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Remarks on the normal bundles of generic rational curves

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Abstract:	In this note we give a different proof of Sacchiero's theorem about the splitting type of the normal bundle of a generic rational curve. Moreover we discuss the the existence and the construction of smooth monomial curves having generic type of normal bundle.
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REMARKS ON THE NORMAL BUNDLES OF GENERIC RATIONAL CURVES

ALBERTO ALZATI AND RICCARDO RE

ABSTRACT. In this note we give a different proof of Sacchiero's theorem about the splitting type of the normal bundle of a generic rational curve. Moreover we discuss the existence and the construction of smooth monomial curves having generic type of the normal bundle.

1. INTRODUCTION

Let $C = \mathrm{Im}(f)$ be a degree d rational curve in $\mathbb{P}^s(\mathbb{C})$ $(d > s \geq 3)$ where $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^s(\mathbb{C})$ is a birational morphism. Let us assume that C is smooth, hence C admits a well defined normal bundle $\mathcal{N}_C$, splitting as the direct sum of line bundles. In [S] Sacchiero proved that, for a generic C as above, the splitting type of $\mathcal{N}_C$ is uniquely determined.

In [A-R2] we developed a general method to get the splitting type of $\mathcal{N}_C$ based on the fact that C is always a suitable projection of the rational normal curve Γ_d of degree d in $\mathbb{P}^d(\mathbb{C})$ from a projective linear space of dimension $d - s - 1$. This method was previously used in [A-R1] to get the splitting type of the restricted tangent bundle of C. In [A-R-T] an explicit formula is given when C is a monomial curve, i. e. when f is given by monomials of degree d in two variables.

Here we will use the method developed in [A-R2] to prove Sacchiero's Theorem and we will establish a range in which a monomial curve C can be considered generic from the point of view of the splitting type of $\mathcal{N}_C$.

In §2 we fix notations and we recall some known results. In §3 we give our proof. In §4 we consider the case of monomial curves. In §5 we take the opportunity to give a corrective remark to [A-R2] suggested by the referee.

2. NOTATION AND BACKGROUND

As above, a rational curve $C \subset \mathbb{P}^s(\mathbb{C})$ will be the target of a birational morphism $f : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^s(\mathbb{C})$. We will work always over $\mathbb{C}$. We will always assume that C is not contained in any hyperplane and that it is smooth. Let us put $d := \deg(C) > s \geq 3$. Let $\mathcal{I}_C$ be the ideal sheaf of C, then $\mathcal{N}_C := \mathcal{H}om_{\mathcal{O}_C}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C)$ as usual and, taking the differential of f, we get:

$$0 \to \mathcal{T}_{\mathbb{P}^1} \to f^*\mathcal{T}_{\mathbb{P}^s} \to f^*\mathcal{N}_C \to 0$$

where $\mathcal{T}$ denotes the tangent bundle. Of course we can write:

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$$\mathcal{N}_f := f^*\mathcal{N}_C = \bigoplus_{i=1}^{s-1} \mathcal{O}_{\mathbb{P}^1}(c_i + d + 2)$$

for suitable integers $c_i \geq 0$ (see [S] Proposizione 1, see also Proposition 10 of [A-R2]).

Every curve C is, up to a projective transformation, the projection in $\mathbb{P}^s$ of a d-Veronese embedding Γ_d of $\mathbb{P}^1$ in $\mathbb{P}^d := \mathbb{P}(V)$ from a $(d-s-1)$-dimensional projective space $\mathbb{P}(T)$ where V and T are vector spaces of dimension, respectively, $d+1$ and $e+1 := d-s$. Of course we require that $\mathbb{P}(T) \cap \Gamma_d = \emptyset$ as we want that f is a regular map.

Let us denote by $U = \langle x, y \rangle$ a fixed 2-dimensional vector space such that $\mathbb{P}^1 = \mathbb{P}(U)$, then we can identify V with S^dU (d-th symmetric power) in such a way that the rational normal degree d curve Γ_d can be considered as the set of pure tensors of degree d in $\mathbb{P}(S^dU)$ and the d-Veronese embedding is the map

$$\alpha x + \beta y \to (\alpha x + \beta y)^d \qquad (\alpha : \beta) \in \mathbb{P}^1.$$

From now on, any degree d rational curve C, will be determined (up to projective equivalences which are not important in our context) by the choice of a proper subspace $T \subset S^dU$ such that $\mathbb{P}(T) \cap \Gamma_d = \emptyset$.

