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Solutions of $\mathcal{N} = 2$ gauged supergravity in 4, 5 and 6 dimensions

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Supervisore: Prof. Dietmar KLEMM

Coordinatore: Prof. Francesco RAGUSA

Tesi di Dottorato di:

Camilla SANTOLI

Anno Accademico 2016-2017

External Referees:

Prof. Patrick Meessen (Universidad de Oviedo)

Prof. Sergio Cacciatori (Università degli studi dell'Insubria)

Commission of the final examination:

Prof. Dietmar Klemm (Università degli studi di Milano)

Prof. Tomás Ortín (Instituto de Física Teórica, Madrid)

Prof. Stefan Vandoren (Univerisiteit Utrecht)

Final examination:

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Università degli Studi di Milano, Dipartimento di Fisica, Milano, Italy

Abstract

The main goal of the work we are presenting is the exploration of some sectors of the space of the supersymmetric solutions of supergravity theories with 8 supercharges, in 4, 5 and 6 dimensions, which led to the new solutions originally presented in [1–3].

We are firstly reviewing the supergravity theories of our interest, paying attention to their geometrical structure and to their possible gaugings. While discussing the 4-dimensional case, we are also introducing a couple of solutions generating techniques, originally developed in [4] and [5].

We then introduce the idea of dimensionally reducing a theory and its solutions on a circle, as the necessary tool to relate 4, 5 and 6-dimensional ungauged supergravity theories among themselves. An interesting feature emerges from this treatment: there are two classes of 6-dimensional theories that lead, when compactified, to the same 5-dimensional model; the two 6-dimensional theories are therefore dual. In the present context, the main reason to treat dimensional reduction is given by the possibility of generating new solutions, reducing or uplifting known ones.

Finally, we apply these techniques to generate the first 4-dimensional solution of $\mathrm{U}(1)$-Fayet-Iliopoulos gauged supergravity coupled to vector multiplets, whose scalars parametrize a non-homogeneous Kähler manifold. It is a black hole solution, with AdS_4 asymptotes [1]. We then explore the $\mathrm{SU}(2)$-Fayet-Iliopoulos gauged sector, in 4 dimensions. Once a simple model -known as $\overline{\mathbb{CP}}^3$- is chosen, some solutions for this setting are found [3]. More solutions for the same theory are obtained through dimensional reduction -that has been generalized to relate gauged theories- of a couple of known solutions for the 6-dimensional, FI-gauged theory. 5-dimensional solutions are obtained in the procedure, as well. Finally, a known 5-dimensional solution, an extremal black hole sourced by a BPST instanton, is uplifted to 6 dimensions [2].

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Introduction

Among the most studied but still unsolved problems that modern physics is facing, a relevant role is surely taken by the existing gap between the two fundamental theories that build up modern physics itself: general relativity and quantum mechanics.
The first of them describes to high accuracy the gravitational phenomena as the behavior of space and time, while the second one, especially in its relativistic formulation -quantum field theory and the standard model- provides an excellent, extremely predictive description of particles and their interactions. This is true at least up to the scale that are nowadays experimentally accessible, i.e. in a realm where the effects of gravity are definitely negligible with respect to the other forces.
There are, however, situations where this distinction breaks down and gravitational phenomena are as important as quantum effects. Black holes and the primordial, inflationary universe are well known examples thereof. In particular, black holes emerged as classical, macroscopic solutions in general relativity, but the singularity in their interior signals the breakdown of the classical theory. It is widely believed that a more profound, universal, quantum theory of gravitation will resolve these singularities, possibly providing a precise, finite description of the black hole interior and of the quantum phenomena occuring in this regime.
Moreover, Hawking [6] showed how these objects possess some quantum properties as well -they have a temperature, an entropy and can evaporate-, leading to inconsistencies and paradoxes which seem to be unsolvable within the standard picture developed so far by modern physics.

Many attempts have been made to develop a unified theory, but the problem has not been solved yet. The most widely studied approach is surely represented by string theory, whose low energy limit is a field theory called supergravity. There are five versions of perturbative string theory, related by dualities: a given physical situation may admit more than one theoretical formulation and it can turn out that the respective levels of difficulty in analyzing these distinct, dual, formulations can be wildly different. Hard questions to answer from one perspective can turn into far easier questions to answer in another [7]. The existence of these dualities led to the conjecture of a non perturbative 11-dimensional theory that unifies all the consistent versions of superstring theory,

named M-theory, whose low energy sector should be described by 11-dimensional supergravity. In the same way, the low energy limit of type IIA and type IIB superstring theories are 10-dimensional supergravity theories, called $\mathcal{N} = 2A$ and $\mathcal{N} = 2B$. When compactified on a circle, these theories give rise to the same 9-dimensional supergravity: they are, in this sense, related by T-duality, reflecting the connection between the respective complete string theories. Moreover, when the 10-dimensional supergravity theories are compactified on 6-dimensional Calabi-Yau manifolds, the resulting 4-dimensional supergravities are not maximally symmetric anymore: they are exactly the $\mathcal{N} = 2$ theories we will discuss in the present thesis. If the 10-dimensional theories are compactified on mirror Calabi-Yau manifolds, a manifestation of duality appears at the 4-dimensional level, too, where it is called mirror symmetry and relates theories with an interchanged number of vector multiplets and hypermultiplets. Another manifestation of duality, at the level of supergravity theories, will be discussed in chapter 2.

As low energy limit of superstring theory, supergravity is a field theory of supersymmetric matter coupled to gravity. This theory is invariant under local supersymmetry transformations, whose local parameters must be understood as diffeomorphisms; this means that local supersymmetry automatically requires the inclusion of gravity. On the other hand, if we are interested in constructing a theory of gravity enjoying supersymmetry, the latter should be realized locally.

The possible theories of supergravity are classified in terms of the dimension of the spacetime on which they are defined and of the amount of supersymmetry under which they are invariant. These two parameters determine the field content of each theory. The so called minimal supergravities are theories whose field content only consists of the multiplet in which the graviton is included, together with its superpartner, a fermion called gravitino. Supersymmetric matter multiplets, whose nature again depends on the mentioned parameters, can be coupled to this basic setting. The more supersymmetry the theory is respecting, the more the matter couplings are constrained. On the other hand, symmetries turn out to be really helpful in simplifying the equations that need to be solved in order to find solutions of a given theory.

In this context, the theories we are going to deal with in the present thesis, characterized by 8 supercharges, are considered to be a good compromise: they still admit many different matter couplings, but they have enough supersymmetry to provide solvable equations.
In fact, many solutions of these theories are known (see, for example [8–10]) and in some cases, a classification of the solutions has been obtained [11–13].

Another interesting feature is the possibility of deforming these theories, introducing a gauge group. Vector fields must be present and matter is charged under them. An intriguing outcome of this gauging procedure is the appearance, in some of the so called gauged supergravity theories, of AdS vacua and asymptotically AdS solutions, including AdS black holes.

These latter deserve to be studied in particular from a gauge/gravity duality point of view. The motivation is twofold: it has been discovered that these solutions represent the gravity dual of certain strongly coupled field theories, describing relevant condensed matter systems such as, for example, the quark gluon plasma or the high-temperature superconductors [14, 15]. Thus, providing new gravity solutions of this kind could help in understanding the behavior of their dual systems. On the other hand, exact computations can be performed on the field theory side, which could provide an insight into the fundamental and still unknown properties of black holes, as the microscopic origin of their entropy [16].

Many solutions of the gauged theories are known as well [1,3,5,17–31] and solutions in 4 and 5 dimensions have been classified [4,32–35].

Another way of interpreting the interest that the search for new solutions raises, considers that the solutions of supergravity theories can always be understood as solutions of general relativity coupled to matter, neglecting their supersymmetric behavior and exploiting supersymmetry as a solution generating technique.

The main goal of this thesis is the exploration of some sectors of the space of the supersymmetric solutions of supergravity theories which had not been explored before, as one of the most elementary steps that can be taken to get a more complete understanding of its structure.
New solutions of $\mathcal{N} = 2$ supergravities, some of which represent black objects and some are asymptotically AdS, have been presented in [1, 3]. This thesis is mostly based on these works, together with [2], where the connection between the theories in 5 and 6 dimensions has been studied.

The present thesis is organized as follows: after a brief general introduction, the first chapter is devoted to a review of the supergravity theories with 8 supercharges in 4, 5 and 6 dimensions. Special attention will be paid to their geometrical structure and to the related topic of their possible gaugings. As far as the 4-dimensional theory is concerned, we are also detailing some example of these geometries, characterizing well known models.
While discussing the 4-dimensional case, we are also introducing a couple of solutions generating techniques, originally developed in [4] and [5]; the first one, in particular, classifies all the time-like supersymmetric solutions of the 4-dimensional theory.
Although similar treatments exists in the 5-dimensional case [34,36], we decided not to present them here, since they are very similar and we are not going to use them explicitly.
The solutions of the 6-dimensional theory have been only partially classified in [37,38].

The second chapter introduces the idea of dimensionally reducing a theory and its solutions on a circle, following the method developed in [39].
After a general discussion, the well known example of how 5-dimensional ungauged supergravity can be reduced to certain models of the 4-dimensional ungauged theory is presented. The relations between the fields in the two theories is explicitly given, so

that reducing a solution with one isometry is just a matter of applying the general rules. Analogously, solutions can be uplifted from 4 to 5 dimensions.
The original result we are presenting concerns a similar procedure relating 5 to 6-dimensional supergravity theories; the problem is slightly more involved, due to the absence of a general covariant action for the 6-dimensional theories, which involves chiral 2-form fields. An interesting feature emerges from this treatment: there are two classes of 6-dimensional theories that lead, when compactified, to the same 5-dimensional model. This property has been interpreted as the manifestation of a duality between the two 6-dimensional theories, analogous to the one connecting the two maximal 10-dimensional supergravities, $\mathcal{N} = 2A$ and $\mathcal{N} = 2B$: when compactified on a circle, they give rise to the same 9-dimensional maximal supergravity, which is unique.
In the present context, the main reason to treat dimensional reduction is given by the possibility of generating new solutions, once one is known in a certain theory. We have constructed all the instruments we need to reduce solutions with isometries from 6 to 5 and to 4 dimensions, and to uplift 4 and 5-dimensional solutions to solutions of two different 6-dimensional theories. Although only ungauged theories have been considered so far, the relations are still applicable as long as the gauge group does not change in the process of dimensional reduction, as detailed in chapter 3.

Chapter 3 is finally devoted to the application of these techniques to generate new solutions. In particular, we are presenting the first 4-dimensional solution of $\mathrm{U}(1)$-Fayet-Iliopoulos gauged supergravity coupled to vector multiplets, whose scalars parametrize a non-homogeneous Kähler manifold. It is a black hole solution, with AdS_4 asymptotes [1].
Another sector that had not been explored before is the $\mathrm{SU}(2)$-Fayet-Iliopoulos gauged one, in 4 dimensions. Once a simple model, $\overline{\mathbb{CP}}^3$, is chosen, some solutions in this peculiar setting are found [3].
More solutions for the same theory are obtained through dimensional reduction of a couple of known solutions for the 6-dimensional, FI-gauged theory. As outlined previously, the reduction proceeds by steps, so that 5-dimensional solutions are originated in the procedure, as well.
Finally, a known 5-dimensional solution involving non-Abelian fields, an extremal black hole sourced by a BPST instanton, is uplifted to the two possible 6-dimensional theories [2]. The solutions that emerge in this way represent a superposition of a dyonic string and a wave, with an additional BPST instanton, and a superposition of a selfdual string and a *pp*-wave, with a non-Abelian contribution that can be interpreted as an instanton.

CHAPTER 1

$\mathcal{N} = 2$ gauged supergravity in 4, 5 and 6 dimensions

The supergravity theories with 8 real supercharges provide a very interesting arena for the construction and study of supersymmetric solutions, because they have enough symmetry to be tractable and exhibit interesting properties such as the attractor mechanism of their black-hole and black-string solutions [8, 40–43] but not so much symmetry that only a few models are permitted. They can also be coupled to matter, which is the case we are interested in. Matter comes in multiplets, containing the same number of bosonic and fermionic degrees of freedom, whose field content varies with the dimension of the theory. The presence of vector fields give rise to the possibility of gauging the theory, charging the matter fields under Abelian and non-Abelian gauge groups.

Most of the work in these theories has been devoted to the 4-and 5-dimensional ones for different reasons: for a given matter content, many models are possible; they are the effective theories of type II superstrings and of the conjectured M-theory, respectively, compactified on Calabi-Yau 3-folds (moreover, the compactification of all the 5-dimensional theories in a circle gives rise to 4-dimensional supergravities); they have rich geometrical structures known as Special Geometry (Kähler in $d = 4$, real in $d = 5$); they admit supersymmetric black-hole solutions and some of them are asymptotically AdS, gaining interest from a gauge/gravity duality perspective.
In fact, most of those supersymmetric solutions have been classified in [4, 11, 24, 32, 33, 44, 45] and [12, 13, 34, 35, 46–49] respectively.

Much less work has been done in the 6-dimensional theories (often called $\mathcal{N} = (2, 0), d = 6$ supergravities because they have chiral fermions), whose structure is not as rich. However, it has been shown [50,51] that compactifications of F-theory on elliptic Calabi-Yau 3-folds generate a large class of 6-dimensional theories and it has been conjectured that all UV-consistent 6-dimensional chiral supergravity theories can be realized in string theory.

The present chapter is devoted to a brief introduction to these theories, which is not aimed to be complete, but should provide useful definitions and a general overview to contextualize the original results that are presented in the rest of this thesis. A general and deep review of all these theories can be found in [52] and for the 4-dimensional case

only, in [53]. The 4- and 5-dimensional ones are also reviewed in [54], with emphasis on the supersymmetric bosonic solutions.

The rest of this chapter is organized as follows: in section 1.1 we present some feature that are common to all the theories we are interested in, $\mathcal{N} = 2$ supergravity in 4 and 5 dimensions and the 6-dimensional $\mathcal{N} = (2, 0)$ theory; we will describe the matter multiplets to which they can be coupled and introduce the concepts of R-symmetry group, gauged supergravity, scalar geometry and of the amount of supersymmetry preserved by a solution.
In section 1.2 we will give a detailed description of the 4-dimensional theory, with a particular emphasis on the geometries defined by the scalars in its vector and hypermultiplets, the special and quaternionic Kähler manifolds and of the isometries that can be gauged. We will explicitly give the action, the equations of motion and the supersymmetry transformations for these theories and we will present two techniques to generate solutions, both based on the requirement that the solutions preserve a certain amount of supersymmetry: solving the full set of the equations of motion is really difficult, due to the presence of the non linear Einstein equations, while the constraints imposed by supersymmetry greatly simplify the problem. These techniques go under the name of "bilinear method" and "squaring of the action". We will finally give some examples of models involving different special Kähler manifolds, to illustrate how the quantities that enter the Lagrangian can be obtained once the scalar geometry is known, and to pave the way for the specific situations that will be considered in chapter 3.
Similarly, in the remaining sections 1.3-1.4, we will give an overview of the 5 and 6-dimensional theories, involving the description of the possible matter multiplets, of the scalar geometries (real Kähler in 5 dimensions) and of the possible gaugings. In the context of the present work, these theories are mostly relevant because of their relation, through dimensional reduction, with the 4-dimensional one. As explained in chapter 2, this means that solutions of one of these theories can be, under certain conditions, dimensionally uplifted of reduced, to give rise to new solutions for the other theories. This mechanism will be exploited as a solution generating technique in chapter 3.

1.1 General features

Supergravity theories are theories with gauged, or local, supersymmetry; the action of these theories is invariant under supersymmetry transformations in which the spinor parameters are arbitrary functions of the spacetime coordinates. The supersymmetry algebra is then involving local translation parameters, which must be viewed as diffeomorphisms. Therefore, we can say that local supersymmetry requires gravity. On the other hand, in any supersymmetric theory which includes gravity, supersymmetry must be realized locally. A minimal supergravity theory is an interacting field theory involving the gravity multiplet only. Eventually, other matter multiplets of the underlying

d=4	$\mathcal{N} = 2$	SU(2)×U(1)
d=5	$\mathcal{N} = 2$	SU(2)
d=6	$\mathcal{N} = (2,0)$	SU(2)

Table 1.1: R-symmetry automorphism group

global supersymmetry algebra can be coupled to minimal supergravity, giving rise to matter coupled supergravity theories, still respecting the invariance under local supersymmetry transformation.

Supergravity theories that are symmetric under more than one supersymmetry transformation are called extended supergravities; this is the case we are interested in, specifically in theories including two spinor supercharges -8 supercharges in total-, which are denoted as $\mathcal{N} = 2$ supergravities in 4 and 5 dimensions, or as $\mathcal{N} = (2,0)$ in 6 dimensions, where both supercharges have the same chirality.[1]

In the gauging of the theory that we are going to perform, a relevant role will be played by the R-symmetry. The main characteristic of this symmetry is that it commutes with Lorentz and translation generators, but it does not commute with the supercharges. This distinguishes it from the gauge symmetries of the theory, which do commute with supersymmetry. Upon investigation of the commutation rules, it turns out that the R-symmetry groups of the theories we are dealing with are those listed in table 1.1.

The field content of these theories is determined by the massless particle representation of the corresponding superalgebra.
The supergravity multiplet, in which the graviton $g_{\mu\nu}$ and its superpartners, the gravitinos $\psi^i{}_\mu$ are included, is always present and can be coupled to different kinds of matter multiplets: vector multiplets, hypermultiplets and, as far as the 6-dimensional theory is concerned, tensor multiplets. The field content of these multiplets, in each dimension, is given in table 1.2.
When present, the vector field in the supergravity multiplet $A^0{}_\mu$ is dubbed graviphoton. In chapter 2, we are discussing how the theories in different dimensions are related by dimensional reduction; this topic is going to clarify the origin of the scalars in the vector multiplets, which are absent in 6 dimensions, real ϕ in 5 and complex Z in 4. Vector multiplets always involve vector fields A_μ and gauginos λ^i.
Since dimensional reduction for scalars and spin-$\frac{1}{2}$ fermions leads to the same type of particles in lower dimensions, the hypermultiplets are always including real scalars (q) and spinors (ξ^i) and their properties do not depend crucially on dimension.
In 6 dimensions, an antisymmetric tensor can have (anti)selfdual properties and a (anti) selfdual field-strength; these fields appear in the so called antisymmetric tensor multiplets as $B^+_{\mu\nu}$ -whose field strength is selfdual- and in the supergravity multiplet as $B^-_{\mu\nu}$

[1] In literature, the 5-dimensional theory is sometimes denoted as $d = 5$, $\mathcal{N} = 1$ supergavity, while the 6-dimensional one is also known as $\mathcal{N} = (1,0)$.

d	supergravity	tensor	vector	hyper
4	$g_{\mu\nu}, \psi^i{}_\mu, A^0{}_\mu$	-	A_μ, λ^i, Z	ξ^i, q
5	$g_{\mu\nu}, \psi^i{}_\mu, A^0{}_\mu$	-	A_μ, λ^i, ϕ	ξ^i, q
6	$g_{\mu\nu}, \psi^i{}_\mu, B^-_{\mu\nu}$	$B^+_{\mu\nu}, \chi^i, \varphi$	A_μ, λ^i	ξ^i, q

Table 1.2: Multiplets

4	special Kähler	$\times$ quaternionic Kähler
5	special real	$\times$ quaternionic Kähler
6	$\frac{\mathrm{O}(1,n_T)}{\mathrm{O}(n_T)}$	$\times$ quaternionic Kähler

Table 1.3: Scalar manifolds

-whose field strength is anti-selfdual-. The tensor multiplets involve, in addition to the 2-form $B^+_{\mu\nu}$, a scalar φ and a tensorino χ^i.

For a given matter content, i.e. once the number of vector n_V, tensor n_T and hypermultiplets n_H has been fixed, the kinetic terms of these theories still depend on one or more arbitrary functions. These latter determine the metric of the non-linear σ-models that characterize the kinetic term for the scalars. These metrics are part of the data that define a particular theory, they fix the geometry of the scalar manifold.
The total scalar manifold of a theory has always the form of a direct product of the manifold of the scalars in the hypermultiplets (quaternionic Kähler manifolds) and those of tensor multiplets in 6 dimensions or vector multiplets in 5 (special real manifolds) and 4 dimensions (special Kähler manifolds), as summarized in table 1.3. The involved geometries belong to a class called special geometries, which are of particular interest. We are discussing them in what follows.

We are now going to introduce deformations of the basic theories which are obtained introducing a gauge group. A gauged supergravity is a theory in which vector fields gauge a subgroup of the global symmetries. The number of generators of the gauge group cannot exceed the number of vector fields that are present in the theory, including vectors in the vector multiplets and eventually those in the supergravity multiplet. In fact, the kinetic terms of these vectors are generally mixed together in the action. Matter fields are charged under the gauge group.
The deformed theory differs in various ways from the original one:

- the supersymmetry transformations of the fermions acquire new terms, called fermion shifts;
- covariant derivatives appear instead of ordinary ones;
- the field strengths are redefined properly;

- a scalar potential is generated, which can be expressed as sums of squares of the fermion shifts;
- there are new terms in the action, such as fermion masses.

Gauged supergravities can also be approached from the point of view of the isometry group of the scalar sector: the manifold, which is defined by the metric of the kinetic term for the scalars, has isometries that are global symmetries and nearly always extend to global symmetries of the full supergravity action. The gauged supergravity is then obtained by gauging a subgroup of these global symmetries.

Once the action of a specific supergravity theory has been given, we will be looking for solutions of the classical equations of motion. In particular, we are interested in solutions that can be interpreted as classical backgrounds, or vacua, above which fluctuations can be treated quantum mechanically. The backgrounds we are considering have vanishing values for every fermionic fields and are determined by configurations of the bosonic fields only. Therefore, from now on, we are only dealing with the bosonic sector of each theory.
The solution should possibly be invariant under global supersymmetries that are a subset of the local supersymmetries of the supergravity action, i.e. it should preserve some supersymmetry. Preserved supersymmetries of a solution are determined by the vanishing of the supersymmetry transformations of the fermions.
The other way around, supersymmetry can be considered as a solution generating technique, since the constraints that its preservation imposes, the vanishing of the supersymmetry transformations for the fermions, allows to greatly simplify the equations of motion that have to be solved to find new solutions. An example is given in section 1.2.5.

We are now ready to enter the details of the bosonic sector of the 4, 5 and 6-dimensional theories, separately, while in chapter 2 we are examining the relations among them and their solutions.

1.2 The 4-dimensional theory

$\mathcal{N} = 2$, $d = 4$ supergravity is the simplest theory of extended supergravity. Even though it has no direct phenomenological relevance to particle physics due to the impossibility of reproducing the chiral structure of the standard model, it is a very interesting theory on its own.
Among the reasons for this interest, there is the origin of many of the models in this theory, that arise from the compactification of type-II theories in Calabi–Yau 3-folds, which means that many of the results and solutions obtained in this framework can be embedded in full superstring theory.

As already stated, there are many possible matter couplings, even for the same matter content. All of them are different realizations of a common and very rich mathematical

structure, known as special Kähler geometry, that governs the couplings of the vector multiplets and the supergravity one and of another structure, known as quaternionic Kähler geometry, that governs the coupling of the hypermultiplets. In particular, special Kähler geometry contains information on the symplectic structure present in the couplings between scalars and vector fields.

We quickly recall that the supergravity multiplet includes the following fields,

$$\left\{e^a{}_\mu, \psi^I{}_\mu, A^0{}_\mu\right\} \tag{1.1}$$

and it is coupled to n_V vector multiplets and n_H hypermultiplets. The field content of each of the n_V vector multiplets, labeled by the index $i = 1, \dots n_V$, is

$$\left\{A^i{}_\mu, \lambda^{iI}, Z^i\right\} \ , \tag{1.2}$$

where λ^{iI} is a pair of gauginos and Z^i is a complex scalar which, in the coupled theory, will be interpreted as a complex coordinate in a special Kähler manifold. Since duality rotations will mix the graviphoton field with the matter vector fields, we conveniently use a common notation for all of them: the index $\Lambda = (0, i)$, which takes values $\Lambda = 0, \dots, n_V$.

Each hypermultiplet consists of 4 real hyperscalars and 2 hyperinos. If n_H hypermultiplets are involved, their fields can collectively be labeled by $u = 1, \dots, 4n_H$ and $\alpha = 1, \dots, 2n_H$

$$\{q^u, \xi_\alpha\} \ . \tag{1.3}$$

The hyperscalars parametrize a quaternionic Kähler manifold.

Specifying the choice for the 2 involved geometries is enough to fully determine the ungauged model. We are now presenting its bosonic action, which involves many geometrical quantities; they are defined in sections 1.2.1 and 1.2.2, where the properties of the special Kähler and quaternionic Kähler manifolds, respectively, are depicted.

$$\begin{aligned} S = \int d^4x \sqrt{|g|} \Big[& R + 2\mathcal{G}_{ij^*} \partial_\mu Z^i \partial^\mu Z^{*\,j^*} + 2\mathsf{H}_{uv} \partial_\mu q^u \partial^\mu q^v \\ & + 2\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^\Sigma{}_{\mu\nu} - 2\Re\mathfrak{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} \star F^\Sigma{}_{\mu\nu} \Big] \ . \end{aligned} \tag{1.4}$$

In the previous expression, $\mathcal{G}_{ij^*}$ is the metric of the special Kähler manifold, parametrized by the complex scalars Z^i, while H_{uv} is the metric of the quaternionic Kähler space, parametrized by the hyperscalars q^u. The period matrix $\mathcal{N}_{\Lambda\Sigma}$ is a function of the Z^i and is determined by the choice of special Kähler geometry.

It can be noticed that the hyperscalars are decoupled from the vector supermultiplets, so that in the ungauged theory the hypermultiplets can always be consistently truncated.

We are now reporting the Lagrangian of the most general gauged theory,

$$
\begin{aligned}
S = \int d^4x\sqrt{|g|}\,\Big[& R + 2\mathcal{G}_{ij^*}\mathfrak{D}_\mu Z^i \mathfrak{D}^\mu Z^{*\,j^*} + 2\mathsf{H}_{uv}\mathfrak{D}_\mu q^u \mathfrak{D}^\mu q^v \\
& + 2\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^{\Sigma}{}_{\mu\nu} - 2\Re\mathfrak{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} \star F^{\Sigma}{}_{\mu\nu} - \mathbf{V}(Z, Z^*, q)\Big]\,,
\end{aligned}
\tag{1.5}
$$

where the scalar potential $\mathbf{V}(Z, Z^*, q)$ is given by

$$
\begin{aligned}
\mathbf{V}(Z, Z^*, q) = & -\tfrac{1}{4} g^2 (\Im\mathfrak{m}\mathcal{N})^{-1|\Lambda\Sigma} \mathcal{P}_\Lambda \mathcal{P}_\Sigma \\
& + 2g^2 \mathsf{H}_{uv} \mathsf{k}_\Lambda{}^u \mathsf{k}^v_\Sigma \mathcal{L}^{*\Lambda} \mathcal{L}^\Sigma + \tfrac{1}{2} g^2 \left(\mathcal{G}^{ij^*} f^\Lambda{}_i f^{*\,\Sigma}{}_{j^*} - 3\mathcal{L}^{*\,\Lambda} \mathcal{L}^\Sigma \right) \mathsf{P}_\Lambda{}^x \mathsf{P}_\Sigma{}^x\,.
\end{aligned}
\tag{1.6}
$$

$\mathcal{P}_\Lambda$ are known as holomorphic momentum maps, while $\mathsf{P}_\Sigma{}^x$ are the triholomorphic momentum maps. The definition and the origin of these objects, together with the form of the covariant derivatives appearing in the action, will be clarified in section 1.2.3, where we are discussing all the possible gaugings of the ungauged theories and depicting how the Lagrangian (1.5) was obtained. $\mathcal{L}^\Lambda$ and $f^\Lambda{}_i$ are instead geometrical quantities, whose definition is given in section 1.2.1.

1.2.1 Special Kähler geometries

Special Kähler geometry is the structure that dictates the couplings between the fields in the vector multiplets and in the supergravity multiplet. These couplings are encoded in several functions of the complex scalars appearing in the action (1.4). Supersymmetry requires that all these objects are related in a very specific way. These relations are the essential content of special Kähler geometry that we are now presenting.

We are considering a complex manifold[2] with an Hermitian metric $\mathcal{G}_{ij^*}$,

$$
ds^2 = \mathcal{G}_{ij^*} dZ^i dZ^{*j^*}\,. \tag{1.7}
$$

We can define the fundamental 2-form

$$
\mathcal{J} = i\mathcal{G}_{ij^*} dZ^i \wedge dZ^{*j^*}\,. \tag{1.8}
$$

If $\mathcal{J}$ is closed, $d\mathcal{J} = 0$, it is called a Kähler form and the manifold is a Kähler manifold. This condition implies the existence in every coordinate patch of a real function $\mathcal{K}(Z, Z^*)$, the Kähler potential, such that the metric is locally given by

$$
\mathcal{G}_{ij^*} = \partial_i \partial_{j^*} \mathcal{K}\,. \tag{1.9}
$$

The Kähler potential is not uniquely defined, since a Kähler transformation

$$
\mathcal{K}(Z, Z^*) \to \mathcal{K}(Z, Z^*) + \lambda(Z) + \lambda^*(Z^*)\,, \tag{1.10}
$$

[2] In this section and in the next one we are using the notation of [54].

where λ is an holomorphic function, leaves the metric (1.9) invariant.
The Kähler connection 1-form is defined as

$$\mathcal{A} = \frac{1}{2i}\left(\partial - \partial^*\right)\mathcal{K} \tag{1.11}$$

and is not invariant under Kähler transformations.
In general, an object Ψ is said to have Kähler weight (p, q) if, under Kähler transformations (1.10), it behaves as

$$\Psi \to e^{-\frac{1}{2}(p\lambda + q\lambda^*)}\Psi\,. \tag{1.12}$$

The Kähler covariant derivative is given by

$$\mathcal{D}_i = \nabla_i + \frac{1}{2}p\partial_i\mathcal{K}\,, \qquad \mathcal{D}_{i^*} = \nabla_{i^*} + \frac{1}{2}q\partial_{i^*}\mathcal{K} \tag{1.13}$$

where ∇ is the standard covariant derivative associated to the Hermitian connection, due to the tensorial nature of Ψ.

If $p = -q = 1$, the Kähler transformations for the field Ψ are $\mathrm{U}(1)$, Z dependent transformations

$$\Psi \to e^{-i\Im\mathfrak{m}\lambda(Z)}\Psi\,. \tag{1.14}$$

The structure that supports these fields is an $\mathrm{U}(1)$ bundle, associated to a complex line bundle $L_1 \to \mathcal{M}$ over the Kähler manifold $\mathcal{M}$. This construction is consistent only if the first Chern class of the bundle equals the Kähler class, i.e. the cohomology class defined by $\mathcal{J}$. Kähler manifolds satisfying this requirement are known as Kähler–Hodge manifolds.

Consider now a Kähler-Hodge manifold $\mathcal{M}_{KH}$ of complex dimension n_V and a flat $2(n_V + 1)$ vector bundle $E \to \mathcal{M}_{KH}$, with structure group $\mathrm{Sp}(2(n_V + 1); \mathbb{R})$.
$\mathcal{M}_{KH}$ is a special Kähler manifold if there is a covariantly holomorphic symplectic section $\mathcal{V}$ of the product bundle $E \otimes L_1 \to \mathcal{M}_{KH}$ satisfying certain properties.
The section $\mathcal{V}$ has Kähler weight $p = -q = 1$. Its holomorphic Kähler-covariant derivative $\mathcal{U}_i$ is given by

$$\mathcal{U}_i = \mathcal{D}_i\mathcal{V} = \left(\partial_i + \frac{1}{2}\partial_i\mathcal{K}\right)\mathcal{V}\,, \tag{1.15}$$

in terms of which the properties that guarantees the special Kähler nature of the base of the symplectic bundle are written as

$$\langle\mathcal{V}|\mathcal{V}^*\rangle = -i\,, \tag{1.16}$$

$$\mathcal{D}_{i^*}\mathcal{V} = 0\,, \tag{1.17}$$

$$\langle\mathcal{U}_i|\mathcal{V}\rangle = 0\,. \tag{1.18}$$

As we will see, the symplectic section $\mathcal{V}$ describes the scalars Z^i, but encodes also all the information about the geometry of the model under consideration. In particular, the objects that are needed to determine the Lagrangian (1.4) can be obtained from $\mathcal{V}$; let us start with its components and those of its covariant derivative, carrying a $\Lambda = 0, \ldots, n_V$ index, which can be in upper or lower position,

$$\mathcal{V} = \begin{pmatrix} \mathcal{L}^\Lambda \\ \mathcal{M}_\Lambda \end{pmatrix} , \qquad \mathcal{U}_i = \begin{pmatrix} f^\Lambda{}_i \\ h_{\Lambda i} \end{pmatrix} . \tag{1.19}$$

The symplectic inner product, in terms of these components, is given by

$$\langle \mathcal{A} | \mathcal{B} \rangle = \mathcal{B}^\Lambda \mathcal{A}_\Lambda - \mathcal{B}_\Lambda \mathcal{A}^\Lambda . \tag{1.20}$$

From the basic definitions (1.16-1.18) and the properties of the covariant derivatives, the following identities can be obtained,

$$\begin{aligned} \langle \mathcal{U}_i | \mathcal{U}^*{}_{j^*} \rangle &= i \mathcal{G}_{ij^*} , \\ \langle \mathcal{U}_i | \mathcal{V}^* \rangle &= 0 , \\ \langle \mathcal{U}_i | \mathcal{U}_j \rangle &= 0 . \end{aligned} \tag{1.21}$$

These identities tell us that the sections $\mathcal{V}$, $\mathcal{U}_i$ and their complex conjugates are linearly independent and allow us to write the completeness relation for a generic symplectic section $\mathcal{A}$ as

$$\mathcal{A} = i \langle \mathcal{A} | \mathcal{V}^* \rangle \mathcal{V} - i \langle \mathcal{A} | \mathcal{V} \rangle \mathcal{V}^* + i \langle \mathcal{A} | \mathcal{U}_i \rangle \mathcal{G}^{ij^*} \mathcal{U}^*{}_{j^*} - i \langle \mathcal{A} | \mathcal{U}^*{}_{j^*} \rangle \mathcal{G}^{ij^*} \mathcal{U}_i . \tag{1.22}$$

The period matrix $\mathcal{N}_{\Lambda\Sigma}$, that appears in the Lagrangian (1.4) and determines the kinetic term of the vector fields -including the graviphoton-, is a function of the scalars Z^i, Z^{*i} and it is defined in special Kähler geometry by the following identities, involving the quantities we mentioned in (1.19),

$$\mathcal{M}_\Lambda = \mathcal{N}_{\Lambda\Sigma} \mathcal{L}^\Sigma , \qquad h_{\Lambda i} = \mathcal{N}^*{}_{\Lambda\Sigma} f^\Sigma{}_i . \tag{1.23}$$

This definition guarantees some interesting properties to $\mathcal{N}_{\Lambda\Sigma}$, which make it suitable to appear in a kinetic term: it is symmetric and its imaginary part $\Im\mathfrak{m}\, \mathcal{N}_{\Lambda\Sigma}$ is invertible and negative definite.

Another symplectic section can be introduced,

$$\Omega = e^{-\frac{\mathcal{K}}{2}} \mathcal{V} , \tag{1.24}$$

that is holomorphic and has Kähler weight $(2, 0)$. The defining properties (1.16-1.18) take then the form

$$\langle\Omega|\Omega^*\rangle = -ie^{-\mathcal{K}}\,, \qquad \langle\Omega|\mathcal{D}_i\Omega\rangle = \langle\Omega|\partial_i\Omega\rangle = 0 \tag{1.25}$$

providing a way to evaluate the Kähler potential.
The section $\Omega(Z)$ is locally represented by a symplectic vector of the form

$$\Omega(Z) = \begin{pmatrix} \mathcal{X}^\Lambda(Z) \\ F_\Lambda(Z) \end{pmatrix} \tag{1.26}$$

If we assume that the components F_Λ depend on Z^i only through $\mathcal{X}^\Lambda(Z)$, the second equation in (1.25) becomes

$$\partial_i\mathcal{X}^\Lambda\left(2F_\Lambda - \partial_\Lambda\left(\mathcal{X}^\Sigma F_\Sigma\right)\right) = 0\,, \tag{1.27}$$

which is satisfied if there is a holomorphic homogeneous function of second degree $F(\mathcal{X})$, called a prepotential, such that $F_\Lambda = \partial_\Lambda F$.
In general a prepotential may not exist, but it is always possible, once a special Kähler manifold is given, to do a symplectic transformation to a frame in which it exists [55].
The prepotential provides an alternative way to characterize a given special Kähler geometry. All we need is a choice of the coordinates Z^i to express the $\mathcal{X}^\Lambda$ as holomorphic functions of them. A common choice is given by

$$Z^i = \frac{\mathcal{X}^\Lambda}{\mathcal{X}^0}\,, \qquad \mathcal{X}^0 = 1\,. \tag{1.28}$$

We can then evaluate all the relevant geometrical quantities in terms of the prepotential only: F_Λ in Ω are given by its derivatives, the Kähler potential is obtained as in (1.25), and the period matrix is

$$\mathcal{N}_{\Lambda\Sigma} = F^*{}_{\Lambda\Sigma} + 2i\frac{\Im\mathfrak{m}F_{\Lambda\Gamma}\mathcal{X}^\Gamma\Im\mathfrak{m}F_{\Sigma\Omega}\mathcal{X}^\Omega}{\Im\mathfrak{m}F_{\Delta\Psi}\mathcal{X}^\Delta\mathcal{X}^\Psi}\,, \tag{1.29}$$

where $F_{\Lambda\Sigma} = \partial_\Lambda\partial_\Sigma F$.

1.2.2 Quaternionic Kähler geometries

A quaternionic Kähler manifold is a $4n_H$-dimensional Riemannian manifold whose holonomy group is a subgroup of $\mathrm{USp}(2n_H)\times\mathrm{SU}(2)$.

An equivalent characterization defines a quaternionic Kähler manifold as a $4n_H$ dimensional Riemannian manifold satisfying the following properties:

- it admits a triplet J^x, $x = 1, 2, 3$, of almost complex structures satisfying the quaternion relation

$$\mathsf{J}^1\mathsf{J}^2 = \mathsf{J}^3 \qquad \text{(and cyclic permutations)}\,; \tag{1.30}$$

- the Riemmanian metric H_{uv} is Hermitian with respect to each of the complex structures

$$\mathsf{H}_{uv} = \mathsf{J}^x{}_u{}^w \mathsf{J}^x{}_v{}^t \mathsf{H}_{wt} \qquad \text{(no sum over } x\text{)}\,. \tag{1.31}$$

  As in any Hermitian manifold, a Kähler 2-form can be defined for each complex structure as

$$\mathsf{K}^x{}_{uv} = \mathsf{J}^x{}_{uv} \equiv \mathsf{H}_{uv} \mathsf{J}^x{}_u{}^w\,. \tag{1.32}$$

  The triplet of Kähler 2-forms is known as hyper-Kähler 2-form;

- the hyper-Kähler 2-form should satisfy

$$\nabla_u \mathsf{K}^x{}_{vw} + \epsilon^{xyz} \mathsf{A}^y{}_u \mathsf{K}^z{}_{vw} = 0\,, \tag{1.33}$$

  where ∇_u is the covariant derivative with Levi-Civita connection and A^x is the SU(2) connection 1-form.

A quaternionic-Kähler manifold $(n_H > 1)$ is necessarily Einstein and the SU(2) curvature is proportional to the complex structures,

$$R_{uv} = \frac{1}{4n_H} \mathsf{H}_{uv} R \tag{1.34}$$

$$\mathsf{F}^x \equiv d\mathsf{A}^x + \frac{1}{2}\varepsilon^{xyz} \mathsf{A}^y \wedge \mathsf{A}^z = \varkappa\, \mathsf{K}^x\,, \tag{1.35}$$

$$\varkappa \equiv \frac{R}{4n_H(n_H+2)}\,. \tag{1.36}$$

The quaternionic Kähler manifolds that appear in supergravity must have a negative constant $\varkappa$ in (1.36), which implies that they have negative scalar curvature.

It is customary to decompose the tangent indexes with respect to the holonomy group and to introduce frame fields $\mathsf{U}^u{}_{\alpha I}(q)$ connecting the scalar fields q^u, that are coordinates on the quaternionic Kähler manifold, to the fermions ζ^α. The index α runs from 1 to $2n_H$, while $I = 1, 2$, so that the $\mathsf{U}^u{}_{\alpha I}(q)$ can be seen as $4n_H \times 4n_H$ invertible matrices when (αI) is considered as a single index. The inverse is written as $\mathsf{U}^{\alpha I}{}_u(q)$,

$$\mathsf{U}^{\alpha I}{}_u \mathsf{U}^v{}_{\alpha I} = \delta^v_u\,, \qquad \mathsf{U}^{\alpha I}{}_u \mathsf{U}^u{}_{\beta J} = \delta^I_J \delta^\alpha_\beta\,. \tag{1.37}$$

The reality condition

$$\mathsf{U}_{\alpha I u} \equiv \left(\mathsf{U}^{\alpha I}{}_u\right)^* = \varepsilon_{IJ}\, \mathbb{C}_{\alpha\beta}\, \mathsf{U}^{\beta J}{}_u\,, \qquad \mathsf{U}^{u\alpha I} \equiv \left(\mathsf{U}^u{}_{\alpha I}\right)^* = \varepsilon^{IJ}\, \mathbb{C}^{\alpha\beta}\, \mathsf{U}^u{}_{\beta J}\,, \tag{1.38}$$

must be satisfied, where $\mathbb{C}_{\alpha\beta}$ is a non-degenerate tensor satisfying

$$\mathbb{C}_{\alpha\beta} = -\mathbb{C}_{\beta\alpha}\,, \qquad \mathbb{C}_{\alpha\beta}\,\mathbb{C}^{\beta\gamma} = -\delta_\alpha^\gamma\,, \qquad \mathbb{C}^{\alpha\beta} = (\mathbb{C}_{\alpha\beta})^*\ . \tag{1.39}$$

The metric is given by

$$\mathsf{H}_{uv} = \mathsf{U}^{\alpha I}{}_u\,\varepsilon_{IJ}\,\mathbb{C}_{\alpha\beta}\,\mathsf{U}^{\beta J}{}_v\,, \tag{1.40}$$

so that $\varepsilon_{IJ}\,\mathbb{C}_{\alpha\beta}$ can be interpreted as a metric in tangent space.

1.2.3 Possible gaugings

The global symmetries of a theory of $\mathcal{N}=2, d=4$ supergravity coupled to vector supermultiplets are the holomorphic isometries of the Kähler metric that also preserve the rest of the special Kähler structure; in particular, they must act as transformations of the symplectic group $\mathrm{Sp}(2n+2,\mathbb{R})$ on the symplectic section and, as a consequence, on the period matrix. Moreover, there is the R-symmetry group $\mathrm{U}(2)$ which only acts on the fermion fields in the fundamental representation.
If hypermultiplets are present, the global symmetries include also the isometry group of the quaternionic Kähler manifold parametrized by the real scalars. Only isometries that respect the quaternionic Kähler structure are global symmetries of the theory and can be gauged.

Therefore, $\mathcal{N}=2, d=4$ supergravities admit several kinds of gaugings:[3]

1. a non-Abelian subgroup of the isometry group of the special Kähler manifold of the complex scalars belonging to the vector multiplets[4] can simply be gauged. This is the simplest possibility: it does not involve the hypermultiplets and trying to gauge an Abelian isometry only would have no effect since all the terms that would have to be added (proportional, for instance, to the Killing vector) vanish identically. In absence of hypermultiplets, these theories have been called $\mathcal{N}=2, d=4$ Super-Einstein-Yang-Mills (SEYM) [23, 24], because they are the simplest $\mathcal{N}=2$ supersymmetrization of the Einstein-Yang-Mills theories;

2. a general subgroup of the isometry group of the quaternionic Kähler manifold of the scalars belonging to the hypermultiplets can be gauged. Since this requires the coupling to a set of gauge vector fields transforming in the adjoint of the gauge group, and since the available vectors come in supermultiplets that also contain scalars in a special Kähler manifold, the gauge group must also be a subgroup of the isometry group of the special Kähler manifold and must necessarily act on the hypermultiplets and the vector multiplets simultaneously. It must act in the adjoint representation on the latter.

[3] See, for instance, [52–54] for a general review on these theories with references to the original literature.

[4] Only isometries that respect the complete Special Geometry structure are global symmetries of the theory and can be gauged.

This case can be considered as an extension of the previous one, in which the hypermultiplets are not mere spectators. There is, however, a very important difference: in this setting, where the quaternionic Kähler sector is involved as well, Abelian gaugings are not trivial any more;

3. in absence of hypermultiplets, the complete SU(2) factor of the R-symmetry group, or just a U(1) subgroup of that SU(2) factor, can be gauged by introducing what would be constant triholomorphic momentum maps if there were hypermultiplets. These constants are usually called, respectively, SU(2) or U(1) Fayet–Iliopoulos (FI) terms and the theories that are obtained in this way are called SU(2) or U(1)-FI gauged $\mathcal{N} = 2$, $d = 4$ supergravities, respectively.

   In particular:

   (a) we are dealing with the U(1)-FI gauged theory to present a new solution in section 3.1.
   Please note that there is no contradiction with the impossibility of gauging Abelian isometries of the special Kähler manifold stated above because, in this case, the global symmetry being gauged is a U(1) subgroup of the SU(2) factor of the R-symmetry group U(2) =U$(1)\times$SU(2);[5]

   (b) the SU(2)-FI-gauged theories can be seen as deformations of the $\mathcal{N} = 2$, $d = 4$ SEYM theories in which the SU(2) factor of the R-symmetry group is gauged simultaneously with an SU(2) subgroup of the isometry group of the special Kähler manifold. Gauging this latter is necessary for gauging the SU(2) factor of the R-symmetry group because the global symmetry to be gauged has to act on the gauge fields in the adjoint representation and, for the gauging to respect supersymmetry, it must act on the complete vector supermultiplets, including the scalars. This action must then be an isometry of their metric. As far as we known, no solutions of these theories where found before [3], whose results are reported in sections 3.2.1 and 3.4.

Super Einstein-Yang-Mills theories

We are now describing in some more detail the first of the situations listed above.
Assume that the metric $\mathcal{G}_{ij^*}$ admits a set of Killing vectors $\{K_\Lambda = k_\Lambda{}^i \partial_i + k^{*}_\Lambda{}^{i^*} \partial_{i^*}\}$ satisfying the Lie algebra

$$[K_\Lambda, K_\Sigma] = -f_{\Lambda\Sigma}{}^{\Omega} K_\Omega \,, \tag{1.41}$$

of the group that we want to gauge. Hermiticity and the ij, i^*j^* components of the Killing equation imply that the components $k_\Lambda{}^i$ and $k^{*}_\Lambda{}^{i^*}$ are respectively holomorphic and anti-holomorphic and satisfy the above Lie algebra separately. In this notation the

[5] As explained in a while, the U(1) factor cannot be gauged if the special Kähler structure is preserved.

generators of the gauge group carry the same indexes as the fundamental vector fields Λ. It is understood that the generators, Killing vectors, structure constants etc. vanish in the directions which remain ungauged.

To gauge the theory, the scalar and vector field strengths are modified in the standard way, to make them covariant under the local transformations

$$\mathfrak{D}_\mu Z^i = \partial_\mu Z^i + g A^\Lambda{}_\mu k_\Lambda{}^i \,, \tag{1.42}$$

$$F^\Lambda{}_{\mu\nu} = 2\partial_{[\mu} A^\Lambda{}_{\nu]} + g f_{\Sigma\Omega}{}^\Lambda A^\Sigma{}_{[\mu} A^\Omega{}_{\nu]} \,. \tag{1.43}$$

Here g is the gauge coupling constant.
The replacement of partial with covariant derivatives makes the σ-model, describing the kinetic term of the scalars in the action, gauge invariant. Furthermore, supersymmetry requires the addition of the scalar potential (1.6); in absence of hypermultiplets, it is expressed in terms of the quantities $\mathcal{P}_\Lambda$, which arise when imposing that the transformations generated by the Killing vectors preserve the Kähler 2-form,

$$0 = \pounds_\Lambda \mathcal{J} = (\imath_{k_\Lambda} d + d\imath_{k_\Lambda})\, \mathcal{J} = d\imath_{k_\Lambda} \mathcal{J} = -id \left(k_{\Lambda i} dZ^i - k_{\Lambda i^*} dZ^{*i^*} \right) \tag{1.44}$$

where we exploited the closedness of $\mathcal{J}$ and $\imath_{k_\Lambda}$ is the interior derivative with respect to k_Λ. Since $\imath_{k_\Lambda}\mathcal{J}$ is a closed form, a real function $\mathcal{P}_\Lambda(Z, Z^*)$ exists, called a *momentum map*, such that locally,

$$\imath_{k_\Lambda} \mathcal{J} = -d\mathcal{P}_\Lambda \,. \tag{1.45}$$

This means that a holomorphic Killing vector is given by

$$k_\Lambda{}^i = i\mathcal{G}^{ij^*} \partial_{j^*} \mathcal{P}_\Lambda \,, \tag{1.46}$$

from which the momentum maps can be determined as

$$\mathcal{P}_\Lambda = -\frac{i}{2} \left(k_\Lambda{}^i \partial_i \mathcal{K} - k_\Lambda{}^{i^*} \partial_{i^*} \mathcal{K} \right) - \frac{i}{2} \left(\lambda_\Lambda(Z) - \lambda^*_\Lambda(Z^*) \right) \,, \tag{1.47}$$

where the holomorphic functions λ_Λ generate the Kähler transformations.

