ψ_2, ψ_4) into "complex fields" $\psi^\pm_\omega$:

$$\psi^\pm_{\mathbf{x},1} := \frac{1}{\sqrt{2}}(\psi_{\mathbf{x},1} \mp i\psi_{\mathbf{x},3}); \qquad \psi^\pm_{\mathbf{x},-1} := \pm\frac{i}{\sqrt{2}}(\psi_{\mathbf{x},2} \mp i\psi_{\mathbf{x},4}), \tag{2.30}$$

which is inverted (recall (2.22)) as

$$\psi_{\mathbf{x}} = \sqrt{2}\cdot \begin{cases} \psi^-_{\mathbf{x},1} + \psi^-_{\mathbf{x},-1} & \text{if } (x_1, x_2) = (\text{even}, \text{even}), \\ \psi^+_{\mathbf{x},1} - \psi^+_{\mathbf{x},-1} & \text{if } (x_1, x_2) = (\text{even}, \text{odd}), \\ \psi^+_{\mathbf{x},1} + \psi^+_{\mathbf{x},-1} & \text{if } (x_1, x_2) = (\text{odd}, \text{even}), \\ \psi^-_{\mathbf{x},1} - \psi^-_{\mathbf{x},-1} & \text{if } (x_1, x_2) = (\text{odd}, \text{odd}). \end{cases} \tag{2.31}$$

Here $\psi^+_{\mathbf{x},\omega}$ formally plays the role of complex conjugate of $\psi^-_{\mathbf{x},\omega}$. Using its definition we see that $\langle \psi^+_{\mathbf{x},\omega}\psi^+_{\mathbf{y},\omega'}\rangle = \langle \psi^-_{\mathbf{x},\omega}\psi^-_{\mathbf{y},\omega'}\rangle = 0$, while

$$\begin{pmatrix} \langle \psi^-_{\mathbf{x},1}\psi^+_{\mathbf{y},1}\rangle & \langle \psi^-_{\mathbf{x},1}\psi^+_{\mathbf{y},-1}\rangle \\ \langle \psi^-_{\mathbf{x},-1}\psi^+_{\mathbf{y},1}\rangle & \langle \psi^-_{\mathbf{x},-1}\psi^+_{\mathbf{y},-1}\rangle \end{pmatrix} = \begin{pmatrix} G_{++}(\mathbf{x}-\mathbf{y}) & iG_{+-}(\mathbf{x}-\mathbf{y}) \\ -iG_{-+}(\mathbf{x}-\mathbf{y}) & G_{--}(\mathbf{x}-\mathbf{y}) \end{pmatrix}. \tag{2.32}$$

The "complex" nature of the field $\psi^\pm_\omega$ justifies the name "lattice Dirac field," which is used for it. In the following, it will be sometimes convenient to work with Majorana variables and sometimes with Dirac variables.

2.2. *Dimer-dimer correlations*

Applying formula (2.11) together with the Wick rule (2.6), one can easily express the dimer-dimer correlations of the non-interacting model in terms of the free propagator $g^{(\theta\tau)}_\Lambda$. In the infinite volume $L \to \infty$ and massless limit $m \to 0$, using the asymptotics in Proposition 2, one recovers the well-known result:

Proposition 3. *Let $\lambda = 0$. Given two bonds $b = (\mathbf{x}, \mathbf{x} + \hat{e}_j)$ and $b' = (\mathbf{y}, \mathbf{y} + \hat{e}_{j'})$, we have*

$$\langle \mathbb{1}_b; \mathbb{1}_{b'}\rangle_{\lambda=0} = \mathbf{1}_{\mathbf{x}\neq\mathbf{y}}\Bigg[-\frac{1}{2\pi^2}(-1)^{\mathbf{x}-\mathbf{y}}\,\mathrm{Re}\,\frac{(i)^{j+j'}}{((x_1-y_1)+i(x_2-y_2))^2} + \delta_{j,j'}\frac{1}{2\pi^2}(-1)^{x_j-y_j}\frac{1}{|\mathbf{x}-\mathbf{y}|^2}\Bigg] + R_{j,j'}(\mathbf{x}-\mathbf{y}), \tag{2.33}$$

with $|R_{j,j'}(\mathbf{x}-\mathbf{y})| \le C(1+|\mathbf{x}-\mathbf{y}|)^{-3}$.

(This is re-derived, as a by-product, also in Section 3.2 below.)

2.2.1. *Multi-scale decomposition of the free propagator*
An important tool in constructive RG is a multi-scale decomposition of the free propagator G as a sum of terms $G^{(h)}$, $h \le 0$, each one collecting contributions at a given distance $\simeq 2^h$ (in Fourier space) from the singularities $\mathbf{p_1}, \dots, \mathbf{p_4}$.

We start from $G(\mathbf{x})$ defined in (2.24) or, better, defined as the finite volume counterpart of (2.24) with boundary conditions (θ, τ), in which case the integrals over $\mathbf{k}$ are sums[3] in $\mathcal{D}^{(11)}_\Lambda$. Let $h^* = \lfloor \log_2 m \rfloor$ and recall that $L^{-1} \ll m \ll 1$. Recall that $\bar\chi(\cdot)$ is the cut-off function appearing in (2.24), that should be thought of as a smoothed version of $\mathbf{1}_{\|\mathbf{k}\|_\infty \le \pi/2}$, see the explicit definition (C.2) in Appendix C. Let $\chi(\cdot)$ be another positive, C^∞ cut-off function, that

[3] From now on, unless explicitly stated, we shall write integrals over $\mathbf{k}$ just as shorthands for the corresponding finite volume sums. All the equations and estimates written formally in the thermodynamic limit are valid at finite volume as well, uniformly in Λ.

we require to be rotationally invariant as a function on the Brillouin zone $[-\pi,\pi]^2$, see explicit definition (C.3). One should think of $\chi(\cdot)$ as a smoothed version of $\mathbf{1}_{\|\mathbf{k}\|\le\pi/2}$, with $\|\cdot\|$ the Euclidean norm.

We decompose $G(\mathbf{x})$ as

$$G(\mathbf{x}) = \sum_{h=h^*+1}^{0} G^{(h)}(\mathbf{x}) + G^{(\le h^*)}(\mathbf{x}), \tag{2.34}$$

where $G^{(h)}$ (resp. $G^{(\le h^*)}$) is as in (2.24), except that $\bar\chi(\mathbf{k})$ is replaced by

$$f_h(\mathbf{k}) := \chi_h(\mathbf{k}) - \chi_{h-1}(\mathbf{k}) \tag{2.35}$$

(resp. by $\chi_{h^*}(\mathbf{k})$); here $\chi_0(\mathbf{k}) := \bar\chi(\mathbf{k})$, while

$$\chi_h(\mathbf{k}) := \chi\big(2^{-h}\mathbf{k}\big), \quad \forall h<0. \tag{2.36}$$

Observe that f_h (resp. χ_{h^*}) has compact support contained in

$$S_h := \big\{\mathbf{k}\in\mathbb{T}^2 : c2^h \le \|\mathbf{k}\|^2 = k_1^2+k_2^2 \le C2^h\big\}, \tag{2.37}$$

(resp. in $\bigcup_{h\le h^*} S_h$) for suitable constants $c, C>0$, and that $\sum_{h=h^*+1}^{0} f_h(\mathbf{k}) + \chi_{h^*}(\mathbf{k}) = \bar\chi(\mathbf{k})$. One easily checks that, if ε in (C.3) is small enough, then

$$f_{h_1}(\mathbf{k}) f_{h_2}(\mathbf{k}) = 0, \quad \text{if } |h_1-h_2|>1. \tag{2.38}$$

At $m=0$, the decomposition (2.29) induces a similar decomposition for the single-scale propagator: $G^{(h)}(\mathbf{x}) = \mathfrak{g}^{(h)}(\mathbf{x}) + R^{(h)}(\mathbf{x})$. Finally we have:

Lemma 2. *For $h^*<h\le 0$ and $n_1,n_2\ge 0$, the matrix $G^{(h)}(\mathbf{x})$ satisfies, for a suitable $C_{n_1,n_2}>0$,*

$$\big\|\partial_1^{n_1}\partial_2^{n_2} G^{(h)}(\mathbf{x})\big\| \le C_{n_1,n_2} 2^{h(1+n_1+n_2)} e^{-c\sqrt{2^h|\mathbf{x}|}} \tag{2.39}$$

with ∂_j the right discrete derivative in the j direction. The off-diagonal elements of $G^{(h)}$ satisfy a better estimate:

$$\big|\partial_1^{n_1}\partial_2^{n_2} G^{(h)}_{+-}(\mathbf{x})\big| \le C_{n_1,n_2} m 2^{h(n_1+n_2)} e^{-c\sqrt{2^h|\mathbf{x}|}} \le C' 2^{h^*+h(n_1+n_2)} e^{-c\sqrt{2^h|\mathbf{x}|}}. \tag{2.40}$$

The propagator $G^{(\le h^*)}$ satisfies the same estimates, with h replaced by h^*. If $m=0$, the propagator $\mathfrak{g}^{(h)}$ satisfies the same estimates as $G^{(h)}$ in (2.39), while $R^{(h)}$ satisfies an improved estimate:

$$\big\|\partial_1^{n_1}\partial_2^{n_2} R^{(h)}(\mathbf{x})\big\| \le C_{n_1,n_2} 2^{h(2+n_1+n_2)} e^{-c\sqrt{2^h|\mathbf{x}|}}. \tag{2.41}$$

See Appendix C for a sketch of proof.

2.3. *The interacting model*

2.3.1. *Partition function and dimer correlations*

Our goal here is to rewrite the partition function (1.1) and the correlation functions of the interacting model as a Grassmann integral. For the partition function we have:

Proposition 4. *Let $\alpha = e^\lambda - 1$. We have*

$$Z_\Lambda(\lambda,m) = \frac{1}{2}\sum_{\theta,\tau} C_{\theta,\tau} \int_{(\theta\tau)} \Big[\prod_{\mathbf{x}\in\Lambda} d\psi_{\mathbf{x}}\Big] e^{S(\psi)+V_\Lambda(\psi)}, \tag{2.42}$$

with

$$V_\Lambda(\psi) = \sum_{\gamma\subset\Lambda} \xi(\gamma), \tag{2.43}$$

where, if $b_1, \dots, b_k$ are adjacent parallel bonds (with $k \geq 2$) and $\gamma = \{b_1, \dots, b_k\}$, then

$$\xi(\gamma) = \xi\big(\{b_1, \dots, b_k\}\big) = (-1)^k \alpha^{k-1} E^{(m)}_{b_1} \cdots E^{(m)}_{b_k} \tag{2.44}$$

and $\xi(\gamma) = 0$ otherwise. Recall that $E^{(m)}_b$ was defined just after (2.11).

Proof of Proposition 4. We re-write

$$\begin{aligned} Z_\Lambda(\lambda, m) &= \sum_{M\in\mathcal{M}_\Lambda} \Big[\prod_{b\in M} t^{(m)}_b\Big] \prod_{P\subset\Lambda} \big(1 + \alpha N_P(M)\big) \\ &= \sum_{M\in\mathcal{M}_\Lambda} \Big[\prod_{b\in M} t^{(m)}_b\Big] \prod_{\langle b,b'\rangle\subset\Lambda} \big(1 + \alpha \mathbb{1}_b(M)\mathbb{1}_{b'}(M)\big), \end{aligned} \tag{2.45}$$

where the product $\prod_{\langle b,b'\rangle\subset\Lambda}$ runs over pairs of neighboring parallel bonds b, b' (i.e., such that the union of the four vertices of b and b' are the four vertices of a plaquette in Λ). In the second identity we used the fact that, if P is the plaquette with sites $\mathbf{x}$, $\mathbf{x}+\hat{e}_1$, $\mathbf{x}+\hat{e}_2$, $\mathbf{x}+\hat{e}_1+\hat{e}_2$, then

$$N_P = \mathbb{1}_{(\mathbf{x},\mathbf{x}+\hat{e}_1)}\mathbb{1}_{(\mathbf{x}+\hat{e}_2,\mathbf{x}+\hat{e}_1+\hat{e}_2)} + \mathbb{1}_{(\mathbf{x},\mathbf{x}+\hat{e}_2)}\mathbb{1}_{(\mathbf{x}+\hat{e}_1,\mathbf{x}+\hat{e}_1+\hat{e}_2)}$$

and

$$\mathbb{1}_{(\mathbf{x},\mathbf{x}+\hat{e}_1)}\mathbb{1}_{(\mathbf{x}+\hat{e}_2,\mathbf{x}+\hat{e}_1+\hat{e}_2)}\mathbb{1}_{(\mathbf{x},\mathbf{x}+\hat{e}_2)}\mathbb{1}_{(\mathbf{x}+\hat{e}_1,\mathbf{x}+\hat{e}_1+\hat{e}_2)} = 0$$

as an observable over dimer configurations; therefore,

$$1 + \alpha N_P = (1 + \alpha \mathbb{1}_{(\mathbf{x},\mathbf{x}+\hat{e}_1)}\mathbb{1}_{(\mathbf{x}+\hat{e}_2,\mathbf{x}+\hat{e}_1+\hat{e}_2)})(1 + \alpha \mathbb{1}_{(\mathbf{x},\mathbf{x}+\hat{e}_2)}\mathbb{1}_{(\mathbf{x}+\hat{e}_1,\mathbf{x}+\hat{e}_1+\hat{e}_2)}). \tag{2.46}$$

We now rewrite the last product in (2.45) as

$$\prod_{\langle b,b'\rangle\subset\Lambda} (1 + \alpha \mathbb{1}_b \mathbb{1}_{b'}) = \sum_{n\geq 0} \sum_{\{\gamma_1,\dots,\gamma_n\}\subset\Lambda}^{*} \zeta(\gamma_1)\cdots\zeta(\gamma_n), \tag{2.47}$$

where γ_i are "contours," each consisting of a sequence of 2 or more adjacent parallel bonds $\sum^*_{\{\gamma_1,\dots,\gamma_n\}}$ runs over unordered *compatible* n-ples of contours (here we say that $\{\gamma_1, \dots, \gamma_n\}$ is compatible if $\gamma_i \cap \gamma_j = \varnothing, \forall i \neq j$, where $\gamma_i \cap \gamma_j = \varnothing$ means that the bonds in γ_i are all different from those in γ_j; note that the geometric supports of two compatible contours may overlap). Moreover, if $b_1, \dots, b_k$ are adjacent parallel bonds (with $k \geq 2$) and $\gamma = \{b_1, \dots, b_k\}$, then

$$\zeta(\gamma) = \zeta\big(\{b_1, \dots, b_k\}\big) = \alpha^{k-1} \mathbb{1}_{b_1} \cdots \mathbb{1}_{b_k}. \tag{2.48}$$

Finally, the term with $n = 0$ in the r.h.s. of (2.47) should be interpreted as 1. By inserting (2.47) into (2.45) we find:

$$Z_\Lambda(\lambda, m) = \sum_{n\geq 0} \sum_{\{\gamma_1,\dots,\gamma_n\}\subset\Lambda}^{*} \sum_{M\in\mathcal{M}_\Lambda} \Big[\prod_{b\in M} t^{(m)}_b\Big] \zeta(\gamma_1)\cdots\zeta(\gamma_n). \tag{2.49}$$

Note that each term $\zeta(\gamma_1)\cdots\zeta(\gamma_n)$ in the r.h.s. of (2.49) is proportional to a product of operators $\mathbb{1}_b$ over *different* bonds: actually, having a representation involving only products of $\mathbb{1}_b$ over different bonds was the very purpose

of grouping the bonds into contours and of rewriting the product in the l.h.s. of (2.47) as a sum over compatible collections of contours. Therefore, we can evaluate the sum $\sum_{M\in\mathcal{M}_\Lambda}[\prod_{b\in M} t_b^{(m)}]\zeta(\gamma_1)\cdots\zeta(\gamma_n)$ by using (2.11):

$$Z_\Lambda(\lambda,m)=\frac{1}{2}\sum_{\theta,\tau}C_{\theta,\tau}\sum_{n\geq 0}\sum_{\{\gamma_1,\dots,\gamma_n\}\subset\Lambda}^{*}\int_{(\theta\tau)}\Big[\prod_{\mathbf{x}\in\Lambda}d\psi_{\mathbf{x}}\Big]\xi(\gamma_1)\cdots\xi(\gamma_n)e^{S(\psi)}. \tag{2.50}$$

Finally note that, by the Grassmann anti-commutation rules,

$$\sum_{n\geq 0}\sum_{\{\gamma_1,\dots,\gamma_n\}\subset\Lambda}^{*}\xi(\gamma_1)\cdots\xi(\gamma_n)=e^{\sum_{\gamma\subset\Lambda}\xi(\gamma)}, \tag{2.51}$$

(in the expansion of the exponential, terms containing incompatible contours vanish since $E_b^2=0$) so that (2.50) simplifies into (2.42). □

Remark 6. *It is worth noting that V_Λ can be written as*

$$V_\Lambda(\psi)=\alpha\sum_{\mathbf{x}\in\Lambda}\big(2+2m(-1)^{x_1}+m^2\big)\psi_{\mathbf{x}}\psi_{\mathbf{x}+\hat{e}_1}\psi_{\mathbf{x}+\hat{e}_2}\psi_{\mathbf{x}+\hat{e}_1+\hat{e}_2}+W_{\geq 6}(\psi), \tag{2.52}$$

where $W_{\geq 6}(\psi)$ is a sum over Grassmann monomials of order larger or equal than 6, *whose kernels decay exponentially in space if $\lambda<\log 2$ (with rate $\kappa=-\log|\alpha|>0$).*

Besides the partition function $Z_\Lambda(\lambda,m)$, we are interested in computing *truncated* multipoint dimer correlations (cumulants) of the form

$$\langle\mathbb{1}_{b_1};\dots;\mathbb{1}_{b_k}\rangle_{\Lambda;\lambda,m}=\frac{\partial^k}{\partial A_{b_1}\cdots\partial A_{b_k}}\log\mathcal{Z}_\Lambda(\lambda,m,\mathbf{A})\Big|_{\mathbf{A}=\mathbf{0}}, \tag{2.53}$$

with

$$\mathcal{Z}_\Lambda(\lambda,m,\mathbf{A}):=\sum_{M\in\mathcal{M}_\Lambda}\Big[\prod_{b\in M}t_b^{(m)}\Big]e^{\lambda W_\Lambda(M)+\sum_{b\subset\Lambda}A_b\mathbb{1}_b} \tag{2.54}$$

and $b_1,\dots,b_k$ a k-ple of bonds. Moreover, $\mathbf{A}=\{A_b\}_{b\subset\Lambda}$. The modified partition function $\mathcal{Z}_\Lambda(\lambda,m,\mathbf{A})$ can be expressed in the form of a Grassmann integral, by proceeding in the same way that we followed for $Z_\Lambda(\lambda,m)$. The result is:

Proposition 5.

$$\mathcal{Z}_\Lambda(\lambda,m,\mathbf{A})=\frac{1}{2}\sum_{\theta,\tau=0,1}C_{\theta\tau}\int_{(\theta\tau)}\Big[\prod_{\mathbf{x}\in\Lambda}d\psi_{\mathbf{x}}\Big]e^{S(\psi)+V_\Lambda(\psi)+\mathcal{B}_\Lambda(\psi,\mathbf{J})}, \tag{2.55}$$

where $\mathbf{J}=\mathbf{J}(\mathbf{A})=\{J_b(A_b)\}_{b\subset\Lambda}$ with $J_b=e^{A_b}-1$, and

$$\mathcal{B}_\Lambda(\psi,\mathbf{J})=\sum_{k\geq 1}\sum_{\gamma=\{b_1,\dots,b_k\}\subset\Lambda}\sum_{\varnothing\neq R\subset\gamma}\tilde{\xi}(\gamma;R), \tag{2.56}$$

$$\tilde{\xi}(\gamma;R)=(-1)^k\alpha^{k-1}\prod_{b\in\gamma}E_b^{(m)}\prod_{b\in R}J_b. \tag{2.57}$$

Here, as above, $b_1,\dots,b_k$ are adjacent parallel bonds.

The proof is analogous to that of Proposition 4, details are left to the reader. Note that once the truncated correlations are known, the standard correlations can be reconstructed via the inversion formula:

$$\langle \mathbb{1}_{b_1} \cdots \mathbb{1}_{b_k} \rangle_{\Lambda;\lambda,m} = \sum_{\{\underline{i}^{(1)},\dots,\underline{i}^{(s)}\}\in\mathcal{P}[1,\dots,k]} \langle \mathbb{1}_{b_{i_1^{(1)}}}; \dots; \mathbb{1}_{b_{i_{k_1}^{(1)}}} \rangle_{\Lambda;\lambda,m} \cdots \langle \mathbb{1}_{b_{i_1^{(s)}}}; \dots; \mathbb{1}_{b_{i_{k_s}^{(s)}}} \rangle_{\Lambda;\lambda,m}, \tag{2.58}$$

where $\underline{i}^{(j)} = \{i_1^{(j)}, \dots, i_{k_j}^{(j)}\} \subset \{1, \dots, k\}$, with $k_j \geq 1$, is a non-empty set of indices, and $\mathcal{P}[1, \dots, k]$ is the set of partitions of $\{1, \dots, k\}$. In (2.58), the single-bond average $\langle \mathbb{1}_b \rangle_{\Lambda;\Lambda'}$, $b \subset \Lambda$, is given by

$$\langle \mathbb{1}_b \rangle_{\Lambda;\lambda,m} = \frac{\partial}{\partial A_b} \log \mathcal{Z}_\Lambda(\lambda, m, \mathbf{A})\Big|_{\mathbf{A}=\mathbf{0}}. \tag{2.59}$$

2.3.2. *Rewriting the partition function in terms of Majorana or Dirac fields*
The partition function (and, similarly, the generating function $\mathcal{Z}_\Lambda(\lambda, m, \mathbf{A})$ for dimer correlations) can be rewritten in terms of the Majorana or Dirac fields: going back to (2.42) and (2.55) we get for instance

$$Z_\Lambda(\lambda, m) = \frac{1}{2} \sum_{\theta\tau} C_{\theta\tau} \operatorname{Pf} K_\Lambda^{(\theta\tau)} \int_{(\theta\tau)} \prod_{\gamma=1,\dots,4} P_\Lambda^{(\theta\tau)}(d\psi_\gamma) \exp\{V_\Lambda(\psi)\}, \tag{2.60}$$

with $\psi = \{\psi_{\mathbf{x}}\}_{\mathbf{x}\in\Lambda}$ as in (2.22). We used the addition formula for normalized Grassmann Gaussian integrations, cf. (2.14).

2.4. *Reduction to a single Pfaffian*

We have seen in (2.18) that the propagator $g_\Lambda^{(\theta\tau)}$ loses dependence on (θ, τ) in the limit $\Lambda \nearrow \mathbb{Z}^2$. This holds also for $\operatorname{Pf} K_\Lambda^{(\theta\tau)}$, the normalization of the measure $P_\Lambda^{(\theta\tau)}$. More precisely, while each Pfaffian grows exponentially in L^2, for $m > 0$ one has (cf. Appendix A.2)

$$\lim_{\Lambda\nearrow\mathbb{Z}^2} \frac{\operatorname{Pf} K_\Lambda^{(11)}}{\operatorname{Pf} K_\Lambda^{(\theta\tau)}} = 1 \tag{2.61}$$

and the limit is reached exponentially fast in L. This is a consequence of the fact that at very large distances the propagator decays exponentially (actually this is the main technical reason why we introduced the infrared regularization $m > 0$).

The observation (2.61) implies important simplifications in the thermodynamic limit. Suppose that we want to compute the average of a dimer observable, say $\mathbb{1}_{b_1} \cdots \mathbb{1}_{b_k}$, with $b_1, \dots, b_k$ distinct bonds, for the non-interacting system ($\lambda = 0$). From (2.11) we get

$$\langle \mathbb{1}_{b_1} \cdots \mathbb{1}_{b_k} \rangle_{\Lambda;0,m} = \frac{\sum_{\theta\tau} C_{\theta,\tau} [\operatorname{Pf} K_\Lambda^{(\theta\tau)}] \int_{(\theta\tau)} P_\Lambda^{(\theta\tau)}(d\psi)(-1)^k E_{b_1}^{(m)} \cdots E_{b_k}^{(m)}}{\sum_{\theta\tau} C_{\theta,\tau} [\operatorname{Pf} K_\Lambda^{(\theta\tau)}]}.$$

We have seen above that the free propagator, and therefore the integrals in the numerator, become independent of $(\theta\tau)$ when $\Lambda \nearrow \mathbb{Z}^2$. Together with (2.61), this implies that

$$\lim_{\Lambda\nearrow\mathbb{Z}^2} \langle \mathbb{1}_{b_1} \cdots \mathbb{1}_{b_k} \rangle_{\Lambda;0,m} = \lim_{\Lambda\nearrow\mathbb{Z}^2} \int_{(11)} P_\Lambda^{(11)}(d\psi)(-1)^k E_{b_1}^{(m)} \cdots E_{b_k}^{(m)} \tag{2.62}$$

$$= \int P(d\psi)(-1)^k E_{b_1}^{(m)} \cdots E_{b_k}^{(m)}, \tag{2.63}$$

with $P(d\psi)$ the Gaussian Grassmann measure with propagator $g(\mathbf{x},\mathbf{y})$. That is, it is sufficient to consider (11) boundary conditions in the Grassmann integrations (these are more convenient than (00) conditions since even for $m=0$ the denominator in (2.16) is never singular for $\mathbf{k}\in\mathcal{D}_\Lambda^{(11)}$).

An analogous fact holds also for the interacting model ($\lambda\neq 0$), as a consequence of the fact that the interacting propagator also decays exponentially as long as $m>0$ (the model remains off-critical even in the presence of interactions, see Remark 13 below). More precisely, for $m>0$ the following holds: given $n\geq 1$ distinct $\mathbf{x}_1,\dots,\mathbf{x}_n$,

$$\lim_{\Lambda\nearrow\mathbb{Z}^2}\frac{\int_{(\theta\tau)}P_\Lambda^{(\theta\tau)}e^{V_\Lambda(\psi)}\psi_{\mathbf{x}_1}\cdots\psi_{\mathbf{x}_n}}{\int_{(11)}P_\Lambda^{(11)}e^{V_\Lambda(\psi)}\psi_{\mathbf{x}_1}\cdots\psi_{\mathbf{x}_n}}=1 \tag{2.64}$$

and actually the limit is reached exponentially fast in L. The proof is a corollary of the multiscale construction described in Section 5 below, and goes along the same lines as [48, Appendix G].

As a consequence of (2.64) and (2.61), using (2.55), we see that

$$\lim_{\Lambda\nearrow\mathbb{Z}^2}\frac{\partial^n}{\partial A_{b_1}\cdots\partial A_{b_n}}\log\mathcal{Z}_\Lambda(\lambda,m,\mathbf{A})\Big|_{\mathbf{A}=\mathbf{0}} \tag{2.65}$$

$$=\lim_{\Lambda\nearrow\mathbb{Z}^2}\frac{\partial^n}{\partial A_{b_1}\cdots\partial A_{b_n}}\log\mathcal{Z}_\Lambda^{(11)}(\lambda,m,\mathbf{A})\Big|_{\mathbf{A}=\mathbf{0}} \tag{2.66}$$

with

$$\mathcal{Z}_\Lambda^{(11)}(\lambda,m,\mathbf{A})=\int_{(11)}P_\Lambda^{(11)}(d\psi)e^{V_\Lambda(\psi)+\mathcal{B}_\Lambda(\psi,\mathbf{J})}. \tag{2.67}$$

3. Height fluctuations in the non-interacting model: Proof of Theorem 1 for $\lambda=0$

As a warm-up, and in order to introduce some basic ideas that will be important later, here we prove Theorem 1 in the special but important non-interacting case, $\lambda=\alpha=0$. The strategy we use is convenient for the subsequent generalization to the interacting case.

3.1. *Grassmann representation for height function fluctuations*

Let us start with some considerations that hold both for $\lambda=0$ and $\lambda\neq 0$. We are interested in computing the height fluctuations, i.e., the n-point truncated self-correlations, $n\geq 2$:

$$\lim_{m\to 0}\lim_{\Lambda\nearrow\mathbb{Z}^2}\langle\underbrace{h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}};\dots;h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}}}_{n\text{ times}}\rangle_{\Lambda;\lambda,m}. \tag{3.1}$$

For lightness we will write here $\langle\cdot\rangle_\Lambda$ instead of $\langle\cdot\rangle_{\Lambda;\lambda,m}$. The definition (1.4) allows us to re-express (3.1) in terms of sums of multipoint dimer correlations:

$$\langle\underbrace{h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}};\dots;h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}}}_{n\text{ times}}\rangle_\Lambda=\sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}\cdots\sum_{b_n\in\mathcal{C}^{(n)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}\sigma_{b_1}\cdots\sigma_{b_n}\langle\mathbb{1}_{b_1};\dots;\mathbb{1}_{b_n}\rangle_\Lambda, \tag{3.2}$$

where $\mathcal{C}^{(j)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ are paths on $(\mathbb{Z}^2)^*$ from $\boldsymbol{\xi}$ to $\boldsymbol{\eta}$, which we assume not to wind around the torus and to be independent of L. The n-point dimer correlation in the r.h.s. of (3.2) can be computed via (2.53), so that

$$\langle\underbrace{h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}};\dots;h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}}}_{n\text{ times}}\rangle_\Lambda$$

$$=\sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}\cdots\sum_{b_n\in\mathcal{C}^{(n)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}\sigma_{b_1}\cdots\sigma_{b_n}\frac{\partial^n}{\partial A_{b_1}\cdots\partial A_{b_n}}\log\mathcal{Z}_\Lambda(\lambda,m,\mathbf{A})|_{\mathbf{A}=\mathbf{0}}. \tag{3.3}$$

Finally, one takes the limit $\lim_{m\to 0}\lim_{\Lambda\nearrow\mathbb{Z}^2}$ of the expression thus obtained. Since the limit $\Lambda\nearrow\mathbb{Z}^2$ is taken keeping $\boldsymbol{\xi}$, $\boldsymbol{\eta}$ fixed, in view of (2.65) we are allowed to replace $\mathcal{Z}_\Lambda(\lambda, m, \mathbf{A})$ in the r.h.s. of (3.3) by $\mathcal{Z}_\Lambda^{(11)}(\lambda, m, \mathbf{A})$, modulo an error term that is negligible in the thermodynamic limit and that we will simply forget in the following formulas.

Using the Grassmann representation discussed in Section 2, we can rewrite the r.h.s. of (3.3) in terms of expectations of Grassmann variables. Let $\mathcal{E}^T$ indicate the truncated expectation with respect to $P_\Lambda^{(11)}(d\psi)$, i.e.,

$$\mathcal{E}^T\big(X_1(\psi);\dots;X_s(\psi)\big)=\frac{\partial^s}{\partial\lambda_1\cdots\partial\lambda_s}\log\int_{(11)}P_\Lambda^{(11)}(d\psi)e^{\lambda_1X_1(\psi)+\cdots+\lambda_sX_s(\psi)}\Big|_{\lambda_i=0}. \tag{3.4}$$

In particular,

$$\log\int_{(11)}P_\Lambda^{(11)}(d\psi)e^{X(\psi)}=\sum_{s\ge1}\frac{1}{s!}\mathcal{E}^T\big(\underbrace{X(\psi);\dots;X(\psi)}_{s\text{ times}}\big). \tag{3.5}$$

Therefore, recalling (2.67), we get

$$\begin{aligned}&\log\mathcal{Z}_\Lambda^{(11)}(\lambda,m,\mathbf{A})\\&\quad=\sum_{s\ge1}\frac{1}{s!}\mathcal{E}^T\big(\underbrace{V_\Lambda(\psi)+\mathcal{B}_\Lambda(\psi,\mathbf{J});\dots;V_\Lambda(\psi)+\mathcal{B}_\Lambda(\psi,\mathbf{J})}_{s\text{ times}}\big)\\&\quad=E_\Lambda(\lambda,m)+\sum_{k\ge1}\sum_{\{b_1,\dots,b_k\}\subset\Lambda}\Big[\prod_{j=1}^kJ_{b_j}\Big]S_{\Lambda,k}(b_1,\dots,b_k),\end{aligned} \tag{3.6}$$

where the third line is the definition of $E_\Lambda(\lambda,m)$ and of $S_{\Lambda,k}(b_1,\dots,b_k)$, i.e., $E_\Lambda(\lambda,m)$ (resp. $S_{\Lambda,k}(b_1,\dots,b_k)\times J_{b_1}\cdots J_{b_k}$) collects all the terms in the second line that are independent of $\mathbf{J}$ (resp. are proportional to $J_{b_1}\cdots J_{b_k}$ but are independent of the other J_b's). In the last line, the sum over $\{b_1,\dots,b_k\}$ does not run just over k-ples of different bonds. Rather, $\{b_1,\dots,b_k\}=:B$ is a *bond configuration* in which some bonds are allowed to coincide. Formally, one such configuration is a function $b\to B(b)$ with nonnegative integer values such that $\sum_bB(b)=N(B)<+\infty$. The number $B(b)$ has the meaning of *multiplicity* of b in B. Given B, we denote by $\tilde B$ the set of bonds b such that $B(b)>0$; hence $\tilde B$ is the *support* of B, and it consists of the bonds that are in B, each counted without taking multiplicity into account.

Let us remark that the fermionic truncated expectations in the previous equations can be computed explicitly, by using the definition (3.4) and the fermionic Wick rule (2.6). In order to understand how to evaluate (3.4), assume that the functions $X_i(\psi)$ are Grassmann monomials (which is not a restrictive assumption, since the operator $\mathcal{E}^T$ is multilinear in its arguments), i.e.,

$$X_i(\psi)=c_i\psi_{\mathbf{x}_1^{(i)}}\dots\psi_{\mathbf{x}_{n_i}^{(i)}}. \tag{3.7}$$

Then Eq. (3.4) admits the following diagrammatical representation:

(1) draw s vertices, each representing one of the monomials X_i, with a number of "legs" equal to the order n_i of the corresponding monomial; each leg is associated with a label $\mathbf{x}_j^{(i)}$, which we will think of as the point which the leg exits from (or is anchored to);
(2) contract in all possible *connected* ways the legs, by pairing them two by two and by graphically representing every such pair by a line (here a contraction, or pairing, is called connected if the s vertices are geometrically connected by the contracted lines).

In this way, each pairing is in one-to-one correspondence with its diagrammatical representation, called Feynman diagram, and (3.4) can be computed as follows (see e.g. [34, Appendix A3.1]):

Proposition 6 (Wick rule for truncated expectations). *If the functions X_i are as in* (3.7), *the truncated expectation* (3.4) *is equal to the sum over connected Feynman diagrams of their values, where the value of a diagram is: the*

product $\prod_{i=1}^{s} c_i$ *of the "kernels"* $c_1, \dots, c_s$ *of* $X_1, \dots, X_s$, *times the product of the propagators associated with the contracted lines, times a sign, which is equal to the sign of the permutation required for placing next to each other the contracted Grassmann fields, starting from their original ordering in* $X_1(\psi) \cdots X_s(\psi)$, *times a combinatorial factor* $1/s!$.

For instance, if $s = 2$ and $X_i(\psi) = \psi_{\mathbf{x}_{2i-1}} \psi_{\mathbf{x}_{2i}}$, $i = 1, 2$, and we contract the leg associated with $\mathbf{x}_1$ with $\mathbf{x}_3$ and $\mathbf{x}_2$ with $\mathbf{x}_4$, the value of the corresponding Feynman diagram is $(-1/2) g_{\Lambda}^{(11)}(\mathbf{x}_1, \mathbf{x}_3) g_{\Lambda}^{(11)}(\mathbf{x}_2, \mathbf{x}_4)$, where -1 is the signature of the permutation that transforms 1234 into 1324. This diagrammatical representation, if applied to (3.6), leads to the Feynman diagram expansion for the height fluctuations, discussed in Section 5.1 below.

If $\lambda = 0$, then (3.3)–(3.6) lead to the following explicit representation (observe that in this case $V_{\Lambda}(\psi) = 0$ and $\mathcal{B}_{\Lambda}(\psi, \mathbf{J}) = -\sum_b J_b E_b^{(m)}$):

$$\begin{aligned}
&\langle \underbrace{h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}; \dots; h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}}_{n \text{ times}} \rangle_{\Lambda;\lambda=0,m} \\
&= \sum_{b_1 \in \mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \cdots \sum_{b_n \in \mathcal{C}^{(n)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \sigma_{b_1} \cdots \sigma_{b_n} \sum_{m_1=1}^{B(b'_1)} \cdots \sum_{m_s=1}^{B(b'_s)} (-1)^{m_1+\cdots+m_s} \\
&\quad \times P_{m_1}\big(B\big(b'_1\big)\big) \cdots P_{m_s}\big(B\big(b'_s\big)\big) \mathcal{E}^T\big(\underbrace{E^{(m)}_{b'_1}; \dots; E^{(m)}_{b'_1}}_{m_1 \text{ times}}; \dots; \underbrace{E^{(m)}_{b'_s}; \dots; E^{(m)}_{b'_s}}_{m_s \text{ times}}\big),
\end{aligned} \tag{3.8}$$

where $\{b_1, \dots, b_n\} =: B$ should be thought of as a bond configuration (possibly with repetitions), $\tilde{B} = \{b'_1, \dots, b'_s\}$ as the support of B and $B(b'_i)$ as the multiplicity of b'_i, see the discussion after (3.6). Moreover,

$$P_k(N) := \frac{\partial^N}{\partial A^N} \frac{(e^A - 1)^k}{k!}\bigg|_{A=0}. \tag{3.9}$$

Then, we take the limit $\lim_{m\to 0} \lim_{\Lambda \nearrow \mathbb{Z}^2}$: this simply means that in the computations of the averages $\mathcal{E}^T(\cdots)$ all propagators $g_{\Lambda}^{(11)}(\mathbf{x}, \mathbf{y})$ are replaced by $\lim_{m\to 0} g(\mathbf{x}, \mathbf{y})$ and $E_b^{(m)} = t_b^{(m)} E_b$ is replaced by E_b.

Let us now discuss how to evaluate (3.8), separately for the cases $n = 2$ (the variance) and $n > 2$.