By arguing in this way, the elements of a base of T can be thought as homogeneous, degree d, polynomials in x, y. In [A-R1] and [A-R2] we related the polynomials of any base of T with the splitting type of $\mathcal{T}_f$ and $\mathcal{N}_f$. To describe this relation we need some additional definitions.

Let us indicate by $\langle \partial_x, \partial_y \rangle$ the dual space $U^*$of U, where ∂_x and ∂_y indicate the partial derivatives with respect to x and y.

Definition 1. *Let T be any proper subspace of S^dU. Then:*

$$\partial T := \langle \omega(T) | \omega \in U^* \rangle .$$

Note that Definition 1 allows to define also $\partial^k T$ for any integer $k \geq 1$, by induction.

To get the splitting type of $\mathcal{N}_f$ the following Proposition is useful:

Proposition 1. *In the above notations, for any integer $k \geq 0$, let us call $\varphi(k) := h^0(\mathbb{P}^1, \mathcal{N}_f(-d-2-k))$. Then the splitting type of $\mathcal{N}_f$ is completely determined by $\Delta^2[\varphi(k)] := \varphi(k+2) - 2\varphi(k+1) + \varphi(k)$.*

Proof. We know that $\mathcal{N}_f(-d-2) = \bigoplus_{i=1}^{s-1} \mathcal{O}_{\mathbb{P}^1}(c_i)$, so that we have only to determine the integers c_i. By definition, $\Delta^2[\varphi(k)]$ is exactly the number of integers c_i which are equal to k. ∎

From Proposition 1 it follows that to know the splitting type of $\mathcal{N}_f$ it suffices to know $\varphi(k)$ for any $k \geq 0$.

Let us consider the linear operators $D_k : S^kU \otimes S^dU \to S^{k-1}U \otimes S^{d-1}U$, such that $D_k := \partial_x \otimes \partial_y - \partial_y \otimes \partial_x$, and $D_k^2 : S^kU \otimes S^dU \to S^{k-2}U \otimes S^{d-2}U$. Of course, as $T \subset S^dU$, we can restrict D_k^2 to $S^kU \otimes T$ and we get a linear map $\psi_k := D^2_{k|S^kU\otimes T} : S^kU \otimes T \to S^{k-2}U \otimes \partial^2 T$; let us define:

$$T_k := \ker(\psi_k).$$

Then we have the following:

Theorem 1. *In the above notations:*

$\varphi(0) = d + e$

$\varphi(1) = 2(e+1)$
$\varphi(2) = 3(e+1) - \dim(\partial^2 T)$
$\varphi(k) = \dim(T_k)$ *for any* $k \geq 2$.
Moreover the number of integers c_i *such that* $c_i = 0$ *is* $d - 1 - \dim(\partial^2 T)$.

Proof. See Theorem 1 and Proposition 11 of [A-R2]. ∎

To calculate $\varphi(k)$, for $k \geq 2$, it is very useful the following

Proposition 2. *In the above notations, let us assume that* T *is decomposed as* $T = T^1 \oplus T^2 \oplus .... \oplus T^q$ *in such a way that* $\partial^2 T = \partial^2 T^1 \oplus \partial^2 T^2 \oplus .... \oplus \partial^2 T^q$ *for some* $q \geq 1$. *Then, if we put* $K^i := \ker(D_k^2 : S^k U \otimes T^i \to S^{k-2} U \otimes \partial^2 T^i)$ *for any* $i = 1, ..., q$, *we have that* $\varphi(k) = \dim(K^1) + ... + \dim(K^q)$.

Proof. See Lemma 13 of [A-R2] or Proposition 3 of [A-R-T]. ∎

Now let us state the Sacchiero's theorem we want to re-prove (see [S] pag. 33):

Theorem 2. *(G. Sacchiero) Let* C *be a smooth, generic, rational curve of degree* d *in* $\mathbb{P}^s(\mathbb{C})$ $(d > s \geq 3)$. *Then*

$$\mathcal{N}_f = \mathcal{O}_{\mathbb{P}^1}(d+1+c)^{\oplus s-1-\rho} \bigoplus \mathcal{O}_{\mathbb{P}^1}(d+2+c)^{\oplus \rho},\ 0 \leq \rho < s-1$$

where c *and* ρ *are, respectively, the quotient and remainder of the euclidean division of* $2d - s - 1$ *by* $s - 1$.