Moreover, on a special Kähler manifold, the isometries should also conserve the special structure, i.e. they should be embedded in the symplectic group. Therefore, the invariant section $\mathcal{V}$, with Kähler weight $(1, -1)$, should respect

$$K_\Lambda \mathcal{V} = \left(\mathcal{S}_\Lambda - \frac{1}{2} \left(\lambda_\Lambda - \lambda^*_\Lambda \right) \right) \mathcal{V} \,, \tag{1.48}$$

where the constant matrices $\mathcal{S}_\Lambda = \begin{pmatrix} a_\Lambda & b_\Lambda \\ c_\Lambda & -a_\Lambda{}^T \end{pmatrix}$, $b_\Lambda = b_\Lambda{}^T$, $c_\Lambda = c_\Lambda{}^T$ generate the transformations of the $\mathrm{Sp}(2n+2, \mathbb{R})$ group and they must provide a representation of the Lie algebra of the symmetry group we are gauging, $[\mathcal{S}_\Lambda, \mathcal{S}_\Sigma] = f_{\Lambda\Sigma}{}^\Omega \mathcal{S}_\Omega$.

We are considering only those elements such that $b_\Lambda = 0$, otherwise the symplectic transformation is not a symmetry of the action, and $c_\Lambda = 0$, to avoid complicated Chern-Simons terms, so $a_\Lambda{}^\Omega{}_\Sigma = f_{\Lambda\Sigma}{}^\Omega$.

On the other hand, $\mathcal{V}$ is covariantly holomorphic,

$$\begin{aligned} K_\Lambda \mathcal{V} &= k_\Lambda{}^i \mathcal{U}_i - \frac{1}{2} k_\Lambda{}^i \partial_i \mathcal{K} \mathcal{V} + \frac{1}{2} k_\Lambda{}^{i^*} \partial_{i^*} \mathcal{K} \mathcal{V} \\ &= k_\Lambda{}^i \mathcal{U}_i - i \mathcal{P}_\Lambda \mathcal{V} - \frac{1}{2} \left(\lambda_\Lambda - \lambda_\Lambda^* \right) \mathcal{V} \,, \end{aligned} \tag{1.49}$$

where we made use of (1.47). Comparing (1.48) and 1.49), we obtain a new expression for the momentum maps and the holomorphic Killing vectors

$$\mathcal{P}_\Lambda = \langle \mathcal{V}^* | \mathcal{S}_\Lambda \mathcal{V} \rangle \tag{1.50}$$

$$k_\Lambda{}^i = i \partial^i \mathcal{P}_\Lambda = i \left\langle \mathcal{V} | \mathcal{S}_\Lambda \mathcal{U}^{*i} \right\rangle = i f_{\Lambda\Sigma}{}^\Omega \left(f^{*i\Sigma} \mathcal{M}_\Omega + \mathcal{L}^\Sigma h^i_\Omega \right) \,. \tag{1.51}$$

As stated above, an Abelian gauging will not produce any effect on the theory. Moreover, (1.51) shows that the Killing vectors are completely determined, with no arbitrary constants and motivates the impossibility of gauging through Fayet-Iliopoulos terms the $\mathrm{U}(1)$ factor of the R-symmetry group. A $\mathrm{U}(1)$ subgroup of the $\mathrm{SU}(2)$ factor in the R-symmetry group, or the entire $\mathrm{SU}(2)$ can instead be FI-gauged, as illustrated in the next section.

The result of the gauging of a subgroup of the isometries of the Kähler manifold which respect all the conditions mentioned so far is the minimal $\mathcal{N} = 2$ supersymmetrization of the bosonic Einstein-Yang-Mills theory for that gauge group. These theories were called $\mathcal{N} = 2$ Super-Einstein-Yang-Mills (SEYM) theories and their action is

$$\begin{aligned} S = \int d^4x \sqrt{|g|} \Big[& R + 2 \mathcal{G}_{ij^*} \mathfrak{D}_\mu Z^i \mathfrak{D}^\mu Z^{*\,j^*} + 2 \Im\mathfrak{m} \mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^\Sigma{}_{\mu\nu} \\ & - 2 \Re\mathfrak{e} \mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} \star F^\Sigma{}_{\mu\nu} - \tfrac{1}{4} g^2 (\Im\mathfrak{m}\mathcal{N})^{-1|\Lambda\Sigma} \mathcal{P}_\Lambda \mathcal{P}_\Sigma \Big] \,, \end{aligned} \tag{1.52}$$

where the covariant derivatives and the field strength were defined in (1.42) and (1.43) and we assumed to have no hypermultiplets.
The time-like supersymmetric solutions of these theories were characterized in [24] and studied in [17,23,29,30].

R-symmetry gauging through Fayet-Iliopoulos terms

Gauging a subgroup of the R-symmetry group seems to be a different choice, and, indeed it is if the subgroup is Abelian (as in case 2.a), because, as we mentioned above, Abelian holomorphic isometries of the Kähler manifold cannot be gauged. In absence of

hypermultiplets, the gauging is done via Fayet-Iliopoulos (FI) terms. The supersymmetric solutions of these theories have been classified and studied in [32, 33].

However, when the subgroup of the R-symmetry to be gauged is non-Abelian (SU(2) is the only possibility, case 2.b), it turns out that that choice is actually not so different from the SEYM setting: to gauge it we need gauge vector fields transforming in the adjoint representation of the gauge group. This implies that the whole supermultiplets, and, in particular the complex scalars, must transform in the adjoint representation leaving the whole special Kähler structure (and, in particular, the Kähler metric) invariant. Thus, if we gauge an SU(2) subgroup of the R-symmetry group we have to gauge at the same time an SU(2) isometry subgroup of the global symmetry group and the resulting theory can be seen as a deformation, via FI terms, of a $\mathcal{N}=2$ SEYM theory with a gauge group that includes a SU(2) factor. Therefore, for a subset of the vector indexes $\Lambda, \Sigma, \ldots$ that we are going to denote with the indexes $x, y, \ldots$, that only take 3 possible values, the structure constants are those of SU(2),

$$f_{xy}{}^{z} = -\varepsilon_{xyz} \,. \tag{1.53}$$

The time-like supersymmetric solutions of these theories were characterized as part of the general case in [4].

To understand the origin of the Fayet-Iliopoulos gauging, we have to enter into the Quaternionic Kähler geometry in some more detail. They indeed arise from the quaternionic sector, although the theories we are interested in contain no hypermultiplets.
To be gauged, the isometries of the quaternionic Kähler manifold should preserve the quaternionic Kähler structure, i.e. they should preserve the complex structures K^x up to rotations,

$$\pounds_{\mathsf{k}_\Lambda} \mathsf{K}^x = (\imath_{\mathsf{k}_\Lambda} d + d\imath_{\mathsf{k}_\Lambda})\, \mathsf{K}^x = -\varepsilon^{xyz} \mathsf{W}_\Lambda{}^y \mathsf{K}^z \,, \tag{1.54}$$

where k_Λ are the quaternionic Killing vectors generating the isometries and

$$\mathsf{W}_\Lambda{}^x = \mathsf{k}_\Lambda{}^u \mathsf{A}^x_u - \mathsf{P}_\Lambda{}^x \tag{1.55}$$

introduces the triholomorphic momentum maps $\mathsf{P}_\Lambda{}^x$. The defining property of K^x guarantees that $d\mathsf{K}^x = -\varepsilon^{xyz} \mathsf{A}^y \wedge \mathsf{K}^z$, so

$$d\imath_{\mathsf{k}_\Lambda} \mathsf{K}^x + \varepsilon^{xyz} \mathsf{A}^y \wedge \imath_{\mathsf{k}_\Lambda} \mathsf{K}^z = \varepsilon^{xyz} \mathsf{P}_\Lambda{}^y \mathsf{K}^z \,. \tag{1.56}$$

Taking the external derivative of (1.56) leads to

$$\varepsilon^{xyz} \left(d\mathsf{P}_\Lambda{}^y + \varepsilon^{yvw} \mathsf{A}^v \mathsf{P}_\Lambda{}^w + \varkappa\, \imath_{\mathsf{k}_\Lambda} \mathsf{K}^y \right) \wedge \mathsf{K}^z = 0 \tag{1.57}$$

and to

$$\partial_u \mathsf{P}_\Lambda{}^x + \varepsilon^{xyz} \mathsf{A}^y{}_u \mathsf{P}_\Lambda{}^z = \varkappa \mathsf{K}^x{}_{uv} \mathsf{k}_\Lambda{}^v \,. \tag{1.58}$$

The latter equation, which is the analogous of (1.46) for the quaternionic Kähler manifold, clarifies how non-vanishing triholomorphic momentum maps can exist even if no hypermultiplets are present and there are no isometries to be generated by the Killing vectors[6]. This is the way in which Fayet-Iliopoulos gaugings arise, for constant momentum maps.

The Killing vectors satisfy the Lie algebra of the group we are gauging, $[\mathsf{k}_\Lambda, \mathsf{k}_\Sigma] = -f_{\Lambda\Sigma}{}^\Omega \mathsf{k}_\Omega$, and the Lie derivatives should respect $[\mathcal{L}_\Lambda, \mathcal{L}_\Sigma] = \mathcal{L}_{[\mathsf{k}_\Lambda, \mathsf{k}_\Sigma]}$. When applied to (1.54), the following relation is found for the field $\mathsf{W}_\Lambda{}^x$,

$$2\mathcal{L}_{[\Lambda} \mathsf{W}_{\Sigma]}{}^x + \varepsilon^{xyz} \mathsf{W}_\Lambda{}^y \mathsf{W}_\Sigma{}^z = -f_{\Lambda\Sigma}{}^\Omega \mathsf{W}_\Omega{}^x \,, \tag{1.59}$$

which, exploiting (1.55) and (1.58) gives rise to the *equivariance relation* for the momentum maps

$$\varepsilon^{xyz} \mathsf{P}_\Lambda{}^y \mathsf{P}_\Sigma{}^z - \varkappa \mathsf{k}_\Lambda{}^u \mathsf{K}^x{}_{uv} \mathsf{k}_\Sigma{}^v = f_{\Lambda\Sigma}{}^\Omega \mathsf{P}_\Omega{}^x \,. \tag{1.60}$$

In what follows, we are solving (1.60) for constant momentum maps, considering the two possible Fayet-Iliopoulos gaugings: a $\mathrm{U}(1)$ subgroup of the $\mathrm{SU}(2)$ factor in the R-symmetry group and the entire $\mathrm{SU}(2)$ factor.

The $\mathrm{U}(1)$-Fayet-Iliopoulos gauged theory

If no hypermultiplets are present, $n_H = 0$, there are still two possibilities of solving the equivariance condition (1.60) for the momentum maps $\mathsf{P}_\Lambda{}^x$. The first one leads to the Fayet-Iliopoulos gauging of a $\mathrm{U}(1)$ subgroup of the R-symmetry group.
In this situation, where $f_{\Lambda\Sigma}{}^\Omega = 0$, the equation (1.60) becomes $\varepsilon^{xyz} \mathsf{P}_\Lambda{}^y \mathsf{P}_\Sigma{}^z = 0$, whose solution is

$$\mathsf{P}^x_\Lambda = e^x \xi_\Lambda \,. \tag{1.61}$$

Here $\vec{e}$ is a vector in the $\mathrm{SU}(2)$ space and ξ_Λ are constants[7]. Without loss of generality, we can make the choice $e^x = \delta^x_3$, that can always be achieved by a global $\mathrm{SU}(2)$ rotation, since $\mathrm{SU}(2)$ is a global symmetry of the theory.
We are interested in models where no further gaugings are performed, i.e. the isometries of the special Kähler manifold are not gauged, $k_\Lambda{}^i = 0\,, \mathcal{P}_\Lambda = 0$. The resulting theory has

[6] Although we are not interested in the case in which hypermultiplets are present, we recall that the quaternionic Killing vectors $\mathsf{k}_\Lambda{}^u$ we just introduced are present in the generic Lagrangian (1.5) in the covariant derivatives

$$\mathfrak{D}_\mu q^u = \partial_\mu q^u + g A^\Lambda{}_\mu \mathsf{k}_\Lambda{}^u$$

and in the potential (1.6).

[7] In what follows, we are often denoting $g_\Lambda = g\xi_\Lambda$ as gauging parameters, since the momentum maps appear in the action multiplied by the coupling constant g.

a gauged U(1) subgroup of the SU(2) factor in the R-symmetry group, with gauge field $\xi_\Lambda A^\Lambda$. Its action is explicitly given by

$$S = \int d^4x \sqrt{|g|} \left[R + 2\mathcal{G}_{ij^*} \partial_\mu Z^i \partial^\mu Z^{*\,j^*} + 2\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^{\Sigma}{}_{\mu\nu} - 2\Re\mathfrak{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} \star F^{\Sigma}{}_{\mu\nu} \right.$$
$$\left. -\tfrac{1}{2} g^2 \left(\mathcal{G}^{ij^*} f^\Lambda{}_i f^{*\,\Sigma}{}_{j^*} - 3\mathcal{L}^{*\,\Lambda} \mathcal{L}^\Sigma \right) \xi_\Lambda \xi_\Sigma \right] . \tag{1.62}$$

These theories have been studied in [33], where their time-like supersymmetric solutions where classified using the Killing spinor techniques. Various explicit examples of black hole solutions arose from this classification, for many different models and prepotentials [5,18–22,25,26,28,31]. All of them involve models in which the Kähler manifold is homogeneous. The only exception is given by the solution presented in [1], as discussed in section 3.1.

Some of the solutions listed above are obtained in a symplectic covariant framework, where the gauging parameters ξ_Λ are extended to a symplectic vector

$$G = g(\xi^\Lambda, \xi_\Lambda) = (g^\Lambda, g_\Lambda) , \tag{1.63}$$

introducing a set of dual gauging parameters ξ^Λ in addition to the ξ_Λ. It is possible, then, using the duality-complete vector of gauging parameters G, to give a duality invariant definition of the scalar potential as

$$\mathbf{V} = |\mathcal{D}L|^2 - 3\,|L|^2 , \tag{1.64}$$

by means of the new symplectic invariant quantity, $L = \langle G, \mathcal{V} \rangle$.

The SU(2)-Fayet-Iliopoulos gauged theory

The other possibility we have, in absence of hypermultiplets, is to gauge the entire SU(2) factor in the R-symmetry group through FI terms. This choice is more involved than the previous one and not every model admits this setting. In fact, it involves both a Fayet-Iliopoulos term and the gauging of an SU(2) subgroup of the isometries of the special Kähler manifold, as we explained earlier.
The triholomorphic momentum maps $\mathsf{P}_\Lambda{}^x$, $x, y, \ldots = 1, 2, 3$, are assumed to be of the form

$$\mathsf{P}_\Lambda{}^x = e_\Lambda{}^x \xi , \tag{1.65}$$

where[8] $\xi = 0, 1$ and $e_\Lambda{}^x$ is a constant tensor, which differs from zero when Λ is in the range of the SU(2) factor and satisfies the equivariance condition

[8]The role of this unphysical parameter is to underline which terms are specifically due to the FI terms, while the $\mathcal{N} = 2, d = 4$ SEYM theories are recovered setting it to zero.

$$\varepsilon_{xyz} e_\Lambda{}^y e_\Sigma{}^z = f_{\Lambda\Sigma}{}^\Omega e_\Omega{}^x \,, \tag{1.66}$$

or, taking (1.53) into account,

$$\varepsilon_{xy'z'} e_y{}^{y'} e_z{}^{z'} = -\varepsilon_{xyz'} e_{z'}{}^x \,. \tag{1.67}$$

With no loss of generality we will choose the simplest solution

$$e_x{}^{x'} = -\delta_x{}^{x'} \,. \tag{1.68}$$

With this choice, the scalar potential (1.6) takes the simple form

$$\mathbf{V}(Z,Z^*) = -\tfrac{1}{4} g^2 (\Im\mathfrak{m}\mathcal{N})^{-1|\Lambda\Sigma} \mathcal{P}_\Lambda \mathcal{P}_\Sigma + \tfrac{1}{2}\xi^2 g^2 \left(\mathcal{G}^{ij^*} f^x{}_i f^{*\,x}{}_{j^*} - 3\mathcal{L}^{*\,x}\mathcal{L}^x \right) \,. \tag{1.69}$$

Observe that the first term may contain the contribution of other, necessarily non-Abelian, gauge factors apart from the SU(2) one labeled by $x, y, \ldots$ In the examples that we are considering, that possibility is not included and, therefore, the sum over indexes $\Lambda, \Sigma, \ldots$ is restricted to a sum over the SU(2) induces $x, y, \ldots$
The action is then

$$\begin{aligned} S = \int d^4x \sqrt{|g|} \Big[& R + 2\mathcal{G}_{ij^*} \mathfrak{D}_\mu Z^i \mathfrak{D}^\mu Z^{*\,j^*} + 2\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} F^\Sigma{}_{\mu\nu} \\ & -2\Re\mathfrak{e}\mathcal{N}_{\Lambda\Sigma} F^{\Lambda\,\mu\nu} \star F^\Sigma{}_{\mu\nu} - \mathbf{V}(Z,Z^*) \Big] \,, \end{aligned} \tag{1.70}$$

where

$$\begin{aligned} \mathfrak{D}_\mu Z^x &= \partial_\mu Z^x - g\varepsilon^x{}_{yz} A^y{}_\mu Z^z \,, \\ F^\Lambda{}_{\mu\nu} &= 2\partial_{[\mu} A^0{}_{\nu]} \qquad\qquad \text{if} \quad \Lambda \neq x \,, \\ F^x{}_{\mu\nu} &= 2\partial_{[\mu} A^x{}_{\nu]} - g\varepsilon^x{}_{yz} A^y{}_{[\mu} A^z{}_{\nu]} \,. \end{aligned} \tag{1.71}$$

The time-like supersymmetric solutions of the most general $\mathcal{N}=2, d=4$ supergravities, with the most general matter content and the most general gauging, were characterized in [4], whose content is summarized in what follows, section 1.2.5. That work is built on previous results about the supersymmetric solutions of the general $\mathcal{N}=2, d=4$ ungauged theories with vector multiplets and hypermultiplets [11, 44, 45], the U(1)-FI-gauged $\mathcal{N}=2, d=4$ theories with no hypermultiplets [32, 33, 56, 57] and on the $\mathcal{N}=2, d=4$ SEYM theories [23, 24].

Many solutions of the ungauged, U(1)-FI-gauged and SU(2) SEYM theories have been constructed in the literature. In the present work, we are giving some new examples: a black hole solution of the U(1)-FI-gauged theory is presented in section 3.1,

which is the first one involving a non-homogeneous special Kähler manifold, and a solution for the 6-dimensional SU(2) SEYM theory is obtained by dimensional uplifting a well known 5-dimensional one (see section 3.3), that in turn had been obtained in [30] exploiting the relation between the 4 and 5-dimensional theories.
So far, no supersymmetric solution of SU(2)-FI-gauged theories with no hypermultiplets was explicitly known. This was due to the complexity of the theories and of the equations that need to be solved to construct supersymmetric solutions. We are here presenting the first examples of such solutions; some of them are obtained considering the simplest model that can be SU(2) gauged, the $\overline{CP}^3$ model (section 3.2), while others come from dimensional reduction of known 6-dimensional solutions (section 3.4).

Maximally supersymmetric vacua

The SU(2)-FI gauged theory has a peculiarity concerning its maximally supersymmetric solutions. As we will show in what follows, they do not exist, due to the specific form of the triholomorphic moment maps.

According to the results of [58], the supersymmetric solutions of these theories, if any, must be of the same kind as those of the corresponding ungauged theories: in absence of electromagnetic fluxes, Minkowski spacetime M_4 or anti-de Sitter spacetime AdS_4 and, in presence of fluxes, Bertotti-Robinson spacetimes $AdS_2\times S^2$ [59, 60] or Kowalski-Glikman homogeneous pp-wave spacetimes KG_4 [61]. Furthermore, maximally supersymmetric solutions in gauged supergravities are characterized by the vanishing of all the fermion shifts and of the R-symmetry connection [58].

For the $\mathcal{N} = 2, d = 4$ theories, the different possibilities were analyzed in detail in [62]. The maximally supersymmetric solutions with zero curvature (M_4, $AdS_2\times S^2$ and KG_4) must have identically vanishing triholomorphic momentum maps $\mathsf{P}_\Lambda{}^x = 0$, which is not possible in the case we are considering. The remaining possibility is the only maximally supersymmetric solution with negative curvature, i.e. AdS_4. The following conditions have to be satisfied in this case

$$\begin{aligned} \mathsf{P}_\Lambda{}^x \mathsf{P}_\Sigma{}^{*\,x} \mathcal{L}^\Lambda \mathcal{L}^{*\,\Sigma} &\neq 0\,, \\ k_\Lambda{}^i \mathcal{L}^{*\,\Lambda} &= 0\,, \\ \mathsf{P}_\Lambda{}^x f^\Lambda{}_i &= 0\,, \\ \varepsilon^{xyz} \mathsf{P}_\Lambda{}^y \mathsf{P}_\Sigma{}^{*\,z} \mathcal{L}^\Lambda \mathcal{L}^{*\,\Sigma} &= 0\,. \end{aligned} \tag{1.72}$$

With our choice of FI terms (1.65),(1.68) these conditions take the form

$$\mathcal{L}^x \mathcal{L}^{*\,x} \neq 0\,, \tag{1.73}$$

$$k_x{}^i \mathcal{L}^{*\,x} = 0\,, \tag{1.74}$$

$$f^x{}_i = 0\,, \tag{1.75}$$

$$\varepsilon^{xyz}\mathcal{L}^{y}\mathcal{L}^{*\,z}=0\,. \tag{1.76}$$

Choosing the coordinates as $Z^i=\mathcal{X}^i/\mathcal{X}^0$ and the gauge $\mathcal{X}^0=1$, it is not difficult to see, from the definition $f^{\Lambda}{}_i=e^{\frac{\mathcal{K}}{2}}\mathcal{D}_i\mathcal{X}^{\Lambda}$ that it is not possible to satisfy all the equations (1.75) at the same time.

We conclude that these theories do not admit maximally supersymmetric vacua.

1.2.4 Equations of motion and supersymmetry variations

For later convenience, we are reporting here, following [4], the bosonic equations of motion obtained from the action (1.5),

$$\begin{aligned}
\mathcal{E}_a{}^{\mu} &\equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta e^a{}_{\mu}}=0\,,\\
\mathcal{E}_i &\equiv -\frac{1}{2\sqrt{|g|}}\frac{\delta S}{\delta Z^i}=0\,,\\
\mathcal{E}_{\Lambda}{}^{\mu} &\equiv \frac{1}{8\sqrt{|g|}}\frac{\delta S}{\delta A^{\Lambda}{}_{\mu}}=0\,,\\
\mathcal{E}^u &\equiv -\frac{1}{4\sqrt{|g|}}\mathsf{H}^{uv}\frac{\delta S}{\delta q^v}=0\,,
\end{aligned} \tag{1.77}$$

and the Bianchi identities for the vector field strengths,

$$\mathcal{B}^{\Lambda\,\mu}\equiv\mathfrak{D}_{\nu}\star F^{\Lambda\,\nu\mu}\,, \tag{1.78}$$

where

$$\begin{aligned}
\mathcal{E}_{\mu\nu} &= G_{\mu\nu}+8\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda+}{}_{\mu}{}^{\rho}F^{\Sigma-}{}_{\nu\rho}+2\mathcal{G}_{ij^*}[\mathfrak{D}_{(\mu}Z^i\mathfrak{D}_{\nu)}Z^{*j^*}-\frac{1}{2}g_{\mu\nu}\mathfrak{D}_{\rho}Z^i\mathfrak{D}^{\rho}Z^{*j^*}]\\
&\quad+2\mathsf{H}_{uv}\,[\mathfrak{D}_{\mu}q^u\mathfrak{D}_{\nu}q^v-\frac{1}{2}g_{\mu\nu}\mathfrak{D}_{\rho}q^u\mathfrak{D}_{\rho}q^v]+\frac{1}{2}g_{\mu\nu}\mathbf{V}(Z,Z^*,q)\,,\\
\mathcal{E}_{\Lambda}{}^{\mu} &= \mathfrak{D}_{\nu}\star F_{\Lambda}{}^{\nu\mu}+\frac{1}{4}g(k_{\Lambda\,i^*}\mathfrak{D}^{\mu}Z^{*j^*}+k^*{}_{\Lambda\,i}\mathfrak{D}^{\mu}Z^i)+\frac{1}{2}g\mathsf{k}_{\Lambda\,u}\mathfrak{D}^{\mu}q^u\,,\\
\mathcal{E}^i &= \mathfrak{D}^2Z^i+\partial^iF_{\Lambda}{}^{\mu\nu}\star F^{\Lambda}{}_{\mu\nu}+\frac{1}{2}\partial^i\mathbf{V}(Z,Z^*,q)\,,\\
\mathcal{E}^u &= \mathfrak{D}^2q^u+\frac{1}{4}\partial^u\mathbf{V}(Z,Z^*,q)\,,
\end{aligned}$$

and the dual field strengths F_{Λ} are defined as

$$F_{\Lambda\,\mu\nu}\equiv-\frac{1}{4\sqrt{|g|}}\frac{\delta S}{\delta\star F^{\Lambda}{}_{\mu\nu}}=\Re\mathfrak{e}\mathcal{N}_{\Lambda\Sigma}F^{\Sigma}{}_{\mu\nu}+\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma}\star F^{\Sigma}{}_{\mu\nu}\,. \tag{1.79}$$

The supersymmetry transformation rules for the bosons are the same in the ungauged case and in the gauged one,

$$
\begin{aligned}
\delta_\epsilon e^a{}_\mu &= -\frac{i}{4}\bar{\psi}_{I\,\mu}\gamma^a\epsilon^I + \text{c.c.}\,,\\
\delta_\epsilon A^\Lambda{}_\mu &= \frac{1}{4}\mathcal{L}^{*\Lambda}\varepsilon^{IJ}\bar{\psi}_{I\,\mu}\epsilon_J + \frac{i}{8}f^\Lambda{}_i\varepsilon_{IJ}\bar{\lambda}^{Ii}\gamma_\mu\epsilon^J + \text{c.c.}\,,\\
\delta_\epsilon Z^i &= \frac{1}{4}\bar{\lambda}^{Ii}\epsilon_I\,,\\
\delta_\epsilon q^u &= \frac{1}{4}\mathsf{U}_{\alpha I}{}^u\bar{\zeta}^\alpha\epsilon^I + \text{c.c.}\,,
\end{aligned}
\tag{1.80}
$$

while for the fermions, when the fermionic fields are vanishing, we have

$$
\begin{aligned}
\delta_\epsilon\psi_{I\,\mu} &= \mathfrak{D}_\mu\epsilon_I + \left[T^+{}_{\mu\nu}\varepsilon_{IJ} - \frac{1}{2}S^x\eta_{\mu\nu}\varepsilon_{IK}(\sigma^x)^K{}_J\right]\gamma^\nu\epsilon^J\,,\\
\delta_\epsilon\lambda^{Ii} &= i\not{\mathfrak{D}}Z^i\epsilon^I + \left[\left(\not{G}^{i+} + W^i\right)\varepsilon^{IJ} + \frac{i}{2}W^{i\,x}\,(\sigma^x)^I{}_K\varepsilon^{KJ}\right]\epsilon_J\,,\\
\delta_\epsilon\zeta_\alpha &= i\mathsf{U}_{\alpha I\,u}\not{\mathfrak{D}}q^u\epsilon^I + N_\alpha{}^I\epsilon_I\,,
\end{aligned}
\tag{1.81}
$$

where the covariant derivative acts on spinors as

$$
\mathfrak{D}_\mu\epsilon_I = \left[\nabla_\mu + \frac{i}{2}\left(\mathcal{A}_\mu + gA^\Lambda{}_\mu\mathcal{P}_\Lambda\right)\right]\epsilon_I + \frac{i}{2}\left(\mathsf{A}^x{}_u\partial_\mu q^u + gA^\Lambda{}_\mu\mathsf{P}_\Lambda{}^x\right)\sigma^x{}_I{}^J\epsilon_J\,,
\tag{1.82}
$$

and $\mathcal{A}_\mu$ is the pullback to spacetime of the Kähler connection (1.11), while σ^x are the Pauli matrices.
The quantities $S^x, W^i, W^{i\,x}$ and $N_\alpha{}^I$ are the fermion shifts, originated by the gauging. They are defined as

$$
\begin{aligned}
S^x &= \frac{1}{2}g\mathcal{L}^\Lambda\mathsf{P}_\Lambda{}^x\,,\\
W^i &= \frac{1}{2}g\mathcal{L}^{*\Lambda}k_\Lambda{}^i = -\frac{i}{2}g\,\mathcal{G}^{ij^*}f^{*\Lambda}{}_{j^*}\mathcal{P}_\Lambda\,,\\
W^{i\,x} &= g\,\mathcal{G}^{ij^*}f^{*\Lambda}{}_{j^*}\mathsf{P}_\Lambda{}^x\,,\\
N_\alpha{}^I &= g\,\mathsf{U}_\alpha{}^I{}_u\mathcal{L}^{*\Lambda}\mathsf{k}_\Lambda{}^u\,,
\end{aligned}
\tag{1.83}
$$

while $T_{\mu\nu}$ and $G^i{}_{\mu\nu}$ are respectively the graviphoton and matter vector field strengths, defined by

$$
\begin{aligned}
T_{\mu\nu} &\equiv 2i\mathcal{L}^\Sigma\Im\mathfrak{m}\mathcal{N}_{\Sigma\Lambda}F^\Lambda{}_{\mu\nu}\,,\\
G^i{}_{\mu\nu} &\equiv -\mathcal{G}^{ij^*}f^{*\Sigma}{}_{j^*}\Im\mathfrak{m}\mathcal{N}_{\Sigma\Lambda}F^\Lambda{}_{\mu\nu}\,.
\end{aligned}
\tag{1.84}
$$

1.2.5 Characterization of supersymmetric solutions

To find supersymmetric solutions of a supergravity theory, a set of first order differential equations, called Killing spinor equations, has to be solved. This is typically easier than

directly solving the equations of motion, which include the second order Einstein equations. In this sense, supersymmetry can be regarded as a solution generating technique, since the increased amount of symmetry provides a simplified system of equations to be solved and therefore an easier way to find new solutions.
The time-like bosonic supersymmetric solutions of the $\mathcal{N} = 2$, $d = 4$ theories have been characterized in [4], where the most general coupling -involving both vector multiplets and hypermultiplets- and gauging of these theories was considered. This purpose was achieved with the so called bilinear method [12].
We are going to summarize here the results obtained in [4], while in section 3.2 we are using this work, after specifying it to a particular case of interest, to obtain the minimal system of equations that has to be solved in order to find new solutions.

In what follows, we are going to introduce the Killing spinor identities for $\mathcal{N} = 2$ 4-dimensional supergravity, obtaining the minimal set of equations of motion that must be imposed on a supersymmetric configuration to ensure that all the equations of motion are satisfied.
The Killing spinor equations are then going to provide us with the equations characterizing every supersymmetric field configurations.
Once the minimal set of equations of motion is imposed, the complete set of equations that the fields of a supersymmetric solution have to satisfy is found.

We are finally specifying these results to the case of an SU(2)-FI gauged theory and we are explicitly giving the equations that will be solved in section 3.2.

Let us consider a generic supersymmetric action $S = S[\phi^b, \phi^f]$, involving bosonic (ϕ^b) and fermionic (ϕ^f) fields, and its bosonic supersymmetric field configurations, i.e. bosonic configurations satisfying $\delta_{\varepsilon_K}\phi^f \mid_{\phi^f=0} = 0$ for some supersymmetry parameter ε_K, called a Killing spinor[9]. It can then be proven that the following Killing spinor identities hold,

$$\left.\frac{\delta(\delta_{\varepsilon_K} S)}{\delta\phi^f}\right|_{\phi^f=0} = \sum_b \frac{\delta S}{\delta\phi^b}\left.\frac{\delta(\delta_{\varepsilon_K}\phi^b)}{\delta\phi^f}\right|_{\phi^f=0} = 0\,, \tag{1.85}$$

relating the bosonic equations of motion through the variation with respect to the fermionic fields of the supersymmetry variation of the bosonic fields. This means, in general, that only a subset of the equations of motion has to be imposed on a bosonic supersymmetric configuration, in order to ensure that all the equations of motion are satisfied.

As far as $\mathcal{N} = 2, d = 4$ supergravity is concerned, the conditions $\delta_{\varepsilon_K}\phi^f \mid_{\phi^f=0} = 0$ take the form

[9] In what follows, the supersymmetry parameter ε_K -that will be involved in the construction of the spinor bilinears- will be treated as a commuting spinor.

$$
\begin{aligned}
\mathfrak{D}_\mu \epsilon_I + \left[T^+{}_{\mu\nu} \varepsilon_{IJ} - \frac{1}{2} S^x \eta_{\mu\nu} \varepsilon_{IK} (\sigma^x)^K{}_J \right] \gamma^\nu \epsilon^J &= 0\,, \\
i \not{\mathfrak{D}} Z^i \epsilon^I + \left[\left(\not{G}^{i\,+} + W^i \right) \varepsilon^{IJ} + \frac{i}{2} W^{i\,x} \, (\sigma^x)^I{}_K \varepsilon^{KJ} \right] \epsilon_J &= 0\,, \\
i \mathsf{U}_{\alpha I\, u} \not{\mathfrak{D}} q^u \epsilon^I + N_\alpha{}^I \epsilon_I &= 0\,,
\end{aligned}
\tag{1.86}
$$

as can be inferred from the supersymmetry transformations (1.81). Exploiting the explicit equations of motion (1.77,1.79) we gave in the previous section, the Killing spinor identities are summarized as

$$
\mathcal{E}_a{}^\mu \gamma^a \epsilon^I - 4i\, \varepsilon^{IJ} \mathcal{L}^\Lambda \mathcal{E}_\Lambda{}^\mu \epsilon_J = 0\,, \tag{1.87}
$$

$$
\mathcal{E}^i \epsilon^I - 2i \varepsilon^{IJ} \bar{f}^{i\Lambda} \not{\mathcal{E}}_\Lambda \epsilon_J = 0\,, \tag{1.88}
$$

$$
\mathcal{E}^u \, \mathsf{U}^{\alpha I}{}_u \epsilon_I = 0\,, \tag{1.89}
$$

or, in their formally electric-magnetic duality-covariant version, as

$$
\mathcal{E}_a{}^\mu \gamma^a \epsilon_I - 4i \langle \, \mathcal{E}^\mu | \, \mathcal{V} \, \rangle \varepsilon_{IJ} \epsilon^J = 0\,, \tag{1.90}
$$

$$
\mathcal{E}^i \epsilon^I + 2i \langle \not{\mathcal{E}} \, | \, \bar{\mathsf{U}}^i \, \rangle \varepsilon^{IJ} \epsilon_J = 0\,, \tag{1.91}
$$

$$
\mathcal{E}^u \, \mathsf{U}^{\alpha I}{}_u \epsilon_I = 0\,, \tag{1.92}
$$

where the Maxwell equations and Bianchi identities have been collected in the symplectic vector $\mathcal{E}^\mu$,

$$
\mathcal{E}^\mu \equiv \begin{pmatrix} \mathcal{B}^{\Lambda\,\mu} \\ \mathcal{E}_\Lambda{}^\mu \end{pmatrix} . \tag{1.93}
$$

The vector bilinear $V^a \equiv i \bar{\epsilon}^I \gamma^a \epsilon_I$, constructed with the Killing vectors, can be either a null or a time-like vector. We are only considering the time-like class of configurations. In this case we can choose an orthonormal frame, whose time component e^0 is given by $V/|V|$. Acting on the identities (1.90–1.92) with gamma matrices and conjugate spinors, they can be set in the form

$$
\mathcal{E}^{0m} = \mathcal{E}^{mn} = 0\,, \tag{1.94}
$$

$$
\langle \, \mathcal{V}/X | \, \mathcal{E}^0 \, \rangle = \frac{1}{4} |X|^{-1} \mathcal{E}^{00}\,, \tag{1.95}
$$

$$
\langle \, \mathcal{V}/X | \, \mathcal{E}^m \, \rangle = 0\,, \tag{1.96}
$$

$$
\langle \, \mathcal{U}^*{}_{i^*} | \, \mathcal{E}^0 \, \rangle = \frac{1}{2} e^{-i\alpha} \mathcal{E}_{i^*}\,, \tag{1.97}
$$

$$
\langle \, \mathcal{U}^*{}_{i^*} | \, \mathcal{E}^m \, \rangle = 0\,, \tag{1.98}
$$

$$
\mathcal{E}^u = 0\,, \tag{1.99}
$$

where $X \equiv e^{i\alpha}|X| \equiv \frac{1}{2}\varepsilon^{IJ}\bar{\epsilon}_I\epsilon_J$ is the scalar bilinear.

Using the special geometry completeness relation (1.22), these identities imply that every time-like supersymmetric configuration automatically satisfies all the equations of motion except $\mathcal{E}^{00} = 0$, $\mathcal{E}^i = 0$ and $\mathcal{E}^0 = 0$. They also imply that, in order to guarantee that all the equations of motion are satisfied, we only have to impose the vanishing of the time components of the Maxwell equations and Bianchi identities, $\mathcal{E}^0 = 0$.

Before imposing these Maxwell and Bianchi equations of motion, we should require that the field configurations are supersymmetric, i.e. that the supersymmetry variations of the fermionic fields vanish: the equations $\delta_\epsilon\psi_{I\,\mu} = \delta_\epsilon\lambda^{Ii} = \delta_\epsilon\zeta_\alpha = 0$, which are first order differential equations for the supersymmetry parameters, have to admit at least one solution ϵ_I. These equations are known as Killing Spinor Equations and their solutions as Killing spinors.

We are for the moment considering the field strengths F^Λ and the vector potentials A^Λ as independent fields; they will become related once the Bianchi identities is imposed.

In terms of the bilinears

$$X = \frac{1}{2}\varepsilon^{IJ}\bar{\epsilon}_I\epsilon_J\,, \quad V_a = i\bar{\epsilon}^I\gamma_a\epsilon_I\,, \quad V^x{}_a = i\sigma^x{}_I{}^J\bar{\epsilon}^I\gamma_a\epsilon_J\,, \quad \Phi^x{}_{ab} = i\sigma^{xIJ}\bar{\epsilon}_I\gamma_{ab}\epsilon_J\,, \tag{1.100}$$

constructed out of Killing spinors, the gravitino supersymmetry transformation rule gives rise to the independent equations

$$\mathfrak{D}_\mu X = iV^\nu T^+{}_{\nu\mu} + \frac{i}{\sqrt{2}}S^x V^x{}_\mu\,, \tag{1.101}$$

$$\nabla_{(\mu}V_{\nu)} = 0\,, \tag{1.102}$$

$$dV = 4iX\bar{T}^- - \sqrt{2}\bar{S}^x\Phi^x + \text{c.c.}\,, \tag{1.103}$$

$$\mathfrak{D}_{(\mu}V^x{}_{\nu)} = \bar{T}^-{}_{(\mu|\rho}\Phi^x{}_{|\nu)}{}^\rho + \frac{i}{\sqrt{2}}X\bar{S}^x g_{\mu\nu} + \text{c.c.}\,, \tag{1.104}$$

$$\mathfrak{D}V^x = -i\epsilon^{xyz}\bar{S}^y\Phi^z + \text{c.c.}\,, \tag{1.105}$$

where V, V^x and Φ^x are the differential forms associated to the corresponding bilinears, and the SU(2)-covariant derivative is given by

$$\mathfrak{D}V^x = dV^x + \epsilon^{xyz}\mathsf{A}^y \wedge V^z\,. \tag{1.106}$$

From the gauginos transformation, we get the equation

$$\begin{aligned} &i\bar{X}\varepsilon^{KI}\mathfrak{D}^\mu Z^i + i\Phi^{KI\,\mu\nu}\mathfrak{D}_\nu Z^i - 4i\varepsilon^{IJ}G^{i+\mu}{}_\nu V^K{}_J{}^\nu \\ &- iW^i\varepsilon^{IJ}V^K{}_J{}^\mu - iW^{i\,IJ}V^K{}_{J\,\mu} = 0\,, \end{aligned} \tag{1.107}$$

while the rule for the hyperinos gives

$$V^I{}_K{}^\mu \mathfrak{D}_\mu q^u - i\mathsf{K}^{x\,u}{}_v \sigma^x{}_J{}^I V^J{}_K{}^\mu \mathfrak{D}_\mu q^v + gX\delta^I{}_K \bar{\mathcal{L}}^\Lambda k_\Lambda{}^u + \frac{i}{2} gX\bar{\mathcal{L}}^\Lambda \mathfrak{D}^u \mathsf{P}_\Lambda{}^x \sigma^{x\,I}{}_K = 0\,. \quad (1.108)$$

V^μ, which was assumed to be time-like, is a Killing vector (1.102), as usual in supergravity, while V^x are not in general, due to (1.104).

Consistency of equation (1.101) requires

$$V^\mu \mathfrak{D}_\mu X = 0\,. \quad (1.109)$$

The antisymmetric part of equation (1.107) gives

$$V^\nu G^{i\,+}{}_{\nu\mu} = \frac{1}{2}\bar{X}\mathfrak{D}_\mu Z^i + \frac{1}{4}W^i V_\mu - \frac{i}{4\sqrt{2}} W^{ix} V^x{}_\mu\,, \quad (1.110)$$

which implies

$$V^\mu \mathfrak{D}_\mu Z^i + 2XW^i = 0\,. \quad (1.111)$$

Thanks to special geometry identities,

$$F^{\Lambda\,+} = i\bar{\mathcal{L}}^\Lambda T^+ + 2f^\Lambda{}_i G^{i\,+}\,, \quad (1.112)$$

and substituting equations (1.101) and (1.110) in (1.112), the following relation can be obtained

$$V^\nu F^{\Lambda\,+}{}_{\nu\mu} = \bar{\mathcal{L}}^\Lambda \mathfrak{D}_\mu X + \bar{X}\mathfrak{D}_\mu \mathcal{L}^\Lambda + \frac{i}{8} g \Im\mathrm{m}(\mathcal{N})^{-1|\Lambda\Sigma}(\mathcal{P}_\Sigma V_\mu + \sqrt{2}\,\mathsf{P}_\Sigma{}^x V^x{}_\mu)\,, \quad (1.113)$$

which leads to the following expression for the field strengths F^Λ

$$\begin{aligned} F^\Lambda = &-\frac{1}{2}\mathfrak{D}[\mathcal{R}^\Lambda V] \\ &-\frac{1}{2}\star\left\{V\wedge\left[\mathfrak{D}\mathcal{I}^\Lambda + \sqrt{2}g\left(\mathcal{R}^\Lambda\mathcal{R}^\Sigma\mathsf{P}^x_\Sigma - \frac{1}{8|X|^2}\Im\mathrm{m}(\mathcal{N})^{-1|\Lambda\Sigma}\mathsf{P}^x_\Sigma\right)V^x\right]\right\}\,, \end{aligned} \quad (1.114)$$

in terms of the zero Kähler weight sections

$$\frac{\mathcal{V}^M}{X} = \mathcal{R}^M + i\mathcal{I}^M\,. \quad (1.115)$$

For any model of $\mathcal{N}=2, d=4$ supergravity, the components[10] $\mathcal{R}^M$ can, in principle, be expressed entirely in terms of the components $\mathcal{I}^M$, although, in practice, this can be very hard to do for certain models. This is often referred to as "solving the stabilization

[10]The index M can take $2(n_V+1)$ values, running on the possible values of both the upper and lower index Λ.

equations" or as "solving the Freudenthal duality equations". We shall simply assume that this has been done. Then, the symplectic product $\mathcal{R}_M \mathcal{I}^M = \langle \mathcal{R} \mid \mathcal{I} \rangle = \mathcal{R}_\Lambda \mathcal{I}^\Lambda - \mathcal{R}^\Lambda \mathcal{I}_\Lambda$ is a function of the $\mathcal{I}^M$ only. It is homogeneous of degree two in the $\mathcal{I}$-s and it is called the *Hesse potential*,

$$W(\mathcal{I}) \equiv \mathcal{R}_M(\mathcal{I})\mathcal{I}^M \,. \tag{1.116}$$

The trace of equation (1.108) is

$$V^\mu \mathfrak{D}_\mu q^u - i\sqrt{2}\mathsf{K}^{x\,u}{}_v V^{x\,\mu}\mathfrak{D}_\mu q^v + 2gX\mathcal{L}^{*\Lambda} k_\Lambda{}^u = 0 \,, \tag{1.117}$$

whose real and imaginary parts are

$$V^\mu \mathfrak{D}_\mu q^u + 2g|X|^2 \mathcal{R}^\Lambda k_\Lambda{}^u = 0 \,, \tag{1.118}$$

$$\mathsf{K}^{x\,u}{}_v V^{x\,\mu}\mathfrak{D}_\mu q^v + \sqrt{2}g|X|^2 \mathcal{I}^\Lambda k_\Lambda{}^u = 0 \,. \tag{1.119}$$

To proceed further, we introduce a time coordinate t associated to the time-like Killing vector V,

$$V^\mu \partial_\mu \;\equiv\; \sqrt{2}\partial_t \,. \tag{1.120}$$

The gauge choice

$$V^\mu A^\Lambda{}_\mu = \sqrt{2}A^\Lambda{}_t = -2|X|^2 \mathcal{R}^\Lambda \,. \tag{1.121}$$

is then always possible. In this gauge, equations (1.109), (1.111) and (1.118) reduce to the requirement of time-independence for all the scalar fields and the bilinear X,

$$\partial_t Z^i = \partial_t X = \partial_t q^u = 0 \,, \tag{1.122}$$

which of course also implies the time-independence of the $\mathcal{R}$ and $\mathcal{I}$ sections.

The definition (1.120) and the Fierz identity $V^2 = 4|X|^2$ imply that the 1-form V takes the form

$$V = 2\sqrt{2}|X|^2(dt + \omega) \,, \tag{1.123}$$

where ω is a time-independent spatial 1-form, since V is Killing, satisfying by definition

$$d\omega = \frac{1}{2\sqrt{2}} d\left(\frac{V}{|X|^2}\right) \,, \tag{1.124}$$

or, exploiting (1.101) and (1.103),

$$d\omega = -\frac{i}{2\sqrt{2}} \star \left[\left(X\mathfrak{D}\bar{X} - \bar{X}\mathfrak{D}X + ig\sqrt{2}|X|^2 \mathcal{R}^\Lambda \mathsf{P}_\Lambda{}^x V^x\right) \wedge \frac{V}{|X|^4}\right] \,. \tag{1.125}$$

The $\hat{V}^x$ are mutually orthogonal and also orthogonal to $\hat{V}$, which means that they can be used as a Dreibein for a 3-dimensional Euclidean metric

$$\delta_{xy}\hat{V}^x \otimes \hat{V}^y \;\equiv\; \gamma_{\underline{mn}} dx^m dx^n \,, \tag{1.126}$$

where we introduced the remaining 3 spatial coordinates x^m ($m = 1, 2, 3$). The 4-dimensional metric takes the coordinate-form

$$ds^2 \;=\; 2|X|^2(dt + \hat{\omega})^2 - \frac{1}{2|X|^2}\gamma_{\underline{mn}} dx^m dx^n \,. \tag{1.127}$$

In what follows we will use the Vierbein basis

$$e^0 = \frac{1}{2|X|}\hat{V} \,, \qquad e^x = \frac{1}{\sqrt{2}|X|}\hat{V}^x \,, \tag{1.128}$$

that is

$$(e^a{}_\mu) = \begin{pmatrix} \sqrt{2}|X| & \sqrt{2}|X|\omega_{\underline{m}} \\ 0 & \frac{1}{\sqrt{2}|X|}V^x{}_{\underline{m}} \end{pmatrix} , \qquad (e^\mu{}_a) = \begin{pmatrix} \frac{1}{\sqrt{2}|X|} & -\sqrt{2}|X|\omega_x \\ 0 & \sqrt{2}|X|V_x{}^{\underline{m}} \end{pmatrix} , \tag{1.129}$$

where $V_x{}^{\underline{m}}$ is the inverse Dreibein $V_x{}^{\underline{m}}V^y{}_{\underline{m}} = \delta^y{}_x$ and $\omega_x = V_x{}^{\underline{m}}\omega_{\underline{m}}$. Observe that we can raise and lower flat 3-dimensional indexes with δ_{xy} and δ^{xy}, whence their position is rather irrelevant. We shall also adopt the convention that, from now on, all objects with flat or curved 3-dimensional indexes refer to the above Dreibein and the corresponding metric.

The 3-dimensional form of (1.125) is then

$$(d\omega)_{xy} = 2\,\varepsilon_{xyz}\left\{\langle\mathcal{I}|\tilde{\mathfrak{D}}_z\mathcal{I}\rangle - \frac{g}{2\sqrt{2}|X|^2}\mathcal{R}^\Lambda \mathsf{P}^z_\Lambda\right\} , \tag{1.130}$$

where $\tilde{\mathfrak{D}}$ is the covariant derivative with respect to the effective 3-dimensional gauge connection

$$\tilde{A}^\Lambda{}_m \equiv A^\Lambda{}_m - \omega_m A^\Lambda{}_t = A^\Lambda{}_m + \sqrt{2}|X|^2\mathcal{R}^\Lambda\omega_m \,. \tag{1.131}$$

The spatial 3-dimensional metric is further constrained by (1.105). Its purely spatial part takes the form

$$dV^x + \epsilon^{xyz}\tilde{\tilde{A}}^y \wedge V^z + T^x = 0 \,, \tag{1.132}$$

with

$$\begin{aligned} \tilde{\tilde{A}}^x{}_m &\equiv \mathsf{A}^x{}_m - g\,\tilde{A}^\Lambda_m \mathsf{P}^x_\Lambda \,, \\ T^x &= -\frac{g}{\sqrt{2}}\,\mathcal{I}^\Lambda \mathsf{P}^y_\Lambda\, V^y \wedge V^x \,. \end{aligned} \tag{1.133}$$

Equation (1.132) can be interpreted as Maurer-Cartan's first structure equation for the Dreibein V^x.