3.2. *The height variance*

In this section we prove:

Theorem 4. *Let* $\lambda = 0$. *There exists a uniformly bounded function* $R(\boldsymbol{\xi})$ *such that*

$$\big\langle (h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}})^2 \big\rangle_{\lambda=0} = \frac{1}{\pi^2} \log |\boldsymbol{\xi} - \boldsymbol{\eta}| + R(\boldsymbol{\xi} - \boldsymbol{\eta}). \tag{3.10}$$

Proof. We assume for simplicity that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ have the same parity. We choose the two paths $\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$, $\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ in such a way that: (1) they are completely distinct, i.e., the bonds in $\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ are all different from those in $\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$; (2) they are both of length comparable with $|\boldsymbol{\xi} - \boldsymbol{\eta}|$; (3) they consist of a union of straight portions (i.e. horizontal or vertical portions), each of which is of even length. Moreover, we assume that $\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$, $\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ are "well-separated," in the following sense. Fix $c, c' > 0$. Inside balls of radius $c|\boldsymbol{\xi} - \boldsymbol{\eta}|$ around $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, the two paths are portions of length $c|\boldsymbol{\xi} - \boldsymbol{\eta}|$ of infinite periodic paths (that is, they are portions of straight paths – apart from lattice discretization – see [47, Definition 2.1]) and have mutually different asymptotic directions, say opposite. Outside of these balls the paths stay at distance at

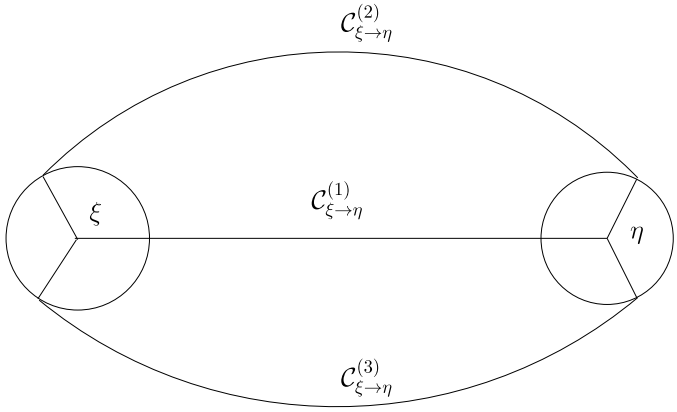

Fig. 3. A schematic view of the paths $\mathcal{C}^{(i)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ for $n=3$. Near $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, paths are essentially linear for a length proportional to $|\boldsymbol{\xi}-\boldsymbol{\eta}|$, with non-zero mutual angles.

least $c'|\boldsymbol{\xi}-\boldsymbol{\eta}|$ of each other and their length is of order $|\boldsymbol{\xi}-\boldsymbol{\eta}|$. See Figure 3. Using (3.8) for $n=2$ and the above assumptions on the paths, we can rewrite the variance as

$$\lim_{m\to 0}\lim_{\Lambda\nearrow\mathbb{Z}^2}\big\langle(h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}})^2\big\rangle_{\Lambda;\lambda=0,m}=\sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}\sum_{b_2\in\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}\sigma_{b_1}\sigma_{b_2}\mathcal{E}^T(E_{b_1};E_{b_2}),\tag{3.11}$$

where the expectation $\mathcal{E}^T$ has propagator $\lim_{m\to 0} g(\mathbf{x},\mathbf{y})$. Let $b=(\mathbf{x},\mathbf{x}+\hat{e}_j)$ be a bond crossed say by $\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ and observe that, since we assumed that the white sites are on the even-even and odd-odd sub-lattice (and letting $(-1)^{\mathbf{x}}:=(-1)^{x_1+x_2}$),

$$\sigma_b=\alpha_b(-1)^{\mathbf{x}}(-1)^j,\tag{3.12}$$

where α_b is $+1/-1$, depending on whether the bond b is crossed by the oriented path $\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ in the positive/negative direction (the positive direction is upwards for vertical portions of the paths, and rightwards for horizontal portions).

Next, we have to rewrite $\mathcal{E}^T(E_{b_1};E_{b_2})$ and we start by expressing $E_{\mathbf{x},\mathbf{x}+\hat{e}_j}=i^{(j-1)}\psi_{\mathbf{x}}\psi_{\mathbf{x}+\hat{e}_j}$ in terms of Dirac variables:

(1) we replace each of the two fields by a combination of Dirac fields using (2.31);
(2) whenever $\psi^{\pm}_{\mathbf{x}+\hat{e}_j,\omega}$ appears we replace it by $\psi^{\pm}_{\mathbf{x},\omega}+\partial_j\psi^{\pm}_{\mathbf{x},\omega}$ with ∂_j the (right) discrete derivative in the j direction.

In this way we obtain (we skip lengthy but straightforward computations):

$$E_{\mathbf{x},\mathbf{x}+\hat{e}_1}=\mathcal{A}_{\mathbf{x},\mathbf{x}+\hat{e}_1}+\mathcal{R}_{\mathbf{x},\mathbf{x}+\hat{e}_1}\tag{3.13}$$

$$:=-2(-1)^{\mathbf{x}}\sum_{\omega}\psi^{+}_{\mathbf{x},\omega}\psi^{-}_{\mathbf{x},\omega}-2(-1)^{x_1}\sum_{\omega}\psi^{+}_{\mathbf{x},\omega}\psi^{-}_{\mathbf{x},-\omega}+\mathcal{R}_{\mathbf{x},\mathbf{x}+\hat{e}_1},\tag{3.14}$$

$$E_{\mathbf{x},\mathbf{x}+\hat{e}_2}=\mathcal{A}_{\mathbf{x},\mathbf{x}+\hat{e}_2}+\mathcal{R}_{\mathbf{x},\mathbf{x}+\hat{e}_2}\tag{3.15}$$

$$:=-2i(-1)^{\mathbf{x}}\sum_{\omega}\omega\psi^{+}_{\mathbf{x},\omega}\psi^{-}_{\mathbf{x},\omega}-2i(-1)^{x_2}\sum_{\omega}\omega\psi^{+}_{\mathbf{x},\omega}\psi^{-}_{\mathbf{x},-\omega}+\mathcal{R}_{\mathbf{x},\mathbf{x}+\hat{e}_2},\tag{3.16}$$

where $\mathcal{R}$ is a linear combinations of terms of the type $\psi^{\varepsilon}_{\mathbf{x},\omega}\partial_j\psi^{\varepsilon'}_{\mathbf{x},\omega'}$, with $\varepsilon,\varepsilon'=\pm$. Let us consider first the "local parts" $\mathcal{A}_{\mathbf{x},\mathbf{x}+\hat{e}_i}$, i.e. let us neglect for the moment $\mathcal{R}$. When the path crosses the bond b, the change of position Δz_b in the complex plane is $i\alpha_b$ if b is horizontal and α_b is b is vertical. Therefore,

$$\sigma_b\mathcal{A}_b=-2i\,\Delta z_b\Big[\sum_{\omega}\psi^{+}_{\mathbf{x},\omega}\psi^{-}_{\mathbf{x},\omega}+(-1)^{x_2}\sum_{\omega}\psi^{+}_{\mathbf{x},\omega}\psi^{-}_{\mathbf{x},-\omega}\Big]\tag{3.17}$$

if b is horizontal, and

$$\sigma_b \mathcal{A}_b = -2i\,\Delta z_b \Big[\sum_{\omega} \omega \psi^+_{\mathbf{x},\omega} \psi^-_{\mathbf{x},\omega} + (-1)^{x_1} \sum_{\omega} \omega \psi^+_{\mathbf{x},\omega} \psi^-_{\mathbf{x},-\omega} \Big] \tag{3.18}$$

if b is vertical. At this point we can write, assuming for the moment that both b_1 and b_2 are horizontal bonds, i.e., $b_1 = (\mathbf{x}, \mathbf{x} + \hat{e}_1)$ and $b_2 = (\mathbf{y}, \mathbf{y} + \hat{e}_1)$,

$$\sigma_{b_1}\sigma_{b_2}\mathcal{E}^T(\mathcal{A}_{b_1}; \mathcal{A}_{b_2}) = -8\,\mathrm{Re}\big[\Delta z_{b_1}\Delta z_{b_2}\mathcal{E}^T\big(\psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},+}; \psi^+_{\mathbf{y},+}\psi^-_{\mathbf{y},+}\big)\big] \tag{3.19}$$

$$- 8(-1)^{x_2+y_2}\,\mathrm{Re}\big[\Delta z_{b_1}\Delta z_{b_2}\mathcal{E}^T\big(\psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},-}; \psi^+_{\mathbf{y},-}\psi^-_{\mathbf{y},+}\big)\big]. \tag{3.20}$$

Here we used the fact that $\Delta z_{b_1}\Delta z_{b_2}$ is real, that

$$\mathcal{E}^T\big(\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}; \psi^+_{\mathbf{y},\omega'}\psi^-_{\mathbf{y},-\omega'}\big) = 0$$

(because $\langle \psi^-_{\mathbf{x},-}\psi^+_{\mathbf{y},+}\rangle = -iG_{-+}(\mathbf{x}-\mathbf{y})$ vanishes when the mass m is zero as is the case here: recall in fact that we already sent $m \to 0$, see the l.h.s of (3.11)) and that

$$\mathcal{E}^T\big(\psi^+_{\mathbf{x}_1,-1}\psi^-_{\mathbf{x}_1,-1}; \psi^+_{\mathbf{x}_2,-1}\psi^-_{\mathbf{x}_2,-1}\big) = \mathcal{E}^T\big(\psi^+_{\mathbf{x}_1,1}\psi^-_{\mathbf{x}_1,1}; \psi^+_{\mathbf{x}_2,1}\psi^-_{\mathbf{x}_2,1}\big)^*$$

(cf. (2.32) and the first of (2.25)). Using the Wick rule (Proposition 6) and the first and third of (2.26) we have, assuming that $b_1 = (\mathbf{x}, \mathbf{x} + \hat{e}_1)$ and $b_2 = (\mathbf{y}, \mathbf{y} + \hat{e}_1)$,

$$\sigma_{b_1}\sigma_{b_2}\mathcal{E}^T(\mathcal{A}_{b_1}; \mathcal{A}_{b_2}) = -8\,\mathrm{Re}\big[\Delta z_{b_1}\Delta z_{b_2}\big(G_{++}(\mathbf{x}-\mathbf{y})\big)^2\big] \tag{3.21}$$

$$- 8(-1)^{x_2+y_2}\Delta z_{b_1}\Delta z_{b_2}\big|G_{++}(\mathbf{x}-\mathbf{y})\big|^2. \tag{3.22}$$

In the general case $b_1 = (\mathbf{x}, \mathbf{x} + \hat{e}_{j_1})$, $b_2 = (\mathbf{y}, \mathbf{y} + \hat{e}_{j_2})$ one finds with similar computations

$$\sigma_{b_1}\sigma_{b_2}\mathcal{E}^T(\mathcal{A}_{b_1}; \mathcal{A}_{b_2}) = -8\,\mathrm{Re}\big[\Delta z_{b_1}\Delta z_{b_2}\big(G_{++}(\mathbf{x}-\mathbf{y})\big)^2\big] \tag{3.23}$$

$$+ 8\delta_{j_1=j_2}(-1)^{j_1}(-1)^{x_{3-j_1}+y_{3-j_1}}\Delta z_{b_1}\Delta z_{b_2}\big|G_{++}(\mathbf{x}-\mathbf{y})\big|^2. \tag{3.24}$$

Using Proposition 2 to express G at $m = 0$ as $\mathfrak{g}$ plus a fast decaying remainder, we have

$$\sigma_{b_1}\sigma_{b_2}\mathcal{E}^T(\mathcal{A}_{b_1}; \mathcal{A}_{b_2}) = -\,\mathrm{Re}\Big[\Delta z_{b_1}\Delta z_{b_2}\frac{1}{2\pi^2}\frac{\mathbf{1}_{\mathbf{x}\neq\mathbf{y}}}{(z_{\mathbf{y}}-z_{\mathbf{x}})^2}\Big] \tag{3.25}$$

$$+ \delta_{j_1=j_2}(-1)^{j_1}(-1)^{x_{3-j_1}+y_{3-j_1}}\Delta z_{b_1}\Delta z_{b_2}\frac{1}{2\pi^2}\frac{\mathbf{1}_{\mathbf{x}\neq\mathbf{y}}}{|z_{\mathbf{y}}-z_{\mathbf{x}}|^2} + R_{j_1,j_2}(\mathbf{x}-\mathbf{y}), \tag{3.26}$$

where $z_{\mathbf{x}} = x_1 + ix_2$ and $|R_{j_1,j_2}(\mathbf{x}-\mathbf{y})| \le (\text{const.})(1+|\mathbf{x}-\mathbf{y}|)^{-3}$. Now we can sum over b_i in the paths $\mathcal{C}^{(i)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$, $i = 1, 2$. The contribution from R_{j_1,j_2} is of order 1 uniformly in $\boldsymbol{\xi}$, $\boldsymbol{\eta}$: to see this, use the properties of the paths spelled out at the beginning of this subsection. The same holds for the second term, this time because of the oscillating factor $(-1)^{x_{3-j_1}+y_{3-j_1}}$, that in the sum has the effect of a discrete derivative in the direction $3-j_1$: in fact, recall that the paths are assumed to consist of unions of straight portions of even length; on each such portion, the sum over $\mathbf{x}$ and $\mathbf{y}$ of $(-1)^{x_{3-j_1}+y_{3-j_1}}|z_{\mathbf{x}}-z_{\mathbf{y}}|^{-2}$ is of the same order as the sum of $\partial_{x_{3-j_1}}\partial_{y_{3-j_1}}|z_{\mathbf{x}}-z_{\mathbf{y}}|^{-2}$, which decays at large distances like $|\mathbf{x}-\mathbf{y}|^{-4}$. As for the first term, it produces the Riemann approximation to the integral

$$-\frac{1}{2\pi^2}\,\mathrm{Re}\int_{z_{\boldsymbol{\xi}}}^{z_{\boldsymbol{\eta}}} dz \int_{z'_{\boldsymbol{\xi}}}^{z'_{\boldsymbol{\eta}}} dw\,\frac{1}{(z-w)^2} \tag{3.27}$$

(here $z'_{\boldsymbol{\xi}}$ and $z'_{\boldsymbol{\eta}}$ are points at a distance $O(1)$ from $z_{\boldsymbol{\xi}}$ and $z_{\boldsymbol{\eta}}$, respectively), and differs from it by a constant, independent of $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. This integral is the same found in [45] (see the second equation at p. 1043); it can be explicitly evaluated and gives (see the third and fourth equation at p. 1043 of [45]):

$$\frac{1}{2\pi^2}\operatorname{Re}\log\frac{(z'_{\boldsymbol{\eta}}-z_{\boldsymbol{\xi}})(z'_{\boldsymbol{\xi}}-z_{\boldsymbol{\eta}})}{(z'_{\boldsymbol{\eta}}-z_{\boldsymbol{\eta}})(z'_{\boldsymbol{\xi}}-z_{\boldsymbol{\xi}})}=\frac{1}{\pi^2}\log|z_{\boldsymbol{\xi}}-z_{\boldsymbol{\eta}}|+O(1). \tag{3.28}$$

It remains to study the contribution coming from the error terms $\mathcal{R}_b$ in (3.13), (3.15), that we disregarded so far.

Remark 7. *Each remainder $\mathcal{R}_{b_i}$ is a linear combination of terms like $\psi^{\varepsilon}_{\omega}\partial\psi^{\varepsilon'}_{\omega'}$, all localized in the vicinity of $\mathbf{x}_i$, where $\mathbf{x}_i$ is such that $b_i=(\mathbf{x}_i,\mathbf{x}_i+\hat{e}_{j_i})$. Therefore, we can symbolically write the contribution to the height variance from all the terms containing at least one term $\mathcal{R}_{b_i}$ as*

$$\tilde{R}(\boldsymbol{\xi}-\boldsymbol{\eta})=\sum_{\substack{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}},\\ b_2\in\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}}\sum_{\substack{\omega_1,\dots,\omega'_2\\ \varepsilon_1,\dots,\varepsilon'_2}}\sum_{\substack{\alpha_1,\alpha_2\in\{0,1\}:\\ \alpha_1+\alpha_2\geq 1}}\mathcal{E}^T\Big(\big(\psi^{\varepsilon_1}_{\omega_1}\partial^{\alpha_1}\psi^{\varepsilon'_1}_{\omega'_1}\big)(\mathbf{x}_1);\big(\psi^{\varepsilon_2}_{\omega_2}\partial^{\alpha_2}\psi^{\varepsilon'_2}_{\omega'_2}\big)(\mathbf{x}_2)\Big). \tag{3.29}$$

The writing is symbolical, in the sense that the terms in the sum should in general be multiplied by extra factors, depending on all the indices we are summing over, which are not written explicitly just for lightness of notation. Moreover, the discrete derivatives have an index depending on the orientation of the bonds b_1, b_2, which is not written explicitly, again for lightness.

Using (2.32) to express the propagator of the Dirac fields $\psi^{\pm}_{\omega}$ in terms of the propagator $G(\mathbf{x})$ of the Majorana fields ψ_γ and the decay properties of $G(\mathbf{x})$ stated in Proposition 2, we can bound the expression in square brackets by a constant times $(1+|\mathbf{x}_1-\mathbf{x}_2|)^{-3}$ (the discrete derivative of G decays like $1/|\mathbf{x}|^2$), so that, recalling that $b_i=(\mathbf{x}_i,\mathbf{x}_i+\hat{e}_{j_i})$,

$$\big|\tilde{R}(\boldsymbol{\xi}-\boldsymbol{\eta})\big|\leq\sum_{\substack{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}},\\ b_2\in\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}}\frac{C}{(1+|\mathbf{x}_1-\mathbf{x}_2|)^3}\leq C', \tag{3.30}$$

for suitable constants $C, C'>0$. Putting all together, we find

$$\Big|\langle h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}};h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}}\rangle_{\lambda=0}-\frac{1}{\pi^2}\log|z_{\boldsymbol{\xi}}-z_{\boldsymbol{\eta}}|\Big|\leq C'', \tag{3.31}$$

as desired, since $|z_{\boldsymbol{\xi}}-z_{\boldsymbol{\eta}}|=|\boldsymbol{\xi}-\boldsymbol{\eta}|$. □

To get Proposition 3, just recall (3.25), (3.12) plus the discussion above on the contribution of the $\mathcal{R}$ terms appearing in (3.13) and (3.15).

3.3. *The nth cumulant*

Here we prove:

Theorem 5. *Let $\lambda=0$. For every $n\geq 3$ there exists a constant C_n such that, uniformly in $\boldsymbol{\xi}$, $\boldsymbol{\eta}$,*

$$\big|\underbrace{\langle h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}};\dots;h_{\boldsymbol{\xi}}-h_{\boldsymbol{\eta}}\rangle}_{n\ \text{times}}{}_{\lambda=0}\big|\leq C_n. \tag{3.32}$$

Proof. Again, we assume for simplicity that $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ have the same parity. As in the case of the variance, we fix $c_n, c_n' > 0$, and we assume that the n paths satisfy the following: (i) inside balls of radius $c_n|\boldsymbol{\xi}-\boldsymbol{\eta}|$ around $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, the n paths are portions of length $c_n|\boldsymbol{\xi}-\boldsymbol{\eta}|$ of infinite periodic paths and have mutually different asymptotic directions, say $(\cos\theta_j, \sin\theta_j)$, with $\theta_j = 2\pi j/n$, and $j = 0, \dots, n-1$; (ii) outside of these balls they stay at distance at least $c_n'|\boldsymbol{\xi}-\boldsymbol{\eta}|$ of each other and their length is of order $|\boldsymbol{\xi}-\boldsymbol{\eta}|$. See Figure 3.

Moreover, we require that the n paths consist of unions of straight portions of even length. Note that if $b_i = b_j$ with $i \neq j$ in (3.8), then b_i is at a distance smaller than r_n from $\boldsymbol{\xi}$, or from $\boldsymbol{\eta}$. Here and below $C_n, C_n', \dots$, and $c_n, c_n', \dots$ denote n-dependent constants, which might change from line to line. If we drop the index n, it means that the constants can be chosen independent of n.

We rewrite (3.8) as the contribution from the bonds $b_1, \dots, b_n$ that are all outside the balls $B_{r_n}(\boldsymbol{\xi})$ and $B_{r_n}(\boldsymbol{\eta})$ of radius r_n around $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$, plus a rest (and the limit $\lim_{m\to 0}\lim_{\Lambda\nearrow\mathbb{Z}^2}$ has been already taken):

$$\langle \underbrace{h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}; \dots; h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}}_{n \text{ times}} \rangle_{\lambda=0} = D_n(\boldsymbol{\xi}, \boldsymbol{\eta}) + R_n(\boldsymbol{\xi}, \boldsymbol{\eta})$$

$$:= \sum_{b_1 \in \mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \cdots \sum_{b_n \in \mathcal{C}^{(n)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \sigma_{b_1}\cdots\sigma_{b_n}(-1)^n \mathcal{E}^T(E_{b_1}; E_{b_2}; \dots; E_{b_n}) + R_n(\boldsymbol{\xi}, \boldsymbol{\eta}), \tag{3.33}$$

where the $*$ on the sums indicate the constraints that the b_i's are at a distance larger than r_n from $\boldsymbol{\xi}$ and from $\boldsymbol{\eta}$, and we used the fact that such constrained sums involve n-ples of bonds that are all distinct from each other. The rest $R_n(\boldsymbol{\xi}, \boldsymbol{\eta})$ contains all the remaining contributions, including those where some of the bonds are coinciding.

We start by analyzing the dominant term, namely $D_n(\boldsymbol{\xi}, \boldsymbol{\eta})$. With the notations of (3.13), we write

$$D_n(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_{b_1 \in \mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \cdots \sum_{b_n \in \mathcal{C}^{(n)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} (-1)^n \sigma_{b_1}\cdots\sigma_{b_n} \mathcal{E}^T(\mathcal{A}_{b_1}; \dots; \mathcal{A}_{b_n}) + D_n'(\boldsymbol{\xi}, \boldsymbol{\eta}), \tag{3.34}$$

where $D_n'(\boldsymbol{\xi}, \boldsymbol{\eta})$ collects all the terms containing at least one remainder term $\mathcal{R}_{b_i}$ and can be symbolically written (in the sense of Remark 7) as

$$D_n'(\boldsymbol{\xi}, \boldsymbol{\eta}) = \sum_{b_1 \in \mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \cdots \sum_{b_n \in \mathcal{C}^{(n)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \sum_{\substack{\omega_1,\dots,\omega_n' \\ \varepsilon_1,\dots,\varepsilon_n'}} \sum_{\substack{\alpha_1,\dots,\alpha_n: \\ \sum_i \alpha_i > 0}} \mathcal{E}^T\Big(\big(\psi^{\varepsilon_1}_{\omega_1} \partial^{\alpha_1} \psi^{\varepsilon_1'}_{\omega_1'}\big)(\mathbf{x}_1); \dots; \big(\psi^{\varepsilon_n}_{\omega_n} \partial^{\alpha_n} \psi^{\varepsilon_n'}_{\omega_n'}\big)(\mathbf{x}_n)\Big), \tag{3.35}$$

where in the last sum $\alpha_i \in \{0, 1\}$ and, once again, $\mathbf{x}_i$ is one of the sites of bond b_i. In the spirit of the diagrammatical rules explained after (3.7), we can graphically represent every monomial $(\psi^{\varepsilon_i}_{\omega_i} \partial^{\alpha_i} \psi^{\varepsilon_i'}_{\omega_i'})(\mathbf{x}_i)$ by a two-legged *vertex* v_i, consisting of two solid half-lines (indexed by ε_i, ω_i and $\varepsilon_i', \omega_i'$, respectively) exiting from the point $\mathbf{x}_i$, one of which has a derivative ∂^{α_i} on top. It is customary to draw an extra dotted line (external field) exiting from the vertex v_i, thus representing it as in Figure 4(a). Using the rules explained after (3.7), we find that the truncated expectation in the r.h.s. of (3.35) is equal to the sum of "sun diagrams," as in Figure 4(b). Since $m = 0$, the allowed contractions involve pairs of legs with opposite ε indices and equal ω indices, see (2.32) (recall that, if $m = 0$, $G_{+-} = G_{-+} = 0$). Therefore, the value of every allowed sun diagram is equal (up to a sign) to

$$\partial^{\bar\alpha_1} G_{\bar\omega_1\bar\omega_1}(\mathbf{x}_1 - \mathbf{x}_{\pi(2)}) \cdots \partial^{\bar\alpha_n} G_{\bar\omega_{\pi(n)}\bar\omega_{\pi(n)}}(\mathbf{x}_{\pi(n)} - \mathbf{x}_1), \tag{3.36}$$

for suitable indices $\bar\alpha_i \in \{0, 1, 2\}$ (such that $\sum_i \bar\alpha_i = \sum_i \alpha_i$), $\bar\omega_i \in \{\pm\}$ and a suitable permutation π of $\{2, \dots, n\}$.

As for $\mathcal{E}^T(\sigma_{b_1}\mathcal{A}_{b_1}; \dots; \sigma_{b_n}\mathcal{A}_{b_n})$, going back to (3.17)–(3.18) we see that we can distinguish two contributions: one that collects all terms without oscillating pre-factors $(-1)^{x_i}$, $i = 1, 2$ and one that contains at least one term with oscillating factor.

Let us look at the latter first. When we sum over $b_1, \dots, b_n$, we remarked in Section 3.2 that the effect of an oscillating factor $(-1)^{x_j}$ is the same as a discrete derivative ∂_j acting on a propagator. Therefore, the contribution to the nth cumulant, to be called $D_n''(\boldsymbol{\xi}, \boldsymbol{\eta})$, can be symbolically written exactly like $D_n'(\boldsymbol{\xi}, \boldsymbol{\eta})$ in (3.35).

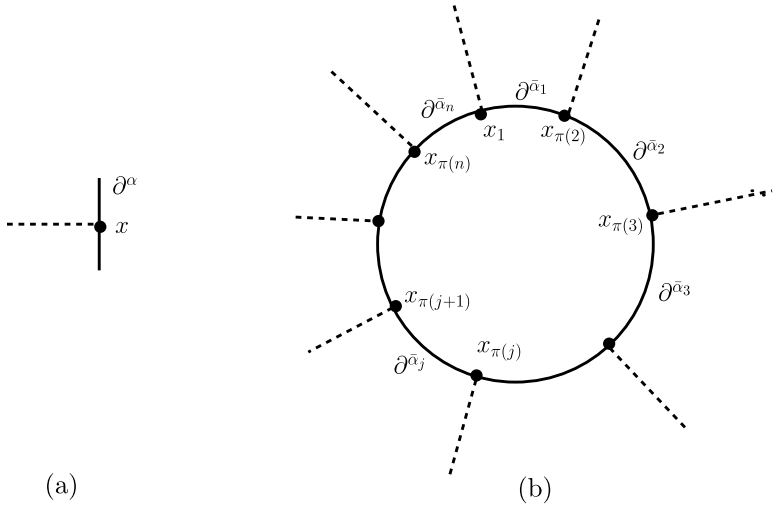

Fig. 4. (a) A vertex of type $\psi\partial^\alpha\psi$. (b) A sun diagram obtained by contracting n vertices of type $\psi\partial^\alpha\psi$.

Next, we look at the term without oscillating factors. In analogy with the derivation of the first term in the r.h.s. of (3.19), one can check that we get

$$2^n \cdot 2 \cdot \mathrm{Re}\big[(-i)^n \Delta z_{b_1} \cdots \Delta z_{b_n} \mathcal{E}^T\big(\psi^+_{\mathbf{x}_1,1}\psi^-_{\mathbf{x}_1,1}; \dots; \psi^+_{\mathbf{x}_n,1}\psi^-_{\mathbf{x}_n,1}\big)\big]. \tag{3.37}$$

The truncated expectation in (3.37) can be evaluated via Wick's rule (Proposition 6) as:

$$\begin{aligned}&\mathcal{E}^T\big(\psi^+_{\mathbf{x}_1,1}\psi^-_{\mathbf{x}_1,1}; \dots; \psi^+_{\mathbf{x}_n,1}\psi^-_{\mathbf{x}_n,1}\big)\\ &\quad = -\sum_{\pi \text{ on } \{2,\dots,n\}} G_{++}(\mathbf{x}_1 - \mathbf{x}_{\pi(2)}) \cdots G_{++}(\mathbf{x}_{\pi(n)} - \mathbf{x}_1).\end{aligned} \tag{3.38}$$

Plugging the decomposition (2.27) into (3.38) gives

$$-\sum_{\pi \text{ on } \{2,\dots,n\}} \mathfrak{g}_{++}(\mathbf{x}_1 - \mathbf{x}_{\pi(2)}) \cdots \mathfrak{g}_{++}(\mathbf{x}_{\pi(n)} - \mathbf{x}_1) + R'(\mathbf{x}_1, \dots, \mathbf{x}_n),$$

where R' collects all the terms involving at least one factor $R(\mathbf{x} - \mathbf{x}')$ from (2.27). Now, a well known combinatorial identity (see e.g. [29, Eq. (D.29)]) states that, if $n \geq 3$ and $\mathbf{x}_1, \dots, \mathbf{x}_n$ are all distinct, then

$$\sum_{\pi \text{ on } \{2,\dots,n\}} \mathfrak{g}_{++}(\mathbf{x}_1 - \mathbf{x}_{\pi(2)}) \cdots \mathfrak{g}_{++}(\mathbf{x}_{\pi(n)} - \mathbf{x}_1) = 0. \tag{3.39}$$

Therefore, the only non-vanishing contributions to the expression in (3.37) come from the terms involving at least one factor $R(\mathbf{x} - \mathbf{x}')$. These terms can be represented by sun diagrams similar to those in Figure 4(b), with the difference that the lines can be either associated with a propagator $\mathfrak{g}$ or with R, and there must be at least one propagator of type R. They give a contribution to the nth cumulant of the height that we denote by $D'''_n(\boldsymbol{\xi}, \boldsymbol{\eta})$.

In order to evaluate $D'_n(\boldsymbol{\xi}, \boldsymbol{\eta})$, $D''_n(\boldsymbol{\xi}, \boldsymbol{\eta})$, $D'''_n(\boldsymbol{\xi}, \boldsymbol{\eta})$, we resort to a multiscale decomposition and a tree expansion that are typical of constructive quantum field theory. While in the non-interacting case $\lambda = 0$ this could be avoided, this is the right approach that can be generalized to the interacting case. Let us focus on $D'_n(\boldsymbol{\xi}, \boldsymbol{\eta})$ first, the discussion for $D''_n(\boldsymbol{\xi}, \boldsymbol{\eta})$ and $D'''_n(\boldsymbol{\xi}, \boldsymbol{\eta})$ being completely analogous. We expand the value of every sun diagram (3.36) by using the multiscale decomposition for G in (2.34) (recall that $m = 0$, so that $h^* = -\infty$ in that formula), so that (3.36) is replaced by

$$\sum_{h_1,\dots,h_n \leq 0} \partial^{\bar\alpha_1} G^{(h_1)}_{\bar\omega_1\bar\omega_1}(\mathbf{x}_1 - \mathbf{x}_{\pi(2)}) \cdots \partial^{\bar\alpha_n} G^{(h_n)}_{\bar\omega_{\pi(n)}\bar\omega_{\pi(n)}}(\mathbf{x}_{\pi(n)} - \mathbf{x}_1). \tag{3.40}$$

Diagrammatically, every such contribution is associated with a labelled sun diagram, similar to the one in Figure 4(b), with extra scale labels h_i attached to every solid line. Using (2.39), we can bound every factor in (3.40) as

$$\left|\partial^{\bar{\alpha}_k} G^{(h_k)}_{\tilde{\omega}_{\pi(k)}\tilde{\omega}_{\pi(k+1)}}(\mathbf{x}_{\pi(k)} - \mathbf{x}_{\pi(k+1)})\right| \leq C 2^{\bar{\alpha}_k h_k} 2^{h_k} e^{-c\sqrt{2^{h_k}|\mathbf{x}_{\pi(k)} - \mathbf{x}_{\pi(k+1)}|}}$$

which implies the following bound on $D'_n(\boldsymbol{\xi}, \boldsymbol{\eta})$:

$$\left|D'_n(\boldsymbol{\xi}, \boldsymbol{\eta})\right| \leq C_n \sum_{b_1 \in \mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \cdots \sum_{b_n \in \mathcal{C}^{(n)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \sum_{\pi \text{ on } \{2,\dots,n\}} \sum_{h_1,\dots,h_n \leq 0} 2^{\bar{h}} \left[\prod_{k=1}^{n} 2^{h_k} e^{-c\sqrt{2^{h_k}|\mathbf{x}_{\pi(k)} - \mathbf{x}_{\pi(k+1)}|}}\right]. \tag{3.41}$$

Here: (i) C_n is a suitable positive constant, (ii) $\bar{h} = \max_{i=1,\dots,n} h_j$, and $2^{\bar{h}}$ is an upper bound on $\prod_i 2^{\bar{\alpha}_i h_i}$, (iii) $\pi(1)$ and $\pi(n+1)$ should be interpreted as being equal to 1.

Now we can sum over $b_1, \dots, b_n$ (which is the same as summing over $\mathbf{x}_1, \dots, \mathbf{x}_n$), observing that each sum is one-dimensional (b_i and, therefore, $\mathbf{x}_i$ runs along the path $\mathcal{C}^{(i)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$) and that, thanks to the way the paths were chosen, $|\mathbf{x}_i - \mathbf{x}_j| \geq c_n(d_i + d_j)$, with $d_i = \min\{d(\mathbf{x}_i, \boldsymbol{\xi}), d(\mathbf{x}_i, \boldsymbol{\eta})\}$ and e.g. $d(\mathbf{x}_i, \boldsymbol{\xi})$ the distance between $\boldsymbol{\xi}$ and $\mathbf{x}_i$ along $\mathcal{C}^{(i)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$. Then,

$$\prod_{k=1}^{n} e^{-c\sqrt{2^{h_k}|\mathbf{x}_{\pi(k)} - \mathbf{x}_{\pi(k+1)}|}} \leq \prod_{k=1}^{n} e^{-c'_n\sqrt{d_{\pi(k)}(2^{h_k} + 2^{h_{k-1}})}}, \tag{3.42}$$

where h_0 should be interpreted as being equal to h_n. The kth factor can now be easily summed over $b_{\pi(k)}$ and gives:

$$\sum_{b_{\pi(k)} \in \mathcal{C}^{(\pi(k))}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} e^{-c'_n\sqrt{d_{\pi(k)}(2^{h_k} + 2^{h_{k-1}})}} \leq 2\sum_{d=0}^{\infty} e^{-c'_n\sqrt{d\cdot(2^{h_k} + 2^{h_{k-1}})}} \leq C'_n 2^{-\max\{h_k, h_{k-1}\}}. \tag{3.43}$$

Plugging these bounds into (3.41) gives

$$\left|D'_n(\boldsymbol{\xi}, \boldsymbol{\eta})\right| \leq C''_n \sum_{h_1,\dots,h_n \leq 0} 2^{\bar{h}} \left[\prod_{k=1}^{n} 2^{h_k} 2^{-\max\{h_k, h_{k-1}\}}\right]. \tag{3.44}$$

The sum over the h_i's in the r.h.s. of (3.44) can be performed in various ways. We follow a specific strategy (possibly not the most straightforward), which admits a natural generalization to the interacting case. We think, once again, of the scale labels as being associated with the propagators of a labelled sun diagram. Every choice of $(h_1, \dots, h_n)$ produces a hierarchical organization of the vertices of the sun diagram into *clusters*, defined as follows. We say that a group of vertices forms a cluster on scale h if:

- the vertices are connected in the sub-graph where only lines on scale $h' \geq h$ are drawn;
- the group of vertices is maximal (i.e. no other vertex can be added while keeping the first property).

With this definition, every cluster contains at least 2 vertices. Note that the same group of vertices can be a cluster on various different scales. Every choice of $(h_1, \dots, h_n)$ defines a set of clusters, which are partially ordered in the natural sense induced by the subset relation: if a cluster v on scale h *strictly* contains a cluster v' on scale h', then $h' > h$. If v on scale h contains a cluster v' on scale $h' > h$, we say that v' follows v. In this sense, every choice of $(h_1, \dots, h_n)$ defines a cluster structure. An example is shown in Figure 5.

The partial ordering introduced above allows to represent a cluster structure as a tree, see Figure 6. The tree can be drawn on a grid of vertical lines, each associated with its scale label, and ordered from left to right, from the scale of the root (which is by convention one unit smaller than $\min_j h_j$) to 1. Vertices v_i correspond to endpoints (leaves) of the tree, which are all drawn by convention on the vertical line of scale 1. The intersections between the vertical lines and the tree are called nodes. All the nodes followed by at least two endpoints correspond to clusters: the cluster of scale h_v associated with such a node v is the set of endpoints following v on τ; in terms of this definition, it is natural

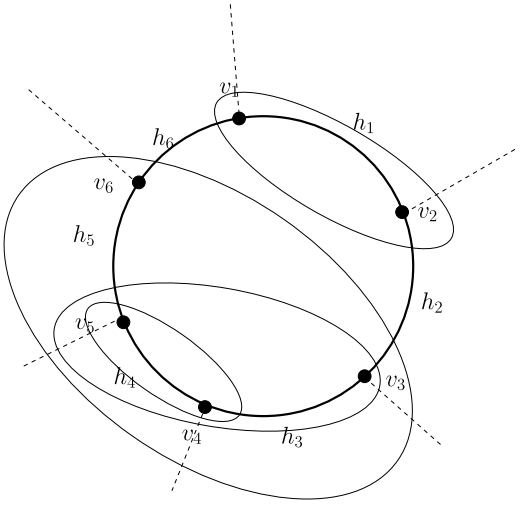

Fig. 5. An example of a labelled sun diagram with (part of) its cluster structure. In the picture it is assumed that $h_6 < h_2 < h_1$ and $h_2 < h_5 < h_3 < h_4$. The cluster on scale h_2 is not indicated explicitly.

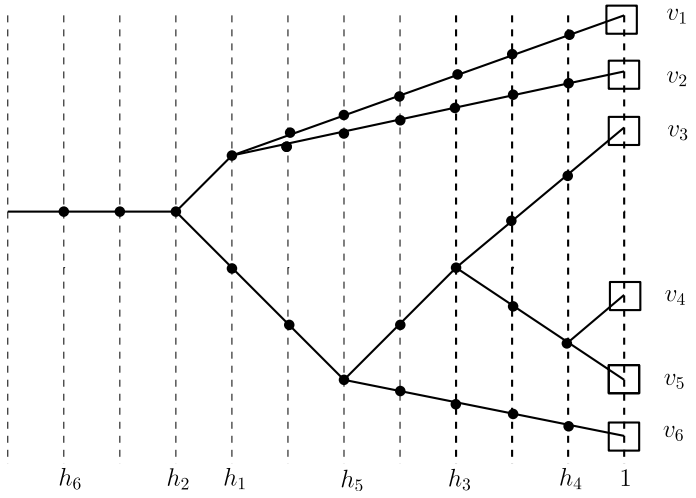

Fig. 6. The tree representing the hierarchical cluster structure of the labelled graph in Figure 5. Dots are nodes, squares are endpoints.

to think of the endpoints, as well as of the nodes followed by just one endpoint, as (trivial) clusters. Given the tree, the cluster structure can be reconstructed unambiguously. If we identify trees obtained from each other by pivoting the branches on the branching points, then the trees are in one-to-one correspondence with the cluster structures.