To translate the original Sacchiero's notations we have to put $(n, r, \delta) = (d, s-1, c+1)$. Note also that the above result was generalized to some reducible rational curves by Ran in [R].

3. Our proof of Sacchiero's theorem

Let us recall that $\varphi(k)$ is a strictly monotone decreasing function for $k \geq 0$, by definition. Let us remark that, if C is generic, T must be generated by $e+1$ generic degree d polynomials p_i with $i = 1, ..., e+1$.

Let us divide the proof of Theorem 2 into two cases.

First case: $d \geq 3e+4$. Then $\dim(\partial^2 T) = 3(e+1)$ and $\varphi(2) = 0$. In fact every polynomial p_i gives rise to a plane in $\mathbb{P}^{d-2}$ (generated by $\partial_x\partial_x p_i$, $\partial_x\partial_y p_i$, $\partial_y\partial_y p_i$), so that we have $e+1$ generic planes in $\mathbb{P}^{d-2}$ generating a projective space $\mathbb{P}(\partial^2 T) \subseteq \mathbb{P}^{d-2}$ of dimension $3(e+1) - 1 \leq d-2$. As $\varphi(2) = 0$ we get $\varphi(k) = 0$ for any $k \geq 2$ and, by using Theorem 1 we have

$\Delta^2[\varphi(0)] = d - 3e - 4$,
$\Delta^2[\varphi(1)] = 2e + 2$,
$\Delta^2[\varphi(k)] = 0$ otherwise.

If $d - 3e - 4 > 0$ we have $\mathcal{N}_f(-d-2) = \mathcal{O}_{\mathbb{P}^1}^{\oplus d-3e-4} \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2e+2}$ by Proposition 1, hence $\mathcal{N}_f = \mathcal{O}_{\mathbb{P}^1}(d+2)^{\oplus d-3e-4} \oplus \mathcal{O}_{\mathbb{P}^1}(d+3)^{\oplus 2e+2}$. By putting $c = 1$ and $\rho = 2e+2$, we have $d - 3e - 4 = s - 1 - \rho$ and noticing that $2d - s - 1 = 1(s-1) + \rho$ with $2e + 2 < s - 1$, we have proved Theorem 2 in this case.

If $d - 3e - 4 = 0$ we have $\mathcal{N}_f(-d-2) = \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2e+2}$ by Proposition 1, hence $\mathcal{N}_f = \mathcal{O}_{\mathbb{P}^1}(d+3)^{\oplus 2e+2}$. By putting $c = 2$ and $\rho = 0$, we have $2e + 2 = s - 1$ and noticing that $2d - s - 1 = 2(s-1) + 0$, we have proved Theorem 2 also in this case.

Second case: $d < 3e+4$, i.e. $d = 3e+4-v$ with $v \in [1, 2e]$, in fact $d - e - 1 = s \geq 3$, hence $3e + 4 - v - e - 1 \geq 3$ implies $v \leq 2e$.

In this case $\mathbb{P}(\partial^2 T)$ is always generated in $\mathbb{P}^{d-2}$ by $e+1$ planes π_i $i = 1, ..., e+1$, in general position, however now $3(e+1) - 1 > d-2$, hence $\mathbb{P}(\partial^2 T) = \mathbb{P}^{d-2}$, $\dim(\partial^2 T) = d - 1 = 3(e+1) - v$ and $\varphi(2) = v \geq 1$ by Theorem 1.

Let us consider $\varphi(k)$ for $k \geq 3$. The number $\varphi(k)$ is the dimension of the kernel of a linear map $\psi_k : S^k U \otimes T \to S^{k-2} U \otimes \partial^2 T$; as C (hence T) is generic the map will be generic too, hence it will have maximal rank. In fact, it is possible to choose a suitable basis for $\partial^2 T$, such that the matrix representing ψ_k has maximal rank and it is sufficient to prove this fact when $v = 2e$, i.e. when $\partial^2 T$ has the minimal possible dimension. When $v = 2e$ this fact is true by Proposition 15 of [A-R2], showing the existence of suitable spaces T for which $rk(\psi_k)$ is maximal; by semicontinuity the same is true for the generic T.

It follows that $\varphi(k) = \dim(S^k U \otimes T) - \dim(S^{k-2} U \otimes \partial^2 T)$ if this number is positive and $\varphi(k) = 0$ otherwise.