So far we have shown that a time-like bosonic supersymmetric field configuration, with the gauge choice (1.121), necessarily satisfies equations (1.127), (1.130), (1.132), (1.122), (1.114) and (1.119). At this stage, $\mathcal{R}$, $\mathcal{I}$ and consequently the complex scalars Z^i, are only constrained to be t-independent.

These necessary conditions are also sufficient to guarantee supersymmetry, as proven in [4]. In fact, for any such configuration, there is always a Killing spinor of the form

$$\epsilon_I = X^{1/2}\eta_I \,, \tag{1.134}$$

where η_I is a constant spinor satisfying

$$\eta^I \;+\; i\gamma^0\,\varepsilon^{IJ}\eta_J = 0 \qquad \text{and} \qquad \eta_I \;+\; \gamma^{0(x)}\sigma^{(x)\,J}{}_I\eta_J = 0 \qquad \text{(no sum over } x\text{)}\,. \tag{1.135}$$

Each of the four compatible constraints in (1.135) is able to project out half of the components of η_I. However only three of the constraints are independent, so that one of the eight real components always survives. The configurations are then preserving at least $\frac{1}{8}$ of the original supersymmetry.

We still have to impose that the configuration fulfills the equations of motion which are not identically satisfied, i.e. the time components of the Maxwell equations and of the Bianchi identities.

The time component of the Hodge dual of the Bianchi identities is just the Bianchi identity of the effective 3-dimensional field strength $\tilde{F}^\Lambda$, which has the following 3-dimensional expression,

$$\tilde{F}^\Lambda{}_{xy} \equiv -\frac{1}{\sqrt{2}}\varepsilon_{xyz}\{\tilde{\mathfrak{D}}_z\mathcal{I}^\Lambda + g\mathcal{B}^\Lambda{}_z\}\,, \tag{1.136}$$

where

$$\mathcal{B}^\Lambda{}_z \equiv \sqrt{2}\left[\mathcal{R}^\Lambda\mathcal{R}^\Sigma + \frac{1}{8|X|^2}\Im\mathfrak{m}(\mathcal{N})^{\Lambda\Sigma}\right]\mathsf{P}_\Sigma{}^z\,. \tag{1.137}$$

Given $\tilde{A}^\Lambda{}_{\underline{m}}$, $\mathcal{I}^\Lambda$ and $\mathcal{B}^\Lambda{}_x$ solving that equation, then we find a $\tilde{A}^\Lambda{}_{\underline{m}}$ that gives rise to the field strength $\tilde{F}^\Lambda{}_{\underline{mn}}$ with the form prescribed by supersymmetry and the 3-dimensional Bianchi identity and, therefore, the 0-th component of the 4-dimensional one, are automatically satisfied.

The integrability equation of (1.136) takes the form of a generalized gauge covariant Laplace equation for the $\mathcal{I}^\Lambda$,

$$\tilde{\mathfrak{D}}^2\mathcal{I}^\Lambda + g\tilde{\mathfrak{D}}_x\mathcal{B}^\Lambda{}_x = 0\,, \tag{1.138}$$

where the covariant derivatives include both the gauge connection and the spin connection for the 3-dimensional base space with metric $\gamma_{\underline{mn}}$.

The time component of the Maxwell equations takes the form of a sort of Bianchi identity for the dual field strengths F_Λ, which can be written as

$$\begin{aligned} -\frac{1}{\sqrt{2}}\varepsilon_{xyz}\tilde{\mathfrak{D}}_x\tilde{F}_{\Lambda\, yz} = \frac{1}{\sqrt{2}}g\langle\mathcal{I}|\tilde{\mathfrak{D}}_x\mathcal{I}\rangle\mathsf{P}^x_\Lambda + \frac{1}{2}g^2 f_{\Lambda(\Omega}{}^\Gamma f_{\Delta)\Gamma}{}^\Sigma\, \mathcal{I}^\Omega\mathcal{I}^\Delta\mathcal{I}_\Sigma \\ + \frac{g^2}{4|X|^2}\mathcal{R}^\Sigma\left[\mathsf{k}_{\Lambda\, u}\mathsf{k}_\Sigma{}^u - \mathsf{P}_\Lambda{}^x\mathsf{P}_\Sigma{}^x\right], \end{aligned} \tag{1.139}$$

where $\tilde{F}_\Lambda$ is defined -there are no dual 1-forms A_Λ in this formulation- by

$$\tilde{F}_{\Lambda\, xy} \equiv -\frac{1}{\sqrt{2}}\varepsilon_{xyz}\{\tilde{\mathfrak{D}}_z\mathcal{I}_\Lambda + g\mathcal{B}_{\Lambda\, z}\}, \tag{1.140}$$

with

$$\mathcal{B}_{\Lambda\, x} \equiv \sqrt{2}\left[\mathcal{R}_\Lambda\mathcal{R}^\Sigma + \frac{1}{8|X|^2}\Re\mathfrak{e}\mathcal{N}_{\Lambda\Gamma}\Im\mathfrak{m}(\mathcal{N})^{-1|\Gamma\Sigma}\right]\mathsf{P}^x_\Sigma\,. \tag{1.141}$$

A summary of this analysis is given in the following paragraph, restricted to the absence of hypermultiplets and to a SU(2)-FI gauging, which is the case we are interested in.

Equations in the $SU(2)$-Fayet-Iliopoulos gauged case

In this section we are going to particularize the results of [4] to the case presenting an SU(2) gauge group and a Fayet-Iliopoulos term given by (1.65) and (1.68).
We are here summarizing the form that the fields of the time-like supersymmetric solutions, $\{g_{\mu\nu}, A^\Lambda{}_\mu, Z^i\}$, should take in the case under study:

- the metric can always be written in the conformastationary form

$$ds^2 = e^{2U}(dt+\hat{\omega})^2 - e^{-2U}\gamma_{\underline{mn}}dx^m dx^n\,, \tag{1.142}$$

  where

  - the metric function e^{-2U} is given by the Hesse potential

$$e^{-2U} = W(\mathcal{I}) = \frac{1}{2|X|^2}\,; \tag{1.143}$$

  - the 3-dimensional metric $\gamma_{\underline{mn}}$ can be expressed in terms of Dreibein $\hat{V}^x$, $x = 1,2,3$

$$\gamma_{\underline{mn}} = V^x{}_{\underline{m}}V^y{}_{\underline{n}}\delta_{xy}\,, \tag{1.144}$$

that must satisfy the equation

$$d\hat{V}^x - \xi g \epsilon^{xyz} \hat{\tilde{A}}^y \wedge \hat{V}^z + \hat{T}^x = 0\,, \tag{1.145}$$

where $\hat{\tilde{A}}^\Lambda$ is the effective 3-dimensional gauge connection

$$\tilde{A}^\Lambda{}_{\underline{m}} \equiv A^\Lambda{}_{\underline{m}} + \tfrac{1}{\sqrt{2}} e^{2U} \mathcal{R}^\Lambda \omega_{\underline{m}}\,, \tag{1.146}$$

and

$$\hat{T}^x = \tfrac{1}{\sqrt{2}} \xi g \mathcal{I}^y \hat{V}^y \wedge \hat{V}^x\,; \tag{1.147}$$

– the 1-form $\hat{\omega}$ satisfies the equation (in tangent 3-dimensional space)

$$(d\hat{\omega})_{xy} = 2\varepsilon_{xyz} \left\{ \mathcal{I}_M \tilde{\mathfrak{D}}_z \mathcal{I}^M + \tfrac{1}{\sqrt{2}} \xi e^{-2U} \mathcal{R}^z \right\}\,, \tag{1.148}$$

where $\tilde{\mathfrak{D}}$ is the covariant derivative w.r.t. the effective 3-dimensional gauge connection

$$\tilde{\mathfrak{D}}_z \mathcal{I}^x = \partial_z \mathcal{I}^x - g \varepsilon_{yw}{}^x \tilde{A}^y{}_z \mathcal{I}^w\,, \tag{1.149}$$

$$\tilde{\mathfrak{D}}_z \mathcal{I}_x = \partial_z \mathcal{I}_x - g \varepsilon_{xy}{}^w \tilde{A}^y{}_z \mathcal{I}_w\,, \tag{1.150}$$

$$\tilde{\mathfrak{D}}_z \mathcal{I}^M = \partial_z \mathcal{I}^M\,, \quad \text{when} \quad M \neq x\,, \quad \text{(ungauged directions);} \tag{1.151}$$

- the time-component of the vector fields has been gauge-fixed to

$$A^\Lambda{}_t = -\tfrac{1}{\sqrt{2}} e^{2U} \mathcal{R}^\Lambda\,, \tag{1.152}$$

and the space components $A^\Lambda{}_x$ together with the functions $\mathcal{I}^M$ are determined by

$$\tilde{F}^\Lambda{}_{xy} = -\tfrac{1}{\sqrt{2}} \varepsilon_{xyz} \left\{ \tilde{\mathfrak{D}}_z \mathcal{I}^\Lambda - \sqrt{2} \xi g \left[\mathcal{R}^\Lambda \mathcal{R}^z + \tfrac{1}{4} e^{-2U} (\Im\mathfrak{m}\mathcal{N})^{-1|\Lambda\, z} \right] \right\}\,, \tag{1.153}$$

and

$$-\tfrac{1}{\sqrt{2}} \varepsilon_{xyz} \tilde{\mathfrak{D}}_x \tilde{F}_{\Lambda\, yz} = \tfrac{1}{2} g \delta_\Lambda{}^x \left[g \left(\mathcal{I}^x \mathcal{I}^y \mathcal{I}_y - \mathcal{I}_x \mathcal{I}^y \mathcal{I}^y \right) - \tfrac{1}{\sqrt{2}} \xi \varepsilon_{xyz} (d\hat{\omega})_{yz} \right]\,, \tag{1.154}$$

where we have defined

$$\tilde{F}_{\Lambda\, xy} \equiv -\tfrac{1}{\sqrt{2}} \varepsilon_{xyz} \left\{ \tilde{\mathfrak{D}}_z \mathcal{I}_\Lambda - \sqrt{2} g \xi \left[\mathcal{R}_\Lambda \mathcal{R}^z + \tfrac{1}{4} e^{-2U} \Re\mathfrak{e}\mathcal{N}_{\Lambda\Gamma} (\Im\mathfrak{m}\mathcal{N})^{-1|\Gamma\, z} \right] \right\}\,; \tag{1.155}$$

- finally, the scalars are given by

$$Z^i = \frac{\mathcal{R}^i + i\mathcal{I}^i}{\mathcal{R}^0 + i\mathcal{I}^0}\,. \tag{1.156}$$

1.2.6 BPS rewriting of the action

In [5], the 4-dimensional $\mathcal{N} = 2$ supergravity theories with $\mathrm{U}(1)$ Fayet-Iliopoulos gauging, involving vector multiplets only, were considered and a completely covariant approach was developed, where magnetic gauges of the form (1.63) are allowed. The purpose was to seek dyonic black hole solutions and [5] provides a powerful effective procedure to obtain them, which was further exploited in [1] to derive the solution we are presenting in section 3.1. Since a fully duality covariant action has not been built yet, certain simplifying assumptions have been done, to get to a set of first order flow equations, driven by a superpotential $\mathcal{W}$, whose solutions can be proven to be solutions of the equations of motions and of the Killing spinor equations.

The potential should be written in the covariant form (1.64) and, since we are interested in static black holes with radial symmetry, the metric Ansatz

$$\mathrm{d}s^2 = -e^{2U(r)}\mathrm{d}t^2 + e^{-2U(r)}(\mathrm{d}r^2 + e^{2\psi(r)}\mathrm{d}\Omega_\kappa^2)\,, \tag{1.157}$$

will be employed, where $\mathrm{d}\Omega_\kappa^2 = \mathrm{d}\theta^2 + f_\kappa^2(\theta)\,\mathrm{d}\phi^2$ is the metric on the two-surfaces $\Sigma = \{\mathrm{S}^2, \mathbb{E}^2, \mathrm{H}^2\}$ of constant scalar curvature $R = 2\kappa$, with $\kappa = \{1, 0, -1\}$ respectively. Here the function $f_\kappa(\theta)$ is given by

$$f_\kappa(\theta) = \begin{cases} \sin\theta\,, & \kappa = 1\,, \\ \theta\,, & \kappa = 0\,, \\ \sinh\theta\,, & \kappa = -1\,. \end{cases} \tag{1.158}$$

The scalars are assumed to depend on the radial coordinate r only, $Z^i = Z^i(r)$, while the gauge fields should have an appropriate profile to satisfy

$$p^\Lambda = \frac{1}{\mathrm{vol}(\Sigma)}\int_\Sigma F^\Lambda\,, \qquad q_\Lambda = \frac{1}{\mathrm{vol}(\Sigma)}\int_\Sigma G_\Lambda\,, \tag{1.159}$$

with p^Λ and q_Λ being the magnetic and electric charges associated to the black hole and G_Λ denoting the dual field strength,

$$G_\Lambda = \frac{\delta\mathscr{L}}{\delta F^\Lambda}\,. \tag{1.160}$$

The symplectic invariant central charge is given by

$$Z = \langle \mathcal{Q}, \mathcal{V}\rangle\,, \tag{1.161}$$

where we introduced the vector of magnetic and electric charges, $\mathcal{Q} = (p^\Lambda, q_\Lambda)$.

Following the procedure outlined in [5], the previous Ansätze are plugged into the action and give rise to an effective 1-dimensional action involving the scalar fields and the warp functions $U(r), \psi(r)$,

$$S_{1d} = \int dr \left\{ e^{2\psi} \left[(U' - \psi')^2 + 2\psi'^2 + \mathcal{G}_{ij^*} Z^{i\prime} Z^{*j^*\prime} + e^{2U-4\psi} \mathbf{V}_{BH} + e^{-2U} \mathbf{V}_g \right. \right. \\ \left. \left. + 2\psi'' - U'' \right] - 1 \right\} , \tag{1.162}$$

which, after an integration by parts, can be written as

$$S_{1d} = \int \mathrm{d}r \left\{ e^{2\psi} \left[U'^2 - \psi'^2 + \mathcal{G}_{ij^*} Z^{i\prime} Z^{*j^*\prime} + e^{2U-4\psi} \mathbf{V}_{\mathrm{BH}} + e^{-2U} \mathbf{V}_g \right] - 1 \right\} \\ + \int \mathrm{d}r \frac{\mathrm{d}}{\mathrm{d}r} \left[e^{2\psi} (2\psi' - U') \right] . \tag{1.163}$$

Here $\mathbf{V}_{\mathrm{BH}}$ denotes the so-called black hole potential [43], defined as

$$\mathbf{V}_{\mathrm{BH}} = |\mathcal{D}Z|^2 + |Z|^2 \tag{1.164}$$

or equivalently as

$$\mathbf{V}_{\mathrm{BH}} = -\frac{1}{2} \mathcal{Q}^T M \mathcal{Q} , \tag{1.165}$$

where

$$M = \begin{pmatrix} \Im\mathfrak{m}\mathcal{N} + \Re\mathfrak{e}\mathcal{N} (\Im\mathfrak{m}\mathcal{N})^{-1} \Re\mathfrak{e}\mathcal{N} & -\Re\mathfrak{e}\mathcal{N} (\Im\mathfrak{m}\mathcal{N})^{-1} \\ -(\Im\mathfrak{m}\mathcal{N})^{-1} \Re\mathfrak{e}\mathcal{N} & (\Im\mathfrak{m}\mathcal{N})^{-1} \end{pmatrix} , \tag{1.166}$$

and $\mathcal{N}$ is the period matrix.

The effective 1-dimensional action (1.163) can be rewritten as a sum of squares of first order differential expressions and a constraint. Setting each of these terms to zero provides the same equations that emerge from the direct analysis of the supersymmetry transformations, as we are showing in a while.
The rewriting is obtained by means of many special geometry identities, which descend from the following basic ones

$$\begin{aligned} &\mathcal{V}^{*T} M \mathcal{V} = i \mathcal{V}^{*T} \Omega \mathcal{V} = i \langle \mathcal{V}^*, \mathcal{V} \rangle = -1 , \\ &\mathcal{U}_i^T M \mathcal{U}_{j^*}^* = i \mathcal{U}_i^T \Omega \mathcal{U}_{j^*}^* = i \langle \mathcal{U}_i, \mathcal{U}_{j^*}^* \rangle = -\mathcal{G}_{ij^*} , \\ &\mathcal{V}^{T\prime} M \mathcal{V}^{*\prime} = \mathcal{G}_{ij^*} Z^{i\prime} Z^{*j^*\prime} + \mathcal{A}_r^2 , \end{aligned} \tag{1.167}$$

where

$$\mathcal{A}_\mu = \Im\mathfrak{m}(\partial_\mu Z^i (\partial_i \mathcal{K})) \tag{1.168}$$

is the connection associated to the Kähler transformations (1.11). We are also defining

$$e^{2i\alpha} = \frac{Z - i e^{2(\psi - U)} L}{Z^* + i e^{2(\psi - U)} L^*} . \tag{1.169}$$

The effective action (1.163) becomes then

$$
\begin{aligned}
S_{1d} = \int dr \Big\{ & -\frac{1}{2} e^{2(U-\psi)} \mathcal{E}^T M \mathcal{E} - e^{2\psi} \left[(\alpha' + \mathcal{A}_r) + 2e^{-U} \, \Re\mathfrak{e}(e^{-i\alpha} L) \right]^2 \\
& - e^{2\psi} \left[\psi' - 2e^{-U} \, \Im\mathfrak{m}(e^{-i\alpha} L) \right]^2 - (\kappa + \langle G, \mathcal{Q} \rangle) \\
& -2 \frac{d}{dr} \left[e^{2\psi - U} \, \Im\mathfrak{m}(e^{-i\alpha} L) + \, e^{U} \, \Re\mathfrak{e}(e^{-i\alpha} Z) \right] \Big\} ,
\end{aligned}
\tag{1.170}
$$

where

$$
\begin{aligned}
\mathcal{E}^T = & 2e^{2\psi} \left(e^{-U} \Im\mathfrak{m} \left(e^{-i\alpha} \mathcal{V} \right) \right)'^{\,T} - e^{2(\psi - U)} G^T \Omega M^{-1} \\
& + 4e^{-U} \left(\alpha' + \mathcal{A}_r \right) \Re\mathfrak{e} \left(e^{-i\alpha} \mathcal{V} \right)^T + \mathcal{Q}^T .
\end{aligned}
\tag{1.171}
$$

If the charges satisfy the condition

$$
\langle G, \mathcal{Q} \rangle = -\kappa \, , \tag{1.172}
$$

the effective action (1.163) has been rewritten as a sum of squares of first order differential conditions and a boundary term. Setting to zero each of these terms, a system of first order equations is obtained,

$$
\begin{aligned}
& 2e^{2\psi} \left(e^{-U} \mathrm{Im}(e^{-i\alpha} \mathcal{V}) \right)' + e^{2(\psi - U)} \Omega M G + 4e^{-U} (\alpha' + \mathcal{A}_r) \mathrm{Re}(e^{-i\alpha} \mathcal{V}) + \mathcal{Q} = 0 \, , \\
& \psi' = 2e^{-U} \mathrm{Im}(e^{-i\alpha} L) \, , \\
& \alpha' + \mathcal{A}_r = -2e^{-U} \mathrm{Re}(e^{-i\alpha} L) \, .
\end{aligned}
\tag{1.173}
$$

These equations are all symplectic covariant and so every solution, obtained for a specific choice of charges and Fayet-Iliopoulos terms, can be mapped to a different solution, with new $\mathcal{Q}$ and G, by a duality transformation.

Superpotential and BPS flow

A simple rewriting of the equations (1.173) leads to the identification of a superpotential function $\mathcal{W}$ driving the BPS flow. Projecting the equation $\mathcal{E} = 0$ on $\mathcal{V}$ and $\mathcal{U}_i$, and defining

$$
\mathcal{W} = e^{U} \left| Z - i e^{2(\psi - U)} L \right| \, , \tag{1.174}
$$

the BPS conditions (1.173) can be expressed as flow equations for the physical degrees of freedom, the warp factors ψ and U and the scalar fields Z^i,

$$
\begin{aligned}
U' &= -g^{UU} \partial_U \mathcal{W} \, , \\
\psi' &= -g^{\psi\psi} \partial_\psi \mathcal{W} \, ,
\end{aligned}
\tag{1.175}
$$

$$Z^{i\prime} = -2g^{ij^*}\partial_{j^*}\mathcal{W}\,,$$

where $g_{UU} = -g_{\psi\psi} = e^{2\psi}$ and $g_{ij^*} = e^{2\psi}\mathcal{G}_{ij^*}$.

Supersymmetry equations

In order to be able to claim that the solutions of the equations (1.173) provide supersymmetric configurations, the supersymmetry variations have been analyzed in [5]. It has been proven that, when two independent projector conditions are imposed, the vanishing of the supersymmetry variations reproduces the flow equations (1.175).

The variations that should vanish are those in (1.81), which in the U(1) Fayet-Iliopoulos gauged theory reduce to

$$\begin{aligned}\delta\psi_{\mu I} &= \mathfrak{D}_\mu\epsilon_I - 2i\varepsilon_{IJ}\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma}\mathcal{L}^\Sigma F^{-\Lambda}{}_{\mu\nu}\gamma^\nu\epsilon^J - \frac{i}{2}L\delta_{IJ}\gamma^\nu\eta_{\mu\nu}\epsilon^J\,,\\ \delta\lambda^{iI} &= i\partial\!\!\!/ Z^i\epsilon^I - \mathcal{G}^{ij^*}f^{*\Sigma}{}_{j^*}\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma}F^{-\Lambda}{}_{\mu\nu}\gamma^{\mu\nu}\varepsilon^{IJ}\epsilon_J + \mathcal{D}^{*i}L\delta^{IJ}\epsilon_J\,,\end{aligned} \tag{1.176}$$

where $F^{-\Lambda}{}_{\mu\nu} = \frac{1}{2}\left(F^\Lambda{}_{\mu\nu} - \frac{i}{2}\epsilon_{\mu\nu\rho\sigma}F^{\Lambda\rho\sigma}\right)$.

The Ansatz on the metric, on the radial dependence of the scalars and on the gauge fields

$$\begin{aligned}F^\Lambda{}_{tr} &= \frac{e^{2(U-\psi)}}{2}\left(\Im\mathfrak{m}\mathcal{N}^{-1}\right)^{\Lambda\Sigma}\left(\Re\mathfrak{e}\mathcal{N}_{\Sigma\Omega}p^\Omega - q_\Sigma\right)\,,\\ F^\Lambda{}_{\theta\phi} &= -\frac{1}{2}p^\Lambda\sin\theta\,,\end{aligned} \tag{1.177}$$

have to be employed to conclude that the vanishing of (1.176) reproduces the flow equations (1.173) for a Killing spinor of the form

$$\epsilon_I = e^{\frac{U}{2}+\frac{i}{2}\int\left\{\mathcal{A}_r+\left[e^{U-2\psi}\Im\mathfrak{m}\left(e^{-i\alpha}Z\right)+e^{-U}\Re\mathfrak{e}\left(e^{-i\alpha}\right)\right]\right\}dr}\chi_I \tag{1.178}$$

where χ_I is a constant spinor subject to the following constraints,

$$\gamma^0\chi_I = i\varepsilon_{IJ}\chi^J\,, \qquad \gamma^1\chi_I = \delta_{IJ}\chi^J\,. \tag{1.179}$$

Two independent conditions have been imposed on the Killing spinor, each halving the number of preserved supersymmetries. The solutions of (1.173) are therefore $1/4$-BPS.

1.2.7 Some well known models

The aim of this paragraph is to present some examples of widely studied models of $\mathcal{N}=2$, $d=4$ supegravity coupled to vector multiplets. The kinetic term of the scalars in the vector multiplets is a σ-model, whose target manifold is a special Kähler one. The

cases we are presenting involve simple and interesting geometries, although they are not necessarily trivial. In particular, one of the cases under study has the peculiarity of being a non-homogeneous manifold, i.e. it cannot be described as a coset.

These examples are going to be relevant in what follows, since they have been used to find the explicit solutions we are constructing in the last chapter. We are here discussing the prepotential from which they arise and reporting some relevant features of their geometry.

Cubic models

A broad class of widely studied models are those giving rise to the so-called very special Kähler manifolds, that can be obtained by dimensional reduction from the vector multiplets' scalar geometries coupled to minimal supergravity in $d = 5$, known as special real manifolds.

All the models originating from this kind of geometry are described (cf. e.g. [63, 64]) by a cubic prepotential of the form

$$F = d_{ijk} \frac{\mathcal{X}^i \mathcal{X}^j \mathcal{X}^k}{\mathcal{X}^0} \,, \tag{1.180}$$

where d_{ijk} is a real and symmetric tensor and the corresponding special Kähler space is usually dubbed a d-space [64]. This class of models is very rich and interesting because d-geometries naturally arise in the compactification of type-II superstring theories on complex 3-dimensional Calabi–Yau manifolds.

Among the models in this class, those that have been studied in more depth are the homogeneous ones. For a homogeneous manifold, the isometry group acts transitively, which means that any two points of the manifold are related by an isometry transformation. The orbit swept out by the action of the isometry group G from any given point is locally isomorphic to the coset space G/H, where H is the isotropy group of that point.
After [64] and [65], homogeneous special Kähler d-spaces, either symmetric or non-symmetric, have been classified in terms of the corresponding d-tensor, which uniquely determines their geometry. For the non-symmetric spaces the isometry group G is not semisimple and the isotropy group H is always its maximal compact subgroup.
No homogeneous, non-symmetric, special Kähler (non-compact, Riemannian) spaces which are not based on cubic prepotentials (1.180) are known, although there is no proof that they do not exist, as far as we know.

As stated in [54], a property that characterizes the d-tensors of symmetric cubic models is

$$d^{m(ij} d^{kl)n} d_{mnp} = \frac{4}{3} \delta^{(i}_{p} d^{jkl)} \tag{1.181}$$

or, alternatively [22], the fact that the contraction $d_{ijk} y^k$ with some object y^k defines an

invertible matrix. Another peculiarity is the constancy of the tensor

$$\hat{d}^{lmn} = \frac{\mathcal{G}^{il}\mathcal{G}^{jm}\mathcal{G}^{kn}}{(d_{pqr}\lambda^p\lambda^q\lambda^r)^2} d_{ijk} \,. \tag{1.182}$$

where $Z^i = x^i + i\lambda^i$ are the special coordinates defined by $\mathcal{X}^\Lambda = \begin{pmatrix} 1 \\ Z^i \end{pmatrix}$. Once these coordinates have been chosen, in the notation given by (1.200), the relevant geometrical quantities are given by

$$\Omega = \begin{pmatrix} 1 \\ Z^i \\ -F(Z) \\ 3d_{Z,i} \end{pmatrix} , \qquad e^{-\mathcal{K}} = 8d_\lambda \,, \qquad \mathcal{G}_{ij} = -\frac{3}{2}\frac{d_{\lambda,ij}}{d_\lambda} + \frac{9}{4}\frac{d_{\lambda,i}d_{\lambda,j}}{d_\lambda^2} \,. \tag{1.183}$$

Two of the following examples are symmetric cubic models, the stu model and the family of $ST[2,n]$ models. Moreover, we are entering a detailed description of a cubic non-homogeneous and non-symmetric Kähler manifold, giving rise to what we have called nh-stu model.

The stu model

The stu model provides a particularly manageable framework and has therefore been used in many cases when seeking solutions, as in the pioneer paper [28]. It involves 3 vector multiplets, whose scalar Z^i are usually named s, t and u, accounting for the name of the model. The cubic prepotential is defined by the tensor $d_{ijk} = \frac{1}{6}|\varepsilon_{ijk}|$ and is explicitly given by

$$F = \frac{\mathcal{X}^1\mathcal{X}^2\mathcal{X}^3}{\mathcal{X}^0} \,. \tag{1.184}$$

As can be directly obtained from the previous, general formulation, the relevant geometrical quantities are

$$\Omega = \begin{pmatrix} 1 \\ s \\ t \\ u \\ -stu \\ tu \\ su \\ st \end{pmatrix} , \qquad e^{-\mathcal{K}} = 8\Im\mathfrak{m}s\Im\mathfrak{m}t\Im\mathfrak{m}u \,, \qquad \mathcal{G}_{ij^*} = \frac{\delta_{ij^*}}{4(\Im\mathfrak{m}Z^i)^2} \,, \tag{1.185}$$

where there is no sum over the index i in the last expression. The Kähler metric is the product of the metrics of three $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$ coset spaces. The reality of the Kähler potential, together with the symmetry between the three scalars, implies that the imaginary parts $\Im\mathfrak{m} Z^i$ are positive.

Two truncations of the stu model are also very well known: the one with $t = u$ and prepotential $F = \frac{\mathcal{X}^1(\mathcal{X}^2)^2}{\mathcal{X}^0}$ and the simplest, one modulus model, where $s = t = u$ and the prepotential is $F = \frac{(\mathcal{X}^1)^3}{\mathcal{X}^0}$, in which the scalar parametrizes the coset space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$.

The nh-stu model

We are here treating the special Kähler 3-moduli model based on the holomorphic prepotential[11]

$$F = \frac{\mathcal{X}^1\mathcal{X}^2\mathcal{X}^3}{\mathcal{X}^0} - \frac{A}{3}\frac{\left(\mathcal{X}^3\right)^3}{\mathcal{X}^0}\,, \tag{1.186}$$

where A is an arbitrary real constant. For $A = -1$, the prepotential reads

$$F = \frac{\mathcal{X}^1\mathcal{X}^2\mathcal{X}^3}{\mathcal{X}^0} + \frac{1}{3}\frac{\left(\mathcal{X}^3\right)^3}{\mathcal{X}^0}\,, \tag{1.187}$$

which has been constructed in the context of Type IIA string theory compactified on Calabi-Yau manifolds in [68]. In particular, analyzing string vacua with three complex moduli (section 3.2 therein), different bases for the toric construction of such a model have been considered; (1.187) corresponds to the basis $\mathbb{F}_0$ of [68], while other toric constructions determine the same model in different symplectic frames. The prepotential (1.187) can also be obtained as the $c = 0$ limit of the heterotic prepotential appearing in [69] and the corresponding one-loop prepotential V_{GS} is given by considering its $c = 0$ limit.

In absence of gauging, the BPS attractor equations for this model have been discussed in [69]; a solution for a generic supporting black hole charge configuration was obtained in this context and, as a consequence, the BPS black hole entropy was determined as a function of the charges.

A full-fledged, explicit determination of the BPS black hole entropy of the model based on (1.187) was later given by Shmakova in the investigation of BPS attractor equations for black holes based on Calabi-Yau cubic prepotentials [70]. We report here the expression of the ungauged BPS black hole entropy, for later convenience:

$$\frac{S_{\mathrm{BH}}}{\pi} = \frac{\sqrt{f\left(\mathcal{Q}\right)}}{3p^0}, \tag{1.188}$$

[11] Black holes of type IIA Calabi-Yau compactifications in the presence of perturbative quantum corrections, leading to a prepotential of the form $F = d_{ijk}\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k/\mathcal{X}^0 + ic(\mathcal{X}^0)^2$ (for some constant c), were constructed and studied in [66, 67].

where

$$
\begin{aligned}
& f(\mathcal{Q}) := \\
& 2\left\{\left(p^1p^2+\left(p^3\right)^2-p^0q_3\right)\left[\left(p^1p^2+\left(p^3\right)^2-p^0q_3\right)^2+12\left(p^2p^3-p^0q_1\right)\left(p^1p^3-p^0q_2\right)\right]\right. \\
& \left.+\left[\left(p^1p^2+\left(p^3\right)^2-p^0q_3\right)^2-4\left(p^2p^3-p^0q_1\right)\left(p^1p^3-p^0q_2\right)\right]^{3/2}\right\} \\
& -9\left[p^0\left(p^0q_0+p^1q_1+p^2q_2+p^3q_3\right)-2p^1p^2p^3-\frac{2}{3}\left(p^3\right)^3\right]^2,
\end{aligned}
\tag{1.189}
$$

and the conditions $f(\mathcal{Q}) > 0$ and $p^0 > 0$ define the BPS-supporting black hole charge vector $\mathcal{Q}$. It is immediate to check that (1.188) and (1.189) imply the entropy S_{BH} to be homogeneous of degree two in the charges, as it must be in four dimensions for 0-branes.

The model (1.186) under consideration, where A has to be considered a parameter, belongs to the class of the very special Kähler manifolds. In particular, the model (1.186) is defined by $d_{123} = 1/6$ and $d_{333} = -A/3$.

It is worth pointing out that the d-space corresponding to (1.186) is neither symmetric nor homogeneous [65, 71]. In particular, it does not fall within the class of symmetric models examined in [22], that are characterized by a constant tensor[12] $\hat{d}^{ijk}$ defined in (1.182).

In fact, it can be easily checked that the prepotential (1.186) implies a non-constant $\hat{d}^{lmn}$. For this reason, we will henceforth dub the cubic model (1.186) as a *non-homogeneous deformation* of the homogeneous and symmetric stu model (shortly, nh-stu), to which it reduces when $A = 0$.

The vector multiplets' scalar manifold of the nh-stu model is neither symmetric nor homogeneous; namely, the non-compact Riemannian space endowed with the special Kähler geometry specified by the cubic holomorphic prepotential (1.186) (with non-vanishing A) cannot be described as a coset G/H, where H is a local, compact isotropy group (linearly realized on the scalar fields, which generally sit in its representations) and G is a global, non-compact symmetry group (non-linearly realized by the scalar fields, but linearly realized by the vectors).

In theories of Abelian Maxwell fields, the group G describes the electric-magnetic duality symmetry, and its non-compactness in presence of scalar fields was firstly discussed by Gaillard and Zumino in [73].

Linearly realized electric-magnetic duality (U-duality[13]) plays a key role in Einstein-

[12] For some considerations on the completely contravariant d-tensor in generic d-spaces (and the corresponding definition of the so-called E-tensor for non-symmetric special Kähler spaces), cf. e.g. [72], and references therein.

[13] Here, U-duality is referred to as the 'continuous' symmetries of [74]. Their discrete versions are the U-duality non-perturbative string theory symmetries introduced by Hull and Townsend [75].

Maxwell theories coupled to scalar fields in presence of local supersymmetry, and consequently in their regular solutions, such as the dyonic black holes we are going to discuss. Even if the scalar manifold is not a coset G/H, a global U-duality symmetry group G always exists, even if it may be non-reductive or also discrete in generic, (semi-)realistic models of string compactifications.

A general feature of Einstein-Maxwell theories coupled to non-linear sigma models of scalar fields in four dimensions is the symplectic structure of the field strength 2-forms and of their duals, which in turn allows to define the symplectic inner scalar product. It results in the maximal, generally non-symmetric embedding [73]

$$G \subset \mathrm{Sp}(2n,\mathbb{R}) \tag{1.190}$$

$$\mathbf{R} = \mathbf{2n}\,, \tag{1.191}$$

where n is the number of vector fields, $\mathbf{2n}$ is the fundamental representation of $\mathrm{Sp}(2n,\mathbb{R})$ and $\mathbf{R}$ is the representation of G, not necessarily irreducible.
Thus, it is interesting to determine the (continuous, Lie component of the) U-duality algebra $\mathfrak{g}_{\text{nh-stu}}$ of the nh-stu model of $\mathcal{N}=2$, $d=4$ supergravity. In this case we have $n=4$, since one graviphoton and three vectors from the vector multiplets are present. We aim to explicitly find the realization of the maximal, non-symmetric embedding

$$\mathfrak{g}_{\text{nh-stu}} \subset \mathfrak{sp}(8,\mathbb{R})\,. \tag{1.192}$$

This is worth also in view of the fact that $G_{\text{nh-stu}}$, the Lie group generated by $\mathfrak{g}_{\text{nh-stu}}$, does not have a transitive action on the non-linear sigma model described by the $\mathcal{N}=2$ holomorphic prepotential (1.186).

Since in the ungauged theory the semiclassical Bekenstein-Hawking entropy is generally invariant under linearly realized global symmetries, $\mathfrak{g}_{\text{nh-stu}}$ can be determined by finding all infinitesimal symplectic transformations which leave the BPS black hole entropy S_{BH} (1.188)-(1.189) invariant.
Let us choose $A=-1$. From (1.188)-(1.189), the infinitesimal invariance condition reads

$$\begin{aligned}\delta S_{\text{BH}} &= \frac{1}{2S_{\text{BH}}}\delta S_{\text{BH}}^2 \\ &= \frac{1}{2S_{\text{BH}}}\left[\left(-\frac{2f}{p^0}+\frac{\partial f}{\partial p^0}\right)\delta p^0+\frac{\partial f}{\partial p^i}\delta p^i+\frac{\partial f}{\partial q_0}\delta q_0+\frac{\partial f}{\partial q_i}\delta q_i\right]=0\,,\end{aligned} \tag{1.193}$$

or equivalently

$$-6f\delta p^0+p^0\delta f=0, \tag{1.194}$$

where $\delta f=\frac{\partial f}{\partial \mathcal{Q}}\delta\mathcal{Q}$ and

$$\delta\mathcal{Q} = \left(\delta p^0, \delta p^i, \delta q_0, \delta q_i\right)^T = \mathcal{S}\mathcal{Q}, \tag{1.195}$$

with $\mathcal{S}$ belonging to the symplectic Lie algebra. It is an 8×8 matrix which can generically be written in blocks as

$$\mathfrak{sp}\left(8,\mathbb{R}\right) \ni \mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A^T = -D\,, \quad B^T = B\,, \quad C^T = C\,, \tag{1.196}$$

where each block is a 4×4 matrix. Thus, $\mathcal{S}$ depends on ten real parameters.

By solving (1.194) for a BPS-supporting configuration with generic charges $\mathcal{Q}$ satisfying $f\left(\mathcal{Q}\right) > 0$ and $p^0 > 0$, the symplectic embedding of the U-duality Lie algebra $\mathfrak{g}_{\text{nh-stu}}$ of the nh-stu model into $\mathfrak{sp}(8,\mathbb{R})$ is realized by the following 4-dimensional, lower triangular matrix subalgebra (cf. (1.192); $a,b,c\in\mathbb{R}$, $\phi\in\mathbb{R}_0^+$)

$$\mathcal{S}_{\text{nh-stu}}(a,b,c,\phi) = \begin{pmatrix} -3\phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & -\phi & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & -\phi & 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & -\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3\phi & -a & -b & -c \\ 0 & 0 & c & b & 0 & \phi & 0 & 0 \\ 0 & c & 0 & a & 0 & 0 & \phi & 0 \\ 0 & b & a & 2c & 0 & 0 & 0 & \phi \end{pmatrix} \in \mathfrak{g}_{\text{nh-stu}} \subset \mathfrak{sp}(8,\mathbb{R})\,. \tag{1.197}$$

For a generic A, this can be generalized as follows:

$$\mathcal{S}_{\text{nh-stu}}(a,b,c,\phi;A) = \begin{pmatrix} -3\phi & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ a & -\phi & 0 & 0 & 0 & 0 & 0 & 0 \\ b & 0 & -\phi & 0 & 0 & 0 & 0 & 0 \\ c & 0 & 0 & -\phi & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3\phi & -a & -b & -c \\ 0 & 0 & c & b & 0 & \phi & 0 & 0 \\ 0 & c & 0 & a & 0 & 0 & \phi & 0 \\ 0 & b & a & -2Ac & 0 & 0 & 0 & \phi \end{pmatrix} \in \mathfrak{g}_{\text{nh-stu}} \subset \mathfrak{sp}(8,\mathbb{R})\,. \tag{1.198}$$

It should be noticed that (1.198) (which reduces to (1.197) for $A=-1$) determines a maximal Abelian subalgebra of $\mathfrak{sp}(8,\mathbb{R})$, whose four generators commute. Moreover, the generators corresponding to a,b,c in (1.197) span an axionic Peccei-Quinn translational 3-dimensional algebra, which is nilpotent of degree four. Indeed, the part of (1.197) generated by a,b,c can be recast in the following generic, d-parametrized form [76]

$$\mathcal{S} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a^j & 0 & 0 & 0 \\ 0 & 0 & 0 & -a^i \\ 0 & d_{a,ij} & 0 & 0 \end{pmatrix} \subset \mathfrak{sp}(2n, \mathbb{R}) \,, \tag{1.199}$$

where $(i = 1, ..., n-1)$

$$\begin{aligned} & d_{a,ij} := d_{ijk} a^k \,, \qquad d_{a,i} := d_{ijk} a^j a^k \,, \qquad d_a := d_{ijk} a^i a^j a^k \,, \\ & a^1 := 6a \,, \qquad a^2 := 6b \,, \qquad a^3 := 6c \,. \end{aligned} \tag{1.200}$$

$\mathcal{S}$ in (1.199) can be easily checked to be nilpotent of degree four[14],

$$\mathcal{S}^4 = 0 \Rightarrow \exp(\mathcal{S}) = \mathbb{I}_{2n} + \mathcal{S} + \frac{1}{2}\mathcal{S}^2 + \frac{1}{3!}\mathcal{S}^3, \tag{1.201}$$

yielding, at group level [77,78],

$$\exp(\mathcal{S}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ a^j & \mathbb{I}_{n-1} & 0 & 0 \\ -\frac{1}{6} d_a & -\frac{1}{2} d_{a,i} & 1 & -a^i \\ \frac{1}{2} d_{a,j} & d_{a,ij} & 0 & \mathbb{I}_{n-1} \end{pmatrix} \subset \mathrm{Sp}(2n, \mathbb{R}) \,. \tag{1.202}$$

Such an $(n-1)$-dimensional Abelian global symmetry algebra/group, as discussed in [78] (see also references therein, in particular [79]), characterizes *every* model of $d = 4$ supergravity based on a cubic scalar geometry, even not of special Kähler type (i.e. the scalar geometries of $N = 4$, 6 and 8 supergravity theories, dubbed 'generalized d-geometries' in [78]): the representation of axions in $d = 4$ is *always* nilpotent of degree four.

Besides the $(n-1)$-dimensional axionic Peccei-Quinn translational algebra, the universal sector of the electric-magnetic duality algebra of every (generalized) d-geometry (also cf. [80]) is given by the $2n \times 2n$ generalization of the ϕ-parametrized part of (1.198), where ϕ can be thus regarded as the Kaluza-Klein radius/real dilaton of the Kaluza-Klein (KK) $\mathfrak{so}_{\mathrm{KK}}(1,1)$,

$$K(\phi) = \begin{pmatrix} -3\phi & 0 & 0 & 0 \\ 0 & -\phi \mathbb{I}_{n-1} & 0 & 0 \\ 0 & 0 & 3\phi & 0 \\ 0 & 0 & 0 & \phi \mathbb{I}_{n-1} \end{pmatrix} \in \mathfrak{so}_{\mathrm{KK}}(1,1) \subset \mathfrak{sp}(2n, \mathbb{R}) \,. \tag{1.203}$$

Therefore, the $2n \times 2n$ matrix realization of the universal sector of the global electric-magnetic duality symmetry of an Einstein-Maxwell theory whose scalar manifold is endowed with a 'generalized d-geometry' can be written at the Lie algebra level as [77,78]

[14] $\mathbb{I}_d$ denotes the $d \times d$ identity matrix throughout.

$$\mathcal{S}(a)+K(\phi)=\begin{pmatrix} -3\phi & 0 & 0 & 0 \\ a^j & -\phi\delta_i^j & 0 & 0 \\ 0 & 0 & 3\phi & -a^i \\ 0 & d_{a,ij} & 0 & \phi\delta_j^i \end{pmatrix} \subset \mathfrak{sp}(2n,\mathbb{R})\,, \tag{1.204}$$

and at the Lie group level as [77,78]

$$\exp(\mathcal{S}(a))\exp(K(\phi))=\begin{pmatrix} e^{-3\phi} & 0 & 0 & 0 \\ a^j & e^{-\phi}\delta_i^j & 0 & 0 \\ -\frac{1}{6}d_a & -\frac{1}{2}d_{a,i} & e^{3\phi} & -a^i \\ \frac{1}{2}d_{a,j} & d_{a,ij} & 0 & e^{\phi}\delta_j^i \end{pmatrix} \subset \mathrm{Sp}(2n,\mathbb{R})\,. \tag{1.205}$$

Consistently, for $A=0$ the expression (1.205) enhances to an 8-dimensional U-duality group, given by the $\mathrm{SL}(2,\mathbb{R})^{\otimes 3}$ group of the stu model.

When considering $\mathcal{N}=2$, $d=4$ theories, this result for special Kähler d-geometries was known since [64]. In [78], (1.205) was shown also to pertain to the universal sector of axionic and KK coordinates in the scalar manifolds of $d=4$ theories based on 'generalized d-geometries' (for non-homogeneous $\mathcal{N}=2$ very special Kähler geometries, the same parametrization provides a general description of the generic element of the flat symplectic bundle over the vector multiplets' scalar manifold [78,79]).
Thus, in this sense, one can conclude that the nh-stu model has the *smallest possible* electric-magnetic duality algebra, consistent with its cubic nature (and thus with its uplifitability to $\mathcal{N}=2$, $d=5$ supergravity).

When obtaining a solution (see section 3.1), we have only considered the axion-free case, thus parameterizing the purely imaginary scalar fields as $Z^i=-i\lambda^i$, with λ^i real and positive ($i=1,2,3$); we have also chosen the projective coordinates as

$$\frac{\mathcal{X}^1}{\mathcal{X}^0}=-i\lambda^1\,, \qquad \frac{\mathcal{X}^2}{\mathcal{X}^0}=-i\lambda^2\,, \qquad \frac{\mathcal{X}^3}{\mathcal{X}^0}=-i\lambda^3\,. \tag{1.206}$$

Thus, the symplectic sections $\mathcal{V}$ become ($\Lambda=0,1,2,3$)

$$\begin{aligned} \mathcal{L}^\Lambda &= e^{\mathcal{K}/2}\left(1,-i\lambda^1,-i\lambda^2,-i\lambda^3\right)^T\,, \\ \mathcal{M}_\Lambda &= e^{\mathcal{K}/2}\left(-i\left(\lambda^1\lambda^2\lambda^3-\frac{A}{3}(\lambda^3)^3\right),-\lambda^2\lambda^3,-\lambda^1\lambda^3,-\lambda^1\lambda^2+A(\lambda^3)^2\right)^T\,, \end{aligned} \tag{1.207}$$

while the Kähler potential reads

$$e^{-\mathcal{K}}=8\left(\lambda^1\lambda^2\lambda^3-\frac{A}{3}(\lambda^3)^3\right)\,. \tag{1.208}$$

For vanishing axions, the special Kähler metric takes the form

$$\mathcal{G}_{ij^*} = \frac{1}{4\left(\lambda^1\lambda^2\lambda^3 - \frac{A}{3}(\lambda^3)^3\right)^2} \begin{pmatrix} (\lambda^2)^2(\lambda^3)^2 & \frac{A}{3}(\lambda^3)^4 & -\frac{2}{3}A\lambda^2(\lambda^3)^3 \\ \frac{A}{3}(\lambda^3)^4 & (\lambda^1)^2(\lambda^3)^2 & -\frac{2}{3}A\lambda^1(\lambda^3)^3 \\ -\frac{2}{3}A\lambda^2(\lambda^3)^3 & -\frac{2}{3}A\lambda^1(\lambda^3)^3 & (\lambda^1)^2(\lambda^2)^2 + \frac{A^2}{3}(\lambda^3)^4 \end{pmatrix} . \tag{1.209}$$

The symplectic matrix $\mathcal{N}_{\Lambda\Sigma}$ has, in the axion-free case under consideration, vanishing real part, while $\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma}$ is given by

$$\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma} = -\frac{1}{8}e^{-\mathcal{K}} \begin{pmatrix} 1 & 0 \\ 0 & 4\mathcal{G}_{ij^*} \end{pmatrix} , \tag{1.210}$$

which is thus consistently negative definite.

The $ST[2, n_V]$ models

The $ST[2, n_V]$ models are cubic models involving $n_V = n + 1$ vector multiplets. In the next chapter we are showing how they arise when dimensionally reducing along a circle the 6-dimensional theory of supergravity to 5 and then to 4 dimensions.
It admits an SU(2) gauging if $n \geq 4$, so we have been dealing with it when uplifting or reducing solutions of the SEYM theories with SU(2) gauging (section 3.3) and of the SU(2)-FI gauged theory (section 3.4).