In Section 5, when analyzing the interacting model, we will need a more general class of trees.

Every labelled tree can be naturally thought of as a "topological" (i.e., unlabeled) tree, together with its scale labels. The idea is to reinterpret the sum over $(h_1, \dots, h_n)$ in (3.44) as a sum over trees, to be performed by first summing over the scale labels at fixed topological tree, and then over the topological trees. This can be done very easily: calling $\mathcal{T}^0_{h;n}$ the family of labelled trees with n endpoints and root on scale h that we just introduced, (3.44) implies

$$\left|D'_n(\boldsymbol{\xi},\boldsymbol{\eta})\right| \le C'''_n \sum_{h<0} \sum_{\tau\in\mathcal{T}^0_{h;n}} 2^{\bar{h}_\tau} \prod_{v\in V(\tau)} 2^{h_v \tilde{n}_v} \prod_{v\in V_{nt}(\tau)} 2^{-h_v \bar{m}^J_v}, \tag{3.45}$$

where: (i) $V(\tau)$ is the set of nodes of τ that are neither endpoints nor the root; (ii) $V_{nt}(\tau)$ is the set of branching points of τ; (iii) $\bar{h}_\tau = \max_{v\in V_{nt}(\tau)} h_v$; (iv) $\tilde{n}_v$ is the number of propagators contained in the cluster v but not in any other

cluster $v' > v$ [we say that a propagator is contained in a cluster v of scale h_v if it connects two endpoints in v, and if its scale is $\geq h_v$]; (v) if $v \in V_{nt}(\tau)$, then $\bar{m}_v^J$ is the number of endpoints contained in the cluster v but not in any other cluster $v' \in V_{nt}(\tau)$ such that $v' > v$. The exponent in the last product can be rewritten as follows. First note that, given a function f_v on $V(\tau)$ one has

$$\sum_{v \in V(\tau)} h_v f_v = h \sum_{v \in V(\tau)} f_v + \sum_{v \in V(\tau)} \sum_{\substack{w \in V(\tau): \\ w \geq v}} f_w \tag{3.46}$$

with h as usual the scale of the root. Similarly,

$$\sum_{v \in V_{nt}(\tau)} h_v f_v = h \sum_{v \in V_{nt}(\tau)} f_v + \sum_{v \in V_{nt}(\tau)} (h_v - h_{v'}) \sum_{\substack{w \in V_{nt}(\tau): \\ w \geq v}} f_w, \tag{3.47}$$

where, given $v \in V_{nt}(\tau)$, we denoted by v' the rightmost node in $V_{nt}(\tau)$ preceding v on τ (if v is the leftmost node in $V_{nt}(\tau)$, then we let $h_{v'} = h$). On the other hand, if n_v^e is the number of solid lines exiting from the cluster v in the Feynman diagram, see Figure 5, and m_v^J is the number of endpoints following v, one has

$$\sum_{\substack{v \in V(\tau): \\ v \geq w}} \tilde{n}_v = m_w^J - \frac{n_w^e}{2}, \tag{3.48}$$

which can be easily proved by induction. Similarly, if $w \in V_{nt}(\tau)$, then $\sum_{\substack{v \in V_{nt}(\tau): \\ v \geq w}} \bar{m}_v^J = m_w^J$. Then, one deduces

$$\sum_{v \in V(\tau)} h_v \tilde{n}_v - \sum_{v \in V_{nt}(\tau)} h_v \bar{m}_v^J = - \sum_{w \in V^*(\tau)} n_w^e/2, \tag{3.49}$$

where $V^*(\tau) = \{v \in V(\tau) : m_v^J > 1\}$ and we used the fact that $\sum_{v \in V(\tau)} \tilde{n}_v = \sum_{v \in V_{nt}(\tau)} \bar{m}_v^J$. Moreover, $n_v^e = 2$ for every cluster except the one at scale $h + 1$ (just look at Figure 5). Therefore, plugging (3.49) back into (3.45) gives

$$\left| D'_n(\boldsymbol{\xi}, \boldsymbol{\eta}) \right| \leq 2 C'''_n \sum_{h<0} \sum_{\tau \in \mathcal{T}^0_{h;n}} 2^{\bar{h}_\tau} \prod_{v \in V^*(\tau)} 2^{-1}, \tag{3.50}$$

which readily shows that the sum over the scale labels is convergent (first sum over the scale labels h_v at fixed h^*_τ, and then over $h^*_\tau \leq 0$): finally, we multiply by the number of topological trees with n endpoints, which is a constant depending only on n, so that

$$\left| D'_n(\boldsymbol{\xi}, \boldsymbol{\eta}) \right| \leq C''''_n, \tag{3.51}$$

as desired. The bounds on $D''_n(\boldsymbol{\xi}, \boldsymbol{\eta})$ and $D'''_n(\boldsymbol{\xi}, \boldsymbol{\eta})$ are completely analogous, because both quantities can be bounded as in (3.41). This is obvious for $D''_n(\boldsymbol{\xi}, \boldsymbol{\eta})$, for what already observed a few lines above (3.37). For what concerns $D'''_n(\boldsymbol{\xi}, \boldsymbol{\eta})$, recall that every contribution to it comes from a sun diagram whose lines are either of type $\mathfrak{g}$ or R, and there is at least one rest propagator R. After a multiscale decomposition of the propagators, we use the dimensional estimates on $\mathfrak{g}^{(h)}$ and $R^{(h)}$ stated in Lemma 2, and note that dimensionally $R^{(h)}$ behaves exactly like $\partial G^{(h)}$. This implies the analogue of (3.41) for $D'''_n(\boldsymbol{\xi}, \boldsymbol{\eta})$.

We are left with the rest $R_n(\boldsymbol{\xi}, \boldsymbol{\eta})$ in (3.33), which is easier to analyze. In order to estimate it, we do not even need to use the cancellation (3.39). Proceeding as above,[4] we find the analogue of (3.41):

$$\left| R_n(\boldsymbol{\xi}, \boldsymbol{\eta}) \right| \leq C_n \overset{\circ}{\sum_{\substack{b_i \in \mathcal{C}^{(i)}_{\boldsymbol{\xi} \to \boldsymbol{\eta}} \\ i=1,\dots,n}}} \sum_{\pi \text{ on } \{2,\dots,n\}} \sum_{h_1,\dots,h_n \leq 0} \left[\prod_{k=1}^n 2^{h_k} e^{-c\sqrt{2^{h_k} |\mathbf{x}_{\pi(k)} - \mathbf{x}_{\pi(k+1)}|}} \right], \tag{3.52}$$

[4] To be precise, when applying (3.8) one should take into account the multiplicity of the coinciding bonds. Since these multiplicities are bounded by n, this only changes the constants C_n below.

where the $\circ$ on the sum indicates the constraint that at least one coordinate belongs to $B_{r_n}(\boldsymbol{\xi}) \cup B_{r_n}(\boldsymbol{\eta})$. Note that, as compared to (3.41), the (good) factor $2^{\max_i h_i}$ is now absent. After summing over $b_1, \dots, b_n$, we get

$$\big|R_n(\boldsymbol{\xi}, \boldsymbol{\eta})\big| \le C_n'' \sum_{h_1,\dots,h_n \le 0} 2^{\max_j h_j} \left[\prod_{k=1}^{n} 2^{h_k} 2^{-\max\{h_k, h_{k-1}\}} \right], \tag{3.53}$$

where the gain factor $2^{\max_j h_j}$ arises from the fact that at least one of the coordinates $\mathbf{x}_i$ is not summed over (or, more precisely, is summed over a region of size r_n) and, therefore, at least one of the factors $2^{-\max\{h_k, h_{k-1}\}}$ in the r.h.s. of (3.53) in reality should not be there (in fact, recall that these factors come from (3.43); if the sum over d from 0 to ∞ in (3.43) is replaced by a sum over a finite set of nonnegative integers, then the r.h.s. of (3.43) can be replaced by a constant C_n'). The r.h.s. of (3.53) is the same as (3.44) and, therefore, leads to the analogue of (3.51): $|R_n(\boldsymbol{\xi}, \boldsymbol{\eta})| \le C_n$. This concludes the proof of (3.32) and of Theorem 1 in the case $\lambda = 0$. □

4. The height variance in the interacting case

In the proof of Theorem 1 for $\lambda = 0$, a crucial role was the sharp asymptotic behavior of multi-dimer correlations, see in particular Proposition 3 for the two-point function. We need analogous estimates for $\lambda \neq 0$. In particular, for the proof of (1.7) (logarithmic divergence of the height variance) we need the sharp asymptotic estimate on the two-point dimer correlation, provided by Theorem 2 (which is proved in Section 6.4). Given this, the proof of (1.7) is immediate and is presented here. The height variance can be written as

$$\langle h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}; h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}} \rangle = \sum_{b_1 \in \mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \sum_{b_2 \in \mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}} \sigma_{b_1}\sigma_{b_2} \langle \mathbb{1}_{b_1}; \mathbb{1}_{b_2} \rangle_\lambda, \tag{4.1}$$

with $\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$, $\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ chosen as explained after (3.10). Plugging (1.9) into (4.1), we obtain

$$\begin{aligned} \langle h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}}; h_{\boldsymbol{\xi}} - h_{\boldsymbol{\eta}} \rangle = \sum_{\substack{b_1 \in \mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}} \\ b_2 \in \mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}} \sigma_{b_1}\sigma_{b_2} \Bigg\{ \mathbf{1}_{\mathbf{x}_1 \neq \mathbf{x}_2} \Bigg[-\frac{K}{2\pi^2} (-1)^{\mathbf{x}_1 - \mathbf{x}_2} \operatorname{Re} \frac{(i)^{j_1+j_2}}{(z_{\mathbf{x}_1} - z_{\mathbf{x}_2})^2} \\ + \delta_{j_1,j_2} \frac{\tilde{K}}{2\pi^2} \frac{(-1)^{(\mathbf{x}_1-\mathbf{x}_2)j_1}}{|\mathbf{x}_1 - \mathbf{x}_2|^{2\kappa}} \Bigg] + R_{j_1,j_2}(\mathbf{x}_1 - \mathbf{x}_2) \Bigg\}, \end{aligned} \tag{4.2}$$

where $\mathbf{x}_1$, $\mathbf{x}_2$, j_1, j_2 are such that $b_1 = (\mathbf{x}_1, \mathbf{x}_1 + \hat{e}_{j_1})$ and $b_2 = (\mathbf{x}_2, \mathbf{x}_2 + \hat{e}_{j_2})$, and $z_{\mathbf{x}} = x_1 + i x_2$ is the complex number associated with $\mathbf{x}$. Recall now that $\sigma_{b_i} = \alpha_{b_i} (-1)^{\mathbf{x}_i} (-1)^{j_i}$, with α_b defined just after (3.12). Using this explicit expression for σ_b into (4.2), we can rewrite the term proportional to K as:

$$-\frac{K}{2\pi^2} \sum_{\substack{b_1 \in \mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}} \\ b_2 \in \mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}} \sigma_{b_1}\sigma_{b_2} \operatorname{Re} \frac{(-1)^{\mathbf{x}_1-\mathbf{x}_2}(i)^{j_1+j_2}}{(z_{\mathbf{x}_1} - z_{\mathbf{x}_2})^2} = -\frac{K}{2\pi^2} \sum_{\substack{b_1 \in \mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}} \\ b_2 \in \mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}} \operatorname{Re} \frac{\Delta z_{b_1} \Delta z_{b_2}}{(z_{\mathbf{x}_1} - z_{\mathbf{x}_2})^2}, \tag{4.3}$$

where Δz_{b_i} is the displacement associated with the elementary portion of the path $\mathcal{C}^{(i)}$ crossing b_i, thought of as a complex vector of modulus 1. This term is the $\lambda \neq 0$ analog of the first term in the r.h.s. of (3.25), which referred to the case $\lambda = 0$. Exactly like in the $\lambda = 0$ situation, the r.h.s. of (4.3) is equal to K times the integral in (3.27) (which is the desired dominant contribution to the variance of the height), plus a rest that is uniformly bounded in $|\boldsymbol{\xi} - \boldsymbol{\eta}|$.

Let us now estimate the contributions to the variance coming from the other two terms in the r.h.s. of (4.2). The last term, i.e., the sum over b_1, b_2 of $\sigma_{b_1}\sigma_{b_2}R_{j_1,j_2}(\mathbf{x}_1-\mathbf{x}_2)$, leads to a contribution that is uniformly bounded in $|\boldsymbol{\xi}-\boldsymbol{\eta}|$, thanks to the decay estimate on R_{j_1,j_2}: $|R_{j,j'}(\mathbf{x}-\mathbf{y})| \le C_\theta(1+|\mathbf{x}-\mathbf{y}|)^{-2-\theta}$, for some $\frac{1}{2} \le \theta < 1$ and $C_\theta > 0$. Regarding the term proportional to $\tilde{K}$, note that

$$\sigma_{b_1}\sigma_{b_2}(-1)^{(\mathbf{x}_1-\mathbf{x}_2)_{j_1}} = \alpha_{b_1}\alpha_{b_2}(-1)^{(\mathbf{x}_1-\mathbf{x}_2)_{3-j_1}}.$$

Namely, the oscillatory factor $\sigma_{b_1}\sigma_{b_2}$ *does not* compensate the oscillatory factor $(-1)^{(\mathbf{x}_1-\mathbf{x}_2)_{j_1}}$. Once summed over the path, and using the fact that the paths $\mathcal{C}^{(i)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ consist of union of straight portions, each of which is formed by an even number of bonds, we see that the oscillatory factor $(-1)^{(\mathbf{x}_1-\mathbf{x}_2)_{3-j_1}}$ has the same effect as a discrete derivative (we are sketchy here, but the very same argument was used in the non-interacting model just after (3.25)):

$$\Big|\sum_{\substack{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}\\ b_2\in\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}} \alpha_{b_1}\alpha_{b_2}\delta_{j_1,j_2}\frac{(-1)^{(\mathbf{x}_1-\mathbf{x}_2)_{3-j_1}}}{|\mathbf{x}_1-\mathbf{x}_2|^{2+2\eta_2'}}\Big| \le c\sum_{\substack{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}\\ b_2\in\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}} \Big|\partial_{3-j_1}\frac{1}{|\mathbf{x}_1-\mathbf{x}_2|^{2+2\eta_2'}}\Big|, \tag{4.4}$$

which shows that also this term is bounded uniformly in $|\boldsymbol{\xi}-\boldsymbol{\eta}|$. This concludes the proof of (1.7), i.e., of Theorem 1 for $n=2$. □

The proof of Theorem 2, which is, as we just saw, the crucial ingredient behind the proof of (1.7), is very hard. It is based on a renormalized, convergent, perturbative expansion for the generating function $\mathcal{Z}_\Lambda(\lambda, m, \mathbf{A})$ to be discussed in Section 6 below. The renormalized expansion $\mathcal{Z}_\Lambda(\lambda, m, \mathbf{A})$ induces a convergent expansion for the multi-point dimer correlations, which is the key ingredient in the computation of the cumulants of the height fluctuations of order 3 or higher, to be discussed in Section 7.

5. The interacting case: Formal perturbation theory

Before explaining the renormalized, convergent, expansion for the generating function for dimer correlations, we make a digression to explain why naive perturbation theory in λ does not work to get results like (1.9). This discussion will help the non-expert reader understand the meaning of the renormalized perturbation expansion of Section 6, which is behind e.g. Theorem 2. Since strictly speaking the present section is not necessary for the proof, our exposition here is more informal than in the rest of the article.

Remark 8 (Warning on the literature). *Here and in Section 6 we will often appeal to results from the literature on constructive RG, notably* [8,12–15,34]. *These works do not study exactly the same model as ours: however, they all study models that can be written as two-dimensional interacting Majorana or Dirac fermions, with potentials having the same symmetry and decay properties as ours. The results we refer to can be easily extended to our context.*

5.1. *The Feynman diagrams expansion of the height fluctuations*

We restart from (3.6). We emphasize that, since $\mathcal{Z}_\Lambda(\lambda, m, \mathbf{A})$ is a polynomial in λ for finite Λ and it equals 1 when $\lambda = 0$, $\mathbf{A} = \mathbf{0}$, the sums in the second and third lines of (3.6) are convergent for sufficiently small λ, $\mathbf{A}$. However, proving that the radius of convergence in λ does not shrink to zero as $\Lambda \nearrow \mathbb{Z}^2$ is a highly non-trivial task. Of course, before even attempting to prove uniform convergence, we need at least to understand how to compute the r.h.s. of (3.6) formally, i.e., order by order in λ. A possible way of computing the perturbation series in λ for the generating function is in terms of Feynman diagrams, as explained after (3.7), see Proposition 6. In particular, $E_\Lambda(\lambda, m)$ equals the sum of all possible connected Feynman diagrams obtained by contracting vertices of type $\xi(\gamma)$ (coming from $V_\Lambda(\psi)$, see (2.43)), where $\gamma = \{b_1, \dots, b_k\} \subset \Lambda$ is a collection of $k \ge 2$ parallel adjacent bonds, see (2.44); in order to graphically represent $\xi(\gamma)$, we imagine to represent $E^{(m)}_{(\mathbf{x},\mathbf{x}+\hat{e}_j)} = (i)^{j-1}(1+\delta_{j,1}m(-1)^{x_1})\psi_{\mathbf{x}}\psi_{\mathbf{x}+\hat{e}_j}$ as a pair of solid half-lines, each of which can be contracted with another solid half-line to form a solid line (a propagator), while the α's can be thought of as wiggly lines from b_1 to b_2, etc., to b_k, see Figure 7.

Moreover, $[\prod_{j=1}^{k} J_{b_j}]S_{\Lambda,k}(b_1,\dots,b_k)$ is the sum of all possible connected Feynman diagrams obtained by contracting vertices of type $\xi(\gamma)$ and of type $\tilde{\xi}(\gamma;R)$ (coming from $\mathcal{B}_\Lambda(\psi,\mathbf{J})$), with the obvious constraint that the product of the J_b factors involved produces exactly $\prod_{j=1}^{k} J_{b_j}$. See Figure 8. For example, one of the diagrams contributing to $S_{\Lambda,4}(b_1,\dots,b_4)$ is shown in Figure 9. The diagram in Figure 9 is obtained from a contraction of the vertices depicted in Figure 10.

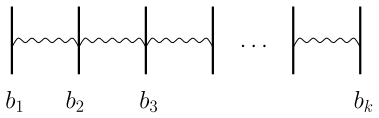

Fig. 7. Graphical representation of a vertex of type $\xi(\{b_1,\dots,b_k\})$.

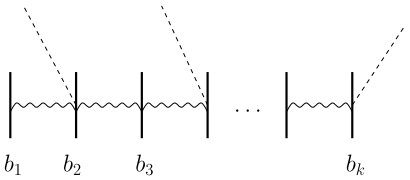

Fig. 8. Graphical representation of a vertex of type $\tilde{\xi}(\{b_1,\dots,b_k\};R)$, with $R=\{b_2,b_4,b_k\}$. The dotted lines represent the external fields J_{b_i}. If $|\gamma|=|R|=1$ the vertex is said to be of type $-J_b E_b^{(m)}$ (see also Figure 4(a)).

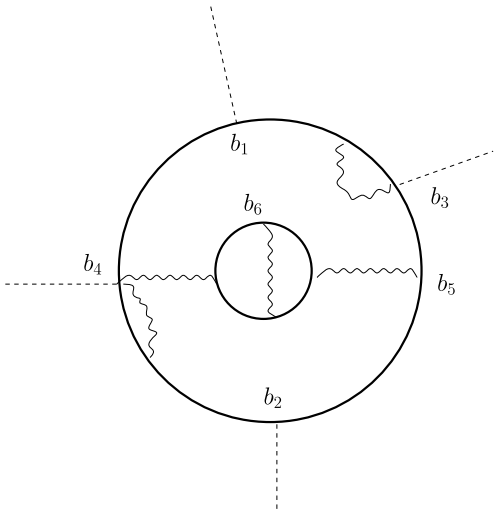

Fig. 9. A diagram contributing to $S_{\Lambda,4}(b_1,\dots,b_4)$.

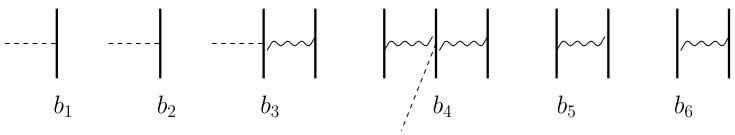

Fig. 10. The vertex elements producing the diagram in Figure 9, after a suitable contraction of the solid half-lines.

5.2. *Failure of the Feynman diagram expansion*

In order to bound the Feynman diagram expansion for the kernels $S_{\Lambda,n}(b_1,\dots,b_n)$ in (3.6), we can try to proceed as follows (the strategy is similar to the one used in Section 3.3). We decompose each of the propagators G appearing in the values of the Feynman diagrams as in (2.34) and in this way we obtain labelled Feynman graphs with solid lines (propagators) each carrying a scale label $h^* \le h \le 0$, the label $h = h^*$ corresponding to $G^{(\le h^*)}$. Any labelled graph has a corresponding cluster structure, in the sense explained after (3.44), which can be conveniently represented by a tree analogous to those in Figure 6; in the interacting case, these trees are known as *Gallavotti–Nicolò* (GN) trees, first introduced in [33] for studying the renormalization theory of the φ^4_4 Quantum Field Theory (QFT), and later applied to several other problems in statistical mechanics and field theory (for a detailed derivation of the tree expansion, see e.g. [32] and the more recent reviews [34,36,50]; a description of its main features is summarized below, for completeness). It is now tempting to bound the value of every labelled Feynman diagram by using Lemma 2, then sum the resulting bound over the scale labels at fixed cluster structure, and then sum over the cluster structures, exactly as we did in Section 3.3. Natural as it appears, this strategy *does not work*, and actually perturbation expansion in Feynman diagrams does not provide any information on the interacting dimer correlations. As this is a key point in order to understand the motivations of the more elaborate analysis in the following sections, it is convenient to explain why the power series expansion in Feynman diagrams does not work, i.e., it cannot be proved directly to be convergent.

5.2.1. *The tree and the labelled Feynman diagram expansions*

In contrast with the trees we introduced for the non-interacting model, GN trees have endpoins of different type, depending on whether they are associated with a vertex of type $V_\Lambda(\psi)$ (i.e., of type $\xi(\gamma)$, see (2.43)), in which case the endpoints will be called "normal," or of type $\mathcal{B}_\Lambda(\psi,\mathbf{J})$ (i.e., of type $-J_b E_b^{(m)}$ or of type $\tilde\xi(\gamma;R)$ with $|\gamma|\ge 2$, see (2.57)), in which case they will be called "special." Note that in the non-interacting case $\alpha = 0$ we had $V_\Lambda(\psi) = 0$ and $\mathcal{B}_\Lambda(\psi,\mathbf{J}) = -\sum_{b\subset\Lambda} J_b E_b^{(m)}$, so that all the endpoints were special. It is important to realize that, given a labeled tree (including possibly the labels that specify the order in α of the endpoints), there may be many Feynman diagrams compatible with it, see e.g. Figure 11.

To explain precisely how to express $E_\Lambda(\lambda,m)$ and $S_{\Lambda,k}(b_1,\dots,b_k)$ as a sum over trees and over Feynman diagrams compatible with the trees, we need to make a small detour about the main features and definitions of the GN trees. The trees introduced in this section are called "non-renormalized trees," as opposed to the "renormalized trees" that will be introduced in Section 6. Let us also remark that some of the conventions introduced here are slightly different from those used in Section 3, such as the rule for identifying trees, and the meaning of the word "vertex."

(1) Consider the family of all trees which can be constructed by joining a point r, the *root*, with an ordered set of $N + n \ge 1$ points, the *endpoints* of the *unlabeled tree*, so that r is not a branching point. The endpoints can be

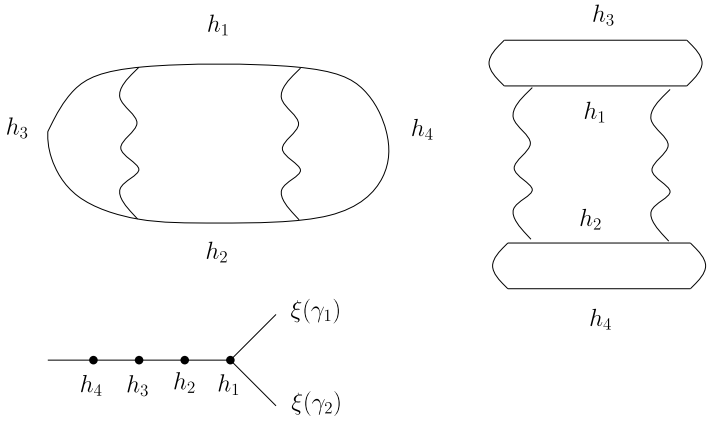

Fig. 11. Two different labelled Feynman graphs, coming from the contractions of two vertices of type $\xi(\gamma_i)$, $i = 1, 2$ with $|\gamma_i| = 2$, giving the same tree. Here, $h_4 < h_3 < h_2 < h_1$.

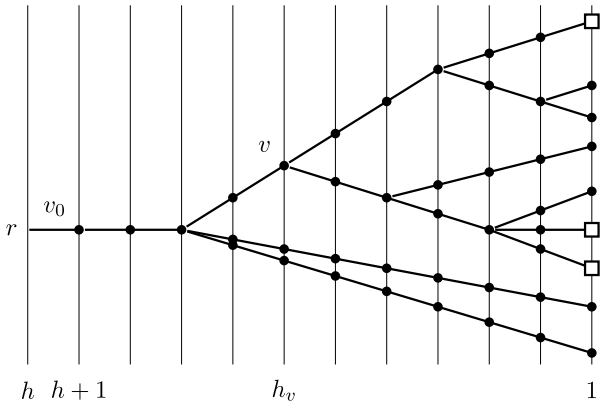

Fig. 12. A tree $\tau \in \tilde{\mathcal{T}}^{(h)}_{N,n}$ with $N = 6$ and $n = 3$: the root is on scale h and the endpoints are all on scale 1.

of two types, either normal or special, the former drawn as dots, the latter as squares, see Figure 12; N and n are the number of normal and special endpoints, respectively. The branching points will be called the *non-trivial vertices*. The unlabeled trees are partially ordered from the root to the endpoints in the natural way; we shall use the symbol $<$ to denote the partial order. Two unlabeled trees are identified if they can be superposed by a suitable continuous deformation, so that the endpoints with the same index coincide. We shall also consider the *labelled trees* (to be called simply trees in the following); they are defined by associating some labels with the unlabeled trees, as explained in the following items.

(2) We associate a label $h^* - 1 \le h < 0$ with the root and we denote by $\tilde{\mathcal{T}}^{(h)}_{N,n}$ the corresponding set of labelled trees with N normal and n special endpoints (the tilde in $\tilde{\mathcal{T}}^{(h)}_{N,n}$ reminds that the trees are non-renormalized). Moreover, we introduce a family of vertical lines, labeled by an integer taking values in $[h, 1]$, and we represent any tree $\tau \in \tilde{\mathcal{T}}^{(h)}_{N,n}$ so that, if v is an endpoint, it is contained in the vertical line with index $h_v = 1$, while if it is a non-trivial vertex, it is contained in a vertical line with index $h < h_v \le 0$, to be called the *scale* of v; the root r is on the line with index h. In general, the tree will intersect the vertical lines in set of points different from the root, the endpoints and the branching points; these points will be called *trivial vertices*. The set of the *vertices* will be the union of the endpoints, of the trivial vertices and of the non-trivial vertices; note that the root is not a vertex. Every vertex v of a tree will be associated to its scale label h_v, defined, as above, as the label of the vertical line whom v belongs to. Note that, if v_1 and v_2 are two vertices and $v_1 < v_2$, then $h_{v_1} < h_{v_2}$.

(3) There is only one vertex immediately following the root, called v_0 and with scale label equal to $h + 1$.

(4) Given a vertex v of $\tau \in \tilde{\mathcal{T}}^{(h)}_{N,n}$ that is not an endpoint, we can consider the subtrees of τ with root v, which correspond to the connected components of the restriction of τ to the vertices $w \ge v$. If a subtree with root v contains only v and one endpoint on scale $h_v + 1$, it will be called a *trivial subtree* (and in this case $h_v = 0$).

(5) If v is not and endpoint, the cluster associated with it is the set of endpoints following v on τ; if v is an endpoint, it is itself a (trivial) cluster. The tree provides an organization of endpoints into a labelled hierarchy of clusters (the cluster structure).

(6) Normal endpoints are associated with (one of the monomials contributing to) $V_\Lambda(\psi)$, while special endpoints are associated with (one of the monomials contributing to) $\mathcal{B}_\Lambda(\psi, \mathbf{J})$, both thought of as functions of the Majorana fields $\psi_{\mathbf{x},\gamma}$, with $\mathbf{x} \in \Lambda$ and $\gamma = 1, \dots, 4$.

In order to distinguish the various contributions arising from the choices of the monomials in the factors $V_\Lambda(\psi)$ and $\mathcal{B}_\Lambda(\psi, \mathbf{J})$ associated with the endpoints, as well as the scale at which each field in these monomials is contracted,[5] we need a few more definitions. We introduce a *field label* f to distinguish the field variables appearing in the monomials associated with the endpoints; the set of field labels associated with the endpoint v will be called I_v; if v is not an endpoint, we shall call I_v the set of field labels associated with the endpoints following the vertex v. Note that every

[5] A remark on nomenclature: we refer to both Majorana variables $\psi_{\mathbf{x},\gamma}$ and to $J_{\mathbf{x},j}$ as "fields" (ψ fields and J fields respectively).

field can be either of type J or ψ: correspondingly, we denote by I_v^J and I_v^ψ the set of field labels of type J and ψ, respectively, associated with v. Furthermore, we denote by $\mathbf{x}(f)$ the spatial coordinate of the field variable with label f; if $f \in I_v^J$, we denote by $b(f)$ the bond label of the corresponding J field, and we let $\mathbf{x}(f)$ and $j(f)$ be such that $b(f) = (\mathbf{x}(f), \mathbf{x}(f) + \hat{e}_{j(f)})$; if $f \in I_v^\psi$, we denote by $\gamma(f)$ the Majorana label of the corresponding Grassmann field. Similarly, we let $\mathbf{x}_v := \bigcup_{f \in I_v} \mathbf{x}(f)$, etc.

Remark 9 (Kernels of endpoints). *Given an endpoint v and the labels I_v, $\mathbf{x}_v$, etc., the value of the endpoint is uniquely specified, and we denote it by $K_v(\mathbf{x}_v, I_v) J(I_v^J) \psi(I_v^\psi)$, where K_v is the* kernel *of v, while $J(I) = \prod_{f \in I} J_{b(f)}$ and $\psi(I) = \prod_{f \in I} \psi_{\mathbf{x}(f),\gamma(f)}$. For instance, if v is an endpoint of type $\xi(\gamma)$ with γ a collection of k adjacent vertical bonds then $K_v = (-1)^k i^k \alpha^{k-1}$, see (2.44).*

We associate with any vertex v of the tree a subset P_v^ψ of I_v^ψ, the *set of external fermionic fields* of v. They correspond to the lines exiting from the cluster v, in the same sense discussed after (3.47); in particular, their cardinality is the analogue of the quantity n_v^e introduced there. These subsets must satisfy various constraints. First of all, if v is not an endpoint and $v_1, \dots, v_{s_v}$ are the $s_v \geq 1$ vertices immediately following it on τ, then $P_v^\psi \subset \bigcup_i P_{v_i}^\psi$; if v is an endpoint, $P_v^\psi = I_v^\psi$. If v is not an endpoint, we shall denote by $Q_{v_i}^\psi$ the intersection of P_v^ψ and $P_{v_i}^\psi$; this definition implies that $P_v^\psi = \bigcup_i Q_{v_i}^\psi$. The union of the subsets $P_{v_i}^\psi \setminus Q_{v_i}^\psi$ is, by definition, the set of the *internal fields* of v, and is non-empty if $s_v > 1$. For convenience, in the following we shall also indicate $P_v^J := I_v^J$, $P_v := P_v^\psi \cup P_v^J$ and $Q_v := Q_v^\psi \cup P_v^J$. Given $\tau \in \tilde{\mathcal{T}}_{N,n}^{(h)}$, there are many possible choices of the subsets P_v, $v \in \tau$, compatible with all the constraints. We shall denote by $\mathcal{P}_\tau$ the family of all these choices and by $\mathbf{P}$ the elements of $\mathcal{P}_\tau$. For every τ and $\mathbf{P} \in \mathcal{P}_\tau$, we let $\Gamma(\mathbf{P}, \tau)$ be the set of labelled Feynman diagrams compatible with the tree and the choice of the field labels.

In terms of these trees and labels, the generating function for correlations in (3.6) can be written as (see e.g. [34, Section 6])

$$\log \mathcal{Z}_\Lambda^{(11)}(\lambda, m, \mathbf{A}) = \sum_{\substack{N,n \geq 0: \\ N+n \geq 1}} \sum_{h=h^*-1}^{-1} \sum_{\tau \in \tilde{\mathcal{T}}_{N,n}^{(h)}} \sum_{\substack{\mathbf{P} \in \mathcal{P}_\tau: \\ P_{v_0}^\psi = \varnothing}}^{*} \sum_{\mathbf{x}_{v_0}} J\big(P_{v_0}^J\big) \times \Big[\prod_{v \in E(\tau)} K_v(\mathbf{x}_v, P_v) \Big] \Big[\prod_{v \in V(\tau)} \frac{1}{s_v!} \mathcal{E}_{h_v}^T \big(\psi(P_{v_1} \setminus Q_{v_1}), \dots, \psi(P_{v_{s_v}} \setminus Q_{v_{s_v}}) \big) \Big], \tag{5.1}$$

where the $*$ on the sum over $\mathbf{P}$ indicates the constraint that $P_{v_0}^\psi = \varnothing$ and the set of internal fields of v_0 is non-empty. Moreover, $\mathcal{E}_h^T$ indicates truncated expectation with respect to the propagator $G^{(h)}$, if $h > h^*$, or $G^{(\leq h^*)}$, if $h = h^*$. Finally, $E(\tau)$ is the set of endpoints of τ, $V(\tau)$ is the set of vertices of τ that are not in $E(\tau)$; for each $v \in V(\tau)$, we indicated by $v_1, \dots, v_{s_v}$ the vertices immediately following v on τ. After re-expressing the truncated expectations in the r.h.s. as a sum over Feynman diagrams, we obtain the desired representation of the generating function in terms of a double sum over trees and labelled Feynman diagrams:

$$\log \mathcal{Z}_\Lambda^{(11)}(\lambda, m, \mathbf{A}) = \sum_{\substack{N,n \geq 0: \\ N+n \geq 1}} \sum_{h=h^*-1}^{-1} \sum_{\tau \in \tilde{\mathcal{T}}_{N,n}^{(h)}} \sum_{\substack{\mathbf{P} \in \mathcal{P}_\tau: \\ P_{v_0}^\psi = \varnothing}}^{*} \sum_{\mathbf{x}_{v_0}} \sum_{\mathcal{G} \in \Gamma(\tau, \mathbf{P})} J\big(P_{v_0}^J\big) \operatorname{Val}(\mathcal{G}) \tag{5.2}$$

with $\operatorname{Val}(\mathcal{G})$ the value of the graph $\mathcal{G}$, including the combinatorial factor $\prod 1/(s_v!)$. To obtain the multiscale expansion for $E_\Lambda(\lambda, m)$ it is enough to compute this expression for $\mathbf{J} = \mathbf{0}$, so that $n = 0$. Similarly, to obtain $S_{\Lambda,k}(b_1, \dots, b_k)$ it is enough to derive with respect to $J_{b_1}, \dots, J_{b_k}$ and then take $\mathbf{J} = \mathbf{0}$.

5.2.2. *Dimensional estimates*

At this point we can discuss how to obtain estimates on the generic term of the non-renormalized expansion just introduced, and see whether the resulting upper bound is summable or not over all the labels and the trees.

Let us consider for simplicity a contribution to $E_\Lambda(\lambda, m)$. That is, consider $\mathcal{G} \in \Gamma(\tau, \mathbf{P})$, where $\tau \in \tilde{\mathcal{T}}^{(h)}_{N,0}$ and $P_{v_0} = \varnothing$; note that $P_v = P_v^\psi$, because $n = 0$. In order to estimate $\mathrm{Val}(\mathcal{G})$ we use that, from Lemma 2,

$$\big\|G^{(h)}(\cdot)\big\|_1 := \sum_{\mathbf{x}\in\Lambda} \big\|G^{(h)}(\mathbf{x})\big\| \le C2^{-h}, \qquad \big\|G^{(h)}(\cdot)\big\|_\infty \le C2^h. \tag{5.3}$$

Moreover, given $v \in E(\tau)$ and an arbitrary field label $f^* \in P_v$,

$$\sum_{\mathbf{x}_v \setminus \mathbf{x}(f^*)} \big|K_v(\mathbf{x}_v, P_v)\big| \le C^{|P_v|}\alpha^{|P_v|/2-1}, \tag{5.4}$$

as it follows from the very definition (2.44) of the kernel v. Therefore,

$$\sum_{\mathbf{x}_{v_0}} \big|\mathrm{Val}(\mathcal{G})\big| \le |\Lambda| \Big[\prod_{v\in E(\tau)} \big(C'\big)^{|P_v|}\alpha^{|P_v|/2-1}\Big]\Big[\prod_{v\in V(\tau)} \frac{1}{s_v!} 2^{h_v\tilde{n}_v - 2h_v(s_v-1)}\Big], \tag{5.5}$$

where $\tilde{n}_v = (\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|)/2$ was already introduced after (3.45), i.e., it is the number of propagators contained in v but not in any $v' > v$ or, equivalently, the number of propagators obtained by contracting the internal fields of v. We also recall that s_v is the number of vertices immediately following v on τ (i.e., the number of clusters contained in v but not in any other cluster $w > v$). To understand (5.5) note that the factor $|\Lambda|$ in (5.5) comes from translation invariance (i.e. from the sum over the location of the cluster at scale $h+1$) and that the factor associated with the product over the endpoints comes from (5.4). Moreover, the factor associated with the product over $V(\tau)$ comes from the following argument: for any vertex $v \in V(\tau)$ with s_v descendants $v_1, \dots, v_{s_v}$, we select a minimal number, $s_v - 1$, of propagators at scale h_v connecting them; all the non-selected lines are estimated in the ℓ_∞ norm and give $C2^{h_v}$ each, by the second of (5.3); when the relative positions of $v_1, \dots, v_{s_v}$ are summed over, each selected line gives instead $C2^{-h_v}$ by the first of (5.3).