By considering that $\dim(\partial^2 T) = 3(e+1) - v$ we have:

$\varphi(k) = k[v - 2(e+1)] + 4(e+1) - v$ if $k[v - 2(e+1)] + 4(e+1) - v > 0$

$\varphi(k) = 0$ if $k[v - 2(e+1)] + 4(e+1) - v \leq 0$.

Note that the above formula is true also for $k = 0, 1, 2$ by Theorem 1:

$\varphi(0) = d + e = 4(e+1) - v > 0$

$\varphi(1) = 2(e+1) > 0$

$\varphi(2) = v > 0$.

As $\varphi(k)$ is a linear function (when it is strictly positive), there exists a unique integer $\overline{k} \geq 2$ such that $\varphi(\overline{k}) > 0$ and $\varphi(k) = 0$ for any $k \geq \overline{k} + 1$.

It follows that:

$\Delta\varphi(k) = 2(e+1) - v > 0$ for $k \in [0, \overline{k} - 1]$,

$\Delta\varphi(\overline{k}) = \varphi(\overline{k})$,

$\Delta\varphi(k) = 0$ for $k \geq \overline{k} + 1$.

Hence:

$\Delta^2\varphi(\overline{k}) = \varphi(\overline{k}) > 0$,

$\Delta^2\varphi(\overline{k} - 1) = 2(e+1) - v - \varphi(\overline{k}) = [2(e+1) - v](\overline{k} - 1) - 4(e+1) + v > 0$,

$\Delta^2\varphi(k) = 0$ otherwise.

By definition of $\overline{k}$, we have that $(\overline{k} + 1)[v - 2(e+1)] + 4(e+1) - v \leq 0$, i.e. $\varphi(\overline{k}) \leq 2(e+1) - v = s - 1$.

If $2(e+1) - v - \varphi(\overline{k}) > 0$, by Proposition 1 we get

$$\mathcal{N}_f(-d-2) = \mathcal{O}_{\mathbb{P}^1}(\overline{k} - 1)^{\oplus 2(e+1) - v - \varphi(\overline{k})} \oplus \mathcal{O}_{\mathbb{P}^1}(\overline{k})^{\oplus \varphi(\overline{k})},$$
$$\text{hence } \mathcal{N}_f = \mathcal{O}_{\mathbb{P}^1}(d + 1 + \overline{k})^{\oplus 2(e+1) - v - \varphi(\overline{k})} \oplus \mathcal{O}_{\mathbb{P}^1}(d + 2 + \overline{k})^{\oplus \varphi(\overline{k})}.$$

By putting $c = \overline{k}$ and $\rho = \varphi(\overline{k})$, we have $2(e+1) - v - \varphi(\overline{k}) = s - 1 - \rho$ and noticing that $2d - s - 1 = \overline{k}(s-1) + \varphi(\overline{k})$ with $\varphi(\overline{k}) < s - 1$, we have proved Theorem 2 in this case.

If $2(e+1) - v - \varphi(\overline{k}) = 0$, by Proposition 1 we get

$$\mathcal{N}_f(-d-2) = \mathcal{O}_{\mathbb{P}^1}(\overline{k})^{\oplus \varphi(\overline{k})},$$
$$\text{hence } \mathcal{N}_f = \mathcal{O}_{\mathbb{P}^1}(d + 2 + \overline{k})^{\oplus \varphi(\overline{k})}.$$

By putting $c = \overline{k} + 1$ and $\rho = 0$, we have $s - 1 = \varphi(\overline{k})$ and noticing that $2d - s - 1 = (\overline{k} + 1)(s - 1) + 0$, we have proved Theorem 2 also in this case.

4. Monomial curves

In this section we consider monomial degree d rational curves in $\mathbb{P}^s$, i.e. curves such that the morphism $f : \mathbb{P}^1 \to \mathbb{P}^s$ is given by monomials of degree d in two variables, as follows:

$$(*) \qquad f(x:y) = (x^{h_0} : x^{h_1}y^{d-h_1} : ... : x^{h_i}y^{d-h_i} : ... : x^{h_s}y^{d-h_s}),$$

with $i = 0, ..., s$ and $h_0 > h_1 > ... > h_s \geq 0$. Let us recall that, as we are considering smooth curves, it is necessary and sufficient that : $h_0 = d$, $h_1 = d-1, h_{s-1} = 1$, $h_s = 0$ (see for instance Lemma 3.1 of [C-R]).