The $ST[2, n_V]$ model is described by the prepotential

$$F = -\frac{1}{2}\eta_{\alpha\beta}\frac{\mathcal{X}^1\mathcal{X}^\alpha\mathcal{X}^\beta}{\mathcal{X}^0} , \tag{1.211}$$

where $\eta_{\alpha\beta} = diag(+-\cdots-)$ and α, β take n values from 2 to n_V. The first scalar $Z^1 = \frac{\mathcal{X}^1}{\mathcal{X}^0}$ plays a different role with respect to the other scalars in the model, and parametrizes a SL$(2,\mathbb{R})$/SO(2) coset space. The remaining n scalars $Z^\alpha = \frac{\mathcal{X}^\alpha}{\mathcal{X}^0}$ parametrize instead a SO$(2,n)$/(SO(2)×SO(n)) coset space. The metric is the product of that of the two spaces. In the usual $\mathcal{X}^0 = 1$ gauging, the symplectic section, the Kähler potential and the metric are given by

$$\Omega = \begin{pmatrix} 1 \\ Z^1 \\ Z^\alpha \\ \frac{1}{2}Z^1\eta_{\alpha\beta}Z^\alpha Z^\beta \\ -\frac{1}{2}\eta_{\alpha\beta}Z^\alpha Z^\beta \\ -Z^1\eta_{\alpha\beta}Z^\beta \end{pmatrix} , \qquad e^{-\mathcal{K}} = 4\Im\mathfrak{m}Z^1\eta_{\alpha\beta}\Im\mathfrak{m}Z^\alpha\Im\mathfrak{m}Z^\beta ,$$
$$\mathcal{G}_{11^*} = \frac{1}{4(\Im\mathfrak{m}Z^1)^2} , \qquad \mathcal{G}_{\alpha\beta^*} = \frac{\eta_{\alpha\gamma}\Im\mathfrak{m}Z^\gamma\eta_{\beta\delta}\Im\mathfrak{m}Z^\delta}{(\eta_{\epsilon\phi}\Im\mathfrak{m}Z^\epsilon\Im\mathfrak{m}Z^\phi)^2} - \frac{\eta_{\alpha\beta}}{2\eta_{\epsilon\phi}\Im\mathfrak{m}Z^\epsilon\Im\mathfrak{m}Z^\phi} . \tag{1.212}$$

The reality of the Kähler potential has to be required, giving rise to two different branches

$$\Im\mathfrak{m} Z^1 > 0\,,\ \eta_{\alpha\beta}\Im\mathfrak{m} Z^\alpha \Im\mathfrak{m} Z^\beta > 0 \qquad \text{and} \qquad \Im\mathfrak{m} Z^1 < 0\,,\ \eta_{\alpha\beta}\Im\mathfrak{m} Z^\alpha \Im\mathfrak{m} Z^\beta < 0\,. \tag{1.213}$$

The group SO$(2,n)$ does not act linearly on the special coordinates Z^α; with a symplectic transformation another formulation can be obtained, in which the invariance is evident. This formulation cannot, however, be derived from any prepotential.

When dealing with theories involving an SU(2) gauged subgroup of the isometries of the special Kähler manifold, we should note that the group SO(3) acts in the adjoint on the coordinates $\alpha = 3, 4, 5$, if $n \geq 4$.

The $\overline{\mathbb{CP}}^n$ models

The $\overline{\mathbb{CP}}^n$ family contains models with $n_V = n$ vector multiplets. We are particularly interested in the $\overline{\mathbb{CP}}^3$ case, because it provides the simplest example of a model admitting a SU(2) gauging.

The $\overline{\mathbb{CP}}^n$ models do not belong to the class of the cubic models. Their prepotential is in fact quadratic,

$$F = -\tfrac{i}{4}\eta_{\Lambda\Sigma}\mathcal{X}^\Lambda\mathcal{X}^\Sigma, \qquad (\eta_{\Lambda\Sigma}) = \mathrm{diag}(+-\cdots-)\,. \tag{1.214}$$

We can define the n complex scalars, which parametrize a U$(1,n)/$(U$(1)\times$U$(n))$ coset space, by

$$Z^i \equiv \mathcal{X}^i/\mathcal{X}^0\,. \tag{1.215}$$

It is advantageous to add to these $Z^0 \equiv 1$ and to use Z^Λ and Z_Λ

$$(Z^\Lambda) \equiv \left(\mathcal{X}^\Lambda/\mathcal{X}^0\right) = (1, Z^i)\,, \qquad (Z_\Lambda) \equiv (\eta_{\Lambda\Sigma}Z^\Sigma) = (1, Z_i) = (1, -Z^i)\,. \tag{1.216}$$

The Kähler potential, the Kähler metric (which is the standard Bergman metric for the symmetric space U$(1,n)/$(U$(1)\times$U$(n))$ [81]) and its inverse in the $\mathcal{X}^0 = 1$ gauge are given by

$$\begin{aligned} \mathcal{K} &= -\log\left(Z^{*\Lambda}Z_\Lambda\right), \\ \mathcal{G}_{ij^*} &= e^{\mathcal{K}}\left(\delta_{ij^*} + e^{\mathcal{K}}Z_i^*Z_{j^*}\right), \\ \mathcal{G}^{ij^*} &= e^{-\mathcal{K}}\left(\delta^{ij^*} - Z^iZ^{*j^*}\right), \end{aligned} \tag{1.217}$$

which implies that the complex scalars are constrained to the region

$$0 \leq \sum_i |Z^i|^2 < 1\,. \tag{1.218}$$

The covariantly holomorphic symplectic section $\mathcal{V}$, its Kähler-covariant derivative $\mathcal{U}_i = \mathcal{D}_i\mathcal{V}$ and the period matrix are given by

$$\mathcal{V} = e^{\mathcal{K}/2}\begin{pmatrix} Z^\Lambda \\ \\ -\frac{i}{2}Z_\Lambda \end{pmatrix}\,,\quad \mathcal{U}_i = e^{\mathcal{K}/2}\begin{pmatrix} -e^{\mathcal{K}}Z_i^* Z^\Lambda + \delta_i{}^\Lambda \\ \\ \frac{i}{2}(e^{\mathcal{K}}Z_i^* Z_\Lambda - \eta_{i\Lambda}) \end{pmatrix}\,,\quad \mathcal{N}_{\Lambda\Sigma} = \tfrac{i}{2}\left[\eta_{\Lambda\Sigma} - 2\frac{Z_\Lambda Z_\Sigma}{Z^\Gamma Z_\Gamma}\right]\,. \tag{1.219}$$

For later use we also quote

$$\begin{aligned} \Im\mathfrak{m}\,\mathcal{N}_{\Lambda\Sigma} &= \tfrac{1}{2}\left[\eta_{\Lambda\Sigma} - \left(\frac{Z_\Lambda Z_\Sigma}{Z^\Gamma Z_\Gamma} + \text{c.c}\right)\right]\,, \\ (\Im\mathfrak{m}\,\mathcal{N})^{-1|\Lambda\Sigma} &= 2\left[\eta^{\Lambda\Sigma} - \left(\frac{Z^\Lambda Z^{*\,\Sigma}}{Z^\Gamma Z^*_\Gamma} + \text{c.c}\right)\right]\,, \end{aligned} \tag{1.220}$$

and the Hesse potential

$$\mathsf{W}(\mathcal{I}) = \tfrac{1}{2}\eta_{\Lambda\Sigma}\mathcal{I}^\Lambda\mathcal{I}^\Sigma + 2\eta^{\Lambda\Sigma}\mathcal{I}_\Lambda\mathcal{I}_\Sigma\,, \tag{1.221}$$

where $\mathcal{I}^\Lambda$ were defined in (1.115).

When $n = 3$, the metric is invariant under $\mathrm{U}(1,3) = \mathrm{U}(1) \times \mathrm{SU}(1,3)$, admitting the gauging of the subgroup $\mathrm{SO}(3) \in \mathrm{SU}(1,3)$.

1.3 The 5-dimensional theory

In the context of the present thesis, we are mostly interested in higher dimensional supergravity theories because their compactification gives rise to the 4-dimensional theories we presented so far. Since our purpose is to find new supersymmetric solutions of these latter, the technique of dimensional reduction (see chapter 2) will allow us to generate new 4-dimensional solutions exploiting known higher dimensional ones.
However, the inverse procedure is also possible, and new higher dimensional solutions can be obtained from known 4-dimensional ones -an example is given in section 3.3-. In fact, 4-dimensional solutions are better understood and widely studied [17,23,24,29,82,83].
Moreover, the higher dimensional theories we are interested in naturally arise in superstring or M-theory compactifications and deserve therefore interest on their ones. Many of them can be obtained from compactifications of 11-dimensional supergravity on Calabi–Yau 3-folds.

The supersymmetry parameter and the fermions are given in this theory by just one minimal spinor with eight real components and the theory is in this sense minimal.

However, the spinors are usually taken to be pairs of symplectic Majorana spinors, with symplectic index $i = 1, 2$, therefore the theory deserves the name of $\mathcal{N} = 2,\ d = 5$ supergravity. It involves 8 supercharges, as the $\mathcal{N} = 2,\ d = 4$ theory to which it is related by compactification on a circle.
More details about this theory can be found in [54, 84, 85].

We begin by recalling that the supergravity multiplet consists of the graviton ${e_a}^\mu$, the graviphoton vector fields ${A^0}_\mu$ and the gravitino ${\psi^i}_\mu$. We can couple to it n_V vector supermultiplets, labeled by $x = 1, \ldots, n_V$; each of them includes a vector field ${A^x}_\mu$, a gaugino λ^{xi} and a real scalar ϕ^x.
The most general symmetry of the equations of motion is a subgroup of GL(n_V+1), rotating the graviphoton into the matter vector fields. It is therefore convenient to introduce an index $I = 0, \ldots, n_V$, to treat all the vector fields on equal footing as ${A^I}_\mu = ({A^0}_\mu, {A^x}_\mu)$. The hypermultiplets are exactly the same as in the 4-dimensional theory, see section 1.2.2.

The peculiarity of the 5-dimensional $\mathcal{N} = 2$ supergravity theories is that, if no hypermultiplets are present, the entire Lagrangian for a certain model can be determined once the constant and completely symmetric tensor C_{IJK} is known. As will be detailed in the next chapter, when the 5-dimensional theory is compactified on a circle, a 4-dimensional cubic model is obtained, whose characteristic d_{ijk} tensor is determined (2.40) by the C_{IJK} tensor we just introduced.

In the 5-dimensional theory, the scalars ϕ^x parametrize a real special manifold with σ-model metric $g_{xy}(\phi)$, but they transform non-linearly under the symmetries of the theory; therefore, a redundant parametrization is preferred, in terms of $n_V + 1$ functions of the scalars, $h^I(\phi)$, transforming in the vector representation. These functions can be seen as coordinates of an $n_V + 1$-dimensional Riemannian space with a metric $a_{IJ}(\phi)$, while the scalar manifold is the n_V-dimensional hypersurface in $\mathbb{R}^{n_V+1}$ defined by the cubic equation

$$C_{IJK} h^I(\phi) h^J(\phi) h^K(\phi) = 1, \tag{1.222}$$

where C_{IJK} is the fully symmetric real constant tensor we mentioned before: once it has been fixed, the σ-model metric $g_{xy}(\phi)$ is given by

$$g_{xy} \equiv 3 a_{IJ} \frac{\partial h^I}{\partial \phi^x} \frac{\partial h^J}{\partial \phi^y} = -2 C_{IJK} \frac{\partial h^I}{\partial \phi^x} \frac{\partial h^J}{\partial \phi^y} h^K \,, \tag{1.223}$$

while the metric on the Riemannian space, a_{IJ}, which occurs in the theory as the kinetic matrix of the vector fields is given by

$$a_{IJ} = -2 C_{IJK} h^K + 3 h_I h_J \,, \tag{1.224}$$

where the $h_I(\phi)$ are defined by

$$h_I \equiv C_{IJK} h^J h^K \,. \tag{1.225}$$

The tensor C_{IJK} itself determines the Chern–Simons terms in the supergravity action.6

The bosonic action of the ungauged theory coupled to n_V vector multiplets and n_H hypermultiplets is entirely determined by the tensor C_{IJK} and by the metric of the quaternionic Kähler manifold H_{uv}. It is given by

$$S = \int d^5x \sqrt{|g|} \Big\{ R + \tfrac{1}{2} g_{xy} \partial_\mu \phi^x \partial^\mu \phi^y + \frac{1}{2} \mathsf{H}_{uv} \partial_\mu q^u \partial^\mu q^v \\ - \tfrac{1}{4} a_{IJ} F^I F^J + \frac{\epsilon}{12\sqrt{3}\sqrt{|g|}} C_{IJK} F^I F^J A^K \Big\} . \tag{1.226}$$

1.3.1 Real special geometry

Real special geometry arises from the need to integrate in a single structure the Riemannian metric of the σ-model parametrized by the n_V real scalars ϕ^x with the GL$(n_V + 1)$ structure that controls their coupling to the vector fields, via the kinetic matrix a_{IJ}.

To this purpose, the redundant description of the scalars in terms of the functions h^I -constrained by (1.222)- is introduced. Another possible set of variables is given by h_I in (1.225), such that $h^I h_I = 1$. The sets are related by the metric a_{IJ}, that raises and lowers the I indexes

$$h_I \equiv a_{IJ} h^J , \qquad h^I \equiv a^{IJ} h_J , \qquad a_{IJ} a^{JK} = \delta^I_K . \tag{1.227}$$

The metric of the hypersurface $C_{IJK} h^I h^J h^K = 1$, on which the physical scalars live, is given by g_{xy} in (1.223) and it is the pullback of a_{IJ}. It can be used to raise and lower the x indexes.

1.3.2 The gauged theory

The analysis of the possible gaugings of the 5-dimensional theory is quite similar to that of the 4-dimensional case, developed in section 1.2.3. The symmetries of the action are in fact the isometries of the real special manifold and of the quaternionic Kähler manifold preserving the geometrical structure, and a SU(2) R-symmetry group.

- To gauge the isometries of the real special manifold, we have to require that the Killing vectors of the σ-model metric $g_{xy}(\phi)$, ${k_A}^x(\phi)$, respect the real special structure. To realize it, two conditions have to be met. Firstly, the functions h^I must be invariant under the isometries we are considering up to GL$(n_V + 1)$ rotations. Their covariant Lie derivative should vanish,

$$\mathbb{L}_A h^I \equiv (\mathcal{L}_A - T_A) h^I = {k_A}^x \partial_x h^I - (T_A)^I{}_J h^J = 0 , \tag{1.228}$$

where T_A are a matrix representation of the Lie algebra isometry group

$$[k_A, k_B] = -f_{AB}{}^C k_C \,, \qquad [T_A, T_B] = f_{AB}{}^C T_C \,. \tag{1.229}$$

It can be demonstrated, on the base of special geometrical identities, that this implies that the Killing vectors are given by

$$k_A{}^x = -\sqrt{3}(T_A)^I{}_J g^{xy} \partial_y h_I h^J \,. \tag{1.230}$$

The second condition imposes that the tensor C_{IJK} must be invariant under the compensating $\mathrm{GL}(n_V + 1)$ transformations

$$\mathbb{L}_A C_{IJK} = 3(T_A)^L{}_{(I} C_{JK)L} = 0 \,. \tag{1.231}$$

The two conditions imply that the kinetic matrix a_{IJ} is also invariant,

$$\mathbb{L}_A a_{IJ} = (\mathcal{L}_A - T_A) a_{IJ} = k_A{}^x \partial_x a_{IJ} + 2(T_A)^K{}_{(I} a_{J)K} = 0 \,. \tag{1.232}$$

- Since the hypermultiplets are the same as in the 4-dimensional theory, the gauging of the isometries of the quaternionic Kähler manifold proceeds as already reviewed in section 1.2.3. Again, even if no hypermultiplets are present, non-vanishing constant triholomorphic moment maps lead to the appearance of Fayet-Iliopoulos terms, that gauge the whole $\mathrm{SU}(2)$ R-symmetry group or a $\mathrm{U}(1)$ subgroup only.

The gauge covariant derivatives of the scalars and the vector field strengths are given by the standard expressions

$$\begin{aligned} \mathfrak{D}_\mu \phi^x &= \partial_\mu \phi^x + g A^I{}_\mu k_I{}^x \,, \\ \mathfrak{D}_\mu q^u &= \partial_\mu q^u + g A^I{}_\mu \mathsf{k}_I{}^u \,, \\ F^I{}_{\mu\nu} &= 2\partial_{[\mu} A^I{}_{\nu]} + g f_{JK}{}^I A^J{}_{[\mu} A^K{}_{\nu]} \,. \end{aligned} \tag{1.233}$$

The generic Lagrangian of the gauged theory is then given by

$$\begin{aligned} S = \int d^5x \sqrt{g} \Bigg\{ & R + \frac{1}{2} g_{xy} \mathfrak{D}_\mu \phi^x \mathfrak{D}^\mu \phi^y + \frac{1}{2} \mathsf{H}_{uv} \mathfrak{D}_\mu q^u \mathfrak{D}^\mu q^v - \mathbf{V}(\phi, q) - \frac{1}{4} a_{IJ} F^I F^J \\ & + \frac{1}{12\sqrt{3}} C_{IJK} \frac{\epsilon}{\sqrt{g}} \Big(F^I F^J A^K - \frac{1}{2} g f_{LM}{}^I F^J A^K A^L A^M \\ & + \frac{1}{10} g^2 f_{LM}{}^I f_{NP}{}^J A^K A^L A^M A^N A^P \Big) \Bigg\} \,, \end{aligned} \tag{1.234}$$

where

$$\begin{aligned}\mathbf{V}(\phi,q) = &-g^2\Big(\left(4h^I h^J - 2g^{xy}\partial_x h^I \partial_y h^J\right)\vec{\mathsf{P}}_I\vec{\mathsf{P}}_J \\ &-\frac{3}{2}h^I h^J\left(k_I{}^x k_J{}^y g_{xy} + \mathsf{k}_I{}^u \mathsf{k}_J{}^v \mathsf{H}_{uv}\right)\Big)\,.\end{aligned} \tag{1.235}$$

The main difference with respect to the 4-dimensional case is due to the presence of the Chern-Simons term, containing not only the vector field strengths, but also the vector fields themselves. The gauge covariantization of this term provides more terms to the Lagrangian of the gauged theory.

The supersymmetric solutions of general models of gauged $\mathcal{N}=2, d=5$ supergravity were classified in [34, 35], but the construction of explicit examples in the theories with non-Abelian gaugings has only been successfully completed recently in [30, 86]. These theories are the simplest supersymmetrization of the Einstein-Yang-Mills (EYM) theory and have been called $\mathcal{N}=2, d=5$ Super-Einstein-Yang-Mills (SEYM) theories in [30].

1.4 The 6-dimensional theory

The minimal supergravity theory, first constructed in [87] by dimensional reduction from 11-dimensional supergravity [88], contains the graviton $e_a{}^\mu$, the gravitino $\psi^i{}_\mu$ and a 2-form $B^-{}_{\mu\nu}$ with anti-selfdual 3-form field strength $H^-{}_{\mu\nu\rho}$ and it does not admit a covariant action, which makes it more complicated to work with.

This theory can be coupled to different matter multiplets:

- n_V vector multiplets, which have no scalars. Each multiplet includes a vector field $A^i{}_\mu$ and a gaugino $\lambda^{\alpha i}$, where $i=1,\ldots,n_V$;

- n_T tensor multiplets, which have real scalars $\varphi^{\underline{r}}, \underline{r}=1,\ldots n_T$, always parameterizing the symmetric space $\mathrm{SO}(1,n_T)/\mathrm{SO}(n_T)$, tensorinos $\chi^{\alpha\underline{r}}$ and 2-forms $B^{+\underline{r}}{}_{\mu\nu}$ whose 3-form field strengths $H^{\underline{r}}{}_{\mu\nu\rho}$ are selfdual;

- n_H hypermultiplets, whose scalars parametrize arbitrary quaternionic-Kähler manifolds, as in lower dimensional theories.

The scalar fields $\varphi^{\underline{r}}$ can be seen as coordinates in the coset space $\mathrm{SO}(1,n_T)/\mathrm{SO}(n_T)$. In describing the couplings of the tensor multiplets, it is useful to introduce the coset representatives L_r and $L_r{}^R$, which together form an $(n_T+1)\times(n_T+1)$ matrix obeying the properties of an $\mathrm{SO}(n_T,1)$ group element. The index $r=0,\ldots,n_T$ labels the

fundamental representation of SO$(n_T, 1)$ while the index $R = 1, \cdots, n_T$ labels the fundamental representation of SO(n_T). Denoting the components of the inverse matrix by L^r and $L_R{}^r$, they obey the relations

$$L_r L^r = 1\,, \qquad L_R{}^r L_r = 0\,, \qquad L_r{}^R L^r = 0\,. \tag{1.236}$$

The SO$(n_T, 1)$ invariant constant metric

$$\eta_{rs} = L_r L_s - L_r{}^R L_{sR}\,, \qquad \eta_{rs} = \mathit{diag}(1, -1, \ldots, -1)\,, \tag{1.237}$$

can be used to raise and lower the SO$(n_T, 1)$ vector indexes,

$$L_r = \eta_{rs} L^s\,, \qquad L_{rR} = -\eta_{rs} L_R{}^s\,. \tag{1.238}$$

Another useful quantity is the symmetric SO$(1, n_T)$ matrix $\mathcal{M}_{rs}$, defined by

$$\mathcal{M}_{rs} = L_r L_s + L_r{}^R L_{sR} = 2 L_r L_s - \eta_{rs}\,, \tag{1.239}$$

that depends on the coordinates $\varphi^{\underline{r}}$.

An SO$(1, n_T)$-symmetric σ-model for the scalars $\varphi^{\underline{r}}$ can be constructed as

$$L_s{}^r \partial_a L^s{}_t L_u{}^t \partial^a L^u{}_r = -\partial_a L^r \partial^a L_r\,, \tag{1.240}$$

where we have used the above properties of the coset representative and $L_r = L_r{}^0$.
A simple parametrization of the functions L^r in terms of the physical scalars is provided by

$$L^0 = (1 - \varphi^{\underline{r}}\varphi^{\underline{r}})^{-1/2}\,, \qquad L^{\underline{r}} = \varphi^{\underline{r}}(1 - \varphi^{\underline{s}}\varphi^{\underline{s}})^{-1/2} \quad \Rightarrow \quad \varphi^{\underline{r}} = \frac{L^{\underline{r}}}{L^0}\,. \tag{1.241}$$

The matter and supergravity 2-forms are combined into a single SO$(1, n_T)$ vector $(B^r) = (B^0, B^{\underline{r}})$, with 3-form field strengths $H^r = \frac{1}{3!} H^r{}_{\mu\nu\rho}\, dx^\mu \wedge dx^\nu \wedge dx^\rho$ defined by

$$H^r = dB^r + \tfrac{1}{2} c^r{}_{ij} F^i \wedge A^j\,, \tag{1.242}$$

where $c^r{}_{ij}$ is an array of constant positive-definite matrices. They have to satisfy the (anti)selfduality constraint

$$\mathcal{M}_{rs} H^s = -\eta_{rs} \star H^s\,. \tag{1.243}$$

Using this constraint in the Bianchi identity of the 3-form field strengths

$$dH^r - \tfrac{1}{2} c^r{}_{ij} F^i \wedge F^j = 0\,, \tag{1.244}$$

the equation of motion for the 2-forms is obtained,

$$d\left(\mathcal{M}_{rs} \star H^s\right) + \tfrac{1}{2} c_{r\,ij} F^i \wedge F^j = 0\,. \tag{1.245}$$

Although a general covariant action does not exists, the bosonic equations of motion can be derived from the pseudoaction

$$\begin{aligned}\hat{S} = \int d^6x \sqrt{|g|} \Big\{ & R - \partial_a L^r \partial^a L_r + \tfrac{1}{3} \mathcal{M}_{rs} H^r{}_{abc} H^{s\,abc} + \frac{1}{2} \mathsf{H}_{uv} \partial_\mu q^u \partial^\mu q^v \\ & - L_r c^r{}_{ij} F^i{}_{ab} F^{j\,ab} - \tfrac{1}{4} c_{r\,ij} \epsilon^{abcdef} B^r{}_{ab} F^i{}_{cd} F^j{}_{ef} \Big\}\,.\end{aligned} \tag{1.246}$$

once the (anti)selfduality constraint (1.243) has been imposed.

One way to avoid the complications of having to deal with chiral 2-forms[15] is to consider supergravity theories coupled to just one tensor multiplet so the two chiral 2-forms of opposite chiralities combine into one unconstrained 2-form. In what follows, we have been calling them $\mathcal{N} = 2A\,, d = 6$ supergravity theories. They can describe the effective theory of the truncated, toroidally compactified Heterotic String, whose field content consists of the metric, a Kalb-Ramond 2-form and a dilaton. These theories, coupled to vector multiplets and hypermultiplets were constructed in [89–91].
The $\mathcal{N} = 2A, d = 6$ theories can be gauged in essentially two ways:

- We could just gauge a subgroup of the $\mathrm{SO}(n_V)$ group that rotates the vector fields among themselves. The only fermion fields this global symmetry acts on are the gaugini, which carry the same indexes as the vector fields and an $\mathrm{Sp}(1) \sim \mathrm{SU}(2)$ R-symmetry index which remains inert under these transformations. Observe that the only scalar of the theory, the dilaton, is also inert.

- We can gauge the whole R-symmetry group, $\mathrm{SO}(3)$ or a $\mathrm{SO}(2)$ subgroup of it using Fayet-Iliopoulos terms, as in [38]. Observe that vectors transforming in the same fashion are needed. Thus, in this case one would be gauging $\mathrm{SO}(3)$ or a $\mathrm{SO}(2)$ subgroup of $\mathrm{SO}(n_V)$ which, on top of acting on some the $\mathrm{SO}(n_V)$ indexes of the vectors and gaugini, would also act on the R-symmetry indexes of all the fermions of the theory, which would now be charged.

The theories we are calling $\mathcal{N} = 2B$ are instead coupled to an arbitrary number of tensor multiplets. They were described in [92] and have attracted much less attention because they have not been identified as the effective field theory of some string or M-theory compactification yet and they cannot be gauged, at least in any conventional sense, because they do not have vectors that can be used as gauge fields. A meaning for the "gauged" $\mathcal{N} = 2B$ theory will however emerge from dimensional reduction, as explained in section 3.3.2. The coupling to tensors, vectors and hypermultiplets with some gaugings was described in [93].

[15] That is: 2-form potentials with selfdual or anti-selfdual 3-form field strengths.

The supersymmetric solutions of most of $\mathcal{N} = (2,0), d = 6$ supergravity theories have not been classified yet. The only ones that have been considered are the pure supergravity theory in [37,94] and a theory with one tensor multiplet and a triplet of vector multiplets with $\mathrm{SU}(2)$ and $\mathrm{U}(1)$ gaugings via Fayet-Iliopoulos terms in [38].

CHAPTER 2

Dimensional reduction

The present chapter is devoted to the presentation of the technique of dimensional reduction on a circle and to its application to the theories we have presented so far, the 6, 5 and 4-dimensional $\mathcal{N} = 2$ supergravities without hypermultiplets. The underlying idea is that certain appropriate models in these theories are related among themselves by dimensional reduction and, when admitting isometries, so are their solutions.
We are initially developing this technique to connect ungauged theories, but we are later explaining how certain gaugings do not affect the treatment we are presenting. In particular, if the gauge group does not change in the procedure of dimensional reduction, Super-Einstein-Yang-Mills theories can be reduced or uplifted exploiting exactly the same rules we have obtained in the ungauged case, giving rise to the same kind of theory in different dimensions. Analogously, a Fayet-Iliopoulos term only produces a potential in the action, which can be independently reduced to produce the potential due to the same FI-gauging in one dimension less.

In the framework of the present thesis, the interest for this procedure is determined by the possibility of generating new solutions in diverse dimensions, once a certain solution is known. In general, the supersymmetric solutions of theories related by dimensional reduction are also related: all the solutions of the lower dimensional theory can be uplifted to supersymmetric solutions of the higher dimensional theory while all the supersymmetric solutions of the higher dimensional theory admitting translational isometries [95] can be reduced along the associated directions to supersymmetric solutions of the lower dimensional theories[1]. Thus, one can get new supersymmetric solutions of one of the theories from known supersymmetric solutions of the other one. Of course, the same can be done with non-supersymmetric solutions.

Two conditions have to be met in order to apply this simple solution-generating technique:

[1]In the case of toroidal compactification. The general condition is that the Killing spinors of the higher dimensional solutions can also be understood as spinors of the lower dimensional theory. This requires the spinors to have a particular dependence (or independence) on the coordinates of the compactification manifold which, in turn, requires the solution to meet certain conditions. In toroidal compactifications the isometries associated to the circles must act without fixed points (be translational isometries). In more general cases the conditions have not been studied. Dimensional reduction can, in general, break symmetries.

1. We need to know which theories are related by dimensional reduction.
2. The detailed relation ("dictionary") between the fields of the higher and lower dimensional theories must also be known.

This way of obtaining new solutions discloses its power when other methods, as those presented in the previous chapter, fail; this can happen if no classification of the solutions is available, as for the 6-dimensional theory, or if the equations are particularly involved. In the 4-dimensional SU(2)-FI-gauged theory, for example, few solutions were found through direct methods and we have been able to handle the simplest model only (see section 3.2). However, thanks to the method that we are presenting here, new solutions were generated from previously known 6-dimensional ones [38], for the 5 and 4-dimensional theories, in section 3.4.

The rest of the present chapter is dedicated to summarize the general principles of the dimensional reduction on a circle (section 2.1), followed by the explicit derivation of the rules that relate the 5-dimensional $\mathcal{N} = 2$ supergravity theory to the 4-dimensional cubic models (section 2.2). The results in the remaining sections were presented in [2]: the purpose is to dimensionally reduce the 6-dimensional theory to 5-dimensions and to identify which models in the lower dimensional case can be obtained in this way. Of course, the absence of a general covariant action in 6-dimensions has to be taken into account, and the reduction has to be performed on the pseudo-action and on the (anti)selfduality constraints, as is going to be explained in full detail in section 2.3. A peculiarity that emerges in this analysis is the existence of two different 6-dimensional models related to the same 5-dimensional one. This fact has been interpreted as the existence of a duality between the two theories, analogous to the one connecting the two maximal 10-dimensional supergravities, $\mathcal{N} = 2A$ and $\mathcal{N} = 2B$: when compactified on a circle, they give rise to the same 9-dimensional maximal supergravity, which is unique. Through the lower dimensional theory, we can give the rules transforming a solution of one of the 6-dimensional theories admitting one isometry into another solution of the other theory, also admitting one isometry (section 2.5).

2.1 Compactification on a circle

The present paragraph is devoted to the description of the formalism developed by Scherk and Schwarz in [39] to perform the dimensional reduction of the gravitational field from $\hat{d}$ to $d = \hat{d} - 1$ dimensions. The first condition that has to be met, in order to proceed with this program, is that the dynamics of the problem under study is irrelevant in at least one direction, that we are denoting as[2] z.

One of the reasons that raised interest in theories involving extra dimensions is the possibility of interpreting gauge symmetries in the lower dimensional theories as spacetime

[2] All $\hat{d}$-dimensional objects carry a hat, whereas $d = (\hat{d} - 1)$-dimensional ones do not. The $\hat{d}$-dimensional curved indexes split into $\hat{\mu} = (\mu, \underline{z})$, while the tangent space ones are $\hat{a} = (a, z)$.

symmetries in the extra dimensions, unifying in this way all the symmetries. In the original theory of compactification by Kaluza and Klein, the extra dimensions are curled up in a small compact manifold, the simplest example of which is a circle, and the motion of particles along it is not observable.
In the full theory of compactification, all modes of the involved field -the initial and the reduced ones- should be taken into account, both massless and massive. However, the effective theory describing the low-energy behavior of the full theory is obtained when all massive modes are ignored and only the massless spectrum is kept. This is equivalent to ignoring all dynamics in the internal dimensions and is called dimensional reduction. This is the only consistent truncation of the full theory.
We would like to point out that, as a general rule, a truncation cannot follow from setting some field to a constant value at the level of the action, since not all the truncated equations of motion can be inferred from a truncated action.

A guiding principle for this technique is that the number of degrees of freedom, or helicity states, should be conserved by dimensional reduction. Thus, for example, a massless mode of the graviton in 5 dimensions $\hat{g}_{\hat{\mu}\hat{\nu}}$, that has 5 helicity states, gives rise in 4 dimensions to a massless graviton $g_{\mu\nu}$ with 2 helicity states, a massless vector A_μ carrying 2 helicity states and a massless scalar k with 1 helicity state. The massless spectrum of the reduced theory is then given by $\{g_{\mu\nu}, A_\mu, k\}$; the vector and the scalar are often dubbed Kaluza-Klein (KK) vector and scalar, respectively.
In a more general case, a $\hat{d}$-dimensional massless graviton has $\frac{\hat{d}(\hat{d}-3)}{2}$ helicity states, while a $(p+1)$-form has $\frac{(\hat{d}-2)!}{(p+1)!(\hat{d}-p-3)!}$ helicity states. In particular, a massless vector $(p=0)$ has $\hat{d}-2$ degrees of freedom and a scalar particle $(p=-1)$ always has one. Therefore, a $\hat{d}$-dimensional massless graviton is decomposed, in d dimensions, into a massless graviton, a massless vector field and a scalar; a massless $\hat{d}$-dimensional $(p+1)$-form in a d-dimensional $(p+1)$-form and in a p-form; scalars are essentially unaffected.

We are now reducing the massless mode of the $\hat{d}$-dimensional graviton field to $d = \hat{d}-1$ dimensions following [39]. It cannot depend on the compact coordinate z. We are also identifying the d-dimensional fields in which the graviton has split.
We start from a z-independent metric $\hat{g}_{\mu\nu}$, which admits a Killing vector $\hat{k}^{\hat{\mu}}$,

$$\hat{\nabla}_{(\hat{\mu}}\hat{k}_{\hat{\nu})} = 0\,. \tag{2.1}$$

The d-dimensional spacetime for the lower dimensional theory is defined by those hypersurfaces that are orthogonal to the Killing vector. The induced metric on them is

$$\hat{\Pi}_{\hat{\mu}\hat{\nu}} = \hat{g}_{\hat{\mu}\hat{\nu}} - \left(\hat{k}^{\hat{\lambda}}\hat{k}_{\hat{\lambda}}\right)^{-1}\hat{k}_{\hat{\mu}}\hat{k}_{\hat{\nu}}\,. \tag{2.2}$$

In adapted coordinates, where $\hat{k}^{\hat{\mu}} = \delta^{\hat{\mu}}_{\underline{\hat{z}}}$, and keeping in mind that $\hat{\Pi}$ and $\hat{k}$ are orthogonal, we have

$$k = \left|\hat{k}^{\hat{\lambda}}\hat{k}_{\hat{\lambda}}\right|^{\frac{1}{2}} = \left|\hat{g}_{\underline{zz}}\right|^{\frac{1}{2}} , \qquad \hat{\Pi}_{\hat{\mu}\underline{z}} = 0 , \qquad g_{\mu\nu} \equiv \hat{\Pi}_{\mu\nu} . \tag{2.3}$$

Taking into account that the fields should have the right transformation properties, the lower dimensional fields are given by the following natural combinations of higher dimensional components of the metric

$$\begin{aligned} g_{\mu\nu} &= \hat{g}_{\mu\nu} - \frac{\hat{g}_{\mu\underline{z}}\hat{g}_{\nu\underline{z}}}{\hat{g}_{\underline{zz}}} , \\ A_\mu &= \frac{\hat{g}_{\mu\underline{z}}}{\hat{g}_{\underline{zz}}} , \\ k &= \left|\hat{g}_{\underline{zz}}\right|^{\frac{1}{2}} . \end{aligned} \tag{2.4}$$

The inverse transformations, that express the higher dimensional fields in terms of the lower dimensional ones, can immediately be obtained

$$\begin{aligned} \hat{g}_{\mu\nu} &= g_{\mu\nu} - k^2 A_\mu A_\nu , \\ \hat{g}_{\mu\underline{z}} &= -k^2 A_\mu , \\ \hat{g}_{\underline{zz}} &= -k^2 , \end{aligned} \tag{2.5}$$

or, equivalently

$$d\hat{s}^2 = ds^2 - k^2 \left(dz + A\right)^2 . \tag{2.6}$$

In general, after dimensional reductions, global internal transformations give rise to global symmetries of the lower dimensional theory, which rescale and/or rotate the fields among themselves. In particular, these symmetries act on the scalar fields and thus scalars naturally parametrize a σ-model. In the case under study, the scalar k parametrizes a σ-model with target space $\mathbb{R}^+$.

The relations (2.4) and (2.5) provide the first example of what we have been calling a *dictionary*, a complete set of rules relating the fields of two different theories, which allows mapping the equations of motion of one theory to the equations of motion of the other one. The procedure we followed automatically ensures that any field configuration that solves the lower dimensional equations of motion also solves the higher dimensional equations of motion, once the relations for the fields are used.
However, performing the dimensional reduction on the equations of motion is in general a quite lengthy calculation. To avoid it, Scherk and Schwarz proposed in [39] a systematic procedure to dimensional reduce the action. Moreover, this procedure employs the Vielbein formalism, so it can also be applied to fermions. In what follows we will summarize their procedure.

The relations (2.4, 2.5) we found are conveniently translated to the Vielbein formalism as

$$\left(\hat{e}_{\hat{\mu}}{}^{\hat{a}}\right) = \begin{pmatrix} e_\mu{}^a & kA_\mu \\ 0 & k \end{pmatrix}, \qquad \left(\hat{e}_{\hat{a}}{}^{\hat{\mu}}\right) = \begin{pmatrix} e_a{}^\mu & -A_a \\ 0 & k^{-1} \end{pmatrix}. \tag{2.7}$$

We use d-dimensional tangent space indexes for d-dimensional fields that have been contracted with the d-dimensional Vielbeins, as in $A_a = e_a{}^\mu A_\mu$.
Local Lorentz rotations assures the generality of the upper triangular form, which is however preserved by a d-dimensional Lorentz subgroup only. In presence of other symmetries, we would have to add compensating Lorentz transformations to preserve this choice for the Vielbein.

In order to reduce the action

$$\hat{S} = \int d^{\hat{d}}\hat{x}\sqrt{|\hat{g}|}\hat{R}\,, \tag{2.8}$$

we first eliminate the derivatives of the spin connection in

$$\hat{R} = 2\hat{e}_{\hat{a}}{}^{\hat{\mu}}\hat{e}_{\hat{b}}{}^{\hat{\nu}}\partial_{[\hat{\mu}}\hat{\omega}_{\hat{\nu}]}{}^{\hat{a}\hat{b}} + \hat{\omega}_{\hat{a}}{}^{\hat{a}\hat{c}}\hat{\omega}_{\hat{b}}{}^{\hat{b}}{}_{\hat{c}} + \hat{\omega}_{\hat{b}}{}^{\hat{a}\hat{c}}\hat{\omega}_{\hat{a}\hat{c}}{}^{\hat{b}} \tag{2.9}$$

by integrating by parts, to obtain the Palatini identity

$$\begin{aligned}\int d^{\hat{d}}\hat{x}\sqrt{|\hat{g}|}K\hat{R} = \int d^{\hat{d}}\hat{x}\sqrt{|\hat{g}|}K\big(&-2\partial_{[\hat{\mu}|}\left(\hat{e}_{\hat{a}}{}^{\hat{\mu}}\hat{e}_{\hat{b}}{}^{\hat{\nu}}\right)\hat{\omega}_{|\hat{\nu}]}{}^{\hat{a}\hat{b}} + 2\hat{\omega}_{\hat{a}}{}^{\hat{a}\hat{b}}\left(\partial_{\hat{b}}\ln K\right) \\ &+2\hat{e}_{\hat{b}}{}^{\hat{\mu}}\partial_{\hat{c}}\hat{e}_{\hat{\mu}}{}^{\hat{c}}\hat{\omega}_{\hat{a}}{}^{\hat{a}\hat{b}} - \hat{\omega}_{\hat{b}}{}^{\hat{b}\hat{a}}\hat{\omega}_{\hat{c}}{}^{\hat{c}}{}_{\hat{a}} - \hat{\omega}_{\hat{a}}{}^{\hat{b}\hat{c}}\hat{\omega}_{\hat{c}\hat{b}}{}^{\hat{a}}\big)\,.\end{aligned} \tag{2.10}$$

We evaluate the non-vanishing components of $\hat{\Omega}_{\hat{a}\hat{b}\hat{c}} = \hat{e}^{\hat{\mu}}{}_{\hat{a}}\hat{e}^{\hat{\nu}}{}_{\hat{b}}\partial_{[\hat{\mu}}\hat{e}_{\hat{\nu}]\hat{c}}$ exploiting the form of the Vielbein (2.7)

$$\hat{\Omega}_{abc} = \Omega_{abc}\,, \qquad \hat{\Omega}_{abz} = -\frac{1}{2}kF_{ab}\,, \qquad \hat{\Omega}_{azz} = -\frac{1}{2}\partial_a\ln k\,, \tag{2.11}$$

where $F_{ab} = e_a{}^\mu e_b{}^\nu F_{\mu\nu}$ and $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$, and of the spin connection

$$\begin{aligned}\hat{\omega}_{abc} &= \omega_{abc}\,, & \hat{\omega}_{abz} &= \frac{1}{2}kF_{ab}\,, \\ \hat{\omega}_{zbc} &= -\frac{1}{2}kF_{bc}\,, & \hat{\omega}_{zbz} &= -\partial_b\ln k\,.\end{aligned} \tag{2.12}$$

Moreover

$$\sqrt{|\hat{g}|} = \sqrt{|g|}k\,. \tag{2.13}$$

When all these results are plugged into (2.10), with $K = 1$, the $\hat{d}$-dimensional action is expressed in terms of the d-dimensional quantities as

$$\begin{aligned}\hat{S} = &\int d^{\hat{d}}\hat{x}\sqrt{|\hat{g}|}\hat{R} = \\ &\int dz\int d^dx\sqrt{|g|}k\left(-\omega_b{}^{ba}\omega_c{}^c{}_a - \omega_a{}^{bc}\omega_{bc}{}^a + 2\omega_b{}^{ba}\partial_a\ln k - \frac{1}{4}k^2F^2\right)\,.\end{aligned} \tag{2.14}$$

Nothing depends on the z coordinate and we can integrate over it, obtaining a constant factor that will be ignored. Using the Palatini identity (2.10) in d dimensions, with $K = k$, the d-dimensional action takes a simpler form

$$S = \int d^d x \sqrt{|g|} k \left(R - \frac{1}{4} k^2 F^2 \right) . \tag{2.15}$$

The result is correct up to total derivatives, that have been ignored when deriving (2.10).

Even if the way in which the KK scalar appears in this action may look unconventional, since no kinetic term is present, its equation of motion is the standard one and is implicit in the Einstein's equations.
Another way to see that the KK scalar is dynamical is to rescale the metric to the so-called Einstein conformal frame. By definition, this frame is the one in which the Einstein-Hilbert action has the standard form, with no k factors appearing in front of the Ricci scalar. The metric $g_{\mu\nu}$ is related to the Einstein metric $g_{E\mu\nu}$ by the conformal factor Ω as

$$g_{\mu\nu} = \Omega^2 g_{E\mu\nu} , \tag{2.16}$$

while for the Ricci scalars we can exploit the identity

$$R = \Omega^{-2} \left(R_E + (d-1)(d-2) \left(\partial \ln \Omega \right)^2 + 2(d-1) \nabla_E^2 \ln \Omega \right) . \tag{2.17}$$

If the choice $\Omega = k^{-\frac{1}{d-2}}$ is made, the action in the Einstein frame takes the conventional form

$$S_E = \int d^d x \sqrt{|g_E|} \left(R_E + \frac{d-1}{d-2} k^{-2} \left(\partial k \right)^2 - \frac{1}{4} k^{2\frac{d-1}{d-2}} F^2 \right) . \tag{2.18}$$

To deal with a scalar presenting the standard kinetic term, the change of variables

$$k = e^{\pm \sqrt{2\frac{d-2}{d-1}} \phi} \tag{2.19}$$

can be performed, to give the following action

$$S_E = \int d^d x \sqrt{|g_E|} \left(R_E + 2 \left(\partial \phi \right)^2 - \frac{1}{4} e^{\pm 2 \sqrt{2\frac{d-1}{d-2}} \phi} F^2 \right) . \tag{2.20}$$

In the following sections we are showing how the bosonic actions of 6-dimensional $\mathcal{N} = (2,0)$ and the 5 and 4-dimensional $\mathcal{N} = 2$ supergravity theories are related through dimensional reduction on circle. Thanks to these examples, we are clarifying how the reduction proceeds when fields other than the gravitational one are involved. In fact scalars, vector fields and 2 and 3-form potentials will be present.

2.2 From 5 to 4 dimensions

In the present section[3], our purpose is to dimensionally reduce the ungauged $\mathcal{N} = 2$ 5-dimensional supergravity theory coupled to an arbitrary number n_V of vector multiplets, to 4 dimensions. The first step consists in dimensionally reducing the bosonic action; as far as the Einstein-Hilbert term is concerned, we are relying on the analysis that we have already performed, while we are here detailing how the remaining terms, involving scalars and vector fields, are reduced. Once a 4-dimensional action has been obtained, we have to identify in it the physical fields of 4-dimensional $\mathcal{N} = 2$ supergravity, in presence of vector multiplets. This rewriting of the action in terms of the appropriate degrees of freedom will let us notice that only certain models of 4-dimensional supergravity can be obtained by dimensional reduction, the cubic ones. In particular, the outlined procedure is providing us with a precise relation between the C_{IJK} tensor defining the 5-dimensional model and the d_{ijk} tensor (1.180) in the prepotential of a cubic 4-dimensional model.

As already stated, the dimensional reduction accounts for a compactification and a truncation. If the massless sector only is kept, the 5-dimensional graviton $\hat{g}_{\hat{\mu}\hat{\nu}}$ will give rise to a 4-dimensional graviton $g_{\mu\nu}$, a KK vector field A_μ and a KK scalar k ; every vector field $\hat{A}^I_{\hat{\mu}}$ is going to be reduced in the same way, independently on whether they pertain to the gravity multiplet, $\hat{A}^0_{\hat{\mu}}$, or to one out of the n_{V_5} vector multiplets, providing a vector and a scalar to the lower dimensional theory. Scalars are substantially unaffected. Upon combining these fields to recover the action of a 4-dimensional supergravity, we are left with the set $\left\{g_{\mu\nu}, A^\Lambda_\mu, Z^i\right\}$, where $\Lambda = \{0, i\} = \{0, I+1\}$ takes $n_{V_5} + 2$ values. In other words, the resulting theory has the field content of a 4-dimensional $\mathcal{N} = 2$ supergravity coupled to $n_{V_4} = n_{V_5} + 1$ vector multiplets.

2.2.1 Reduction of the fields

We start from the action (1.226)

$$\hat{S} = \int d^5x \sqrt{|\hat{g}|} \left\{ \hat{R} + \frac{1}{2}\hat{g}_{xy}\partial_{\hat{\mu}}\hat{\phi}^x \partial^{\hat{\mu}}\hat{\phi}^y - \frac{1}{4}\hat{a}_{IJ}\hat{F}^I\hat{F}^J + \frac{\epsilon}{12\sqrt{3}\sqrt{|\hat{g}|}} C_{IJK}\hat{F}^I\hat{F}^J\hat{A}^K \right\}, \tag{2.21}$$

and we assume that the fields do not depend on z. We keep in mind the results for the Einstein-Hilbert term (2.4,2.5).

We now turn our attention to the term involving the non-linear σ-model; exploiting the properties of real special geometry, we can introduce the functions $\hat{h}^I(\hat{\phi})$, where $I = 0, \ldots, n_{V_5}$ while $x = 1, \ldots, n_{V_5}$, with the constraint

$$C_{IJK}\hat{h}^I\hat{h}^J\hat{h}^K = 1 \tag{2.22}$$

[3]We are here following [54]

in order to rewrite the kinetic term for the scalars as

$$\frac{1}{2}\hat{g}_{xy}\partial_{\hat{\mu}}\hat{\phi}^x\partial^{\hat{\mu}}\hat{\phi}^y = \frac{3}{2}\hat{a}_{IJ}\partial_{\hat{\mu}}\hat{h}^I\partial^{\hat{\mu}}\hat{h}^J\,, \tag{2.23}$$

where $\hat{a}_{IJ}$ is the same scalar-dependent kinetic matrix of the vector fields. We recall that the curved indexes are denoted by Greek letters, and split in $\hat{\mu} = \{\mu, \underline{z}\}$, while the flat indexes are $\hat{a} = \{a, z\}$.
Every $\hat{h}^I$ gives rise to a 4-dimensional scalar h^I, such that

$$\hat{h}^I = h^I\,, \qquad \hat{a}_{IJ} = a_{IJ}\,, \qquad \partial_z\hat{h}^I = 0\,, \qquad \partial_{\hat{a}}\hat{h}^I = \partial_a\hat{h}^I = \partial_a h^I\,, \tag{2.24}$$

and the kinetic term in the action is dimensionally reduced to

$$\hat{a}_{IJ}\partial_{\hat{\mu}}\hat{h}^I\partial^{\hat{\mu}}\hat{h}^J = a_{IJ}\partial_\mu h^I\partial^\mu h^J\,. \tag{2.25}$$

As far as the vector fields are concerned, in accord with the Scherk-Schwarz formalism, we use flat indexes to identify the lower dimensional field $A^I_a = \hat{A}^I_a$, which is given by

$$e_a{}^\mu A^I_\mu \equiv \hat{e}_a{}^{\hat{\mu}}\hat{A}^I_{\hat{\mu}} = e_a{}^\mu\left(\hat{A}^I_\mu - \hat{A}^I_{\underline{z}}A_\mu\right)\,, \tag{2.26}$$

while the $\hat{A}^I_{\underline{z}}$ components become d-dimensional massless scalars l^I. If (2.4,2.5) are exploited, the dictionary to reduce and oxidize vector fields is given by

$$\begin{aligned} \hat{A}^I_\mu &= A^I_\mu + l^I A_\mu\,, & \hat{A}^I_{\underline{z}} &= l^I\,, \\ A^I_\mu &= \hat{A}^I_\mu - \hat{A}^I_{\underline{z}}\frac{\hat{g}_{\mu\underline{z}}}{\hat{g}_{\underline{zz}}}\,, & l^I &= \hat{A}^I_{\underline{z}}\,. \end{aligned} \tag{2.27}$$

The d-dimensional field strengths are obtained as

$$\begin{aligned} \hat{F}^I_{ab} &= \hat{e}_a{}^{\hat{\mu}}\hat{e}_b{}^{\hat{\nu}}\hat{F}^I_{\hat{\mu}\hat{\nu}} \\ &= 2\hat{e}_a{}^{\hat{\mu}}\hat{e}_b{}^{\hat{\nu}}\partial_{[\hat{\mu}}\hat{A}^I_{\hat{\nu}]} = 2e_a{}^\mu e_b{}^\nu\partial_{[\mu}\left(A^I_{\nu]} + l^I A_{\nu]}\right) - 2e_a{}^\mu e_b{}^\nu A^I_\nu\partial_\mu l^I \\ &= F^I_{ab} + l^I F_{ab}\,, \end{aligned} \tag{2.28}$$

where (2.7) has been used. In the same way

$$\hat{F}^I_{az} = k^{-1}\partial_a l^I \tag{2.29}$$

is found and the kinetic term for the vector fields in the action can be written in terms of 4-dimensional quantities only,

$$\hat{a}_{IJ}\hat{F}^I{}_{\hat{\mu}\hat{\nu}}\hat{F}^{J\hat{\mu}\hat{\nu}} = a_{IJ}\left(F^I + l^I F\right)_{\mu\nu}\left(F^J + l^J F\right)^{\mu\nu} - 2k^{-2}a_{IJ}\partial_\mu l^I\partial^\mu l^J\,, \tag{2.30}$$

where we made use of (2.24).