Then we proceed as in (3.49), and in particular we use (3.46) for $f_v = \tilde{n}_v - 2(s_v - 1)$ and the analogues of (3.48), namely

$$\begin{aligned} &\sum_{\substack{v\in V(\tau):\\ v\ge w}} \tilde{n}_v = \frac{1}{2} \sum_{\substack{v\in V(\tau):\\ v\ge w}} \Big(\sum_{i=1}^{s_v} |P_{v_i}| - |P_v|\Big) = \frac{1}{2}\big(|I_w| - |P_w|\big), \\ &\sum_{\substack{v\in V(\tau):\\ v\ge w}} (s_v - 1) = m_w - 1, \end{aligned} \tag{5.6}$$

where m_w is the number of normal endpoints following w on τ, and we get

$$\sum_{\mathbf{x}_{v_0}} \big|\mathrm{Val}(\mathcal{G})\big| \le |\Lambda| \big(C'\big)^{|I_{v_0}|}\alpha^{|I_{v_0}|/2-N} 2^{h(2+|I_{v_0}|/2-2m_{v_0})} \prod_{v\in V(\tau)} \frac{1}{s_v!} 2^{2-|P_v|/2+|I_v|/2-2m_v},$$

where we used the fact that P_{v_0} is empty. Next we note that

$$hm_{v_0} + \sum_{v\in V(\tau)} m_v = h|I_{v_0}| + \sum_{v\in V(\tau)} |I_v| = 0, \tag{5.7}$$

thanks to the fact that the vertices immediately preceding the endpoints on τ are all on scale 0 (otherwise, we would have e.g. $hm_{v_0} + \sum_{v\in V(\tau)} m_v = \sum_{v\in E(\tau)} h_{v'}$ with v' the vertex immediately preceding v on τ). Therefore,

$$\sum_{\mathbf{x}_{v_0}} \big|\mathrm{Val}(\mathcal{G})\big| \le |\Lambda| \big(C'\big)^{|I^\psi_{v_0}|}\alpha^{|I^\psi_{v_0}|/2-N} 2^{2h} \Big[\prod_{v\in V(\tau)} \frac{1}{s_v!} 2^{2-|P_v^\psi|/2}\Big], \tag{5.8}$$

where the apex ψ on I_v and P_v is inserted to recall that in the case considered so far $P_v = P_v^\psi$. A similar estimate is valid for the contributions to $S_{\Lambda,k}(b_1, \dots, b_k)$ from the graphs $\Gamma \in \mathcal{G}(\tau, \mathbf{P})$ with $\tau \in \tilde{\mathcal{T}}_{N,n}^{(h)}$, with the only important difference that the *scaling dimension* $2 - |P_v^\psi|/2$ is replaced by $2 - |P_v^\psi|/2 - |P_v^J|$, to be denoted by d_v.

If we could assume that the scaling dimensions $d_v = 2 - |P_v^\psi|/2 - |P_v^J|$ are ≤ 0 for all $v \in V(\tau)$ and strictly negative for $v \in V^*(\tau)$ (here $V^*(\tau)$ is the subset of vertices of $V(\tau)$ that are followed by at least two endpoints), then (5.8) would be summable over the scale labels, and after summation we would get a bound proportional to $\alpha^{|I_{v_0}^\psi|/2-N}$. However, there are trees τ and graphs $\Gamma \in \mathcal{G}(\tau, \mathbf{P})$ with vertices $v \in V(\tau)$ such that d_v is either 0 or 1: this happens for $(|P_v^\psi|, |P_v^J|) = (2, 0), (4, 0), (2, 1)$, in which case (5.8) is not summable, uniformly in L, over the trees in $\tilde{\mathcal{T}}_{N,0}^{(h)}$ and on the scale label $h < 0$. In the Renormalization Group language, clusters with scaling dimension 0 are called *marginal*, and those with scaling dimension 1 are called (linearly) *relevant*. Note that in the non-interacting case there were neither marginal, nor relevant clusters in $V^*(\tau)$, simply because $|P_v^J| \ge 2$ for such vertices; moreover, all vertices v followed by exactly one endpoint had $(|P_v^\psi|, |P_v^J|) = (2, 1)$, so that $d_v = 0$; as a consequence we could safely sum over the scale labels. In the interacting case the presence of trees and graphs containing marginal or relevant clusters is inevitable, and this makes the Feynman diagram expansion useless, because it leads to bounds on e.g. $|E_\Lambda(\lambda, m)|$ that diverge as $m \to 0$ and $L \to \infty$ (recall that, in the sums on scales, $|h_v|$ ranges from 0 to $|h^*| \propto -\log m \gg -\log L$). In other words, the Feynman graph expansion is *not* sufficient for gaining control on the perturbative expansion at $\alpha \neq 0$, not even order by order in α.

On top of the problem of divergence of Feynman diagrams outlined above, there is also a combinatorial issue to be faced: even if we could sum every single Feynman diagram over the scale labels, we should still sum over the Feynman diagrams. However, assuming for definiteness that $n = 0$, the number of Feynman diagrams is at least $(\text{const.})^N (N!)^2$, where N is the total number of normal endpoints and we used the fact that every endpoint is associated with a vertex with 4 or more fermionic fields (i.e. half-lines), as well as the fact that the number of Feynman diagrams is equal to the number of possible Wick contractions of such fields (it is easy to see that the number of possible contractions of the half-lines exiting from N vertices, each with 4 external half-lines, scales like $(\text{const.})^N (N!)^2$, and even faster if we allow vertices with more than 4 external half-lines). On the other hand, the factor $\prod_{v \in V(\tau)} 1/s_v!$ in (5.8) behaves like $1/N!$ at large N, which means that the bound on the total contribution of order N grows like (assuming for simplicity that all endpoints have 4 external lines) $\alpha^N N!$, which is not summable in N, even for α small.

These two problems are the counterparts of analogous difficulties emerging in QFT. The divergence of Feynman diagrams with m as $m \to 0$ is called the *infrared problem*, and it signals that an expansion in α is not suitable for treating the interacting system at hand. Rather, we need to introduce scale-dependent parameters λ_h, Z_h which measure the effective strength of the interaction and of the propagator at scale h (in the language of field theory, Z_h is called "wave function renormalization"). The theory depends analytically on λ_h, Z_h, so that all the potential divergences of the theory are "absorbed" into the definition of the *running coupling constants* λ_h, Z_h, whose behavior can be studied in terms of a *finite-dimensional* discrete flow equation. For example, the iterative equation for λ_h leads a priori (i.e., on the basis of dimensional estimates of the contributions to $\beta_h^\lambda := \lambda_h - \lambda_{h-1}$, which are also expressed as a perturbation series in λ_h, Z_h) to a divergence of λ_h as $h \to -\infty$ (dimensionally, the divergence is linear in $|h|$); however, remarkable cancellations in the *beta function* β_h^λ allow one to show that λ_h reaches a fixed point close to α as $h \to -\infty$. The same cancellations are of course (a posteriori) present also in the original naive power series expansion, but are much less visible there.

Finally, let us comment about the combinatorial divergence due to the large number of diagrams: this divergence indicates that we should not simply expand in a sum over Feynman graphs, but rather over (resummed) families of such diagrams; in the fermionic context, the regrouping of Feynman diagrams into families leads to a *determinant* or Pfaffian expansion, which is better behaved combinatorially than the original expansion (see for instance [34, Section 4]). Roughly speaking, using the signs from the fermionic Wick rule we can regroup families of Feynman diagrams into determinants; the sum over Feynman diagrams is obtained by expanding the determinant along a row or column; however, it is better to estimate the determinant of an $n \times n$ matrix in terms of the maximal eigenvalue, rather than in terms of the sum over the $n!$ terms in the definition of the determinant.

Using well known methods coming from constructive QFT one can solve the above difficulties, as explained in the following section.

6. The interacting case: Non-perturbative multiscale construction

As discussed in the previous section, the perturbation theory in Feynman diagrams for the pressure and correlation functions of the model does not appear to be convergent in α, uniformly in L and m. In this section we show that at finite L and m we can reorganize the expansion, thus obtaining a new series, the *renormalized expansion*, which is not a power series in α anymore and has better convergence properties. In particular, it will allow us to show that the observables of interest are well-defined and analytic in α, uniformly as $L \to \infty$ and $m \to 0$ and to get Theorem 2 (see Section 6.4) and the corresponding statements for multi-dimer correlations (Section 7.2).

The renormalized expansion has been described in detail in several specialized and review papers in the last 20 years, see [9–12,34,50], and is reviewed and adapted to the present case in this section. In order to derive it, we proceed roughly speaking as follows: we first decompose the propagator in a way similar to (2.34) and we integrate step by step the propagator on scale $h = 0, -1, -2, \dots$. At each step, before integrating the next scale, we properly resum the expansion at hand, by isolating the divergent parts of the relevant and marginal contributions from the rest (the irrelevant terms); the relevant and marginal divergent parts are proportional to the running coupling constants, already mentioned at the end of the previous section. Moreover, at each step we express the effect of the integration on scale h in a way similar to (5.1), with the important difference that the truncated expectation in the r.h.s. of (5.1) is not written as a sum over Feynman diagrams, but rather as a sum over Pfaffians, each of which collects several contributions arising from different pairings. The resulting expansion takes the form of a multiscale Pfaffian expansion, expressed in terms of the running coupling constants, rather than in terms of α.

We start from the generating function with anti-periodic boundary conditions on the Grassmann variables in both coordinate directions: dropping, for notational simplicity, the label (11),

$$\mathcal{Z}_\Lambda(\lambda, m, \mathbf{A}) = \int P_\Lambda(d\psi) e^{V_\Lambda^{(0)}(\psi) + \mathcal{B}_\Lambda^{(0)}(\psi, \mathbf{J})}, \tag{6.1}$$

where $V_\Lambda^{(0)} + \mathcal{B}_\Lambda^{(0)}$ is obtained from $V_\Lambda + \mathcal{B}_\Lambda$ by re-expressing the original Grassmann fields in terms of Majorana fields, via (2.22). After the integration of the fields on scales $0, -1, \dots, h+1$, we recast (6.1) into a form similar to (6.1), with $V^{(0)}$ and $\mathcal{B}^{(0)}$ replaced by scale-dependent effective potentials, depending on the *infrared fields* $\psi^{(\le h)}$. This is expressed by the following lemma.

Proposition 7. *For any $h \le 0$, (6.1) can be rewritten as*

$$\mathcal{Z}_\Lambda = e^{E_\Lambda^{(h)} + S_\Lambda^{(h)}(\mathbf{J})} \int P_{Z_h, m_h, \chi_h}\big(d\psi^{(\le h)}\big) e^{V_\Lambda^{(h)}(\sqrt{Z_h}\psi^{(\le h)}) + \mathcal{B}_\Lambda^{(h)}(\sqrt{Z_h}\psi^{(\le h)}, \mathbf{J})}. \tag{6.2}$$

For $h = 0$, $E_\Lambda^{(0)} = S_\Lambda^{(0)}(\mathbf{J}) = 0$, $Z_0 = 1$, $m_0(\mathbf{k}) = m \cos k_1$, and $P_{Z_0, m_0, \chi_0}(d\psi)$ is the same as $P_\Lambda(d\psi)$, once written in the basis of the Majorana fields ψ_γ. If $h < 0$, the Gaussian integration $P_{Z_h, m_h, \chi_h}(d\psi^{(\le h)})$ has propagator

$$\frac{g^{(\le h)}(\mathbf{x} - \mathbf{y})}{Z_h} := \int_{\mathbb{T}^2} \frac{d\mathbf{k}}{(2\pi)^2} \frac{\chi_h(\mathbf{k})}{Z_h} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \begin{pmatrix} \hat{G}_{m_h}(\mathbf{k}) & 0 \\ 0 & \hat{G}_{m_h}(\mathbf{k}) \end{pmatrix}, \tag{6.3}$$

where χ_h was defined in (2.36) and

$$\hat{G}_{m_h}(\mathbf{k}) = \frac{1}{2} \begin{pmatrix} -i \sin k_1 + \sin k_2 & i m_h(\mathbf{k}) \\ -i m_h(\mathbf{k}) & -i \sin k_1 - \sin k_2 \end{pmatrix}^{-1}. \tag{6.4}$$

The constants $E_\Lambda^{(h)}$, Z_h, the functions $m_h(\mathbf{k})$ and the effective potentials $S_\Lambda^{(h)}(\mathbf{J})$, $V_\Lambda^{(h)}(\psi)$, $\mathcal{B}_\Lambda^{(h)}(\psi, \mathbf{J})$ are defined inductively in the course of the proof.

The kernels of the effective potential $V_\Lambda^{(h)}(\psi)$ are defined in terms of the following representation:

$$V_\Lambda^{(h)}(\psi) = \sum_{\substack{n \ge 1: \\ n \text{ even}}} \sum_{\gamma_1, \dots, \gamma_n} \sum_{\mathbf{x}_1, \dots, \mathbf{x}_n} W_{n, \boldsymbol{\gamma}}^{(h)}(\mathbf{x}_1, \dots, \mathbf{x}_n) \Big[\prod_{i=1}^n e^{i \mathbf{p}_{\gamma_i} \mathbf{x}_i} \psi_{\mathbf{x}_i, \gamma_i} \Big], \tag{6.5}$$

where $\boldsymbol{\gamma}$ is a shorthand for $(\gamma_1, \dots, \gamma_n)$, the sums over $\mathbf{x}_1, \dots, \mathbf{x}_n$ run over Λ, and $W^{(h)}_{n,\boldsymbol{\gamma}}$ depends on Λ (weakly, see comments after (6.7) below), but we drop the label Λ for simplicity of notation. Note that if we impose that

$$W^{(h)}_{n,(\gamma_1,\dots,\gamma_n)}(\mathbf{x}_1, \dots, \mathbf{x}_n) = (-1)^\pi W^{(h)}_{n,(\gamma_{\pi(1)},\dots,\gamma_{\pi(n)})}(\mathbf{x}_{\pi(1)}, \dots, \mathbf{x}_{\pi(n)}) \tag{6.6}$$

with π any permutation and $(-1)^\pi$ its signature, then the representation (6.5) is unique. Similarly,

$$\mathcal{B}^{(h)}_\Lambda(\psi, \mathbf{J}) = \sum_{\substack{n\geq 1:\\ n \text{ even}}} \sum_{q\geq 1} \sum_{\substack{\gamma_1,\dots,\gamma_n,\\ j_1,\dots,j_q}} \sum_{\substack{\mathbf{x}_1,\dots,\mathbf{x}_n\\ \mathbf{y}_1,\dots,\mathbf{y}_q}} W^{(h)}_{n,q,\boldsymbol{\gamma},\mathbf{j}}(\mathbf{x}_1, \dots, \mathbf{x}_n; \mathbf{y}_1, \dots, \mathbf{y}_q) \times \left[\prod_{i=1}^n e^{i\mathbf{p}_{\gamma_i}\mathbf{x}_i} \psi_{\mathbf{x}_i,\gamma_i}\right]\left[\prod_{i=1}^q J_{\mathbf{y}_i,j_i}\right], \tag{6.7}$$

where $J_{\mathbf{x},j}$ is an alternative symbol for J_b, with $b = (\mathbf{x}, \mathbf{x} + \hat{e}_j)$. The kernels $W^{(h)}_{q,\mathbf{j}}(\mathbf{y}_1, \dots, \mathbf{y}_q)$ of $S^{(h)}_\Lambda$ are defined analogously. All these kernels satisfy "natural" dimensional estimates that can be deduced from the discussion in Section 6.2.1 below (see in particular (6.71)). In a finite box, the kernels $W^{(h)}_{n,\boldsymbol{\gamma}}$, etc., depend weakly on the volume and mass, in the sense that for any $m > 0$ they reach their infinite volume limit exponentially fast, and these infinite-volume kernels admit a limit as $m \to 0$ (see also comments around (2.64)). Therefore, the finite-volume, finite-mass kernels are all bounded uniformly in L and m, provided $L \gg m^{-1} \gg 1$.

Remark 10 (Translation invariance properties of the kernels). *At the initial step, $h = 0$, the kernels $W^{(0)}_{n,q,\boldsymbol{\gamma},\mathbf{j}}$ and $W^{(0)}_{n,\boldsymbol{\gamma}}$ are obtained from (2.43) and (2.57) after re-expressing the field ψ in the Majorana basis, via (2.22). Because of the factors $t^{(m)}_b$ entering the definition of $E^{(m)}_b$, these kernels are not translation invariant. However, the non-translation invariant terms vanish at $m = 0$ (see e.g. the quartic terms in the r.h.s. of (2.52) as an illustration): therefore, $\mathcal{P}_0 W^{(0)}_{n,q,\boldsymbol{\gamma},\mathbf{j}} := W^{(0)}_{n,q,\boldsymbol{\gamma},\mathbf{j}}|_{m=0}$ is translation invariant (same for $\mathcal{P}_0 W^{(0)}_{n,\boldsymbol{\gamma}}$ and $\mathcal{P}_0 W^{(0)}_{q,\boldsymbol{j}}$), a fact that will be useful in the following. Similarly, for later convenience, we introduce the operator $\mathcal{P}_1$, which extracts the linear part in m from the kernel it acts on: $\mathcal{P}_1 W^{(0)}_{n,q,\boldsymbol{\gamma},\mathbf{j}} := m\partial_m W^{(0)}_{n,q,\boldsymbol{\gamma},\mathbf{j}}|_{m=0}$. It is easy to see that the kernels $\mathcal{P}_1 W^{(0)}_{n,q,\boldsymbol{\gamma},\mathbf{j}}(\mathbf{x}_1, \dots, \mathbf{x}_n; \mathbf{y}_1, \dots, \mathbf{y}_q)$ are translation invariant, up to an overall oscillatory factor $(-1)^{(\mathbf{x}_1)_1}$ (see again (2.52)). The same properties are valid for the kernels at lower scales, as it follows from the induction below.*

6.1. *Multi-scale integration (proof of Proposition 7)*

We proceed inductively. We already discussed the validity of (6.2) at the first step, $h = 0$. We now need to show how to go from scale h to $h - 1$. The first key step that we have to perform at each iteration is the localization procedure, which consists in isolating the potentially divergent contributions in $V^{(h)}_\Lambda$ and $\mathcal{B}^{(h)}_\Lambda$ from the rest (Sections 6.1.1 and 6.1.2); next we will rescale the Grassmann fields and finally we will integrate out the (rescaled) fields on scale h (Section 6.1.3).

6.1.1. *The localization procedure*

We write:

$$V^{(h)}_\Lambda = \mathcal{L}V^{(h)}_\Lambda + \mathcal{R}V^{(h)}_\Lambda, \qquad \mathcal{B}^{(h)}_\Lambda = \mathcal{L}\mathcal{B}^{(h)}_\Lambda + \mathcal{R}\mathcal{B}^{(h)}_\Lambda, \tag{6.8}$$

where $\mathcal{L}$, the localization operator, is a projection operator that acts linearly on the effective potential as described in the following. The operator $\mathcal{R}$ is called the renormalization operator: it extracts from $V^{(h)}_\Lambda + \mathcal{B}^{(h)}_\Lambda$ the well-behaved ("irrelevant") part. For simplicity, in the following we spell out the definitions of $\mathcal{L}$ and $\mathcal{R}$ in the $L \to \infty$ case only, the finite volume case being treatable in a similar, even though notationally more cumbersome, way, see e.g. [12, Eqs (2.74)–(2.75)]. Recall that the only potentially divergent diagrams in the multiscale expansion are those with $(|P^\psi_v|, |P^J_v|) = (2, 0), (4, 0), (2, 1)$ (see the discussion after (5.8)), the $(2, 0)$ terms being relevant, and $(4, 0), (2, 1)$

being marginal: therefore, $\mathcal{L}$ acts non-trivially only on these terms. More precisely, denoting by $V^{(h)}_{n,\Lambda}$ the n-legged contribution to the effective potential, i.e.,

$$V^{(h)}_{n,\Lambda}(\psi) = \sum_{\substack{\mathbf{x}_1,\dots,\mathbf{x}_n \\ \boldsymbol{\gamma}}} W^{(h)}_{n,\boldsymbol{\gamma}}(\mathbf{x}_1,\dots,\mathbf{x}_n)\Big[\prod_{i=1}^{n} e^{i\mathbf{p}_{\gamma_i}\mathbf{x}_i}\psi_{\mathbf{x}_i,\gamma_i}\Big] \tag{6.9}$$

we let (dropping the Λ label to indicate that we are formally giving the definition in the $L\to\infty$ case only)

$$\begin{aligned}\mathcal{L}V^{(h)}_2(\psi) &= \sum_{\substack{\mathbf{x},\mathbf{y}\\ \gamma,\gamma'}} e^{i\mathbf{p}_\gamma\mathbf{x}}\psi_{\mathbf{x},\gamma}\mathcal{P}_0 W^{(h)}_{2,(\gamma,\gamma')}(\mathbf{x},\mathbf{y})e^{i\mathbf{p}_{\gamma'}\mathbf{y}}\big[1+(\mathbf{y}-\mathbf{x})\cdot\hat{\boldsymbol{\partial}}\big]\psi_{\mathbf{x},\gamma'}\\ &\quad+\sum_{\substack{\mathbf{x},\mathbf{y}\\ \gamma,\gamma'}} e^{i\mathbf{p}_\gamma\mathbf{x}}\psi_{\mathbf{x},\gamma}\mathcal{P}_1 W^{(h)}_{2,(\gamma,\gamma')}(\mathbf{x},\mathbf{y})e^{i\mathbf{p}_{\gamma'}\mathbf{y}}\psi_{\mathbf{x},\gamma'}\end{aligned} \tag{6.10}$$

and

$$\mathcal{L}V^{(h)}_4(\psi) = \sum_{\substack{\mathbf{x}_1,\dots,\mathbf{x}_4\\ \gamma_1,\dots,\gamma_4}} \mathcal{P}_0 W^{(h)}_{4,\boldsymbol{\gamma}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)\Big[\prod_{i=1}^{4} e^{i\mathbf{p}_{\gamma_i}\mathbf{x}_i}\psi_{\mathbf{x}_1,\gamma_i}\Big], \tag{6.11}$$

while $\mathcal{L}V^{(h)}_n(\psi)=0$, $\forall n>4$. In the first line of (6.10), $\hat{\boldsymbol{\partial}}$ indicates the symmetric discrete gradient, whose ith component acts on lattice functions as $\hat{\partial}_i f(\mathbf{x}) = \frac{1}{2}(f(\mathbf{x}+\hat{e}_i)-f(\mathbf{x}-\hat{e}_i))$. Note that all the fields appearing in these formulas are *localized* at the same point, or at two points at a distance 1, which justifies the name of *localization operator* for $\mathcal{L}$. The action of $\mathcal{L}$ on the source term is defined similarly (and it acts non-trivially only on the term with two ψ and one J fields):

$$\mathcal{L}\mathcal{B}^{(h)}(\psi) = \sum_{\substack{\mathbf{x},\mathbf{y},\mathbf{z}\\ \gamma,\gamma',j}} e^{i\mathbf{p}_\gamma\mathbf{x}}\psi_{\mathbf{z},\gamma}\mathcal{P}_0 W^{(h)}_{2,1,(\gamma,\gamma'),j}(\mathbf{x},\mathbf{y};\mathbf{z})e^{i\mathbf{p}_{\gamma'}\mathbf{y}}\psi_{\mathbf{z},\gamma'}J_{\mathbf{z},j}. \tag{6.12}$$

The rationale behind the definition of $\mathcal{L}$ is that it guarantees that: (1) the action of $\mathcal{R}=1-\mathcal{L}$ on the kernels produces a dimensional gain, which is enough to make the analogue of the dimensional estimate (5.8) for renormalized graphs (i.e. graphs such that each non-trivial subgraph is renormalized by the action of $\mathcal{R}$) convergent; (2) the algebraic structure of $\mathcal{L}$ is sufficiently simple that the linear space spanned by $\mathcal{L}(V+\mathcal{B})$ is finite dimensional, i.e., it can be parametrized by a finite number of constants. The fact that the action of $\mathcal{R}=1-\mathcal{L}$ on the kernels produces a dimensional gain has been discussed in several books and review papers, see e.g. [10,34]. A heuristic explanation of this point, adapted to the present case, is discussed at the end of the present section, see Section 6.1.4 below.

6.1.2. *The structure of the local terms*

Let us now discuss the explicit structure of the local terms, and let us show that they are parametrized by a finite number of constants. We define

$$K^{(h)}_{2,\boldsymbol{\gamma}}(\mathbf{x}-\mathbf{y}) := \mathcal{P}_0 W^{(h)}_{2,\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y}), \tag{6.13}$$

$$K^{(h)}_{4,\boldsymbol{\gamma}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) := \mathcal{P}_0 W^{(h)}_{4,\boldsymbol{\gamma}}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4), \tag{6.14}$$

$$M^{(h)}_{2,\boldsymbol{\gamma}}(\mathbf{x}-\mathbf{y}) := (-1)^{y_1}\mathcal{P}_1 W^{(h)}_{2,\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y}), \tag{6.15}$$

$$B^{(h)}_{2,1,\boldsymbol{\gamma},j}(\mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{z}) := \mathcal{P}_0 W^{(h)}_{2,1,\boldsymbol{\gamma},j}(\mathbf{x},\mathbf{y};\mathbf{z}), \tag{6.16}$$

so that $K^{(h)}_{2,\boldsymbol{\gamma}}$, $K^{(h)}_{4,\boldsymbol{\gamma}}$ and $B^{(h)}_{2,1,\boldsymbol{\gamma},j}$ are independent of m, while $M^{(h)}_{2,\boldsymbol{\gamma}}$ is linear in m. They are all translation invariant. Let us separately rewrite in a more compact way the contributions to the local part of the effective potential associated with

these kernels. As an illustration, let us consider the contribution to the local part of the effective potential associated with $K^{(h)}_{2,\boldsymbol{\gamma}}$, which can be rewritten as (again, we provide formulas only in the $L \to \infty$ limit; we also add the apex $(\le h)$ to the fields to recall that they are on scale $\le h$)

$$\sum_{\mathbf{x},\mathbf{y}} \sum_{\gamma,\gamma'} e^{i\mathbf{p}_\gamma \mathbf{x}} \psi^{(\le h)}_{\mathbf{x},\gamma} K^{(h)}_{2,(\gamma,\gamma')}(\mathbf{x}-\mathbf{y}) e^{i\mathbf{p}_{\gamma'}\mathbf{y}} \big[1 + (\mathbf{y}-\mathbf{x})\cdot \hat{\boldsymbol{\partial}}\big] \psi^{(\le h)}_{\mathbf{x},\gamma'}$$

$$= \sum_{\gamma,\gamma'} \int \frac{d\mathbf{k} d\mathbf{k}'}{(2\pi)^2} \left(\hat K^{(h)}_{2,(\gamma,\gamma')}(\mathbf{p}_{\gamma'}) + \sum_{j=1}^{2} \sin k'_j \partial_{k_j} \hat K^{(h)}_{2,(\gamma,\gamma')}(\mathbf{p}_{\gamma'}) \right)$$

$$\times \hat\psi^{(\le h)}_{-\mathbf{k},\gamma} \hat\psi^{(\le h)}_{\mathbf{k}',\gamma'} \delta\big(\mathbf{k} + \mathbf{p}_\gamma - \mathbf{k}' - \mathbf{p}_{\gamma'}\big), \tag{6.17}$$

where, as in (2.18), the integrals over $\mathbf{k}, \mathbf{k}'$ run over the torus $\mathbb{T}^2$, δ is a periodic Dirac delta over the torus,

$$\hat\psi^{(\le h)}_{\mathbf{k},\gamma} := \sum_{\mathbf{x}} e^{i\mathbf{k}\mathbf{x}} \psi^{(\le h)}_{\mathbf{x},\gamma} \tag{6.18}$$

and $\hat K^{(h)}_{2,(\gamma,\gamma')}(\mathbf{k}) := \sum_{\mathbf{x}} e^{i\mathbf{k}\mathbf{x}} K^{(h)}_{2,(\gamma,\gamma')}(\mathbf{x})$. In finite volume, integrals are replaced by discrete sums as in (2.16).

Claim 1. *We have*

$$\mathcal{L} V^{(h)}_2(\psi) = \int \frac{d\mathbf{k}}{(2\pi)^2} \hat\psi^T_{-\mathbf{k}} C_h(\mathbf{k}) \hat\psi_{\mathbf{k}}, \tag{6.19}$$

where $\hat\psi_{\mathbf{k}}$ is a column vector with components $\hat\psi_{\mathbf{k},\gamma}$, $\gamma = 1,2,3,4$, and $C_h(\mathbf{k})$ is a block-diagonal matrix of the form:

$$C_h(\mathbf{k}) = \begin{pmatrix} c_h(\mathbf{k}) & 0 \\ 0 & c_h(\mathbf{k}) \end{pmatrix} \tag{6.20}$$

with

$$c_h(\mathbf{k}) = -\begin{pmatrix} z_h(-i\sin k_1 + \sin k_2) & i\sigma_h \\ -i\sigma_h & z_h(-i\sin k_1 - \sin k_2) \end{pmatrix} \tag{6.21}$$

for some $z_h, \sigma_h \in \mathbb{R}$ with $z_0 = \sigma_0 = 0$. Moreover, there exist real constants l_h, $Z^{(1)}_h$, $Z^{(2)}_h$ such that

$$\mathcal{L} V^{(h)}_4(\psi) = l_h \sum_{\mathbf{x}} \psi_{\mathbf{x},1}\psi_{\mathbf{x},2}\psi_{\mathbf{x},3}\psi_{\mathbf{x},4}, \tag{6.22}$$

$$\mathcal{L} \mathcal{B}^{(h)}(\psi) = \frac{Z^{(1)}_h}{Z_h} F^{(1)}(\psi, \mathbf{J}) + \frac{Z^{(2)}_h}{Z_h} F^{(2)}(\psi, \mathbf{J}) \tag{6.23}$$

and

$$F^{(1)}(\psi,\mathbf{J}) = 2i \sum_{\mathbf{x}} (-1)^{x_1+x_2} \big[J_{\mathbf{x},1}(\psi_{\mathbf{x},1}\psi_{\mathbf{x},3} + \psi_{\mathbf{x},2}\psi_{\mathbf{x},4}) + i J_{\mathbf{x},2}(\psi_{\mathbf{x},1}\psi_{\mathbf{x},3} - \psi_{\mathbf{x},2}\psi_{\mathbf{x},4}) \big], \tag{6.24}$$

$$F^{(2)}(\psi,\mathbf{J}) = 2i \sum_{\mathbf{x}} \big[J_{\mathbf{x},1}(-1)^{x_1+1}(\psi_{\mathbf{x},1}\psi_{\mathbf{x},2} + \psi_{\mathbf{x},3}\psi_{\mathbf{x},4}) + J_{\mathbf{x},2}(-1)^{x_2}(\psi_{\mathbf{x},1}\psi_{\mathbf{x},4} + \psi_{\mathbf{x},2}\psi_{\mathbf{x},3}) \big]. \tag{6.25}$$

Z_h is the same as in (6.2)–(6.3) and is inserted in (6.23) for later convenience.

Remark 11. *For future reference, it is useful to give here the expressions of Eqs* (6.22) *through* (6.25) *in terms of Dirac variables, via* (2.30):

$$\mathcal{L}V_4^{(h)}(\psi) = l_h \sum_{\mathbf{x}} \psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},1}\psi^+_{\mathbf{x},-1}\psi^-_{\mathbf{x},-1}, \tag{6.26}$$

$$F^{(1)} = \sum_{\mathbf{x},\omega} J^{(1)}_\omega(\mathbf{x})\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}, \qquad J^{(1)}_\omega(\mathbf{x}) := 2(-1)^{\mathbf{x}}(J_{\mathbf{x},1} + i\omega J_{\mathbf{x},2}), \tag{6.27}$$

$$F^{(2)} = \sum_{\mathbf{x},\omega} J^{(2)}_\omega(\mathbf{x})\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega}, \qquad J^{(2)}_\omega(\mathbf{x}) := 2\big(J_{\mathbf{x},1}(-1)^{x_1} + i\omega J_{\mathbf{x},2}(-1)^{x_2}\big). \tag{6.28}$$

Proof of Claim 1. Consider first the case $h = 0$. Then, $\mathcal{L}V_2^{(0)} = 0$ (so that $z_0 = \sigma_0 = 0$) simply because the effective potential V_Λ contains no bilinear term in the fields, cf. (2.52). Suppose instead that $h \le -1$. Then, since we are assuming the statement of Proposition 7 at scale h, the field $\hat\psi^{(\le h)}_{\mathbf{k},\gamma}$ has the same support as $\chi_h(\mathbf{k}) = \chi(2^{-h}\mathbf{k})$ (in the sense that its propagator has this support, cf. (6.3)). Remember from the discussion after (2.21) that the support of $\chi(\cdot)$ is essentially $\{\mathbf{k} \in \mathbb{T}^2 : \|\mathbf{k}\| \le \pi/2\}$, where $\|\cdot\|$ is the Euclidean distance on $\mathbb{T}^2$. Then the only non-vanishing terms in (6.17) are the diagonal ones, i.e., those with $\gamma = \gamma'$. Note that the term $\hat K^{(h)}_{2,(\gamma,\gamma)}$ gives zero contribution, since $\int d\mathbf{k}\, d\mathbf{k}' \hat\psi_{-\mathbf{k},\gamma}\hat\psi_{\mathbf{k}',\gamma} = 0$ by anticommutation.

In a similar way we find that the contribution to the local part of the effective potential associated with $M^{(h)}_{2,\boldsymbol{\gamma}}$ can be rewritten as

$$\sum_{\mathbf{x},\mathbf{y}}\sum_{\gamma,\gamma'} e^{i\mathbf{p}_\gamma\mathbf{x}}\psi^{(\le h)}_{\mathbf{x},\gamma} M^{(h)}_{2,(\gamma,\gamma')}(\mathbf{x}-\mathbf{y})e^{i(\mathbf{p}_{\gamma'}+(\pi,0))\mathbf{y}}\psi^{(\le h)}_{\mathbf{x},\gamma'}$$
$$= \sum_{\gamma,\gamma'}\int \frac{d\mathbf{k}\, d\mathbf{k}'}{(2\pi)^2}\hat\psi^{(\le h)}_{-\mathbf{k},\gamma}\hat M^{(h)}_{2,(\gamma,\gamma')}\big(\mathbf{p}_{\gamma'} + (\pi,0)\big)\hat\psi^{(\le h)}_{\mathbf{k}',\gamma'}\delta\big(\mathbf{k} + \mathbf{p}_\gamma - \mathbf{k}' - \mathbf{p}_{\gamma'} - (\pi,0)\big). \tag{6.29}$$

Thanks to the above mentioned properties of the support of the field $\hat\psi^{(\le h)}_{\mathbf{k},\gamma}$, the only non-vanishing terms in (6.29) are those with $(\gamma,\gamma') = (1,2), (2,1), (3,4), (4,3)$. In terms of these definitions and properties we can rewrite $\mathcal{L}V_2^{(h)}$ as (6.19), with

$$C_h(\mathbf{k}) = \begin{pmatrix} c_h(\mathbf{k}) & 0 \\ 0 & d_h(\mathbf{k}) \end{pmatrix}, \tag{6.30}$$

$$c_h(\mathbf{k}) = \begin{pmatrix} a^{(h)}_1 \sin k_1 + b^{(h)}_1 \sin k_2 & \sigma^{(h)}_{1,2} \\ \sigma^{(h)}_{2,1} & a^{(h)}_2 \sin k_1 + b^{(h)}_2 \sin k_2 \end{pmatrix}, \tag{6.31}$$

$$d_h(\mathbf{k}) = \begin{pmatrix} a^{(h)}_3 \sin k_1 + b^{(h)}_3 \sin k_2 & \sigma^{(h)}_{3,4} \\ \sigma^{(h)}_{4,3} & a^{(h)}_4 \sin k_1 + b^{(h)}_4 \sin k_2 \end{pmatrix} \tag{6.32}$$

and: $a^{(h)}_\gamma = \partial_{k_1}\hat K^{(h)}_{2,(\gamma,\gamma)}(\mathbf{p}_\gamma)$, $b^{(h)}_\gamma = \partial_{k_2}\hat K^{(h)}_{2,(\gamma,\gamma)}(\mathbf{p}_\gamma)$, $\sigma^{(h)}_{\gamma,\gamma'} = \frac{1}{2}(\hat M^{(h)}_{2,(\gamma,\gamma')}(\mathbf{p}_\gamma) - \hat M^{(h)}_{2,(\gamma',\gamma)}(\mathbf{p}_{\gamma'}))$. Even more: by using the symmetries of the Grassmann action and of the propagator, one can check (see Appendix B for some details) that:

- $a^{(h)}_\gamma$ is independent of γ and purely imaginary: i.e., $a^{(h)}_\gamma = iz_h$ for some real constant z_h;
- $b^{(h)}_\gamma = (-1)^\gamma i a^{(h)}_\gamma$, so that $b^{(h)}_\gamma = (-1)^{\gamma-1}z_h$, for the same constant z_h;
- $\sigma^{(h)}_{1,2} = -\sigma^{(h)}_{2,1} = \sigma^{(h)}_{3,4} = -\sigma^{(h)}_{4,3} = i\sigma_h$, for some real constant σ_h (the fact that $\sigma_{1,2} = -\sigma_{2,1}$ and $\sigma_{3,4} = -\sigma_{4,3}$ is obvious from the definition).

Therefore, in (6.30), $c_h(\mathbf{k}) = d_h(\mathbf{k})$, and $c_h(\mathbf{k})$ is of the form (6.21).