Concerning the generic type of splitting of $\mathcal{N}_f$ for such curves two natural questions arise:

I) Monomial curves are obviously not generic. However, is it possible to give a notion of "genericity" in this case too and what is the "generic splitting type" ? We will see that, in a certain range of d and $e = d-s-1$, namely $d \geq 3e+4$, it is possible to give a notion of "genericity" and to obtain the splitting type of $\mathcal{N}_f$ for such "general" curves and to give some informations when $d < 3e+4$.

II) The splitting type of $\mathcal{N}_f$ described by Theorem 2 is the most general one from the point of view of deformation theory, hence it is interesting to get examples of monomial curves whose normal bundles have generic splitting types. Is it possible to get examples in any case ? We will see that the answer is positive if $s-1$ is even, and in general it is negative if $s-1$ is odd.

I) As in [A-R-T] every monomial curve of degree d can be defined by choosing $r \geq 1$ disjoint intervals of integers $I_i \subseteq [2, d-2]$, $i = 1, ..., r$, each integer z corresponding to the monomial $x^{d-z}y^z$. Recall that $d > s \geq 3$, hence $d \geq 4$ and $[2, d-2] \neq \emptyset$.

If there exists an interval I_i with $length(I_i) \geq 2$ the involved monomials are linked by some relations because they belong to $\partial(x^\alpha y^\beta)$ for a suitable monomial $x^\alpha y^\beta$ of bigger degree. Hence, if we want to get "generic" monomial curves, we are forced to choose $length(I_i) = 1$ for any $i = 1, ..., r$. However this is not possible when $d-3 < 2e+1$. It follows that a generic monomial curve can be defined only if we choose $e+1$ monomials among $\{x^{d-2}y^2, x^{d-3}y^3, ..., x^2y^{d-2}\}$ such that no two of them correspond to consecutive integers in $[2, d-2]$ and assuming that $d \geq 2e+4$.

Now let us consider any monomial curve as above. We can divide the set of $e+1$ monomials generating T into two subsets:

- first type monomials, belonging to some chain of type $\{x^{d-j}y^j, x^{d-j-2}y^{j+2}, x^{d-j-4}y^{j+4}, ...\}$ for some j;
- second type monomials, not belonging to chains of the above type.

If we decompose T as the direct sum of irreducible subspaces T^i according to Proposition 2, we get that every chain of monomials of the first type generates a unique irreducible subspace of T, whose dimension is the length of the chain according to Proposition 5 of [A-R-T]; while every monomial of the second type generates an irreducible one dimensional subspace of T. In any case, by using Proposition 6 and Theorem 4 of [A-R-T], we have that every irreducible subspace of T gives no contribute to $\varphi(k)$ for $k \geq 3$, hence $\varphi(k) = 0$ for $k \geq 3$ by Proposition 2. Therefore the splitting type of $\mathcal{N}_f$ depends only on the three values of $\varphi(k)$ given by Theorem 1 and we have only to calculate $\dim(\partial^2 T)$.

Let us assume that T is generated by p monomials of the second type and by q chains of length $l_1, l_2, ..., l_q$ of monomials of the first type; obviously $l_i \geq 2$ for any i. Each monomial of the second type gives rise to a plane in $\mathbb{P}^{d-2}$ while every chain of monomials of the first type gives rise to a projective space of dimension $2l_i$, contained in $\mathbb{P}^{d-2}$.

All these projective spaces are in general position in $\mathbb{P}^{d-2}$ by Proposition 6 of [A-R-T], hence their span has projective dimension $2\sum_{i=1}^{q} l_i + q + 3p - 1$ and $\dim(\partial^2 T) = 2\sum_{i=1}^{q} l_i + q + 3p$. Note that $e+1 = \sum_{i=1}^{q} l_i + p$, hence we can write $\dim(\partial^2 T) = 2(e+1) + p + q$. It follows that $\varphi(2) = e+1-p-q = \sum_{i=1}^{q}(l_i - 1) \geq 0$ and $\varphi(2) = 0$ if and only if $q = 0$ (i.e. when there are no chains) and $p = e+1$.

By using Theorem 1 we get:
$\varphi(0) = d + e$;
$\varphi(1) = 2(e+1)$;
$\varphi(2) = e+1-p-q$;
and $\varphi(k) = 0$ for $k \geq 3$. Hence:
$\Delta^2\varphi(0) = d - 2e - 3 - p - q$;
$\Delta^2\varphi(1) = 2(p+q)$;
$\Delta^2\varphi(2) = e+1-p-q = \varphi(2)$;
$\Delta^2\varphi(k) = 0$ otherwise.