To reduce the Chern-Simons term, it is convenient to convert it to an expression involving flat indexes only,

$$C_{IJK}\hat{\epsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\tau}}\hat{F}^{I}{}_{\hat{\mu}\hat{\nu}}\hat{F}^{J}{}_{\hat{\rho}\hat{\sigma}}\hat{A}^{K}{}_{\hat{\tau}} = \sqrt{|\hat{g}|}\hat{\epsilon}^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}\hat{F}^{I}{}_{\hat{a}\hat{b}}\hat{F}^{J}{}_{\hat{c}\hat{d}}\hat{A}^{K}{}_{\hat{e}}\,, \tag{2.31}$$

and to relate the 5 and 4-dimensional Levi-Civita symbols via

$$\hat{\epsilon}^{abcdz} \equiv \epsilon^{abcd}\,. \tag{2.32}$$

After performing several integrations by parts, the Chern-Simons term becomes

$$C_{IJK}\hat{\epsilon}\hat{F}^{I}\hat{F}^{J}\hat{A}^{K} = C_{IJK}\epsilon\left(3l^{I}F^{J}F^{K} + 3l^{I}l^{J}F^{K}F + l^{I}l^{J}l^{K}FF\right)\,. \tag{2.33}$$

Collecting all the partial results, rescaling the action to the Einstein frame and defining the Hodge dual field strengths as

$$\star F^{\mu\nu} = \frac{1}{2\sqrt{|g|}}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}\,, \tag{2.34}$$

the following 4-dimensional action is obtained

$$\begin{aligned} S = \int d^4x\sqrt{|g|}\Big\{ & R + \frac{1}{2}\left(3k^{-2}\left(\partial k\right)^2 + 3a_{IJ}\partial h^I\partial h^J + k^{-2}a_{IJ}\partial l^I\partial l^J\right) \\ & - \frac{1}{4}ka_{IJ}\left(F^I + l^IF\right)\left(F^J + l^JF\right) - \frac{1}{4}k^3F^2 \\ & + \frac{1}{2\sqrt{3}}C_{IJK}\left(l^IF^J\star F^K + l^Il^JF^K\star F + \frac{1}{3}l^Il^Jl^KF\star F\right)\Big\}\,. \end{aligned} \tag{2.35}$$

2.2.2 Identification with 4-dimensional supergravity

We now have to rewrite the action we just found as the bosonic action of $\mathcal{N} = 2$ 4-dimensional supergravity

$$\begin{aligned} S = \int d^4x\sqrt{|g|}\Big\{ & R + 2\mathcal{G}_{ij^*}\partial_\mu Z^i\partial^\mu Z^{*\,j^*} \\ & + 2\Im\mathfrak{m}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}F^{\Sigma}{}_{\mu\nu} - 2\Re\mathfrak{e}\mathcal{N}_{\Lambda\Sigma}F^{\Lambda\,\mu\nu}\star F^{\Sigma}{}_{\mu\nu}\Big\}\,, \end{aligned} \tag{2.36}$$

identifying the fields, the metric $\mathcal{G}_{ij^*}$ and the period matrix $\mathcal{N}_{\Lambda\Sigma}$, that have to correspond to those of some special Kähler geometry. The kinetic term for the scalars in (2.35) can be rewritten as

$$\frac{3}{2}k^{-2}a_{IJ}\left(\frac{1}{\sqrt{3}}l^I + ikh^I\right)\left(\frac{1}{\sqrt{3}}l^J - ikh^J\right)\,, \tag{2.37}$$

where the role of the scalar Z^i is played by the combination in brackets $\left(\frac{1}{\sqrt{3}}l^{i-1} + ikh^{i-1}\right)$ and the Kähler metric $\mathcal{G}_{ij^*} \equiv \frac{3}{4}k^{-2}a_{i-1j^*-1}$ is that of a cubic model (1.183) characterized by $d_{ijk} = C_{i-1,j-1,k-1}$, while the Kähler potential is

$$e^{-\mathcal{K}} = 8d_{IJK}\Im\mathfrak{m}Z^i\Im\mathfrak{m}Z^j\Im\mathfrak{m}Z^k = 8k^3C_{IJK}h^Ih^Jh^K = 8k^3\,. \tag{2.38}$$

There are different symplectic equivalent choices for the vector fields. We identify

$$A^\Lambda = \left(\frac{1}{2\sqrt{2}}A\,, -\frac{1}{2\sqrt{6}}A^{I=i-1}\right) \tag{2.39}$$

where A^Λ are the fields in (2.36), while A and A^I are those of (2.35). It can be checked [54] that the period matrix that appears in the action is exactly the expected one, for a cubic model with the given prepotential.

We are finally summarizing the "dictionary" that allows to reduce 5-dimensional solutions with one isometry to solution of 4-dimensional supergravity

$$\begin{aligned}
g_{\mu\nu} &= \left|\hat{g}_{\underline{zz}}\right|^{\frac{1}{2}}\left(\hat{g}_{\mu\nu} - \frac{\hat{g}_{\mu\underline{z}}\hat{g}_{\nu\underline{z}}}{\hat{g}_{\underline{zz}}}\right)\,,\\
A^0 &= \frac{1}{2\sqrt{2}}\frac{\hat{g}_{\mu\underline{z}}}{\hat{g}_{\underline{zz}}}\,,\\
A^i &= -\frac{1}{2\sqrt{6}}\left(\hat{A}^{i-1}_\mu - \hat{A}^{i-1}_{\underline{z}}\frac{\hat{g}_{\mu\underline{z}}}{\hat{g}_{\underline{zz}}}\right)\,,\\
Z^i &= \frac{1}{\sqrt{3}}\hat{A}^{i-1}_{\underline{z}} + i\left|\hat{g}_{\underline{zz}}\right|^{\frac{1}{2}}\hat{h}^{i-1}\,,\\
d_{ijk} &= C_{i-1j-1k-1}\,,
\end{aligned} \tag{2.40}$$

and the inverse one, to uplift to 5-dimension the 4-dimensional solutions

$$\begin{aligned}
\hat{g}_{\underline{zz}} &= -k^2\,,\\
\hat{g}_{\mu\underline{z}} &= -2\sqrt{2}k^2A^0_\mu\,,\\
\hat{g}_{\mu\nu} &= k^{-1}g_{\mu\nu} - 8kA^0_\mu A^0_\nu\,,\\
\hat{A}^I_{\underline{z}} &= \sqrt{3}\Re\mathfrak{e}Z^{I+1}\,,\\
\hat{A}^I_\mu &= -2\sqrt{6}\left(A^{I+1}_\mu - \Re\mathfrak{e}Z^{I+1}A^0_\mu\right)\,,\\
\hat{h}^I &= k^{-1}\Im\mathfrak{m}Z^{I+1}\,.
\end{aligned} \tag{2.41}$$

2.3 From 6 to 5 dimensions

We are now going to study the often disregarded $\mathcal{N} = (2,0), d = 6$ supergravity theories that have several tensor multiplets with or without vector multiplets. We are particularly interested in how its solutions are related to the supersymmetric solutions of the

$\mathcal{N} = 2, d = 5$ theories by dimensional reduction on a circle. In fact, in absence of a classification, uplifting the known, lower dimensional solutions provide a method to construct new supersymmetric solutions of the $\mathcal{N} = (2,0), d = 6$ supergravity.

In the case under study, it does not actually seem to be widely known which models of $\mathcal{N} = 2, d = 5$ supergravity are related by dimensional reduction to which models of $\mathcal{N} = (2,0), d = 6$ supergravity theories. Thus, our first task is to perform the dimensional reduction of a general, ungauged, $\mathcal{N} = (2,0), d = 6$ supergravity theory with an arbitrary number of tensor and vector multiplets to $d = 5$ and to identify to which model of $\mathcal{N} = 2, d = 5$ supergravity it gives rise. Since the hypermultiplets do not couple to the vector and tensor multiplets, their reduction clearly leads to 5-dimensional hypermultiplets with exactly the same quaternionic-Kähler geometry.
A careful identification of the 5-dimensional fields will provide us with the dictionary we need to reduce and uplift solutions.

As we are going to demonstrate, the identification of the 5-dimensional models leads to a surprise: there are two different families of $\mathcal{N} = (2,0), d = 6$ supergravity models related to the same family of $\mathcal{N} = 2, d = 5$ supergravity models: the one with 1 tensor multiplet and n_{V6} vector multiplets (that we have been calling $\mathcal{N} = 2A, d = 6$ theories) and those with only $n_T = n_{V6} + 1$ tensor multiplets (that we are going to call $\mathcal{N} = 2B, d = 6$ theories) give exactly the same family of models of $\mathcal{N} = 2, d = 5$ supergravity coupled to $n_{V5} = n_{V6} + 2$ vector multiplets characterized by a symmetric tensor C_{IJK} with non-vanishing components $C_{0\,r+1\,s+1} = \frac{1}{3!}\eta_{rs}$, where $r, s = 0, \cdots, n_{V6} + 1$ and $\eta_{rs} = \mathit{diag}(1, -1, \ldots, -1)$.
The $\mathcal{N} = 2A, d = 6$ theories are related to the toroidal compactification and truncation of the Heterotic String. We also consider the 6-dimensional theories obtained by dualizing the 3-form field strength, related to the compactification of the type IIA superstring on K3. We call them $\mathcal{N} = 2A^*, d = 6$ theories.

As already stated, this situation is analogous to what happens when we dimensionally reduce the two maximal 10-dimensional supergravities, $\mathcal{N} = 2A$ and $\mathcal{N} = 2B$, on a circle and we find the same 9-dimensional maximal supergravity [96], which is unique. In that case, this coincidence is interpreted as a manifestation -at the effective field theory level- of the T-duality existing between the two type II superstrings [97–99]. The relation between the fields of the two 10-dimensional supergravities and those of the 9-dimensional one leads to a direct relation between the 10-dimensional fields of the two theories: the type II generalization of the Buscher T-duality rules [100–102] that transform a solution of one of the 10-dimensional theories admitting one isometry into another solution of the other theory (also admitting one isometry) [96].

In the present case it is not clear which is the superstring theory associated to the $\mathcal{N} = 2B, d = 6$ theories (if any), but the relation we have found leads to a new generalization of the Buscher rules for 6-dimensional solutions of these theories admitting one isometry.

It is normally convenient to work with the action of a theory but, in general, these

theories do not have a covariant action, due to (anti-) selfduality constraints satisfied by the 3-forms [87]. Nevertheless, it is sometimes possible to construct *pseudoactions* [103] which give the correct equations of motion of the theory upon use of the (anti-) selfduality constraints in the Euler-Lagrange equations that follow from them. The action of the dimensionally reduced theory can then be derived performing the following steps:

- Dimensionally reduce the pseudoaction and the (anti-) selfduality constraints in the standard way.
- Poincaré-dualize the highest-rank potentials arising from the (anti-) selfdual potentials in the dimensionally-reduced pseudoaction.
- Identify the resulting potentials with the lowest-rank potentials arising from the (anti-) selfdual potentials. This identification should be completely equivalent to the use of the dimensionally reduced (anti-) selfduality constraint in the action.

A well-known example of this procedure is the dimensional reduction to $d = 9$ of the $\mathcal{N} = 2B, d = 10$ supergravity theory [104–106] carried out in [107]: in this case there is a RR 4-form potential $\hat{C}^{(4)}$ whose 5-form field strength $\hat{G}^{(5)}$ is selfdual $\hat{G}^{(5)} = \star_{10}\hat{G}^{(5)}$ and the equations of motion can be derived from the pseudoaction constructed in [103] by imposing a selfduality constraint. The dimensional reduction of the 4-form potential $\hat{C}^{(4)}$ gives rise to a 4- and a 3-form $C^{(4)}, C^{(3)}$ potentials whose 5- and 4-form field strengths $G^{(5)}$ and $G^{(4)}$ are related by the dimensionally reduced selfduality constraint $G^{(5)} \sim \star G^{(4)}$. Following the above recipe, in [107] the pseudoaction and selfduality constraint were reduced to $d = 9$ first. Then, the 9-dimensional 4-form potential $C^{(4)}$ was Poincaré-dualized into a 9-dimensional 3-form potential $\tilde{C}^{(3)}$ in the pseudoaction. At this point the theory has two different 3-form potentials $\tilde{C}^{(3)}$ and $C^{(3)}$ and the selfduality constraint takes the form $\tilde{G}^{(4)} = G^{(4)}$ indicating that the two 3-forms are one and the same $\tilde{C}^{(3)} = C^{(3)}$. Making this identification in the pseudoaction gives the correct 9-dimensional action.

In the case at hands, the bosonic equations of motion can be found by varying the pseudoaction (1.246) with no hypermultiplets

$$\begin{aligned}\hat{S} = \int d^6\hat{x}\sqrt{|\hat{g}|}\Big\{ &\hat{R} - \partial_{\hat{a}}\hat{L}^r\partial^{\hat{a}}\hat{L}_r + \tfrac{1}{3}\mathcal{M}_{rs}\hat{H}^r{}_{\hat{a}\hat{b}\hat{c}}\hat{H}^{s\,\hat{a}\hat{b}\hat{c}} \\ &- \hat{L}_r c^r{}_{ij}\hat{F}^i{}_{\hat{a}\hat{b}}\hat{F}^{j\,\hat{a}\hat{b}} - \tfrac{1}{4}c_{r\,ij}\hat{\epsilon}^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{f}}\hat{B}^r{}_{\hat{a}\hat{b}}\hat{F}^i{}_{\hat{c}\hat{d}}\hat{F}^j{}_{\hat{e}\hat{f}}\Big\}\,.\end{aligned} \tag{2.42}$$

and imposing on the resulting equations of motion the (anti-) selfduality conditions (1.243). However, due to the Chern-Simons term, this action is gauge invariant if and only if the following condition holds [108]

$$\eta_{rs}c^r{}_{i(j}c^s{}_{kl)} = 0\,. \tag{2.43}$$

We are assuming it to hold throughout. Only in this way, consistent 5-dimensional theories can be obtained.

2.3.1 Reduction of the fields

Having described the bosonic sector of the theories we want to study, we are now ready to reduce them to $d = 5$.

We are going to follow the standard procedure proposed in [39] with the particular conventions of [54] we have been using so far. Thus, we assume that none of the fields depend explicitly on the compact coordinate z. We split the world and tangent-space indexes as we have done in the previous sections,

$$\hat{\mu} = (\mu, \underline{z}) \ , \qquad \hat{a} = (a, z) \ , \tag{2.44}$$

and we decompose the components of the Sechsbein basis $\hat{e}^{\hat{a}}{}_{\hat{\mu}}$, which we have chosen to be upper-triangular, into those of a Fünfbein $e^a{}_\mu$, a Kaluza-Klein (KK) vector A_μ and a KK scalar k as in (2.7).

The scalars are the same z-independent functions in both dimensions. In particular, $\hat{L}_r = L_r$.

The vector fields $\hat{A}^i$ decompose into vector fields A^i and scalar fields l^i as

$$\hat{A}^i{}_a \equiv A^i{}_a \ \Leftrightarrow \ \hat{A}^i{}_\mu = A^i{}_\mu + l^i A_\mu \ , \tag{2.45}$$

$$\hat{A}^i{}_z \equiv k^{-1} l^i \ \Leftrightarrow \ \hat{A}^i{}_{\underline{z}} = l^i \ . \tag{2.46}$$

This leads to the following decomposition of the vector field strengths

$$\hat{F}^i{}_{ab} = \mathcal{F}^i{}_{ab} = F^i{}_{ab} + l^i F_{ab} \ , \tag{2.47}$$

$$\hat{F}^i{}_{az} = k^{-1} \partial_a l^i \ , \tag{2.48}$$

where F^i and F are the 5-dimensional field strengths

$$F^i \equiv dA^i \ , \qquad F \equiv dA \ . \tag{2.49}$$

Each 2-form $\hat{B}^r$ produces a 2- and 1-form in five dimensions (B^r and A^r respectively); they will be related by the (anti-) selfduality constraints. It turns out that the following definitions give potentials with good gauge transformation properties

$$\hat{B}^r{}_{\mu\underline{z}} \equiv A^r{}_\mu + \tfrac{1}{2} c^r{}_{ij} l^i A^j{}_\mu \ , \tag{2.50}$$

$$\hat{B}^r{}_{\mu\nu} \equiv B^r{}_{\mu\nu} - A_{[\mu} A^r{}_{\nu]} - c^r{}_{ij} A_{[\mu} A^i{}_{\nu]} l^j \ . \tag{2.51}$$

The 3-form field strengths $\hat{H}^r$ decompose in

$$\hat{H}^r{}_{abc} \equiv H^r{}_{abc} \ , \tag{2.52}$$

$$\hat{H}^r{}_{abz} \equiv k^{-1}\mathcal{F}^r{}_{ab} \equiv k^{-1}\left[F^r + c^r{}_{ij}l^iF^j + \tfrac{1}{2}c^r{}_{ij}l^il^jF\right] , \tag{2.53}$$

where

$$H^r = dB^r - \tfrac{1}{2}F\wedge A^r - \tfrac{1}{2}F^r\wedge A + \tfrac{1}{2}c^r{}_{ij}F^i\wedge A^j , \tag{2.54}$$

$$F^r = dA^r . \tag{2.55}$$

This completely fixes the reduction of fields and field strengths. Plugging these decompositions in the pseudoaction (2.42) together with the decomposition of the Levi-Civita symbol

$$\hat{\epsilon}^{abcdez} \equiv \epsilon^{abcde} , \tag{2.56}$$

we get the 5-dimensional pseudoaction

$$\begin{aligned} S = \int d^5x\sqrt{|g|}k\Big\{ & R - \tfrac{1}{4}k^2F^2 - \partial_\mu L^r\partial^\mu L_r + 2k^{-2}L_rc^r{}_{ij}\partial_\mu l^i\partial^\mu l^j \\ & + \tfrac{1}{3}\mathcal{M}_{rs}H^rH^s - k^{-2}\mathcal{M}_{rs}\mathcal{F}^r\mathcal{F}^s - L_rc^r{}_{ij}\mathcal{F}^i\mathcal{F}^j \\ & + \frac{k^{-1}\epsilon}{6\sqrt{|g|}}c_{r\,ij}\left[H^r(\mathcal{F}^il^j - 2\partial l^iA^j) - 3\mathcal{F}^r\mathcal{F}^iA^j\right]\Big\} , \end{aligned} \tag{2.57}$$

where the indexes are assumed to be contracted in the obvious way: $\mathcal{F}^r\mathcal{F}^s \equiv \mathcal{F}^r{}_{\mu\nu}\mathcal{F}^{s\,\mu\nu}$, $\epsilon H^rc_{r\,ij}(\mathcal{F}^il^j - 2\partial l^iA^j) = \epsilon^{\mu\nu\rho\kappa\sigma}H^r{}_{\mu\nu\rho}c_{r\,ij}(\mathcal{F}^i{}_{\kappa\sigma}l^j - 2\partial_{[\kappa}l^iA^j{}_{\sigma]})$, etc.

Finally, we perform a rescaling of the metric in order to express the action in the Einstein frame, with metric $g_{E\,\mu\nu}$

$$g_{\mu\nu} = k^{-2/3}g_{E\,\mu\nu} , \tag{2.58}$$

and redefine the KK scalar k in order to give it a kinetic term with standard normalization

$$k = e^{\sqrt{3/8}\phi} . \tag{2.59}$$

The result, up to total derivatives, is the pseudoaction

$$\begin{aligned} S = \int d^5x\sqrt{|g_E|}\Big\{ & R_E + \tfrac{1}{2}(\partial\phi)^2 - \partial_\mu L^r\partial^\mu L_r + 2e^{-\sqrt{3/2}\phi}L_rc^r{}_{ij}\partial_\mu l^i\partial^\mu l^j - \tfrac{1}{4}e^{\sqrt{8/3}\phi}F^2 \\ & - e^{-\sqrt{2/3}\phi}\mathcal{M}_{rs}\mathcal{F}^r\mathcal{F}^s - L_rc^r{}_{ij}e^{\phi/\sqrt{6}}\mathcal{F}^i\mathcal{F}^j + \tfrac{1}{3}e^{\sqrt{2/3}\phi}\mathcal{M}_{rs}H^rH^s \\ & + \frac{\epsilon}{6\sqrt{|g_E|}}c_{r\,ij}\left[H^r(\mathcal{F}^il^j - 2\partial l^iA^j) - 3\mathcal{F}^r\mathcal{F}^iA^j\right]\Big\} . \end{aligned} \tag{2.60}$$

The reduction of the (anti-) selfduality constraints offers no problems and becomes a duality relation between the 2- and 1-form potentials B^r, A^r

$$\mathcal{M}_{rs}H^s = -e^{-\sqrt{2/3}\phi}\eta_{rs} \star \mathcal{F}^s \,. \tag{2.61}$$

The equations of motion of the 5-dimensional theory can be obtained by varying the above pseudoaction and imposing the duality constraints. However, in order to identify the resulting 5-dimensional theories with known models of $\mathcal{N} = 2, d = 5$ supergravity coupled to vector multiplets, it is convenient to eliminate this constraint. We are accomplishing this task thanks to the procedure we outlined previously.

2.3.2 Dualization

As already explained, we are going to Poincaré dualize the 2-forms B^r into 1-forms $\tilde{A}_r$. First, we are going replace the 2-forms B^r by their 3-form field strengths H^r as variables of the pseudoaction (2.60). This is possible because the pseudoaction only depends on the 2-forms through their field strengths. However, we have to add a Lagrange-multiplier term to enforce the Bianchi identities of the H^r, which have the form

$$4\partial_{[\mu}H^r{}_{\nu\rho\sigma]} + 6F^r{}_{[\mu\nu}F_{\rho\sigma]} - 3c^r{}_{ij}F^i{}_{[\mu\nu}F^j{}_{\rho\sigma]} = 0 \,. \tag{2.62}$$

The term to be added takes the form

$$\frac{\epsilon}{\sqrt{|g_E|}}\tilde{A}_r\Big(\partial H^r + \tfrac{3}{2}F^rF - \tfrac{3}{4}c^r{}_{ij}F^iF^j\Big) \,, \tag{2.63}$$

where the Lagrange multiplier is the 1-form field $\tilde{A}_r$.

Adding this term to the pseudoaction and integrating it by parts, we get

$$\begin{aligned} S = \int d^5x\sqrt{|g_E|}\Big\{ & R_E + \tfrac{1}{2}(\partial\phi)^2 - \partial_\mu L^r\partial^\mu L_r + 2e^{-\sqrt{3/2}\phi}L_rc^r{}_{ij}\partial_\mu l^i\partial^\mu l^j - \tfrac{1}{4}e^{\sqrt{8/3}\phi}F^2 \\ & - e^{-\sqrt{2/3}\phi}\mathcal{M}_{rs}\mathcal{F}^r\mathcal{F}^s - L_rc^r{}_{ij}e^{\phi/\sqrt{6}}\mathcal{F}^i\mathcal{F}^j + \tfrac{1}{3}e^{\sqrt{2/3}\phi}\mathcal{M}_{rs}H^rH^s \\ & + \frac{\epsilon}{6\sqrt{|g_E|}}\big[c_{r\,ij}H^r(\mathcal{F}^il^j - 2\partial l^iA^j) - 3c_{r\,ij}\mathcal{F}^r\mathcal{F}^iA^j \\ & + 3\tilde{F}_r(H^r + \tfrac{3}{2}FA^r + \tfrac{3}{2}F^rA - \tfrac{3}{2}c^r{}_{ij}F^iA^j)\big]\Big\}, \end{aligned} \tag{2.64}$$

where

$$\tilde{F}_r \equiv d\tilde{A}_r \,. \tag{2.65}$$

Since in this pseudoaction H^r is an independent field, we can compute its field equation, which relates it to $\tilde{F}_r$. It is given by

$$\mathcal{M}_{rs}H^s = -\tfrac{1}{2}e^{-\sqrt{2/3}\phi} \star \left[c_{r\,ij}(\mathcal{F}^i l^j - 2\partial l^i A^j) + 3\tilde{F}_r\right]. \tag{2.66}$$

We are using this equation to eliminate H^r completely from the pseudoaction and from the duality relation (2.61). After this operation, the 2-forms B^r will disappear from both, having been replaced by the dual 1-forms $\tilde{A}_r$. After this replacement, the constraint reads

$$\tilde{F}_r = \tfrac{2}{3}\Big(\eta_{rs}F^s + c_{r\,ij}\partial(l^i A^j)\Big)\,, \tag{2.67}$$

which implies the following algebraic relation between potentials

$$\tilde{A}_r = \tfrac{2}{3}\eta_{rs}A^s + \tfrac{1}{3}c_{r\,ij}l^i A^j\,, \tag{2.68}$$

that can be used in the pseudoaction to eliminate $\tilde{A}_r$ completely. In this way, the only fields remaining from the reduction of the 2-forms B^r are the 1-forms A^r. Furthermore, there are no constraints to be imposed, so that the pseudoaction has become a standard action

$$\begin{aligned} S = \int d^5x\sqrt{|g_E|}\Bigg\{ & R_E + \tfrac{1}{2}(\partial\phi)^2 - \partial_\mu L^r\partial^\mu L_r + 2e^{-\sqrt{3/2}\phi}L_r c^r{}_{ij}\partial_\mu l^i\partial^\mu l^j \\ & - \tfrac{1}{4}e^{\sqrt{8/3}\phi}F^2 - 2e^{-\sqrt{2/3}\phi}\mathcal{M}_{rs}\mathcal{F}^r\mathcal{F}^s - L_r c^r{}_{ij}e^{\phi/\sqrt{6}}\mathcal{F}^i\mathcal{F}^j \\ & + \frac{\epsilon}{\sqrt{|g_E|}}\Big(\eta_{rs}F^rF^sA - c_{r\,ij}F^iF^jA^r\Big)\Bigg\}. \end{aligned} \tag{2.69}$$

2.3.3 Identification with 5-dimensional supergravity

The next step is to identify the previous theory as a model of $\mathcal{N}=2, d=5$ supergravity coupled to n_{V5} vector multiplets. As already explained in section 1.3, these theories contain $n_{V5}+1$ 1-form fields A^I, $I,J,\ldots = 0,1,\cdots,n_{V5}$ and n_{V5} scalars ϕ^x, $x,y,\ldots = 1,\cdots,n_{V5}$.

In order to identify the models corresponding to the theories we have obtained by dimensional reduction, we start by rescaling the vector fields

$$A \to \tfrac{1}{\sqrt{12}}A\,, \qquad A^r \to \tfrac{1}{\sqrt{12}}A^r\,, \qquad A^i \to \tfrac{1}{\sqrt{12}}A^i\,, \tag{2.70}$$

so that the action becomes

$$
\begin{aligned}
S = \int d^5x\sqrt{|g_E|}\Bigg\{ & R_E + \tfrac{1}{2}(\partial\phi)^2 - \partial_\mu L^r \partial^\mu L_r + 2e^{-\sqrt{3/2}\phi} L_r c^r{}_{ij}\partial_\mu l^i \partial^\mu l^j \\
& - \tfrac{1}{48}e^{\sqrt{8/3}\phi}F^2 - \tfrac{1}{12}L_r c^r{}_{ij} e^{\phi/\sqrt{6}}(F^i{}_{\mu\nu} + l^i F_{\mu\nu})(F^j{}_{\mu\nu} + l^j F_{\mu\nu}) \\
& - \tfrac{1}{6}e^{-\sqrt{2/3}\phi}\mathcal{M}_{rs}\Big(F^r{}_{\mu\nu} + c^r{}_{ij}l^i F^j{}_{\mu\nu} + \tfrac{1}{2}c^r{}_{ij}l^i l^j F_{\mu\nu}\Big)\Big(F^s{}_{\mu\nu} + c^s{}_{ij}l^i F^j{}_{\mu\nu} + \tfrac{1}{2}c^s{}_{ij}l^i l^j F_{\mu\nu}\Big) \\
& + \frac{\epsilon}{12\sqrt{3}\sqrt{|g_E|}}\Big(\tfrac{1}{2}\eta_{rs}F^r F^s A - \tfrac{1}{2}c_{r\,ij}F^i F^j A^r\Big)\Bigg\} .
\end{aligned}
\tag{2.71}
$$

Comparing this theory to (1.226), we immediately notice that there is a total of $n_T + n_V + 2$ 1-forms, therefore we are in presence of a theory coupled to $n_{V5} = n_T + n_V + 1$ vector multiplets. We can decompose the 5-dimensional index I as $I = \{0, r{+}1, i{+}n_T{+}1\}$ where the indexes take the values $r = 0, \ldots, n_T$, $i = 1, \ldots, n_V$, and we identify

$$
A^0 = A\,, \qquad A^{I=r+1} = A^r\,, \qquad A^{I=i+n_T+1} = A^i\,, \tag{2.72}
$$

where the fields in the l.h.s.'s are those of (1.226), while the fields in the r.h.s.'s are those of (2.71).

We can also identify the components of the C_{IJK} tensor that characterizes the model of $\mathcal{N} = 2, d = 5$ supergravity

$$
C_{0\,r+1\,s+1} = \tfrac{1}{3!}\eta_{rs}\,, \qquad C_{r+1\,i+n_T+1\,j+n_T+1} = -\tfrac{1}{3!}c_{r\,ij}\,. \tag{2.73}
$$

We are discussing the properties of these models later on, focusing on two particular subfamilies. Since the tensor C_{IJK} has been given, the expected forms of a_{IJ} and g_{xy} are known, so that we are able to identify the scalar fields of (2.71) with the scalar functions h^I and with the physical scalars ϕ^x.

The components of a_{IJ} in (2.71) are

$$
\begin{aligned}
a_{00} &= \tfrac{1}{12}\Big[e^{2\phi/\sqrt{6}} + 2L_r\xi^r e^{-\phi/\sqrt{6}}\Big]^2, \\
a_{0\,r+1} &= \tfrac{1}{3}\mathcal{M}_{rs}\xi^s e^{-\sqrt{2/3}\phi}, \\
a_{0\,i+n_T+1} &= \tfrac{1}{3}L_r c^r{}_{ij} l^j e^{-\phi/\sqrt{6}}\Big(e^{2\phi/\sqrt{6}} + 2L_s\xi^s e^{-\phi/\sqrt{6}}\Big), \\
a_{r+1\,s+1} &= \tfrac{2}{3}e^{-\sqrt{2/3}\phi}\mathcal{M}_{rs}, \\
a_{r+1\,i+n_T+1} &= \tfrac{2}{3}e^{-\sqrt{2/3}\phi}\mathcal{M}_{rs}c^s{}_{ij}l^j, \\
a_{i+n_T+1\,j+n_T+1} &= \tfrac{2}{3}e^{-\sqrt{2/3}\phi}\mathcal{M}_{rs}c^r{}_{ik}c^s{}_{jl}l^k l^l + \tfrac{1}{3}e^{\phi/\sqrt{6}}L_r c^r{}_{ij},
\end{aligned}
\tag{2.74}
$$

where $\xi^r \equiv c^r{}_{ij}l^i l^j$ and we have made some simplifications by using the properties $L^r L_r = 1$, $\xi^r\xi_r = 0$, $\xi^r c_{r\,ij}l^i = 0$ and $\mathcal{M}_{rs} = 2L_r L_s - \eta_{rs}$. Finally, when the set $(\phi^x) =$

$(\phi^1, \cdots, \phi^{n_V+n_T+1}) = (\phi, \varphi^{\underline{\alpha}}, l^i)$ is identified as the set of the physical scalar fields, we deduce from (2.71) that the block-diagonal components of g_{xy} only are non-vanishing,

$$\begin{aligned} g_{11} &= 1\,, \\ g_{\underline{\alpha}+1\,\underline{\beta}+1} &= -2\partial_{\underline{\alpha}} L^r \partial_{\underline{\beta}} L_r, \\ g_{i+n_T+1\,j+n_T+1} &= 4e^{-\sqrt{3/2}\phi} L_r c^r{}_{ij}\,. \end{aligned} \tag{2.75}$$

Comparing these expressions with the formulas (1.224) and (1.223) for the theories with symmetric tensor given by (2.73) we conclude that the scalar functions h^I are given by

$$h^0 = 2e^{-2\phi/\sqrt{6}}\,, \qquad h^r = L^r e^{\phi/\sqrt{6}} + \xi^r e^{-2\phi/\sqrt{6}}\,, \qquad h^i = -2e^{-2\phi/\sqrt{6}} l^i\,. \tag{2.76}$$

For the sake of convenience we also give the h_I, defined as in (1.225)

$$h_0 = \tfrac{1}{6}\left(e^{2\phi/\sqrt{6}} + 2\xi_r L^r e^{-\phi/\sqrt{6}}\right), \qquad h_r = \tfrac{2}{3} L_r e^{-\phi/\sqrt{6}}\,, \qquad h_i = \tfrac{2}{3} e^{-\phi/\sqrt{6}} c_{r\,ij} L^r l^j\,. \tag{2.77}$$

In what follows, we will be interested in two particular cases, which correspond to models of the same family with all the scalars in symmetric spaces SO$(1,n)$/SO(n) for some value of n:

- $n_V = 0$, which has SO$(1, n_T+1)$/SO(n_T+1);
- $n_T = 1$, which has SO$(1, n_V+2)$/SO(n_V+2).

The next paragraphs consist of a close review of these models.

Case $n_V = 0$

If the starting point is a 6-dimensional theory with an arbitrary number n_T of tensor multiplets and no vector multiplets, the 5-dimensional model that is obtained when going through the procedure outlined so far presents $n_{V5} = n_T + 1$ vector multiplets and is characterized by

$$C_{0rs} = \tfrac{1}{3!}\eta_{rs}\,. \tag{2.78}$$

We are using the parametrization

$$h^0 = 2e^{-2\phi^1/\sqrt{6}}\,, \qquad h^r = e^{\phi^1/\sqrt{6}} L^r\,, \tag{2.79}$$

where $L^r = L^r(\phi^2, \cdots, \phi^{n_T+1})$.

The $n_{V5} = n_T+1$ scalars of these models parametrize the coset SO$(1, n_T+1)$/SO(n_T+1).

Upon further dimensional reduction, the $ST[2, n_T+2]$ model of $\mathcal{N}=2, d=4$ supergravity is obtained. It is coupled to $n_{V4} = n_{V5}+1 = n_T+2$ vector multiplets, parameterizing the coset space $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)} \times \frac{\mathrm{SO}(2,n_T+1)}{\mathrm{SO}(2)\times\mathrm{SO}(n_T+1)}$.

Case $n_T = 1$

If we start instead from a 6-dimensional theory with $n_T = 1$, an arbitrary number of vector multiplets n_V and choose the coefficients $c_{r\,ij}$ to be

$$c_{0\,ij} = c_{1\,ij} = \delta_{ij}\,, \tag{2.80}$$

which is a particularly simple solution of the constraint (2.43), we are then dealing with theories containing two 2-forms of opposite selfduality that can be combined into a single, unconstrained, 2-form that can be identified with the Kalb-Ramond field, a single scalar that can be identified with the dilaton field and a set of Abelian vector fields. These theories can be obtained by toroidal compactification of the Heterotic String theory to 6 dimensions and subsequent truncation, assuming that the number of Abelian vectors does not exceed 16.

We are now showing how, upon dimensional reduction, these theories end up belonging to the same family as those of the previous $n_V = 0$ case.

With the above choice of coefficients, the parametrization of $\tilde{h}^i$ is given by[4]

$$\begin{aligned} \tilde{h}^0 &= 2e^{-2\phi/\sqrt{6}}\,, & \tilde{h}^1 &= \tilde{L}^0 e^{\phi/\sqrt{6}} + l^2 e^{-2\phi/\sqrt{6}}\,, \\ \tilde{h}^2 &= \tilde{L}^1 e^{\phi/\sqrt{6}} - l^2 e^{-2\phi/\sqrt{6}}\,, & \tilde{h}^i &= -2e^{-2\phi/\sqrt{6}} l^i\,. \end{aligned} \tag{2.81}$$

These functions satisfy the equation

$$1 = \tilde{C}_{IJK}\tilde{h}^I\tilde{h}^J\tilde{h}^K = \tfrac{1}{2}\tilde{h}^0\big[(\tilde{h}^1)^2 - (\tilde{h}^2)^2\big] - \tfrac{1}{2}(\tilde{h}^1+\tilde{h}^2)\tilde{h}^i\tilde{h}^i\,. \tag{2.82}$$

However, we are free to make linear transformations of the $\tilde{h}^I$ and A^I in order to obtain equivalent theories. In particular, if we perform the following transformation $(\tilde{h}^0, \tilde{h}^1, \tilde{h}^2, \tilde{h}^i) \to (h^0, h^r)$, with $r = 1, 2, i+2$,

$$\begin{aligned} \tilde{h}^0 &= h^1 + h^2, \\ \tilde{h}^1 &= \tfrac{1}{2}(h^0 + h^1 - h^2), \\ \tilde{h}^2 &= \tfrac{1}{2}(h^0 - h^1 + h^2), \\ \tilde{h}^i &= h^{i+2}\,, \end{aligned} \tag{2.83}$$

the new variables still satisfy

$$1 = \tfrac{1}{2}h^0\big((h^1)^2 - (h^2)^2 - h^{i+2}h^{i+2}\big) = \tfrac{1}{2}h^0 h^r h^s \eta_{rs} \equiv C_{IJK}h^I h^J h^K\,, \tag{2.84}$$

[4]We are going to denote the objects of these theories with tildes.

so these models are equivalent to those with $C_{0rs} = \frac{1}{3!}\eta_{rs}$.

We conclude that $\mathcal{N} = (2,0), d = 6$ supergravity coupled to n_T tensor multiplets gives the same 5-dimensional supergravity model as $\mathcal{N} = (2,0), d = 6$ supergravity coupled to just 1 tensor multiplet and and $n_V = n_T - 1$ vector multiplets. Furthermore, the 5-dimensional theory that is obtained by dimensional reducing these two 6-dimensional theories can be embedded in Heterotic String theory.

These two 6-dimensional supergravity theories, dimensionally reduced on a circle, are dual in the same sense in which the 10-dimensional $\mathcal{N} = 2A$ and $\mathcal{N} = 2B$ supergravity theories are T-dual [96], a fact related to the T-duality of the type IIA and IIB superstring theories compactified on circles of dual radii [97–99]. Before interpreting this duality between supergravity theories in the context of superstring theory as a large-small radii or coupling constant duality, we need to find the dictionary that relates the fields of both 6-dimensional theories. This dictionary will be the analogous of the Buscher rules for T-duality [96, 100–102, 109] and it will allow us to transform any solution of one of these theories admitting one isometry into a solution of the dual theory.

The initial step to derive this dictionary consists in finding out how each solution of the 5-dimensional theory can be oxidized to two different solutions of the two different 6-dimensional theories: one which only contains chiral 2-forms and one with a non-chiral 2-form and vector fields.

We recall that, in what follows, we are going to rename as $\mathcal{N} = 2A$ the 6-dimensional supergravity theories with just one tensor multiplet, n_V vector multiplets and $c_{0\,ij} = c_{1\,ij} = \delta_{ij}$, while the dual theories with $n_T = n_V + 1$ tensor multiplets and no vector multiplets are going to be called $\mathcal{N} = 2B$ theories.

We are now focusing on the 5-dimensional theories with $n_{V5} = n_V + 2$ vector multiplets which have these two possible 6-dimensional origins.

2.4 Uplifting solutions to 6 dimensions

Let us consider the family of $\mathcal{N} = 2, d = 5$ theories coupled to $n_{V5} = n_V + 2$ vector multiplets and characterized by a tensor C_{IJK}, $I = 0, \cdots, n_V + 2$ of the form $C_{0rs} = \frac{1}{3!}\eta_{r+1\,s+1}$, $r, s, \ldots = 0, \cdots, n_V + 1$.
The scalar functions h^I can be parametrized in terms of the physical scalars as

$$h^0 = 2e^{-2\phi^1/\sqrt{6}}\,, \qquad h^{r+1} = L^r e^{\phi^1/\sqrt{6}}\,, \tag{2.85}$$

where the functions L^r only depend on the scalars $\phi^2, \cdots, \phi^{n_V+2}$, and satisfy

$$L^r L^s \eta_{rs} = 1\,. \tag{2.86}$$

The action can be written in terms of these functions plus the scalar ϕ^1. It takes the form

$$S = \int d^5x\sqrt{|g|}\Big\{R + \tfrac{1}{2}(\partial\phi^1)^2 - \partial_\mu L^r\partial^\mu L_r - \tfrac{1}{48}e^{4\phi^1/\sqrt{6}}F^0F^0 \\ - \tfrac{1}{6}e^{-2\phi^1/\sqrt{6}}\mathcal{M}_{rs}F^{r+1}F^{s+1} + \frac{\epsilon}{24\sqrt{3}\sqrt{|g|}}\eta_{rs}F^{r+1}F^{s+1}A^0\Big\}, \tag{2.87}$$

where we denoted

$$L_r = \eta_{rs}L^s \qquad \text{and} \qquad \mathcal{M}_{rs} = 2L_rL_s - \eta_{rs}\,. \tag{2.88}$$

For our purposes, though, it is convenient to express everything in terms of the auxiliary functions h^I, so that

$$L^r = h^{r+1}\sqrt{h^0/2}\,, \qquad L_r = h_{r+1}/\sqrt{h^0/2}\,, \qquad \mathcal{M}_{rs} = 4\frac{h_{r+1}h_{s+1}}{h_0} - \eta_{rs}. \tag{2.89}$$

According to our previous discussion, this theory can be uplifted to two different 6-dimensional theories. The explicit form it takes is presented in the following sections.

2.4.1 Uplift to $\mathcal{N} = 2B, d = 6$ supergravity

$\mathcal{N} = 2B, d = 6$ supergravity is the name we have given to the theories of $\mathcal{N} = (2,0), d = 6$ supergravity coupled to $n_T = n_V + 1$ tensor multiplets only. The equations of motion of this theory can be obtained form the pseudoaction

$$\hat{S} = \int d^6\hat{x}\sqrt{|\hat{g}|}\Big\{\hat{R} - \partial_{\hat{a}}\hat{L}^r\partial^{\hat{a}}\hat{L}_r + \tfrac{1}{3}\hat{\mathcal{M}}_{rs}\hat{H}^r{}_{\hat{a}\hat{b}\hat{c}}\hat{H}^{s\ \hat{a}\hat{b}\hat{c}}\Big\}, \tag{2.90}$$

supplemented by the (anti-) selfduality conditions

$$\hat{\mathcal{M}}_{rs}\hat{H}^r = -\eta_{rs}\star\hat{H}^s\,. \tag{2.91}$$

Then, according to our previous results, the 6-dimensional fields of this theory can be expressed in terms of those of the 5-dimensional theory (2.90) as follows.

Scalars

The physical scalars $\hat{\varphi}^{\underline{\alpha}}$ and the functions $\hat{L}^r$, where $\underline{\alpha} = 1, \cdots, n_V{+}1$ and $r = 0, \cdots, n_V{+}1$, are given by

$$\hat{\varphi}^{\underline{\alpha}} = \phi^{\underline{\alpha}+1}\,, \\ \hat{L}^r(\varphi^{\underline{\alpha}}) = h^{r+1}\left(\frac{h^0}{2}\right)^{1/2}. \tag{2.92}$$

Metric

The 6-dimensional metric components are

$$\begin{aligned}
\hat{g}_{\underline{z}\underline{z}} &= -\left(\frac{h^0}{2}\right)^{-3/2} , \\
\hat{g}_{\mu\underline{z}} &= -\tfrac{1}{\sqrt{12}}\left(\frac{h^0}{2}\right)^{-3/2} A^0{}_\mu , \\
\hat{g}_{\mu\nu} &= \left(\frac{h^0}{2}\right)^{1/2} g_{\mu\nu} - \tfrac{1}{12}\left(\frac{h^0}{2}\right)^{-3/2} A^0{}_\mu A^0{}_\nu ,
\end{aligned} \tag{2.93}$$

or, equivalently

$$d\hat{s}^2 = -\left(\frac{h^0}{2}\right)^{-3/2}\left[dz + \tfrac{1}{\sqrt{12}}A^0\right]^2 + \left(\frac{h^0}{2}\right)^{1/2} ds^2 . \tag{2.94}$$

2-forms

We only need the $\hat{B}^r{}_{\mu\underline{z}}$ components of the 2-forms, because the rest of them are determined through the duality relations (2.91). We have

$$\hat{B}^r{}_{\mu\underline{z}} = \tfrac{1}{\sqrt{12}}A^{r+1}{}_\mu . \tag{2.95}$$

The expression of the 3-form field strengths in the Vielbein basis could also be useful

$$\begin{aligned}
\hat{H}^r{}_{abz} &= \tfrac{1}{\sqrt{12}}\left(\frac{h^0}{2}\right)^2 F^{r+1}{}_{ab} , \\
\hat{H}^r{}_{abc} &= -\tfrac{1}{2\sqrt{12}}\left(\frac{h^0}{2}\right)\mathcal{M}^r{}_s \epsilon_{abcde} F^{s+1\ de} ,
\end{aligned} \tag{2.96}$$

where $F^{s+1\ de}$ and ϵ_{abcde} are 5-dimensional quantities.

2.4.2 Uplift to $\mathcal{N} = 2A, d = 6$ supergravity

$\mathcal{N} = 2A, d = 6$ supergravity is the name we have given to the theories of $\mathcal{N} = (2,0), d = 6$ supergravity coupled to $n_T = 1$ tensor multiplets and n_V vector multiplets, with $c_{0\ ij} = c_{1\ ij} = \delta_{ij}, i = 1, \cdots, n_V$. Since in this case the two 2-forms have opposite chiralities, they can be combined into a single, unrestricted, 2-form that we are going to denote by $\tilde{B}$ (no indexes) and a standard covariant action exists, from which the equations of motion can be directly derived. It takes the form

$$\tilde{S} = \int d^6\tilde{x}\sqrt{|\tilde{g}|}\left\{\tilde{R} + \tfrac{1}{2}(\partial\tilde{\varphi})^2 + \tfrac{1}{3}e^{\sqrt{2}\tilde{\varphi}}\tilde{H}^2 - e^{\tilde{\varphi}/\sqrt{2}}\tilde{F}^i\tilde{F}^i\right\} , \tag{2.97}$$

where we are now using tildes instead of hats in order to distinguish these fields from those in the $\mathcal{N} = 2B$ theory and from the 5-dimensional ones. In this action, $i = 1, \cdots, n_V$ and the 3-form field strength is defined as

$$\tilde{H} = d\tilde{B} + \tilde{F}^i \wedge \tilde{A}^i \,. \tag{2.98}$$

This theory is obtained when the functions $\tilde{L}^r$, $r = 0, 1$ are parametrized as

$$\tilde{L}^0 = \cosh\left(\frac{\tilde{\varphi}}{\sqrt{2}}\right), \qquad \tilde{L}^1 = \sinh\left(\frac{\tilde{\varphi}}{\sqrt{2}}\right), \tag{2.99}$$

and $\tilde{H}$ and $\tilde{B}$ are related to the fields $\tilde{H}^r$ and $\tilde{B}^r$ appearing in (2.42) by

$$\tilde{B} = \tilde{B}^0 - \tilde{B}^1 \,, \qquad \tilde{H} = \tilde{H}^0 - \tilde{H}^1 \,. \tag{2.100}$$

Such a theory is obtained from the compactification of $\mathcal{N} = 1, d = 10$ supergravity coupled to vector multiplets (the effective field theory of the Heterotic String) on T^4 followed by a truncation. In particular, the scalar $\tilde{\varphi}$ is related to the dilaton field of the Heterotic String by

$$\tilde{\varphi} = -\sqrt{2}\,\phi_{\mathfrak{het}} \,. \tag{2.101}$$

As we have already pointed out, this theory gives (2.87) when reduced to five dimensions, exactly as in the $\mathcal{N} = 2B$ case.
To express the 6-dimensional fields in terms of the 5-dimensional ones, the linear transformation (2.83) have to be taken into account. In this way, the transformation rules for the vector fields are straightforwardly obtained. The relations among the scalar fields arise once the parameterizations (2.85) and (2.81) are considered. The outcome is given by the following expressions.