As far as the quartic and source local terms are concerned, we find (6.22) with

$$l_h = \sum_{\substack{\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4 \\ \pi\in S_4}} (-1)^\pi K^{(h)}_{4,(\pi(1),\pi(2),\pi(3),\pi(4))}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4) \prod_{j=1}^4 e^{i\mathbf{p}_{\pi(j)}\mathbf{x}_j}, \tag{6.33}$$

where S_4 is the set of permutations of $(1,2,3,4)$, and

$$\mathcal{LB}^{(h)}(\psi) = \sum_{\gamma<\gamma'}\sum_j Z_{h;(\gamma,\gamma'),j} \sum_{\mathbf{z}} e^{i(\mathbf{p}_\gamma+\mathbf{p}_{\gamma'})\mathbf{z}} \psi_{\mathbf{z},\gamma}\psi_{\mathbf{z},\gamma'} J_{\mathbf{z},j}, \tag{6.34}$$

where the constants $Z_{h;(\gamma,\gamma'),j}$ are

$$Z_{h;(\gamma,\gamma'),j} = \hat B^{(h)}_{2,1,(\gamma,\gamma'),j}(\mathbf{p}_{\gamma'}, \mathbf{p}_\gamma - \mathbf{p}_{\gamma'}) - \hat B^{(h)}_{2,1,(\gamma',\gamma),j}(\mathbf{p}_\gamma, \mathbf{p}_{\gamma'} - \mathbf{p}_\gamma). \tag{6.35}$$

Using again the symmetries of the Grassmann action given in Appendix B we find that the constant l_h is real, while the constants $Z_{h;(\gamma,\gamma'),j}$ are such that the source term takes the form (6.23) where $Z_h^{(1)}$, $Z_h^{(2)}$ are real. □

Remark 12. *Note that $\mathcal{L}V_2^{(h)}$ has the same structure as the inverse of the propagator in (2.23)–(2.24), and it is parametrized just by two real constants z_h and σ_h. In conclusion, thanks to the way $\mathcal{L}$ is defined and to the symmetry of the theory, the local part of the effective potential is parametrized by 5 real constants, namely z_h, σ_h, l_h, $Z_h^{(1)}$, $Z_h^{(2)}$. These constants are all independent of m, except σ_h, which is exactly linear in m. As we shall see in the following, the terms proportional to z_h and σ_h are inserted step by step into the Gaussian integration, thus "dressing" iteratively the propagator at scale h.*

6.1.3. *The integration of the fields on scale h*

We resume the proof of Proposition 7 and we proceed with the inductive proof of (6.2). We assume the representation to be valid at scale h; since $\mathcal{L}V^{(h)}_{2,\Lambda}$ is bilinear in the fields and has antisymmetric kernel, we can apply (2.15) to write

$$P_{Z_h,m_h,\chi_h}\big(d\psi^{(\le h)}\big) e^{\mathcal{L}V^{(h)}_{2,\Lambda}(\sqrt{Z_h}\psi^{(\le h)})} = e^{t^{(h)}_\Lambda} P_{\tilde Z_{h-1},m_{h-1},\chi_h}\big(d\psi^{(\le h)}\big), \tag{6.36}$$

where $\exp(t^{(h)}_\Lambda)$ corresponds to the factor $\sqrt{\det(1-MV)}$ in (2.15), with $V = 2\mathcal{L}V_2^{(h)}(\sqrt{Z_h}\psi^{(\le h)})$, and accounts for the change in the normalization of the two Gaussian Grassmann integrations. To be allowed to apply (2.15) we have to check that $\det(1-\mu MV) > 0$ for every $\mu\in[0,1]$. It is not hard to check that this is satisfied if $\mathcal{L}V_2^{(h)}$ has the symmetry structure summarized in Claim 1, with z_h small and $m_{h-1}(\mathbf{0})/m_h(\mathbf{0})$ close to 1, uniformly in h. We will see later (Remark 18) that this is indeed the case, provided λ is small enough.

The matrix M' appearing in (2.15) can be computed immediately in Fourier space, and we obtain after some algebra that the "dressed" measure $P_{\tilde Z_{h-1},m_{h-1},\chi_h}(d\psi^{(\le h)})$ has a propagator similar to (6.3), namely

$$\int_{\mathbb{T}^2} \frac{d\mathbf{k}}{(2\pi)^2} \frac{\chi_h(\mathbf{k})}{\tilde Z_{h-1}(\mathbf{k})} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \begin{pmatrix} \hat G_{m_{h-1}}(\mathbf{k}) & 0 \\ 0 & \hat G_{m_{h-1}}(\mathbf{k}) \end{pmatrix} =: \frac{\tilde g^{(\le h)}(\mathbf{x}-\mathbf{y})}{Z_{h-1}}, \tag{6.37}$$

where

$$\tilde Z_{h-1}(\mathbf{k}) := Z_h\big(1+z_h\chi_h(\mathbf{k})\big), \tag{6.38}$$

$$m_{h-1}(\mathbf{k}) := \frac{Z_h}{\tilde Z_{h-1}(\mathbf{k})}\big(m_h(\mathbf{k}) + \sigma_h\chi_h(\mathbf{k})\big), \tag{6.39}$$

$$Z_{h-1} := \tilde Z_{h-1}(\mathbf{0}) = Z_h(1+z_h), \tag{6.40}$$

and r.h.s. of (6.37) defines $\tilde{g}^{(\le h)}$. The constants z_h, σ_h are computed from the effective potential at scale h, following the procedure explained in the proof of Claim 1. We can therefore rewrite (6.2) as

$$\mathcal{Z}_\Lambda = e^{E_\Lambda^{(h)}+t_\Lambda^{(h)}+S_\Lambda^{(h)}(\mathbf{J})}\int P_{\tilde{Z}_{h-1},m_{h-1},\chi_h}\big(d\psi^{(\le h)}\big) \times e^{\mathcal{L}V_{4,\Lambda}^{(h)}(\sqrt{Z_h}\psi^{(\le h)})+\mathcal{L}\mathcal{B}_\Lambda^{(h)}(\sqrt{Z_h}\psi^{(\le h)},\mathbf{J})+\mathcal{R}V_\Lambda^{(h)}(\sqrt{Z_h}\psi^{(\le h)})+\mathcal{R}\mathcal{B}_\Lambda^{(h)}(\sqrt{Z_h}\psi^{(\le h)},\mathbf{J})}. \tag{6.41}$$

Remark 13. *Inductively, we see that $m_h(\mathbf{k})$ is linear in m, simply because σ_h is linear in m, and $m_0(\mathbf{k}) = m\cos k_1$. Therefore, the propagator at $m = 0$ is massless: this is an instance of the fact that our theory remains critical at $m = 0$, irrespective of the value of the interaction λ.*

We now apply the "addition formula" (2.14) to split $P_{\tilde{Z}_{h-1},m_{h-1},\chi_h}$ as:

$$P_{\tilde{Z}_{h-1},m_{h-1},\chi_h}\big(d\psi^{(\le h)}\big) = P_{Z_{h-1},m_{h-1},\chi_{h-1}}\big(d\psi^{(\le h-1)}\big)P_{Z_{h-1},m_{h-1},\tilde{f}_h}\big(d\psi^{(h)}\big), \tag{6.42}$$

where the propagator of $P_{Z_{h-1},m_{h-1},\tilde{f}_h}$ is

$$\frac{g^{(h)}(\mathbf{x}-\mathbf{y})}{Z_{h-1}} = \int_{\mathbb{T}^2}\frac{d\mathbf{k}}{(2\pi)^2}\frac{\tilde{f}_h(\mathbf{k})}{Z_{h-1}}e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}\begin{pmatrix}\hat{G}_{m_{h-1}}(\mathbf{k}) & 0\\ 0 & \hat{G}_{m_{h-1}}(\mathbf{k})\end{pmatrix} \tag{6.43}$$

and

$$\tilde{f}_h(\mathbf{k}) = Z_{h-1}\left[\frac{\chi_h(\mathbf{k})}{\tilde{Z}_{h-1}(\mathbf{k})} - \frac{\chi_{h-1}(\mathbf{k})}{Z_{h-1}}\right]. \tag{6.44}$$

(To prove (6.42), just check that $\tilde{g}^{(\le h)}(\mathbf{x}-\mathbf{y})/Z_{h-1} = g^{(\le h-1)}(\mathbf{x}-\mathbf{y})/Z_{h-1} + g^{(h)}(\mathbf{x}-\mathbf{y})/Z_{h-1}$).

Note that $\tilde{f}_h(\mathbf{k})$ has the same support as $f_h(\mathbf{k}) = \chi_h(\mathbf{k}) - \chi_{h-1}(\mathbf{k})$ defined in (2.35), in fact (using (6.40) and the fact that $\chi_{h-1}(\mathbf{k})\chi_h(\mathbf{k}) = \chi_{h-1}(\mathbf{k})$)

$$\tilde{f}_h(\mathbf{k}) = f_h(\mathbf{k})\frac{1+z_h}{1+z_h\chi_h(\mathbf{k})} \ge 0. \tag{6.45}$$

Note also that $g^{(h)}$ satisfies the same estimate as in Lemma 2 (with m replaced by $m_h(\mathbf{0})$ and possibly with different constants C, c) provided that z_h in (6.45) stays uniformly small for all scales $h \le 0$. We now rescale the fields and define

$$\begin{aligned}\widehat{V}_\Lambda^{(h)}\big(\sqrt{Z_{h-1}}\psi^{(\le h)}\big) &:= \mathcal{L}V_{4,\Lambda}^{(h)}\big(\sqrt{Z_h}\psi^{(\le h)}\big) + \mathcal{R}V_\Lambda^{(h)}\big(\sqrt{Z_h}\psi^{(\le h)}\big),\\ \widehat{\mathcal{B}}_\Lambda^{(h)}\big(\sqrt{Z_{h-1}}\psi^{(\le h)}\big) &:= \mathcal{B}_\Lambda^{(h)}\big(\sqrt{Z_h}\psi^{(\le h)}\big).\end{aligned} \tag{6.46}$$

It follows that

$$\mathcal{L}\widehat{V}_\Lambda^{(h)}(\psi) = \lambda_h\sum_{\mathbf{x}}\psi_{\mathbf{x},1}\psi_{\mathbf{x},2}\psi_{\mathbf{x},3}\psi_{\mathbf{x},4} =: \lambda_h F_\lambda(\psi), \tag{6.47}$$

$$\mathcal{L}\widehat{\mathcal{B}}_\Lambda^{(h)}(\psi) = \frac{Z_h^{(1)}}{Z_{h-1}}F^{(1)}(\psi,\mathbf{J}) + \frac{Z_h^{(2)}}{Z_{h-1}}F^{(2)}(\psi,\mathbf{J}), \tag{6.48}$$

with

$$\lambda_h = \left(\frac{Z_h}{Z_{h-1}}\right)^2 l_h. \tag{6.49}$$

A simple computation (simply based on (2.22) and (2.52), plus the observation that $Z_{-1} = Z_0 = 1$: recall from Claim 1 that $z_0 = 0$) shows that

$$\lambda_0 = l_0 = -32\alpha = -32\big(e^{\lambda} - 1\big). \tag{6.50}$$

Similarly, we find $Z_0^{(1)} = Z_0^{(2)} = 1$. We now define (recall the decomposition $\psi^{(\le h)} = \psi^{(h)} + \psi^{(\le h-1)}$ as in (6.42))

$$\begin{aligned} & e^{V_\Lambda^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)}) + \mathcal{B}_\Lambda^{(h-1)}(\sqrt{Z_{h-1}}\psi^{(\le h-1)}, \mathbf{J}) + \tilde{E}_\Lambda^{(h-1)} + \tilde{S}_\Lambda^{(h-1)}(\mathbf{J})} \\ & \quad = \int P_{Z_{h-1}, m_{h-1}, \tilde{f}_h}\big(d\psi^{(h)}\big) e^{\lambda_h F_\lambda(\sqrt{Z_{h-1}}\psi^{(\le h)}) + \sum_{j=1}^2 Z_h^{(j)} F_j(\psi^{(\le h)}, \mathbf{J})} \\ & \qquad \times e^{\mathcal{R}\widehat{V}_\Lambda^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)}) + \mathcal{R}\widehat{\mathcal{B}}_\Lambda^{(h)}(\sqrt{Z_{h-1}}\psi^{(\le h)}, \mathbf{J})} \end{aligned} \tag{6.51}$$

with the constants fixed by the convention that $\mathcal{B}_\Lambda^{(h-1)}(\psi, \mathbf{0}) = V_\Lambda^{(h-1)}(0) = \tilde{S}_\Lambda^{(h-1)}(\mathbf{0}) = 0$, which proves (6.2) with h replaced by $h-1$, if one sets $E_\Lambda^{(h-1)} = E_\Lambda^{(h)} + t_\Lambda^{(h)} + \tilde{E}_\Lambda^{(h-1)}$ and $S_\Lambda^{(h-1)}(\mathbf{J}) = S_\Lambda^{(h)}(\mathbf{J}) + \tilde{S}_\Lambda^{(h-1)}(\mathbf{J})$.

Using (6.51) and the definition of truncated expectation (cf. e.g. (3.5)), we can rewrite

$$\begin{aligned} & V_\Lambda^{(h-1)}(\sqrt{Z_{h-1}}\psi) + \mathcal{B}_\Lambda^{(h-1)}(\sqrt{Z_{h-1}}\psi, \mathbf{J}) + \tilde{E}_\Lambda^{(h-1)} + \tilde{S}_\Lambda^{(h-1)}(\mathbf{J}) \\ & \quad = \sum_{s \ge 1} \frac{1}{s!} \mathcal{E}_h^T \big(\underbrace{\widehat{V}_\Lambda^{(h)}\big(\sqrt{Z_{h-1}}\big(\psi + \psi^{(h)}\big), \mathbf{J}\big); \dots; \widehat{V}_\Lambda^{(h)}\big(\sqrt{Z_{h-1}}\big(\psi + \psi^{(h)}\big), \mathbf{J}\big)}_{s \text{ times}} \big), \end{aligned} \tag{6.52}$$

where $\mathcal{E}_h^T$ is the truncated expectation with respect to the propagator $g^{(h)}/Z_{h-1}$ of the field $\psi^{(h)}$ (cf. (6.43)), and $\widehat{V}_\Lambda^{(h)}(\psi, \mathbf{J})$ is a shorthand for $\widehat{V}_\Lambda^{(h)}(\psi) + \widehat{\mathcal{B}}_\Lambda^{(h)}(\psi, \mathbf{J})$. This concludes the proof of Proposition 7. □

Remark 14 (The beta function). *The above procedure allows us to write the* effective constants $\xi_h := (\lambda_h, Z_h, m_h(\mathbf{0}), Z_h^{(1)}, Z_h^{(2)})$ *with* $h \le 0$, *in terms of* ξ_k *with* $h < k \le 0$:

$$\begin{aligned} & \lambda_{h-1} = \lambda_h + \beta_h^\lambda, \qquad \frac{Z_{h-1}}{Z_h} = 1 + \beta_h^Z, \qquad \frac{m_{h-1}(\mathbf{0})}{m_h(\mathbf{0})} = 1 + \beta_h^m, \\ & \frac{Z_{h-1}^{(1)}}{Z_h^{(1)}} = 1 + \beta_h^{Z,1}, \qquad \frac{Z_{h-1}^{(2)}}{Z_h^{(2)}} = 1 + \beta_h^{Z,2}, \end{aligned} \tag{6.53}$$

where $\beta_h^\# = \beta_h^\#(\xi_h, \dots, \xi_0)$ *is the so-called* beta function. *By construction,* β_h^λ *and* β_h^Z *depend only on* (λ_k, Z_k), $k \ge h$. *Therefore, the first two equations can be solved independently of the others and their solution can be plugged into the other three. Note also that, applying iteratively* (6.53), *and recalling that* $\xi_0 = (-32(e^\lambda - 1), 1, m, 1, 1)$, *one can also see* $\beta_h^\#$ *as a function of* λ *and* (*a priori*) m. *However, by definition and the fact that* $m_{h-1}(\mathbf{0})$ *is linear in* $\{m_k(\mathbf{0})\}_{k \ge h}$, *we see that* $\beta_h^\#$ *is independent of* m, *that is, it only depends on* λ.

Proposition 7 is valid for all $h < 0$. However, it is convenient to use it only for scales $h \ge h^*$, where h^* is the first scale (with respect to the ordering $h = 0, -1, -2, \dots$) such that[6] $m_{h-1}(\mathbf{0}) > 2^h$. When we reach scale h^*, we note that the propagator $\tilde{g}^{(\le h^*)}/Z_{h^*-1}$ (see (6.37)) admits the same dimensional estimates as $g^{(h^*)}/Z_{h^*-1}$ of (6.43). The two propagators differ mainly because in the former the cut-off function is $\chi(2^{-h^*}\mathbf{k})$ and in the latter it is $f_{h^*}(\mathbf{k}) \simeq \chi(2^{-h^*}\mathbf{k}) - \chi(2^{-h^*+1}\mathbf{k})$, so that momenta below 2^{h^*} are absent in the second. However, the mass $m_{h^*-1}(\mathbf{0})$ is bounded from below by 2^{h^*} and it effectively cuts-off momenta below 2^{h^*} also in $\tilde{g}^{(\le h^*)}/Z_{h^*-1}$. Therefore, one can integrate all at once all the scales $\le h^*$, thus obtaining the contribution to the pressure and to the generating function from this last step, $\tilde{E}_\Lambda^{(h^*-1)} + \tilde{S}_\Lambda^{(h^*-1)}(\mathbf{J})$.

[6] Note that this definition is slightly different from the one given in Section 5.2, which referred to the non-renormalized expansion, where the mass was not modified iteratively under the RG flow. The correct one, used from now on and keeping into account the mass renormalization, is the current one. With some abuse of notation we indicate it by the same symbol.

6.1.4. *Dimensional gains associated with the action of $\mathcal{R}$*

Let us now turn to the discussion (promised after (6.12)) of why the localization procedure produces the right dimensional gains, required for making the multiscale expansion of the effective potentials convergent. Recall from Section 5.2.2 that the possible divergences in the tree expansion come from vertices $v \in V(\tau)$ with $(|P_v^{\psi}|, |P_v^{J}|) = (2,0), (4,0), (2,1)$. We focus on the case $(4,0)$ (quartic kernels). Consider the combination

$$V_4^{(h_v)}\big(\psi^{(\le h_v)}\big) = \sum_{\substack{\mathbf{x}_1,\dots,\mathbf{x}_4 \\ \gamma_1,\dots,\gamma_4}} W_{4,\boldsymbol{\gamma}}^{(h_v)}(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_4)\Big[\prod_{i=1}^{4} e^{i\mathbf{p}_{\gamma_i}\mathbf{x}_i}\psi_{\mathbf{x}_i,\gamma_i}^{(\le h_v)}\Big]. \tag{6.54}$$

Such a term appears in the computation of the effective potentials at scale $h_v - 1$, see (6.52). In the multiscale integration procedure, the "external fields" $\psi_{\mathbf{x}_1,\gamma_1}^{(\le h_v)},\dots,\psi_{\mathbf{x}_4,\gamma_4}^{(\le h_v)}$ will be contracted on scales $h_1,\dots,h_4$ smaller or equal to h_v. We let $h^- := \max(h_1,\dots,h_4)$. By proceeding in a way similar to the one described in Section 5.2, $W_{4,\boldsymbol{\gamma}}^{(h_v)}$ can be written as a sum over trees τ_v with root v at scale h_v and over $\mathbf{P} \in \mathcal{P}_{\tau_v}$ of terms $W_{4,\boldsymbol{\gamma}}^{(h_v)}(\tau_v,\mathbf{P};\mathbf{x}_1,\dots,\mathbf{x}_4)$, where τ_v specifies the cluster structure of the labelled diagrams contributing to it, while $\mathbf{P} = \{P_w\}_{w\in V(\tau_v)\cup E(\tau_v)}$ specifies the field labels associated with the vertices w of τ_v (recall that $|P_w|$ represents the number of fields external to the subdiagram associated with w). We denote by $V_4^{(h_v)}(\tau_v,\mathbf{P};\psi)$ the analogue of (6.54) at fixed τ_v and $\mathbf{P}$. The kernel $W_{4,\boldsymbol{\gamma}}^{(h_v)}(\tau_v,\mathbf{P};\underline{\mathbf{x}})$, with $\underline{x} := (\mathbf{x}_1,\dots,\mathbf{x}_4)$, is a combination of propagators, each having a scale strictly larger than h_v. We let $h^+ > h_v$ denote the smallest such scale. Since a propagator at scale k decays over a length scale of order 2^{-k}, $W_{4,\boldsymbol{\gamma}}^{(h_v)}(\tau_v,\mathbf{P};\underline{\mathbf{x}})$ is essentially zero whenever two variables $\mathbf{x}_i,\mathbf{x}_j$ are at distance (much) larger than 2^{-h^+}.

Recall that the quartic kernels have scaling dimension zero. In other words, go back to (5.8): if τ there is a tree that contains the vertex v we are looking at (and τ_v is the sub-tree starting from v), the contribution to the r.h.s. from the portion of the tree τ from scale h^- to h^+ containing the cluster v is $1 = 2^{0\times(h^- - h^+)}$. In order to make the sum over trees τ convergent, we would need to improve this bound by $2^{\theta(h^- - h^+)}$ for some $\theta > 0$. To see that the dimensional estimate of $\mathcal{R}V_4^{(h)}(\tau_v,\mathbf{P};\psi)$ is better than the one of $V_4^{(h)}(\tau_v,\mathbf{P};\psi)$ precisely by such a factor (with $\theta = 1$), we rewrite (denoting $\underline{\mathbf{x}} = (\mathbf{x}_1,\dots,\mathbf{x}_4)$ and omitting for lightness the index $(\le h_v)$ on the fields)

$$\begin{aligned}\mathcal{R}V_4^{(h_v)}(\tau_v,\mathbf{P};\psi) = \sum_{\underline{\mathbf{x}},\boldsymbol{\gamma}}\Big[\prod_{i=1}^{4} e^{i\mathbf{p}_{\gamma_i}\mathbf{x}_i}\Big]\Big\{&(1-\mathcal{P}_0)W_{4,\boldsymbol{\gamma}}^{(h_v)}(\tau_v,\mathbf{P};\underline{\mathbf{x}})\Big[\prod_{i=1}^{4}\psi_{\mathbf{x}_i,\gamma_i}\Big]\\ &+\mathcal{P}_0 W_{4,\boldsymbol{\gamma}}^{(h_v)}(\tau_v,\mathbf{P};\underline{\mathbf{x}})\big[\psi_{\mathbf{x}_1,\gamma_1}\psi_{\mathbf{x}_2,\gamma_2}\psi_{\mathbf{x}_3,\gamma_3}(\psi_{\mathbf{x}_4,\gamma_4}-\psi_{\mathbf{x}_1,\gamma_4})+\cdots\\ &+\psi_{\mathbf{x}_1,\gamma_1}(\psi_{\mathbf{x}_2,\gamma_2}-\psi_{\mathbf{x}_1,\gamma_2})\psi_{\mathbf{x}_1,\gamma_3}\psi_{\mathbf{x}_1,\gamma_4}\big]\Big\}.\end{aligned} \tag{6.55}$$

The kernel in the second line has an operator $1-\mathcal{P}_0$ acting on it, which extracts its m-dependent part, i.e., it extracts the m-dependent part from at least one of the propagators contributing to its value; recalling that every m extracted from a propagator $G^{(k)}$ comes with a dimensional gain of the order 2^{h^*-k} (see Lemma 2; note that the terms linear in m originate necessarily from the non-diagonal part of some propagator), we see that this term has the desired dimensional gain, simply because $2^{h^*-k} \le 2^{h^- - h^+}$. The terms in second and third line of (6.55) involve a difference between two fields at different locations, of the form $\psi_{\mathbf{x}_i,\gamma_i} - \psi_{\mathbf{x}_1,\gamma_i}$, which is formally (i.e., forgetting lattice effects) the same as

$$\psi_{\mathbf{x}_i,\gamma_i} - \psi_{\mathbf{x}_1,\gamma_i} \simeq (\mathbf{x}_i - \mathbf{x}_1)\cdot\int_0^1 ds\,\boldsymbol{\partial}\psi_{\mathbf{x}_1+s(\mathbf{x}_i-\mathbf{x}_1),\gamma_i}. \tag{6.56}$$

Now note that the factor $\mathbf{x}_i - \mathbf{x}_1$ goes together with $\mathcal{P}_0 W_{4,\boldsymbol{\gamma}}^{(h_v)}(\tau_v,\mathbf{P};\underline{\mathbf{x}})$ which, as discussed above, decays over a typical length scale 2^{-h^+}. Therefore, $|\mathbf{x}_i - \mathbf{x}_1|$ can be bounded essentially by 2^{-h^+}. Similarly, the derivative $\boldsymbol{\partial}$ acting on $\psi_{\cdot,\gamma_i}$ corresponds to a dimensional contribution proportional to $2^{h_i} \le 2^{h^-}$, simply because $\psi_{\cdot,\gamma_i}$ is contracted at scale $h_i \le h^-$ and the derivative of $G^{(h_i)}$ satisfies the same qualitative estimates as $G^{(h_i)}$ times an extra 2^{h_i} (see

Lemma 2). Therefore, all the terms appearing in $\mathcal{R}V_4^{(h_v)}(\tau_v, \mathbf{P}; \psi)$ are associated with a gain factor $2^{h^- - h^+}$, which is enough to renormalize the (marginal) quartic terms.

To summarize, the action of $\mathcal{L}$ essentially corresponds to extracting the zero order term in a Taylor expansion of the kernel with respect to m, and of the fields with respect to $\mathbf{x} - \mathbf{x}_1$; conversely, the action of $\mathcal{R}$ corresponds to taking the rest of first order of the same Taylor expansion. The rest of order 1 has an improved estimate by a factor $2^{h^- - h^+}$ as compared to the original kernel. By proceeding similarly, one can show that the rest of order 2 has a dimensional gain $\propto 2^{2(h^- - h^+)}$, etc. The rationale behind the definition of $\mathcal{L}$ should now be clear: if it acts on a marginal term, it extracts the zero-th order term in the aforementioned Taylor expansion, so that the renormalized part has a gain $2^{h^- - h^+}$, which is enough to eliminate the divergences in (5.8) from vertices v with $(|P_v^\psi|, |P_v^J|) = (4, 0)$ or $(2, 1)$; if it acts on a linearly relevant term (i.e., a term with $(|P_v^\psi|, |P_v^J|) = (2, 0)$), it extracts the zero-th plus first order terms in the Taylor expansion (this is precisely the choice done in (6.10)), so that the renormalized part (which is a Taylor rest of order 2) has a gain $2^{2(h^- - h^+)}$.

6.2. *The renormalized tree expansion*

Now that we described the inductive definition of the (renormalized) effective potential, we have to explain why such an expansion is well behaved: that is, we explain how to get estimates on the kernels of the effective potential. As already observed above, see in particular Remark 14, the effective potential on scale h can be thought of as a function of the whole sequence of *effective constants* $\xi_k = (\lambda_k, Z_k, m_k(\mathbf{0}), Z_k^{(1)}, Z_k^{(2)})$, $h < k \le 0$. The sequence $\{\xi_k\}_{k>h}$ is a solution to the beta function equation (6.53) with initial data $(\lambda_0, Z_0, m_0(\mathbf{0}), Z_0^{(1)}, Z_0^{(2)}) = (-32(e^\lambda - 1), 1, m, 1, 1)$ and, therefore, the sequence itself, as well as the effective potential, are just functions of λ and m. Nevertheless, it is convenient to proceed as follows.

We will first think of $\{\xi_k\}_{k>h}$ as an *arbitrary* sequence, not necessarily a solution to the beta function (6.53). The first key result to be discussed, summarized in Proposition 8 and in Eq. (6.70) below, is that the kernels of the effective potential can be written as an *absolutely convergent* series, provided the sequence $\{\xi_k\}_{k>h}$ is such that λ_k, $(Z_k/Z_{k-1} - 1)$ and $(m_k(\mathbf{0})/m_{k-1}(\mathbf{0}) - 1)$ remain small (more precisely, the required assumptions are (6.66)–(6.67) for λ small enough). The proof of this fact requires a combinatorial representation of the expansion in terms of renormalized GN trees, reviewed in this section, and the iterative use of the Pfaffian representation for truncated expectations, recalled in Lemma 3.

Once we know that the kernels of the effective potential are well defined for sequences of effective constants satisfying suitable conditions, the next goal is, of course, to prove that the solution to the beta function equation do satisfy such conditions, i.e., it remains uniformly close to the initial datum for all $h \le 0$. The flow driven by the beta function is very non-trivial and it has been investigated in a series of works from the mid 1990s to the mid 2000s for very similar models (cf. [11,13,14,49,50] among others), by combining the use of the Schwinger–Dyson equation with local Ward Identites. A crucial point is that the beta function can be written as the sum of two terms: one part is "universal," i.e., it is the same for all the models treated in [11,13,14,49,50] and corresponds to the beta function of a reference model (an ultraviolet cut-off version of the Luttinger model [51]); the second part is a model-dependent rest, which is exponentially small and, therefore, summable as $h \to -\infty$. The key point is, therefore, to study the flow under the universal part of the beta function, and to prove that such a flow remains bounded and close to the initial datum for all $h \le 0$. We review the conceptual scheme used to study the flow in Section 6.3 below.

Let us now describe the tree expansion for the effective potential, and let us discuss how to prove its absolute convergence. The definition of the renormalized GN trees arises naturally from the iterative construction described in the proof of Proposition 7 (see (6.52)) and it is described in detail, e.g., in [10,12,34]. The renormalized trees are defined in a way very similar to the one described in Section 5.2.1, with the following important differences.

(1) A renormalized tree τ contributing to $V_\Lambda^{(h)}$, $\mathcal{B}_\Lambda^{(h)}$, $\tilde{E}_\Lambda^{(h)}$, or $\tilde{S}^{(h)}(\mathbf{J})$ has root on scale h and can have endpoints on all possible scales between $h+2$ and $+1$. The endpoints v on scales $h_v \le 0$ are preceded by a node of τ (on scale $h_v - 1$) that is necessarily a branching point.
(2) Normal endpoints on scale h_v are associated with $\lambda_{h_v - 1} F_\lambda$, if $h_v \le 0$; they are associated with (one of the monomials contributing to) $\widehat{V}^{(0)}$, if $h_v = 1$. Similarly, special endpoints on scale h_v are associated with $(Z_{h_v-1}^{(j)}/Z_{h_v-2}) F_j$ (cf. (6.23)), where either $j = 1$ or $j = 2$, if $h_v \le 0$; they are associated with (one of the monomials contributing to) $\widehat{\mathcal{B}}^{(0)}$, if $h_v = 1$.

(3) Each vertex of the tree that is not an endpoint and that is not the special vertex v_0 (the leftmost vertex of the tree, immediately following the root on τ) is associated with the action of an $\mathcal{R}$ operator.

The family of renormalized trees with root on scale h, N normal endpoints and n special endpoints will be denoted by $\mathcal{T}^{(h)}_{N,n}$. In terms of renormalized trees, the l.h.s. of (6.52) can be written as (replacing $h-1$ by h)

$$V^{(h)}_\Lambda\big(\sqrt{Z_h}\psi^{(\le h)}\big)+\mathcal{B}^{(h)}_\Lambda\big(\sqrt{Z_h}\psi^{(\le h)},\mathbf{J}\big)+\tilde{E}^{(h)}_\Lambda+\tilde{S}^{(h)}_\Lambda(\mathbf{J})$$
$$=\sum_{\substack{N,n\ge 0\\ N+n\ge 1}}\sum_{\tau\in\mathcal{T}^{(h)}_{N,n}} V^{(h)}\big(\tau,\sqrt{Z_h}\psi^{(\le h)},\mathbf{J}\big), \tag{6.57}$$

where $V^{(h)}(\tau,\sqrt{Z_h}\psi^{(\le h)},\mathbf{J})$ is defined iteratively: if v_0 is the first vertex of τ, if $\tau_1,\dots,\tau_s$ ($s=s_{v_0}\ge 1$) are the subtrees of τ with root v_0, and if $\mathcal{E}^T_{h+1}$ is the truncated expectation associated with the propagator $Z_h^{-1}g^{(h+1)}$,

$$V^{(h)}\big(\tau,\sqrt{Z_h}\psi^{(\le h)},\mathbf{J}\big)$$
$$=\frac{1}{s!}\mathcal{E}^T_{h+1}\big(\overline{V}^{(h+1)}\big(\tau_1,\sqrt{Z_h}\psi^{(\le h+1)},\mathbf{J}\big);\dots;\overline{V}^{(h+1)}\big(\tau_s,\sqrt{Z_h}\psi^{(\le h+1)},\mathbf{J}\big)\big), \tag{6.58}$$

where $\psi^{(\le h+1)}=\psi^{(\le h)}+\psi^{(h+1)}$ and $\overline{V}^{(h+1)}(\tau_i,\sqrt{Z_h}\psi^{(\le h+1)},\mathbf{J})$:

- is equal to $\mathcal{R}\widehat{V}^{(h+1)}(\tau_i,\sqrt{Z_h}\psi^{(\le h+1)},\mathbf{J})$ if τ_i is non-trivial. Here $\mathcal{R}$ is the linear operator induced by the definitions (6.8)–(6.12), and $\widehat{V}^{(h+1)}(\tau_i,\sqrt{Z_h}\psi^{(\le h+1)},\mathbf{J})$ is defined in analogy with (6.46), that is

$$\widehat{V}^{(h+1)}\big(\tau_i,\sqrt{Z_h}\psi^{(\le h+1)},\mathbf{J}\big):=V^{(h+1)}\big(\tau_i,\sqrt{Z_{h+1}}\psi^{(\le h+1)},\mathbf{J}\big);$$

- is equal to $\lambda_{h+1}F_\lambda(\sqrt{Z_h}\psi^{(\le h+1)})$ if τ_i is trivial, $h<-1$ and the endpoint of τ_i is normal;
- is equal to $\sum_{j=1}^2(Z^{(j)}_{h+1}/Z_h)F_j(\sqrt{Z_h}\psi^{(\le h+1)},\mathbf{J})$ if τ_i is trivial, $h<-1$ and the endpoint of τ_i is special;
- is equal to $\widehat{V}^{(0)}(\sqrt{Z_{-1}}\psi^{(\le 0)})$ (resp. $\widehat{\mathcal{B}}^{(0)}(\sqrt{Z_{-1}}\psi^{(\le 0)},\mathbf{J})$) if τ_i is trivial, $h=-1$ and the endpoint of τ_i is normal (resp. special).

In order to compute as explicitly as possible the tree values $V^{(h)}(\tau,\psi,\mathbf{J})$, we can inductively apply (6.58) and use the *Pfaffian representation* for the truncated expectation in its r.h.s., originally due to Battle, Brydges and Federbush [4,17,18], later improved and simplified [1,19] and re-derived in several review papers, see e.g. [34,36]:

Lemma 3 (Pfaffian representation). *Using a notation similar to* (3.4) *and* (3.7) *we get*

$$\mathcal{E}^T_h\big(X_1(\psi);\dots;X_s(\psi)\big)=c_1\cdots c_s Z_{h-1}^{-p}\sum_{T\in\mathbf{T}}\alpha_T\prod_{\ell\in T}g^{(h)}_\ell\int dP_T(\mathbf{t})\,\mathrm{Pf}\big(M^{h,T}(\mathbf{t})\big). \tag{6.59}$$

Here:

- *the constants c_i are those appearing in the definition* (3.7) *of X_i and $2p=\sum_{i=1}^s n_i$ (recall that n_i is the order of the monomial X_i)*;
- *the first sum runs over set of lines forming a* spanning tree *between the s vertices corresponding to the monomials $X_1,\dots,X_s$, i.e., T is a set of lines that becomes a tree if one identifies all the points in the same clusters*;
- *α_T is a sign (irrelevant for the subsequent bounds)*;
- *$g^{(h)}_\ell$ is a shorthand for $g^{(h)}_{\gamma(\ell),\gamma'(\ell)}(\mathbf{x}(\ell)-\mathbf{x}'(\ell))$, where $\gamma(\ell)$, $\gamma'(\ell)$ and $\mathbf{x}(\ell)$, $\mathbf{x}'(\ell)$ are the γ and $\mathbf{x}$ indices associated with the two ends of the line ℓ, which should be thought of as being obtained from the pairing (contraction) of two fields $\psi_{\mathbf{x}(\ell),\gamma(\ell)}$ and $\psi_{\mathbf{x}'(\ell),\gamma'(\ell)}$*;
- *if $\mathbf{t}=\{t_{i,i'}\in[0,1],1\le i,i'\le s\}$, then $dP_T(\mathbf{t})$ is a probability measure with support on a set of $\mathbf{t}$ such that $t_{i,i'}=\mathbf{u}_i\cdot\mathbf{u}_{i'}$ for some family of vectors $\mathbf{u}_i=\mathbf{u}_i(\mathbf{t})\in\mathbb{R}^s$ of unit norm*;
- *$M^{h,T}(\mathbf{t})$ is an antisymmetric $(2p-2s+2)\times(2p-2s+2)$ matrix, whose elements are given by $M^{h,T}_{f,f'}=t_{i(f),i(f')}g^{(h)}_{\ell(f,f')}$, where: $f,f'\notin\bigcup_{\ell\in T}\{f^1_\ell,f^2_\ell\}$ and f^1_ℓ, f^2_ℓ are two field labels associated with the two (entering*

and exiting) half-lines contracted into ℓ; $i(f) \in \{1, \dots, s\}$ is s.t. $f \in P_{v_{i(f)}}$; $g^{(h)}_{\ell(f,f')}$ is the propagator associated with the line obtained by contracting the two half-lines with indices f and f'.

If $s = 1$ the sum over T is empty, but we can still use Eq. (6.59) by interpreting the r.h.s. as equal to 0 if P_1 is empty and equal to $\operatorname{Pf} M^{h,T}(\mathbf{1})$ otherwise.

Remark 15. *If the Pfaffian is expanded by using its definition (2.3), then (6.59) reduces to the usual representation of the truncated expectation in terms of connected Feynman diagrams. The spanning trees in (6.59) guarantee the minimal connection among the vertices $X_1, \dots, X_s$ and the Pfaffian can be thought of as a resummation of all the Feynman diagrams obtained by pairing (contracting) in all possible ways the fields outside the spanning tree, with the rule that each contracted pair $(\psi_{\mathbf{x},\gamma}, \psi_{\mathbf{y},\gamma'})$ is replaced by $Z_{h-1}^{-1} g^{(h)}_{\gamma,\gamma'}(\mathbf{x}-\mathbf{y})$; the interpolation in $\mathbf{t}$ is necessary in order to avoid an over-counting of the diagrams.*

With respect to the Feynman graph expansion, Eq. (6.59) has the advantage that the Pfaffian $\operatorname{Pf}(M^{h,T}(\mathbf{t}))$ can be bounded by using the Gram–Hadamard inequality [34], *which leads to*

$$\left| \int dP_T(\mathbf{t}) \operatorname{Pf}\big(M^{h,T}(\mathbf{t})\big) \right| \le \big(C2^h\big)^{p-s+1}. \tag{6.60}$$

Here $2(p-s+1)$ is the size of the antisymmetric matrix $M^{h,T}$ and C is the constant appearing in the estimate of $g^{(h)}$, see the lines following (6.45). This is in contrast with the estimate scaling like $(p-s+1)!(C2^h)^{p-s+1}$ that we would get via the Feynman expansion. Morally speaking, recalling that $(\operatorname{Pf} M)^2 = \det M$, the Gram–Hadamard inequality is similar in spirit to bounding the determinant of a $k \times k$ matrix by the largest eigenvalue to the power k (which is combinatorially optimal), rather than by the number of terms in the determinant times the maximum of the matrix elements to the power k. Finally, the number of spanning trees is bounded as

$$|\mathbf{T}| \le s! C^p, \tag{6.61}$$

with p the total number of fields appearing in $X_1, \dots, X_s$ [34, *Appendix* A3.3]. *Note that, if formula (6.59) is applied to the r.h.s. of (6.52), then the number of spanning trees $\propto s!$ is compensated by the factor $1/s!$ appearing there.*

When we apply iteratively (6.58) and Lemma 3, we can naturally distinguish the various contributions arising from the choices of the monomials in the factors $\widehat{V}^{(0)}$ and $\widehat{\mathcal{B}}^{(0)}$ associated with the endpoints on scale 1, as well as the scale at which each field in these monomials is contracted (we can keep track of these informations via the labels $\mathbf{P}$ attached to the trees, as explained in Section 5.2.1).