Note that $d - 2e - 3 - p - q \geq 0$ because $\dim(\partial^2 T) \leq d-1$, due to the fact that $\mathbb{P}(\dim(\partial^2 T)) \subseteq \mathbb{P}^{d-2}$, hence $\dim(\partial^2 T) = 2\sum_{i=1}^{q} l_i + q + 3p = 2(e+1-p) + q + 3p \leq d-1$ implies $d \geq 2e + 3 + p + q$.

If $d \geq 3e+4$, the existence of at least a chain for T is not necessary. Hence, in this range, it is possible to have a reasonable definition of "generic monomial curves" simply by choosing $p = e+1$ monomials in a generic way ($q = 0$). In this case $\varphi(2) = 0$ and, according to Proposition 1, the splitting type of the normal bundle of these curves is the following:

$$\mathcal{N}_f = \mathcal{O}_{\mathbb{P}^1}(d+2)^{\oplus d-3e-4} \oplus \mathcal{O}_{\mathbb{P}^1}(d+3)^{\oplus 2(e+1)}.$$

Note that this is exactly the same splitting type of a generic curve, hence, in this range, generic curves and generic monomial curves are the same from the point of view of the splitting type of $\mathcal{N}_f$ and the above monomial curves are smooth.

If $2e+4 \leq d < 3e+4$ it is not possible that $q = 0$, because at least a chain must exist in T. The splitting type of $\mathcal{N}_f$ depends always on the two integers p and q and not only on d and e, i.e. on d and s, and it is not clear what the "generic" pair (p, q) should be. In any case, for any fixed p and q, by Proposition 1 we have:

$$\mathcal{N}_f = \mathcal{O}_{\mathbb{P}^1}(d+2)^{\oplus d-2e-3-p-q} \oplus \mathcal{O}_{\mathbb{P}^1}(d+3)^{\oplus 2(p+q)} \oplus \mathcal{O}_{\mathbb{P}^1}(d+4)^{\oplus e+1-p-q}.$$

II) Let d and s be two positive integers such that $d > s \geq 3$ and $s-1$ is even. Let c and ρ be, respectively, the quotient and the remainder of the euclidean division of $2d - s - 1$ by $s - 1$. Then there exists a (smooth) monomial curve of degree d in $\mathbb{P}^s$ such that $\mathcal{N}_f$ splits as in Theorem 2. In fact, note that ρ is even and $c \geq 1$; let us put $\alpha := \rho/2 \geq 0$, $\beta := (s-1-2\alpha)/2 > 0$ and $a := c-2$; let us choose α intervals $I \in [2, d-2]$ of length $a+2 \geq 0$ and β intervals I of length $a+1 \geq 0$ generating a subspace $T \subset S^dU$, of dimension $e+1 = \alpha(a+2) + \beta(a+1)$, and defining a rational monomial curve C of degree d. Let us choose the intervals in such a way that between any two intervals there are exactly two monomials, (except of course the first one and the last one) this is possible thanks to the relation: $2d-s-1=c(s-1)+\rho$ implying $d=e+2(\alpha+\beta)+2=e+s+1$, hence C is a monomial rational curve of degree d in $\mathbb{P}^s$.

We have that $\dim(\partial^2 T)=\alpha(a+4)+\beta(a+3)$, then $\mathbb{P}(\partial^2 T)\simeq\mathbb{P}^{d-2}$ and in $\mathbb{P}^{d-2}$ we get α projective subspaces of dimension $a+3$ and β projective subspaces of dimension $a+2$ which are all in general position. Hence all maps ψ_k, for $k\geq 3$, have maximal ranks: by using Proposition 2 we have that every ψ_k is the direct sum of linear maps which are all injective or all surjective, due to the fact that the dimensions of the above projective subspaces differ by only one unit. Then, by arguing as in our proof of Sacchiero's theorem, we get that $\mathcal{N}_f$ splits as in Theorem 2.

If $s-1$ is odd, surprisingly, the situation is very different. In some cases (d,s) there exist monomial curves having $\mathcal{N}_f$ with generic splitting, for instance when $(d,s)=(6,4)$ we can take $T=\langle x^4y^2,x^2y^4\rangle$. However in other cases this is not possible: when $(d,s)=(7,4)\Longrightarrow c=3,\rho=0$, it is easy to see that no choice of T defines a monomial curve whose $\mathcal{N}_f$ splits as $\mathcal{O}_{\mathbb{P}^1}(11)^{\oplus 3}$, the expected type according to Theorem 2.