Scalar

The dilaton is related to the 5-dimensional scalars by

$$e^{\tilde{\varphi}/\sqrt{2}} = 2^{-1/2} h^0 (h^1 + h^2)^{1/2} \,. \tag{2.102}$$

Metric

The KK scalar ϕ and the KK vector A_μ are given by

$$e^{-2\phi/\sqrt{6}} = \tfrac{1}{2}(h^1 + h^2) \,, \qquad A_\mu = \tfrac{1}{\sqrt{12}}(A^1_\mu + A^2_\mu) \,, \tag{2.103}$$

and, therefore, the metric can be expressed as

$$
\begin{aligned}
\tilde{g}_{\underline{zz}} &= -2^{3/2}(h^1+h^2)^{-3/2}\,,\\
\tilde{g}_{\mu\underline{z}} &= -\sqrt{2/3}(h^1+h^2)^{-3/2}(A^1{}_\mu + A^2{}_\mu)\,,\\
\tilde{g}_{\mu\nu} &= \tfrac{1}{\sqrt{2}}(h^1+h^2)^{1/2}g_{\mu\nu} - \tfrac{1}{3\sqrt{2}}(h^1+h^2)^{-3/2}(A^1+A^2)_\mu(A^1+A^2)_\nu\,,
\end{aligned}
\tag{2.104}
$$

or equivalently, as

$$
d\tilde{s}^2 = -2^{3/2}(h^1+h^2)^{-3/2}\left[dz + \tfrac{1}{\sqrt{12}}(A^1+A^2)\right]^2 + 2^{-1/2}(h^1+h^2)^{1/2}ds^2\,. \tag{2.105}
$$

Vectors

The 1-form potentials are given by

$$
\begin{aligned}
\tilde{A}^i{}_{\underline{z}} &= -\frac{h^{i+2}}{h^1+h^2}\,,\\
\tilde{A}^i{}_\mu &= \tfrac{1}{\sqrt{12}}\left[A^{i+2}{}_\mu + \tilde{A}^i{}_{\underline{z}}(A^1{}_\mu + A^2{}_\mu)\right]\,,
\end{aligned}
\tag{2.106}
$$

or equivalently, by

$$
\tilde{A}^i = \tfrac{1}{\sqrt{12}}A^{i+2} - \frac{h^{i+2}}{h^1+h^2}\left[dz + \tfrac{1}{\sqrt{12}}(A^1+A^2)\right]\,. \tag{2.107}
$$

2-form

The components $\tilde{B}_{\mu\underline{z}}$ can be easily found to be

$$
\tilde{B}_{\mu\underline{z}} = \tfrac{1}{\sqrt{12}}(A^1{}_\mu - A^2{}_\mu)\,. \tag{2.108}
$$

The other components, $\tilde{B}_{\mu\nu}$, are independent and have to be explicitly given. They do not have a simple expression, though, and we must content ourselves with the field strength components instead

$$
\begin{aligned}
\tilde{H}_{\mu\nu\underline{z}} &= \tfrac{1}{\sqrt{3}}(h^1+h^2)^{-1}\Big\{\left[h^1 - [h^0(h^1+h^2)]^{-1}\right]F^1{}_{\mu\nu}\\
&\quad - \left[h^2 + [h^0(h^1+h^2)]^{-1}\right]F^2{}_{\mu\nu} + h^iF^i{}_{\mu\nu}\Big\}\,, \qquad i\geq 3\,,\\
\tilde{H}_{\mu\nu\rho} &= -\tfrac{1}{4\sqrt{3}}(h^0)^{-2}\frac{\epsilon_{\mu\nu\rho\alpha\beta}}{\sqrt{|g|}}F^{0\,\alpha\beta} + \tfrac{\sqrt{3}}{2}(A^1{}_{[\rho} + A^2{}_{[\rho})\tilde{H}_{\mu\nu]\underline{z}}\,.
\end{aligned}
\tag{2.109}
$$

2.4.3 Uplift to $\mathcal{N}=2A^*, d=6$ supergravity

The theory that we have been calling $\mathcal{N}=2A, d=6$ supergravity is not uniquely defined. Another theory can be obtained, that we are going to dub $\mathcal{N}=2A^*, d=6$ supergravity, when the field strength $\tilde{H}$ is dualized into $\breve{H}$,

$$\breve{H} = -e^{\sqrt{2}\tilde{\varphi}} \star \tilde{H} \,. \tag{2.110}$$

In the Einstein frame, the 2-form field strength is the only field which varies due to this transformation. However, we are denoting all the fields in this theory with a $\breve{}$ accent. It turns out that this new field strength is an exact 3-form,

$$\breve{H} = d\breve{B} \,, \tag{2.111}$$

where $\breve{H}$ and $\breve{B}$ are related to $\hat{H}^r$ and $\hat{B}^r$ in the theory (2.42) with $n_T = 1$, arbitrary n_V and $c_{0\,ij} = c_{1\,ij} = \delta_{ij}$ by

$$\breve{H} = \hat{H}^0 + \hat{H}^1 \,, \qquad \breve{B} = \hat{B}^0 + \hat{B}^1 \,. \tag{2.112}$$

We would like to point out that the absence of a Chern-Simons term in $\breve{H}$ is due to the cancellation among those in $\hat{H}^0$ and $\hat{H}^1$ and not to the vanishing of the constants $c^r{}_{ij}$.

The action is

$$\breve{S} = \int d^6\breve{x}\sqrt{|\breve{g}|}\left\{\breve{R} + \tfrac{1}{2}(\partial\breve{\varphi})^2 + \tfrac{1}{3}e^{-\sqrt{2}\breve{\varphi}}\breve{H}^2 - e^{\breve{\varphi}/\sqrt{2}}\breve{F}^i\breve{F}^i - \frac{\epsilon}{3\sqrt{|g|}}\breve{H}\breve{F}^i\breve{A}^i\right\}. \tag{2.113}$$

This theory can be obtained from the effective field theory of the type IIA superstrings compactified on K3 [75, 99, 110–112] followed by a truncation. In particular, the scalar $\breve{\varphi}$ (which coincides with $\tilde{\varphi}$), is related to the dilaton of that superstring theory by

$$\breve{\varphi} = \sqrt{2}\,\phi_{IIA} \,. \tag{2.114}$$

The different coupling of the dilaton field to the vector fields, with respect to the $\mathcal{N} = 2A$ case, is mainly due to the fact that RR fields are now present, instead of NSNS ones.

All the fields can be expressed in terms of the 5-dimensional ones as in the previous case (2.102,2.105,2.107), except for the 2-form $\breve{B}$, whose components $\mu\underline{z}$ now are given by

$$\breve{B}_{\mu\underline{z}} = \tfrac{1}{\sqrt{12}}A^0{}_\mu \,. \tag{2.115}$$

The 3-form field strength is

$$\begin{aligned}
\breve{H}_{\mu\nu\underline{z}} &= \tfrac{1}{\sqrt{12}}F^0{}_{\mu\nu} \,, \\
\breve{H}_{\mu\nu\rho} &= -\tfrac{1}{8\sqrt{3}}(h^0)^2(h^1+h^2)\frac{\epsilon_{\mu\nu\rho\alpha\beta}}{\sqrt{|g|}}\Big\{\big[h^1 - [h^0(h^1+h^2)]^{-1}\big]F^{1\,\alpha\beta} \\
&\quad - \big[h^2 + [h^0(h^1+h^2)]^{-1}\big]F^{2\,\alpha\beta} + h^i F^{i\,\alpha\beta}\Big\} + \tfrac{\sqrt{3}}{2}(A^1{}_{[\rho} + A^2{}_{[\rho})\breve{H}_{\mu\nu]\underline{z}} \,, \\
&\qquad i \geq 3 \,.
\end{aligned} \tag{2.116}$$

2.5 Maps between 6-dimensional theories

All the results we presented so far are necessary to generalize the Buscher rules, connecting directly the $\mathcal{N}=2A, 2A^*$ and $2B$ theories. In what follows, we are summarizing the rules to express the fields of one theory in terms of those of another one.

From $\mathcal{N}=2B$ to $\mathcal{N}=2A$

$$
\begin{aligned}
e^{\sqrt{2}\tilde{\varphi}} &= -2\frac{\hat{L}^0+\hat{L}^1}{\hat{g}_{\underline{zz}}}\,,\\
\tilde{g}_{\underline{zz}} &= -2^{3/2}(\hat{L}^0+\hat{L}^1)^{-3/2}|\hat{g}_{\underline{zz}}|^{-1/2}\,,\\
\tilde{g}_{\mu\underline{z}} &= -2^{3/2}(\hat{L}^0+\hat{L}^1)^{-3/2}|\hat{g}_{\underline{zz}}|^{-1/2}(\hat{B}^0+\hat{B}^1)_{\mu\underline{z}}\,,\\
\tilde{g}_{\mu\nu} &= 2^{-1/2}(\hat{L}^0+\hat{L}^1)^{1/2}\Big[|\hat{g}_{\underline{zz}}|^{1/2}\hat{g}_{\mu\nu}+|\hat{g}_{\underline{zz}}|^{-1/2}\hat{g}_{\mu\underline{z}}\hat{g}_{\nu\underline{z}}\Big]\\
&\quad -2^{3/2}(\hat{L}^0+\hat{L}^1)^{-3/2}|\hat{g}_{\underline{zz}}|^{-1/2}(\hat{B}^0+\hat{B}^1)_{\mu\underline{z}}(\hat{B}^0+\hat{B}^1)_{\nu\underline{z}}\,,\\
\tilde{A}^i{}_{\underline{z}} &= -\frac{\hat{L}^{i+1}}{\hat{L}^0+\hat{L}^1}\,,\\
\tilde{A}^i{}_{\mu} &= \hat{B}^{i+1}{}_{\mu\underline{z}}-\frac{\hat{L}^{i+1}}{\hat{L}^0+\hat{L}^1}(\hat{B}^0+\hat{B}^1)_{\mu\underline{z}}\,,\\
\tilde{B}_{\mu\underline{z}} &= (\hat{B}^0-\hat{B}^1)_{\mu\underline{z}}\,.
\end{aligned}
\tag{2.117}
$$

From $\mathcal{N}=2A$ to $\mathcal{N}=2B$

$$
\begin{aligned}
\left|\hat{g}_{\underline{zz}}\right| &= 2^{\frac{3}{2}}e^{-\frac{3}{2\sqrt{2}}\tilde{\varphi}}\left|\tilde{g}_{\underline{zz}}\right|^{-\frac{1}{2}}\,,\\
\hat{g}_{\mu\underline{z}} &= -2^{\frac{3}{2}}e^{-\frac{3}{2\sqrt{2}}\tilde{\varphi}}\left|\tilde{g}_{\underline{zz}}\right|^{-\frac{1}{2}}(\tilde{B}^0+\tilde{B}^1)_{\mu\underline{z}}\,,\\
\hat{g}_{\mu\nu} &= 2^{-\frac{1}{2}}\left|\tilde{g}_{\underline{zz}}\right|^{\frac{1}{2}}e^{\frac{\tilde{\varphi}}{2\sqrt{2}}}\left(\tilde{g}_{\mu\nu}-\frac{\tilde{g}_{\mu\underline{z}}\tilde{g}_{\nu\underline{z}}}{\tilde{g}_{\underline{zz}}}\right)\\
&\quad +2^{\frac{3}{2}}e^{-\frac{3}{2\sqrt{2}}\tilde{\varphi}}\left|\tilde{g}_{\underline{zz}}\right|^{-\frac{1}{2}}(\tilde{B}^0+\tilde{B}^1)_{\mu\underline{z}}(\tilde{B}^0+\tilde{B}^1)_{\nu\underline{z}}\,,\\
\hat{L}^0 &= 2^{-\frac{3}{2}}e^{-\frac{\tilde{\varphi}}{2\sqrt{2}}}\left|\tilde{g}_{\underline{zz}}\right|^{\frac{1}{2}}+2^{-\frac{1}{2}}e^{\frac{\tilde{\varphi}}{2\sqrt{2}}}\left|\tilde{g}_{\underline{zz}}\right|^{-\frac{1}{2}}\left(1+\tilde{A}^r{}_{\underline{z}}\tilde{A}^r{}_{\underline{z}}\right)\,, \qquad r>1\,,\\
\hat{L}^1 &= -2^{-\frac{3}{2}}e^{-\frac{\tilde{\varphi}}{2\sqrt{2}}}\left|\tilde{g}_{\underline{zz}}\right|^{\frac{1}{2}}+2^{-\frac{1}{2}}e^{\frac{\tilde{\varphi}}{2\sqrt{2}}}\left|\tilde{g}_{\underline{zz}}\right|^{-\frac{1}{2}}\left(1-\tilde{A}^r{}_{\underline{z}}\tilde{A}^r{}_{\underline{z}}\right)\,, \qquad r>1\,,\\
\hat{L}^r &= -\sqrt{2}\left|\tilde{g}_{\underline{zz}}\right|^{-\frac{1}{2}}e^{\frac{\tilde{\varphi}}{2\sqrt{2}}}\tilde{A}^{r-1}{}_{\underline{z}}\,, \qquad r\geq 2\,,\\
\hat{B}^0{}_{\mu\underline{z}} &= \frac{1}{2}\left(\tilde{B}_{\mu\underline{z}}+\frac{\tilde{g}_{\mu\underline{z}}}{\tilde{g}_{\underline{zz}}}\right)\,,\\
\hat{B}^1{}_{\mu\underline{z}} &= \frac{1}{2}\left(-\tilde{B}_{\mu\underline{z}}+\frac{\tilde{g}_{\mu\underline{z}}}{\tilde{g}_{\underline{zz}}}\right)\,,\\
\hat{B}^r{}_{\mu\underline{z}} &= \tilde{A}^{r-1}{}_{\mu}-\tilde{A}^{r-1}{}_{\underline{z}}\frac{\tilde{g}_{\mu\underline{z}}}{\tilde{g}_{\underline{zz}}}\,, \qquad r\geq 2\,.
\end{aligned}
\tag{2.118}
$$

From $\mathcal{N}=2B$ to $\mathcal{N}=2A^*$

$$
\begin{aligned}
e^{\sqrt{2}\breve{\varphi}} &= -2\frac{\hat{L}^0+\hat{L}^1}{\hat{g}_{\underline{zz}}}\,,\\
\breve{g}_{\underline{zz}} &= -2^{3/2}(\hat{L}^0+\hat{L}^1)^{-3/2}|\hat{g}_{\underline{zz}}|^{-1/2}\,,\\
\breve{g}_{\mu\underline{z}} &= -2^{3/2}(\hat{L}^0+\hat{L}^1)^{-3/2}|\hat{g}_{\underline{zz}}|^{-1/2}(\hat{B}^0+\hat{B}^1)_{\mu\underline{z}}\,,\\
\breve{g}_{\mu\nu} &= 2^{-1/2}(\hat{L}^0+\hat{L}^1)^{1/2}\Big[|\hat{g}_{\underline{zz}}|^{1/2}\hat{g}_{\mu\nu}+|\hat{g}_{\underline{zz}}|^{-1/2}\hat{g}_{\mu\underline{z}}\hat{g}_{\nu\underline{z}}\Big]\\
&\qquad -2^{3/2}(\hat{L}^0+\hat{L}^1)^{-3/2}|\hat{g}_{\underline{zz}}|^{-1/2}(\hat{B}^0+\hat{B}^1)_{\mu\underline{z}}(\hat{B}^0+\hat{B}^1)_{\nu\underline{z}}\,,\\
\breve{A}^i{}_{\underline{z}} &= -\frac{\hat{L}^{i+1}}{\hat{L}^0+\hat{L}^1}\,,\\
\breve{A}^i{}_{\mu} &= \hat{B}^{i+1}{}_{\mu\underline{z}}-\frac{\hat{L}^{i+1}}{\hat{L}^0+\hat{L}^1}(\hat{B}^0+\hat{B}^1)_{\mu\underline{z}}\,,\\
\breve{B}_{\mu\underline{z}} &= \frac{\hat{g}_{\mu\underline{z}}}{\hat{g}_{\underline{zz}}}\,.
\end{aligned}
\tag{2.119}
$$

From $\mathcal{N}=2A^*$ to $\mathcal{N}=2B$

$$
\begin{aligned}
\left|\hat{g}_{\underline{zz}}\right| &= 2^{\frac{3}{2}}e^{-\frac{3}{2\sqrt{2}}\breve{\varphi}}\left|\breve{g}_{\underline{zz}}\right|^{-\frac{1}{2}}\,,\\
\hat{g}_{\mu\underline{z}} &= -2^{\frac{3}{2}}e^{-\frac{3}{2\sqrt{2}}\breve{\varphi}}\left|\breve{g}_{\underline{zz}}\right|^{-\frac{1}{2}}\breve{B}_{\mu\underline{z}}\,,\\
\hat{g}_{\mu\nu} &= 2^{-\frac{1}{2}}\left|\breve{g}_{\underline{zz}}\right|^{\frac{1}{2}}e^{\frac{\breve{\varphi}}{2\sqrt{2}}}\left(\breve{g}_{\mu\nu}-\frac{\breve{g}_{\mu\underline{z}}\breve{g}_{\nu\underline{z}}}{\breve{g}_{\underline{zz}}}\right)+2^{\frac{3}{2}}e^{-\frac{3}{2\sqrt{2}}\breve{\varphi}}\left|\breve{g}_{\underline{zz}}\right|^{-\frac{1}{2}}\breve{B}_{\mu\underline{z}}\breve{B}_{\nu\underline{z}}\,,\\
\hat{L}^0 &= 2^{-\frac{3}{2}}e^{-\frac{\breve{\varphi}}{2\sqrt{2}}}\left|\breve{g}_{\underline{zz}}\right|^{\frac{1}{2}}+2^{-\frac{1}{2}}e^{\frac{\breve{\varphi}}{2\sqrt{2}}}\left|\breve{g}_{\underline{zz}}\right|^{-\frac{1}{2}}\left(1+\breve{A}^r{}_{\underline{z}}\breve{A}^r{}_{\underline{z}}\right)\,,\qquad r>1\,,\\
\hat{L}^1 &= -2^{-\frac{3}{2}}e^{-\frac{\breve{\varphi}}{2\sqrt{2}}}\left|\breve{g}_{\underline{zz}}\right|^{\frac{1}{2}}+2^{-\frac{1}{2}}e^{\frac{\breve{\varphi}}{2\sqrt{2}}}\left|\breve{g}_{\underline{zz}}\right|^{-\frac{1}{2}}\left(1-\breve{A}^r{}_{\underline{z}}\breve{A}^r{}_{\underline{z}}\right)\,,\qquad r>1\,,\\
\hat{L}^r &= -\sqrt{2}\left|\breve{g}_{\underline{zz}}\right|^{-\frac{1}{2}}e^{\frac{\breve{\varphi}}{2\sqrt{2}}}\breve{A}^{r-1}_{\underline{z}}\,,\qquad r\geq 2\,,\\
\hat{B}^0{}_{\mu\underline{z}} &= \tfrac{1}{2}\left(\tilde{B}_{\mu\underline{z}}+\frac{\breve{g}_{\mu\underline{z}}}{\breve{g}_{\underline{zz}}}\right)\,,\\
\hat{B}^1{}_{\mu\underline{z}} &= \tfrac{1}{2}\left(-\tilde{B}_{\mu\underline{z}}+\frac{\breve{g}_{\mu\underline{z}}}{\breve{g}_{\underline{zz}}}\right)\,,\\
\hat{B}^r{}_{\mu\underline{z}} &= \breve{A}^{r-1}{}_{\mu}-\breve{A}^{r-1}{}_{\underline{z}}\frac{\breve{g}_{\mu\underline{z}}}{\breve{g}_{\underline{zz}}}\,,\qquad r\geq 2\,.
\end{aligned}
\tag{2.120}
$$

CHAPTER 3

Solutions: some examples

So far, we have been presenting in some detail three techniques to generate solutions of $\mathcal{N} = 2$ gauged supergravities.
In 4 dimensions, the classification presented in [4], which we reported in section 1.2.5, is available. Exploiting the existence of Killing spinors, it provides first order equations that have to be solved in order to find supersymmetric solutions. This approach is very general and has been applied to a number of different models, but the resulting equations can, under certain circumstances, be really involved and difficult to be solved.
Therefore, we have also mentioned a less general way to obtain first order equations, under specific assumptions, called the "squaring" of the action (section 1.2.6). It turned out to be a powerful tool to find new solutions, an example of which is given in section 3.1.
We finally explained how some models of $\mathcal{N} = 2$ supergravity in 4 and 5 dimensions and $\mathcal{N} = (2, 0)$ supergravity in 6 dimensions are related through dimensional reduction, and so are their solution, under certain assumptions. It is therefore evident that, once a solution for one of these theories is known, the connection to the other theories can be exploited to generate new solutions. In particular, this techniques has been used to uplift solutions of the well known 4-dimensional gauged supergravity, producing new solutions in 5 and in 6 dimensions, where a general classification is missing. On the other hand, it provided the dictionary to reduce some 6-dimensional solutions to those of the SU(2)-FI gauged 4-dimensional theory. In this latter case, the "direct" method of solving the equations coming from the general classification [4] was hardly tractable, and dimensional reduction constituted an interesting alternative.

The present chapter is entirely devoted to the presentation of some new solutions we found in [1], [2] and [3] and to their physical properties.

The first solution we mentioned has been obtained with the method of the "squaring of the action". We considered $\mathcal{N} = 2$, 4-dimensional supergravity coupled to vector multiplets only, with a U(1) Fayet-Iliopoulos gauging. Black hole solutions in this theory had already been widely studied, for example in [5,18–22,25,26,28,31]. However, all the known solutions involved models in which the Kähler manifold parametrized by the

scalars is symmetric and homogeneous, such as the stu model and its truncations. As far as we know, the solution that we found in [1] and that we are presenting in section 3.1, has been the first one involving a non-homogeneous special Kähler manifold. As already mentioned, we named the model nh-stu, since the prepotential by which it is defined is a deformation of the one giving rise to the stu model and it produces a non-homogeneous geometry. The interest for the nh-stu model is motivated by its stringy origin, as explained for example in [68]. The FI gauging constitutes another reason of interest, in fact it leads to a scalar potential with two critical points corresponding to AdS vacua. One of these extremizes also the superpotential and is thus supersymmetric, while the other vacuum breaks supersymmetry.
Moreover, the solution we found is asymptotically AdS_4 and therefore suitable for being studied from a AdS/CFT perspective. This program has been accomplished in [113], where fluid/gravity correspondence has been used to study the dual holographic superconductor.

In section 3.2 we are turning our attention to the 4-dimensional theory with an $\mathrm{SU}(2)$-FI gauging. This theory had not been studied before, and the cause has probably to be found in the difficulty of considering at the same time the potential originated by the Fayet-Iliopoulos term and the non-Abelian gauge fields that are necessarily present, as we have already explained. We considered the simplest model that admits this kind of gauging, the $\overline{\mathbb{CP}}^3$ model, specified the equations given in [4] and solved them. The results were presented in [3] and they include an exact $\mathrm{AdS}_2\times\mathrm{S}^2$ solution which is $\frac{1}{8}$-BPS and a $\mathbb{R}\times\mathbb{H}^3$ geometry, supported by a 2-form field strength.

More solutions were found for the same theory and a different, cubic model, thanks to the results in [2]. They allowed us to dimensionally reduce to 4 dimensions the 6-dimensional solutions of [38], which presented an $\mathrm{SU}(2)$-FI gauging. The results can be found in section 3.4, where solutions to the 5-dimensional theory have also been presented, since they constitute a necessary step in the reduction procedure.

Finally, section 3.3 presents an explicit example for the procedure we outlined in chapter 2: in [30], some solutions of the 5-dimensional SEYM theory were obtained, thanks to the existence of a general classification [34, 35] and to the simplification that was achieved by considering the correlation to 4-dimensional SEYM theories. We uplifted the simplest of those solutions to 6 dimensions, finding a new solution for every theory that -as we have demonstrated in the previous chapter- can be reduced to the same 5-dimensional model, $\mathcal{N}=2A$, $\mathcal{N}=2A^*$ and $\mathcal{N}=2B$.
With respect to the treatment we have given in the previous chapter 2, a peculiarity has here been added: the non-Abelian gauge fields characterizing Super-Einstein-Yang-Mills theories. As far as the $\mathcal{N}=2A$ and $\mathcal{N}=2A^*$ theories are concerned, a general classification for this kind of 6-dimensional solutions is not known and our procedure gives rise to otherwise unknown solutions. The $\mathcal{N}=2B$ theory cannot be gauged, at least not in a conventional way; comments concerning the meaning of a gauged $\mathcal{N}=2B$, $d=6$

theory can be found in section 3.3.2.

3.1 A non-homogeneous deformation of the stu model

We are now presenting a black hole solution for the nh-stu model in 4-dimensional, $\mathcal{N}=2$ supergravity with a U(1) Fayet-Iliopoulos gauging. It has been obtained thanks to the squaring procedure described in section 1.2.6, i.e. solving the equations

$$\begin{aligned}
&2e^{2\psi}\left(e^{-U}\Im\mathfrak{m}(e^{-i\alpha}\mathcal{V})\right)' + e^{2(\psi-U)}\Omega MG + 4e^{-U}(\alpha'+\mathcal{A}_r)\Re\mathfrak{e}(e^{-i\alpha}\mathcal{V}) + \mathcal{Q} = 0\,,\\
&\psi' = 2e^{-U}\Im\mathfrak{m}(e^{-i\alpha}L)\,,\\
&\alpha' + \mathcal{A}_r = -2e^{-U}\Re\mathfrak{e}(e^{-i\alpha}L)\,,
\end{aligned} \tag{3.1}$$

where the metric has been chosen of the form

$$\mathrm{d}s^2 = -e^{2U(r)}\mathrm{d}t^2 + e^{-2U(r)}(\mathrm{d}r^2 + e^{2\psi(r)}\mathrm{d}\Omega_\kappa^2)\,, \tag{3.2}$$

and we remind that α is defined as

$$e^{2i\alpha} = \frac{Z - ie^{2(\psi-U)}L}{Z^* + ie^{2(\psi-U)}L^*}\,. \tag{3.3}$$

As is the case for many other known solutions [22,28,71,114], we shall assume vanishing axions. This is realized by purely imaginary scalars (with $\lambda^i > 0$),

$$Z^i = x^i - i\lambda^i\,, \qquad x^i = 0\,. \tag{3.4}$$

The advantage of this choice will become evident in what follows: for some values of the FI parameters G, it indeed simplifies the equations of motion (3.1), setting α to a constant.

3.1.1 Dyonic Fayet-Iliopoulos gaugings and near-horizon analysis

To proceed further, we shall assume a specific form for the FI parameters G. The choice

$$G^T = (0, g^1, g^2, g^3, g_0, 0, 0, 0)^T\,, \tag{3.5}$$

together with the vanishing axion condition (3.4), fixes the phase α in (3.1) to the constant value[1] $\alpha = \pm\pi/2$. This can be checked from the explicit expressions of the symplectic invariants Z and L,

[1] Another possible choice yielding the same constant value for α is $G^T = (g^0, 0, 0, 0, 0, g_1, g_2, g_3)^T$, which would in turn require $\mathcal{Q}$ to assume the (magnetic) form $\mathcal{Q}^T = (0, p^1, p^2, p^3, q_0, 0, 0, 0)^T$.

$$
\begin{aligned}
Z &= ie^{\mathcal{K}/2}\left(p^0\left(\lambda^1\lambda^2\lambda^3 - \frac{A}{3}\lambda^3\right) - q_1\lambda^1 - q_2\lambda^2 - q_3\lambda^3\right) , \\
L &= e^{\mathcal{K}/2}\left(g_0 + g^1\lambda^2\lambda^3 + g^2\lambda^1\lambda^3 + g^3(\lambda^1\lambda^2 - A(\lambda^3)^2)\right) .
\end{aligned}
\tag{3.6}
$$

As can be inferred from the BPS equations (3.1), the choice (3.5) requires some charges to vanish, so that the vector $\mathcal{Q}$ takes the form

$$
\mathcal{Q}^T = (p^0, 0, 0, 0, 0, q_1, q_2, q_3)^T . \tag{3.7}
$$

With the choice (3.5), the scalar potential becomes

$$
\begin{aligned}
\mathbf{V} \;=\; & -g^2g^3\lambda^1 - g^1g^3\lambda^2 - \left(g^1g^2 - A(g^3)^2\right)\lambda^3 \\
& -\frac{g_0}{\lambda^1\lambda^2\lambda^3 - \frac{A}{3}(\lambda^3)^3}\left(g^2\lambda^1\lambda^3 + g^1\lambda^2\lambda^3 + g^3\left(\lambda^1\lambda^2 - A(\lambda^3)^2\right)\right) ,
\end{aligned}
\tag{3.8}
$$

which,for $A = 0$, matches the known expression for the stu model [9,28]. In what follows we shall assume that all gauge coupling constants g_0, g^i are positive. Then the potential (3.8) has two critical points, namely one for

$$
\lambda^1 = \frac{g^1}{g^3}\lambda^3 , \qquad \lambda^2 = \frac{g^2}{g^3}\lambda^3 , \qquad \lambda^3 = \sqrt{\frac{g_0g^3}{g^1g^2 - \frac{A}{3}(g^3)^2}} , \tag{3.9}
$$

and the other for

$$
\lambda^1 = \frac{g^1}{g^3}\lambda^3 , \qquad \lambda^2 = -\frac{1}{g^1g^3}\left(g^1g^2 - \frac{2}{3}A(g^3)^2\right)\lambda^3 , \qquad \lambda^3 = \sqrt{\frac{g_0g^3}{g^1g^2 - \frac{A}{3}(g^3)^2}} . \tag{3.10}
$$

The first has $\mathbf{V} = -3\ell^{-2}$, and the second $\mathbf{V} = -\ell^{-2}$, with ℓ defined in (3.33), so both correspond to AdS vacua. It can easily be shown that (3.9) is also a critical point of the superpotential L, while (3.10) is not. The vacuum (3.9) is thus supersymmetric, whereas (3.10) breaks supersymmetry. Moreover, reality and positivity of the scalars λ^i imply that the second vacuum exists only in the range

$$
\frac{3}{2}\frac{g^1g^2}{(g^3)^2} < A < 3\frac{g^1g^2}{(g^3)^2} , \tag{3.11}
$$

in particular it is not present for zero deformation parameter A.

Owing to the constancy of α, the equations of motion (3.1) boil down to

$$
\begin{aligned}
& 2e^{2\psi}\left(e^{-U}\Re\mathfrak{e}\mathcal{V}\right)' + e^{2(\psi-U)}\Omega MG + \mathcal{Q} = 0 , \\
& (e^{\psi})' = 2e^{\psi-U}\Re\mathfrak{e}L .
\end{aligned}
\tag{3.12}
$$

The near-horizon geometry is required to be $\mathrm{AdS}_2 \times \Sigma$, i.e., the metric functions in (3.2) should take the form

$$e^U = \frac{r}{R_1}\,, \qquad e^\psi = r\,\frac{R_2}{R_1}\,, \tag{3.13}$$

while the scalar fields $Z^i(r) = -i\lambda^i(r)$ assume a constant value on the horizon. Under this assumption, the BPS equations (3.12) simplify to

$$\begin{aligned} &\mathcal{Q} + R_2^2 \Omega M G = -4\Im\mathfrak{m}\left(Z^* \mathcal{V}\right)\,, \\ &Z = i\,\frac{R_2^2}{2R_1}\,. \end{aligned} \tag{3.14}$$

In addition, the constraint

$$\langle G, \mathcal{Q}\rangle = -\kappa\,. \tag{3.15}$$

has to be imposed.

Following the procedure described in [115][2], the BPS equations in the near-horizon limit (3.14) provide a set of equations for the variables $\{R_1, R_2, \lambda^i\}$ as functions of the gaugings g_0, g^i and the charges p^0, q_i.

In particular, since R_2 is directly related to the black hole entropy S, this yields an expression for S in terms of the gaugings and charges. In the model described above, the attractor equations (3.14) are implicitly solved by

$$\begin{aligned} &R_2^4\, d_{g,i} + \frac{1}{3}\left(\kappa + \frac{1}{2}\right) p^0 q_i = \frac{1}{36}\left(d_\lambda^{-1}\right)^{ij} q_j\, q_i - \frac{1}{4}\left(p^0\right)^2 d_{\lambda,i}\,, \\ &\lambda^i\left(1 - \frac{\kappa}{2}\right) = \frac{\kappa}{p^0}\left(-R_2^2\, g^i + \frac{1}{6}\left(d_\lambda^{-1}\right)^{ij} q_j\right)\,, \\ &\frac{R_2^2}{R_1} = \left(p^0 e^{-\frac{\kappa}{2}}\left(\kappa - \frac{3}{4}\right) - 2\, e^{\frac{\kappa}{2}}\lambda^j q_j\right)\,, \\ &R_2^6\, d_g + \frac{1}{2} R_2^2\, p^0\left(p^0 g_0 + \kappa g^i q_i\right) = \frac{1}{216}\left(d_\lambda^{-1}\right)^k \left(d_\lambda^{-1}\right)^{ij} q_i\, q_j\, q_k \\ &\qquad + \frac{1}{64} p^0 q_i\, q_j\left(\left(d_\lambda^{-1}\right)^j \lambda^i + 2\left(d_\lambda^{-1}\right)^{ij}\right) + \frac{1}{8}\left(p^0\right)^2\left(\lambda^i q_i + p^0 d_\lambda\right)\,, \end{aligned} \tag{3.16}$$

where the contractions of the tensor d_{ijk} are defined as in (1.200). Note that the non-homogeneity enters through $(d_\lambda^{-1})^{ij}$, that depends on the special Kähler metric, since

$$\mathcal{G}^{ij} = -\frac{2}{3} d_\lambda (d_\lambda^{-1})^{ij} + 2\lambda^i \lambda^j\,,$$

cf. equation (A.6) of [115].

An explicit solution to (3.16) cannot be obtained by applying the analysis developed in [115] for the case of symmetric special Kähler manifolds, because the model under consideration is neither symmetric nor homogeneous, as explained in section 1.2.7.

[2]The equations (3.16) are based on [115], with some misprints corrected.

3.1.2 A black hole solution

In this paragraph, we are presenting an exact black hole solution for the nh-stu model. In order to simplify the BPS equations (3.12), we introduce the functions[3]

$$\begin{aligned} H^0 &= \frac{e^{-U}}{\sqrt{2}}\left(\lambda^1\lambda^2\lambda^3 - \frac{A}{3}(\lambda^3)^3\right)^{-\frac{1}{2}}, \\ H_1 &= \lambda^2\lambda^3 H^0\,, \qquad H_2 = \lambda^1\lambda^3 H^0\,, \qquad H_3 = (\lambda^3)^2 H^0\,. \end{aligned} \tag{3.17}$$

In terms of these latter, equations (3.12) become

$$\begin{aligned} &(H^0)' + 2g_0(H^0)^2 = -e^{-2\psi}p^0\,, \\ &H_1'^1 H_1^2 + \frac{2}{3}Ag^2H_3^2 - \frac{4}{3}Ag^3H_1H_3 = e^{-2\psi}q_1\,, \\ &H_2'^2 H_2^2 + \frac{2}{3}Ag^1H_3^2 - \frac{4}{3}Ag^3H_2H_3 = e^{-2\psi}q_2\,, \\ &H_3' + 2H_3(g^1H_1 + g^2H_2) - 2g^3\left(H_1H_2 + \frac{A}{3}H_3^2\right) = \\ &\quad = e^{-2\psi}\frac{H_3}{H_1H_2 + AH_3^2}(q_1H_2 + q_2H_1 - q_3H_3)\,, \\ &\psi' = g_0H^0 + g^1H_1 + g^2H_2 + g^3\left(\frac{H_1H_2}{H_3} - AH_3\right)\,. \end{aligned} \tag{3.18}$$

A remarkable feature of the nh-stu model is that, contrary to e.g. the case considered in [28], the equations (3.18) cannot be decoupled, due to the non-diagonal terms in the metric (1.209). Following the strategy of [28], we introduce the Ansatz

$$\begin{aligned} &\psi = \log\left(a\,r^2 + c\right)\,, \\ &H^0 = e^{-\psi}\left(\alpha^0 r + \beta^0\right)\,, \\ &H_i = e^{-\psi}\left(\alpha_i r + \beta_i\right)\,, \qquad i = 1,2,3\,. \end{aligned} \tag{3.19}$$

The solution for the fields is then expressed in terms of the functions H^0, H_i by inverting the relations (3.17). This yields

$$e^{2U} = \frac{1}{2}\left(\frac{H_3}{H^0}\right)^{\frac{1}{2}}\left(H_1H_2 - \frac{A}{3}H_3^2\right)^{-1}\,, \tag{3.20}$$

and

[3] A common choice for the functions H_i is to make them coincide with the components of the symplectic sections. For the present situation, we preferred to choose H_3 in a different way, in order to simplify the structure of the equations.

$$\lambda^1 = H_2 \left(H_3 H^0\right)^{-\frac{1}{2}} , \qquad \lambda^2 = H_1 \left(H_3 H^0\right)^{-\frac{1}{2}} , \qquad \lambda^3 = \left(\frac{H_3}{H^0}\right)^{\frac{1}{2}} , \tag{3.21}$$

for the warp factor and the scalars respectively.

By means of the Ansatz (3.19), the differential equations (3.12) boil down to a system of algebraic conditions on the parameters and the charges characterizing the solution, i.e. $\{\alpha^0, \alpha_i, \beta^0, \beta_i, a, c, p^0, q_i\}$. The set of equations obtained in this way reduces, after some algebraic manipulations, to

$$\begin{aligned}
\alpha^0 &= \frac{a}{2g_0} , \qquad \alpha_1 = \frac{g^2}{g^3}\,\alpha_3 , \qquad \alpha_2 = \frac{g^1}{g^3}\,\alpha_3 , \qquad \alpha_3 = \frac{a\,g^3}{2\left(g^1 g^2 - \frac{A}{3}\left(g^3\right)^2\right)} , \\
\beta_1 &= \frac{g^2}{g^3}\,\beta_3 , \qquad \beta_2 = -\frac{1}{2}\,\beta_3 \left(\frac{g^1}{g^3} - A\frac{g^3}{g^2}\right) - \frac{1}{2}\,\beta^0\,\frac{g_0}{g^2} , \\
q_1 &= 2\,\beta_3^2\,\frac{g^2}{\left(g^3\right)^2}\left(g^1 g^2 - \frac{A}{3}\left(g^3\right)^2\right) + g^2\,\frac{ac}{2\left(g^1 g^2 - \frac{A}{3}\left(g^3\right)^2\right)} , \\
q_2 &= \frac{1}{2g^2}\left(\beta^0 g_0 + \beta_3\,\frac{g^1 g^2}{g^3}\right)^2 + g^1\,\frac{ac}{2\left(g^1 g^2 - \frac{A}{3}\left(g^3\right)^2\right)} \\
&\quad + \frac{A}{3}\,\beta_3\,\frac{g^3}{g^2}\left(\beta_3\,\frac{g^1 g^2}{g^3} - \beta^0 g_0 - \frac{A}{2}\beta_3 g^3\right) , \\
q_3 &= \frac{g^2}{g^3}\,q_2 - A\,\frac{g^3}{g^2}\,q_1 , \qquad p^0 = -\frac{ac}{2g_0} - 2g_0\left(\beta^0\right)^2 .
\end{aligned} \tag{3.22}$$

The solution for the scalars is obtained by plugging the parameters (3.22) into the expressions (3.21). In this way, the scalars assume the explicit form

$$\begin{aligned}
\lambda^1 &= \frac{a\,\frac{g^1}{g^3}\left(\lambda^3_\infty\right)^2 r - g_0\,\beta_3\left(\frac{g^1}{g^3} - A\,\frac{g^3}{g^2}\right) - \beta^0\,\frac{g_0^2}{g^2}}{\sqrt{\left(2g_0\,\beta^0 + a\,r\right)\left(2g_0\,\beta_3 + a\,r\left(\lambda^3_\infty\right)^2\right)}} , \\
\lambda^2 &= \frac{g^2}{g^3}\lambda^3 , \qquad \lambda^3 = \lambda^3_\infty\sqrt{\frac{ar + \frac{2\,g_0}{(\lambda^3_\infty)^2}\,\beta_3}{ar + 2g_0\beta^0}} ,
\end{aligned} \tag{3.23}$$

where λ^3_∞ is the asymptotic value of λ^3,

$$\lambda^3_\infty = \sqrt{\frac{g_0 g^3}{g^1 g^2 - \frac{A}{3}(g^3)^2}} . \tag{3.24}$$

The warp factor in the metric reads

$$\mathrm{e}^{2U} = \frac{2g_0 g^3 (ar^2+c)^2}{\lambda_\infty^3 \left(ar - g_0\beta^0 - \frac{g_0}{(\lambda_\infty^3)^2}\beta_3\right)\sqrt{(ar+2g_0\beta^0)\left(ar+\frac{2g_0}{(\lambda_\infty^3)^2}\beta_3\right)}}\,. \tag{3.25}$$

This solution represents a black hole, with a horizon at the largest zero of e^{2U}, i.e., at $r_{\mathrm{h}} = \sqrt{-c/a}$, where we assumed $a > 0$ and $c < 0$. The curvature invariants diverge where the angular component of the metric $e^{2\psi-2U}$ vanishes. Note that all the scalar fields λ_i should be well-defined and positive outside the horizon. Moreover, we still have to impose the condition (3.15), i.e.,

$$g_0 p^0 - g^i q_i = -\kappa \tag{3.26}$$

on the solution (3.22). We checked that these requirements are compatible with all of the three possible choices for $\kappa = 0\,, \pm 1$, i.e. the horizon topology can be either spherical, flat or hyperbolic.

The Dirac quantization condition (3.26) fixes one of the four parameters $\{a, c, \beta^0, \beta_3\}$ that determine the solution, for example c . Furthermore, it is easily seen that the solution enjoys the scaling symmetry

$$(t, r, \theta, \phi, a, c, \beta^0, \beta_3, \kappa) \mapsto (t/s, sr, \theta, \phi, a/s, sc, \beta^0, \beta_3, \kappa)\,, \qquad s \in \mathbb{R}\,, \tag{3.27}$$

that can be used to set $a = 1$ without loss of generality. Consequently, there are only two physical parameters left, on which the solution depends, β^0 and β_3.
Notice that the solution (3.22) is characterized by the proportionality between the scalars λ_2 and λ_3, as is evident from (3.23). However, it is worth stressing that this fact does not trivialize our results, since the locus $\lambda^2 = \frac{g^2}{g^3}\lambda^3$ in the scalar manifold does not yield a consistent two-moduli truncation for the model (1.186). In other words, the Kähler geometry that can be derived from the truncated model $F\left(\mathcal{X}^1, \mathcal{X}^2, \mathcal{X}^3\right)\big|_{\lambda^2 \propto \lambda^3}$ is not equivalent to the 2-dimensional one characterized by the prepotential

$$F = \frac{\tilde{\mathcal{X}}^1 \left(\mathcal{X}^3\right)^2}{\mathcal{X}^0}\,, \qquad \text{with} \qquad \tilde{\mathcal{X}}^1 = \mathcal{X}^1 - \frac{A}{3}\,\mathcal{X}^3\,, \tag{3.28}$$

which is homogeneous and symmetric (the so-called st^2 model, cf. e.g. [116] and references therein). This difference is evident, for example, in terms of the Kähler metric, that turns out to be

$$\mathcal{G}^{(3)}_{ij} d\lambda^i d\lambda^j|_{\lambda_2 \propto \lambda_3} \neq \mathcal{G}^{(2)}_{MN} d\lambda^M d\lambda^N\,, \qquad i,j = 1,2,3\,, \qquad M,N = 1,2\,, \tag{3.29}$$

where the left-hand side is the line element obtained with the metric (1.209) when the condition $\lambda_2 \propto \lambda_3$ is imposed, while the right-hand side describes the geometry associated to the prepotential (3.28).

We conclude with a comment on the behavior of the solution for $A = 0$. Due to the particular definition of H_3 we have chosen (with respect to the more common one used for example in [5,22,28]), setting $A = 0$ and $\lambda^2 = \frac{g^2}{g^3}\lambda^3$ is not sufficient to match exactly the stu black hole solution with two independent parameters, known as st^2 solution, that can be derived from [28]. However, the parameters in (3.19) can be redefined as

$$\alpha_3' = \frac{\alpha_1\alpha_2}{\alpha_3} - \frac{A}{3}\alpha_3\,, \qquad \beta_3' = \frac{\beta_1\beta_2}{\beta_3} - \frac{A}{3}\beta_3\,, \tag{3.30}$$

in terms of which the solution (3.22) matches explicitly the known one when $A = 0$. This redefinition of the parameters is a way to recover the choice for the functions that is usually made when solving the BPS equations (3.1), whose analogue for the present case would have been

$$H_3' = \left(\lambda^1\lambda^2 - A(\lambda^3)^2\right) H^0 \qquad \text{or} \qquad H_3' = e^{-\psi}(\alpha_3' r + \beta_3')\,. \tag{3.31}$$

3.1.3 Physical properties

We are here going to investigate some properties of our solution, like the near-horizon limit, the entropy and the area-product formula.

In the asymptotic limit $r \to \infty$, the metric (3.25) becomes AdS$_4$, i.e., at leading order,

$$\mathrm{d}s^2 \to -\frac{r^2}{\ell^2}\mathrm{d}t^2 + \ell^2\frac{\mathrm{d}r^2}{r^2} + r^2\mathrm{d}\Omega_\kappa^2\,, \tag{3.32}$$

where we defined the asymptotic AdS$_4$ curvature radius ℓ by

$$\ell^2 = \frac{\lambda_\infty^3}{2g_0g^3}\,, \tag{3.33}$$

and we rescaled the coordinates according to $t \to \ell t$, $r \to r/\ell$. Notice that the asymptotic value of the cosmological constant is

$$\Lambda = -\frac{3}{\ell^2} = -\frac{6g_0g^3}{\lambda_\infty^3}\,. \tag{3.34}$$

On the other hand, when r approaches the horizon r_h, the functions U and ψ assume, after shifting $r \to r + r_\mathrm{h}$, the form (3.13), with R_1 and R_2 given by

$$R_1^2 = -\frac{\lambda_\infty^3 f(r_\mathrm{h})}{8g_0g^3c}\,, \qquad R_2^2 = \frac{\lambda_\infty^3 f(r_\mathrm{h})}{2g_0g^3}\,, \tag{3.35}$$

where

$$f(r_\mathrm{h}) \equiv \left(r_\mathrm{h} - g_0\beta^0 - \frac{g_0}{(\lambda_\infty^3)^2}\beta_3\right)\sqrt{(r_\mathrm{h} + 2g_0\beta^0)\left(r_\mathrm{h} + \frac{2g_0}{(\lambda_\infty^3)^2}\beta_3\right)}\,.$$

In this limit, the spacetime becomes AdS$_2 \times \Sigma$, with metric

$$\mathrm{d}s^2 = -\frac{r^2}{R_1^2}\mathrm{d}t^2 + \frac{R_1^2}{r^2}\mathrm{d}r^2 + R_2^2\mathrm{d}\Omega_\kappa^2 \,. \tag{3.36}$$

The Bekenstein-Hawking entropy is given by

$$S_{\mathrm{BH}} = \frac{A_{\mathrm{h}}}{4} = \frac{R_2^2\,\mathrm{vol}(\Sigma)}{4} \,. \tag{3.37}$$

This expression can be written in terms of the charges p^0, q_i and the gaugings g_0, g^i only. To this aim, the equations (3.22) need to be inverted, in order to use the charges p^0, q_1, q_2 as parameters. This result sustains the presence of an *attractor mechanism* also in the case under consideration, which is a non-trivial statement, due to the non-homogeneity of the model we have been discussing.
Finally, the product of the areas of all the horizons $r = r_I$, $I = 1, \ldots, 4$ (i.e., all the roots of e^{2U}) assumes the remarkably simple form

$$\prod_{I=1}^{4} A(r_I) = -\frac{36}{\Lambda^2}\frac{\mathrm{vol}(\Sigma)^4 g^2}{g^3} p^0 q_1 \tilde{q}_2^2 \,, \tag{3.38}$$

where we have defined

$$\tilde{q}_2 \equiv q_2 - \frac{A}{3}\left(\frac{g^3}{g^2}\right)^2 q_1 \,. \tag{3.39}$$

Note that (3.38) depends only on the charges and the gauge parameters. Similar formulas have been proven to be true in a number of examples (see for instance [10, 21, 27, 31, 72, 117, 118]), a fact that calls for an underlying microscopic interpretation.

3.2 The $\overline{\mathbb{CP}}^3$ model with SU(2) Fayet-Iliopoulos gauging

We are now turning our attention to the $\mathcal{N} = 2$, 4-dimensional supergravity theory with a Fayet-Iliopoulos gauging of the entire $\mathrm{SU}(2)$ subgroup of the R-symmetry group. As we have already explained, this kind of gauging should be associated with the gauging of an $\mathrm{SU}(2)$ subgroup of the isometries of the Kähler manifold. The action of the theory is then (1.70). In this paragraph, we are going to present some solutions for it, which have been obtained by explicitly solving the equations that arise from the general classification [4], see section 1.2.5.

We consider the simplest model with enough symmetry to admit the necessary $\mathrm{SU}(2)$ gauging of the isometries of the Kähler manifold, the $\overline{\mathbb{CP}}^3$ model. In what follows, we are specifying all the quantities entering the Lagrangian (1.70) for this specific configuration. The equations that have to be solved are (1.142-1.156), which are here rewritten for the model under consideration.

Since the scalars of the $\overline{\mathbb{CP}}^3$ model parametrize the symmetric space $\mathrm{U}(1,3)/(\mathrm{U}(1)\times\mathrm{U}(3))$, the metric (and, indeed, the whole model) is invariant under global $\mathrm{U}(1,3)$ transformations. We are interested in the $\mathrm{SU}(1,3) \subset \mathrm{U}(1,3)$ subgroup whose $\mathrm{SO}(3)$ subgroup is going to be gauged.