The resulting formula has a natural structure, slightly complicated by the presence of the $\mathcal{R}$ operators acting at all vertices of the tree that are not endpoints. Therefore, in order to make it as transparent as possible, let us *temporarily neglect the action of the renormalization operator*, i.e., let us temporarily pretend that the action of $\mathcal{R}$ on the nodes of τ is replaced by the identity. Then the result of the iteration would lead to the following relation (the reader can easily convince himself of the formula by induction, or consult the aforementioned reviews for more details, see in particular [12,34]):

$$V^{(h)}_*(\tau, \sqrt{Z_h}\psi, \mathbf{J}) = \sum_{\mathbf{P} \in \mathcal{P}_\tau} \sqrt{Z_h}^{|P^\psi_{v_0}|} \sum_{T \in \mathbf{T}} \sum_{\mathbf{x}_{v_0}} W^*_{\tau,\mathbf{P},\mathbf{T}}(\mathbf{x}_{v_0}) \psi\big(P^\psi_{v_0}\big) J\big(P^J_{v_0}\big), \tag{6.62}$$

where $T = \bigcup_{v \text{ not e.p.}} T_v$ is the union of the spanning trees T_v associated with all the nodes that are not endpoints in τ, which arise from the inductive application of the Pfaffian formula (6.59). The star in V_* is to recall that we are ignoring the renormalization operator. Moreover, $W^*_{\tau,\mathbf{P},\mathbf{T}}$ is given by

$$\begin{aligned} W^*_{\tau,\mathbf{P},\mathbf{T}}(\mathbf{x}_{v_0}) = &\Bigg[\prod_{v \text{ not e.p.}} (Z_{h_v}/Z_{h_v-1})^{|P^\psi_v|/2} \Bigg] \Bigg[\prod_{v \text{ e.p.}} K^{(h_v)}_v(\mathbf{x}_v) \Bigg] \\ &\times \Bigg\{ \prod_{v \text{ not e.p.}} \frac{1}{s_v!} \int dP_{T_v}(\mathbf{t}_v) \operatorname{Pf}\big(M^{h_v,T_v}(\mathbf{t}_v)\big) \Bigg[\prod_{\ell \in T_v} g^{(h_v)}_\ell \Bigg] \Bigg\}, \end{aligned} \tag{6.63}$$

where $K_v^{(h_v)}(\mathbf{x}_v)$ is equal to: λ_{h_v-1}, if v is a normal endpoint on scale $h_v < 1$; $(Z^{(j_v)}_{h_v-1}/Z_{h_v-2})$, if v is a special endpoint on scale $h_v < 1$; the kernel (see Remark 9) of the monomial of $\widehat{V}^{(0)}$ (resp. $\widehat{\mathcal{B}}^{(0)}$) compatible with the assignment of external fields P_v, if v is a normal (resp. special) endpoint on scale $h_v = 1$.

The analogous formula for $V^{(h)}(\tau, \sqrt{Z_h}\psi, \mathbf{J})$, in which we do not neglect the action of $\mathcal{R}$, can be written in the form

$$V^{(h)}(\tau, \sqrt{Z_h}\psi, \mathbf{J}) = \sum_{\mathbf{P}\in\mathcal{P}_\tau} \sqrt{Z_h}^{|P^\psi_{v_0}|} \sum_{T\in\mathbf{T}} \sum_{\beta\in B_T} \sum_{\mathbf{x}_{v_0}} W_{\tau,\mathbf{P},\mathbf{T},\beta}(\mathbf{x}_{v_0}) \big[\psi\big(P^\psi_{v_0}\big)\big]_\beta J\big(P^J_{v_0}\big), \tag{6.64}$$

where β is a multi-index that keeps track of the various terms arising from the action of $\mathcal{R}$: see e.g. (6.55), which shows that the action of $\mathcal{R}$ on a four-legged kernel produces 4 different terms. Moreover,

$$\big[\psi\big(P^\psi_{v_0}\big)\big]_\beta = \prod_{f\in P^\psi_{v_0}} \partial^{q_\beta(f)}_{j_\beta(f)} \psi_{\mathbf{x}(f),\omega(f)}, \tag{6.65}$$

where $j_\beta(f) \in \{1, 2\}$ and q_β is a nonnegative integer ≤ 2; the action of a derivative on the fields arises from the interpolation formula (6.56), see [12] for details. In particular, the kernels $W_{\tau,\mathbf{P},\mathbf{T},\beta}$ admit a representation similar to (6.63), see [12, Eq. (3.81)] for an analogous formula (the parameter β appearing there is equal to L in our case, and our β is called α there).

6.2.1. *Analyticity and dimensional estimates of the kernels*
The expressions (6.62)–(6.64) can be bounded by using (6.60). In the absence of the action of the $\mathcal{R}$ operators the resulting bound has the same structure as the final bound of Section 5.2, see (5.8) and following discussion, modulo an improved combinatorial factor due to the use of the Pfaffians rather than of the Feynman diagrams. If, on the contrary, we take the action of $\mathcal{R}$ into account, the dimensional factors are improved by the gain factors discussed in Section 6.1.4. The net result is:

Proposition 8. *Let $|\lambda| \le \lambda_0$ for λ_0 suitably small. If*

$$\sup_{h'>h} |\lambda_{h'}| \le c_l|\lambda|, \qquad \sup_{h'>h} \left|\frac{Z_{h'}}{Z_{h'-1}}\right| \le 2^{2c_z\lambda^2}, \tag{6.66}$$

$$\sup_{h'>h} \left|\frac{m_{h'}(\mathbf{0})}{m_{h'-1}(\mathbf{0})}\right| \le 2^{c_m|\lambda|}, \qquad \sup_{h'>h} 2^{-h'}\big|m_{h'}(\mathbf{0})\big| \le 1 \tag{6.67}$$

for some λ-independent constants $c_l, c_z, c_m > 0$, then there exists a (λ-independent) constant $C > 0$ such that, if $\tau \in \mathcal{T}^{(h)}_{N,n}$,

$$\frac{1}{|\Lambda|}\sum_{\mathbf{x}_{v_0}} \big|W_{\tau,\mathbf{P},T,\beta}(\mathbf{x}_{v_0})\big| \le C^{N+n}\big(c_l|\lambda|\big)^{1/2|I^\psi_{v_0}|-N} 2^{h(2-1/2|P^\psi_{v_0}|-|P^J_{v_0}|-Q_\beta)} \times \Big[\prod_{v \text{ s.e.p.}} \left|\frac{Z^{(j_v)}_{h_v-1}}{Z_{h_v-1}}\right|\Big]\Big[\prod_{\substack{v \text{ not}\\ \text{e.p.}}} \frac{C^{\sum_{i=1}^{s_v}|P_{v_i}|-|P_v|}}{s_v!} 2^{c_z\lambda^2|P^\psi_v|} 2^{2-1/2|P^\psi_v|-|P^J_v|-z(P_v)}\Big], \tag{6.68}$$

where the first product in the second line runs over the special endpoints, while the second over all the vertices of the tree that are not endpoints. Moreover $Q_\beta = \sum_{f\in P^\psi_{v_0}} q_\beta(f)$ and

$$z(P_v) = \begin{cases} 1 - c_m|\lambda| & \text{if } (P^\psi_v, P^J_v) = (4,0), (2,1),\\ 2(1-c_m|\lambda|) & \text{if } (P^\psi_v, P^J_v) = (2,0),\\ 0 & \text{otherwise.}\end{cases} \tag{6.69}$$

This is the analogue of [12, Eq. (3.110)] and the details of its proof can be found there. To understand the factor $\lambda^{\frac{1}{2}|I^{\psi}_{v_0}|-N}$, observe that if for instance all endpoints are quartic monomials $\lambda\psi_{\mathbf{x}_1}\psi_{\mathbf{x}_2}\psi_{\mathbf{x}_3}\psi_{\mathbf{x}_4}$, then $\lambda^{\frac{1}{2}|I^{\psi}_{v_0}|-N} = \lambda^N$.

Note that the renormalized scaling dimension $d(P_v) := 2 - \frac{1}{2}|P^{\psi}_v| - |P^J_v| - z(P_v)$ appearing at exponent in the last factor of (6.68) satisfies

$$d(P_v) + c_z\lambda^2\big|P^{\psi}_v\big| \le \left(-\frac{1}{6} + c_z\lambda^2\right)\big(\big|P^{\psi}_v\big| + 2\big|P^J_v\big|\big), \tag{6.70}$$

which is negative for λ small. Therefore, if λ is small enough, the product $\prod_{v \text{ not e.p.}} 2^{c_z\lambda^2|P^{\psi}_v|+d(P_v)}$ in the second line of (6.68) produces an exponentially small factor smaller than, e.g., $2^{-\frac{|P_v|}{12}(h_v-h_{v'})}$ for each branch of the tree connecting two vertices v and v', with $v' < v$ and $|P_v|$ constant along the branch. Not surprisingly, this allows to sum over the scale differences $h_v - h_{v'}$, as well as over the choices of the field labels $\{P_v\}_{v\in\tau}$ (see [34, Appendix 6.1] for details about how to perform these summations). Using also the fact that the number of spanning trees in $\mathbf{T}$ is smaller than $(\text{const.})^{N+n}\prod_v s_v!$ (see (6.61)), and that the number of elements of B_T is smaller than $(\text{const.})^{N+n}$, we get

$$\begin{aligned}&\frac{1}{|\Lambda|}\sum_{N\ge 1}\sum_{\tau\in\mathcal{T}^{(h)}_{N,n}}\sum_{\mathbf{P}\in\mathcal{P}_\tau}\sum_{T\in\mathbf{T}}\sum_{\substack{\beta\in B_T:\\ Q_\beta=q}}\sum_{\mathbf{x}_{v_0}}\big|W_{\tau,\mathbf{P},T,\beta}(\mathbf{x}_{v_0})\big|\\ &\le C^{n+1}|\lambda|\left(\sup_{\substack{h'>h\\ j=1,2}}\left|\frac{Z^{(j)}_{h'}}{Z_{h'}}\right|\right)^n 2^{h(2-1/2|P^{\psi}_{v_0}|-|P^J_{v_0}|-q)},\end{aligned} \tag{6.71}$$

for a suitable, λ-independent, $C > 0$. This is the analogue of [12, Theorem 3.12] and further details of its proof can be found there. Eq. (6.71) is the final dimensional estimate on the (renormalized) kernels of the effective potential, promised after (6.7). Absolute summability of the tree expansion immediately implies:

Corollary 1. *The kernels on scale h are* analytic *functions of the* sequences $\{(\lambda_k, Z_k, m_k)\}_{k>h}$ *in the space defined by* (6.66)–(6.67), *for λ small.*

Note that the factors $Z^{(j)}_{h'}/Z_{h'}$, corresponding to special endpoints, may diverge in the infrared limit, i.e. for $h' \to -\infty$ (and in fact this happens for $j = 2$, see Proposition 10 below: the ratio grows like $2^{h'(\eta_2(\lambda)-\eta(\lambda))}$ and $\eta_2 \neq \eta$ in general). When we compute the nth cumulant of the height function (Section 7.2) we will consider diagrams with n special endpoints and such diverging factors do appear. It will however turn out that they are irrelevant in the computation the large-distance asymptotics of height cumulants, since they are compensated by oscillations in the multi-dimer correlation functions.

Remark 16 (The short memory property). *If λ is small then, as discussed above (see lines after (6.70)), every branch of the tree is associated with an exponentially decaying factor smaller than, e.g., $2^{-\frac{|P_v|}{12}(h_v-h_{v'})}$. Therefore, not only the sum over the scale and field labels converges exponentially, but we also have that the sum restricted to the trees τ with root on scale h and at least one vertex on scale $k > h$ is bounded dimensionally by the r.h.s. of (6.71) times a dimensional gain of the form $C_\theta 2^{\theta(h-k)}$, for a suitable $\theta > 0$ (it can be checked, in particular, that any θ in $(0, 1 - 2c_m|\lambda| + 2c_z\lambda^2)$ makes the job). This improved bound is usually referred to as the* short memory property (*i.e., trees with long branches are exponentially suppressed*) *and will play an important role in the following. From now on, θ will be a constant in $(0, 1)$, uniformly bounded away from zero for λ small. One can think for definiteness of $\theta = 1/2$, but as we just discussed one can actually take θ close to 1 when λ is close to zero.*

6.3. *The beta function*

The above procedure implies the absolute summability and analyticity of the tree expansion kernels, provided the effective constants λ_h, Z_h, $m_h(\mathbf{0})$ satisfy the conditions (6.66)–(6.67) of Proposition 8. It is easy to verify that these

conditions hold at the first step, $h = 0$, and that they remain valid for a finite number of steps, provided λ is small enough. The difficult issue is to show that they remain valid for *all* the scales such that $h^* \le h \le 0$, uniformly in h^* (that is, uniformly in m, as $m \to 0$).

An important remark is that, as long as these conditions are verified, the beta function itself, governing the flow of the effective constants via (6.53), is analytic: in fact, the beta function is defined simply in terms of the local parts of the 2- and 4-legged kernels of the effective potential $V^{(h)}$ and of the local part of the 3-legged kernel of $\mathcal{B}^{(h)}$, cf. Eqs (6.33)–(6.39) and (6.47)–(6.49). Therefore, the natural strategy to study the flow of λ_h, Z_h, $m_h(\mathbf{0})$ is the following: write down the Taylor expansion for the beta function, which is convergent as long as (6.66)–(6.67) are verified; truncate the Taylor expansion at lowest non-trivial order, and try to check whether the approximate flow governed by this truncated beta function verifies (6.66)–(6.67); if so, prove that the solution is stable under the addition of higher order Taylor approximations.

In order to understand the difficulty of the problem at hand, consider the flow equation for λ_h, and suppose that the second order truncation of the beta function reads $\lambda_{h-1} = \lambda_h + a_h\lambda_h^2 + \cdots$. The qualitative properties of the flow are encoded in a_h: if, e.g., $a_h \ge a > 0$, uniformly in h, then the truncated flow is divergent as $h \to -\infty$, and the same holds for the non-truncated flow; in this case, the multiscale construction in the form described above would have to be stopped at a critical scale, below which perturbation theory in λ_h is not applicable anymore. If, on the contrary, $a_h \le -a < 0$, uniformly in h, then the truncated flow would be convergent, $\lambda_h \to 0$ as $h \to -\infty$, and the same would hold for the non-truncated flow; such a scenario is usually called *asymptotic freedom* in the Renormalization Group language. Quite remarkably, our case of interest realizes a critical, intermediate, scenario: an explicit computation of the lowest order contribution to the beta function shows that in the case of interacting dimers $a_h = 0$. Therefore, the truncated flow of λ_h remains analytically close to the initial datum λ_0, uniformly in h. The problem, of course, is that, since a_h is vanishing, the truncated flow is unstable, and one needs to show that a similar cancellation takes place at all orders in perturbation theory, which is very hard (if not impossible) to prove by direct computation.

The idea to be pursued is that the beta function of the dimer model is asymptotically close as $h \to -\infty$ to that of several other models, all belonging to a family called, in the RG language, a *universality class* (the Luttinger liquid universality class). Other statistical mechanics or field theory models belonging to the same class are: the Luttinger model [14,51], the Thirring model [6,46,62], the XXZ spin chain [12,63], the repulsive 1D Hubbard model [8], the 8-vertex model [5,15], the Ashkin–Teller model at criticality [37,48], etc. All these are associated with the same *reference model* (an ultraviolet cut-off version of the Luttinger model, whose precise definition is given in Section 6.3.2), which is defined in the two-dimensional continuum, with exactly linear effective dispersion relation for the free propagator (in the sense of (6.73) below). The key fact is that the reference model displays more symmetries than the dimer model or any of the other models in the same universality class: these extra symmetries can be used to show that the beta function for λ_h in the reference model is asymptotically zero; as a consequence, the same property is true for the dimer model, as well as for the other models mentioned above. Let us now describe more technically how this idea is implemented.

6.3.1. *Asymptotic vanishing of the beta function*

At each step of the multiscale integration procedure, we can decompose the single scale propagator (6.43) as the sum of a massless relativistic propagator plus a rest:

$$\frac{g^{(h)}(\mathbf{x})}{Z_{h-1}} = \frac{1}{Z_{h-1}}\big(g_R^{(h)}(\mathbf{x}) + r^{(h)}(\mathbf{x})\big), \tag{6.72}$$

where

$$g_R^{(h)}(\mathbf{x}) = \int \frac{d\mathbf{k}}{(2\pi)^2} e^{-i\mathbf{k}\mathbf{x}} \tilde{f}_h(\mathbf{k})(-ik_1 + Jk_2)^{-1} \tag{6.73}$$

and J is the diagonal matrix with diagonal elements $(1, -1, 1, -1)$. The index R stands for "relativistic," which refers to the fact that the denominator is exactly linear in k. Note that the rest satisfies improved dimensional estimates as compared to $g_R^{(h)}$: i.e., $\|r^{(h)}(\mathbf{x})\|$ satisfies an estimate like (2.39) times an extra (gain) factor that can be bounded proportionally to $2^h + m_h(\mathbf{0})/2^h$. Using (6.67) and the definition of h^* we get

$$\frac{m_h(\mathbf{0})}{2^h} \le 2\frac{m_h(\mathbf{0})}{m_{h^*}(\mathbf{0})}\frac{2^{h^*}}{2^h} \le 2 \cdot 2^{(h^*-h)(1-c_m|\lambda|)}.$$

In conclusion, the rest $r^{(h)}$ has an improved dimensional estimate as compared to $g_R^{(h)}$ by a factor proportional to $2^h + 2^{(h^*-h)(1-c_m|\lambda|)}$.

Remark 17. *Any observable on scale h can be naturally decomposed as the sum of a dominant part plus a rest*: *the dominant part is expressed in terms of GN trees with all the endpoints on scale ≤ 0 and their values computed by replacing all the single-scale propagators by their massless relativistic approximation $g_R^{(h)}$*; *the rest can be written as a sum of trees*, *each of which either has at least one endpoint on scale* 1, *or it has at least one single scale propagator of type $r^{(k)}$ for some $k \geq h$. It is easy to see that the rest satisfies a better dimensional estimate than the dominant part* (*better by an exponential factor $2^{\theta h} + 2^{\theta(h^*-h)}$, with $0 < \theta < 1$ as in Remark 16, in the infrared limit*). *To see this*, *use the estimate above for $||r^{(k)}(\mathbf{x})||$ and the short memory property* (*Remark 16*): *just note that $2^k 2^{\theta(h-k)} \leq 2^{\theta h}$.*

In particular, the beta function can be written as the sum of a dominant part plus a rest, in the sense discussed in this remark:

$$\beta_h^\lambda = \beta_{h,R}^\lambda + r_h^\lambda, \tag{6.74}$$

where, as long as (6.66)–(6.67) are verified, the rest satisfies

$$\left|r_h^\lambda\right| \leq (\text{const.})\lambda_h^2 2^{\theta h}. \tag{6.75}$$

The universal part $\beta_{h,R}^\lambda$ of the beta function has been studied in detail in several works. In particular, [14, Theorem 2 and Eq. (57)] establish the *asymptotic vanishing of the beta function*, which is summarized here.

Proposition 9. *For λ_h small enough, let $\bar{Z}_h(\lambda_h)$ be the solution to the beta function equation for Z_h with the sequence $(\lambda_h, \dots, \lambda_0)$ replaced by $(\lambda_h, \dots, \lambda_h)$ and $\bar{Z}_0(\lambda_h) = 1$. Then $\beta_{h,R}^\lambda((\lambda_h, \bar{Z}_h(\lambda_h)), \dots, (\lambda_h, \bar{Z}_0(\lambda_h)))$ is asymptotically vanishing as $h \to -\infty$, i.e.,*

$$\left|\beta_{h,R}^\lambda\big(\big(\lambda_h, \bar{Z}_h(\lambda_h)\big), \dots, \big(\lambda_h, \bar{Z}_0(\lambda_h)\big)\big)\right| \leq C_\theta |\lambda_h|^2 2^{\theta h}, \tag{6.76}$$

for $0 < \theta < 1$ (see Remark 16) and a suitable $C_\theta > 0$.

Note that, at the nth order in perturbation theory, $\beta_{h,R}^\lambda$ is the sum of $O(2n!)$ Feynman graphs of order $O(|\lambda_h|^n)$, each of which is *not* vanishing as $h \to -\infty$, but a dramatic cancellation implies that their sum is $O(|\lambda_h|^n 2^{\theta h})$, for all $n \geq 2$. A consequence of (6.75) and (6.76) and of a lowest order computation of β_h^Z, β_h^m, $\beta_h^{Z,j}$ is that the flow of the interacting dimer model is exponentially convergent, as summarized in the following proposition (the proof is a simple corollary of Proposition 9, see also the comment following [14, Eq. (57)] and [6, Theorem 2.1]). In reading the following proposition, recall that the beta functions of λ_h, Z_h, $Z_h^{(j)}$, $m_h(\mathbf{0})/m$ are independent of m and, therefore, the corresponding flows can be extrapolated to $h \to -\infty$ (i.e., in the study of their flow we do not need to stop at h^*).

Proposition 10. *For λ small enough, the solution to the beta function equations (6.53) satisfies the following*:

$$\lim_{h \to -\infty} \lambda_h = \lambda_{-\infty}(\lambda) \tag{6.77}$$

with $\lambda_{-\infty}(\lambda)$ analytic in λ and such that

$$\left|\lambda_h(\lambda) - \lambda_{-\infty}(\lambda)\right| \leq C_\theta \left|\lambda^2\right| 2^{\theta h} \tag{6.78}$$

for a suitable $0 < \theta < 1$ as in Remark 16 and $C_\theta > 0$. Moreover, $|\frac{Z_h}{Z_{h-1}}| \leq 2^{2c_z\lambda^2}$ and $|\frac{m_h(\mathbf{0})}{m_{h-1}(\mathbf{0})}| \leq 2^{c_m|\lambda|}$, for suitable constants $c_z, c_m > 0$, uniformly in h. Finally,

$$Z_h \sim 2^{\eta(\lambda)h}, \qquad Z_h^{(i)} \sim 2^{\eta_i(\lambda)h}, \qquad m_h(\mathbf{0}) \sim m 2^{\eta_m(\lambda)h}, \tag{6.79}$$

where $\sim$ means that the ratio of the two sides is bounded from above and below by two universal positive constants, uniformly in h, and $\eta(\lambda)$, $\eta_1(\lambda)$, $\eta_2(\lambda)$ and $\eta_m(\lambda)$ are analytic functions of λ, such that $\eta(0) = \eta'(0) = \eta_1(0) = \eta_2(0) = \eta_m(0) = 0$. Moreover, $\eta_1(\lambda) = \eta(\lambda)$.

This proposition implies, in particular, that (6.66)–(6.67) are satisfied for all $h \in [h^*, 0]$, with $h^* = (\log_2 m)/(1 - \eta_m) + O(1)$, as $m \to 0$. Combining this with Proposition 8 and Corollary 1, we get that the kernels of the effective potential on scale h are analytic in λ, uniformly in h. The last claim in the proposition, i.e., the fact that $\eta_1(\lambda) = \eta(\lambda)$, is proved in [13, Theorem 1] (where the index η_b equals our $2(\eta - \eta_1)$).

Remark 18. *Note that $|Z_h/Z_{h-1}| \le 2^{2c_z\lambda^2}$ and $|m_h(\mathbf{0})/m_{h-1}(\mathbf{0})| \le 2^{c_m|\lambda|}$ say that z_h and σ_h are small with λ, uniformly in h (recall from (6.40) that $Z_{h-1}/Z_h = 1 + z_h$), as was required for the multi-scale integration to be valid (see comment after (6.36)).*

Remark 19. *The flow of the effective constants is stable under small changes in the original energy function of the model; e.g., it remains valid in the presence of a finite range, rather than purely nearest neighbor, interaction. A small analytical change in the weights entering the definition of the model induces a small analytical change in the values of $\lambda_{-\infty}(\lambda)$, $\eta(\lambda)$ and $\eta_i(\lambda)$. In this sense, these functions are* non-universal, *i.e., they are model-dependent. However, the critical exponents $\eta = \eta_1$ and η_2 are* universal (*i.e., model-independent*) *functions of $\lambda_{-\infty}(\lambda)$. The proof these claims goes together with the proof of Propositions 9 and 10 and we will not discuss it in details. However, in Section 6.3.2 below, we will explain more technically some of the ideas behind them.*

6.3.2. *The reference model: Emerging Dirac description*

In this section we define the reference model, which we mentioned so far only vaguely. It is needed both in the proofs of Proposition 10, and in the explicit computation of the dimer-dimer correlation (e.g. Theorem 2), which is required for a sharp estimate of the height fluctuations. The generating function of the reference model are defined by the following Grassmann functional integral (for lightness of notations we give formally the expression in infinite volume and massless limit, but to be precise the model is defined on $[-L, L]^2$ with anti-periodic b.c. on the fields $\psi^{\pm}_{\omega}$ and with an infrared regularization, similar to putting $m > 0$ in the dimer model; see [15] for details):

$$e^{\mathcal{S}_R(\mathbf{J})} = \int P_Z\big(d\psi^{(\le M)}\big) e^{\mathcal{V}(\sqrt{Z}\psi^{(\le M)}) + \mathcal{B}(\sqrt{Z}\psi^{(\le M)}, \mathbf{J})}, \tag{6.80}$$

where the Grassmann field is $\{\psi^{(\le M)\pm}_{\mathbf{x},\omega}\}^{\omega=\pm}_{\mathbf{x}\in\mathbb{R}^2}$, $P_Z(d\psi^{(\le M)})$ is the Grassmann Gaussian integration with propagator

$$\int P_Z\big(d\psi^{(\le M)}\big) \psi^{(\le M)\varepsilon}_{\mathbf{x},\omega} \psi^{(\le M)\varepsilon'}_{\mathbf{y},\omega'} = \frac{\delta_{\varepsilon,-\varepsilon'}\delta_{\omega,\omega'}}{2Z} \int \frac{d\mathbf{k}}{(2\pi)^2} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})} \frac{\chi_M(\mathbf{k})}{-ik_1 + \omega k_2}, \tag{6.81}$$

and the χ_M is an ultraviolet cut-off (coherently with our previous notations, it is a smooth function that vanishes say for $\|\mathbf{k}\| \ge 2^M$), to be eventually removed, $M \to +\infty$. Moreover,

$$\mathcal{V}(\psi) = \lambda_\infty \int d\mathbf{x}\, d\mathbf{y}\, v(\mathbf{x}-\mathbf{y}) \psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},1}\psi^+_{\mathbf{y},-1}\psi^-_{\mathbf{y},-1} \tag{6.82}$$

with $v(\mathbf{x}-\mathbf{y})$ a smooth short-range potential (decaying on a length-scale of order 1), and

$$\mathcal{B}(\psi, \mathbf{J}) = \frac{Z^{(1)}}{Z} \sum_\omega \int d\mathbf{x} J^{(1)}_\omega(\mathbf{x}) \psi^{(\le M)+}_{\mathbf{x},\omega} \psi^{(\le M)-}_{\mathbf{x},\omega} \tag{6.83}$$

$$+ \frac{Z^{(2)}}{Z} \sum_\omega \int d\mathbf{x} J^{(2)}_\omega(\mathbf{x}) \psi^{(\le M)+}_{\mathbf{x},\omega} \psi^{(\le M)-}_{\mathbf{x},-\omega}. \tag{6.84}$$

We denote the correlation functions of the reference model as

$$S^{(j_1,\dots,j_n)}_{R;\omega_1,\dots,\omega_n}(\mathbf{x}_1,\dots,\mathbf{x}_n) := \lim_{M\to\infty} \frac{\partial^n}{\partial J^{(j_1)}_{\omega_1}(\mathbf{x}_1)\cdots\partial J^{(j_n)}_{\omega_n}(\mathbf{x}_n)} \mathcal{S}_R(\mathbf{J})\Big|_{\mathbf{J}=\mathbf{0}}. \tag{6.85}$$

For instance,

$$S^{(1,1)}_{R;\omega,\omega'}(\mathbf{x},\mathbf{y}) = \lim_{M\to\infty} \big(Z^{(1)}\big)^2 \big\langle \psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega}; \psi^+_{\mathbf{y},\omega'}\psi^-_{\mathbf{y},\omega'}\big\rangle^{(M)}_{R,\lambda_\infty}, \tag{6.86}$$

where $\langle\cdot\rangle^{(M)}_{R,\lambda_\infty}$ denotes (the $L\to\infty$ limit of) the average with respect to the measure of density

$$e^{-\mathcal{S}_R(\mathbf{0})} e^{\mathcal{V}(\sqrt{Z}\psi^{(\le M)})} P_Z\big(d\psi^{(\le M)}\big).$$

There is a clear analogy between the ($M\to\infty$ limit of the) reference model and the dimer model with $m=0$, once the latter is re-expressed in terms of Dirac variables $(\psi^\pm_{\mathbf{x},\omega})_{\mathbf{x}\in\Lambda}$ (see (2.30)). Indeed, the corresponding free propagators have the same asymptotic behavior at large distances, see Proposition 2. Also, recall (cf. (6.26)–(6.28)) that the local parts of the interaction potential and of the source term of the dimer model are given, in terms of Dirac variables, by

$$\lambda_0 \sum_{\mathbf{x}} \psi^+_{\mathbf{x},1}\psi^-_{\mathbf{x},1}\psi^+_{\mathbf{x},-1}\psi^-_{\mathbf{x},-1}, \qquad \sum_{\mathbf{x},\omega} \big(J^{(1)}_\omega(\mathbf{x})\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},\omega} + J^{(2)}_\omega(\mathbf{x})\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{x},-\omega}\big),$$

respectively, to be compared with (6.82)–(6.84). The analogy is approximate because the fields of the reference model are defined on the continuum and those of the dimer model on the lattice. However, the large-distance behavior of the correlation functions do turn out to be the same for the two models, see Propositions 11 and 12. For ease of comparison, let us introduce a convenient notation for the dimer correlation functions expressed in terms of Dirac fields: if

$$\mathcal{S}(\mathbf{J}) = \lim_{m\to 0} \lim_{\Lambda\nearrow\mathbb{Z}^2} \log \frac{\mathcal{Z}^{(11)}_\Lambda(\lambda,m,\mathbf{A})}{\mathcal{Z}^{(11)}_\Lambda(\lambda,m,\mathbf{0})} \tag{6.87}$$

is the ($m\to 0$ limit of the $\Lambda\nearrow\mathbb{Z}^2$ limit of the) generating function of correlations for the interacting dimer model, we let

$$S^{(j_1,\dots,j_n)}_{\omega_1,\dots,\omega_n}(\mathbf{x}_1,\dots,\mathbf{x}_n) := \frac{\partial^n}{\partial J^{(j_1)}_{\omega_1}(\mathbf{x}_1)\cdots\partial J^{(j_n)}_{\omega_n}(\mathbf{x}_n)} \mathcal{S}(\mathbf{J})\Big|_{\mathbf{J}=\mathbf{0}} \tag{6.88}$$

be the corresponding correlation functions, where the external fields $J^{(j)}_\omega(\mathbf{x})$ are related to $J_{\mathbf{x},j}$ via (6.27)–(6.28).

The generating and correlation functions $\mathcal{S}_R(\mathbf{J})$ and $S^{(j_1,\dots,j_n)}_{R;\omega_1,\dots,\omega_n}(\mathbf{x}_1,\dots,\mathbf{x}_n)$ of the reference model can be expressed in terms of trees, whose values are the same as those of the dominant trees contributing to the corresponding functions $\mathcal{S}(\mathbf{J})$ and $S^{(j_1,\dots,j_n)}_{\omega_1,\dots,\omega_n}(\mathbf{x}_1,\dots,\mathbf{x}_n)$ of the dimer model (once again, here we call "dominant" the trees where the propagators $g^{(h)}$ are replaced by the relativistic propagators $g^{(h)}_R$, as discussed in Remark 17). In particular, both types of trees are associated only with endpoints of type λ_h, $Z^{(1)}_h$, or $Z^{(2)}_h$, and the single-scale propagators have exactly the same form, once the identification between Dirac fields of continuum and discrete models is used.

A minor difference between the contributions to $\mathcal{S}_R(\mathbf{J})$ and the dominant contributions to $\mathcal{S}(\mathbf{J})$ lies in the fact that the trees contributing to $\mathcal{S}_R(\mathbf{J})$ have endpoints on all scales $\le M$, rather than ≤ 0; moreover, the sequence of running coupling constants $\lambda_{h,R}, Z_{h,R}, Z^{(1)}_{h,R}, Z^{(2)}_{h,R}$ of the reference model, corresponding to the initial data $\lambda_\infty, Z, Z^{(1)}, Z^{(2)}$ is different in general from the corresponding sequence of the dimer model. However, the key observation is the following.

Proposition 11. *The initial data $\lambda_\infty, Z, Z^{(1)}, Z^{(2)}$ of the reference model can be properly adjusted, so that $\lambda_{h,R}, Z_{h,R}, Z^{(1)}_{h,R}, Z^{(2)}_{h,R}$ are asymptotically the same as the constants of the dimer model, as $h\to-\infty$, namely, if $h\le 0$,*

$$|\lambda_h - \lambda_{h,R}| + \left|\frac{Z_h}{Z_{R,h}} - 1\right| \le C_\theta |\lambda|^2 2^{\theta h}, \qquad \left|\frac{Z^{(i)}_h}{Z^{(i)}_{R,h}} - 1\right| \le C_\theta |\lambda| 2^{\theta h}, \tag{6.89}$$

uniformly in M, for some $0 < \theta < 1$ as in Remark 16 and a suitable $C_\theta > 0$, provided λ is sufficiently small. In particular, the infrared fixed point of $\lambda_{h,R}$ is the same as the one of λ_h: $\lambda_{-\infty,R} = \lambda_{-\infty}$. $Z_{R,h}$ and $Z^{(i)}_{R,h}$ satisfy the first two of (6.79) with critical exponents that coincide with those of the dimer model, once all of them are expressed as functions of $\lambda_{-\infty}$.

For the proof, see [15], where a similar statement is proven for a quantum spin chain instead of the interacting dimer model. See in particular [15, Eq. (79)], where $Z^{(th)}_h$ is the same as our $Z_{R,h}$.

As anticipated above, the reason why it is useful to introduce the reference model at all is that it has more symmetries than the dimer model. In particular, its "action" $\mathcal{V} + \mathcal{B}$ is formally covariant under a "local chiral gauge transformation" $\psi^\pm_{\mathbf{x},\omega} \mapsto e^{\pm i\alpha_\omega(\mathbf{x})}\psi^\pm_{\mathbf{x},\omega}$ (here, "local" refers to the fact that the phase transformation depends on the point, "chiral" to the fact that it depends on ω, while "formally" means "up to corrections due to the ultraviolet regularization $\chi_M(\mathbf{k})$"). The latter induces exact identities (known as Ward Identities) between the correlation functions of the reference model, which in turn induce asymptotic identities between the correlations of the dimer model. By playing with these identities one can prove, among other things, Proposition 10, as well as the following equations for the correlation functions.

Proposition 12. *Fix the bare parameters $\lambda_\infty, Z, Z^{(i)}$ as in Proposition 11. Then the correlation functions of the reference and dimer models are asymptotically the same at large distances; more precisely, denoting by $D_{\underline{\mathbf{x}}}$ the diameter of the set $\underline{\mathbf{x}} := \{\mathbf{x}_1, \dots, \mathbf{x}_n\}$, $n \geq 2$, and by $\delta_{\underline{\mathbf{x}}}$ the minimal distance among the points in $\underline{\mathbf{x}}$, if $\delta_{\underline{\mathbf{x}}} \geq \max\{1, c_0 D_{\underline{\mathbf{x}}}\}$ for some $c_0 > 0$, then*

$$\left|S^{(j_1,\dots,j_n)}_{R;\omega_1,\dots,\omega_n}(\mathbf{x}_1,\dots,\mathbf{x}_n) - S^{(j_1,\dots,j_n)}_{\omega_1,\dots,\omega_n}(\mathbf{x}_1,\dots,\mathbf{x}_n)\right| \leq \frac{C_{n,\theta}}{D^{n+\theta}_{\underline{\mathbf{x}}}}, \tag{6.90}$$

for some $0 < \theta < 1$ as in Remark 16 and a suitable $C_{n,\theta} > 0$, which may depend on c_0.

Moreover, there exist functions $K_1(\cdot)$, $K_2(\cdot)$, $\kappa_2(\cdot)$, analytic in their argument in a neighborhood of zero, such that $K_1(0) = K_2(0) = \kappa_2(0) = 1$, and for all $\mathbf{x} \neq \mathbf{y}$

$$S^{(1,1)}_{R;\omega,\omega'}(\mathbf{x},\mathbf{y}) = \frac{\delta_{\omega,\omega'}}{(4\pi)^2}\frac{K_1(\lambda_{-\infty})}{((x_1-y_1)+i\omega(x_2-y_2))^2} + R^{(1)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y}), \tag{6.91}$$

$$S^{(1,2)}_{R;\omega,\omega'}(\mathbf{x},\mathbf{y}) = S^{(2,1)}_{R;\omega,\omega'}(\mathbf{x},\mathbf{y}) = 0, \tag{6.92}$$

$$S^{(2,2)}_{R;\omega,\omega'}(\mathbf{x},\mathbf{y}) = \frac{\delta_{\omega,-\omega'}}{(4\pi)^2}\frac{K_2(\lambda_{-\infty})}{|\mathbf{x}-\mathbf{y}|^{2\kappa_2(\lambda_{-\infty})}} + R^{(2)}_{\omega,\omega'}(\mathbf{x}-\mathbf{y}), \tag{6.93}$$

where, if $n_1, n_2 \geq 0$, the rest $R^{(i)}_{\omega,\omega'}$ satisfies

$$\left|\partial^{n_1}_{x_1}\partial^{n_2}_{x_2} R^{(i)}_{\omega,\omega'}(\mathbf{x})\right| \leq C'_{n_1+n_2,\theta}|\mathbf{x}|^{-2-\theta-n_1-n_2}$$

for $\theta \in (0,1)$ as in Remark 16 and $C'_{n,\theta} > 0$. Moreover, if $q > 2$,

$$S^{(1,1,\dots,1)}_{R;\omega_1,\dots,\omega_q}(\mathbf{x}_1;\dots;\mathbf{x}_q) = 0. \tag{6.94}$$

Eq. (6.90) for $n = 2$ and $j_1 = j_2 = 1$, and Eq. (6.91) are proved in [15]: (6.90) for $n = 2$ and $j_1 = j_2 = 1$ is the same as [15, Eq. (43)], while (6.91) is the same as [15, Eq. (41)], just expressed in real space rather than momentum space. The rest $R^{(1)}_{\omega,\omega'}$ can be written in closed form (as apparent from [15, Eq. (39)]), but we do not write it here explicitly, in order to avoid a further digression that would not be needed for our purposes. The proof of (6.90) for general values of $n, j_1, \dots, j_n$ is a corollary of the proof in [15]. Eq. (6.92) is a trivial consequence of the fact that the propagator (6.81) is diagonal in ω and the interaction (6.82) contains as many fields with $\omega = +$ as fields with $\omega = -$. Eqs (6.93) and

(6.94) are proven in [7, Theorem 1.1][7] in the case where $v(\mathbf{x}-\mathbf{y})$ is replaced by a local delta-like interaction (in this case the reference model is called Thirring model). If instead v is as in (6.82), then (6.93) and (6.94) can be proven by comparing the tree expansions of the Thirring and of the reference model (6.80), in the same spirit as one compares the expansions of the dimer and reference model, see discussion before Proposition 11.