Let us explain this last example in detail. To have $\mathcal{N}_f=\mathcal{O}_{\mathbb{P}^1}(11)^{\oplus 3}$ is equivalent to have $\Delta^2\varphi(2)=3$ and $\Delta^2\varphi(k)=0$ for $k\neq 2$ (see Proposition 1). As $\dim(T)=3$ we have only four possibilities for T :

i) $T=\langle x^5y^2,x^4y^3,x^3y^4\rangle$ *ii)* $T=\langle x^5y^2,x^4y^3,x^2y^5\rangle$

iii) $T=\langle x^5y^2,x^3y^4,x^2y^5\rangle$ *iv)* $T=\langle x^4y^3,x^3y^4,x^2y^5\rangle$.

In cases *i)* and *iv)* we have $\varphi(0)=9$, $\varphi(1)=6$, $\varphi(2)=4$ (see Theorem 1), hence $\Delta^2\varphi(0)=1$. In case *ii)* we have $\varphi(0)=9$, $\varphi(1)=6$, $\varphi(2)=3$, and $\varphi(3)\geq 1$ because $\psi_3(2y^3\otimes x^5y^2+5xy^2\otimes x^4y^3-x^3\otimes x^2y^5)=0$, hence $\Delta^2\varphi(1)\geq 1$. In case *iii)* we have $\varphi(0)=9$, $\varphi(1)=6$, $\varphi(2)=3$, and $\varphi(3)\geq 1$ because $\psi_3(-y^3\otimes x^5y^2+5x^2y\otimes x^3y^4-x^3\otimes x^2y^5)=0$, hence $\Delta^2\varphi(1)\geq 1$.

5. Corrective remark to [A-R2]

Here we correct a wrong statement given in [A-R2] detected by the referee, while reading the first version of this paper where the statement was repeated. In the introduction to [A-R2] and especially in section 1.1 therein, it was wrongly stated that if $C\subset\mathbb{P}^s$ is a rational curve with ordinary singularities, with birational parametrization map $f:\mathbb{P}^1\to C\subset\mathbb{P}^s$, then one may identify $f^*\mathcal{N}_C$ with the quotient $\mathcal{Q}=f^*\mathcal{T}_{\mathbb{P}^s}/df(\mathcal{T}_{\mathbb{P}^1})$ as the differential of f is injective by assumption.

The referee to the present paper has pointed out to us that this is wrong. One simple example is a rational plane cubic $C\subset\mathbb{P}^2$ with one ordinary node, which has $f^*\mathcal{N}_C=\mathcal{O}_{\mathbb{P}^1}(9)$, while $\mathcal{Q}=\mathcal{O}_{\mathbb{P}^1}(7)$, by degree reasons. The same computation works for any plane rational curve with ordinary singularities.

Note however that if C is smooth, then it is true that $\mathcal{Q}=f^*\mathcal{N}_C$ and we recall that all the calculations and results in [A-R2] involve the vector bundle $\mathcal{Q}$ only. Moreover, the main results of [A-R2], namely the construction of an example of Hilbert scheme of the rational curves with a given splitting of $\mathcal{Q}$ given in section 6 and Theorem 7.3 of section 7, involve smooth rational curves. Indeed in section 6 we found two components of the Hilbert scheme of rational curves $C\subset\mathbb{P}^8$ of degree $d=11$ with $\mathcal{Q}\cong\mathcal{O}_{\mathbb{P}^1}(d+4)\oplus\mathcal{O}_{\mathbb{P}^1}(d+3)\oplus\mathcal{O}_{\mathbb{P}^1}(d+2)$, such that the general curve in any of the two components is smooth, as it is explicitly proved in that section. Theorem 7.3 studies smooth rational curves in rational normal scrolls, it characterizes them in terms of their restricted tangent bundle and computes their normal bundles. In conclusion, all the results of [A-R2] remain valid when restricting oneself to smooth curves and in any case for the bundle $\mathcal{Q}$ in the place of $f^*\mathcal{N}_C$.