The special coordinates $\mathcal{X}^\Lambda$ transform in the fundamental representation of $\mathrm{SU}(1,3)$

$$\mathcal{X}'^{\,\Lambda} = \Lambda^\Lambda{}_\Sigma \mathcal{X}^\Sigma\,, \qquad \Lambda^{*\,\Gamma}{}_\Lambda\, \eta_{\Gamma\Delta}\, \Lambda^\Delta{}_\Sigma = \eta_{\Lambda\Sigma}\,, \tag{3.40}$$

and, according to their definition, the complex scalars transform non-linearly, as

$$Z'^{\,\Lambda} = \frac{\Lambda^\Lambda{}_\Sigma Z^\Sigma}{\Lambda^0{}_\Sigma Z^\Sigma}\,, \qquad Z'_\Lambda = \frac{\Lambda_\Lambda{}^\Sigma Z_\Sigma}{\Lambda^0{}_\Sigma Z^\Sigma}\,, \quad \text{where} \quad \Lambda_\Lambda{}^\Sigma \equiv \eta_{\Lambda\Gamma}\Lambda^\Gamma{}_\Omega \eta^{\Omega\Sigma}\,. \tag{3.41}$$

The indexes of the $\mathrm{SU}(1,3)$ transformations $\Lambda^\Lambda{}_\Sigma$ are lowered and raised by the metric $\eta_{\Lambda\Gamma}$ and its inverse.

These transformations leave the Kähler potential invariant up to Kähler transformations $\mathcal{K}' = \mathcal{K} + f + f^*$ with

$$f(Z) = \log\left(\Lambda^0{}_\Sigma Z^\Sigma\right), \tag{3.42}$$

which implies the exact invariance of the Kähler metric.

The symplectic section $\mathcal{V}$ is also left invariant by the combined action of the symplectic transformation that gives the embedding of the group $\mathrm{SU}(1,3)$ in the symplectic group $\mathrm{Sp}(8,\mathbb{R})$

$$(S^M{}_N) = \begin{pmatrix} \Re\mathfrak{e}\,\Lambda^\Lambda{}_\Sigma & -2\Im\mathfrak{m}\,\Lambda^{\Lambda\Sigma} \\ \\ \frac{1}{2}\Im\mathfrak{m}\,\Lambda_{\Lambda\Sigma} & \Re\mathfrak{e}\,\Lambda_\Lambda{}^\Sigma \end{pmatrix}, \tag{3.43}$$

and a Kähler transformation with the parameter $f(Z)$ given in (3.42). This proves the invariance of the whole model of $\mathcal{N}=2, d=4$ supergravity.

The 15 generators $T_m{}^\Lambda{}_\Sigma$ of $\mathfrak{su}(1,3)$, defined by

$$\Lambda^\Lambda{}_\Sigma \sim \delta^\Lambda{}_\Sigma + \alpha^m\, T_m{}^\Lambda{}_\Sigma, \tag{3.44}$$

are traceless and such that $T_{m\,\Lambda\Sigma} \equiv \eta_{\Lambda\Gamma}\, T_m{}^\Gamma{}_\Sigma$ is anti-Hermitian. Then, the corresponding $\mathfrak{su}(1,3)$ generators, whose exponentiation gives the matrix (3.43), are given by

$$(\mathcal{T}_m{}^M{}_N) = \begin{pmatrix} \Re\mathfrak{e} T_m{}^\Lambda{}_\Sigma & -2\Im\mathfrak{m} T_m{}^{\Lambda\Sigma} \\ \\ \frac{1}{2}\Im\mathfrak{m} T_{m\,\Lambda\Sigma} & \Re\mathfrak{e} T_{m\,\Lambda}{}^\Sigma \end{pmatrix}. \tag{3.45}$$

The holomorphic Killing vectors that generate the transformations of the scalars (3.41) can be written in the form

$$Z'^{\,\Lambda} = Z^\Lambda + \alpha^m k_m{}^\Lambda(Z)\,, \qquad k_m{}^\Lambda(Z) = T_m{}^\Lambda{}_\Sigma\, Z^\Sigma - T_m{}^0{}_\Omega\, Z^\Omega Z^\Lambda\,, \tag{3.46}$$

which allows us to show easily that, if the matrices T_m have the commutation relations $[T_m, T_n] = f_{mn}{}^p\, T_p$, where $f_{mn}{}^p$ are the $\mathfrak{su}(1,3)$ structure constants, then the commutation relations of the symplectic generators and the Lie brackets of the holomorphic Killing vectors are given by

$$[\mathcal{T}_m, \mathcal{T}_n] = f_{mn}{}^p\, \mathcal{T}_p\,, \qquad [k_m, k_n] = -f_{mn}{}^p\, k_p\,. \tag{3.47}$$

The holomorphic functions $\lambda_m(Z)$ defined through

$$\mathcal{L}_{K_m}\mathcal{K} = \lambda_m + \lambda_m^*\,, \quad \text{where} \quad K_m = k_m(Z) + k_m^*(Z^*)\,, \tag{3.48}$$

are given by

$$\lambda_m = T_m{}^0{}_\Sigma Z^\Sigma\,, \tag{3.49}$$

and the holomorphic momentum maps $\mathcal{P}_m$, defined through the relation

$$i\mathcal{P}_m = k_m{}^i \partial_i \mathcal{K} - \lambda_m\,, \tag{3.50}$$

are given by

$$\mathcal{P}_m = ie^{\mathcal{K}} \eta_{\Lambda\Omega} T_m{}^\Lambda{}_\Sigma Z^\Sigma Z^{*\,\Omega}\,. \tag{3.51}$$

The SU(2) subgroup that we are going to gauge acts in the adjoint representation on the special coordinates $\mathcal{X}^i$ and on the physical scalars Z^i, leaving $\mathcal{X}^0$ exactly invariant, as well as the prepotential and the Kähler potential (so $f = \lambda = 0$). We are going to use the indexes $x, y, z, \cdots = 1, 2, 3$ to denote the scalars of the gauged directions, instead of $i, j, \cdots$. Thus, the vector fields A^Λ split into A^0 and A^x, the physical scalars are Z^x, the non-vanishing structure constants and the generators are[4]

$$f_{xy}{}^z = -\varepsilon_{xy}{}^z\,, \qquad T_x{}^y{}_z = \varepsilon_x{}^y{}_z\,, \qquad (\mathcal{T}_x{}^M{}_N) = \begin{pmatrix} \varepsilon_x{}^y{}_z & 0 \\ 0 & \varepsilon_{xy}{}^z \end{pmatrix}, \tag{3.52}$$

the holomorphic momentum maps and Killing vectors are given by

$$\mathcal{P}_x = ie^{\mathcal{K}} \varepsilon_{xyz} Z^y Z^{*\,z}\,, \qquad k_x{}^y = \varepsilon_x{}^y{}_z Z^z\,. \tag{3.53}$$

The SU(2) FI terms are obtained once (1.65) and (1.68) have been taken into account. The gauge-covariant derivatives, the vector field strengths and the scalar potential of the model can be read from (1.69) and (1.71), and take the form

[4]The indexes $x, y, \cdots$ are raised and lowered with δ^{xy}, δ_{xy} and, therefore, their actual position is immaterial.

$$\mathfrak{D}_\mu Z^x = \partial_\mu Z^x - g\varepsilon^x{}_{yz} A^y{}_\mu Z^z \,, \tag{3.54}$$

$$F^0{}_{\mu\nu} = 2\partial_{[\mu} A^0{}_{\nu]} \,, \tag{3.55}$$

$$F^x{}_{\mu\nu} = 2\partial_{[\mu} A^x{}_{\nu]} - g\varepsilon^x{}_{yz} A^y{}_{[\mu} A^z{}_{\nu]} \,, \tag{3.56}$$

$$\mathbf{V}(Z, Z^*) = 2g^2 e^{2\mathcal{K}} (\Re\mathfrak{e} Z^x \Re\mathfrak{e} Z^x)(\Im\mathfrak{m} Z^y \Im\mathfrak{m} Z^y) \sin^2\alpha + \tfrac{1}{2} g^2 \xi^2 \left(5 - 2e^{\mathcal{K}}\right) \,, \tag{3.57}$$

where α is the angle between the 3-vectors $\Re\mathfrak{e} Z^x$ and $\Im\mathfrak{m} Z^y$. Observe that the first term in the potential is non-negative but also bounded from above due to the constraint (1.218)

$$0 \leq 2g^2 (\Re\mathfrak{e} Z^x \Re\mathfrak{e} Z^x)(\Im\mathfrak{m} Z^y \Im\mathfrak{m} Z^y) \sin^2\alpha \leq 2g^2 \,, \tag{3.58}$$

but the second, which is associated to the FI terms, is unbounded from below ($e^{\mathcal{K}} \in (1, \infty)$)

$$-\infty \leq \tfrac{1}{2} g^2 \xi^2 \left(5 - e^{\mathcal{K}}\right) \leq 2g^2 \,. \tag{3.59}$$

We have explored the minima of this potential and we have found that there is a minimum when all the scalar fields vanish, when one of them vanishes, when two of them are equal or when two of them are real, but the potential is not negative for any of these minima and, therefore, we have not been able to find any (necessarily non-maximally supersymmetric, see section 1.2.3) AdS_4 vacuum in this theory.

As we have already mentioned, the choice of this specific model is due to its simplicity; in particular, its Freudenthal duality equations can easily be solved,

$$\mathcal{R}_\Lambda = \tfrac{1}{2} \eta_{\Lambda\Sigma} \mathcal{I}^\Sigma \,, \qquad \mathcal{R}^\Lambda = -2\eta^{\Lambda\Sigma} \mathcal{I}_\Sigma \,. \tag{3.60}$$

3.2.1 Time-like supersymmetric solutions

We just have to adapt the equations of the general recipe reviewed in section 1.2.5 to the gauged model described in the previous section. In particular, we use the imaginary part of period matrix (1.220) expressed in terms of the real symplectic vectors $\mathcal{R}^M$ and $\mathcal{I}^M$ and the solution of the Freudenthal duality equations (3.60) to eliminate $\mathcal{R}^M$ from the equations. We are also going to impose

$$\mathcal{I}_\Lambda = 0 \,, \tag{3.61}$$

(so that $\mathcal{R}^\Lambda = 0$) in order to simplify the problem. In particular, with this choice, the form ω is closed, and we set it to zero. The equations that remain to be solved are

$$F^0{}_{xy} = -\tfrac{1}{\sqrt{2}} \varepsilon_{xyz} \left\{ \partial_z \mathcal{I}^0 + \tfrac{1}{\sqrt{2}} g\xi \mathcal{I}^0 \mathcal{I}^z \right\} \,, \tag{3.62}$$

$$F^z{}_{xy} = -\tfrac{1}{\sqrt{2}} \varepsilon_{xyw} \left\{ \mathfrak{D}_w \mathcal{I}^z + \tfrac{1}{\sqrt{2}} g\xi \left[e^{-2U} \delta^{zw} + \mathcal{I}^w \mathcal{I}^z \right] \right\} \,, \tag{3.63}$$

$$\mathfrak{D}_\xi \hat{V}^x = -\tfrac{1}{\sqrt{2}} g\xi \mathcal{I}^y \hat{V}^y \wedge \hat{V}^x \,, \tag{3.64}$$

where

$$\mathfrak{D}_\xi \hat{V}^x \equiv d\hat{V}^x - g\xi\varepsilon^x{}_{yz}\hat{A}^y \wedge \hat{V}^z \,. \tag{3.65}$$

Observe that, for $\xi = 1$, $\mathfrak{D}_\xi \hat{V}^x = \mathfrak{D}\hat{V}^x$ and that for $\xi = 0$, when the FI terms vanish, $\mathfrak{D}_\xi \hat{V}^x = d\hat{V}^x$; this last equation would be solved by choosing coordinates $\hat{V}^x = dx^x$.

The integrability condition of the last equation can be obtained by acting with $\mathfrak{D}$ on both sides and using the Ricci identity ($\xi \neq 0$)

$$\mathfrak{D}\mathfrak{D}_\xi \hat{V}^x = -g\xi\varepsilon^{xyz}\hat{F}^y \wedge \hat{V}^z \,. \tag{3.66}$$

We find, up to the overall factor $g\xi$

$$F^y{}_{xy} + \tfrac{1}{\sqrt{2}}\varepsilon_{xyz}\mathfrak{D}_z\mathcal{I}^y = 0 \,, \tag{3.67}$$

which is satisfied if (3.63) holds.

Hedgehog Ansatz

The first attempt that has been made to solve these equations, was naturally looking for spherically-symmetric solutions. We have adopted the *hedgehog Ansatz* for the gauge field $A^x{}_{\underline{m}}$ and the corresponding[5] "Higgs field" Φ^x, which has proven to be fruitful in the SEYM case,

$$-\tfrac{1}{\sqrt{2}}\mathcal{I}^x = \Phi^x(r) = -x^x f(r) \,, \qquad A^x{}_{\underline{m}} = \varepsilon^x{}_{\underline{mn}}x^n h(r) \,. \tag{3.68}$$

We have also assumed that the 3-dimensional metric $\gamma_{\underline{mn}}$ is conformally flat and chosen Dreibeins of the form

$$V^x{}_{\underline{m}} = \delta^x{}_{\underline{m}}V(r) \,. \tag{3.69}$$

We can also safely assume that

$$-\tfrac{1}{\sqrt{2}}\mathcal{I}^0 = \Phi^0(r) \,. \tag{3.70}$$

The Ansatz for the Abelian vector field $A^0{}_{\underline{m}}$ cannot be spherically symmetric: we know that the potential of the Dirac monopole is not spherically symmetric even though its field strength is. If the unit vector s^m indicates the direction of the Dirac string, the Dirac monopole potential can be written in the form

$$A^0{}_{\underline{m}} = \tfrac{1}{2}p\varepsilon_{mnp}\frac{s^n x^p}{r}k(w) \,, \quad \text{where} \quad w \equiv \frac{s^m x^m}{r} \,, \quad \text{and} \quad k(w) = (1-w)^{-1} \,. \tag{3.71}$$

[5]The signs have been chosen so that the equations originally obtained by Protogenov in [119] coincide with those studied and used in [17, 29, 30, 82].

In this case, we have tried and generalized the previous form for the gauge fields with the Ansatz

$$A^0{}_{\underline{m}} = \varepsilon_{mnp} \frac{s^n x^p}{r^2} k(r, w)\,, \tag{3.72}$$

where the function k can have an additional dependence on r, not only through w.

Substituting this Ansatz into (3.62,3.64) we get the following differential equations,

$$V^{-1}[2h + rh'] - f[1 + gr^2 h] - \tfrac{1}{2} g\xi V \left[(\Phi^0)^2 - r^2 f^2\right] = 0\,, \tag{3.73}$$

$$V^{-1}[rh' - gr^2 h^2] - gr^2 hf + rf' + g\xi V r^2 f^2 = 0\,, \tag{3.74}$$

$$(V^{-1})' + g\xi r[hV^{-1} - f] = 0\,, \tag{3.75}$$

$$x^m \partial_{\underline{m}} k = 0\,, \tag{3.76}$$

$$\Phi^{0\,\prime} + V^{-1} s^m \left(\frac{\partial_{\underline{m}} k}{r} - \frac{2x^m k}{r^3} \right) + g\xi r V \Phi^0 f = 0\,, \tag{3.77}$$

where primes indicate differentiation with respect to r, which is the only argument of the functions Φ^0, f, h, V.

The above equation (3.76) implies that k is a function of w only and we are left with

$$\partial_{\underline{m}} k = k' \left(\frac{s^m}{r} - \frac{w x^m}{r^2} \right)\,, \tag{3.78}$$

and

$$s^m \left(\partial_{\underline{m}} k - \frac{2x^m k}{r^2} \right) = \frac{1}{r} \frac{d}{dw} [(1 - w^2) k]\,. \tag{3.79}$$

This is the only term in (3.77) that depends on s^m and that dependence must disappear because the corresponding equation is spherically symmetric. Therefore, we must require that

$$\frac{d}{dw} [(1 - w^2) k] = C\,, \tag{3.80}$$

for some constant C. This equation can be integrated, to give

$$k = \frac{Cw + D}{1 - w^2}\,, \tag{3.81}$$

for some other integration constant D. The standard form of the Dirac monopole is recovered when $C = D = p/2$. Exploiting these results in (3.77), we obtained

$$\Phi^{0\,\prime} + C \frac{V^{-1}}{r^2} + g\xi r \Phi^0 f = 0\,, \tag{3.82}$$

and we are left with a non-autonomous system of 4 ordinary differential equations for 4 variables f, h, V, Φ^0 that generalizes Protogenov's [119].

The next step consists in trying and rewriting this system as an autonomous system by a change of variables. In the original, Protogenov system, this task is achieved as in [29]. Actually, the same change of variables works here. Defining

$$gr^2 \equiv e^{2\eta}\,, \qquad 1+gr^2h \equiv N\,, \qquad gr^2 f \equiv I\,, \qquad gr^2(\Phi^0)^2 \equiv K^2\,, \qquad C' = g^{1/2}C\,, \tag{3.83}$$

and combining the differential equations, we arrived to the autonomous system

$$\partial_\eta N = V\left[IN - \tfrac{1}{2}\xi V I^2 + \tfrac{1}{2}g\xi V K^2\right]\,, \tag{3.84}$$

$$\partial_\eta I = (N^2-1)V^{-1} + I - \tfrac{1}{2}\xi V I^2 - \tfrac{1}{2}g\xi V K^2\,, \tag{3.85}$$

$$\partial_\eta V^{-1} = -\xi(N-1)V^{-1} + \xi I\,, \tag{3.86}$$

$$\partial_\eta K = K - C'V^{-1} - \xi V K I\,. \tag{3.87}$$

When $\xi = 0$, the third equation is solved by a constant V and, setting that constant to 1, the first two equations become those of the Protogenov system and involve only two variables: N and I. When $\xi = 1$ the four equations are coupled in a non-trivial way and we had to make additional assumptions in order to simplify the system and find solutions.

Observe that there are no solutions with vanishing scalars, that is, with $I = 0$. In fact, setting $I = 0$ in (3.84) and (3.85) and combining them to eliminate K, a differential equation that only involves N is obtained. It can be integrated to give $N = -\tanh\eta + \alpha$ where α is some integration constant. However, in this way (3.85) cannot be satisfied for any real V or K.

A further change of variables, $\mathfrak{I} = VI$ and $\mathfrak{K} = VK$, allowed us to rewrite the system in a simpler way:

$$\partial_\eta N = N\mathfrak{I} - \tfrac{1}{2}\mathfrak{I}^2 + \tfrac{1}{2}g\mathfrak{K}^2\,, \tag{3.88}$$

$$\partial_\eta \mathfrak{I} = N^2 - 1 + N\mathfrak{I} - \tfrac{3}{2}\mathfrak{I}^2 - \tfrac{1}{2}g\mathfrak{K}^2\,, \tag{3.89}$$

$$\partial_\eta \mathfrak{K} = \mathfrak{K}N - C' - 2\mathfrak{K}\mathfrak{I}\,, \tag{3.90}$$

$$\partial_\eta \log V = N - \mathfrak{I} - 1\,. \tag{3.91}$$

This system admits a solution in which N, $\mathfrak{I}$ and $\mathfrak{K}$ are constants: the first three equations are algebraic and the fourth is trivially solved. This allowed us to obtain the first solution of this theory.

Solution 1: $\mathbf{AdS}_2\times\mathbf{S}^2$

With no loss of generality we can assume $\mathfrak{I}$ to be positive. The solution depends on two constants, $\mathfrak{I}$ and v, and is given by

$$
\begin{aligned}
C' &= \pm\sqrt{\frac{\mathfrak{I}}{g}}\left(3\mathfrak{I}+\sqrt{3\mathfrak{I}^2+1}\right)\left(3\mathfrak{I}+2\sqrt{3\mathfrak{I}^2+1}\right)^{\frac{1}{2}}\,,\\
N &= -\mathfrak{I}-\sqrt{3\mathfrak{I}^2+1}\,,\\
\mathfrak{K} &= \mp\sqrt{g}\left(3\mathfrak{I}^2+2\mathfrak{I}\sqrt{3\mathfrak{I}^2+1}\right)^{\frac{1}{2}}\,,\\
V &= vg^{-\mathfrak{I}-\frac{1}{2}-\frac{1}{2}\sqrt{3\mathfrak{I}^2+1}}r^{-2\mathfrak{I}-1-\sqrt{3\mathfrak{I}^2+1}}\,.
\end{aligned}
\tag{3.92}
$$

The physical fields are then recovered, and turn out to be

$$
\begin{aligned}
ds^2 &= \frac{v^2}{2\mathfrak{I}}g^{-2\mathfrak{I}+1-\sqrt{3\mathfrak{I}^2+1}}\left(\mathfrak{I}+\sqrt{3\mathfrak{I}^2+1}\right)^{-1}r^{-4\mathfrak{I}-2\sqrt{3\mathfrak{I}^2+1}}dt^2\\
&\quad -2\mathfrak{I}\left(\mathfrak{I}+\sqrt{3\mathfrak{I}^2+1}\right)\frac{1}{g^2r^2}\left(dr^2+r^2d\Omega^2_{(2)}\right)\,,\\
Z^x &= \pm\frac{x^x}{gr}\mathfrak{I}\left(3\mathfrak{I}^2+2\mathfrak{I}\sqrt{3\mathfrak{I}^2+1}\right)^{\frac{1}{2}}\,,\\
\Phi^0 &= \frac{1}{v}g^{\mathfrak{I}+\frac{1}{2}+\frac{1}{2}\sqrt{3\mathfrak{I}^2+1}}\left(3\mathfrak{I}^2+2\mathfrak{I}\sqrt{3\mathfrak{I}^2+1}\right)^{\frac{1}{2}}r^{2\mathfrak{I}+\sqrt{3\mathfrak{I}^2+1}}\,,\\
A^x{}_{\underline{m}} &= \varepsilon^x{}_{mn}\frac{x^n}{gr^2}\left(-\mathfrak{I}-1-\sqrt{3\mathfrak{I}^2+1}\right)\,.
\end{aligned}
\tag{3.93}
$$

This metric is exactly $\mathrm{AdS}_2\times\mathrm{S}^2$ (with different radii), independently of the value of $\mathfrak{I}$, as becomes evident performing the following change of variables,

$$
\begin{aligned}
\rho &= r^{-2\mathfrak{I}-\sqrt{3\mathfrak{I}^2+1}}\,,\\
\tau &= v\left(7\mathfrak{I}^2+1+4\mathfrak{I}\sqrt{3\mathfrak{I}^2+1}\right)^{-1}g^{-2\mathfrak{I}-1-\sqrt{3\mathfrak{I}^2+1}}\,t\,,
\end{aligned}
\tag{3.94}
$$

which leads to

$$
\begin{aligned}
ds^2 &= \frac{1}{2\mathfrak{I}}\frac{7\mathfrak{I}^2+1+4\mathfrak{I}\sqrt{3\mathfrak{I}^2+1}}{\mathfrak{I}+\sqrt{3\mathfrak{I}^2+1}}g^2\rho^2d\tau^2-2\mathfrak{I}\frac{\mathfrak{I}+\sqrt{3\mathfrak{I}^2+1}}{7\mathfrak{I}^2+1+4\mathfrak{I}\sqrt{3\mathfrak{I}^2+1}}g^{-2}\frac{d\rho^2}{\rho^2}\\
&\quad -2\mathfrak{I}\left(\mathfrak{I}+\sqrt{3\mathfrak{I}^2+1}\right)g^{-2}d\Omega^2_{(2)}\,,\\
Z^i &= \pm\frac{x^i}{g}\mathfrak{I}\left(3\mathfrak{I}^2+2\mathfrak{I}\sqrt{3\mathfrak{I}^2+1}\right)^{\frac{1}{2}}\rho^{\frac{1}{2\mathfrak{I}+\sqrt{3\mathfrak{I}^2+1}}}\,,\\
\Phi^0 &= \frac{1}{v\rho}g^{\mathfrak{I}+\frac{1}{2}+\frac{1}{2}\sqrt{3\mathfrak{I}^2+1}}\left(3\mathfrak{I}^2+2\mathfrak{I}\sqrt{3\mathfrak{I}^2+1}\right)^{\frac{1}{2}}\,,\\
A^x{}_{\underline{m}} &= \varepsilon^x{}_{mn}\frac{x^n}{g}\left(-\mathfrak{I}-1-\sqrt{3\mathfrak{I}^2+1}\right)\rho^{\frac{2}{2\mathfrak{I}+\sqrt{3\mathfrak{I}^2+1}}}\,.
\end{aligned}
\tag{3.95}
$$

The potential (3.57) assumes in this situation a constant value, which can be negative for certain values of the parameter $\mathfrak{I}$

$$\mathbf{V} < 0\,, \quad \Leftrightarrow \quad \mathfrak{I}^2\left(3\mathfrak{I}^2 + 2\mathfrak{I}\sqrt{3\mathfrak{I}^2+1}\right) < g^2 < \tfrac{5}{3}\mathfrak{I}^2\left(3\mathfrak{I}^2 + 2\mathfrak{I}\sqrt{3\mathfrak{I}^2+1}\right)\,. \tag{3.96}$$

By construction this solution is supersymmetric. In order to determine which fraction of the total supersymmetry it preserves (the minimal amount is $\frac{1}{8}$), we take advantage of the analysis performed in [4]: the gaugini Killing Spinor Equation is solved imposing three projection operators, each of which projects out half of the components of the Killing spinor. However, if some gaugini's shifts

$$W^{ix} = g\mathcal{G}^{ij^*} f^{*\Lambda}{}_{j^*} \mathsf{P}_\Lambda{}^x\,, \tag{3.97}$$

vanish identically for the configuration we are examining, the corresponding projector does not need to be imposed, and the preserved supersymmetry can be larger. From (1.217) and (1.219) we get, for the model we are dealing with,

$$W^{ix} = 0 \quad \Leftrightarrow \quad Z^i Z^{*\,x} - \tfrac{1}{2}\delta^{ix} = 0\,, \tag{3.98}$$

which can never be satisfied for the solution we are presenting, where $Z^x \propto x^x$. This solution is therefore only $\frac{1}{8}$-BPS.

Another Ansatz

In order to generalize the Ansatz we exploited so far, we are going to relax the requirement (3.69): in the new Ansatz, the Dreibein is going to have the same form,

$$V^x{}_{\underline{m}} = \delta^x{}_{\underline{m}} V\,, \tag{3.99}$$

but now V can be an arbitrary, not necessarily spherically-symmetric, function of the coordinates $x^{\underline{m}}$.

With this choice, (3.64) is solved by

$$A^x{}_{\underline{m}} = \varepsilon^x{}_{\underline{mn}} h^n\,, \tag{3.100}$$

$$\partial_{\underline{m}} V = gV\left(h^m + V\Phi^m\right) \tag{3.101}$$

for some triplet of arbitrary functions h^m that, in particular, can vanish identically. We consider this possibility first.

Solution 2: $h^m = 0$

The Ansatz (3.100,3.101) is here considered, and some further assumptions are made: the h^m are chosen to vanish identically and all the functions involved depend on a single direction, say x^1, so that

$$A^x{}_{\underline{m}} = 0\,, \qquad \partial_{\underline{1}} V^{-1} = -g\Phi^1\,, \qquad \Phi^2 = \Phi^3 = 0\,. \tag{3.102}$$

This Ansatz is adequate to find domain-wall-type solutions.

Under these assumptions, equation (3.62) implies that the only non-trivial component of ${F^0}_{mn}$ is ${F^0}_{23}$. However, by assumption, the components ${A^0}_{\underline{2},\underline{3}}$ are functions of the coordinate x^1 only, so they have to be constants and the purely spatial components of the field strength ${F^0}_{\underline{mn}}$ should vanish identically.

The equations in (3.62) and (3.63) that remain to be solved are

$$\partial_{\underline{1}} V^{-1} = -g\Phi^1 \,, \tag{3.103}$$

$$\partial_{\underline{1}} \Phi^1 = \tfrac{1}{2} g V \left[(\Phi^0)^2 + (\Phi^1)^2 \right] \,, \tag{3.104}$$

$$\partial_{\underline{1}} \Phi^0 = g\Phi^0\Phi^1 V \,, \tag{3.105}$$

that can be rewritten as

$$\partial_{V^{-1}} \Phi^0 = -\Phi^0 V \,, \tag{3.106}$$

$$\partial_{V^{-1}} \Phi^1 = -\tfrac{1}{2} \frac{V}{\Phi^1} \left[(\Phi^0)^2 + (\Phi^1)^2 \right] \,, \tag{3.107}$$

$$\partial_{\underline{1}} V^{-1} = -g\Phi^1 \,. \tag{3.108}$$

The system can be immediately integrated, giving

$$\begin{aligned}
\Phi^0 =& p^0 V \,, \\
\Phi^1 =& \pm \sqrt{(p^0)^2 V^2 + p^1 V} \,, \\
V =& -2^{\frac{5}{3}} (p^0)^2 (p^1)^2 \left\{ (p^1)^3 \left[16(p^0)^2 - 9(p^1)^4 \left(-gx^1 + v \right)^2 \right]^2 \right. \\
& \left. + 3\sqrt{(p^1)^{10} \left(-gx^1 + v \right)^2 \left[-16(p^0)^2 + 9(p^1)^4 \left(-gx^1 + v \right)^2 \right]^3} \right\}^{-\frac{1}{3}} \\
& - 2^{\frac{1}{3}} \left\{ (p^1)^3 \left[16(p^0)^2 - 9(p^1)^4 \left(-gx^1 + v \right)^2 \right]^2 \right. \\
& \left. + 3\sqrt{(p^1)^{10} \left(-gx^1 + v \right)^2 \left[-16(p^0)^2 + 9(p^1)^4 \left(-gx^1 + v \right)^2 \right]^3} \right\}^{\frac{1}{3}} \cdot \\
& \left[16(p^0)^2 - 9(p^1)^4 \left(-gx^1 + v \right)^2 \right]^{-1}
\end{aligned} \tag{3.109}$$

where p^0, p^1 and v are integration constants.
The metric function for these solutions is

$$e^{-2U} = (\Phi^0)^2 - (\Phi^1)^2 = -p^1 V(x^1) \,, \tag{3.110}$$

while the complete metric takes the form

$$ds^2 = -\frac{1}{p^1 V} dt^2 + p^1 V^3 [(dx^1)^2 + (dx^2)^2 + (dx^3)^2] \,. \tag{3.111}$$

The constant p^0 should differ from zero, otherwise $\Phi^0 = 0$ and the metric function would always be negative. We must require $p^1 V < 0$ in order to have $e^{-2U} > 0$. The profile of e^{-2U} changes dramatically with the integration constants and it is not easy to find physically meaningful solutions.
One of the few simple examples that we have found corresponds to the choice $p^0 = -1$, $p^1 = 1$, $v = 0$. In this case $e^{-2U}(x^1)$ is positive in an interval of the real line, as shown in the figure.

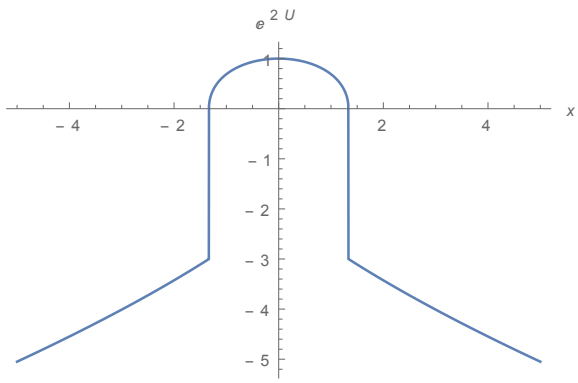

Figure 3.1: The inverse of the metric function, e^{2U}, as a function of the coordinate x^1, with the choice $p^0 = -1$, $p^1 = 1$, $v = 0$, $g = 1$.

At the boundary of that region, e^{-2U} and V blow up and so does the scalar potential, which in this case is given by

$$\mathbf{V} = \tfrac{1}{2} g^2 \left(5 - \frac{2(p^0)^2 V}{p^1} \right) \,. \tag{3.112}$$

On the other hand, the condition $W^{ix} = 0$ cannot be satisfied for any x, meaning that the solution is $\frac{1}{8}$-BPS.

Solution 3: $h^m \neq 0$, an open Einstein universe

If, in the context of the Ansatz (3.100,3.101), where all the involved functions only depend on x^1, the vanishing of h^m is not assumed, then the non-trivial components of equation (3.64) take the form

$$\partial_{\underline{1}} V = gV \left(h^1 - V\Phi^1 \right) \,, \tag{3.113}$$

$$h^{2,3} = -V\Phi^{2,3} \,, \tag{3.114}$$

those of equation (3.63) are

$$\partial_{\underline{1}} A^0{}_{\underline{2}} = -gV\Phi^0\Phi^3\,, \tag{3.115}$$
$$\partial_{\underline{1}} A^0{}_{\underline{3}} = gV\Phi^0\Phi^2\,, \tag{3.116}$$
$$\partial_{\underline{1}} \Phi^0 = gV\Phi^0\Phi^1\,, \tag{3.117}$$

and finally, those of equation (3.62) can be written as

$$\partial_{\underline{1}} \Phi^{2,3} = gh^1\Phi^{2,3}\,, \tag{3.118}$$
$$\Phi^2\Phi^3 = 0\,, \tag{3.119}$$
$$\partial_{\underline{1}} h^1 = -gVh^1\Phi^1 + \tfrac{1}{2}gV^2\left[(\Phi^0)^2 - (\Phi^1)^2 - (\Phi^2)^2 + (\Phi^3)^2\right]\,, \tag{3.120}$$
$$\partial_{\underline{1}} h^1 = -gVh^1\Phi^1 + \tfrac{1}{2}gV^2\left[(\Phi^0)^2 - (\Phi^1)^2 + (\Phi^2)^2 - (\Phi^3)^2\right]\,, \tag{3.121}$$
$$\partial_{\underline{1}} \Phi^1 = -\frac{g}{V}(h^1)^2 + 2gV\Phi^2\Phi^3 + \tfrac{1}{2}gV\left[(\Phi^0)^2 + (\Phi^1)^2 - (\Phi^2)^2 - (\Phi^3)^2\right]\,. \tag{3.122}$$

It is immediate to conclude that

$$\Phi^2 = \Phi^3 = 0\,, \qquad A^0{}_{\underline{2},\underline{3}} = \text{const.}\,, \qquad A^2 = h^1 dx^3\,, \qquad A^3 = -h^1 dx^2\,, \tag{3.123}$$

and the equations that remain to be solved are

$$\partial_{\underline{1}} V = gV\left(h^1 - V\Phi^1\right)\,, \tag{3.124}$$
$$\partial_{\underline{1}} h^1 = -gVh^1\Phi^1 + \tfrac{1}{2}gV^2\left[(\Phi^0)^2 - (\Phi^1)^2\right]\,, \tag{3.125}$$
$$\partial_{\underline{1}} \Phi^0 = gV\Phi^0\Phi^1\,, \tag{3.126}$$
$$\partial_{\underline{1}} \Phi^1 = -\frac{g}{V}(h^1)^2 + \tfrac{1}{2}gV\left[(\Phi^0)^2 + (\Phi^1)^2\right]\,. \tag{3.127}$$

This system of equations can be simplified by setting $\Phi^1 = 0$; in this way, the resulting equations

$$\Phi^0 = \pm\sqrt{2}\frac{h^1}{V} = \text{const.}\,, \tag{3.128}$$
$$\partial_{\underline{1}} V = gVh^1\,, \tag{3.129}$$
$$\partial_{\underline{1}} h^1 = g(h^1)^2\,, \tag{3.130}$$

are easily solved and the solution is determined by the following non-vanishing fields

$$\Phi^0 = \pm\frac{\sqrt{2}}{b}\,, \tag{3.131}$$

$$A^3{}_{\underline{2}} = -A^2{}_{\underline{3}} = \frac{1}{gx^1}\,, \tag{3.132}$$

$$ds^2 = \frac{2}{b^2}dt^2 - \frac{b^4}{2g^2(x^1)^2}dx^m dx^m\,, \tag{3.133}$$

where b is an integration constant.

The spatial part of the metric is that of a 3-dimensional hyperboloid.
This becomes evident when using coordinates that are analogous to the Poincaré coordinates of AdS_3: the hyperboloid is defined as the hypersurface

$$(X^1)^2 + (X^2)^2 + (X^3)^2 - (X^4)^2 = -1\,, \tag{3.134}$$

in $\mathbb{R}^4$ endowed with the metric

$$ds^2 = (dX^1)^2 + (dX^2)^2 + (dX^3)^2 - (dX^4)^2\,; \tag{3.135}$$

If we parametrize it with coordinates x^1, x^2, x^3

$$X^1 + X^4 \equiv -\frac{1}{x^1}\,, \qquad X^{2,3} \equiv \frac{x^{2,3}}{x^1}\,, \tag{3.136}$$

the induced metric is

$$ds^2 = \frac{1}{(x^1)^2}dx^m dx^m\,. \tag{3.137}$$

Therefore, the complete metric (3.133) has the geometry of an open Einstein universe, $\mathbb{R} \times \mathbb{H}^3$, and it is supported by a non-Abelian field only, whose field strength is related to the volume form of $\mathbb{H}^3$ by

$$F^x{}_{yz} = -g\varepsilon_{\underline{x}\underline{y}\underline{z}}\,. \tag{3.138}$$

Usually, p-form field strengths support p- of $(d-p)$-dimensional symmetric spaces. For instance, 2-form field strengths support $AdS_2 \times S^2$ solutions in 4 dimensions and $AdS_2 \times S^3$ or $AdS_3 \times S^2$ solutions in 5 dimensions. In this sense, this solution is exceptional and the exceptionality is related to the rank of the form and to the dimension of the gauge group.

The potential is again equal to a positive constant when this configuration is considered, and the amount of supersymmetry preserved by the solution is $\frac{1}{8}$.

3.3 Solutions via dimensional uplifting: the SU(2)-SEYM theory

We are now going to exploit the results of section 2.4 to construct new supersymmetric solutions of the 6-dimensional theories we have been dealing with ($\mathcal{N} = 2A, 2A^*, 2B$) by uplifting solutions of the $\mathcal{N} = 2, d = 5$ theories they all reduce to. We are going to add a new twist to this story, though. The relations between the fields of two ungauged supergravity theories related by standard dimensional reduction do not change if we gauge

both of them in the same way. Thus, we can use the uplifting formulas of section 2.4 to uplift supersymmetric solutions of the same models of $\mathcal{N}=2, d=5$ supergravity but, now, with non-Abelian gaugings.

We are interested in the simplest supersymmetrization of the Einstein-Yang-Mills (EYM) theory, called $\mathcal{N}=2, d=5$ Super-Einstein-Yang-Mills (SEYM) theories in [30]. The starting point is a recently found solution [30] of these theories. The method that has been used there is essentially the same we are going to use here: the uplifting of solutions of the 4-dimensional non-Abelian gauged theories, which are better understood [17,23,24,29,82,83]. We have considered only the simplest solution in [30], but this turned out to be enough to produce interesting 6-dimensional solutions.

While the uplifting of non-Abelian solutions to the $\mathcal{N}=2A, 2A^*$ theories is well justified, the meaning in the $\mathcal{N}=2B$ case is less evident. In fact, these theories cannot be gauged. However, we believe that a gauged $\mathcal{N}=2B, d=6$ theory can be defined at least when the theory is compactified on a circle. The situation is analogous to that of several coincident M5-branes which, at least when wrapped on a circle, must be described by a non-Abelian theory of chiral 2-forms. We do not know how such a theory should be written down, but we know that, at the massless level, it is effectively described by a standard non-Abelian theory of vector fields in one dimension less, the theory of coincident D4-branes.
Analogously, we do not know how to describe the non-Abelian $\mathcal{N}=2B, d=6$ supergravity theory, which only has chiral 2-forms, but we know that, when compactified on a circle and at the massless level, the theory is described by a standard gauged theory of $\mathcal{N}=2, d=5$ supergravity with 1-forms as gauge fields. It is in this limited sense that the non-Abelian solutions of $\mathcal{N}=2B, d=6$ supergravity that we are going to construct should be interpreted.

3.3.1 The original solution

We are first of all reviewing the original result of [30]. Let us consider the $\mathcal{N}=2, d=5$ SEYM theory with $n_{V5}=5$ vectors labeled by $x=1,\cdots,5$ or $x=1,2,A$ where $A,B,\ldots$, label the three directions that have been gauged with the group $\mathrm{SO}(3)$. The non-vanishing components of C_{IJK} are given by $C_{0xy}=\frac{1}{3!}\eta_{xy}$, $\eta_{xy}=\mathit{diag}(1,-1,-1,-1,-1)$. The solution that we are going to uplift was obtained in a model with one vector multiplet less but here, for a reason that will be explained in what follows, we cannot gauge the first vector multiplet and so we add one more ($x=2$), whose fields will vanish identically.

The metric of the solution we are interested in is static and spherically symmetric

$$ds^2 = f^2 dt^2 - f^{-1}(d\rho^2 + \rho^2 d\Omega^2_{(3)})\,, \tag{3.139}$$

where $d\Omega^2_{(3)}$ is given in (3.154), the metric function f is

$$f^{-1} = 3 \cdot 2^{-1/3} \left\{ L_1^2 \left[L_0 - \frac{9}{2g^2} \left(\rho + \frac{\lambda^2}{4} \rho^3 \right)^{-2} \right] \right\}^{1/3} , \tag{3.140}$$

L_0 and L_1 are two spherically symmetric harmonic functions on $\mathbb{R}^4$

$$L_{0,1} = a_{0,1} + q_{0,1}/\rho^2 , \tag{3.141}$$

$a_{0,1}$ being integration constants and $q_{0,1}$ being electric charges. The integration constants are constrained by the normalization of the metric at infinity, but we are are not going to impose this condition in 5 dimensions.

There is only one non-trivial scalar that can be expressed as h^1/h^0. In terms of the scalar functions h^I we have

$$h^0 = 2^{-1/3} \left[\frac{L_1}{L_0 - \frac{9}{2g^2} \left(\rho + \frac{\lambda^2}{4} \rho^3 \right)^{-2}} \right]^{2/3} , \tag{3.142}$$

$$h^1 = 2^{2/3} \left[\frac{L_1}{L_0 - \frac{9}{2g^2} \left(\rho + \frac{\lambda^2}{4} \rho^3 \right)^{-2}} \right]^{-1/3} , \tag{3.143}$$

$$h^2 = h^A = 0 , \tag{3.144}$$

and

$$\phi^1 = 2 \frac{L_0 - \frac{9}{2g^2} \left(\rho + \frac{\lambda^2}{4} \rho^3 \right)^{-2}}{L_1} . \tag{3.145}$$

Finally, the vector fields of the solution are given by

$$\begin{aligned} A^0 &= -\tfrac{1}{\sqrt{3}} \left[L_0 - \frac{9}{2g^2} \left(\rho + \frac{\lambda^2}{4} \rho^3 \right)^{-2} \right]^{-1} dt , \\ A^1 &= -\tfrac{2}{\sqrt{3}} L_1^{-1} \, dt , \\ A^2 &= 0 , \\ A^A &= -\frac{1}{g} \left(1 + \frac{\lambda^2}{4} \rho^2 \right)^{-1} v^A{}_L , \end{aligned} \tag{3.146}$$

where the $v^A{}_L$ are the left-invariant Maurer-Cartan 1-forms of the Lie group $\mathrm{SU}(2)$, given in our conventions by

$$\begin{aligned} v^1{}_L &= \sin\psi \, d\theta - \sin\theta \cos\psi \, d\phi , \\ v^2{}_L &= -\cos\psi \, d\theta - \sin\theta \sin\psi \, d\phi , \end{aligned} \tag{3.147}$$

$$v^3{}_L = -d\psi - \cos\theta\, d\phi\,.$$

A^A is the potential of the BPST instanton and g is the 5-dimensional gauge coupling constant.

3.3.2 From 5 to 6 dimensions

We are now going to exploit the relations between 5- and 6-dimensional theories that we have uncovered. As already mentioned, we are adding one more twist observing that, if we had dimensionally reduced the *gauged* $\mathcal{N} = 2A, d = 6$ theory we would have obtained a gauged $\mathcal{N} = 2, d = 5$ supergravity theory and the relation between the physical fields of these two gauged theories would be exactly the same we have obtained in the ungauged case. This is true as long as the gauge group does not change in the process of dimensional reduction (as in the case of generalized dimensional reduction [39]). We can therefore use the formulas we have obtained to uplift solutions of the 5-dimensional gauged theories to solutions of the 6-dimensional gauged theories and vice versa.

The dimensional reduction of these gauged 6-dimensional theories would be the models of $\mathcal{N} = 2, d = 5$ supergravity that we have found, characterized by the C_{IJK} tensor with non-vanishing indexes $C_{0rs} = \frac{1}{3!}\eta_{rs}$, with exactly the same kind of gauging (with or without Fayet-Iliopoulos terms). The main difference with respect to the 6-dimensional theories is that, in the non-Abelian case, the gauge group acts on the scalars that originate in the 6th component of the 6-dimensional vector fields and these transformations are isometries of the σ-model metric. The relations between 5- and 6-dimensional fields are still valid in the gauged case, but we have to keep in mind that, in order to get the C_{IJK} tensor exactly in the form $C_{0rs} = \frac{1}{3!}\eta_{rs}$, we performed a linear transformation mixing several different vector fields (2.83). In the gauged case, this can safely be done only if the vector fields have the same transformation properties under the gauge group. Thus, we are only allowed to gauge vector fields that are not involved in these redefinitions, and this is the reason for adding an identically vanishing vector multiplet to the original solution.

The $\mathcal{N} = 2B, d = 6$ theories cannot be gauged, at least not in a conventional way. However, as we have already mentioned, it is believed that there are 6-dimensional gauge theories based on chiral 2-forms associated to coincident M5-branes. The main reason is that, when compactified on a circle, M5-branes behave as D4-branes and the Born-Infeld fields of coincident D4-branes are non-Abelian. This means that, at least, the non-Abelian theory of 2-forms exists when one of the 6 dimensions is compactified on a circle and, under these conditions, the massless modes are essentially non-Abelian 1-forms. There have been, actually, several proposals for non-Abelian theories of 2-forms in 6 dimensions [120–122] and they mainly consider a compactified dimension.

The situation we are facing here is similar and probably directly related to the world volume theories of the M5-branes. It is clear that, when these theories are compactified

on a circle, at least the massless part of the spectrum (1-forms in $d = 5$) can be gauged. We do not know how to formulate the gauging using chiral 2-forms directly in 6 uncompactified dimensions but we do known that, at lowest order, the relation between the 6- and 5-dimensional non-Abelian fields is the same as between the Abelian ones. We can therefore use the uplifting formulas to construct non-Abelian solutions of a "$\mathrm{SO}(3)$-gauged" $\mathcal{N} = 2B, d = 6$ theory whose exact 6-dimensional formulation we do not know. Actually, we can use this relation as a lowest-order formulation of that theory which probably only exists when one of the 6 dimensions is compactified on a circle.

3.3.3 Solutions of the SO(3)-gauged $\mathcal{N} = 2A^*, d = 6$ theory

We are here constructing supersymmetric solutions of the $\mathrm{SO}(3)$-gauged $\mathcal{N} = 2A^*, d = 6$ theory without FI terms by uplifting the supersymmetric solutions of the similarly gauged (no FI terms) $\mathcal{N} = 2, d = 5$ supergravity with no hypermultiplets we have presented in section 3.3.1. In particular, we are going to uplift an extremal black hole sourced by a BPST instanton [123].

We are going to find a solution of the $\mathcal{N} = 2A^*, d = 6$ theory with $n_T = 1$ and $n_V = n_{V5} - 2 = 3$. One of the six 5-dimensional vectors is the KK vector and the other two come from the non-chiral 2-form, while the 3 remaining vectors are the gauge fields of the $\mathrm{SO}(3)$ gauge group[6].