The exponent κ_2 is related in a simple way to the exponents η and η_2 of Z_h and $Z_h^{(2)}$: it is equal to $1+\eta_2-\eta$, once η and η_2 are re-expressed as functions of $\lambda_{-\infty}=\lambda_{-\infty}(\lambda)$, rather than of λ.

Finally, note that Eq. (6.94) is the analogue of the cancellation (3.39) that we already used in the analysis of the non-interacting dimer model.

The usefulness of the formulas for the correlation functions in Proposition 12 is that they can be used to compute sharp estimates for the large distance behavior of the dimer correlation functions. These will be exploited in order to complete the proofs of Theorems 2, 1 and 3.

6.4. *The two-point dimer correlation*: *Proof of Theorem 2*

We are finally in the position of proving Theorem 2. We start from

$$\langle \mathbb{1}_{(\mathbf{x},\mathbf{x}+\hat{e}_j)}; \mathbb{1}_{(\mathbf{y},\mathbf{y}+\hat{e}_{j'})}\rangle_\lambda = \frac{\partial^2}{\partial J_{\mathbf{x},j}\partial J_{\mathbf{y},j'}}\mathcal{S}(\mathbf{J})\Big|_{\mathbf{J}=\mathbf{0}}, \tag{6.95}$$

where $\mathcal{S}(\mathbf{J})$ is defined in (6.88) (see also discussion after (7.2) below). Recalling the definition of $J^{(i)}_\omega(\mathbf{x})$ in terms of $J_{\mathbf{x},j}$ (cf. (6.27)–(6.28)), we can, if desired, re-express (6.95) in terms of the correlation functions for the Dirac fields (6.88). More explicitly,

$$\frac{\partial^2}{\partial J_{\mathbf{x},j}\partial J_{\mathbf{y},j'}}\mathcal{S}(\mathbf{J})\Big|_{\mathbf{J}=\mathbf{0}} = \sum_{\substack{\omega,\omega'=\pm\\ i,i'=1,2}} S^{(i,i')}_{\omega,\omega'}(\mathbf{x},\mathbf{y})\frac{\partial J^{(i)}_\omega(\mathbf{x})}{\partial J_{\mathbf{x},j}}\frac{\partial J^{(i')}_{\omega'}(\mathbf{y})}{\partial J_{\mathbf{y},j'}}. \tag{6.96}$$

Inserting (6.90) in (6.96) and using (6.92), we rewrite:

$$\begin{aligned}&\langle \mathbb{1}_{(\mathbf{x},\mathbf{x}+\hat{e}_j)}; \mathbb{1}_{(\mathbf{y},\mathbf{y}+\hat{e}_{j'})}\rangle_\lambda\\ &\quad=\sum_{i=1}^{2}\sum_{\omega,\omega'=\pm} S^{(i,i)}_{R;\omega,\omega'}(\mathbf{x},\mathbf{y})\frac{\partial J^{(i)}_\omega(\mathbf{x})}{\partial J_{\mathbf{x},j}}\frac{\partial J^{(i)}_{\omega'}(\mathbf{y})}{\partial J_{\mathbf{y},j'}}+\tilde{R}_{j,j'}(\mathbf{x}-\mathbf{y}),\end{aligned} \tag{6.97}$$

where

$$\big|\tilde{R}_{j,j'}(\mathbf{x}-\mathbf{y})\big| \le \frac{C_\theta}{|\mathbf{x}-\mathbf{y}|^{2+\theta}}, \tag{6.98}$$

with $\theta\in(0,1)$ as in Remark 16, for some $C_\theta>0$. Substituting (6.91) and (6.93) into (6.97), and using the definition of $J^{(i)}_\omega(\mathbf{x})$ in (6.27)–(6.28), we obtain (1.9), with $K(\lambda)=K_1(\lambda_{-\infty})$, $\tilde{K}(\lambda)=K_2(\lambda_{-\infty})$ and $\kappa(\lambda)=\kappa_2(\lambda_{-\infty})$. This concludes the proof of Theorem 2, and, therefore, as discussed in Section 4, of Theorem 1 for $n=2$.

7. Height fluctuations in the interacting model: Proof of Theorems 1 and 3 for $\lambda\neq 0$

In this section we use the renormalized tree expansion, the dimensional estimates on the renormalized trees, and the comparison between the dimer and reference models, discussed in the previous section, to complete the proof of our main results.

[7] To get (6.93) and (6.94) from [7, Theorem 1.1] one has to put to zero the parameter ζ there, in which case the Sine-Gordon model appearing in the l.h.s. of [7, Eq. (1.16)] reduces to the massless Gaussian Free Field. Analyticity of the functions $K_i(\cdot)$, $\kappa_-(\cdot)$ is not stated explicitly there, but it follows as byproduct from the proofs.

7.1. *Tree expansion for the correlation functions*

The multiscale construction described in the previous section induces a representation of the multipoint dimer correlation functions in terms of a renormalized tree expansion. We limit ourselves to the discussion of the correlations at distinct bonds, the general case being treatable in a similar manner. Using (2.53) and the discussion in Section 2.4, we find:

$$\langle \mathbb{1}_{b_1}; \dots; \mathbb{1}_{b_q} \rangle_\lambda := \lim_{m\to 0} \lim_{\Lambda \nearrow \mathbb{Z}^2} \langle \mathbb{1}_{b_1}; \dots; \mathbb{1}_{b_q} \rangle_{\Lambda;\lambda,m} \tag{7.1}$$

$$= \lim_{m\to 0} \lim_{\Lambda \nearrow \mathbb{Z}^2} \frac{\partial^k}{\partial A_{b_1} \cdots \partial A_{b_q}} \log \mathcal{Z}^{(11)}_\Lambda(\lambda, m, \mathbf{A}) \Big|_{\mathbf{A}=\mathbf{0}}$$

$$= \frac{\partial^q}{\partial J_{b_1} \cdots \partial J_{b_q}} \mathcal{S}(\mathbf{J}) \Big|_{\mathbf{J}=\mathbf{0}}, \tag{7.2}$$

where $\mathcal{S}(\mathbf{J})$ is defined in (6.88) and it can be computed via the iterative renormalized expansion described in the proof of Proposition 7: in particular, it can be written as $\mathcal{S}(\mathbf{J}) = \sum_{h\le 0} \tilde{S}^{(h)}(\mathbf{J})$, where $\tilde{S}^{(h)}(\mathbf{J})$ is the single-scale contribution to the generating function, see (6.51). Note that in the last line of (7.2) we exchanged a derivative with the limits $\Lambda \nearrow \mathbb{Z}^2$, $m \to 0$. This is justified by the fact that $\mathcal{S}(\mathbf{J})$ can be expressed via an absolutely convergent expansion, uniformly in Λ and m, as already discussed in Section 6.2.1 (see below for more details about the bounds on the tree values contributing to the correlation functions). For what follows, recall that as long as $m > 0$ the sum over h runs from h^* to 0 and that the limit $m \to 0$ corresponds to $h^* \to -\infty$; therefore, in the following formulas, we shall always replace h^* by $-\infty$.

The single-scale contribution $\tilde{S}^{(h)}(\mathbf{J})$ to $\mathcal{S}(\mathbf{J})$ can be written in a way similar to (6.7):

$$\tilde{S}^{(h)}(\mathbf{J}) = \sum_{q\ge 1} \sum_{j_1,\dots,j_q} \sum_{\mathbf{y}_1,\dots,\mathbf{y}_q} S^{(h)}_{q,\mathbf{j}}(\mathbf{y}_1, \dots, \mathbf{y}_q) \prod_{i=1}^{q} J_{\mathbf{y}_i, j_i}, \tag{7.3}$$

where $S^{(h)}_{q;\mathbf{j}}(\mathbf{y}_1, \dots, \mathbf{y}_q)$ collects the contributions to $W^{(h)}_{q;\mathbf{j}}(\mathbf{y}_1, \dots, \mathbf{y}_q)$ involving propagators on scales $> h$ and at least one propagator on scale $h+1$. Therefore,

$$\langle \mathbb{1}_{(\mathbf{y}_1, \mathbf{y}_1+\hat{e}_{j_1})}; \dots; \mathbb{1}_{(\mathbf{y}_q, \mathbf{y}_q+\hat{e}_{j_q})} \rangle_\lambda = q! \sum_{h\le 0} S^{(h)}_{q,\mathbf{j}}(\mathbf{y}_1, \dots, \mathbf{y}_q) \tag{7.4}$$

and, as explained in Section 6.2, $S^{(h)}_{q,\mathbf{j}}(\mathbf{y}_1, \dots, \mathbf{y}_q)$ can be expressed by a sum over trees $\tau \in \mathcal{T}^{(h)}_{N,n}$ with $n \le q$ *special endpoints*[8] and $N \ge 0$ normal end-points:

$$\langle \mathbb{1}_{(\mathbf{y}_1, \mathbf{y}_1+\hat{e}_{j_1})}; \dots; \mathbb{1}_{(\mathbf{y}_q, \mathbf{y}_q+\hat{e}_{j_q})} \rangle_\lambda$$

$$= \sum_{h\le 0} \sum_{N\ge 0} \sum_{n=1}^{q} \sum_{\tau\in\mathcal{T}^{(h)}_{N,n}} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau: \\ |P_{v_0}|=|P^J_{v_0}|=q}} S_{\tau,\mathbf{P}}(\mathbf{y}_1, j_1; \dots; \mathbf{y}_q, j_q). \tag{7.5}$$

Here $S_{\tau,\mathbf{P}}(\mathbf{y}_1, j_1; \dots; \mathbf{y}_m, j_m)$ is the tree value, which can be bounded in a way similar to Eq. (6.68). To give the bound we need a few extra definitions. Given $\tau \in \mathcal{T}^{(h)}_{N,n}$, let us denote by τ^* the minimal subtree of τ connecting all its special endpoints. For each $v \in \tau^*$, let s^*_v be the number of vertices immediately following v on τ^* such that $|P^J_v| \ge 1$ (i.e., the number of descendants of v in τ^*). Moreover, let $V_{nt}(\tau^*)$ be the set of vertices in τ^* with $s^*_v > 1$, which are the branching points of τ^*. For future reference, we also define v^*_0 to be the leftmost vertex on τ^* and h^*_0 its scale. See Figure 13.

[8] The reason why $n \le q$ rather than $n = q$ is that some special endpoints – those on scale 1 – could be associated with monomials of order two or more in the $\mathbf{J}$ fields, i.e. a monomial of type $\tilde{\xi}(\gamma; R)$ with $|R| > 1$, as the one depicted graphically in Figure 8.

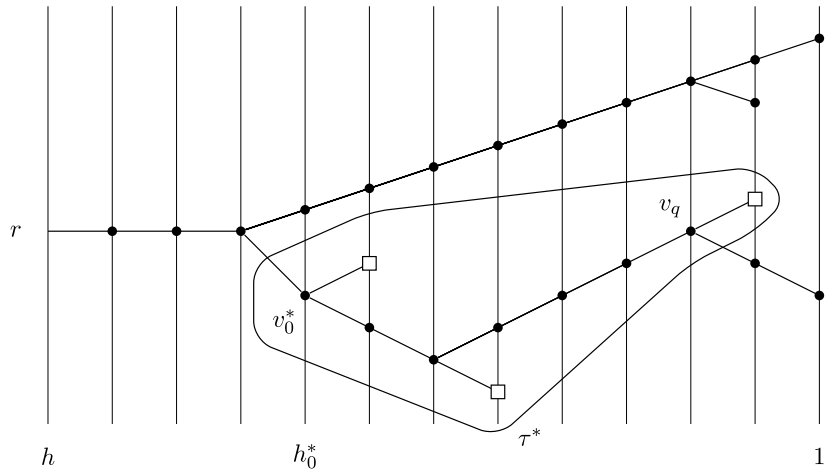

Fig. 13. Example of a tree $\tau \in \mathcal{T}^{(h)}_{N,n}$ appearing in the expansion for the m points correlation function, with $N = 3$ and $n = q = 3$. The subtree τ^* associated with τ is highlighted.

Given these definitions, we can write the bound for $S_{\tau,\mathbf{P}}(\mathbf{y}_1, j_1; \dots; \mathbf{y}_q, j_q)$ as

$$\begin{aligned}
&\left|S_{\tau,\mathbf{P}}(\mathbf{y}_1, j_1; \dots; \mathbf{y}_q, j_q)\right| \\
&\quad \le q! C^{N+n} \left(C|\lambda|\right)^{1/2|I^{\psi}_{v_0}|-N} 2^{h(2-q)} \left[\prod_{v \in V_{nt}(\tau^*)} 2^{2(s^*_v-1)h_v} e^{-c\sqrt{2^{h_v}\delta_v}}\right] \\
&\quad \times \left[\prod_{v \text{ s.e.p.}} \left|\frac{Z^{(j_v)}_{h_v-1}}{Z_{h_v-1}}\right|\right] \left[\prod_{\substack{v \text{ not} \\ \text{e.p.}}} 2^{c\lambda^2|P^{\psi}_v|} 2^{2-1/2|P^{\psi}_v|-|P^J_v|-z(P_v)}\right],
\end{aligned} \tag{7.6}$$

which is very similar to the bound Eq. (6.68) for the renormalized kernels of the effective potential. In particular, $z(P_v)$ is given by (6.69). Note that the assumptions (6.66)–(6.67) are verified, thanks to Proposition 10. In comparison with (6.68), note the presence in (7.6) of the product over the vertices $v \in V_{nt}(\tau^*)$ of $2^{2(s^*_v-1)h_v} e^{-c\sqrt{2^{h_v}\delta_v}}$, where δ_v is the tree distance of the set $\bigcup_{f \in P^J_v} \{\mathbf{x}(f)\}$, i.e. the length of the shortest tree graph on $\mathbb{Z}^2$ connecting its points $(\mathbf{x}(f), f \in P^J_v$, is one of the coordinates $\mathbf{y}_i)$. In this product, the factors $2^{2(s^*_v-1)h_v}$ take into account the dimensional gain coming from the fact that we are *not* summing over the space labels $\mathbf{y}_i$ of the external fields (the gain is meant in comparison with Eq. (6.68) where, on the contrary, we summed over all the field variables). Moreover, the factors $e^{-c\sqrt{2^{h_v}\delta_v}}$ come from the decaying factors $e^{-c\sqrt{2^{h_w}|\mathbf{x}(\ell)-\mathbf{x}'(\ell)|}}$ associated with the propagators $g^{(h_w)}_\ell$, with $w \ge v$ and $\ell \in T$ (the notation $g^{(h_w)}_\ell$ is as in Lemma 3; the exponential factor comes from the estimate on the propagator on scale h_w, cf. Lemma 2 and the comment after (6.45)). See [6, Section 2.3] for a few more details.

Remark 20. *In the following we will actually need improved bounds on the tree values, as compared to* (7.6)*. The improvements will be based on a decomposition of the tree values into a dominant part plus a rest, combined with a crucial cancellation in the dominant part, induced by* (6.94)*. Rather than presenting the improved bounds directly, we prefer to state* (7.6) *first, and then explain how to obtain an extra dimensional gain for the different contributions to* (7.5)*, in order to make the ideas behind the proof of these dimensional gains more transparent.*

As in Section 6.3.1, the tree expansion (7.5) is refined by decomposing the single scale propagators as in (6.72). The "refined tree expansion" brings along an extra set of labels, which distinguishes the fields associated with relativistic propagators $g^{(h)}_R$ from those with non-relativistic propagators $r^{(h)}$. We call "dominant" the contributions from trees with endpoints on scales ≤ 0 and involving only relativistic propagators, as in Remark 17. In the perspective of computing the height fluctuations, it is convenient to distinguish two classes of terms among the dominant ones: those

with all the special endpoints of type $Z_{h_v}^{(1)}$, and the rest. The final decomposition we shall use takes the following form:

$$\langle \mathbb{1}_{(\mathbf{y}_1,\mathbf{y}_1+\hat{e}_{j_1})};\dots;\mathbb{1}_{(\mathbf{y}_q,\mathbf{y}_q+\hat{e}_{j_q})}\rangle_\lambda = \frac{\partial^q}{\partial J_{\mathbf{y}_1,j_1}\cdots\partial J_{\mathbf{y}_q,j_q}}\mathcal{S}(\mathbf{J})\Big|_{\mathbf{J}=\mathbf{0}}$$
$$= \mathcal{S}^{(1)}_{q,\mathbf{j}}(\mathbf{y}_1,\dots,\mathbf{y}_q)+\mathcal{S}^{(2)}_{q,\mathbf{j}}(\mathbf{y}_1,\dots,\mathbf{y}_q)+\mathcal{S}^{(3)}_{q,\mathbf{j}}(\mathbf{y}_1,\dots,\mathbf{y}_q), \tag{7.7}$$

where: $\mathcal{S}^{(1)}$ collects all the dominant contributions from trees whose special endpoints are all of type $Z_h^{(1)}$; $\mathcal{S}^{(2)}$ collects all the dominant contributions from trees with at least one special endpoint of type $Z_h^{(2)}$; $\mathcal{S}^{(3)}$ collects all the subdominant contributions, i.e., the contributions from trees with at least one endpoint on scale 1, or at least one propagator of type $r^{(h)}$.

7.2. *Multipoint dimer correlation and height cumulants*

In order to compute the cumulant of order $q>2$ of the height difference, we start from (3.2), with n replaced by q. Proceeding as in (3.33), in the q-fold sum over the bonds $b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}},\dots,b_q\in\mathcal{C}^{(q)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$, we distinguish a contribution that includes the terms where all the bonds are outside the two balls $B_{r_q}(\boldsymbol{\xi})$, $B_{r_q}(\boldsymbol{\eta})$, from the rest. By construction, the former contribution involves bond configurations such that the bonds are all mutually disjoint, and is the most difficult to bound. For simplicity, we limit our discussion to these terms, leaving the analysis of the rest to the reader. We write them in the form

$$\sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\cdots\sum_{b_q\in\mathcal{C}^{(q)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\big(\mathcal{S}^{(1)}_{q,\mathbf{j}}(\mathbf{x}_1,\dots,\mathbf{x}_q)+\mathcal{S}^{(2)}_{q,\mathbf{j}}(\mathbf{x}_1,\dots,\mathbf{x}_q)+\mathcal{S}^{(3)}_{q,\mathbf{j}}(\mathbf{x}_1,\dots,\mathbf{x}_q)\big), \tag{7.8}$$

where $\mathbf{x}_i$, j_i are such that $b_i=(\mathbf{x}_i,\mathbf{x}_i+\hat{e}_{j_i})$, and the $*$ on the sums indicate the constraint that all the bonds are outside $B_{r_q}(\boldsymbol{\xi})\cup B_{r_q}(\boldsymbol{\eta})$. In the following, we analyze the terms coming from $\mathcal{S}^{(1)}_{q,\mathbf{j}}$ first, and then we discuss the other two contributions.

7.2.1. *The contributions of type* $\mathcal{S}^{(1)}_{q,\mathbf{j}}$

For these terms, we use the cancellation (6.94) for the correlations of the reference model, which implies that the analog of $\mathcal{S}^{(1)}_{q,\mathbf{j}}$ in the reference model, to be called $\mathcal{S}^{(1)}_{R;q,\mathbf{j}}$, is identically zero:

$$\mathcal{S}^{(1)}_{R;q,\mathbf{j}}(\mathbf{x}_1,\dots,\mathbf{x}_q):=\sum_{\omega_1,\dots,\omega_q}S^{(1,\dots,1)}_{R;\omega_1,\dots,\omega_q}(\mathbf{x}_1,\dots,\mathbf{x}_q)\prod_{l=1}^{q}\frac{\partial J^{(1)}_{\omega_l}(\mathbf{x}_l)}{J_{\mathbf{x}_l,j_l}}\equiv 0, \tag{7.9}$$

where, for $\mathbf{x}\in\mathbb{Z}^2$, $J^{(1)}_\omega(\mathbf{x})=(-1)^{\mathbf{x}}(J_{\mathbf{x},1}+i\omega J_{\mathbf{x},2})$, as in (6.27). Therefore, we can add and subtract $\mathcal{S}^{(1)}_{R;q,\mathbf{j}}$, thus finding

$$\mathcal{S}^{(1)}_{q,\mathbf{j}}(\mathbf{x}_1,\dots,\mathbf{x}_q)=\mathcal{S}^{(1)}_{q,\mathbf{j}}(\mathbf{x}_1,\dots,\mathbf{x}_q)-\mathcal{S}^{(1)}_{R;q,\mathbf{j}}(\mathbf{x}_1,\dots,\mathbf{x}_q), \tag{7.10}$$

which implies

$$\Big|\sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\cdots\sum_{b_q\in\mathcal{C}^{(q)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\mathcal{S}^{(1)}_{q,\mathbf{j}}(\mathbf{x}_1,\dots,\mathbf{x}_q)\Big|$$
$$\le\sum_h\sum_{N\ge 0}\sum_{\tau\in\mathcal{T}^{(h)}_{N,q}}^{(1)}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\ |P_{v_0}|=|P^J_{v_0}|=q}}\sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\cdots\sum_{b_q\in\mathcal{C}^{(q)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\big|S^{\mathrm{dom}}_{\tau,\mathbf{P}}(\mathbf{x}_1,j_1;\dots;\mathbf{x}_q,j_q)-S^{R}_{\tau,\mathbf{P}}(\mathbf{x}_1,j_1;\dots;\mathbf{x}_q,j_q)\big|. \tag{7.11}$$

Here, the apex (1) on the sum over the trees recalls that we are summing over the contributions associated with $\mathcal{S}^{(1)}_{q,\mathbf{j}}$. We denoted by $S^{\mathrm{dom}}_{\tau,\mathbf{P}}$ the dominant contribution to the value of the tree τ (i.e. the contribution obtained by replacing each propagator $g^{(h)}$ with $g^{(h)}_R$, see (6.72)), and by $S^R_{\tau,\mathbf{P}}$ the tree value computed in the relativistic reference model. The sum over h ranges between $-\infty$ and M, where M is the ultraviolet cut-off of the reference model, to be eventually sent to infinity.

We distinguish three types of contributions, that we treat separately:

(a) Those associated with the trees with endpoints all on scales ≤ 0, each of which comes in the form of a difference between the dominant contribution of the tree value in the dimer model, and the corresponding tree value in the reference model. These contributions are the same, modulo the fact that the effective constants associated with the endpoints of the tree for the dimer model are λ_h, Z_h, $Z^{(1)}_h$, while those in the tree for the reference model are $\lambda_{R,h}$, $Z_{R,h}$, $Z^{(1)}_{R,h}$. Recall that the difference between these effective constants is bounded as in Proposition 11. Therefore, the contribution associated with each of these trees is bounded in a way similar to (7.6), times an extra factor $2^{\theta h_w}$, with w the right-most endpoint of the tree.

(b) Those associated with the trees that have root at scale $h < 0$ but have at least one endpoint on scale $h_v \ge 1$. Since these terms do not appear (by definition) in $\mathcal{S}^{(1)}_{q,\mathbf{j}}$, we have $|S^{\mathrm{dom}}_{\tau,\mathbf{P}} - S^R_{\tau,\mathbf{P}}| = |S^R_{\tau,\mathbf{P}}|$. These terms will turn out to be negligible due to the short memory property (Remark 16).

(c) Those associated with trees with root at scale $h \ge 0$. Also in this case, $S^{\mathrm{dom}}_{\tau,\mathbf{P}} = 0$.

We claim that the sum in the r.h.s. of (7.11) can be bounded by

$$
\begin{aligned}
& C_q \sum_{h=-\infty}^{+\infty} 2^{h(2-q)} \min\{2^{\theta' h}, e^{-c'\sqrt{2^h \delta_{\min}}}\} \sum_{N\ge 0} C^N |\lambda|^N \\
& \times \sum_{\tau\in\mathcal{T}^{(h)}_{N,q}}^{(1)} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau: \\ |P_{v_0}|=|P^J_{v_0}|=q}} \sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \cdots \sum_{b_q\in\mathcal{C}^{(q)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \Big[\prod_{v \text{ s.e.p.}} \Big|\frac{Z^{(1)}_{R;h_v-1}}{Z_{R;h_v-1}}\Big|\Big]\Big[\prod_{v\in V_{nt}(\tau^*)} 2^{2(s^*_v-1)h_v} e^{-c'\sqrt{2^{h_v}\delta_v}}\Big] \\
& \times \Big[\prod_{v \text{ not e.p.}} 2^{c\lambda^2|P^\psi_v|} 2^{2-\frac{1}{2}|P^\psi_v|-|P^J_v|-z(P_v)+\theta'}\Big]
\end{aligned}
\tag{7.12}
$$

for some positive small $\theta', c' > 0$, where we recall that the "pruned tree" τ^* was defined after (7.5), see Figure 13. Here, if as usual $b_i = (\mathbf{x}_i, \mathbf{x}_i + \hat{e}_{j_i})$,

$$
\delta_{\min} = \min_{1\le i\ne j\le q} \min_{b_i,b_j} |\mathbf{x}_i - \mathbf{x}_j|,
$$

with the minimum taken over all possible locations of b_i, b_j appearing in the sum (7.12) and $\delta_{\min} \ge 1$ since the bonds b_i, b_j are disjoint for $i \ne j$. Let us see why (7.12) holds. First consider the trees of type (a), for which $h < 0$: as we explained, each of these trees satisfies the estimate (7.6), times an extra factor $2^{\theta h_w}$, with w the right-most endpoint of the tree. We can replace $2^{\theta h_w}$ by $2^{\theta' h}$ with some $0 < \theta' \le \theta$, provided we add θ' to the exponent $2 - \frac{1}{2}|P^\psi_v| - |P^J_v| - z(P_v)$ at each vertex that is not an endpoint. Of course, we will choose θ' sufficiently small so that the exponents remain strictly negative at each vertex (recall (6.70)). Also, we have used $h_v \ge h$ and $\delta_v \ge \delta_{\min}$. Next consider trees of type (c), for which $h \ge 0$. In this case, the dimensional gain arises only from the factors $e^{-c\sqrt{2^{h_v}\delta_v}}$ in the second line of (7.6), which are smaller than $e^{-(c/2)\sqrt{2^{h_v}\delta_v}} e^{-(c/2)\sqrt{2^h\delta_{\min}}}$. As for the trees of type (b), the dimensional gain comes from the short memory property. More precisely, since there is at least a vertex on scale 0, we can extract from the bound (7.6) a factor $2^{\theta' h}$ provided we add θ' to every exponent $2 - \frac{1}{2}|P^\psi_v| - |P^J_v| - z(P_v)$.

To prove that the $q > 2$ cumulants of the height differences do not diverge with the distance, it remains to show that (7.12) is bounded by some constant depending only on q. By Propositions 10 and 11, the critical exponent of $Z^{(1)}_{R;h}$ is

equal to the one of $Z_{R;h}$, and the ratios $|Z^{(1)}_{h_v-1}/Z_{h_v-1}|$ can be bounded from above by a constant, independent of h_v. Moreover, by proceeding as in the proof of (3.44), we find that, for a suitable $C'_q > 0$,

$$\sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\cdots\sum_{b_q\in\mathcal{C}^{(q)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\prod_{v\in V_{nt}(\tau^*)}e^{-c'\sqrt{2^{h_v}\delta_v}}\le C'_q\prod_{v\in V_{nt}(\tau^*)}2^{-h_v\bar m^J_v}, \tag{7.13}$$

where $\bar m^J_v$ is the number of special endpoints contained in the cluster v but not in any other cluster $v'>v$ (i.e. the number of special endpoints immediately following v, on scale h_v+1). To get (7.13), we used

$$\delta_v\ge c_q\sum_{f\in P^J_v}\min\bigl(d(\mathbf{x}_f,\boldsymbol{\xi}),d(\mathbf{x}_f,\boldsymbol{\eta})\bigr) \tag{7.14}$$

for some $c_q>0$, see also the comment before (3.42) when $|P^J_v|=2$. In conclusion, (7.11) is bounded by

$$\begin{aligned}&C''_q\sum_{h=-\infty}^{+\infty}2^{h(2-q)}\min\bigl\{2^{\theta' h},e^{-c'\sqrt{2^h\delta_{\min}}}\bigr\}\sum_{N\ge0}C^N|\lambda|^N\\&\quad\times\sum_{\tau\in\mathcal{T}^{(h)}_{N,q}}^{(1)}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\|P_{v_0}|=|P^J_{v_0}|=q}}\Bigl[\prod_{v\in V_{nt}(\tau^*)}2^{h_v(2s^*_v-2-\bar m^J_v)}\Bigr]\Bigl[\prod_{v\text{ not e.p.}}2^{\bar d_v(P_v)}\Bigr],\end{aligned} \tag{7.15}$$

where $\bar d_v(P_v)=2-|P^\psi_v|(1/2-c\lambda^2)-|P^J_v|-z_v+\theta'$ which, from (6.70), is negative and actually smaller than $-1+\varepsilon$, for any $\varepsilon>0$, if λ and θ' are small enough.

By proceeding as in the proof of (3.48), we find

$$\prod_{v\in V_{nt}(\tau^*)}2^{h_v(2s^*_v-2-\bar m^J_v)}=2^{h^*_0(q-2)}\prod_{v\in V(\tau^*)}2^{|P^J_v|-2}, \tag{7.16}$$

where $V(\tau^*)$ is the set of vertices of τ^* that are not endpoints and h^*_0-1 is the scale of the root of τ^*. Note that the factor $2^{h^*_0(q-2)}$, multiplied by the factor $2^{h(2-q)}$ that appears in (7.15), equals the product of $2^{q-2}=2^{|P^J_v|-2}$ over all the vertices on the branch joining the root of τ with the root of τ^*. Therefore, (7.15) is bounded by

$$C''_q\sum_{h=-\infty}^{+\infty}\min\bigl\{2^{\theta' h},e^{-c'\sqrt{2^h\delta_{\min}}}\bigr\}\sum_{N\ge0}C^N|\lambda|^N\sum_{\tau\in\mathcal{T}^{(h)}_{N,q}}^{(1)}\sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\|P_{v_0}|=|P^J_{v_0}|=q}}\prod_{v\text{ not e.p.}}2^{\hat d_v(P_v)}, \tag{7.17}$$

where

$$\hat d_v(P_v)=\begin{cases}-|P^\psi_v|(1/2-c\lambda^2)-z_v+\theta' & \text{if } |P^J_v|>0,\\ \bar d_v(P_v) & \text{otherwise.}\end{cases} \tag{7.18}$$

Note that $\hat d_v\le -a<0$ for every v and a suitable constant a independent of λ, provided λ and θ' are small enough. From this, it follows that (7.17) is summable over $\mathbf{P}$, τ and h (recall $\delta_{\min}\ge1$), the result being a finite, q-dependent, constant, as desired.

7.2.2. *The contributions of type* $\mathcal{S}^{(2)}_{q,\mathbf{j}}$

Let us now consider $\mathcal{S}^{(2)}_{q,\mathbf{j}}$, which is apriori very dangerous, in that each of the trees contributing to it is bounded as in (7.6), without any extra obvious gain (i.e., (6.94) is not true if the upper index is not $(1,\dots,1)$). Nevertheless, as was

the case also for $q = 2$ in Section 7.2, the dimensional gain arises from oscillating factors, when summing over the bonds b_j in the paths $\mathcal{C}^{(j)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$, $j \le q$.

The contribution to the qth cumulant of the height difference from terms of type $\mathcal{S}^{(2)}_{q,\mathbf{j}}$ is of the form

$$\sum_{\substack{h\le 0\\ N\ge 0}} \sum_{\tau\in\mathcal{T}^{(h)}_{N,q}}^{(2)} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\ |P_{v_0}|=|P^J_{v_0}|=q}} \sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \cdots \sum_{b_q\in\mathcal{C}^{(q)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \sigma_{b_1}\cdots\sigma_{b_q} S^{\mathrm{dom}}_{\tau,\mathbf{P}}(\mathbf{x}_1, j_1; \ldots; \mathbf{x}_q, j_q), \tag{7.19}$$

where the notation is analogous to the one used above for the contributions of type $\mathcal{S}^{(1)}_{q,\mathbf{j}}$. An important difference is that here we do not take absolute values, since we want to take advantage of the signs σ_b.

Note in fact that every dominant tree is naturally associated with an oscillatory factor, which is equal to the product of the oscillatory factors $(-1)^{\mathbf{x}}$ or $(-1)^{x_i}$ associated with the special endpoints of these trees (see Eqs (6.23) through (6.25)). The value of a dominant tree equals this oscillatory factor times a "non-oscillatory" value (see below for more details), obtained by contracting via relativistic propagators (which by definition have no oscillatory factors attached) the contributions that are left attached to all the endpoints. Now, it is apparent from (6.24) that all the trees contributing to $\mathcal{S}^{(1)}_{q,\mathbf{j}}(\mathbf{x}_1, \ldots, \mathbf{x}_q)$ have the same oscillatory factor, equal to $(-1)^{\mathbf{x}_1+\cdots+\mathbf{x}_q}$. This compensates *exactly* with the factor $(-1)^{\mathbf{x}_1+\cdots+\mathbf{x}_q}$ from the product of σ_b, see (3.12).

The situation is different for $\mathcal{S}^{(2)}_{q,\mathbf{j}}(\mathbf{x}_1, \ldots, \mathbf{x}_q)$: we recall that the trees involved in this expression have at least one special endpoint of type $Z^{(2)}_h$. If we denote by $\{(\mathbf{x}_i, j_i)\}_{i\in I_2}$ the set of points and directions associated with the endpoints of type $Z^{(2)}_h$ (here $I_2 \subset \{1, \ldots, q\}$ is a suitable *nonempty* index set), then $S^{\mathrm{dom}}_{\tau,\mathbf{P}}(\mathbf{x}_1, j_1; \ldots; \mathbf{x}_q, j_q)$ comes with the oscillatory factor $[\prod_{i\in I_2}(-1)^{(\mathbf{x}_i)_{j_i}}][\prod_{i\in I_2^c}(-1)^{\mathbf{x}_i}]$, where $I_2^c = \{1, \ldots, q\} \setminus I_2$. This means that

$$S^{\mathrm{dom}}_{\tau,\mathbf{P}}(\mathbf{x}_1, j_1; \ldots; \mathbf{x}_q, j_q) = \Big[\prod_{i\in I_2}(-1)^{(\mathbf{x}_i)_{j_i}}\Big]\Big[\prod_{i\in I_2^c}(-1)^{\mathbf{x}_i}\Big]\tilde{S}^{\mathrm{dom}}_{\tau,\mathbf{P}}(\mathbf{x}_1, j_1; \ldots; \mathbf{x}_q, j_q), \tag{7.20}$$

where $\tilde{S}^{\mathrm{dom}}_{\tau,\mathbf{P}}$ is a "non-oscillatory" function, in the sense that it satisfies the following natural scaling properties: if $\mathbf{n} = (n_1, n_2)$ and $\partial^{\mathbf{n}}_{\mathbf{x}} = \partial^{n_1}_{x_1}\partial^{n_2}_{x_2}$ with ∂_{x_i} the discrete derivative in the ith coordinate direction,

$$\begin{aligned}\Big|\Big[\prod_{i=1}^{q}\partial^{\mathbf{n}_i}_{\mathbf{x}_i}\Big]\tilde{S}^{\mathrm{dom}}_{\tau,\mathbf{P}}(\mathbf{x}_1, j_1; \ldots; \mathbf{x}_q, j_q)\Big| &\le q!C^{N+q}|\lambda|^N 2^{h(2-q)}\\ &\times\Big[\prod_{v\ \text{s.e.p.}} C_{n_v}2^{h_v n_v}\frac{Z^{(i_v)}_{h_v-1}}{Z_{h_v-1}}\Big]\Big[\prod_{v\in V_{nt}(\tau^*)} 2^{2(s^*_v-1)h_v}e^{-c\sqrt{2^{h_v}\delta_v}}\Big]\\ &\times\Big[\prod_{v\ \text{not e.p.}} 2^{c\lambda^2|P^\psi_v|}2^{2-1/2|P^\psi_v|-|P^J_v|-z(P_v)}\Big],\end{aligned} \tag{7.21}$$

where, if v is the special endpoint with label $\mathbf{x}_v = \mathbf{x}_i$, with $i \in \{1, \ldots, q\}$, then $n_v = (\mathbf{n}_i)_1 + (\mathbf{n}_i)_2$. This bound differs from (7.6) just by the dimensional factors $2^{h_v n_v}$, which arise from the action of the derivatives $\partial^{\mathbf{n}_i}_{\mathbf{x}_i}$ on a relativistic propagator $g^{(h_v)}_R$, cf.(6.73).