Appendix

After this note was written, the paper [C-R] has appeared on ArXiv. Theorem 3.2 of [C-R] says that, for a monomial curve as in $(*)$,

$$\mathcal{N}_f = \bigoplus_{i=1}^{s-1} \mathcal{O}_{\mathbb{P}^1}(d + h_{i-1} - h_{i+1}) \qquad (C\text{-}R \text{ formula}).$$

By using the above formula it is easy to prove the following theorem, saying exactly for which smooth monomial curves $\mathcal{N}_f$ splits according to Theorem 2.

Theorem 3. *There exists a smooth monomial curve of degree d in $\mathbb{P}^s$, $s \geq 3$, with generic splitting type of $\mathcal{N}_f$ if and only if at least one of the following conditions are satisfied:*

(1) $d < 3s/2$.
(2) $d \geq 3s/2$ *and* $s-1$ *is even.*
(3) $d \geq 3s/2$, $s-1$ *is odd and* $d = a(s/2) + b$, *with* $0 \leq b \leq (s/2) - a$ *and* $a \geq 3$.

Proof. (1) It suffices to show that we can construct a smooth monomial curve with the generic splitting type for $\mathcal{N}_f$ we want. The condition $d < 3s/2$ is equivalent to $d \geq 3e + 4$, hence the monomial curve is "generic" in the sense of I) and we have proved there that the splitting type of $\mathcal{N}_f$ is the generic one according to Theorem 2.

(2) As for (1), it suffices to show that we can construct a smooth monomial curve with the generic splitting type for $\mathcal{N}_f$ we want. We can argue as in II).

$(3)_1$ The condition is necessary. Let us assume that the sequence $h_{i-1} - h_{i+1}$ consists of at most two integers a and $a+1$, with a appearing $q > 0$ times. Let us consider the h_i with even index, which are: $h_0, h_2, ..., h_{s-2}, h_s$, and the sum

$$\sum_{i=1}^{s/2} (h_{2(i-1)} - h_{2i}) = h_0 - h_s = d - 0 = d$$

Similarly, let us consider the h_i with odd index: $h_1, h_3, ..., h_{s-1}$ and the sum

$$\sum_{i=1}^{s/2-1} (h_{2i-1} - h_{2i+1}) = h_1 - h_{s-1} = (d-1) - 1 = d - 2.$$

Suppose that within the even differences $h_{2(i-1)} - h_{2i}$ the number a appears p_1 times, and in the odd differences $h_{2i-1} - h_{2i+1}$ it appears p_2 times, and the number a appears, respectively, q_1 times and q_2 times with $p = p_1 + p_2$ and $q = q_1 + q_2$. Then one has the following relations, which we will call (♣):

$d = a(s/2) + p_1 = q_1 a + p_1(a+1), \qquad 0 \leq p_1 \leq s/2 - 1,\ p_1 + q_1 = s/2$
$d - 2 = a(s/2 - 1) + p_2 = q_2 a + p_2(a+1), \qquad 0 \leq p_2 \leq s/2 - 2,\ p_2 + q_2 = s/2 - 1.$

From the first equality one also has that $d - 2 = a(s/2 - 1) + a + p_1 - 2$, hence one obtains $p_2 = a + p_1 - 2 \leq s/2 - 2$, i.e. $p_1 \leq s/2 - a$. Putting $b = p_1$ we find the stated necessary condition: $d = a(s/2) + b$ with $0 \leq b \leq s/2 - a$. Moreover we also have $s/2 \geq a \geq 2s/(s-2)$, (recall that $s \geq 3$) hence $a \geq 3$.

$(3)_2$ The condition is sufficient. We observe that, if $d = a(s/2) + b$ with $0 \leq b \leq s/2 - a$, then one has $d - 2 = a(s/2 - 1) + a + b - 2$ and one can put $p_1 = b$ and $p_2 = a + b - 2 \leq s/2 - 2$, so one can reproduce the relations (♣). Then one can choose the integers h_{2i} with even indexes so that in the $s/2$ differences $h_{2(i-1)} - h_{2i}$ the number a appears $q_1 := s/2 - p_1$ and the number $a+1$ appears p_1 times. Similarly one can choose the integers h_{2i-1} with odd indexes so that in the $s/2-1$ differences $h_{2i-1} - h_{2i+1}$ the number a appears $q_2 := s/2 - 1 - p_2$ times and the number $a+1$ appears p_2 times. By applying the C-R formula it follows that the constructed sequence $h_0, ..., h_s$ defines a smooth monomial curve with the required splitting type for $\mathcal{N}_f$, because $h_{i-1} - h_{i+1} \in [a, a+1]$. ∎

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