Using the equations (2.102,2.105,2.107,2.109), we straightforwardly obtain the following 6-dimensional fields

$$d\breve{s}^2 = 2\breve{f} du \left[dv' - \tfrac{3}{2}(L_1 - a_1)du\right] - \breve{f}^{-1}(d\rho^2 + \rho^2 d\Omega^2_{(3)}) \,, \tag{3.148}$$

$$\breve{f} = \tfrac{\sqrt{2}}{3}\left\{L_1\left[L_0 - \frac{2}{9g^2}\left(\rho + \frac{\lambda^2}{4}\rho^3\right)^{-2}\right]\right\}^{-1/2} \,, \tag{3.149}$$

$$e^{\sqrt{2}\breve{\varphi}} = \tfrac{1}{2}L_1\left[L_0 - \frac{2}{9g^2}\left(\rho + \frac{\lambda^2}{4}\rho^3\right)^{-2}\right]^{-1} \,, \tag{3.150}$$

$$\breve{A}^A = -\frac{1}{\sqrt{12}g}\left(1 + \frac{\lambda^2}{4}\rho^2\right)^{-1} v^A{}_L \,, \tag{3.151}$$

$$\breve{H} = -\tfrac{1}{6}dv' \wedge du \wedge d\left[L_0 - \frac{2}{9g^2}\left(\rho + \frac{\lambda^2}{4}\rho^3\right)^{-2}\right]^{-1} + \tfrac{3}{2}q_1\omega_3 \,. \tag{3.152}$$

We have renamed the coordinates z and t as u and v, respectively, since they are conjugate null coordinates in 6 dimensions. Then, we have shifted one of them $v = v' + \frac{3}{2}a_1 u$. The null coordinates u and v' can be expressed in terms of time (τ) and space (y) coordinate as

[6]Globally, the instanton solution requires the group to be $\mathrm{SU}(2)$.

$$u = \tfrac{1}{\sqrt{2}}(\tau + y)\,, \qquad v = \tfrac{1}{\sqrt{2}}(\tau - y)\,. \tag{3.153}$$

ω_3 is the volume form of the round 3-sphere of unit radius whose metric is $d\Omega^2_{(3)}$. If, for instance, we use the Euler coordinates (θ, ϕ, ψ) such that

$$d\Omega^2_{(3)} = \tfrac{1}{4}\left[(d\psi + \cos\theta d\phi)^2 + d\theta^2 + \sin^2\theta d\phi^2\right]\,, \tag{3.154}$$

then $\omega_3 = \frac{1}{8}\sin\theta d\theta \wedge d\phi \wedge d\psi$, and the 2-form $\breve{B}$ can be written in this coordinate patch, up to gauge transformations, as

$$\breve{B} = -\tfrac{1}{6}\left[L_0 - \tfrac{2}{9g^2}\left(\rho + \frac{\lambda^2}{4}\rho^3\right)^{-2}\right]^{-1} dv' \wedge du + \tfrac{3}{16}q_1\cos\theta d\psi \wedge d\phi\,. \tag{3.155}$$

Observe that now $\breve{A}^A$ carries a factor of $1/\sqrt{12}$ with respect to the potential of the BPST instanton. The reason behind this apparent inconsistency is that the rescaling of the potentials is harmless in the Abelian case but brings the non-Abelian 2-form field strength to an unconventional form. To bring it back to the standard form we just have to rescale the coupling constant. Thus, the 6-dimensional coupling constant is given in terms of the 5-dimensional one by

$$\breve{g} = \sqrt{12}g\,. \tag{3.156}$$

The metric $d\breve{s}^2$ is that of a typical superposition of a string lying in the z direction and a wave with momentum $\sim q_1$ in the same direction. The 3-form field strength $\breve{H}$ indicates that the string is dyonic, with electric and magnetic charges $\sim q_0, q_1$, respectively. This kind of solutions are very well known as they are particular cases of 3-charge configurations dual to the D1D6W one.[7] The additional ingredient here is the BPST instanton that modifies the metric function $\breve{f}$. The string part of this solution is also clearly related to the "gauge dyonic string" solution of the Heterotic string effective action compactified to 6 dimensions constructed in [124] by adding Yang-Mills instantons in the transverse directions to the dyonic string found in [125] (see also [126]).

We have left the integration constants a_0, a_1 intentionally undetermined, because different choices for them can lead, as we are going to see, to physically inequivalent solutions, depending on whether we demand asymptotic flatness or not.

Asymptotic limit

Let us first consider the $\rho \to \infty$ limit. There are two possibilities:

1. If we choose the two integration constants in the harmonic functions $L_{0,1}$ to be non-vanishing, $a_0 a_1 > 0$, then

[7] Only two out of the three different charges are independent in this solution. This is necessary to have a consistent truncation to minimal supergravity.

$$\breve{f} \sim \frac{\sqrt{2}/3}{\sqrt{a_0 a_1}}\,, \qquad e^{\sqrt{2}\breve{\varphi}_\infty} = \frac{a_1}{2a_0}\,, \qquad \breve{H}_{\rho v' u} \sim -\frac{q_0}{3a_0^2}\frac{1}{\rho^3}\,. \tag{3.157}$$

First of all, we notice that the metric is asymptotically flat. The normalization $\breve{f} = 1$ fixes the integration constants in terms of $\breve{\varphi}_\infty$ only,

$$a_0 = \tfrac{1}{3}e^{-\breve{\varphi}\sqrt{2}}\,, \qquad a_1 = \tfrac{2}{3}e^{+\breve{\varphi}\sqrt{2}}\,. \tag{3.158}$$

This solution describes the superposition of the dyonic string and pp-wave mentioned above. The charges of the string can be easily computed and are given by

$$Q \equiv \tfrac{1}{2\pi^2}\int_{S^3_\infty} e^{-\sqrt{2}\breve{\varphi}} \star \breve{H} = -3q_0\,, \qquad P \equiv \tfrac{1}{2\pi^2}\int_{S^3_\infty} \breve{H} = \tfrac{3}{2}q_1\,. \tag{3.159}$$

The instanton field falls off too fast at infinity to give any contributions to charges, masses or momenta.

2. If only one of the integration constants vanishes, the dilaton would not be well defined. If both of them vanish $a_0 = a_1 = 0$, as long as $q_1\left(q_0 - \frac{8}{3\breve{g}^2}\right) > 0$, $\breve{f}$ remains finite, strictly real and positive for all finite values of ρ and the whole metric is regular. In the $\rho \to \infty$ limit the fields behave as

$$\breve{f} \sim \frac{\rho^2}{R_\infty^2}\,, \qquad e^{\sqrt{2}\breve{\varphi}_\infty} = \frac{q_1}{2q_0}\,, \qquad \breve{H}_{\rho v' u} \sim \frac{1}{3q_0}\rho\,, \tag{3.160}$$

where we have defined the constant

$$R_\infty^2 \equiv \sqrt{\frac{9q_0 q_1}{2}}\,, \tag{3.161}$$

which depends on the charges but not on the modulus $\breve{\varphi}_\infty$. The metric takes a direct product form

$$d\breve{s}^2_\infty = R_\infty^2\left(2du'dv''\rho^2 - 3q_1 du'^{\,2} - \frac{d\rho^2}{\rho^2}\right) - R_\infty^2 d\Omega^2_{(3)}\,, \tag{3.162}$$

where $u = R_\infty^2 u'$ and $v' = R_\infty^2 v''$.

The transverse part of the metric is that of a round 3-sphere of radius R_∞. The rest turns out to be the metric of an AdS_3 space of the same radius R_∞, as becomes evident when computing its Riemann tensor

$$R^{(3)}_{\mu\nu\rho\sigma} = -\frac{2}{R_\infty^2}\, g^{(3)}_{\mu[\rho} g^{(3)}_{\sigma]\nu}\,. \tag{3.163}$$

Thus, the second choice of integration constants gives a solutions which is asymptotically $\mathrm{AdS}_3 \times S^3$ with radii equal to R_∞. Observe that in the Abelian case, which can always be recovered by eliminating the instanton field, the solution would be globally, and not just asymptotically, $\mathrm{AdS}_3 \times S^3$. In the $\rho \to \infty$ limit we recover essentially this Abelian solution because the instanton field vanishes and, in particular, the 3-form field strength $\breve{H}$ takes the form

$$\breve{H} = \tfrac{3}{2} q_1 \left[-\pi_3 + \omega_3\right] , \tag{3.164}$$

where π_3 and ω_3 are the volume forms of unit-radii AdS_3 and S^3, respectively. In the coordinates we are using, the first is given by

$$\pi_3 = \rho d\rho \wedge dv'' \wedge du' . \tag{3.165}$$

Near-horizon limit

For any value of the integration constants a_0, a_1 (i.e., for any of the two different solutions identified above), the $\rho \to 0$ limit guarantees finite values for the Ricci scalar and the Kretschmann invariant of the full metric. Thus, we expect to have a well-defined $\rho \to 0$ metric, which in the asymptotically-flat case will be interpreted as a near-horizon metric. In both cases we have the the following asymptotic expansions:

$$L_{0,1} \sim \frac{q_{0,1}}{\rho^2} + \mathcal{O}(1), \quad \breve{f} = \rho^2/R_{\mathfrak{h}}^2 + \mathcal{O}(\rho^4) , \tag{3.166}$$

where

$$R_{\mathfrak{h}}^2 \equiv \sqrt{\frac{9q_1(q_0 - 8/(3\breve{g}^2))}{2}} , \tag{3.167}$$

which is well defined as long as $q_1(q_0 - 8/(3\breve{g}^2)) > 0$ (in particular, $q_1 \neq 0$). We will assume that this condition holds. Then, rescaling the null coordinates as $u = R_{\mathfrak{h}}^2 u'$, $v' = R_{\mathfrak{h}}^2 v''$ the metric takes the same form we found above

$$d\breve{s}_{\mathfrak{h}}^2 = R_{\mathfrak{h}}^2 \left(2\rho^2 du' dv'' - 3q_1 du'^2 - \frac{d\rho^2}{\rho^2}\right) - R_{\mathfrak{h}}^2 d\Omega_{(3)}^2 , \tag{3.168}$$

which is that of $\mathrm{AdS}_3 \times S^3$ with radii equal to $R_{\mathfrak{h}}$. The fact that this near-horizon limit is the same as in the case of the pure dyonic string solutions, with no pp-wave [127] is somewhat surprising.

In this limit the dilaton takes a constant and finite value,

$$e^{\sqrt{2}\breve{\varphi}} = \frac{q_1}{2(q_0 - \frac{8}{3\breve{g}^2})} , \tag{3.169}$$

while the vectors are simply proportional to the left-invariant Maurer-Cartan 1-forms $\breve{A}^A = -\frac{1}{\tilde{g}}\, v^A{}_L$. Recalling the definition of the left-invariant Maurer-Cartan forms $V = v^A T_A = -u^{-1}du$ for the $\mathrm{SU}(2)$ group representative u and the $\mathfrak{su}(2)$ generators T_A, we conclude that the gauge fields are proportional to a pure gauge configuration, i.e. they describe a meron[8] field, analogous to the one found in [38]. Finally, in the $\rho \to 0$ limit the 3-form field strength $\breve{H}$ takes exactly the same form as in the $\rho \to \infty$ limit (3.164), but we should notice that the coordinates we are using in the AdS_3 are different.

Summarizing, we have found two solutions:

1. The first solution, which is asymptotically flat and has a regular horizon. Asymptotically it cannot be distinguished from the well-known dyonic string solution (plus pp-wave) that can be obtained by eliminating the instanton field. This behavior is similar to that of the colored black holes constructed in [29, 30, 83]. In the near-horizon limit it has an $\mathrm{AdS}_3 \times S^3$ metric with radius $R_{\mathfrak{h}}$ whose value, given in (3.167), does have a contribution from the instanton field.

2. The second solution is a globally regular metric that interpolates between two $\mathrm{AdS}_3 \times S^3$ solutions with radii R_∞ and R_h given, respectively, in (3.161) and (3.167).

3.3.4 Solutions of the SO(3) $\mathcal{N} = 2A, d = 6$ theory

Dualizing the 3-form field strength of the $\mathcal{N} = 2A^*, d = 6$ theory solutions we just obtained according to equation (2.110), we can obtain very similar solutions of the $\mathcal{N} = 2A, d = 6$ theory which will however present very different string-frame metrics[9] and possibly Kalb-Ramond field. We have

$$\tilde{H} = -\tfrac{1}{3} dv \wedge du \wedge dL_1^{-1} - \tfrac{3}{2}\rho^3 \partial_\rho \left[L_0 - \tfrac{2}{9g^2} \left(\rho + \frac{\lambda^2}{4}\rho^3 \right)^{-2} \right] \omega_3 \,. \tag{3.170}$$

Since, in this case, the 3- and 2-form field strengths are defined as

$$\tilde{H} = d\tilde{B} + \tilde{F}^A \wedge \tilde{A}^A + \tfrac{1}{3!}\tilde{g}\varepsilon_{ABC}\tilde{A}^A \wedge \tilde{A}^B \wedge \tilde{A}^C \,, \tag{3.171}$$

$$\tilde{F}^A = d\tilde{A}^A - \tfrac{1}{2}\tilde{g}\varepsilon^A{}_{BC}\tilde{A}^B \wedge \tilde{A}^C \,, \tag{3.172}$$

and the gauge fields are those of the BPS instanton

$$\tilde{A}^A = -\frac{1}{\tilde{g}} \frac{1}{1 + \frac{\lambda^2}{4}\rho^2} v^A{}_L \,, \tag{3.173}$$

we find that

[8] A meron is an Euclidean spacetime solution of the Yang-Mills field equations. It is a singular, localized, non selfdual solution with half unit of topological charge, concentrated at the point where the solution is singular.

[9] In the Einstein frame, the metric is clearly the same as in the $\mathcal{N} = 2A$ theory (3.148).

$$d\tilde{B} = -\tfrac{1}{3} dv \wedge du \wedge dL_1^{-1} + 3q_0\omega_3 \,, \tag{3.174}$$

and using the Euler coordinates as in (3.155), we obtain the 2-form field

$$\tilde{B} = -\tfrac{1}{3} L_1^{-1} dv \wedge du + \tfrac{3}{8} q_0 \cos\theta d\psi \wedge d\phi \,, \tag{3.175}$$

which has no non-Abelian contributions.

3.3.5 Solutions of the "SO(3)-gauged" $\mathcal{N} = 2B, d = 6$ theory

As we have already mentioned, there is no possible gauging in any conventional sense of the $\mathcal{N} = 2B, d = 6$ supergravity theory because it has no vector fields. However, it can be argued that, at least when the theory is compactified in a circle, a gauged $\mathcal{N} = 2B, d = 6$ supergravity theory exists, whose massless (in the 5-dimensional sense) sector is given by a gauged $\mathcal{N} = 2, d = 5$ theory related to the former by dimensional reduction in the Abelian case.

We have also stressed that the relation between the fields of two gauged supergravities is the same as in the ungauged case, as long as their gauge groups are identical. Then, we can use the formulas obtained in the dimensional reduction of the standard $\mathcal{N} = 2B, d = 6$ theory to ungauged $\mathcal{N} = 2, d = 5$ supergravity in order to uplift solutions of the $\mathrm{SO}(3)$-gauged 5-dimensional theory to this conjectured $\mathrm{SO}(3)$-gauged $\mathcal{N} = 2B, d = 6$ supergravity. We are going to apply this idea to the non-Abelian black-hole solution we have considered so far. Eliminating the BPST instanton from the solution, we obtain a solution of the standard, ungauged $\mathcal{N} = 2B, d = 6$ theory.

Thus, using equations (2.92,2.94,2.95), calling u and v the coordinates z and t and shifting $v' = v + 3a_0 u$, we get the following solution

$$\begin{aligned}
d\hat{s}^2 &= \left(\frac{2}{3L_1}\right) 2du \left\{ dv' - 3\left[(L_0 - a_0) - \tfrac{2}{9g^2}\left(\rho + \frac{\lambda^2}{4}\rho^3\right)^{-2} \right] du \right\} \\
&\quad - \left(\frac{2}{3L_1}\right)^{-1} \left(d\rho^2 + \rho^2 d\Omega^2_{(3)} \right) \,, \\
\hat{L}^r &= \delta^r{}_1 \,, \\
\hat{B}^1{}_{uv'} &= \tfrac{1}{3} L_1^{-1} \,, \\
\hat{B}^A{}_{\mu u} dx^\mu &= -\frac{1}{2\sqrt{6}g}\, v^A{}_L \,.
\end{aligned} \tag{3.176}$$

This solution has the typical form of a solution describing the superposition of a selfdual string with charge $\sim q_1$ and a pp-wave with momentum $\sim q_0$, but there is a non-conventional non-Abelian contribution to this wave which can be interpreted as an instanton expressed in 2-form variables. This non-Abelian contribution, as in the previous cases, falls off too fast at infinity to give a contribution to the wave's momentum and, therefore, the solution has the same asymptotic behavior as the standard solution

with no non-Abelian contribution. It also seems to be regular everywhere as long as $L_1 \neq 0$.

In this solution the string charge and the pp-wave momentum are independent and can be set to zero independently. If both are set to zero, a non-standard, purely non-Abelian pp-wave solution is found.

Asymptotic limit

There are two possible choices of the integration constant a_1 which give physically inequivalent solutions, while a_0 has disappeared from the solution:

1. $a_1 = 1$ gives an asymptotically ($\rho \to \infty$ limit) flat metric with the string-plus-wave interpretation mentioned above;

2. $a_1 = 0$ gives a metric that, with the usual rescaling of u and v', takes the form

$$d\hat{s}^2 = R^2 \left\{ \left[2du'dv''\rho^2 - 3\left(q_0 - \tfrac{2}{9g^2}(1+\tfrac{\lambda^2}{4}\rho^2)^{-2}\right) du' - \frac{d\rho^2}{\rho^2} \right] - d\Omega^2_{(3)} \right\} . \quad (3.177)$$

In the $\rho \to \infty$ limit this metric is that of $\mathrm{AdS}_3 \times \mathrm{S}^3$ with radii

$$R^2 = 3q_1/2 \,, \quad (3.178)$$

although, for all finite values of ρ, it differs from it, unless the non-Abelian contribution is eliminated.

Near-horizon limit

For both the solutions $a_1 = 1, 0$, the same metric in the $\rho \to 0$ (near-horizon) limit is obtained: an $\mathrm{AdS}_3 \times \mathrm{S}^3$ whose radii R are again given by (3.178). The difference between this metric and the one obtained in the $\rho \to \infty$ limit for the second solution is that in the near-horizon limit there is a non-Abelian contribution in the g_{uu} component, although this does not affect the value of the radii of the factor spaces.

3.4 Solutions via dimensional reduction: the SU(2) FI-gauged theory

As mentioned previously in section 3.3.2, the supersymmetric solutions of the gauged $\mathcal{N} = 2A, d = 6$ theory with Fayet-Iliopoulos (FI) terms were classified in [38], and some interesting examples were constructed. The results of chapter 2 can be exploited to dimensionally reduce them to 5 and 4 dimensions.

This procedure provides an alternative way to construct solutions of a given theory, when there are theories related to the one we are interested in by these mechanisms

and there are known solutions of them which, if they are to be dimensionally reduced, have enough isometries. It turned out to be particularly fruitful for the SU(2) Fayet-Iliopoulos gauged theory, whose equations of motion (see section 3.2) have proven to be complicated.

As already explained in chapter 2, $\mathcal{N} = 2, d = 4$ supergravity theories are directly related by dimensional reduction or oxidation to other supergravity theories with 8 supercharges.[10] These only exist in $d \leq 6$ and, to the best of our knowledge, theories with SU(2) FI gaugings have only been studied in $\mathcal{N} = (2,0), d = 6$ supergravity coupled to one tensor multiplet and a triplet of vector multiplets in [38]. This theory is unique and describes a truncation of the Heterotic String compactified on T^4. We named it $\mathcal{N} = 2A, d = 6$ in chapter 2, and it includes the metric $\tilde{g}_{\tilde{\mu}\tilde{\nu}}$, a complete (Kalb-Ramond) 2-form $\tilde{B}_{\tilde{\mu}\tilde{\nu}}$, a real scalar (dilaton) $\tilde{\varphi}$ and the three vector fields $\tilde{A}^A_{\tilde{\mu}}$, $A = 1,2,3$. The FI term induces a simple potential for the dilaton, and the action takes the form [2,38]

$$\tilde{S} = \int d^6\tilde{x}\sqrt{|\tilde{g}|}\left\{\tilde{R} + \tfrac{1}{2}(\partial\tilde{\varphi})^2 + \tfrac{1}{3}e^{\sqrt{2}\tilde{\varphi}}\tilde{H}^2 - e^{\tilde{\varphi}/\sqrt{2}}\tilde{F}^i\tilde{F}^i - \tfrac{3}{2}\,g_6^2\,e^{-\tilde{\varphi}/\sqrt{2}}\right\}, \tag{3.179}$$

where g_6 is the 6-dimensional coupling constant.

The results in chapter 2 can be exploited to dimensionally reduce the 6-dimensional solutions found in [38] to solutions of SU(2) FI-gauged $\mathcal{N} = 2, d = 5$ supergravity since the relation between the 6- and 5-dimensional fields of the gauged theories is the same as in the ungauged case, as long as the gauge groups are the same in both theories.

The 5-dimensional model obtained by dimensional reduction is completely characterized by the symmetric tensor $C_{0rs} = \frac{1}{3!}\eta_{rs}$, $r,s = 1,\ldots,5$. The bosonic fields in this theory are the metric $\hat{g}_{\hat{\mu}\hat{\nu}}$, the 6 gauge fields $\hat{A}^I{}_{\hat{\mu}}$, $I = 0,\cdots,5$, 5 of which, $\hat{A}^r{}_{\hat{\mu}}$, correspond to 5 vector multiplets[11], and 5 scalar fields. Due to the reduction procedure, $\hat{A}^{0,1,2}{}_{\hat{\mu}}$ are Abelian fields, while $\hat{A}^{A+2}{}_{\hat{\mu}}$ are the three SU(2) gauge fields. The physical scalars $\hat{\phi}^r$ are encoded in the scalar functions $\hat{h}^I$, constrained by the fundamental relation of real special geometry, which in this case reads

$$C_{IJK}\hat{h}^I\hat{h}^J\hat{h}^K = \tfrac{1}{2}\hat{h}^0\eta_{rs}\hat{h}^r\hat{h}^s = 1\,. \tag{3.180}$$

A convenient parametrization is $\hat{\phi}^r = \hat{h}^r$ so $\hat{h}^0 = 2/(\phi\eta\phi) \equiv \hat{\phi}^0$, where $\phi\eta\phi \equiv \hat{\phi}^r\eta_{rs}\hat{\phi}^s$. In this parametrization, the last 3 scalars $\hat{\phi}^{A+2}$ transform in the adjoint representation of SU(2) and the action of the theory can be written in the compact form

$$\begin{aligned}\hat{S} = \int d^5\hat{x}\sqrt{\hat{g}}\,\Big\{&\hat{R} + \tfrac{3}{2}\hat{a}_{IJ}\hat{\mathfrak{D}}_{\hat{\mu}}\hat{\phi}^I\hat{\mathfrak{D}}^{\hat{\mu}}\hat{\phi}^J - \tfrac{1}{4}\hat{a}_{IJ}\hat{F}^{I\,\hat{\mu}\hat{\nu}}\hat{F}^J{}_{\hat{\mu}\hat{\nu}} - 18\,g_5^2\left(\hat{\phi}^0\right)^{-1}\\ &+ \tfrac{1}{24\sqrt{3}}\eta_{rs}\frac{\hat{\varepsilon}^{\hat{\mu}\hat{\nu}\hat{\rho}\hat{\sigma}\hat{\alpha}}}{\sqrt{\hat{g}}}\hat{A}^0{}_{\hat{\mu}}\hat{F}^r{}_{\hat{\nu}\hat{\rho}}\hat{F}^s{}_{\hat{\sigma}\hat{\alpha}}\Big\}\,,\end{aligned} \tag{3.181}$$

[10] The relation with theories with different number of supercharges must necessarily involve truncations and constraints on the solutions and we will not consider them here.

[11] The reduction of the KR 2-form gives just 2 vector fields.

where

$$\hat{\mathfrak{D}}_{\hat{\mu}}\hat{\phi}^{0,1,2} = \partial_{\hat{\mu}}\hat{\phi}^{0,1,2}\,, \qquad \hat{\mathfrak{D}}_{\hat{\mu}}\hat{\phi}^{A+2} = \partial_{\hat{\mu}}\hat{\phi}^{A+2} - g_5\epsilon^A{}_{BC}\hat{A}^B{}_{\hat{\mu}}\hat{\phi}^{C+2}\,. \tag{3.182}$$

The non-vanishing components of the metric a_{IJ} are

$$a_{00} = \tfrac{1}{12}(\phi\eta\phi)\,, \qquad a_{rs} = \frac{-2\eta_{rs}(\phi\eta\phi) + 4\eta_{rr'}\hat{\phi}^r\eta_{ss'}\hat{\phi}^r}{3(\phi\eta\phi)^2}\,. \tag{3.183}$$

Observe that, as in (3.156), the 6- and 5-dimensional gauge coupling constants are related by

$$g_5 = \frac{1}{\sqrt{12}}g_6\,. \tag{3.184}$$

As explained in section 2.2, the model that arises in the dimensional reduction of the above 5-dimensional model is the ST$[2,6]$ model, which is characterized by the prepotential

$$F = -\tfrac{1}{3!}\frac{d_{ijk}\mathcal{X}^i\mathcal{X}^j\mathcal{X}^k}{\mathcal{X}^0}\,, \tag{3.185}$$

where $i = 1, 2\cdots, 6$ labels the 6 vector multiplets and where the fully symmetric tensor d_{ijk} has as only non-vanishing components,

$$d_{1\alpha\beta} = \eta_{\alpha\beta}\,, \quad \text{where} \quad \eta_{\alpha\beta} = \textit{diag}(1,-1,\ldots,-1) \quad \text{and} \quad \alpha,\beta = 2,\cdots,6\,. \tag{3.186}$$

The 6 complex scalars parametrize the coset space

$$\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)} \times \frac{\mathrm{SO}(2,5)}{\mathrm{SO}(2)\times\mathrm{SO}(5)}\,, \tag{3.187}$$

and the group SO(3) acts in the adjoint on the coordinates $\alpha = 4,5,6$ that we are denoting with $A, B, \ldots$ indexes; these are the directions which are gauged. With our conventions, the $\frac{\mathrm{SL}(2,\mathbb{R})}{\mathrm{SO}(2)}$ factor is parametrized by the scalar Z^1 which is often called the axidilaton field since its real and imaginary parts are, respectively, an axion and a dilaton field.

The scalar potential can be computed using the general formula (1.6), but we obtained it easily by dimensional reduction, using the relation between 5- and 4-dimensional fields that we reported in section 2.2. It takes the extremely simple form

$$\mathbf{V}(Z,Z^*) = -\tfrac{3}{4}\,g_4^2\,\frac{1}{\Im\mathfrak{m}Z^1}\,, \tag{3.188}$$

where now the 5- and 4- coupling constants are related by

$$g_4 = -\sqrt{24}\,g_5 = -\sqrt{2}g_6\,, \tag{3.189}$$

where we used (3.184). It is proportional to the exponential of the dilaton field and, therefore, negative definite.

3.4.1 Rules for dimensional reduction

In the present section no new results are introduced. We are just reporting and summarizing the results we proposed in chapter 2, specifying them to the case under study.

6 → 5

Equations (2.102,2.105,2.107,2.109) are the result of the procedure of dimensional oxidation from the $ST[2,n]$ 5-dimensional model to $\mathcal{N} = 2A$ 6-dimensional theory. We are here reporting the inverse of these rules, specified for the model we are considering. They have been used to reduce the solutions presented in [38].

If we perform the dimensional reduction along the coordinate z, the 5-dimensional fields of can be expressed in terms of the 6-dimensional fields ones as follows:

$$
\begin{aligned}
&\hat{g}_{\hat{\mu}\hat{\nu}} = \tilde{g}_{\hat{\mu}\hat{\nu}} \left|\tilde{g}_{\underline{zz}}\right|^{\frac{1}{3}} + \tilde{g}_{\hat{\mu}\underline{z}}\, \tilde{g}_{\hat{\nu}\underline{z}} \left|\tilde{g}_{\underline{zz}}\right|^{-\frac{2}{3}} ,\\
&\hat{h}^0 = e^{\frac{\tilde{\varphi}}{\sqrt{2}}} \left|\tilde{g}_{\underline{zz}}\right|^{\frac{1}{3}} ,\\
&\hat{h}^1 = \left|\tilde{g}_{\underline{zz}}\right|^{-\frac{2}{3}} \left(1 + \tilde{A}^i{}_{\underline{z}} \tilde{A}^i{}_{\underline{z}}\right) + \frac{1}{2} e^{-\frac{\tilde{\varphi}}{\sqrt{2}}} \left|\tilde{g}_{\underline{zz}}\right|^{\frac{1}{3}} ,\\
&\hat{h}^2 = \left|\tilde{g}_{\underline{zz}}\right|^{-\frac{2}{3}} \left(1 - \tilde{A}^i{}_{\underline{z}} \tilde{A}^i{}_{\underline{z}}\right) - \frac{1}{2} e^{-\frac{\tilde{\varphi}}{\sqrt{2}}} \left|\tilde{g}_{\underline{zz}}\right|^{\frac{1}{3}} ,\\
&\hat{h}^{i+2} = -2 \left|\tilde{g}_{\underline{zz}}\right|^{-\frac{2}{3}} \tilde{A}^i{}_{\underline{z}} ,\\
&\hat{F}^0{}_{\hat{a}\hat{b}} = -4\sqrt{3} \left|\tilde{g}_{\underline{zz}}\right|^{\frac{2}{3}} e^{\sqrt{2}\tilde{\varphi}}\, \epsilon_{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}}\, \tilde{H}^{\hat{c}\hat{d}\hat{e}} ,\\
&\hat{F}^1{}_{\hat{\mu}\hat{\nu}} = \sqrt{3}\, \tilde{H}_{\hat{\mu}\hat{\nu}\underline{z}} + 4\sqrt{3}\, \tilde{A}^i{}_{\underline{z}} \tilde{F}^i{}_{\hat{\mu}\hat{\nu}} + 2\sqrt{3}\, \partial_{[\hat{\mu}} \left[\frac{\tilde{g}_{\hat{\nu}]\underline{z}}}{\tilde{g}_{\underline{zz}}} \left(\tilde{A}^i{}_{\underline{z}} \tilde{A}^i{}_{\underline{z}} + 1\right)\right] ,\\
&\hat{F}^2{}_{\hat{\mu}\hat{\nu}} = -\sqrt{3}\, \tilde{H}_{\hat{\mu}\hat{\nu}\underline{z}} - 4\sqrt{3}\, \tilde{A}^i{}_{\underline{z}} \tilde{F}^i{}_{\hat{\mu}\hat{\nu}} - 2\sqrt{3}\, \partial_{[\hat{\mu}} \left[\frac{\tilde{g}_{\hat{\nu}]\underline{z}}}{\tilde{g}_{\underline{zz}}} \left(\tilde{A}^i{}_{\underline{z}} \tilde{A}^i{}_{\underline{z}} - 1\right)\right] ,\\
&\hat{A}^{i+2}{}_{\hat{\mu}} = \sqrt{12}\, \tilde{A}^i{}_{\hat{\mu}} + 2\sqrt{3}\, \frac{\tilde{g}_{\hat{\mu}\underline{z}}}{\tilde{g}_{\underline{zz}}} \tilde{A}^i{}_{\underline{z}} .
\end{aligned}
\tag{3.190}
$$

5 → 4

Analogously, we are here reporting the results of [30] and making them ready to be used in this case.

If we perform the dimensional reduction along the coordinate y, the 4-dimensional fields can be expressed in terms of the 5-dimensional ones as follows:

$$
\begin{aligned}
g_{\mu\nu} &= \left|\hat{g}_{\underline{yy}}\right|^{\frac{1}{2}} \left[\hat{g}_{\mu\nu} - \frac{\hat{g}_{\mu\underline{y}}\,\hat{g}_{\nu\underline{y}}}{\hat{g}_{\underline{yy}}}\right] , \\
Z^i &= \frac{1}{\sqrt{3}}\hat{A}^{i-1}{}_{\underline{y}} + i\left|\hat{g}_{\underline{yy}}\right|^{\frac{1}{2}} \hat{h}^{i-1} , \\
A^0{}_\mu &= \frac{1}{2\sqrt{2}}\frac{\hat{g}_{\mu\underline{y}}}{\hat{g}_{\underline{yy}}} , \\
A^i{}_\mu &= -\frac{1}{2\sqrt{6}}\left[\hat{A}^{i-1}{}_\mu - \hat{A}^{i-1}{}_{\underline{y}}\frac{\hat{g}_{\mu\underline{y}}}{\hat{g}_{\underline{yy}}}\right] .
\end{aligned}
\tag{3.191}
$$

3.4.2 An Einstein universe

The first solution proposed in [38] that we are going to reduce is, perhaps, the simplest: it is a generalization of the solution with geometry $\mathbb{M}_4\times\mathrm{S}^2$ found by Salam in Sezgin in [128] that has $\mathbb{M}_3\times\mathrm{S}^3$ metric, a constant dilaton field whose value is proportional to the square of the radius of the S^3 and to the square of the coupling constant, a meronic gauge field and vanishing 2-form. The non-vanishing field are given by

$$
\begin{aligned}
d\tilde{s}^2 &= dt^2 - dz^2 - dy^2 - a^2 d\Omega^2_{(3)} , \\
e^{\frac{\tilde{\varphi}}{\sqrt{2}}} &= \frac{a^2\, g_6^2}{2} , \\
\tilde{A}^A &= -\frac{1}{2g_6}\sigma^A ,
\end{aligned}
\tag{3.192}
$$

where the σ^A are the left-invariant Maurer-Cartan 1-forms satisfying $d\sigma^A = \frac{1}{2}\varepsilon^A_{BC}\sigma^B \wedge \sigma^C$, $d\Omega^2_{(3)} = \frac{1}{4}\sigma^A\sigma^A$ and a is a constant parameter.

Reducing along the z coordinate using (3.190), we get a solution of the 5-dimensional theory with the following non-vanishing fields,

$$
\begin{aligned}
d\hat{s}^2 &= dt^2 - dy^2 - a^2 d\Omega^2_{(3)} , \\
\hat{h}^0 &= 6a^2 g_5^2 , \\
\hat{h}^1 &= 1 + \frac{1}{12a^2 g_5^2} , \\
\hat{h}^2 &= 1 - \frac{1}{12a^2 g_5^2} , \\
\hat{A}^{A+2} &= -\frac{1}{2g_5}\sigma^A .
\end{aligned}
\tag{3.193}
$$

This solution belongs to the same class as its 6-dimensional parent: it has constant scalars and a meronic gauge field that support a $\mathbb{M}_2\times\mathrm{S}^3$ geometry.

Reducing further along the y coordinate thanks to (3.191), we obtain a 4-dimensional solution of the same kind with non-vanishing fields

$$
\begin{aligned}
ds^2 &= dt^2 - a^2 d\Omega^2_{(3)}\,, \\
Z^1 &= \frac{i}{4}\, a^2 g_4^2\,, \\
Z^2 &= i\left(1 + \frac{2}{a^2 g_4^2}\right)\,, \\
Z^3 &= i\left(1 - \frac{2}{a^2 g_4^2}\right)\,, \\
A^{A+3} &= -\frac{1}{2\, g_4}\sigma^A\,.
\end{aligned}
\tag{3.194}
$$

The metric of this solution describes a static Einstein universe.

3.4.3 $\mathbf{AdS}_n \times \mathbf{S}^m$ solutions

The second solution we have considered is the *dyomeronic black string* of [38], which corresponds to a black string lying along the z direction with electric and magnetic 3-form and a meronic gauge field in the 4-dimensional transverse space. Its non-vanishing fields are given by

$$
\begin{aligned}
d\tilde{s}^2 &= \frac{r}{\sqrt{Q_1 + \frac{Q_2}{r^2}}}\left(dt^2 - dz^2\right) - \frac{\sqrt{Q_1 + \frac{Q_2}{r^2}}}{r}\left(dr^2 + a^2 r^2\, d\Omega^2_{(3)}\right)\,, \\
e^{\sqrt{2}\tilde{\varphi}} &= \frac{a^4 g_6^4}{4\,(1-a^2)^2} r^2 \left(Q_1 + \frac{Q_2}{r^2}\right)\,, \\
\tilde{A}^i &= -\frac{1-a^2}{2 g_6}\sigma^i\,, \\
\tilde{H} &= \frac{1-a^2}{g_6^2}\left[\frac{a}{4} r\, \sigma^1 \wedge \sigma^2 \wedge \sigma^3 + \frac{2Q_2}{a^2}\frac{1}{r^3\left(Q_1 + \frac{Q_2}{r^2}\right)^2}\, dt \wedge dr \wedge dz\right]\,,
\end{aligned}
\tag{3.195}
$$

where the parameter a satisfies $a^2 < 1$. This solution is not asymptotically AdS (nor some other known vacuum solution) but has a horizon at $r = 0$ and in the near-horizon limit $r \to 0$ the metric is of the form $\mathrm{AdS}_3 \times \mathrm{S}^3$ where the two factors have different radii. Since this limit is equivalent to setting $Q_1 = 0$, the $\mathrm{AdS}_3 \times \mathrm{S}^3$ near-horizon limit is a supersymmetric solution as well.

If we reduce along the z direction, the following 5-dimensional solution is obtained

$$
\begin{aligned}
d\hat{s}^2 &= r^{\frac{4}{3}}\left(Q_1 + \frac{Q_2}{r^2}\right)^{-\frac{2}{3}} dt^2 - r^{-\frac{2}{3}}\left(Q_1 + \frac{Q_2}{r^2}\right)^{\frac{1}{3}}\left(dr^2 + a^2 r^2 d\Omega^2_{(3)}\right)\,, \\
\hat{h}^0 &= \frac{6\, a^2 g_5^2}{1-a^2}\, r^{\frac{4}{3}}\left(Q_1 + \frac{Q_2}{r^2}\right)^{\frac{1}{3}}\,,
\end{aligned}
\tag{3.196}
$$

$$
\begin{aligned}
\hat{h}^1 &= r^{-\frac{2}{3}} \left(Q_1 + \frac{Q_2}{r^2}\right)^{\frac{1}{3}} \left[1 + \frac{1-a^2}{12\,a^2 g_5^2 \left(Q_1 + \frac{Q_2}{r^2}\right)}\right] , \\
\hat{h}^2 &= r^{-\frac{2}{3}} \left(Q_1 + \frac{Q_2}{r^2}\right)^{\frac{1}{3}} \left[1 - \frac{1-a^2}{12\,a^2 g_5^2 \left(Q_1 + \frac{Q_2}{r^2}\right)}\right] , \\
\hat{F}^0 &= 12^2\sqrt{3}\, g_5^2 \frac{a^2}{1-a^2}\, r^{\frac{5}{2}} \left(Q_1 + \frac{Q_2}{r^2}\right)^{-\frac{1}{4}} dt \wedge dr \,, \\
\hat{F}^1 &= -\hat{F}^2 = \frac{1-a^2}{2\sqrt{3}\,a^2 g_5^2} \frac{Q_2}{r^3} \left(Q_1 + \frac{Q_2}{r^2}\right)^{-2} dt \wedge dr \,, \\
\hat{A}^{A+2} &= -\frac{1-a^2}{2\,g_5}\, \sigma^A \,.
\end{aligned}
$$

This solution is singular at $r = 0$ and it is not asymptotically AdS (nor some other known vacuum solution). If we reduce it again along the coordinate ϕ, defined by $d\Omega^2_{(3)} = \frac{1}{4}\left[(d\phi + \cos\theta\, d\psi)^2 + d\theta^2 + \sin^2\theta\, d\psi^2\right]$, we get a 4-dimensional solution which we will refrain from writing explicitly because it has the same problems as the 5-dimensional one.

Of course, we could have performed the reduction from 6 to 5 dimensions with this coordinate ϕ. In this way, we obtain a 5-dimensional solution which presents similar properties to the 6-dimensional case, namely

$$
\begin{aligned}
d\hat{s}^2 &= \left(\frac{a}{2}\right)^{\frac{2}{3}} r^{\frac{4}{3}} \left(Q_1 + \frac{Q_2}{r^2}\right)^{-\frac{1}{3}} \left(dt^2 - dz^2\right) - \left(\frac{a}{2}\right)^{\frac{2}{3}} r^{-\frac{2}{3}} \left(Q_1 + \frac{Q_2}{r^2}\right)^{\frac{2}{3}} dr^2 \\
&\quad - \left(\frac{a}{2}\right)^{\frac{8}{3}} r^{\frac{4}{3}} \left(Q_1 + \frac{Q_2}{r^2}\right)^{\frac{2}{3}} d\Omega^2_{(2)} \,, \\
\hat{h}^0 &= \frac{3 \cdot 2^{\frac{1}{3}} a^{\frac{8}{3}} g_5^2}{1-a^2}\, r^{\frac{4}{3}} \left(Q_1 + \frac{Q_2}{r^2}\right)^{\frac{2}{3}} , \\
\hat{h}^1 &= \left(\frac{2}{a}\right)^{\frac{4}{3}} r^{-\frac{2}{3}} \left(Q_1 + \frac{Q_2}{r^2}\right)^{-\frac{1}{3}} \left[1 + \frac{\left(a^2-1\right)\left(a^2-2\right)}{4 \cdot 12\, g_5^2}\right] , \\
\hat{h}^2 &= \left(\frac{2}{a}\right)^{\frac{4}{3}} r^{-\frac{2}{3}} \left(Q_1 + \frac{Q_2}{r^2}\right)^{-\frac{1}{3}} \left[1 - \frac{\left(a^2-1\right)\left(a^2-2\right)}{4 \cdot 12\, g_5^2}\right] , \\
\hat{h}^{A+2} &= -\frac{1-a^2}{2^{\frac{2}{3}} \cdot 3\, a^{\frac{4}{3}} g_5}\, r^{-\frac{2}{3}} \left(Q_1 + \frac{Q_2}{r^2}\right)^{-\frac{1}{3}} \frac{x^A}{r} \,,
\end{aligned}
\tag{3.197}
$$

$$
\begin{aligned}
\hat{F}^0 &= \frac{3^{\frac{5}{2}} a^6 g_5^2}{1-a^2}\, Q_2\, r^{\frac{3}{2}} \left(Q_1 + \frac{Q_2}{r^2}\right)^{-\frac{1}{4}} \cos\theta\, d\theta \wedge d\psi\,, \\
\hat{F}^1 &= -\,\hat{F}^2 = \left[\frac{\left(1-a^2\right) a}{16\sqrt{3}\, g_5^2}\, r - 2\sqrt{3}\right] \sin\theta\, d\theta \wedge d\psi\,, \\
\hat{A}^3 &= \frac{1-a^2}{2\, g_5} \left(-\sin\psi\, d\theta + \cos\theta \sin\theta \cos\psi\, d\psi\right)\,, \\
\hat{A}^4 &= \frac{1-a^2}{2\, g_5} \left(\cos\psi\, d\theta + \cos\theta \sin\theta \sin\psi\, d\psi\right)\,, \\
\hat{A}^5 &= -\frac{1-a^2}{2\, g_5} \cos\theta \left(1 + \cos\theta\right) d\psi\,,
\end{aligned}
$$

where we have introduced 3 Cartesian coordinates x^A related to the spherical coordinates r, θ, ψ in the standard way.

This solution is regular in the $r \to 0$ limit, where the metric becomes that of the product $\mathrm{AdS}_3 \times \mathrm{S}^2$ with different radii:

$$
d\hat{s}^2 \to \left(\frac{a}{2}\right)^{2/3} \frac{Q_2^{2/3}}{\rho^2} \left(dt^2 - dz^2 - d\rho^2\right) - \left(\frac{a}{2}\right)^{8/3} Q_2^{\frac{2}{3}}\, d\Omega_2^2\,. \tag{3.198}
$$

where $\rho \equiv Q_2^{1/2}/r$. Again, it is not asymptotically AdS.

The $r \to 0$ limit of the complete solution coincides with the solution that one gets by setting $Q_1 = 0$. Thus, there is a globally regular $\mathrm{AdS}_3 \times \mathrm{S}^2$ solution in this theory. It could have been obtained directly by dimensional reduction from the 6-dimensional $\mathrm{AdS}_3 \times \mathrm{S}^3$ solution.

Further reduction along the z coordinate would lead to the same problematic 4-dimensional solution mentioned above.

There is, however, another possibility inspired by the results of [129], where the relation between $\mathrm{AdS}_n \times \mathrm{S}^m$ vacua of the 4-, 5- and 6-dimensional theories with 8 supercharges was studied.

The main point resides in the observation that, just as S^3 can be seen as a $\mathrm{U}(1)$ fibration over S^2, so that S^2 can be obtained by dimensional reduction along that fiber[12], AdS_3 can be seen as a $\mathrm{U}(1)$ fibration over AdS_2 and, by dimensional reduction along that fiber, AdS_2 arises.
Thus, if we had used the $\mathrm{U}(1)$ fiber of the AdS_3 in the $\mathrm{AdS}_3 \times \mathrm{S}^3$ solution -instead of using the coordinate z along which the 6-dimensional string lies- to perform the dimensional reduction, we would have obtained an $\mathrm{AdS}_2 \times \mathrm{S}^3$ solution in 5 dimensions and similarly an $\mathrm{AdS}_2 \times \mathrm{S}^2$ solution in 4 dimensions.

An even more general path is possible: the two $\mathrm{U}(1)$ fibers of the 6-dimensional solution can be rotated among themselves and the dimensional reduction can proceed along one of the rotated fibers. As in the ungauged case studied in [129], the result would be

[12]This is what we have done in section 3.4.3 to go from the $\mathrm{AdS}_3 \times \mathrm{S}^3$ to the $\mathrm{AdS}_3 \times \mathrm{S}^2$ solution.

a solution describing the near-horizon geometry of the BMPV black hole, where the remaining $\mathrm{U}(1)$ would be non-trivially fibered over $\mathrm{AdS}_2 \times \mathrm{S}^2$. This latter space is obtained in 4 dimensions after dimensional reduction along the remaining fiber.

The main difference with the ungauged case, apart from the presence of non-trivial $\mathrm{SU}(2)$ gauge field, is the difference between the radii of the two factors of these metrics.

Carrying out these alternative dimensional reductions is straightforward, albeit quite lengthy and involved, due to the necessity to rewrite the 6-dimensional solution in different coordinates. We don't report it here explicitly, because its main features have already been pointed out.

Conclusions

Exploring the space of the supersymmetric solutions of a supergravity theory is one of the most elementary steps one can take to get a more complete understanding of its structure, providing information about the possible vacua and some of the solitonic objects that can exist on it.

In this thesis, we have been exploring some so far disregarded sectors of the space of solutions of $\mathcal{N} = 2$ gauged supergravity theory, in 4 and 5 dimensions and of $\mathcal{N} = (2, 0)$ in 6[13]. In particular,

- we considered a non-homogeneous deformation of the stu model of 4-dimensional supergravity and computed the symplectic embedding of the electric-magnetic duality algebra. We then focused on a particular FI gauging of this model, that leads to a scalar potential with two AdS critical points, a supersymmetric one, and another that breaks supersymmetry and that exists only when the deformation parameter lies within a specific range. We wrote down the attractor equations for this model, and constructed an explicit BPS black hole solution that interpolates between this attractor geometry and the supersymmetric AdS vacuum at infinity. Various physical properties of this solution were also discussed;

- we applied the method developed in [4] to the SU(2)-FI gauged theory in 4 dimensions, for which no solutions were known. We discussed how this kind of gauging must be associated with an SU(2) gauging of the special Kähler manifold. We showed that no maximally supersymmetric solutions exist in these theories. We chose a particularly simple model, the $\overline{\mathbb{CP}}^3$ model, that admits an SU(2) gauging and implemented various Ansätze to solve the equations; among the new solutions we found, interesting examples present an $\mathrm{AdS}_2 \times \mathrm{S}^2$ and a $\mathbb{R} \times \mathbb{H}^3$ geometry;

[13] In none of these theories we have considered hypermultiplets. Only vector multiplets and, in 6 dimensions, tensor multiplets are involved in the models we have treated.

- we have found a very interesting relation between two families of models of $\mathcal{N} = (2,0), d = 6$ supergravity that can be used to transform solutions of one of them admitting one isometry into solutions of the other. The relation is based on the fact that they reduce to the same family of models of $\mathcal{N} = 2, d = 5$ supergravity, a fact that we have used to construct new 6-dimensional supersymmetric non-Abelian solutions by uplifting a known 5-dimensional non-Abelian black hole solution [30]. As in the 5 and 4-dimensional cases, the non-Abelian fields do not contribute to any of the quantities that can be measured at infinity, like the mass, but they do modify the near-horizon geometry, with a negative contribution to the entropy. This means that, for the same asymptotic data there are several black-body configurations with different entropies and the non-Abelian one, having the least entropy, should be unstable.

- we exploited the relation between 6, 5 and 4-dimensional theories to find more solutions of the SU(2)-FI gauged model, in 4 and 5 dimensions, upon reducing a couple of known 6-dimensional solution [38]. We have found solutions whose geometry is of the form $\mathbb{M}_m \times \mathrm{S}^3$ in 4 and 5 dimensions, descending from a 6-dimensional metric of the same kind and an $\mathrm{AdS}_3 \times \mathrm{S}^2$ solution in 5 dimensions. We proposed a method to originate $\mathrm{AdS}_m \times \mathrm{S}^n$ geometries in 4 and 5 dimensions, reducing the same 6-dimensional solution in a different way.

We are still far from understanding the underlying general structure of the solutions of gauged supergravity (if there is any). It is therefore useful to provide new examples, possibly in different and unexplored models. There are still many sectors of the space of supersymmetric solutions of $\mathcal{N} = 2$ supergravity theories for which no, or few, solutions are known.

A natural question is whether there also exist black holes in this theories that asymptotically yield the non-BPS vacuum. A step in this direction has been taken in [36]. It would also be interesting to investigate solutions of FI-gauged supergravity coupled to hypermultiplets, since there is only one known example, in [130].

Non-Abelian gaugings of the vector multiplets' sector are very little known, especially in relation to the existence and properties of regular black hole solutions, of the related attractor mechanism, of their supersymmetry-preserving features. Moreover, a long-standing problem that remains unsolved as yet is the microscopical interpretation of the entropy of all the black objects with non-Abelian field. We believe that the work presented here will help to find the embedding of these solutions in a string theory, providing the first step to solve it.

As far as the SU(2)-FI gauged theory is concerned, we found a non-maximally supersymmetric solution that can be interpreted as a deformation of the maximally supersymmetric vacua of the ungauged theory, the $\mathrm{AdS}_2 \times \mathrm{S}^2$ solution with its 5-dimensional origin. The existence of deformed versions of the rest of the maximally supersymmetric vacua of $\mathcal{N} = 2, d = 5$ supergravity (Hpp-waves and Gödel spacetimes) seems very

likely. It may be possible to obtain them from the above-mentioned solutions by different limiting procedures. On the other hand, it would be interesting to find complete black-hole and black-string solutions whose near-horizon geometries were precisely the $\mathrm{AdS}_m \times \mathrm{S}^n$ solutions we have discussed.

Finally, another still unexplored sector of $\mathcal{N} = 2, d = 4$ supergravity is the one combining an $\mathrm{U}(1)$-FI gauging to non-Abelian fields and to the gauging of a non-Abelian subgroup of the isometry group of the special Kähler manifold. This setting should provide asymptotic AdS solutions involving non-Abelian fields.

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