Now, using (3.12) and (7.20), we rewrite (7.19) as

$$\begin{aligned}&\sum_{\substack{h\le 0\\ N\ge 0}} \sum_{\tau\in\mathcal{T}^{(h)}_{N,q}}^{(2)} \sum_{\substack{\mathbf{P}\in\mathcal{P}_\tau:\\ |P_{v_0}|=|P^J_{v_0}|=q}} \sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \cdots \sum_{b_q\in\mathcal{C}^{(q)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*} \alpha_{b_1}(-1)^{j_1}\cdots\alpha_{b_q}(-1)^{j_q}\\ &\quad\times\Big[\prod_{i\in I_2}(-1)^{(\mathbf{x}_i)_{3-j_i}}\Big]\tilde{S}^{\mathrm{dom}}_{\tau,\mathbf{P}}(\mathbf{x}_1, j_1; \ldots; \mathbf{x}_q, j_q).\end{aligned} \tag{7.22}$$

Using the fact that the paths $\mathcal{C}^{(i)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$ consist of straight portions, each of which is formed by an even number of bonds, we find that

$$\Big|\sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\cdots\sum_{b_q\in\mathcal{C}^{(q)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\Big[\prod_{i=1}^{q}\alpha_{b_i}(-1)^{j_i}\Big]\Big[\prod_{i\in I_2}(-1)^{(\mathbf{x}_i)_{3-j_i}}\Big]\tilde{S}^{\mathrm{dom}}_{\tau,\mathbf{P}}(\mathbf{x}_1,j_1;\ldots;\mathbf{x}_q,j_q)\Big|$$
$$\le\sum_{b_1\in\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\cdots\sum_{b_q\in\mathcal{C}^{(q)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}}^{*}\Big|\Big[\prod_{i\in I_2}\partial_{(\mathbf{x}_i)_{3-j_i}}\Big]\tilde{S}^{\mathrm{dom}}_{\tau,\mathbf{P}}(\mathbf{x}_1,j_1;\ldots;\mathbf{x}_q,j_q)\Big|. \tag{7.23}$$

Finally, we recognize that the summand in the r.h.s. of this equation can be bounded by the r.h.s. of (7.21), with the factor

$$\Big[\prod_{v\ \mathrm{s.e.p.}}C_{n_v}2^{h_v n_v}\frac{Z^{(i_v)}_{h_v}}{Z_{h_v}}\Big]$$

replaced in this specific case by

$$\Big[\prod_{\substack{v\ \mathrm{s.e.p.:}\\ i_v=1}}\frac{Z^{(1)}_{h_v}}{Z_{h_v}}\Big]\Big[\prod_{\substack{v\ \mathrm{s.e.p.:}\\ i_v=2}}C_1 2^{h_v}\frac{Z^{(2)}_{h_v}}{Z_{h_v}}\Big]\le C^q\Big[\prod_{\substack{v\ \mathrm{s.e.p.:}\\ i_v=2}}2^{h_v(1+\eta_2-\eta)}\Big], \tag{7.24}$$

where we used the fact that, thanks to Proposition 10, $Z^{(1)}_h/Z_h\le(\text{const.})$ and $Z^{(2)}_h/Z_h\le(\text{const.})2^{(\eta_2-\eta)h}$. Since the number of special endpoints of type $Z^{(2)}_h$ is at least 1, the product in the r.h.s. of this equation is smaller than $2^{\theta\bar{h}}$, where θ is a suitable constant between zero and one, and $\bar{h}$ is the largest among the scales of the special endpoints of type $Z^{(2)}_h$. As we did for the terms of type (a) of $\mathcal{S}^{(1)}_{q,\mathbf{j}}$, the dimensional gain $2^{\theta\bar{h}}$ can be "transferred to the root," i.e. transformed into $2^{\theta' h}$ provided θ' is added to the exponent $2-|P^{\psi}_v|(1/2-c\lambda^2)-|P^{J}_v|-z(P_v)$ of each node. At that point, one proceeds like after (7.12).

7.2.3. *The contributions of type* $\mathcal{S}^{(3)}_{q,\mathbf{j}}$

We are finally left with $\mathcal{S}^{(3)}_{q,\mathbf{j}}$, which can be treated in a way similar to (and actually simpler than) the previous cases: the trees contributing to it either contain a non-relativistic propagator $r^{(h_w)}$, which produce an extra factor $2^{\theta h_w}$ (which can be "transferred to the root" by using the short memory property, as for the terms of type (a) of $\mathcal{S}^{(1)}_{q,\mathbf{j}}$); or contain endpoints on scale 1, in which case the short memory property produces an extra factor $2^{\theta h}$. In addition to these gains, one should take into account that all the special endpoints of type 2, possibly appearing in a tree contributing to $\mathcal{S}^{(3)}_{q,\mathbf{j}}$, whose presence produces a dimensional factor $2^{h(\eta_2-\eta)}$ (which may be $\gg 1$, if $\eta_2-\eta<0$), are associated with an oscillatory factor that effectively acts as a derivative operator, thus improving the factor $2^{h(\eta_2-\eta)}$ into $2^{h(1+\eta_2-\eta)}$, precisely as discussed for $\mathcal{S}^{(2)}_{q,\mathbf{j}}$. Details are left to the reader. Summarizing, also the contributions of type $\mathcal{S}^{(3)}_{q,\mathbf{j}}$ give rise to a finite (q-dependent) constant, which concludes the proof of Theorem 1.

7.3. *Proof of Theorem 3*

Since convergence of the moments of a random variable ξ_n to those of a Gaussian random variable ξ implies convergence in law (and therefore in the sense of the characteristic function) of ξ_n to ξ [16, Section 26 and 30], we need only to prove that

$$\lim_{\epsilon\to 0}\big\langle h^{\epsilon}(\phi)\big\rangle_{\lambda}=0, \tag{7.25}$$

$$\lim_{\epsilon\to 0}\big\langle h^{\epsilon}(\phi);h^{\epsilon}(\phi)\big\rangle_{\lambda}=\int\phi(x)\phi(y)G_{\lambda}(x-y)\,dx\,dy \tag{7.26}$$

and

$$\lim_{\epsilon\to 0}\underbrace{\big\langle h^\epsilon(\phi);\dots;h^\epsilon(\phi)\big\rangle_\lambda}_{q\text{ times}}=0,\quad q>2. \tag{7.27}$$

Note that (7.25) is trivial (and does not need the limit $\epsilon\to 0$) since the height is fixed to zero at the central face ("the origin" $\mathbf{0}$) and height gradients have zero expectation by construction, recall (1.4).

Proof of (7.26). Choose a face $\mathbf{p}$ at a distance of order $1/\epsilon$ from the support of $\phi(\epsilon\cdot)$ and rewrite

$$\big\langle h^\epsilon(\phi);h^\epsilon(\phi)\big\rangle_\lambda=\epsilon^4\sum_{\boldsymbol{\eta}_1,\boldsymbol{\eta}_2}\phi(\epsilon\boldsymbol{\eta}_1)\phi(\epsilon\boldsymbol{\eta}_2)\big\langle(h_{\boldsymbol{\eta}_1}-h_{\mathbf{p}});h_{\boldsymbol{\eta}_2}\big\rangle_\lambda \tag{7.28}$$

$$+\epsilon^4\sum_{\boldsymbol{\eta}_1,\boldsymbol{\eta}_2}\phi(\epsilon\boldsymbol{\eta}_1)\phi(\epsilon\boldsymbol{\eta}_2)\langle h_{\mathbf{p}};h_{\boldsymbol{\eta}_2}\rangle_\lambda. \tag{7.29}$$

Let us show first of all that (7.29) is $o(1)$. Indeed, since ϕ is smooth and of zero average,

$$\epsilon^2\sum_{\boldsymbol{\eta}}\phi(\epsilon\boldsymbol{\eta})=O(\epsilon). \tag{7.30}$$

Also, from (1.7) of Theorem 1 and Cauchy–Schwarz it follows that $|\langle h_{\mathbf{p}};h_{\boldsymbol{\eta}_2}\rangle_\lambda|=O(\log(1/\epsilon))$ (write $h_{\boldsymbol{\eta}_2}=h_{\boldsymbol{\eta}_2}-h_{\mathbf{0}}$ and observe that all $\boldsymbol{\eta}_2$ in the sum are at distance $O(1/\epsilon)$ from $\mathbf{0}$; same reasoning for $h_{\mathbf{p}}$). In conclusion, the sum in (7.29) is $O(\epsilon\log(1/\epsilon))=o(1)$. Remark also that in (7.28) one can restrict the sum to $\boldsymbol{\eta}_1\neq\boldsymbol{\eta}_2$. Indeed, again just observe that $|\langle(h_{\boldsymbol{\eta}_1}-h_{\mathbf{p}});h_{\boldsymbol{\eta}_2}\rangle_\lambda|=O(\log(1/\epsilon))$ so the sum (7.28) restricted to $\boldsymbol{\eta}_1=\boldsymbol{\eta}_2$ is $O(\epsilon^2\log(1/\epsilon))=o(1)$.

Let $\mathcal{C}^{(1)}_{\mathbf{p}\to\boldsymbol{\eta}_1}$ (resp. $\mathcal{C}^{(2)}_{\mathbf{0}\to\boldsymbol{\eta}_2}$) be a path from $\mathbf{p}$ to $\boldsymbol{\eta}_1$ (resp. from $\mathbf{0}$ to $\boldsymbol{\eta}_2$) of length at most $a|\mathbf{p}-\boldsymbol{\eta}_1|$ (resp. $a|\boldsymbol{\eta}_2|$), chosen such that the distance between $\mathcal{C}^{(1)}_{\mathbf{p}\to\boldsymbol{\eta}_1}$ and $\mathcal{C}^{(2)}_{\mathbf{0}\to\boldsymbol{\eta}_2}$ is larger than $(1/a)|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2|$ for some positive $a>0$, uniformly in $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$ in the support of $\phi(\epsilon\cdot)$. From the definition (1.4) of height function,

$$\big\langle(h_{\boldsymbol{\eta}_1}-h_{\mathbf{p}});h_{\boldsymbol{\eta}_2}\big\rangle_\lambda=\sum_{b_1\in\mathcal{C}^{(1)}_{\mathbf{p}\to\boldsymbol{\eta}_1},b_2\in\mathcal{C}^{(2)}_{\mathbf{0}\to\boldsymbol{\eta}_2}}\sigma_{b_1}\sigma_{b_2}\langle\mathbb{1}_{b_1};\mathbb{1}_{b_2}\rangle_\lambda. \tag{7.31}$$

The r.h.s. of (7.31) is given by the r.h.s. of (4.2), except that one should read $\mathcal{C}^{(1)}_{\mathbf{p}\to\boldsymbol{\eta}_1}$, $\mathcal{C}^{(2)}_{\mathbf{0}\to\boldsymbol{\eta}_2}$ instead of $\mathcal{C}^{(1)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$, $\mathcal{C}^{(2)}_{\boldsymbol{\xi}\to\boldsymbol{\eta}}$.

The error term R_{j_1,j_2} gives a contribution $O(|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2|^{-\theta})$, and recall that $\theta>0$. The term proportional to $\tilde{K}$, due to the oscillations, gives a contribution $O(|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2|^{-2\kappa+1})$ and recall that κ is close to 1. The one proportional to K instead, looking at (4.3), gives K times the integral

$$-\frac{1}{2\pi^2}\operatorname{Re}\int_{\mathbf{p}}^{\boldsymbol{\eta}_1}dz\int_{\mathbf{0}}^{\boldsymbol{\eta}_2}dw\,\frac{1}{(z-w)^2} \tag{7.32}$$

plus an error $O(|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2|^{-1})$ coming from replacing the Riemann sum with the integral. Altogether, all the error terms are of the form $O(|\boldsymbol{\eta}_1-\boldsymbol{\eta}_2|^{-c'})$ for some positive c'. Once the sum over $\boldsymbol{\eta}_1\neq\boldsymbol{\eta}_2$ in (7.28) is performed, the overall contribution of the error terms is $O(\epsilon^{c'})$.

It remains to compute the integral, that gives (cf. (3.28))

$$\frac{1}{2\pi^2}\log\bigg(\frac{|\boldsymbol{\eta}_2-\mathbf{p}|}{|\boldsymbol{\eta}_2-\boldsymbol{\eta}_1|}\times\frac{|\boldsymbol{\eta}_1|}{|\mathbf{p}|}\bigg). \tag{7.33}$$

Note that

$$\epsilon^4\sum_{\boldsymbol{\eta}_1,\boldsymbol{\eta}_2}\phi(\epsilon\boldsymbol{\eta}_1)\phi(\epsilon\boldsymbol{\eta}_2)\big[\log\big(\epsilon|\boldsymbol{\eta}_2-\mathbf{p}|\big)+\log\big(\epsilon|\boldsymbol{\eta}_1|\big)-\log\big(\epsilon|\mathbf{p}|\big)\big]=o(1). \tag{7.34}$$

Indeed, in all of the logarithms one of the two summed variables is absent, the logarithms are $O(\log(1/\epsilon))$ and $\epsilon^2 \sum_{\boldsymbol{\eta}} \phi(\epsilon\boldsymbol{\eta}) = O(\epsilon)$.

It remains to look at

$$-\frac{K}{2\pi^2}\epsilon^4 \sum_{\boldsymbol{\eta}_1,\boldsymbol{\eta}_2} \phi(\epsilon\boldsymbol{\eta}_1)\phi(\epsilon\boldsymbol{\eta}_2)\log\big(\epsilon|\boldsymbol{\eta}_2-\boldsymbol{\eta}_1|\big) \tag{7.35}$$

that converges to the r.h.s. of (7.26). □

Proof of (7.27). In analogy with the case $q=2$, choose faces $\mathbf{p}_1,\dots,\mathbf{p}_q$ such that all their mutual distances, as well as their distance from the support of $\phi(\epsilon\cdot)$, are in $[c_q/\epsilon, 1/(c_q\epsilon)]$ for some suitably small $c_q>0$, and rewrite

$$\begin{aligned}&\big\langle \underbrace{h^\epsilon(\phi);\dots;h^\epsilon(\phi)}_{q \text{ times}}\big\rangle_\lambda \\ &= \epsilon^{2q} \sum_{\boldsymbol{\eta}_1,\dots,\boldsymbol{\eta}_q}{}' \phi(\epsilon\boldsymbol{\eta}_1)\cdots\phi(\epsilon\boldsymbol{\eta}_q)\big\langle (h_{\boldsymbol{\eta}_1}-h_{\mathbf{p}_1});\dots;(h_{\boldsymbol{\eta}_q}-h_{\mathbf{p}_q})\big\rangle_\lambda + o(1),\end{aligned} \tag{7.36}$$

with sum restricted to configurations such that $|\boldsymbol{\eta}_i-\boldsymbol{\eta}_j|\ge 1/c_q$. The proof of (7.36) is like for $q=2$: use that $|\langle h_{\boldsymbol{\eta}_1};\dots;h_{\boldsymbol{\eta}_q}\rangle_\lambda|$ is bounded by some power of $\log(1/\epsilon)$, as it follows from (1.8) and from a repeated application of Cauchy–Schwarz. Let also $\mathcal{C}^{(i)}_{\mathbf{p}_i\to\boldsymbol{\eta}_i}$, $i\le q$ be paths from $\mathbf{p}_i$ to $\boldsymbol{\eta}_i$ of length at most $(1/c_q)|\mathbf{p}_i-\boldsymbol{\eta}_i|$, with mutual distances at least $c_q \min_{i\ne j}|\boldsymbol{\eta}_i-\boldsymbol{\eta}_j|$.

We will show that, uniformly in $\boldsymbol{\eta}_1,\dots,\boldsymbol{\eta}_q$ in the support of $\phi(\epsilon\cdot)$,

$$\big|\big\langle (h_{\boldsymbol{\eta}_1}-h_{\mathbf{p}_1});\dots;(h_{\boldsymbol{\eta}_q}-h_{\mathbf{p}_q})\big\rangle_\lambda\big| = O\Big(\min_{i\ne j}|\boldsymbol{\eta}_i-\boldsymbol{\eta}_j|^{-\theta}\Big) \tag{7.37}$$

for some $\theta>0$. Given this, it is obvious that (7.36) is $o(1)$.

To get (7.37) replace each $(h_{\boldsymbol{\eta}_i}-h_{\mathbf{p}_i})$ by $\sum_{b_i\in\mathcal{C}^{(i)}_{\mathbf{p}_i\to\boldsymbol{\eta}_i}}\sigma_b\mathbb{1}_b$ and recall the decomposition (7.7). We will consider only the terms of type $\mathcal{S}^{(1)}_{q,\mathbf{j}}$, the two others requiring a very similar argument. The sum over $b_1,\dots,b_q$ of $\mathcal{S}^{(1)}_{q,\mathbf{j}}$ is bounded like in (7.12), except of course that now $b_i\in C^{(i)}_{\mathbf{p}_i\to\boldsymbol{\eta}_i}$. Repeating exactly the steps (7.13)–(7.16), one gets again the upper bound (7.17).

Recall that the sum over N in (7.17) is finite because $\hat{d}_v\le -a<0$. While after (7.17) we used that $\delta_{\min}\ge 1$, here we will take advantage of $\delta_{\min}\ge c_q\min_{i\ne j}|\boldsymbol{\eta}_i-\boldsymbol{\eta}_j|$ because of the way the paths $\mathcal{C}^{(i)}_{\mathbf{p}_i\to\boldsymbol{\eta}_i}$ were chosen. It is then immediate to see that the sum over h in (7.17) is not only finite, but actually decays like

$$C'_q\Big(\min_{i\ne j}|\boldsymbol{\eta}_i-\boldsymbol{\eta}_j|\Big)^{-\theta''} \tag{7.38}$$

for some positive θ''. Eq. (7.37) and therefore also (7.27) are proven. □

Appendix A: The free propagator

Here we compute the free propagator with (θ,τ) boundary conditions, proving Lemma 1. To this purpose, we need to diagonalize the Grassmann quadratic form $S=S(\psi)=-\frac{1}{2}\sum_{\mathbf{x},\mathbf{y}\in\Lambda}\psi_{\mathbf{x}}(K^{(\theta\tau)}_{\mathbf{t}^{(m)}})_{\mathbf{x},\mathbf{y}}\psi_{\mathbf{y}}$. For $\mathbf{k}\in\mathcal{D}^{(\theta\tau)}_\Lambda$, we let

$$\hat{\psi}_{\mathbf{k}} = \sum_{\mathbf{x}\in\Lambda}\psi_{\mathbf{x}}e^{i\mathbf{k}\mathbf{x}} \tag{A.1}$$

so that

$$\psi_{\mathbf{x}} = \frac{1}{L^2} \sum_{\mathbf{k} \in \mathcal{D}_\Lambda^{(\theta,\tau)}} \hat{\psi}_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}}. \tag{A.2}$$

Note that (A.2) holds also when one of the two coordinates of $\mathbf{x}$ equals $L/2+1$, in which case it gives the correct boundary condition $\psi_{(L/2+1,y)} = (-1)^\theta \psi_{(-L/2+1,y)}$ and $\psi_{(x,L/2+1)} = (-1)^\tau \psi_{(x,-L/2+1)}$.

Plugging (A.2) into the definition of S and using the anticommutation relation $\hat{\psi}_{\mathbf{k}}\hat{\psi}_{\mathbf{k}'} = -\hat{\psi}_{\mathbf{k}'}\hat{\psi}_{\mathbf{k}}$ one finds with standard computations

$$S(\psi) = \frac{1}{L^2} \sum_{\mathbf{k} \in \mathcal{D}_\Lambda^{(\theta\tau)}} \left\{ \hat{\psi}_{\mathbf{k}}\hat{\psi}_{-\mathbf{k}}(-i\sin k_1 + \sin k_2) + m\hat{\psi}_{\mathbf{k}}\hat{\psi}_{-\mathbf{k}+(\pi,0)} \cos k_1 \right\}, \tag{A.3}$$

where it is understood that, if $-\mathbf{k}$ does not belong to $\mathcal{D}_\Lambda^{(\theta\tau)}$, one should interpret $-\mathbf{k}$ as $-\mathbf{k} + 2\pi(n_1, n_2) \in \mathcal{D}_\Lambda^{(\theta\tau)}$ for the suitable choice of $n_i \in \mathbb{Z}$ (similarly for $-\mathbf{k} + (\pi, 0)$).

We have rewritten S as $-(1/2)\sum_{\mathbf{k},\mathbf{k}'} \hat{\psi}_{\mathbf{k}} A_{\mathbf{k},\mathbf{k}'} \hat{\psi}_{\mathbf{k}'}$ where the matrix A connects $\mathbf{k}$ only with $-\mathbf{k}$ and with $-\mathbf{k} + (\pi, 0)$. To apply (2.7) it remains only to invert A. It is easy to check that the only non-zero elements of A^{-1} are

$$A^{-1}_{\mathbf{k},-\mathbf{k}} = L^2 \frac{i\sin k_1 + \sin k_2}{2D(\mathbf{k}, m)}, \tag{A.4}$$

$$A^{-1}_{\mathbf{k},-\mathbf{k}+(\pi,0)} = L^2 \frac{m\cos k_1}{2D(\mathbf{k}, m)}. \tag{A.5}$$

Then, formula (2.16) is obtained simply from (A.2), (2.7) and

$$\int_{(\theta\tau)} P_\Lambda^{(\theta\tau)}(d\psi)\hat{\psi}_{\mathbf{k}}\hat{\psi}_{\mathbf{k}'} = A^{-1}_{\mathbf{k},\mathbf{k}'}.$$

A.1. *Large-distance behavior of $G(\mathbf{x})$*

Here we prove Proposition 2. Take $\mathbf{x} \neq \mathbf{0}$, $m = 0$, and let $\hat{G}_{\omega\omega}(\mathbf{k}) = 1/[2(-i\sin k_1 + \omega \sin k_2)]$ and $\hat{\mathfrak{g}}_{\omega\omega}(\mathbf{k}) = 1/[2(-ik_1 + \omega k_2)]$ be the Fourier transforms of the diagonal elements of G and $\mathfrak{g}$, respectively. We have

$$G_{\omega\omega}(\mathbf{x}) = \int_{\mathbb{T}^2} \frac{d\mathbf{k}}{(2\pi)^2} \chi(\mathbf{k}) e^{-i\mathbf{k}\mathbf{x}} \hat{\mathfrak{g}}_{\omega\omega}(\mathbf{k}) + R_{\omega,1}(\mathbf{x}). \tag{A.6}$$

The function $\hat{G}_{\omega\omega}(\mathbf{k}) - \hat{\mathfrak{g}}_{\omega\omega}(\mathbf{k})$ is C^∞ on the support of $\chi(\cdot)$, except at the origin. At $\mathbf{k} = \mathbf{0}$, one can easily check that it has bounded first derivatives and that the second derivatives are bounded by $O(1/|\mathbf{k}|)$. As a consequence (given that $\chi(\mathbf{k})$ is C^∞), an integration by parts argument shows that the remainder $R_{\omega,1}(\mathbf{x})$ decays at least as fast as $|\mathbf{x}|^{-2}$. Next, we rewrite the first term in the r.h.s. of (A.6) as

$$\int_{\mathbb{R}^2} \frac{d\mathbf{k}}{(2\pi)^2} e^{-i\mathbf{k}\mathbf{x}} \hat{\mathfrak{g}}_{\omega\omega}(\mathbf{k}) + R_{\omega,2}(\mathbf{x}), \tag{A.7}$$

where $R_{\omega,2}(\mathbf{x}) = (2\pi)^{-2} \int_{\mathbb{R}^2} d\mathbf{k}(1 - \chi(\mathbf{k}))\hat{\mathfrak{g}}_{\omega\omega}(\mathbf{k})$. Since $(1 - \chi(\mathbf{k}))$ is C^∞ and vanishes in a neighborhood of 0, where $\hat{\mathfrak{g}}_{\omega\omega}$ is singular, again (via integrations by parts) it is easy to see that $R_{\omega,2}(\mathbf{x})$ decays faster than any inverse power of $|\mathbf{x}|$. The sum $R_{\omega,1} + R_{\omega,2}$ produces the diagonal elements of the remainder R in (2.27). Finally, the integral in (A.7) is evaluated explicitly by using the residue theorem, and we obtain (2.28).

A.2. *Finite-size corrections for the non-interacting system*

Here we prove that, as long as $m > 0$, the finite-L corrections to the free propagator $g_\Lambda^{(\theta,\tau)}$ are exponentially small, and that the ratio of Pfaffians (2.61) tends to 1 exponentially fast.

Let us start with the Poisson summation formula, that in our notations we can write as follows: if $\hat F$ is a smooth function on the torus $\mathbb{T}^2$ and L is an even integer, then

$$\frac{1}{L^2}\sum_{\mathbf{k}\in\mathcal{D}_\Lambda^{(\theta,\tau)}}\hat F(\mathbf{k})=\sum_{\ell_1,\ell_2\in\mathbb{Z}}F(\ell_1 L,\ell_2 L)(-1)^{\theta\ell_1+\tau\ell_2}, \tag{A.8}$$

where $\theta,\tau\in\{0,1\}$ and

$$F(\mathbf{x})=\frac{1}{(2\pi)^2}\int_{\mathbb{T}^2}\hat F(\mathbf{k})e^{-i\mathbf{k}\mathbf{x}}\,d\mathbf{k}. \tag{A.9}$$

If $\hat F(\mathbf{k})=e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}N(\mathbf{k},m,y_1)/(2D(\mathbf{k},m))$ then the l.h.s. is exactly $g_\Lambda^{(\theta,\tau)}$, cf. (2.17). The term $(\ell_1,\ell_2)=(0,0)$ in the r.h.s. is $g(\mathbf{x},\mathbf{y})$ (cf. (2.18)), while the terms $(\ell_1,\ell_2)\neq(0,0)$ give a contribution exponentially small in L, since the Fourier transform of the analytic function $\hat F(\mathbf{k})$ decays exponentially.

As for (2.61), from the definition of Pfaffian and the explicit form (A.3) of $K_\Lambda^{(\theta\tau)}$,

$$\frac{1}{L^2}\log\mathrm{Pf}\,K_\Lambda^{(\theta\tau)}=\frac{1}{4L^2}\sum_{\mathbf{k}\in\mathcal{D}_\Lambda^{(\theta\tau)}}\log\bigl[4D(\mathbf{k},m)\bigr]. \tag{A.10}$$

Using again the Poisson summation formula and the smoothness of $D(\mathbf{k},m)$ on the torus, the r.h.s. gives

$$\frac{1}{4(2\pi)^2}\int_{\mathbb{T}^2}d\mathbf{k}\log\bigl[4D(\mathbf{k},m)\bigr]+O\bigl(\exp\bigl(-c(m)L\bigr)\bigr) \tag{A.11}$$

for some $c(m)>0$ and the claim on the ratios of Pfaffians follows.

Appendix B: Symmetry properties

In this Appendix we list the symmetry properties of the Grassmann action required for proving the properties of the coefficients $a_\gamma^{(h)}$, $b_\gamma^{(h)}$, $\sigma_{\gamma,\gamma'}^{(h)}$, l_h and $Z_{h;(\gamma,\gamma'),j}$ listed after (6.20), after (6.33) and after (6.35). It is straightforward to check that the Gaussian integration $P_\Lambda(d\psi)$, the interaction $V_\Lambda^{(0)}$ and the source term $\mathcal{B}_\Lambda^{(0)}$ are separately invariant under the following symmetry transformations, irrespective of the Grassmann boundary conditions.

(1) *Parity*: $\psi_{\mathbf{x},\gamma}\to i\psi_{-\mathbf{x},\gamma}$, $m\to -m$ and $J_{\mathbf{x},j}\to J_{-\mathbf{x}-\hat e_j,j}$.
(2) *Reflections around the horizontal axis*: First change $\kappa\to\kappa^*$, where κ is a generic coefficient in the polynomials $V_\Lambda^{(0)}$, $\mathcal{B}_\Lambda^{(0)}$, and in the quadratic action entering the definition of $P_\Lambda(d\psi)$; then

$$(\psi_{\mathbf{x},1},\psi_{\mathbf{x},2},\psi_{\mathbf{x},3},\psi_{\mathbf{x},4})\mapsto(\psi_{\tilde{\mathbf{x}},1},-\psi_{\tilde{\mathbf{x}},2},-\psi_{\tilde{\mathbf{x}},3},\psi_{\tilde{\mathbf{x}},4}),\quad\text{with }\tilde{\mathbf{x}}=(x_1,-x_2), \tag{B.1}$$

and $J_{\mathbf{x},1}\to J_{\tilde{\mathbf{x}},1}$, $J_{\mathbf{x},2}\to J_{\tilde{\mathbf{x}}-\hat e_2,2}$.
(3) *Quasi-particle interchange* #1:

$$(\psi_{\mathbf{x},1},\psi_{\mathbf{x},2},\psi_{\mathbf{x},3},\psi_{\mathbf{x},4})\mapsto(-\psi_{\mathbf{x},3},-\psi_{\mathbf{x},4},\psi_{\mathbf{x},1},\psi_{\mathbf{x},2}) \tag{B.2}$$

while $J_{\mathbf{x},j}$ is left unchanged.
(4) *Quasi-particle interchange* #2:

$$(\psi_{\mathbf{x},1},\psi_{\mathbf{x},2},\psi_{\mathbf{x},3},\psi_{\mathbf{x},4})\mapsto(-\psi_{\tilde{\mathbf{x}},2},\psi_{\tilde{\mathbf{x}},1},-\psi_{\tilde{\mathbf{x}},4},\psi_{\tilde{\mathbf{x}},3}),\quad\text{with }\tilde{\mathbf{x}}=(x_1,-x_2), \tag{B.3}$$

and $J_{\mathbf{x},1}\to J_{\tilde{\mathbf{x}},1}$, $J_{\mathbf{x},2}\to J_{\tilde{\mathbf{x}}-\hat e_2,2}$.

(5) If in addition $m = 0$, invariance under *reflections in a diagonal line*: First change $\kappa \to \kappa^*$, then transform the Grassmann fields as:

$$(\psi_{\mathbf{x},1}, \psi_{\mathbf{x},2}, \psi_{\mathbf{x},3}, \psi_{\mathbf{x},4}) \mapsto \sqrt{i}(\psi_{\tilde{\mathbf{x}},1}, -i\psi_{\tilde{\mathbf{x}},4}, -\psi_{\tilde{\mathbf{x}},3}, -i\psi_{\tilde{\mathbf{x}},2}), \quad \text{with } \tilde{\mathbf{x}} = (x_2, x_1), \tag{B.4}$$

and the external fields as $J_{\mathbf{x},1} \to J_{\tilde{\mathbf{x}},2}$, $J_{\mathbf{x},2} \to J_{\tilde{\mathbf{x}},1}$.

It is easy to check that the symmetries above are preserved by the multiscale integration. This is based on the observation that if

$$e^{V'(\psi)} = \int P(d\phi) e^{V(\psi+\phi)} \tag{B.5}$$

and if both V and P are invariant under the above symmetries, then V' is also invariant. Therefore, the effective potentials and effective source terms on scales $h = -1, -2$, etc., are also invariant under the same symmetries. We can then use these symmetries in order to infer suitable symmetry properties of the kernels of $V^{(h)}$ and $\mathcal{B}^{(h)}$, which in particular imply the desired properties listed after (6.20), (6.33) and (6.35).

As an illustration of the general method used to infer properties on the renormalization constants from the symmetries above, let us discuss the consequences of symmetry (5) on the structure of the diagonal terms in $\mathcal{L}V_2^{(h)}$. The diagonal terms in $\mathcal{P}_0 V_2^{(h)}(\psi)$ have the form

$$\sum_{\gamma} \int \frac{d\mathbf{k}}{(2\pi)^2} \hat{\psi}_{-\mathbf{k},\gamma} \hat{K}^{(h)}_{2,(\gamma,\gamma)}(\mathbf{k} + \mathbf{p}_\gamma) \hat{\psi}_{\mathbf{k},\gamma} \tag{B.6}$$

and they are left invariant by symmetry (5), which is applicable since in $\mathcal{P}_0 V_2^{(h)}(\psi)$ the mass m is set to zero. That is, (B.6) is equal to

$$\sum_{\gamma} i(-1)^{\gamma+1} \int \frac{d\mathbf{k}}{(2\pi)^2} \hat{\psi}_{\tilde{\mathbf{k}},\tilde{\gamma}} \big[\hat{K}^{(h)}_{2,(\gamma,\gamma)}(\mathbf{k} + \mathbf{p}_\gamma)\big]^* \hat{\psi}_{-\tilde{\mathbf{k}},\tilde{\gamma}}, \tag{B.7}$$

where $\tilde{1} = 1$, $\tilde{2} = 4$, $\tilde{3} = 3$, $\tilde{4} = 2$ and $\tilde{\mathbf{k}} = (k_2, k_1)$. Therefore,

$$\hat{K}^{(h)}_{2,(\gamma,\gamma)}(\mathbf{k} + \mathbf{p}_\gamma) = i(-1)^{\gamma} \big[\hat{K}^{(h)}_{2,(\gamma,\gamma)}(\tilde{\mathbf{k}} + \mathbf{p}_{\tilde{\gamma}})\big]^*, \tag{B.8}$$

which implies that $a^{(h)}_\gamma = i(-1)^\gamma [b^{(h)}_{\tilde{\gamma}}]^*$. To give another example, symmetry (2) implies $a^{(h)}_\gamma = -[a^{(h)}_\gamma]^*$ and $b^{(h)}_\gamma = [b^{(h)}_\gamma]^*$. The remaining reality and symmetry properties of the renormalization constants can be derived similarly. More precisely (details are left to the reader):

- symmetry (3) implies $a^{(h)}_1 = a^{(h)}_3$, $b^{(h)}_1 = b^{(h)}_3$ as well as $a^{(h)}_2 = a^{(h)}_4$, $b^{(h)}_2 = b^{(h)}_4$ and $\sigma^{(h)}_{(1,2)} = \sigma^{(h)}_{3,4}$;
- symmetry (4) implies $a^{(h)}_1 = a^{(h)}_2$, $a^{(h)}_3 = a^{(h)}_4$ as well as $b^{(h)}_1 = -b^{(h)}_2$, $b^{(h)}_3 = -b^{(h)}_4$;
- symmetry (2) implies $(\sigma^{(h)}_{(1,2)})^* = -\sigma^{(h)}_{(1,2)}$ and $l_h \in \mathbb{R}$;
- symmetry (3) implies $Z_{h;(1,2),j} = Z_{h;(3,4),j}$ and $Z_{h;(1,4),j} = Z_{h;(2,3),j}$, $j = 1, 2$ (recall (6.33));
- symmetry (4) for $j = 1$ gives $Z_{h;(1,3),1} = Z_{h;(2,4),1}$ and $Z_{h;(1,4),1} = Z_{h;(2,3),1} = 0$;
- symmetries (1) and (4) for $j = 2$ give $Z_{h;(1,3),2} = -Z_{h;(2,4),2}$ and $Z_{h;(1,2),2} = Z_{h;(3,4),2} = 0$ (symmetry (1) is applicable since in the computation of $\mathcal{L}\mathcal{B}$ one puts the mass m to zero);
- symmetry (2) for $j = 1$ gives $Z_{h;(1,2),1}$, $Z_{h;(1,3),1}$, $Z_{h;(2,4),1}$, $Z_{h;(3,4),1}$ are purely imaginary;
- symmetries (1) and (2) for $j = 2$ give $Z_{h;(1,3),2}$ and $Z_{h;(2,4),2}$ are real, while $Z_{h;(1,4),2}$ and $Z_{h;(2,3),2}$ are purely imaginary;
- symmetry (5) gives $Z_{h;(1,2),1} = -Z_{h;(1,4),2}$ and $Z_{h;(1,3),2} = iZ_{h;(1,3),1}$.

It is important to observe that, to prove the desired properties of $\sigma^{(h)}_{\gamma,\gamma'}$, which depend on the mass m, one needs neither symmetry (5), that holds only for $m = 0$, nor symmetry (1) which requires $m \to -m$.

Appendix C: Gevrey class cut-off functions

We assume that the cut-off functions $\bar\chi(\cdot)$, $\chi(\cdot)$ are in the Gevrey class of order 2. We recall that a $C^\infty(\mathbb{R}^d)$ function f is said to be in the Gevrey class of order s if on every compact $K\subset\mathbb{R}^d$ there are two constants $A=A(K,f)$ and $\mu=\mu(K,f)$ so that for any non-negative integers $n_1,\dots,n_d$

$$\big\|\partial_1^{n_1}\cdots\partial_d^{n_d} f\big\|_\infty \le A\mu^{n_1}\cdots\mu^{n_d}(n_1!\cdots n_d!)^s. \tag{C.1}$$

For $s=1$ we have a common characterization of real-analytic functions; the class of order 2 includes the function $e^{-1/x_1}\mathbf{1}_{x_1\ge 0}$. A useful feature of Gevrey functions is the fact that the Fourier transform $\tilde f(\mathbf{x})$ of a compactly supported Gevrey function $f(\mathbf{k})$ of order s decays at large distances like a stretched exponential $e^{-(\text{const.})|\mathbf{x}|^{1/s}}$.

The example of a function $\bar\chi(\mathbf{k})$ to keep in mind is the following. Given $\varepsilon>0$, let $f_\varepsilon(k):=e^{-(1-k^2/\varepsilon^2)^{-1}}\mathbf{1}_{|k|\le\varepsilon}$ and $F_\varepsilon(k)=\int_{-\infty}^k f_\varepsilon(t)\,dt/\int_{-\infty}^{+\infty} f_\varepsilon(t)\,dt$. Note that F_ε is a smoothed version of the Heaviside step function. It is Gevrey of order 2, and such that $F_\varepsilon(k)+F_\varepsilon(-k)=1$. For $\varepsilon<\pi/2$ we let $\theta_\varepsilon(k)=F_\varepsilon(k+\pi/2)F_\varepsilon(-k+\pi/2)$. We also define $\tilde\chi(k):=\sum_{n\in\mathbb{Z}}\theta_\varepsilon(k+2\pi n)$, which we can naturally think as a function on the circle $\mathbb{T}:=\mathbb{R}\setminus(2\pi\mathbb{Z})$. It is straightforward to check that

$$\bar\chi(\mathbf{k}):=\tilde\chi(k_1)\tilde\chi(k_2) \tag{C.2}$$

satisfies all the properties required, in particular it is positive, symmetric under $\mathbf{k}\to-\mathbf{k}$, $\sum_{i=1}^4\chi(\mathbf{k}-\mathbf{p_i})=1$ and the support of $\bar\chi(\mathbf{k})$ does not include $(0,\pi)$, $(\pi,0)$, (π,π), provided ε is chosen small enough.

We also introduce

$$\chi(\mathbf{k}):=\sum_{\mathbf{n}\in\mathbb{Z}^2}\theta_\varepsilon\big(|\mathbf{k}+2\pi\mathbf{n}|\big), \tag{C.3}$$

which is (as a function on the torus) a rotationally invariant version of $\bar\chi(\mathbf{k})$.

To prove Lemma 2 one should start from the definition of $G^{(h)}$, given as in (2.24) with $\bar\chi(\cdot)$ replaced by $f_h(\cdot)$ and make a change of variables $\mathbf{k}\to\mathbf{k}2^h$ (the prefactor $2^{h(1+n_1+n_2)}$ in (2.39) comes from the change of variables plus the following facts: (i) $D(\mathbf{k},m)\sim|\mathbf{k}|^2$ on the support of f_h, (ii) every discrete derivative ∂_j corresponds to multiplication by $e^{-ik_j}-1$ in Fourier space, and $|e^{-ik_j}-1|\le(\text{const.})2^h$ on the support of f_h). At that point, one can see $G^{(h)}(\mathbf{x})$ as the Fourier transform of a Gevrey function of order 2 computed at $2^h\mathbf{x}$, whence the decay factor $\exp(-c\sqrt{2^h\mathbf{x}})$ in (2.39). The argument is similar for the off-diagonal elements and for $h=h^*$ (thanks to the presence of the infrared cut-off induced by the mass $m\sim 2^{h^*}$), as well as for the propagators $\mathfrak{g}^{(h)}$ and $R^{(h)}$.

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